

# lecture 4: codebooks and kernels

## deep learning for vision

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# outline

**bag of words**

**codebooks**

**beyond codebooks**

**pyramid matching**

**discussion**

# bag of words

# image matching

- so far, we have a representation that is very robust in matching different views of the same object or scene—same **instance**—to be used e.g. for **retrieval**
- the same representation can be used in matching views of different instances of the same category—same **class**—to be used e.g. for **classification** or **detection**
- main differences

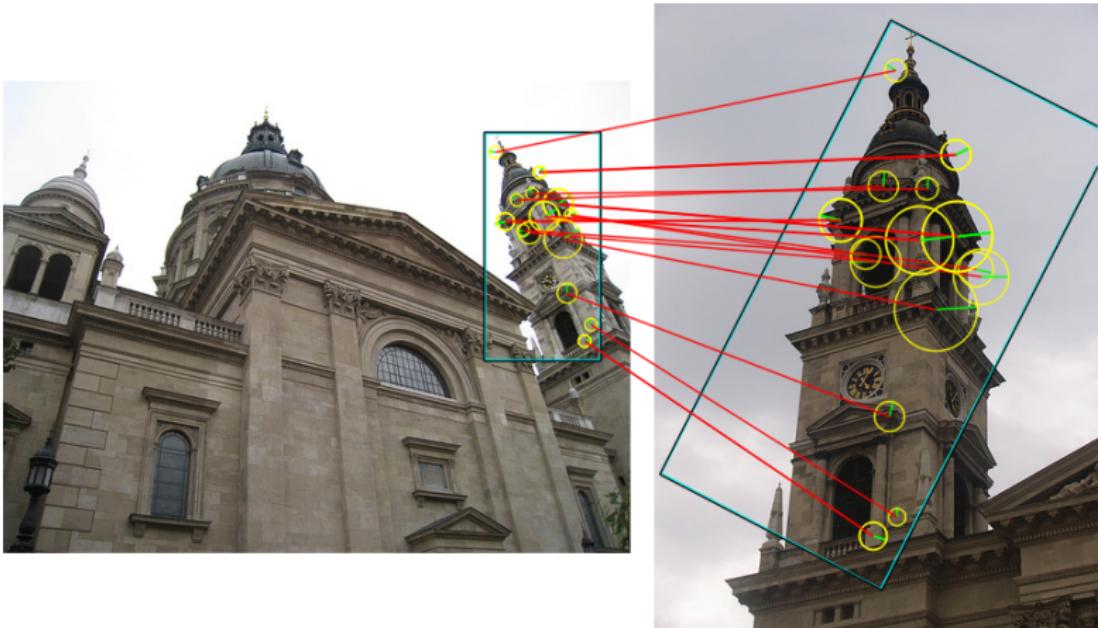
|             | instance | class    |
|-------------|----------|----------|
| features    | sparse   | dense    |
| descriptors |          | same     |
| vocabulary  | fine     | coarse   |
| geometry    | rigid    | flexible |

# image matching

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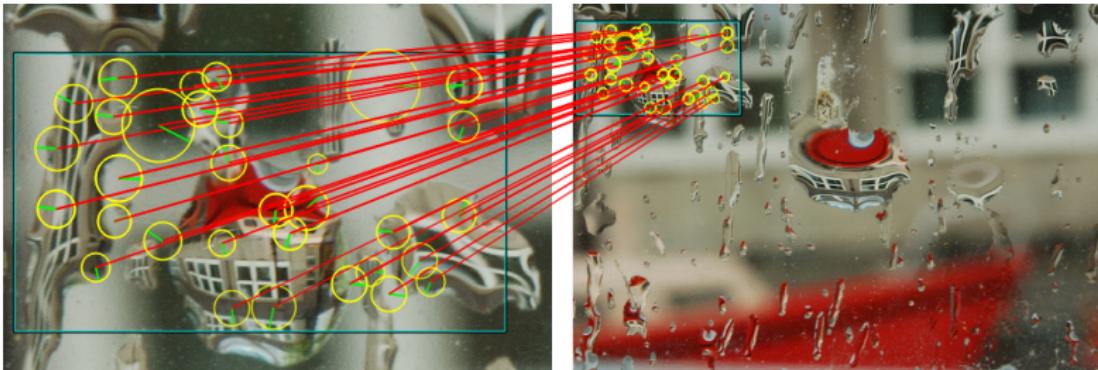
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## spatial matching—same instance



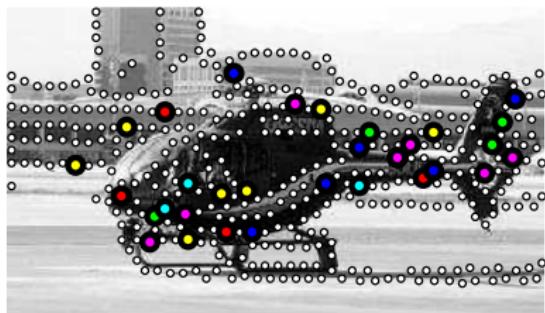
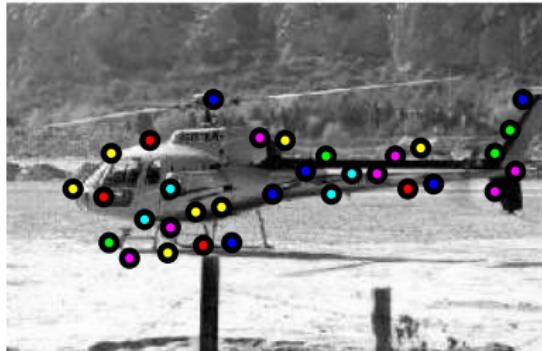
- now robust to scale, viewpoint, occlusion, clutter, lighting
- and very fast

## spatial matching—same instance



- now robust to scale, viewpoint, occlusion, clutter, lighting
- and very fast

## spatial matching—same class



- solve for feature correspondence, flexible transformation and outliers on all possible **correspondence pairs** by joint optimization
- very expensive

## spatial matching—same class



- solve for feature correspondence, flexible transformation and outliers on all possible **correspondence pairs** by joint optimization
- very expensive and error prone

# geometry

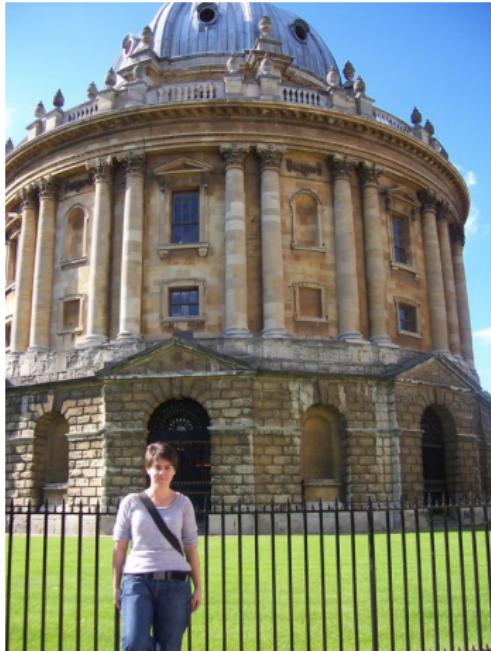
- spatial matching on **same instance** is robust, but expensive
- we can
  - encode position, e.g. with dense features; easier to match, but we loose invariance
  - discard geometry altogether and use a global representation; even easier, we maintain invariance, but loose discriminative power
  - discard geometry as a first step, then bring it back
  - make it more efficient?
- rigid transformations won't work for **classification**, and matching is even more expensive
  - make it more flexible?
  - make it more efficient?
  - maintain invariance?

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  - make it more efficient?
  - maintain invariance?

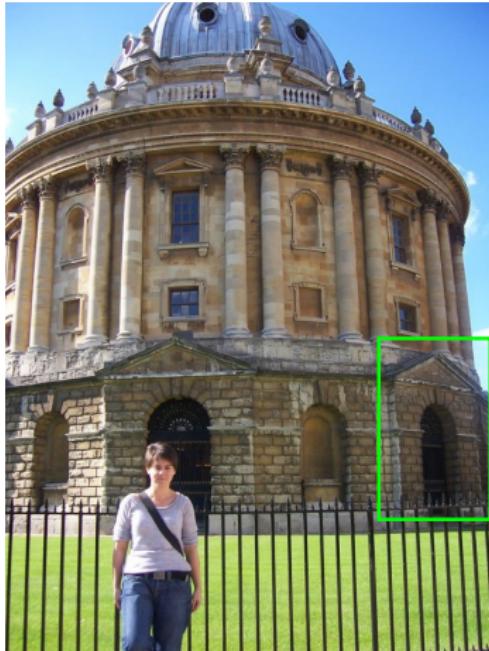
# matching discriminative local features

[Lowe, ICCV 1999]

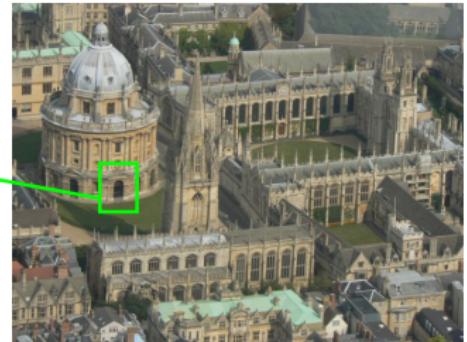


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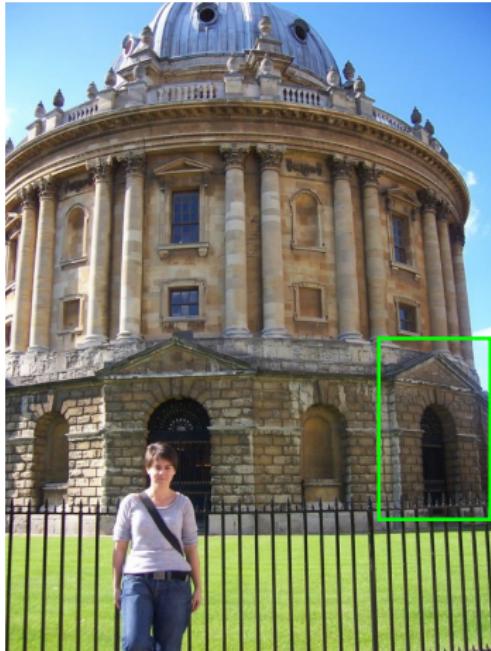


features

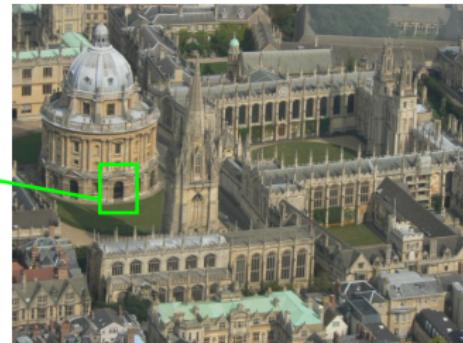
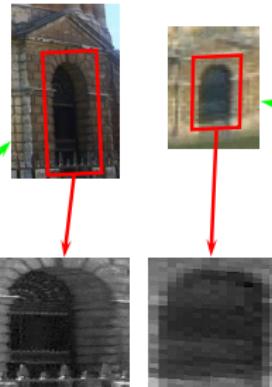


# matching discriminative local features

[Lowe, ICCV 1999]



features



normalized features

# appearance

- matching appearance via descriptors should be easier than geometry
- but
  - if we have positions e.g. with dense features, we know what to match (but we loose invariance)
  - otherwise, we need to find correspondences (expensive)
  - we can apply some **pooling** in image space or in descriptor space; more efficient; it may help or not
  - **global** pooling is the most efficient (but is not as discriminative)
  - local descriptors take up a lot of space; with pooling or not, we can **compress** them

# forget about geometry: bag-of-words

[Sivic and Zisserman 2003]



- in fact, discarding geometry (**bag**) is one thing and quantizing descriptors (**words**) is another

# vector quantization → visual words



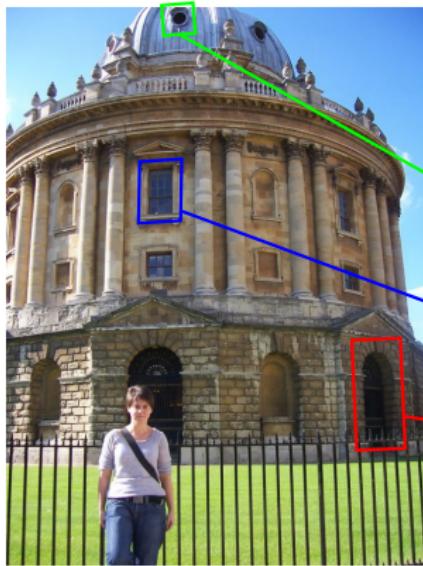
query



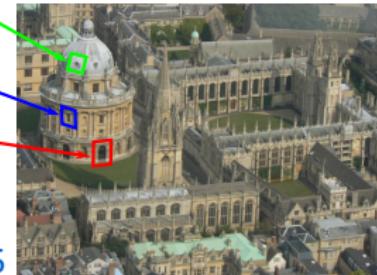
15

- query vs. dataset image

# vector quantization → visual words



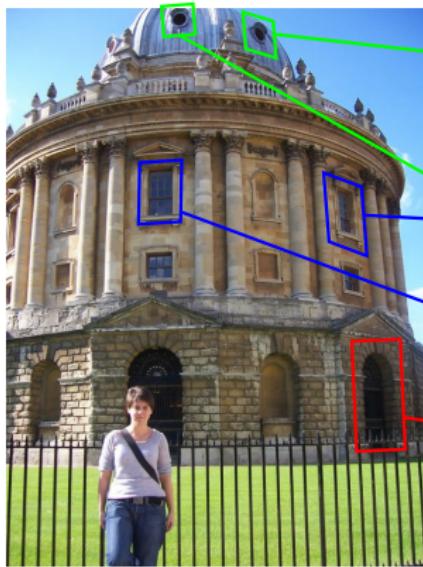
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15

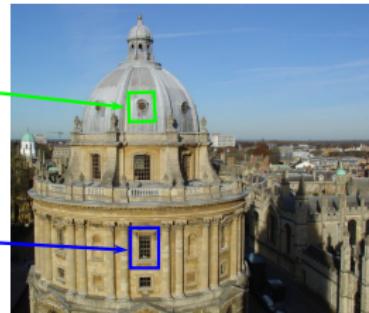
- pairwise descriptor matching

## vector quantization → visual words

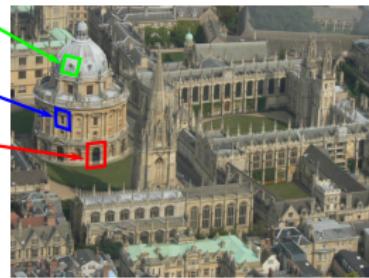


query

19

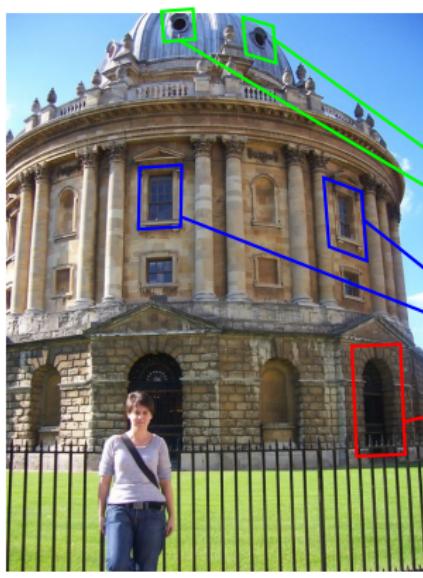


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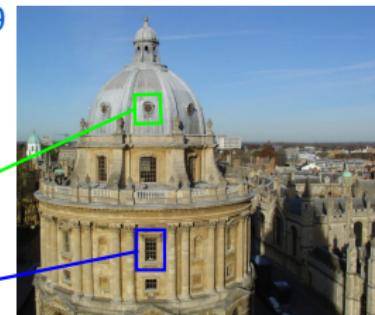
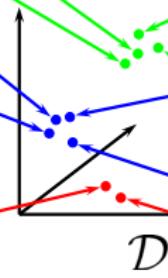


- pairwise descriptor matching for **every** dataset image

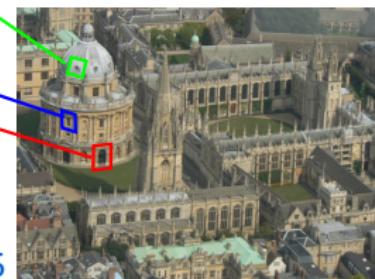
## vector quantization → visual words



## query



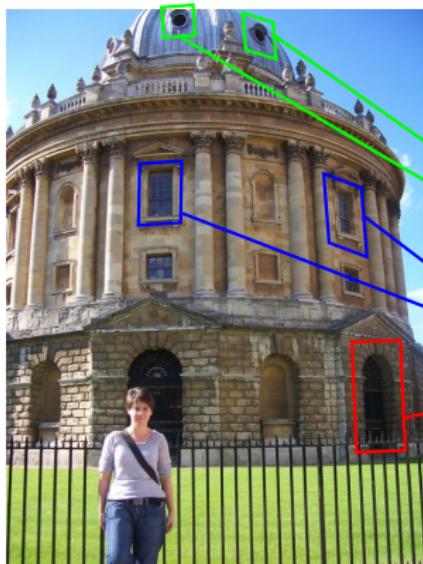
19



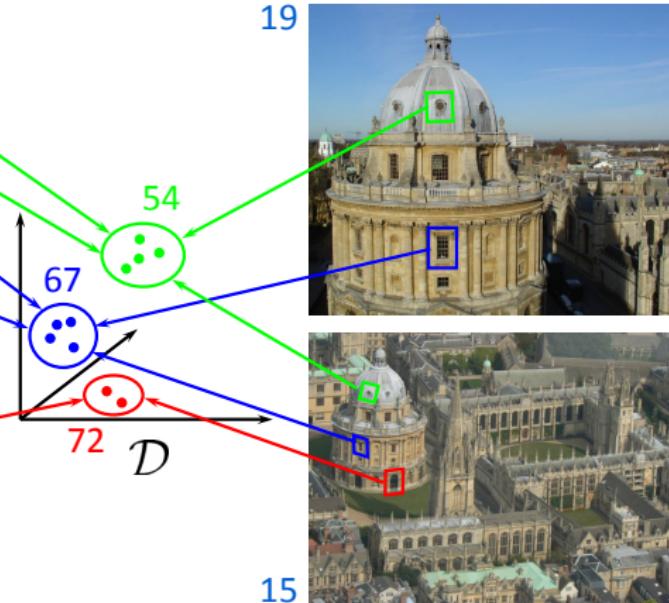
15

- similar descriptors should all be nearby in the descriptor space

# vector quantization → visual words

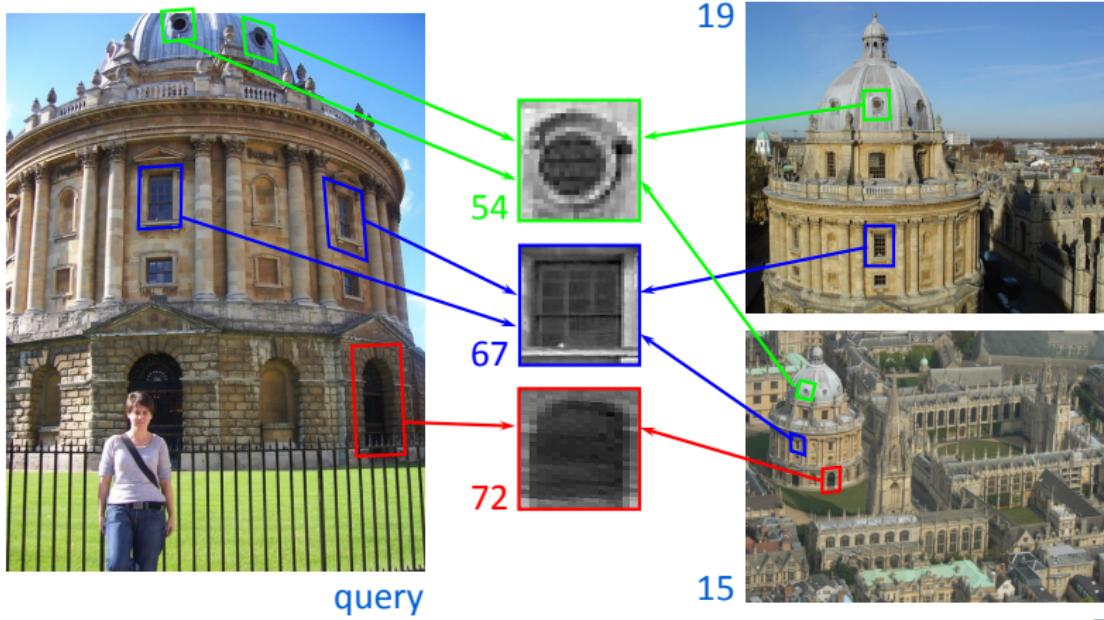


query



- let's quantize them into visual words

## vector quantization → visual words



- now visual words act as a proxy; no pairwise matching needed

## bag-of-words and “cosine” similarity

- each image is represented by a single vector  $\mathbf{z} \in \mathbb{R}^k$ , where  $k$  is the size of the codebook
- each element  $z_i = w_i n_i$  where  $w_i$  fixed weight per visual word (e.g. inverse document frequency) and  $n_i$  the number of occurrences of this word in the image
- this vector then typically normalized, e.g.  $\|\mathbf{z}\|_1 = 1$  or  $\|\mathbf{z}\|_2 = 1$
- given two images represented by  $\mathbf{z}, \mathbf{y}$ , similarity is usually measured by dot product

$$s_{\text{BoW}}(\mathbf{z}, \mathbf{y}) := \mathbf{z}^\top \mathbf{y}$$

- with  $\ell_2$  normalization, this is equivalent to measuring Euclidean distance  $\|\mathbf{z} - \mathbf{y}\|$  because  $\|\mathbf{z} - \mathbf{y}\|^2 = 2(1 - \mathbf{z}^\top \mathbf{y})$

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## bag-of-words for retrieval

- given a set of  $n$  images represented by matrix  $Z \in \mathbb{R}^{k \times n}$  (each image as a column) and query image  $\mathbf{q}$ , we need a vector of similarities

$$\mathbf{s} = S_{\text{BoW}}(Z, \mathbf{q}) := Z^\top \mathbf{q}$$

and then sort  $\mathbf{s}$  by descending order

- when  $k \gg p$ , where  $p$  is the number of features per image on average,  $Z$  and  $\mathbf{q}$  are sparse
- rather than whether a word is contained in an image, ask which images contain a given word

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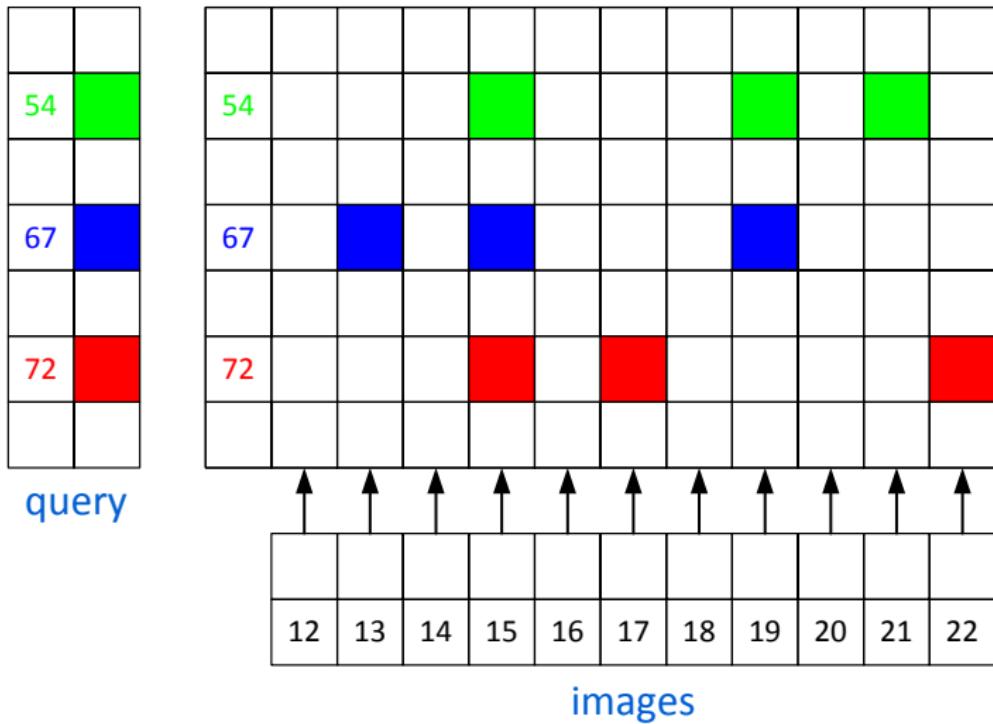
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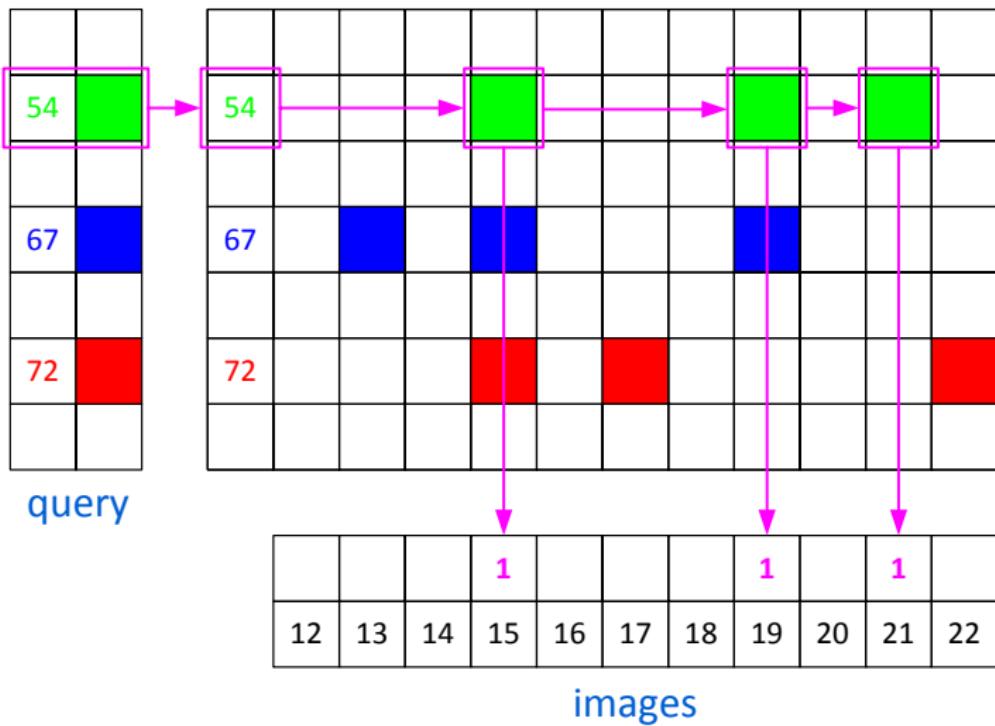
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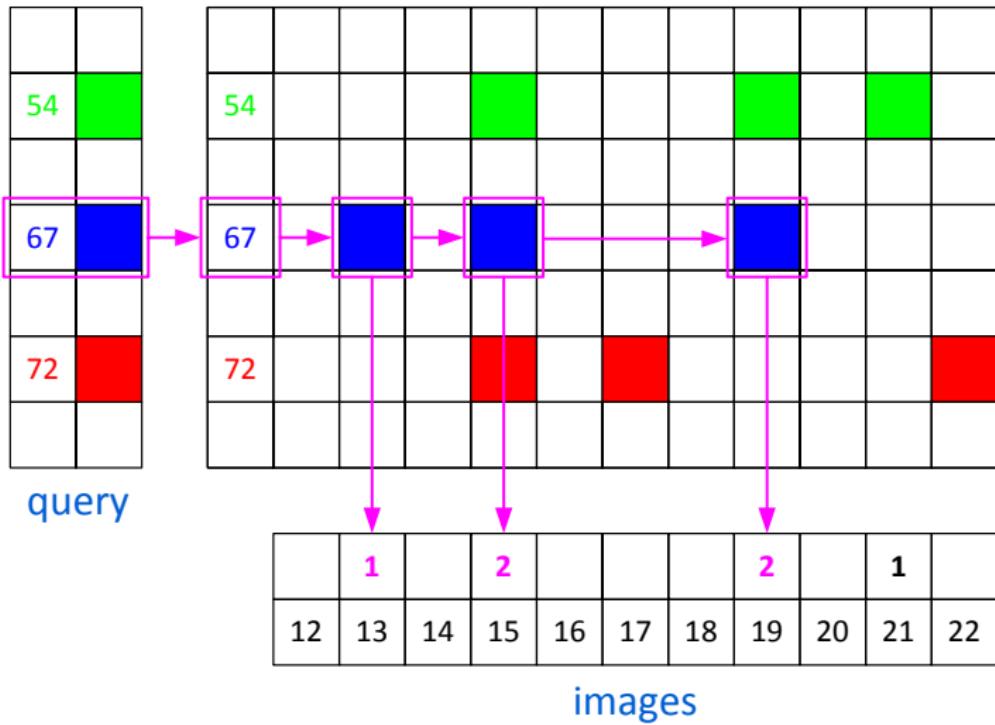
# **inverted file indexing**



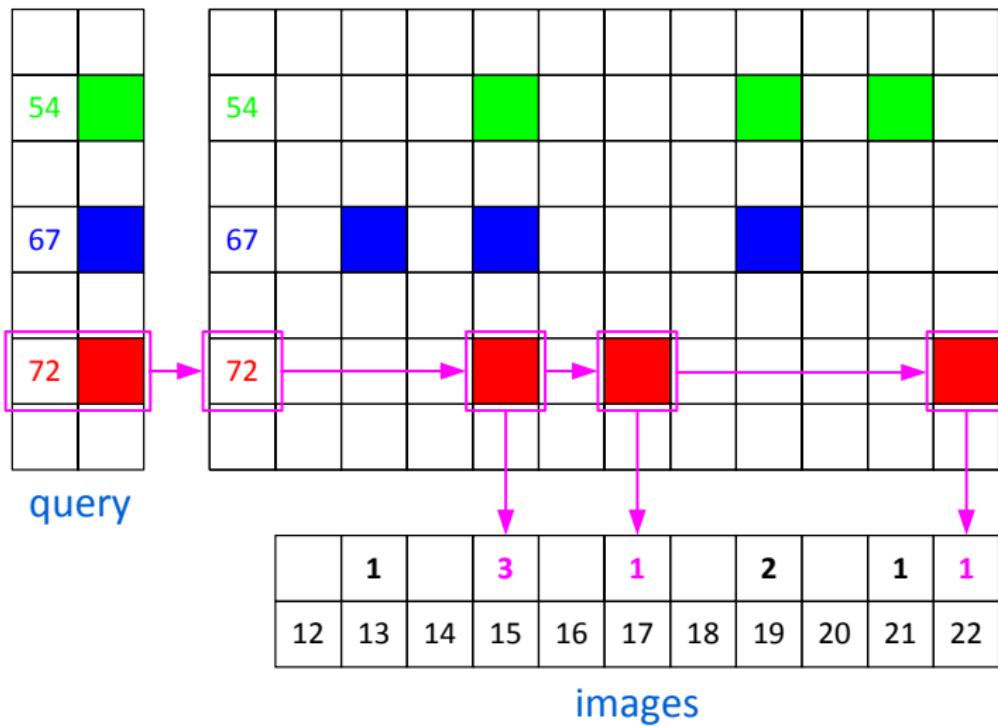
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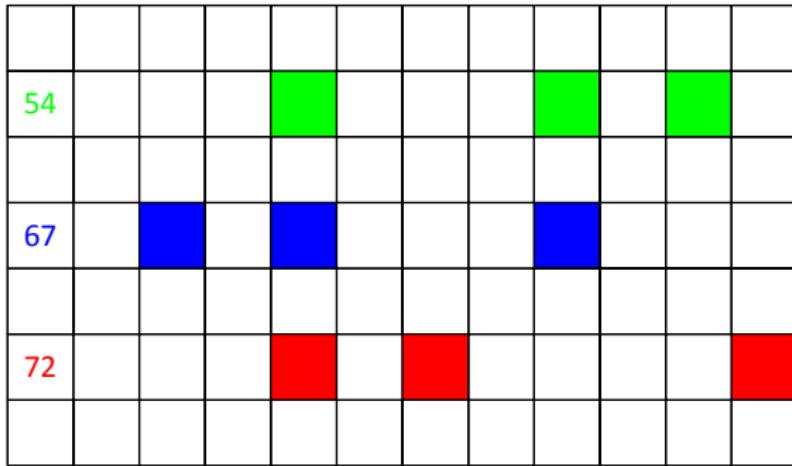
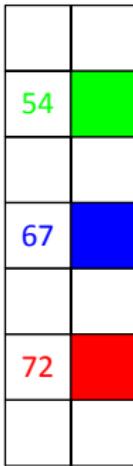
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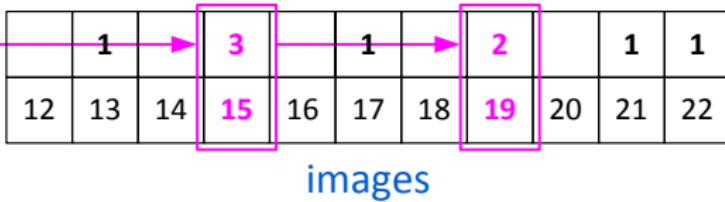


# **inverted file indexing**



## query

## ranked shortlist



## back to geometry: re-ranking

- dot product similarity is fast but quantized descriptors are not discriminative enough; performs poorly in the presence of distractors
- perform spatial matching only on **top-ranking** images, and re-ranking according to a score based on geometry, e.g. number of inliers
- but to save space, **descriptors are not available**: tentative correspondences are based on visual words, and there are too many (too many features are in correspondence if they are assigned to the same visual word)

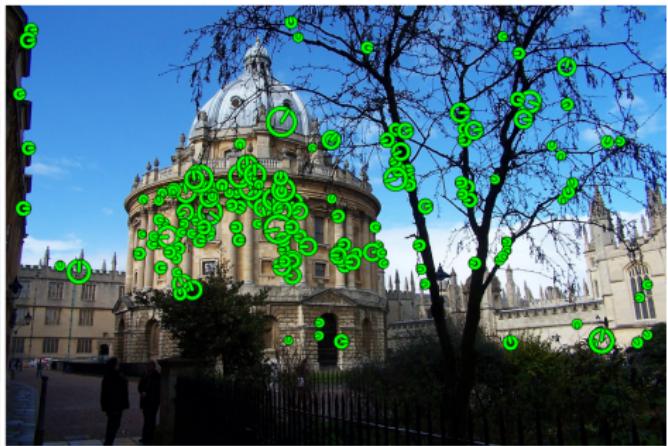
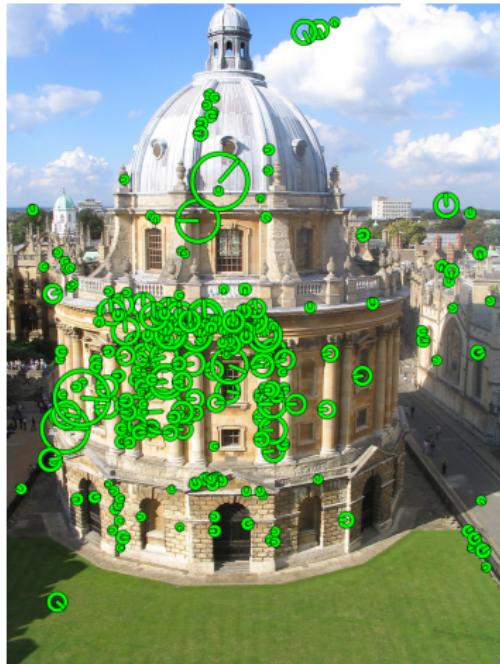
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original images

Fischler and Bolles. CACM 1981. Random Sample Consensus: A Paradigm for Model Fitting With Applications to Image Analysis and Automated Cartography.

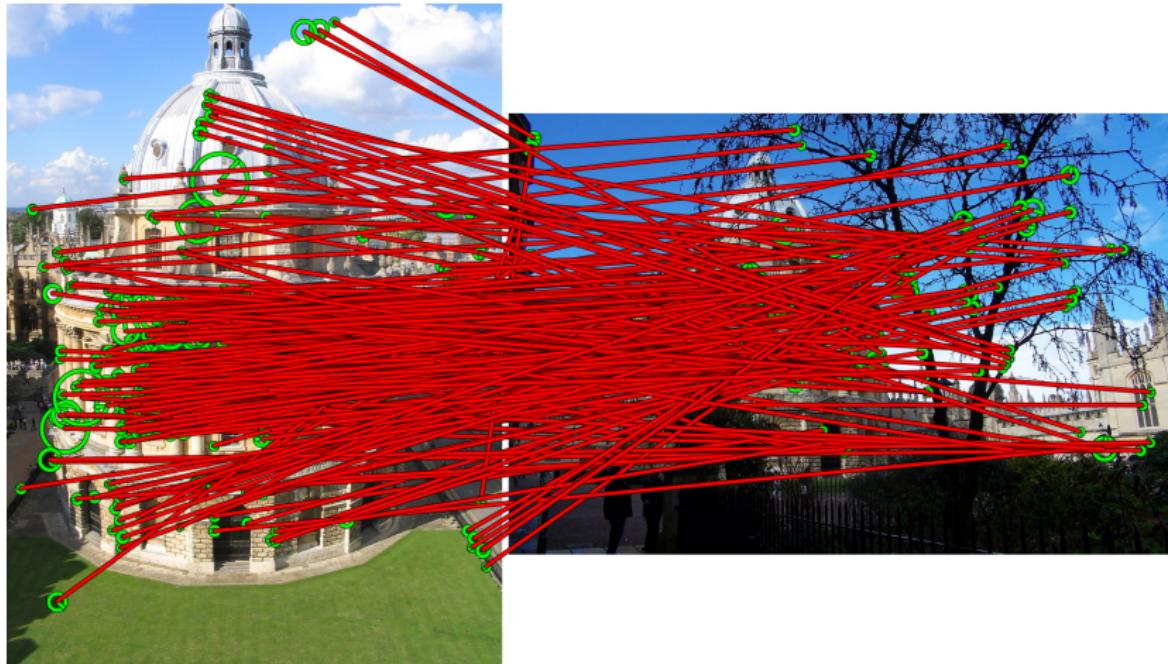
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local features

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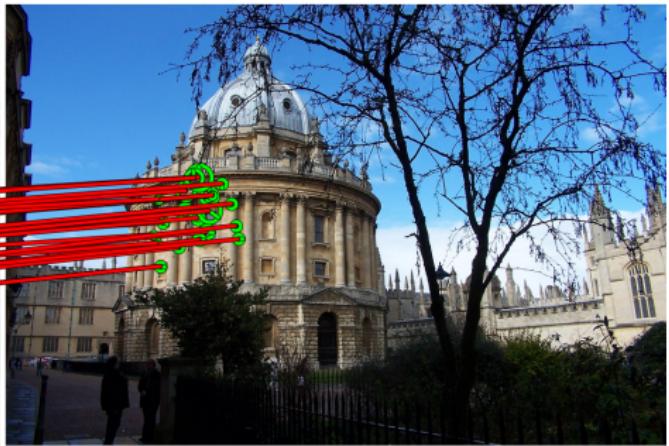
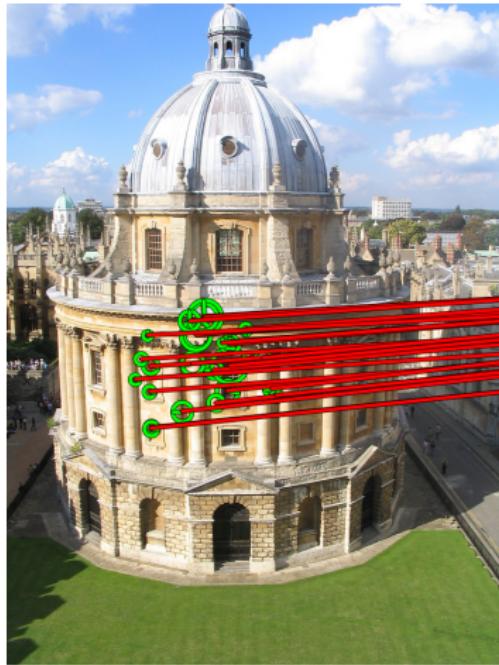
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tentative correspondences: too many

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# back to geometry: re-ranking



inliers: now more expensive to find

## bag of words for classification

- each image represented by  $\mathbf{z} \in \mathbb{R}^k$ ; each element  $z_i$  the number of occurrences of visual word  $c_i$  in the image
- Naïve Bayes: chose maximum posterior probability of class  $C$  given image  $\mathbf{z}$  assuming features are independent → linear classifier with parameters estimated by visual word statistics on training set
- support vector machine (SVM): images  $\mathbf{z}, \mathbf{y}$  compared by kernel function  $\kappa(\mathbf{z}, \mathbf{y})$ ; if  $\kappa(\mathbf{z}, \mathbf{y}) = \mathbf{z}^\top \mathbf{y}$ , this is again a linear classifier and is a standard choice at large scale

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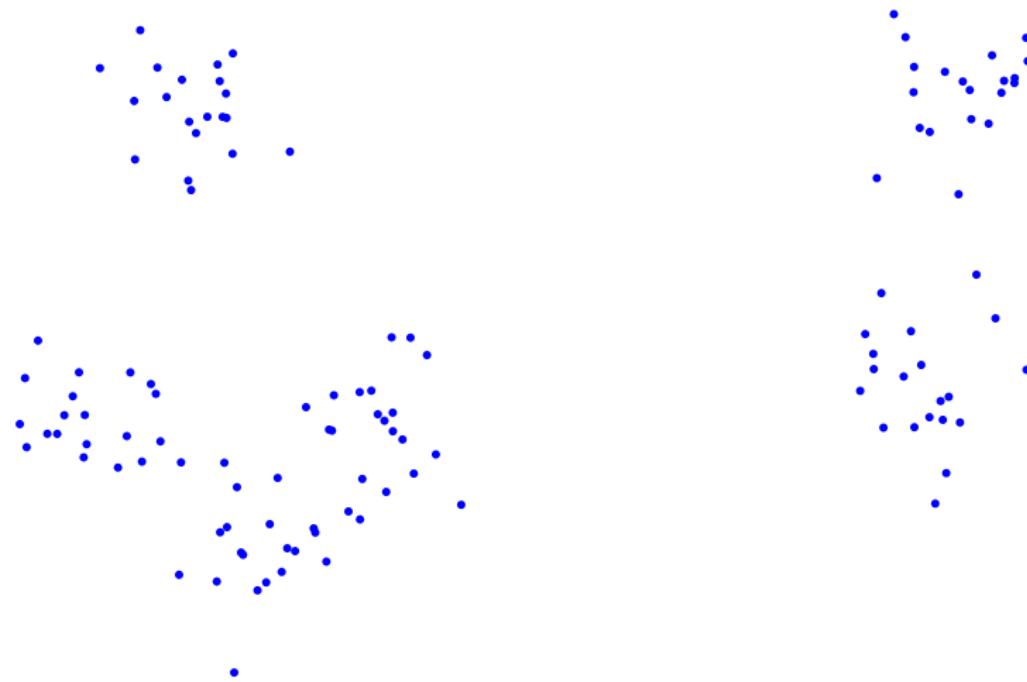
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# codebooks

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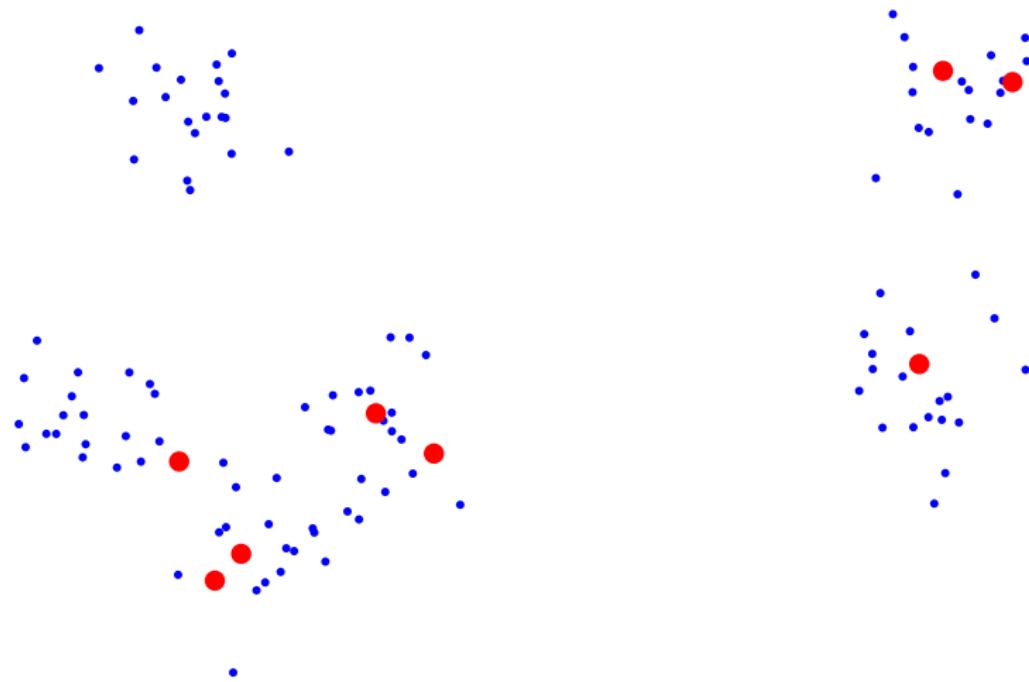
[MacQueen 1967]



## dataset

## vector quantization: *k*-means clustering

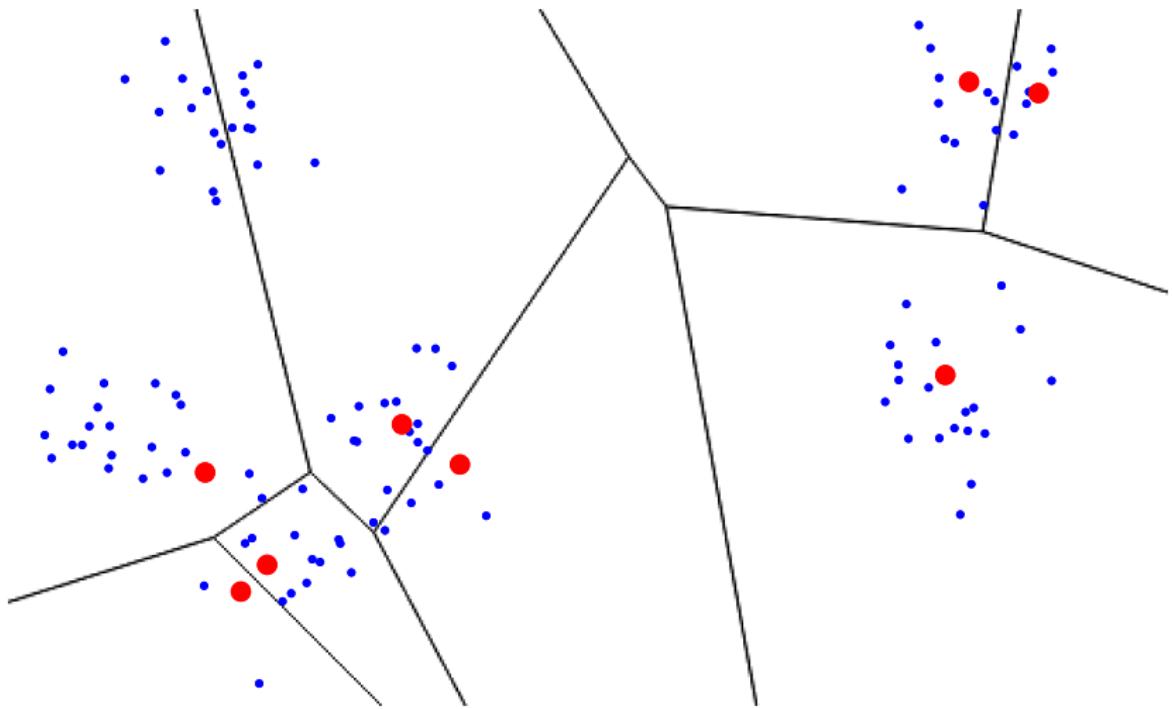
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## initial centroids

## vector quantization: $k$ -means clustering

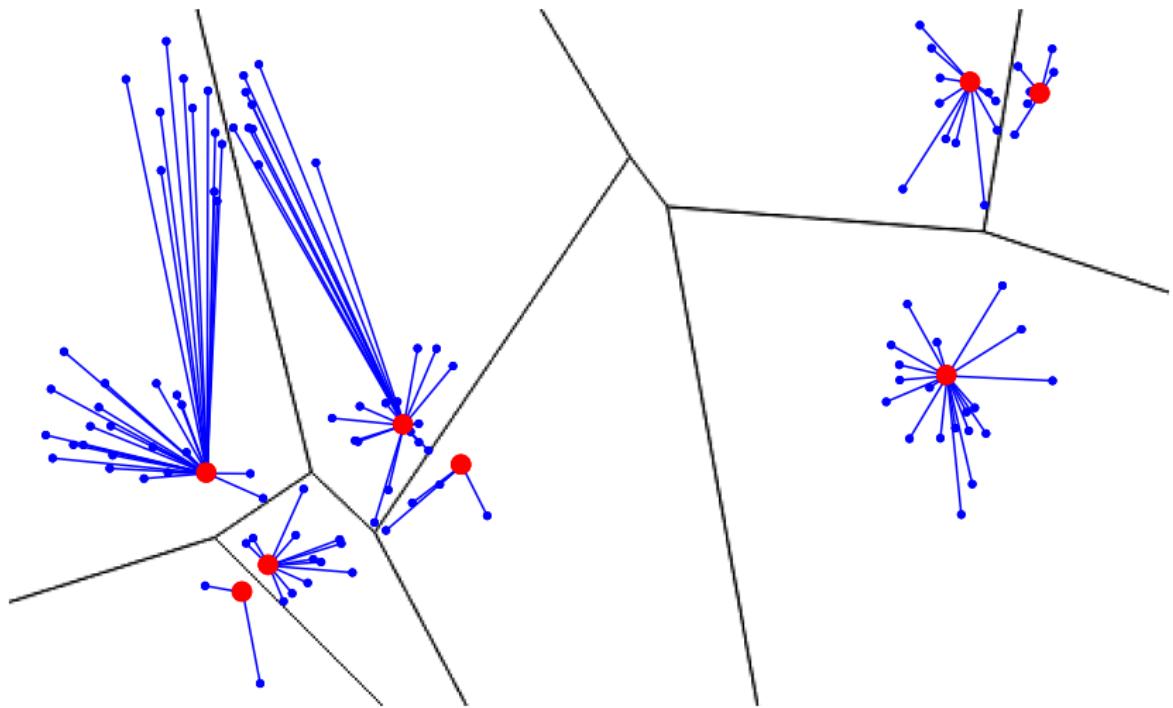
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## Voronoi cells

## vector quantization: $k$ -means clustering

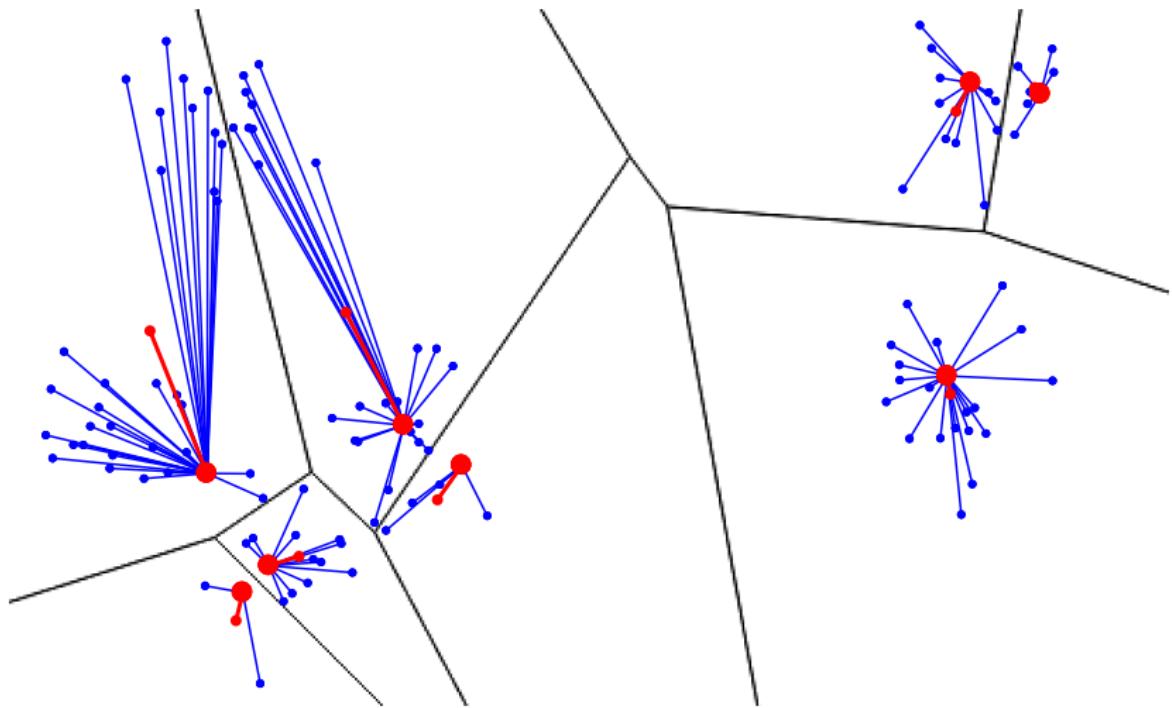
[MacQueen 1967]



points assigned to nearest centroids

## vector quantization: $k$ -means clustering

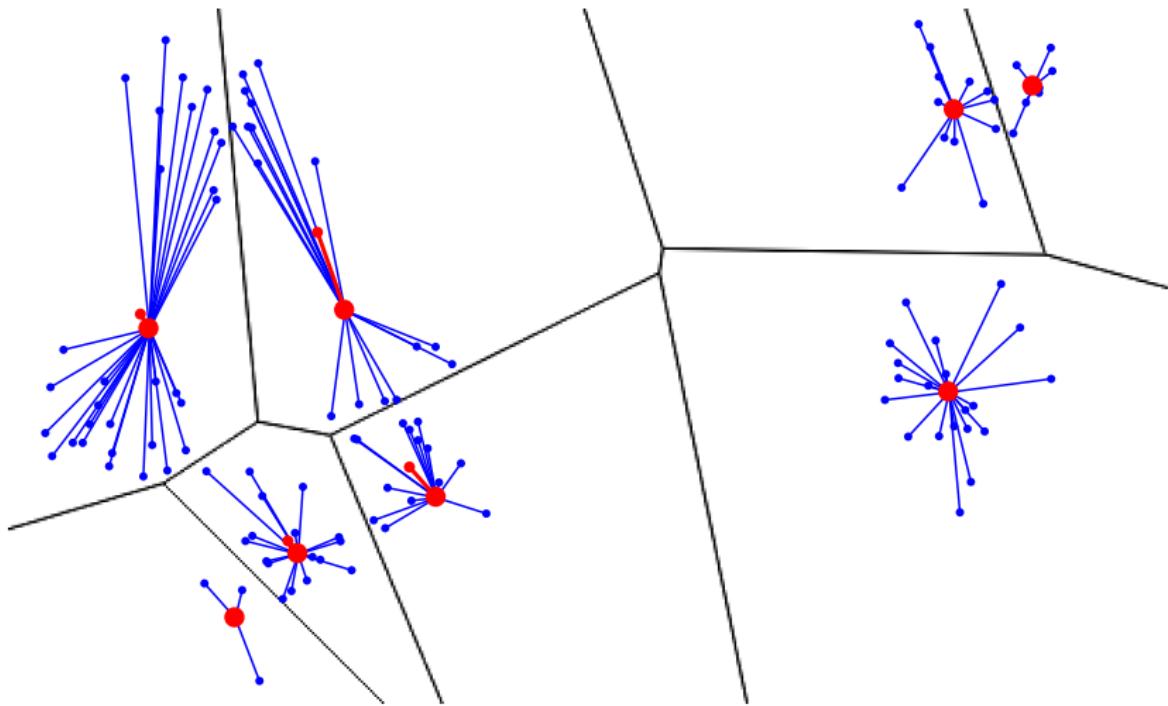
[MacQueen 1967]



centroids move to mean per cell

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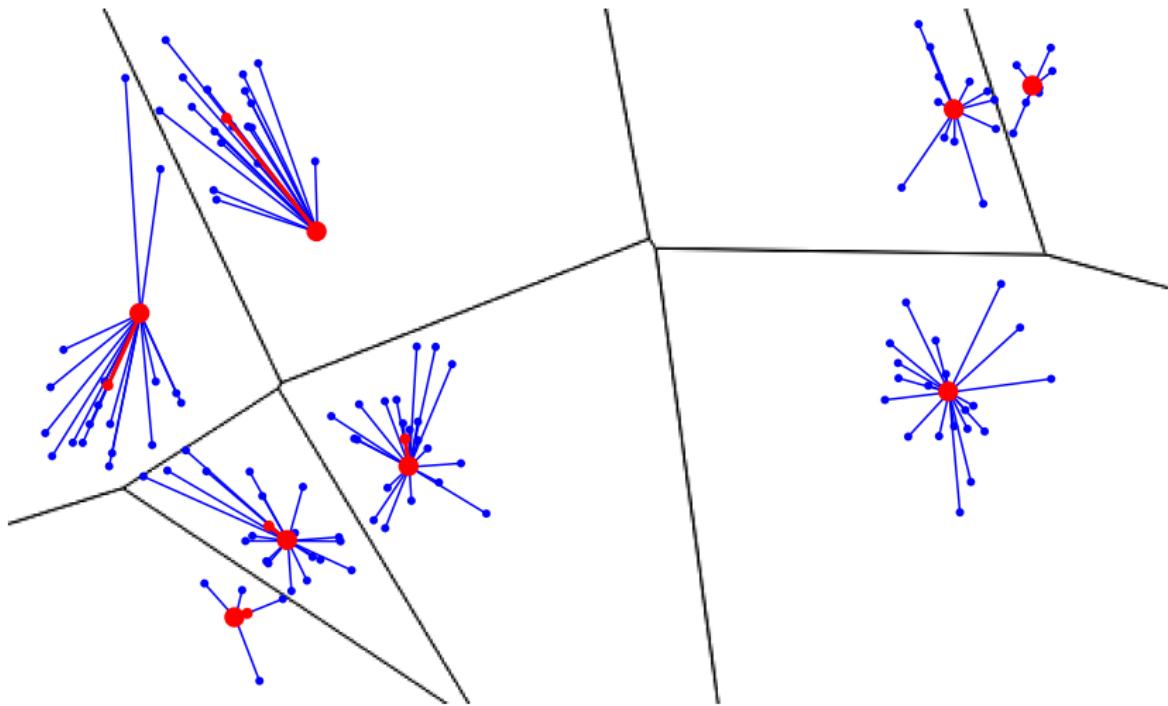
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iterate until convergence

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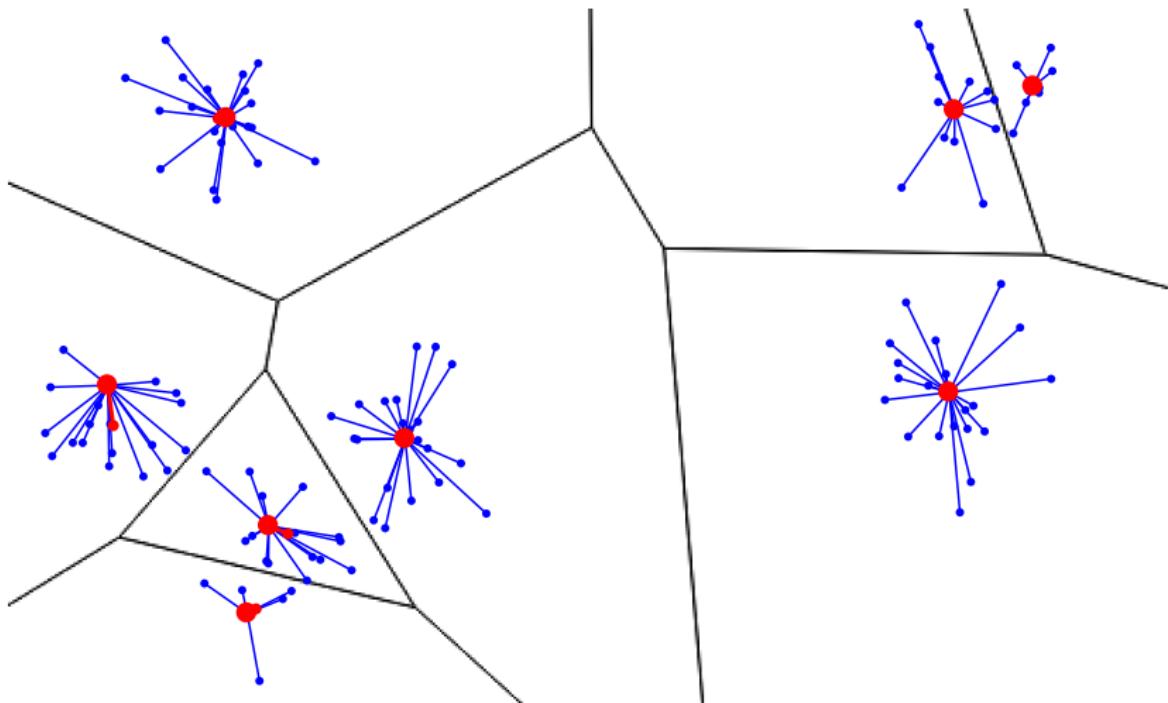
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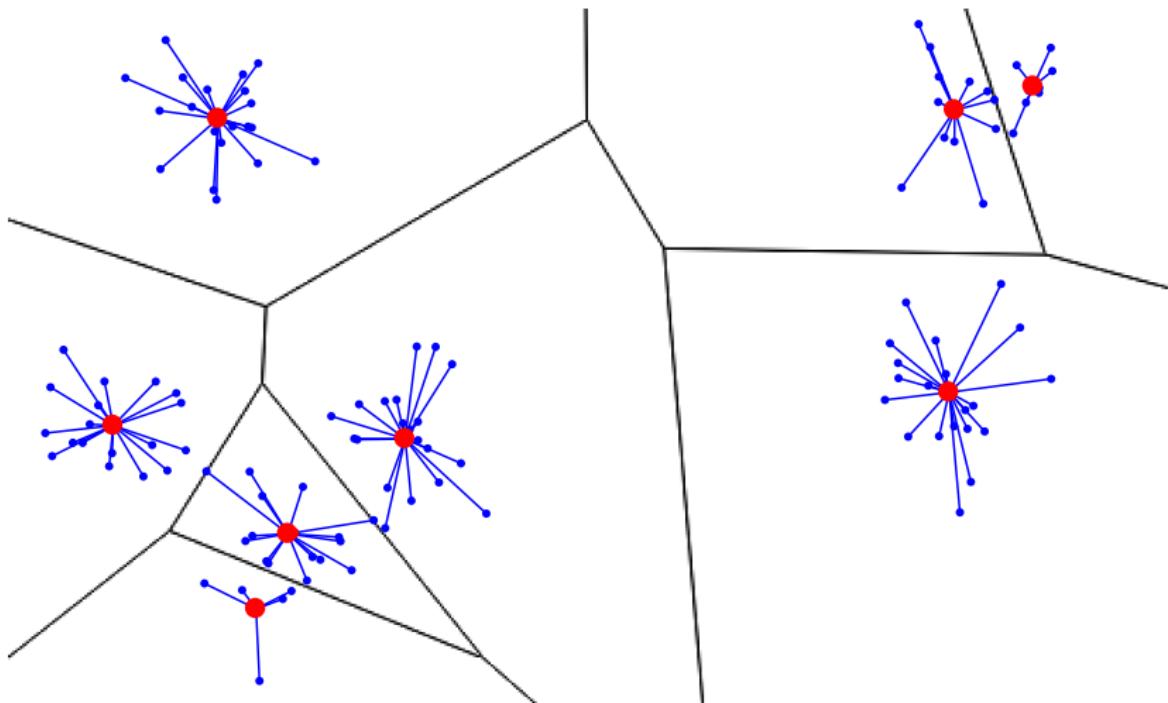
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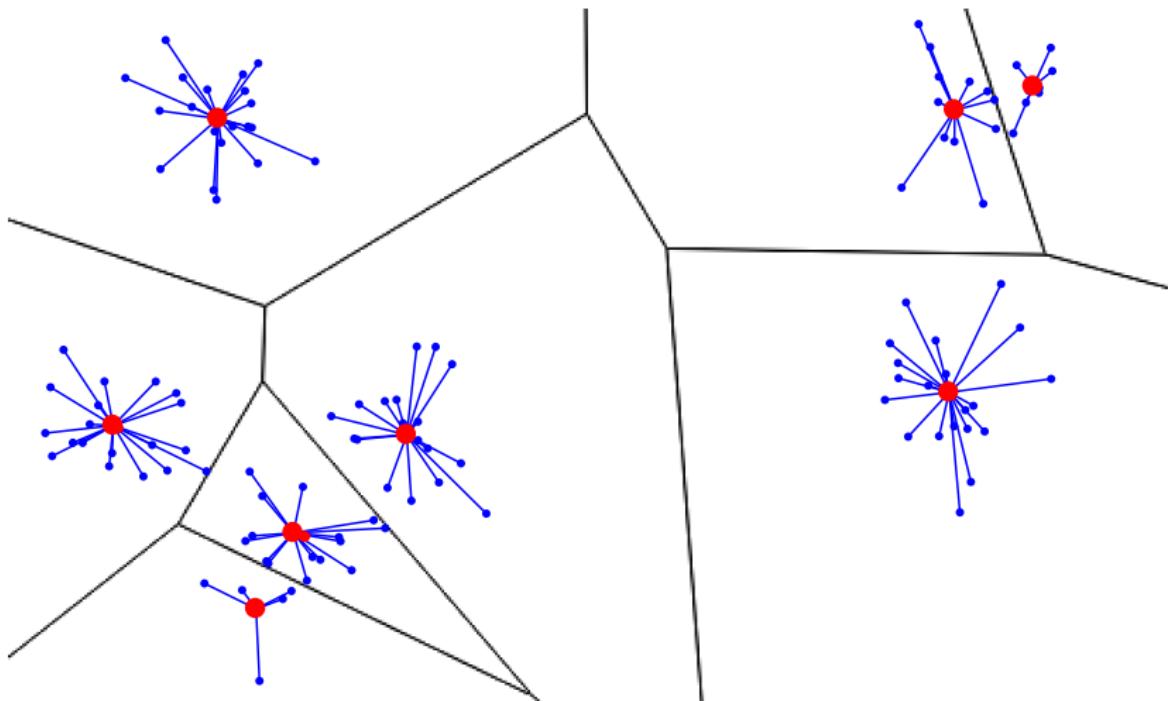
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iterate until convergence

## vector quantization: $k$ -means clustering

[MacQueen 1967]



iterate until convergence

## vector quantization: *k*-means clustering

- objective: given dataset  $X \subset \mathbb{R}^d$ , find codebook  $C \subset \mathbb{R}^d$ , with  $|C| = k$ , and quantizer function  $q : \mathbb{R}^d \rightarrow C$ , minimizing distortion

$$E(C, q) := \sum_{x \in X} \|x - q(x)\|^2$$

- regardless of  $C$ ,  $q$  should map vector  $x$  to its nearest centroid

$$q(x) = \arg \min_{c \in C} \|x - c\|$$

- algorithm: at each iteration, given the set  $X_c = \{x \in X : q(x) = c\}$  of points assigned to centroid  $c$ , (**assignment step**),  $c$  moves to their mean (**update step**)

$$c \leftarrow \frac{1}{|X_c|} \sum_{x \in X_c} x$$

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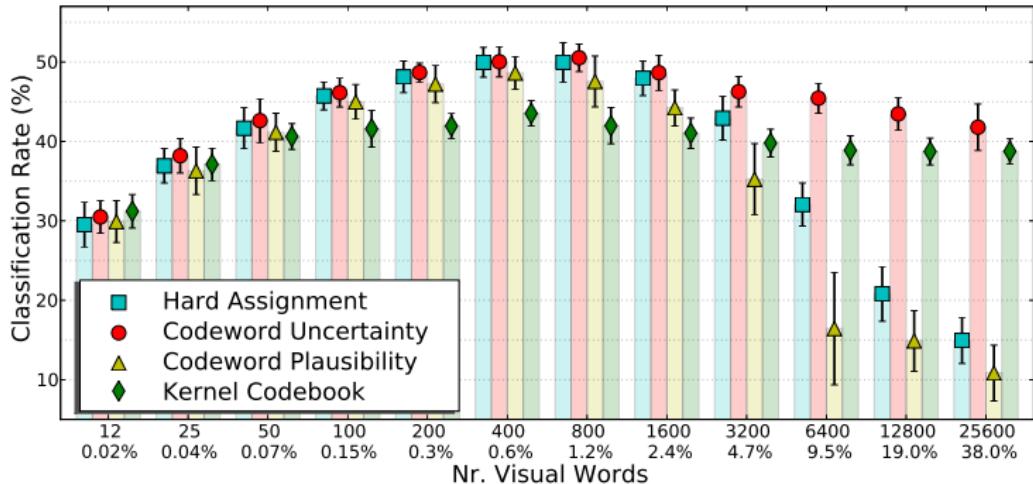
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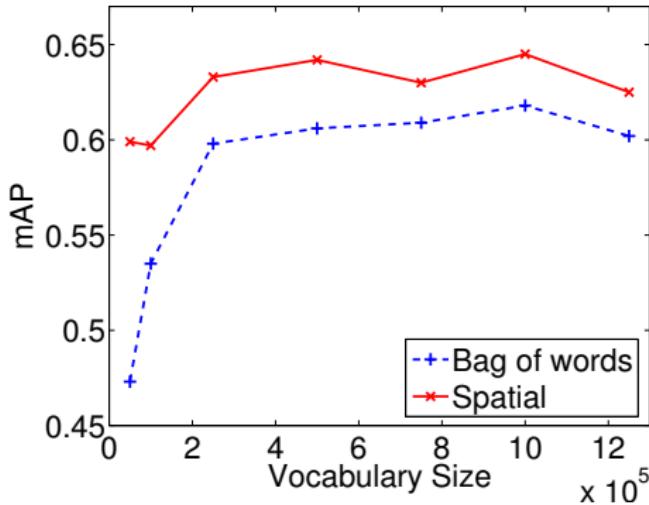
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# codebook size



- classification: **thousands**
- depends on a lot of factors e.g. the number of features in the image representation and size and variability of the dataset

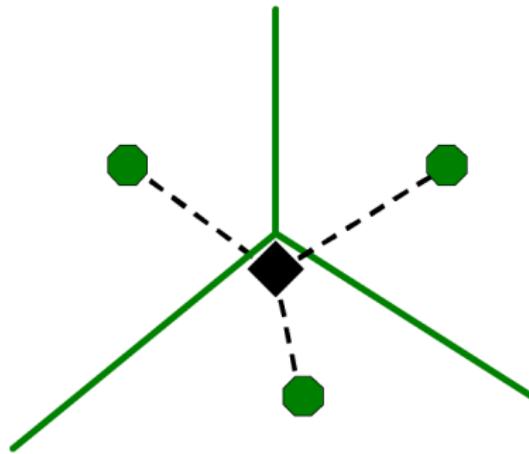
## codebook size



- instance retrieval: millions
- depends on a lot of factors e.g. the number of features in the image representation and size and variability of the dataset

# hierarchical $k$ -means (HKM)\*

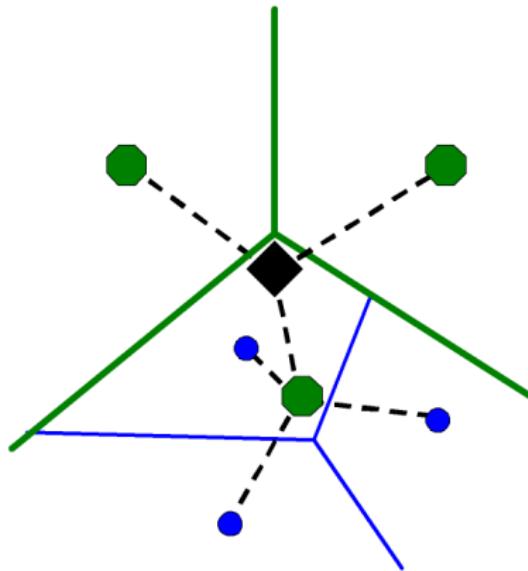
[Fukunaga and Narendra 1975]



- partition data into  $b$  clusters using  $k$ -means

# hierarchical $k$ -means (HKM)\*

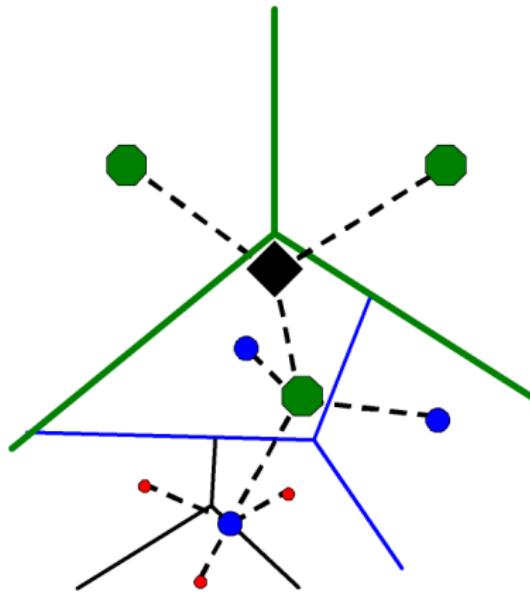
[Fukunaga and Narendra 1975]



- within each cluster, partition data into  $b$  clusters

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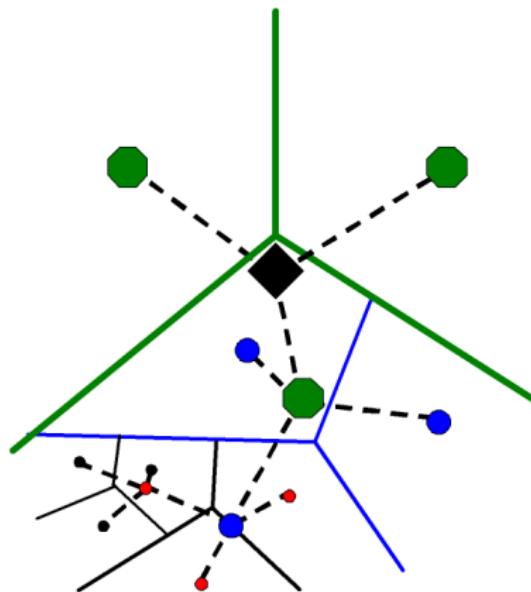
[Fukunaga and Narendra 1975]



- and repeat;  $b$  is called the **branching factor**

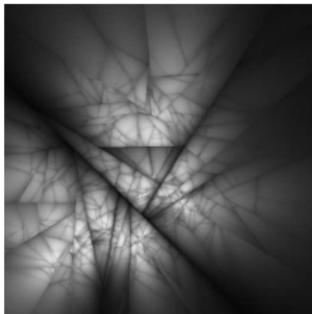
## hierarchical $k$ -means (HKM)\*

[Fukunaga and Narendra 1975]

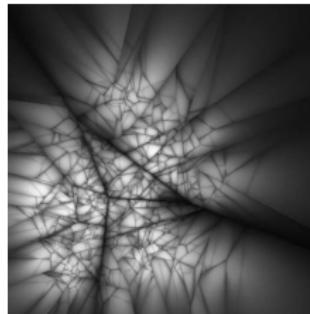


- at  $\ell$  levels, there are  $b^\ell$  total clusters

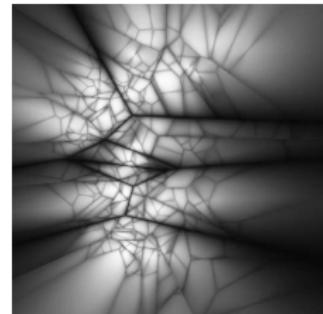
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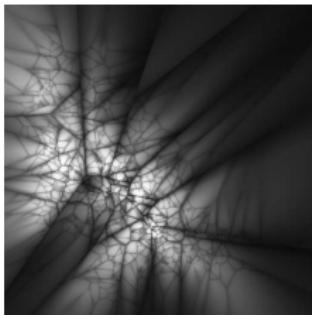
$b = 2$



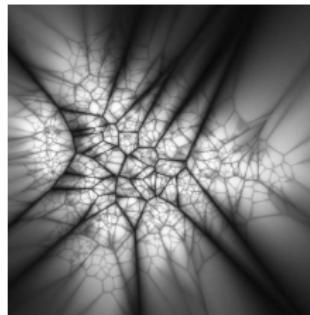
$b = 4$



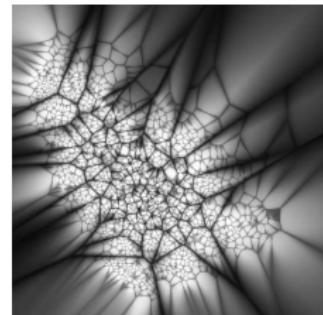
$b = 8$



$b = 16$



$b = 32$

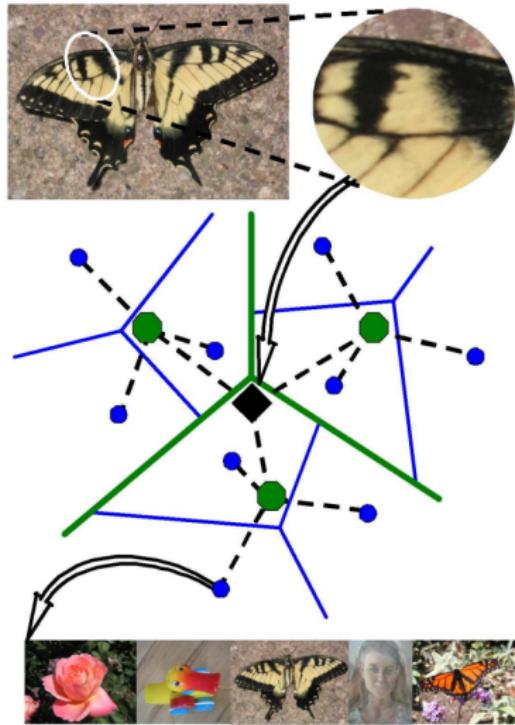


$b = 128$

- intensity: ratio of first to second neighbor distance

## vocabulary tree\*

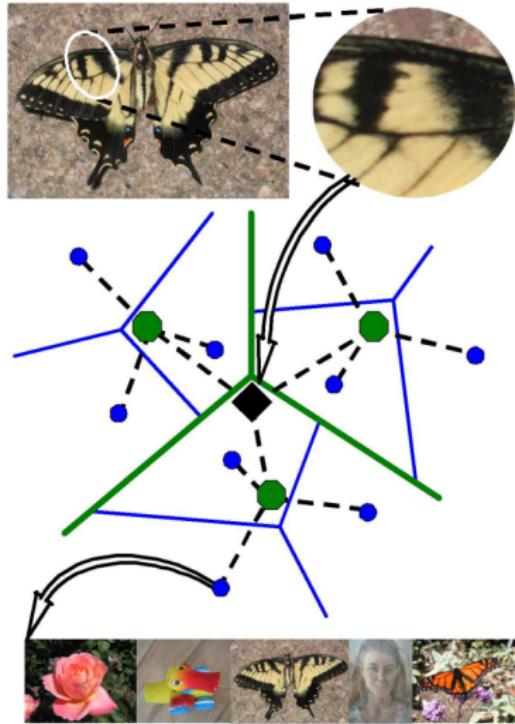
[Nister and Stewenius. CVPR 2006]



- apply  $k$ -means hierarchically and build a fine partition tree
  - descriptors descend from root to leaves by finding nearest node at each level
  - image represented by  $x_i = w_i n_i$  as in BoW, but now there is one element per node including internal nodes
  - dataset searched by inverted files at leaves

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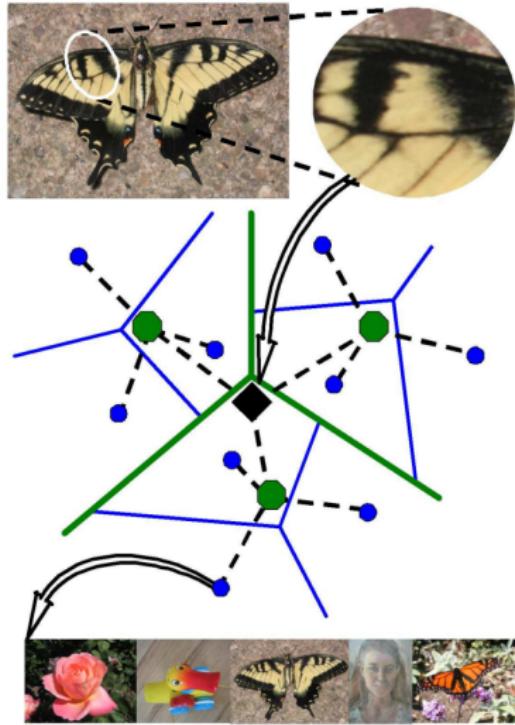
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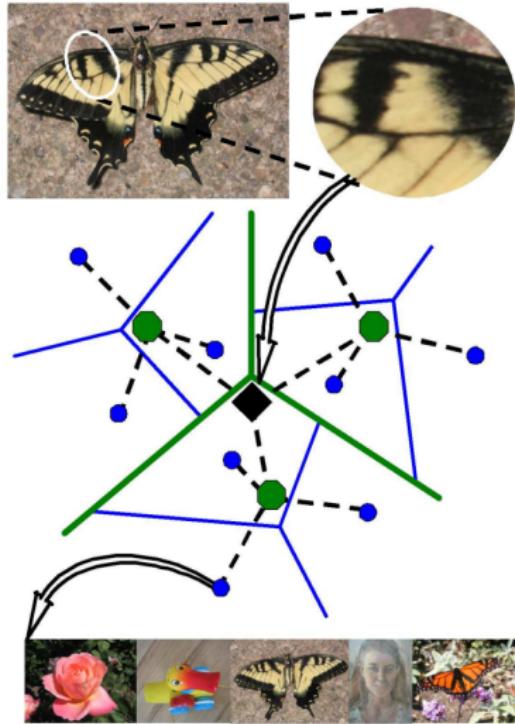


however:

- no principled way of defining  $w_i$  across levels
- distortion minimized only locally; points get assigned to leaves that are not globally nearest

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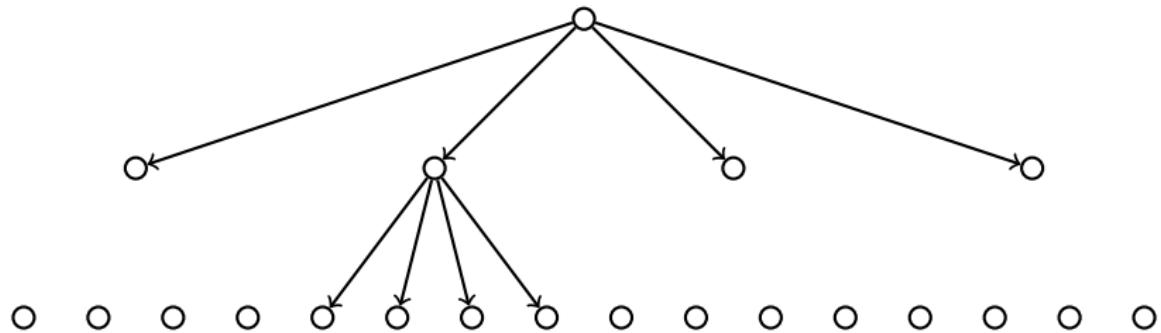


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# approximate $k$ -means (AKM)\*

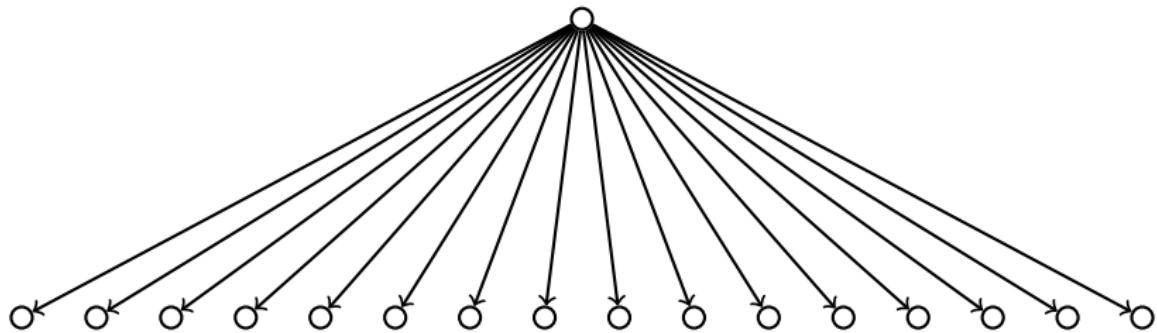
[Philbin et al. 2007]



- with branching factor  $b = 10$  and  $\ell = 6$  levels, HKM yields  $k = 10^6$  visual words; complexity is  $O(nbl)$
- search through multiple randomized trees (comparison to HKM in color)

# approximate $k$ -means (AKM)\*

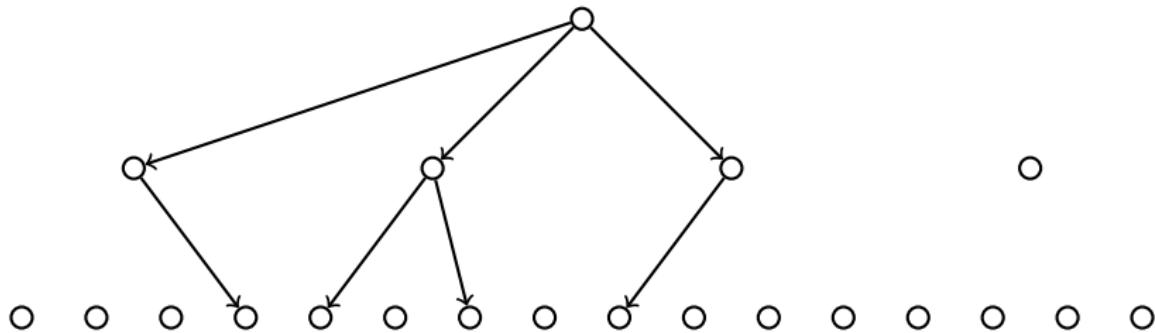
[Philbin et al. 2007]



- flat  $k$ -means with e.g.  $n = 10^7$  points and  $k = 10^6$  centroids is prohibitive; complexity is  $O(nk)$ , because each assignment is  $O(k)$
- search through multiple randomized trees (comparison to HKM in color)

# approximate $k$ -means (AKM)\*

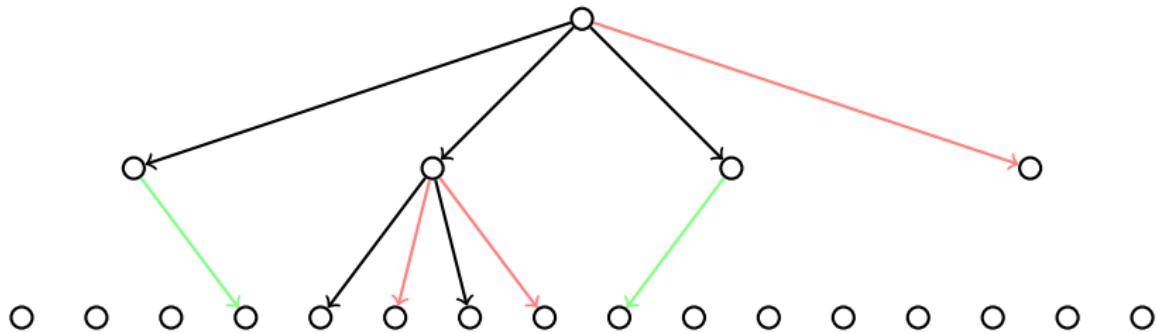
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- **approximate nearest neighbor search** to find the nearest centroid: each assignment is now  $O(\log k)$ , and complexity drops to  $O(n \log k)$
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## approximate $k$ -means (AKM)\*

- if the sole purpose of the hierarchy is to **accelerate assignment**, both at learning and at search, it is better to use a flat vocabulary combined with a more principled nearest neighbor search method
- however, with appropriate **node weighting**, a hierarchical structure can help (see pyramid matching later on)

# pipeline, again

- given **codebook**  $C = \{c_1, \dots, c_k\} \subset \mathbb{R}^d$
- given image with descriptors  $x_i \in \mathbb{R}^d$  at positions  $y_i \in \mathbb{R}^2$ ,  
 $i = 1, \dots, n$
- encode each descriptor  $x_i$  into  $\mathbf{a}_i \in \mathbb{R}^k$

$$\mathbf{a}_i := F(x_i; C) := (f(x_i, c_1; C), \dots, f(x_i, c_k; C))$$

- pool each spatial region  $R_j, j = 1, \dots, m$  into  $\mathbf{z}^j \in \mathbb{R}^k$

$$\mathbf{z}^j := g(\{\mathbf{a}_i : y_i \in R_j\})$$

- concatenate into  $\mathbf{z} \in \mathbb{R}^{km}$

$$\mathbf{z} := (\mathbf{z}^1; \dots; \mathbf{z}^m)$$

- global pooling is just  $m = 1$

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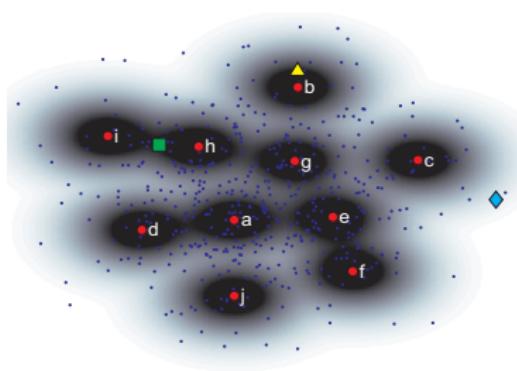
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# soft assignment

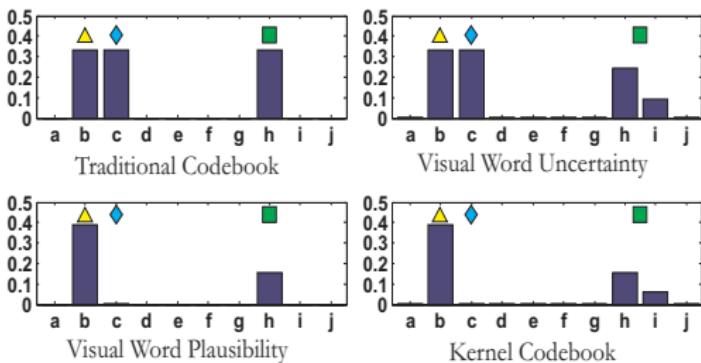
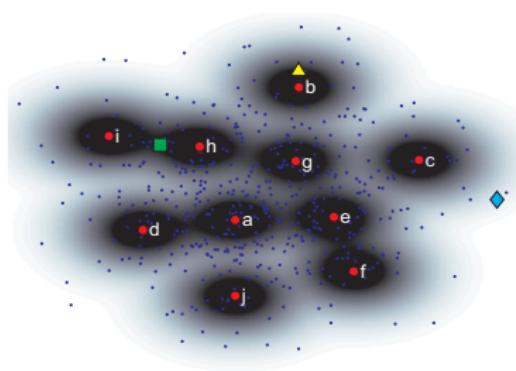
[van Gemert et al. 2008]



- ▲: ok; ■: ambiguous; ◆: not represented
- left: assigned to nearest neighbor; right: to all visual words with different weights
- top: total weight normalized to one; bottom: depends on distance

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# soft assignment

- $r$ -nearest neighbors of  $x$  in  $C$ :  $\text{NN}_C^r(x)$
- kernel function

$$h(x) = h_G(x; \sigma) := \mathcal{N}(\mathbf{0}, \sigma^2 \mathbf{I})(x) \propto \exp\left(-\frac{\|x\|^2}{2\sigma^2}\right)$$

- encoding descriptor  $x$  into visual word  $c$

| $f_C(x, c)$     | visual word  |   |
|-----------------|--|---|
|                 | nearest  | all   |
| fixed weight    | $\mathbb{1}[c \in \text{NN}_C^1(x)]$<br>“BoW”                  | $\frac{h(x-c)}{\sum_j h(x-c_j)}$<br>“uncertainty” |
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- on classification: best model is “uncertainty”

$$f_C(x, c) = \frac{h(x - c)}{\sum_j h(x - c_j)}$$

- it is better to contribute to visual words even if all are far away
- we shall see this is the softmax of negative distances  $-\|x - c\|^2$
- it is also the responsibility of visual word  $c$  for descriptor  $x$  in a Gaussian mixture model with  $C$  as components

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# soft assignment

[Liu et al. 2011]

- on classification: it turns out, it is better to limit contributions to  $r$  nearest neighbors

$$f_C(x, c) = \mathbb{1}[c \in \text{NN}_C^r(x)] \frac{h(x - c)}{\sum_j h(x - c_j)}$$

- this is attributed to respecting the manifold structure of the data, and it superior to more expensive sparse coding that have been proposed in the meantime

# soft assignment

[Philbin et al. 2008]

- on retrieval: “kernel” is followed on  $r$  nearest neighbors

$$f_C(x, c) = \mathbb{1}[c \in \text{NN}_C^r(x)]h(x - c)$$

- it is better to discard descriptors if they are not well represented
- $r$  should be small: this applies to dataset images and increases the required index space and query time (including spatial matching) by  $r$

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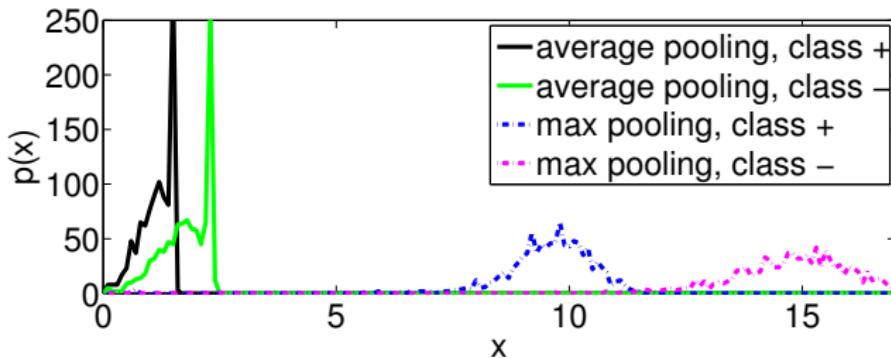
# multiple assignment

[Jégou et al. 2010]

- on retrieval: same as before, but now applies only to query images
- $f_C(x, c)$  further limited to visual words at distance  $\leq \alpha d_1$  from  $x$ , where  $d_1$  is the distance of  $\text{NN}_C^1(x)$
- index space maintained as in standard hard assignment, but query time is still increased by  $r$

# max pooling vs. average pooling

[Boureau et al. 2010]



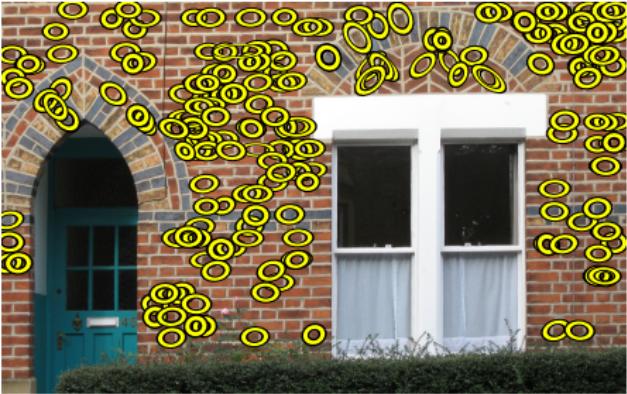
- on classification: max-pooling superior to average pooling

$$g_{\max}(A) = \left( \max_{\mathbf{a} \in A} a_1, \dots, \max_{\mathbf{a} \in A} a_k \right) \quad g_{\text{avg}}(A) = \frac{1}{|A|} \sum_{\mathbf{a} \in A} \mathbf{a}$$

- with max-pooling, SVM with linear and nonlinear kernel perform nearly the same

# burstiness

[Jégou et al. 2009]

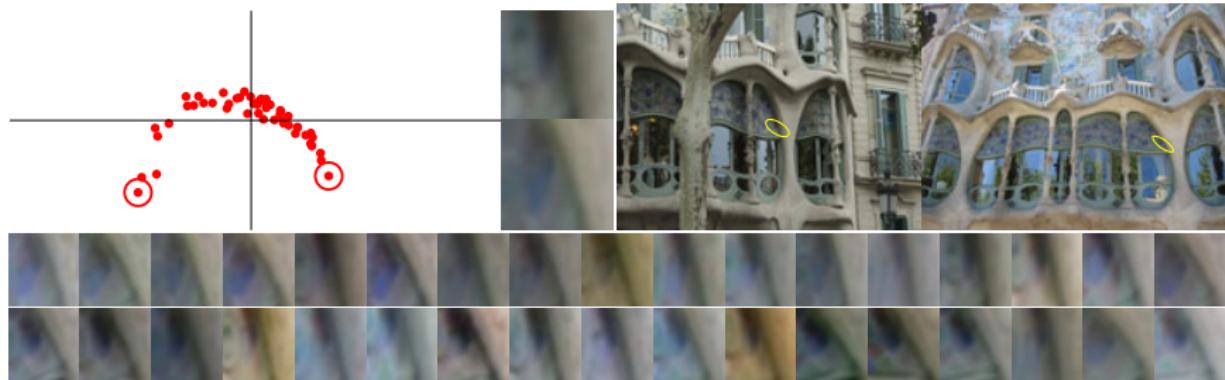


- **burstiness**: descriptors appear more frequently than a statistically independent model predicts; it hurts performance because bursty features dominate the image similarity
  - **on retrieval**: the situation is more complex here; max-pooling would be like keeping only one representative per cell, average pooling like keeping all, but none is the best choice

# beyond codebooks

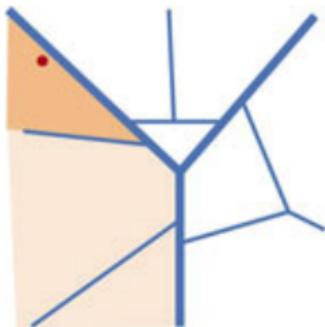
# learning cell shapes

[Mikulik et al. 2010]

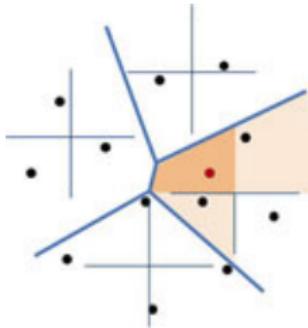


- **on retrieval:** matched across images in an entire dataset, features are connected into **feature tracks**
- feature tracks have curved shape in descriptor space, contrary to the Gaussian assumption—an example of **manifold structure**
- even if such structure cannot be captured by  $k$ -means, cells can still be connected via feature tracks → vocabulary of **16M words**

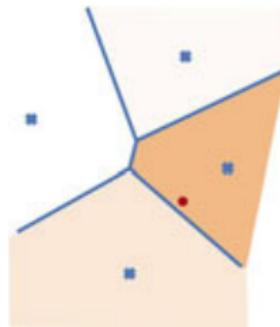
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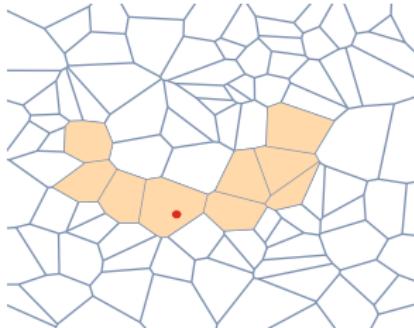
HKM



## Hamming



## soft assignment



learned

## descriptor matching

- **on retrieval:** given two images with descriptors  $X, Y \subset \mathbb{R}^d$ , and recalling  $X_c = \{x \in X : q(x) = c\}$ , bag-of-words similarity on  $C$  is

$$\begin{aligned}s_{\text{BoW}}(X, Y) &\propto \sum_{c \in C} w_c |X_c| |Y_c| \\&= \sum_{c \in C} w_c \sum_{x \in X_c} \sum_{y \in Y_c} 1\end{aligned}$$

- if descriptors are available in some form (**more space**), it is better to use a more general function of the form

$$K(X, Y) := \gamma(X)\gamma(Y) \sum_{c \in C} w_c M(X_c, Y_c)$$

where  $M$  is a **within-cell** matching function and  $\gamma(X)$  serves for normalization

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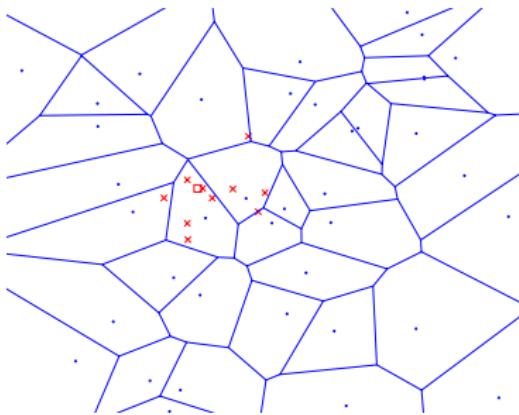
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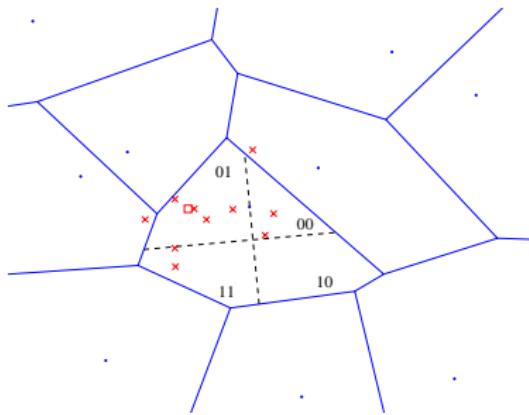
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# Hamming embedding (HE)

[Jégou et al. 2008]



fine vocabulary



Hamming embedding

- each descriptor  $x$  is **binarized** into  $b(x) \in \{0, 1\}^d$
- pairs within cells are kept only if **Hamming distance** is at most  $\tau$

$$M_{\text{HE}}(X_c, Y_c) := \sum_{x \in X_c} \sum_{y \in Y_c} \mathbb{1}[d_{\text{H}}(b(x), b(y)) \leq \tau]$$

# aggregated selective match kernel (ASMK)\*

[Tolias et al. 2013]

- borrow from HE the idea that descriptor pairs are **selected** by a nonlinear function

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$$M_{\text{VLAD}}(X_c, Y_c) := V(X_c)^{\top} V(Y_c) = \sum_{x \in X_c} \sum_{y \in Y_c} r(x)^{\top} r(y)$$

- combine pooling **within** cells with selectivity **between** cells

$$M_{\text{ASMK}}(X_c, Y_c) := \sigma_{\alpha}(\hat{V}(X_c)^{\top} \hat{V}(Y_c))$$

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- apart from saving space, pooling and normalizing per cell helps fight **burstiness**
- still, unlike VLAD, due to the nonlinearity we cannot have a low dimensional embedding
- it is targeting large vocabularies, which, together with compressed descriptors (as in HE), takes up a lot of space

Tolias, Avrithis and Jegou. ICCV 2013. To Aggregate or not to Aggregate: Selective Match Kernels for Image Search.  
Jegou, Douze and Schmid. CVPR 2009. On the burstiness of visual elements.

# efficient match kernels (EMK)

[Bo and Sminchisescu. NIPS 2009]

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- use a continuous function  $\kappa(x, y)$  instead, with no codebook

$$K(X, Y) := \gamma(X) \gamma(Y) \sum_{x \in X} \sum_{y \in Y} \kappa(x, y)$$

- derive an approximate finite-dimensional feature map  $\phi$  such that  $\kappa(x, y) = \phi(x)^\top \phi(y)$ , and

$$K(X, Y) = \left( \gamma(X) \sum_{x \in X} \phi(x) \right)^\top \left( \gamma(Y) \sum_{y \in Y} \phi(y) \right) = \Phi(X)^\top \Phi(Y)$$

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# efficient match kernels (EMK)

[Bo and Sminchisescu. NIPS 2009]

- on classification: given two images with descriptors  $X, Y \subset \mathbb{R}^d$ , bag-of-words similarity on  $C$  is

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- given a function  $K(X, Y)$  on sets  $X, Y$  in the form of a pairwise sum of **nonlinear** functions  $\kappa(x, y)$  of the elements  $x \in X, y \in Y$ , we can decompose it into an inner product of  $\Phi(X), \Phi(Y)$
- this can be done by
  - **encoding**  $x \mapsto \phi(x)$
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- this is always possible for **positive-definite** functions  $\kappa$  but  $\phi$  may be infinite-dimensional; in nonlinear SVM, it is only implicit through  $\kappa$
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# pyramid matching

# histogram intersection

[Swain and Ballard 1991]

- the sum  $\sum_{x \in X_c} \sum_{y \in Y_c} 1$  appearing in  $s_{\text{BoW}}(X, Y)$  implies an **all-all** matching; it is often preferable to have an **one-one** matching instead



- given two histograms  $x, y$  of  $b$  bins, their **histogram intersection** is

$$\kappa_{\text{HI}}(x, y) = \sum_{i=1}^b \min(x_i, y_i)$$

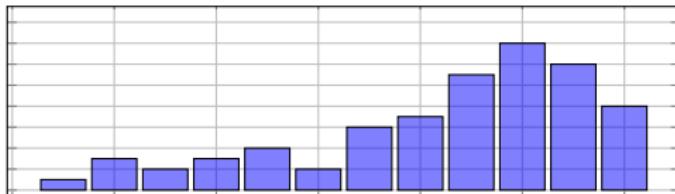
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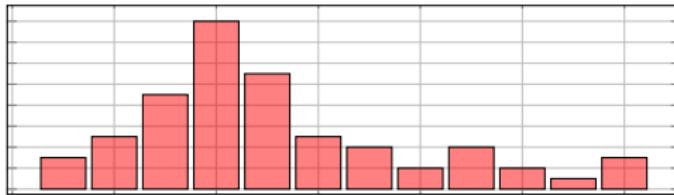
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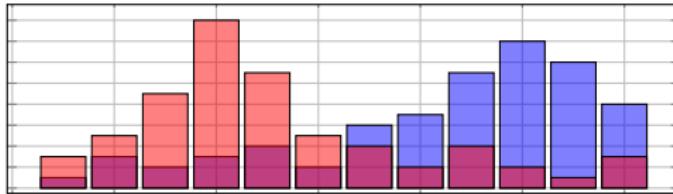
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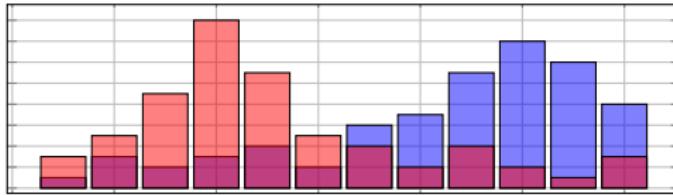
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# pyramid match kernel (PMK)

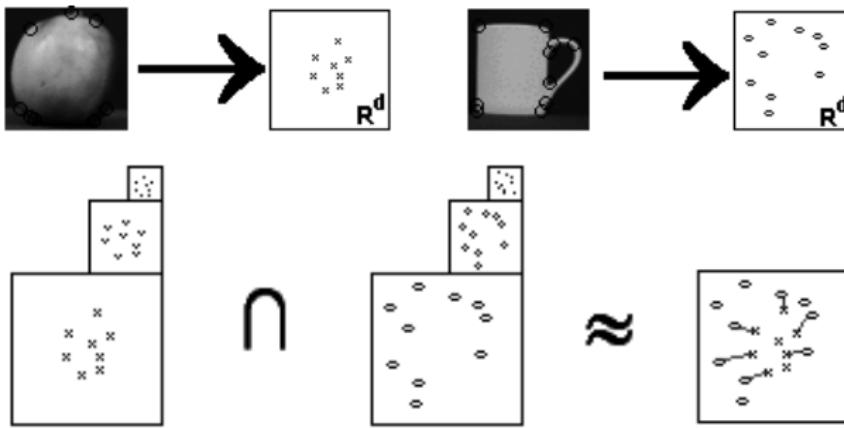
[Grauman and Darrell 2005]



- given the descriptors of two images as point sets  $X, Y$  in  $\mathbb{R}^d$
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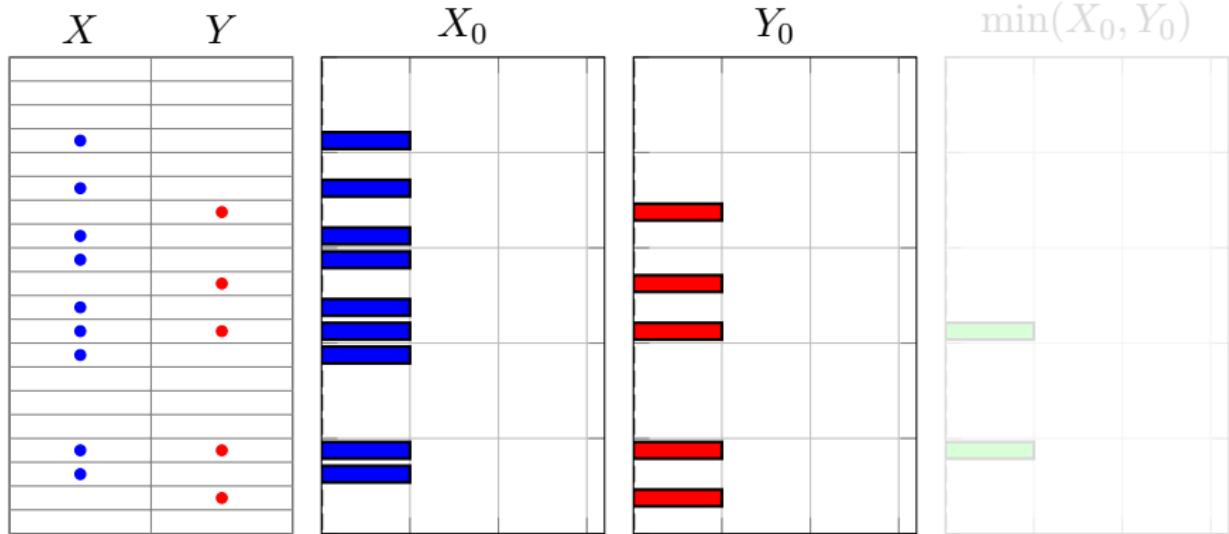
# pyramid match kernel (PMK)



- 1d point sets  $X, Y$  on grid of size 1



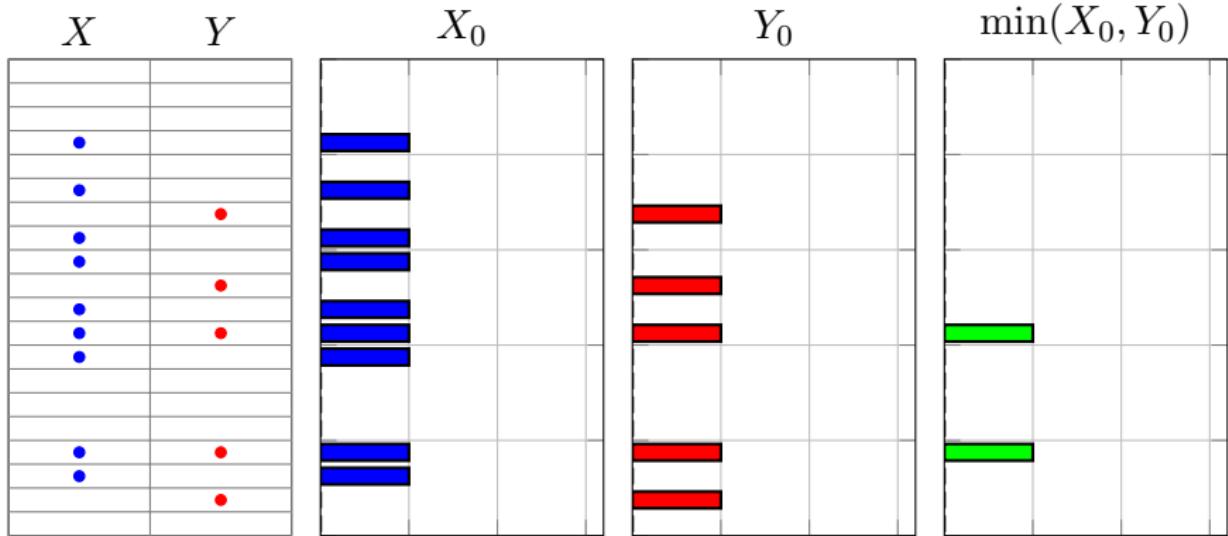
# pyramid match kernel (PMK)



- 1d point sets  $X, Y$  on grid of size 1 - level 0 histograms



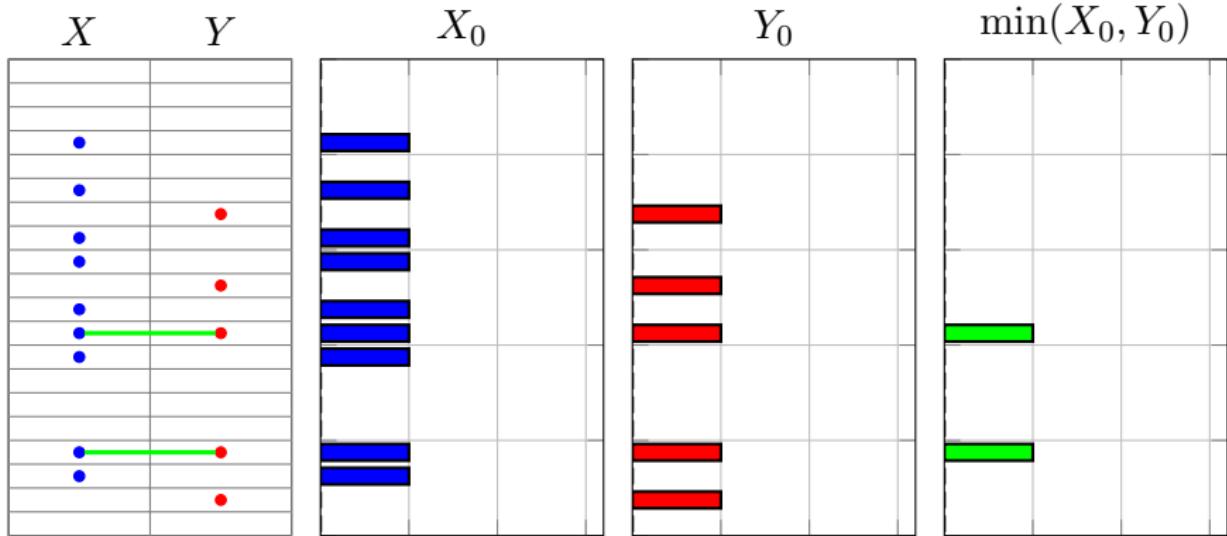
# pyramid match kernel (PMK)



- 1d point sets  $X, Y$  on grid of size 1 - level 0 histograms - intersection

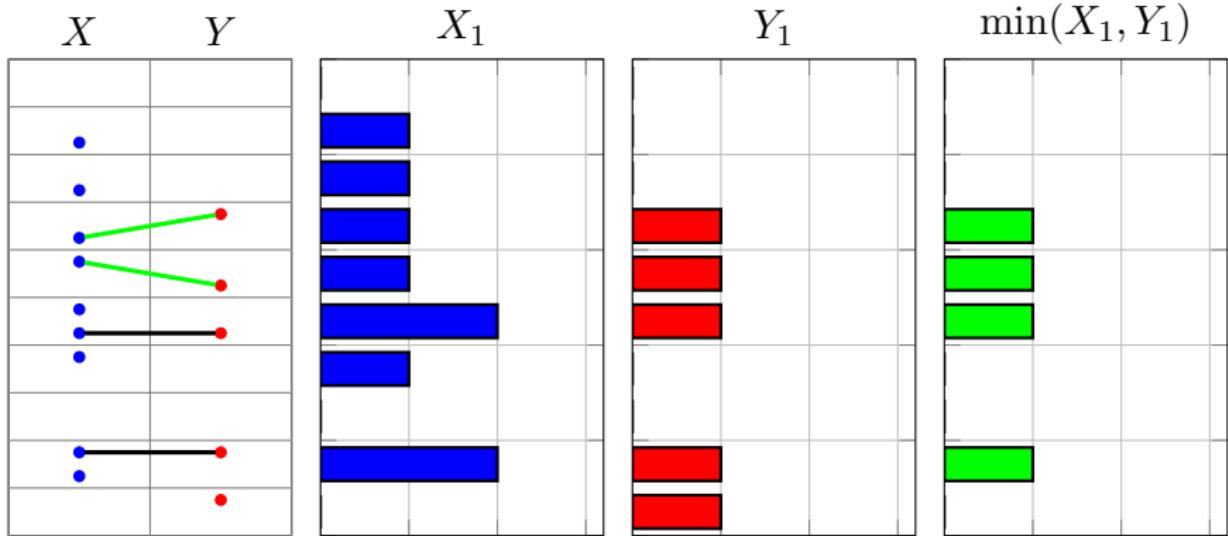


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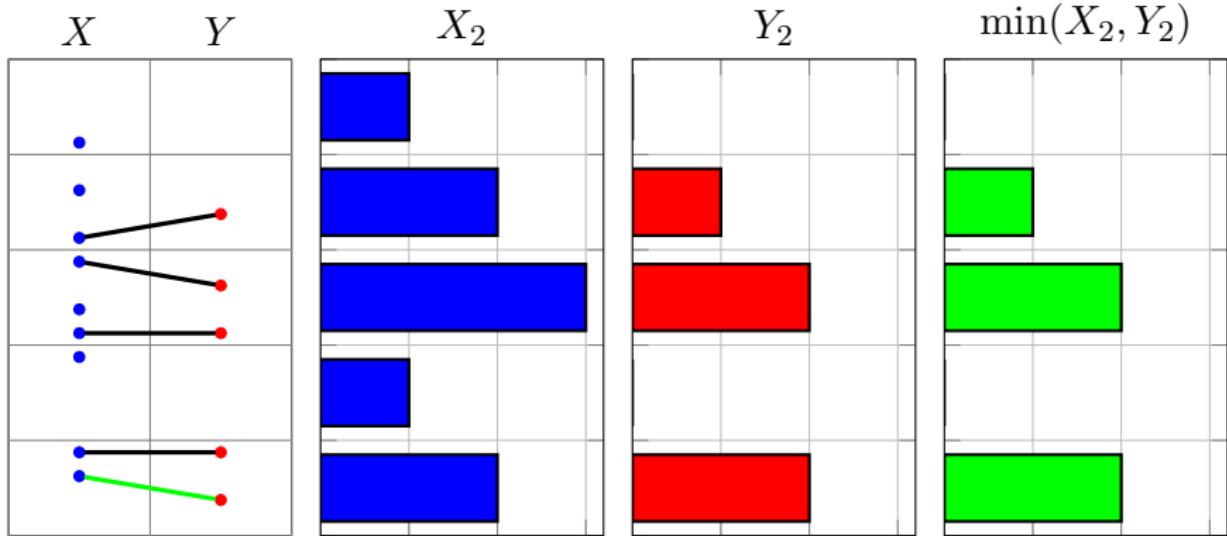
- 1d point sets  $X, Y$  on grid of size 1 - level 0 histograms - intersection
  - (2 matches weighted by 1)
  - total score  $2 \times 1$

# pyramid match kernel (PMK)



- 1d point sets  $X, Y$  on grid of size 2 - level 1 histograms - intersection
- (2 matches weighted by 1) + (2 weighted by  $\frac{1}{2}$ )
- total score  $2 \times 1 + 2 \times \frac{1}{2}$

# pyramid match kernel (PMK)



- 1d point sets  $X, Y$  on grid of size 4 - level 2 histograms - intersection
- $(2 \text{ matches weighted by } 1) + (2 \text{ weighted by } \frac{1}{2}) + (1 \text{ weighted by } \frac{1}{4})$
- total score  $2 \times 1 + 2 \times \frac{1}{2} + 1 \times \frac{1}{4}$

## pyramid match kernel (PMK)

- given a set  $X = \{x_1, \dots, x_n\} \subset \mathbb{R}^d$ , where distances of elements range in  $[1, D]$
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- $i$  ranges from  $-1$ , where each bin has at most one element, to  $L = \lceil \log_2 D \rceil$ , where  $X$  is contained in a single bin
- given two images with descriptors  $X, Y \subset \mathbb{R}^d$ , their **pyramid match** is

$$\begin{aligned} K_{\Delta}(X, Y) &= \gamma(X)\gamma(Y) \sum_{i=0}^L \frac{1}{2^i} (\kappa_{\text{HI}}(X_i, Y_i) - \kappa_{\text{HI}}(X_{i-1}, Y_{i-1})) \\ &= \gamma(X)\gamma(Y) \left( \frac{1}{2^L} \kappa_{\text{HI}}(X_L, Y_L) + \sum_{i=0}^{L-1} \frac{1}{2^{i+1}} \kappa_{\text{HI}}(X_i, Y_i) \right) \end{aligned}$$

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# PMK is a positive-definite kernel

- $\kappa_{\Delta}$  can be written as a weighted sum of  $\kappa_{HI}$  terms, with nonnegative coefficients
- $\kappa_{HI}$  can be written as a sum of min terms
- min can be written as a dot product:

| $x$              | $\phi(x)$ |   |   |   |   |   |   |   |
|------------------|-----------|---|---|---|---|---|---|---|
| 3                | 1         | 1 | 1 | 0 | 0 | 0 | 0 | 0 |
| 5                | 1         | 1 | 1 | 1 | 1 | 0 | 0 | 0 |
| $\min(x, y) = 3$ | 1         | 1 | 1 | 0 | 0 | 0 | 0 | 0 |

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# PMK as an embedding

[Indyk and Thaper 2003]

- there is an explicit embedding for  $\kappa_{\text{HI}}$ , therefore also for  $\kappa_{\Delta}$
- if  $|X| \leq |Y|$  and  $\pi : X \rightarrow Y$  is one-to-one, then  $K_{\Delta}(X, Y)$  approximates the optimal pairwise matching

$$\max_{\pi} \sum_{x \in X} \|x - \pi(x)\|_1^{-1}$$

- this was first shown on the earth mover's distance

$$\min_{\pi} \sum_{x \in X} \|x - \pi(x)\|_1$$

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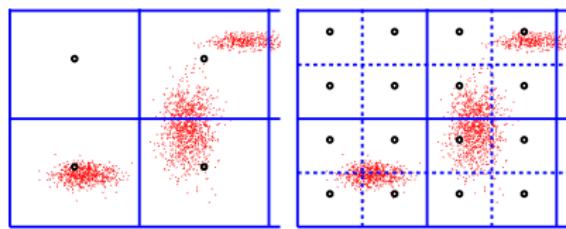
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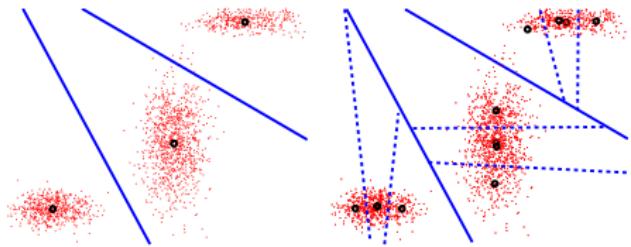
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# PMK and vocabulary tree

[Grauman and Darrell 2007]



uniform bins

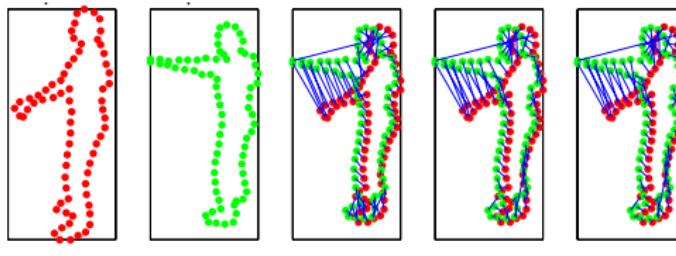


vocabulary-guided bins

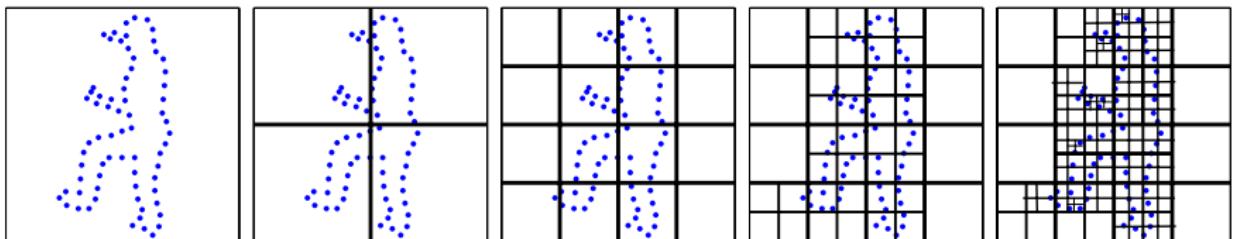
- replace regular grid with hierarchical vocabulary cells
- compared to vocabulary tree, there is a principle in assigning cell weights
- still, its approximation quality suffers at high dimensions

# PMK and spatial matching

[Grauman and Darrell 2004]



optimal matching

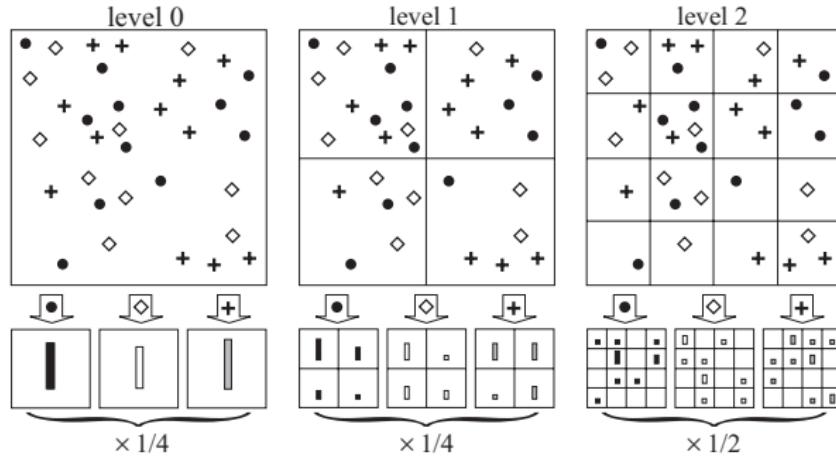


representation

- same idea, applied to image 2d coordinate space for spatial matching
- matching cost is only based on point coordinates; no appearance

# spatial pyramid matching (SPM)

[Lazebnik et al. 2006]

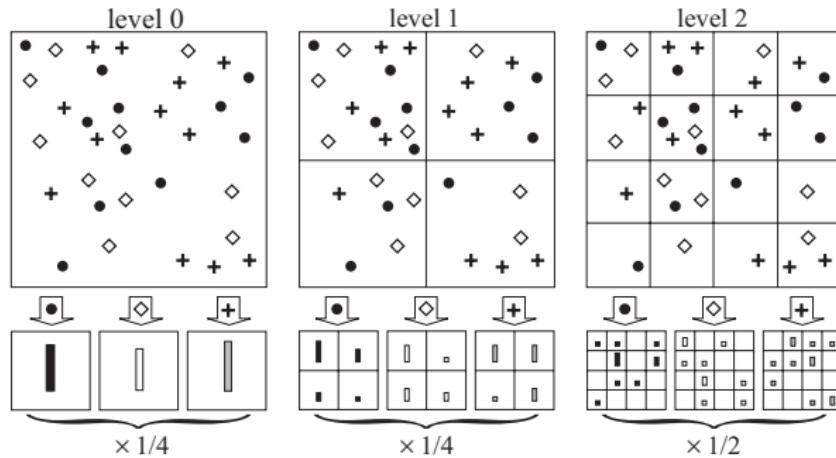


- if  $X^{(j)}, Y^{(j)}$  are the feature coordinates of images  $X, Y$  with descriptors assigned to visual word  $j$ ,

$$K_{\text{SP}}(X, Y) = \sum_{j=1}^k K_{\Delta}(X^{(j)}, Y^{(j)})$$

# spatial pyramid matching (SPM)

[Lazebnik et al. 2006]



- coupled with BoW, it is a set of joint appearance-geometry histograms
- robust to deformation but not invariant to transformations; applied to global scene classification

# Hough pyramid matching (HPM)\*

[Tolias and Avrithis 2011]

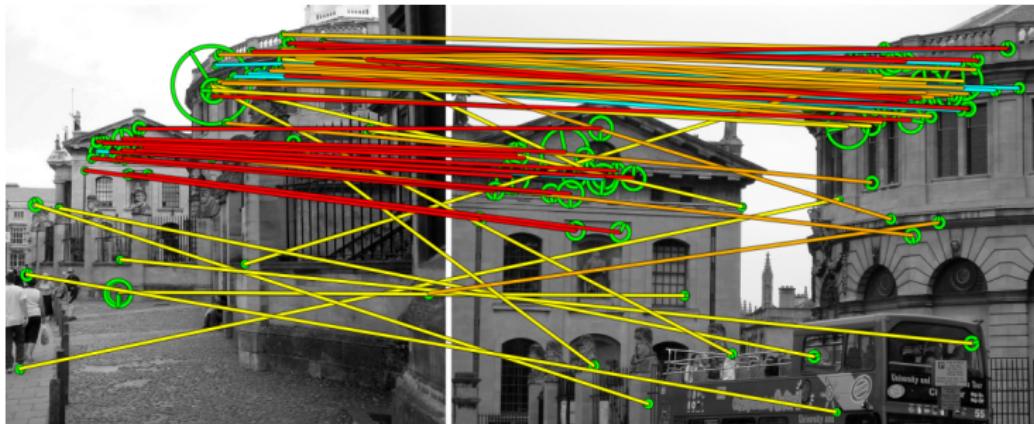


## fast spatial matching

- work with a single set of correspondences instead of two sets of features
- determine a transformation hypothesis by a pair of features and then use histograms to collect votes in the transformation space

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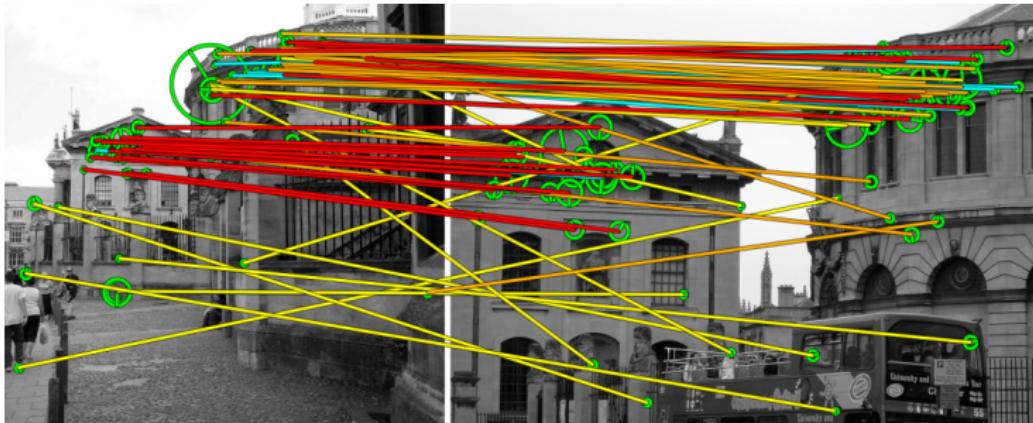


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# Hough pyramid matching (HPM)\*

- a **local feature**  $p$  in image  $P$  has position  $\mathbf{t}(p)$ , scale  $s(p)$  and orientation  $\theta(p)$  given by matrix  $R(p) \in \mathbb{R}^{2 \times 2}$

$$F(p) = \begin{pmatrix} s(p)R(p) & \mathbf{t}(p) \\ \mathbf{0}^\top & 1 \end{pmatrix}$$

- a **correspondence**  $c = (p, q)$  is a pair of features  $p \in P, q \in Q$  of two images  $P, Q$  and determines relative similarity transformation from  $p$  to  $q$

$$F(c) = F(q)F(p)^{-1} = \begin{pmatrix} s(c)R(c) & \mathbf{t}(c) \\ \mathbf{0}^\top & 1 \end{pmatrix}$$

with translation  $\mathbf{t}(c) = \mathbf{t}(q) - s(c)R(c)\mathbf{t}(p)$ , relative scale  $s(c) = s(q)/s(p)$  and rotation  $R(c) = R(q)R(p)^{-1}$  or  
 $\theta(c) = \theta(q) - \theta(p)$

# Hough pyramid matching (HPM)\*

- the 4-dof relative transformation represented by 4d vector

$$f(c) = (\mathbf{t}(c), s(c), \theta(c))$$

- to enforce one-to-one mapping, two correspondences  $c = (p, q)$ ,  $c' = (p', q')$  are **conflicting** if they refer to the same feature on either image, i.e.  $p = p'$  or  $q = q'$

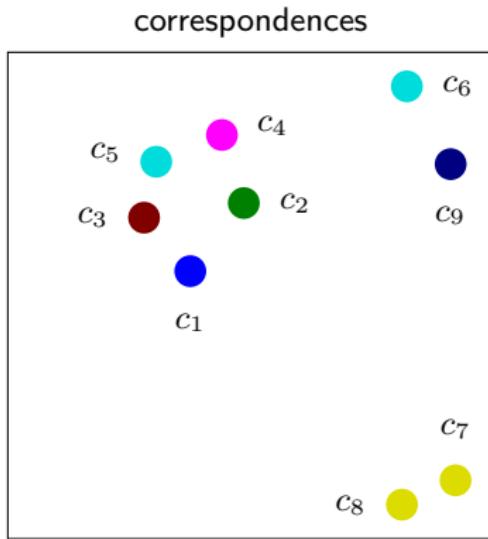
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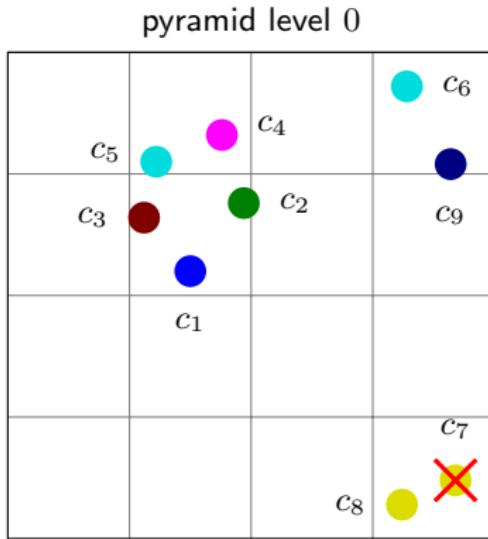
# Hough pyramid matching (HPM)\*



|       | $p$ | $q$ | similarity score                          |
|-------|-----|-----|---|
| $c_1$ |     |     | $(2 + \frac{1}{2}2 + \frac{1}{4}2)w(c_1)$ |
| $c_2$ |     |     | $(2 + \frac{1}{2}2 + \frac{1}{4}2)w(c_2)$ |
| $c_3$ |     |     | $(2 + \frac{1}{2}2 + \frac{1}{4}2)w(c_3)$ |
| $c_4$ |     |     | $(1 + \frac{1}{2}3 + \frac{1}{4}2)w(c_4)$ |
| $c_5$ |     |     | $(1 + \frac{1}{2}3 + \frac{1}{4}2)w(c_5)$ |
| $c_6$ |     |     | 0   |
| $c_7$ |     |     | 0   |
| $c_8$ |     |     | $(\frac{1}{4}6)w(c_8)$                    |
| $c_9$ |     |     | $(\frac{1}{4}6)w(c_9)$                    |

- correspondence  $c$  contributes by  $w(c)$ , based e.g. on visual word
- conflicting correspondences in the same bin  $b$  are erased
- in a bin  $b$  with  $n_b$  correspondences, each groups with  $[n_b - 1]_+$  others
- level 0 weight 1

# Hough pyramid matching (HPM)\*

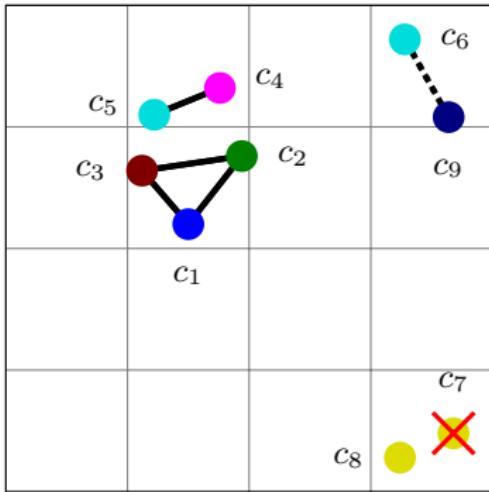


|       | $p$ | $q$ | similarity score                          |
|-------|-----|-----|---|
| $c_1$ |     |     | $(2 + \frac{1}{2}2 + \frac{1}{4}2)w(c_1)$ |
| $c_2$ |     |     | $(2 + \frac{1}{2}2 + \frac{1}{4}2)w(c_2)$ |
| $c_3$ |     |     | $(2 + \frac{1}{2}2 + \frac{1}{4}2)w(c_3)$ |
| $c_4$ |     |     | $(1 + \frac{1}{2}3 + \frac{1}{4}2)w(c_4)$ |
| $c_5$ |     |     | $(1 + \frac{1}{2}3 + \frac{1}{4}2)w(c_5)$ |
| $c_6$ |     |     | 0   |
| $c_7$ |     |     | 0   |
| $c_8$ |     |     | $(\frac{1}{4}6)w(c_8)$                    |
| $c_9$ |     |     | $(\frac{1}{4}6)w(c_9)$                    |

- correspondence  $c$  contributes by  $w(c)$ , based e.g. on visual word
- conflicting correspondences in the same bin  $b$  are erased
- in a bin  $b$  with  $n_b$  correspondences, each groups with  $[n_b - 1]_+$  others
- level 0 weight 1

# Hough pyramid matching (HPM)\*

pyramid level 0

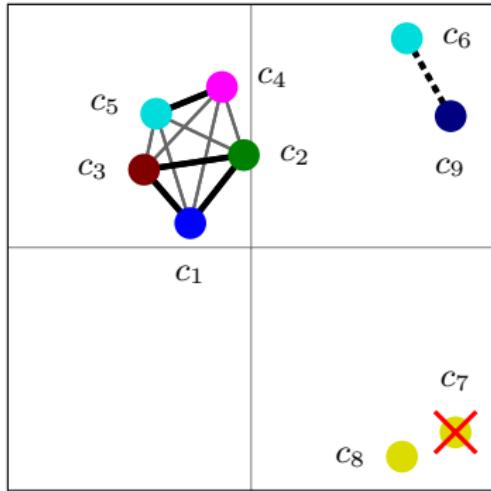


|       | $p$ | $q$ | similarity score                          |
|-------|-----|-----|---|
| $c_1$ |     |     | $(2 + \frac{1}{2}2 + \frac{1}{4}2)w(c_1)$ |
| $c_2$ |     |     | $(2 + \frac{1}{2}2 + \frac{1}{4}2)w(c_2)$ |
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- level 0 weight 1

# Hough pyramid matching (HPM)\*

pyramid level 1

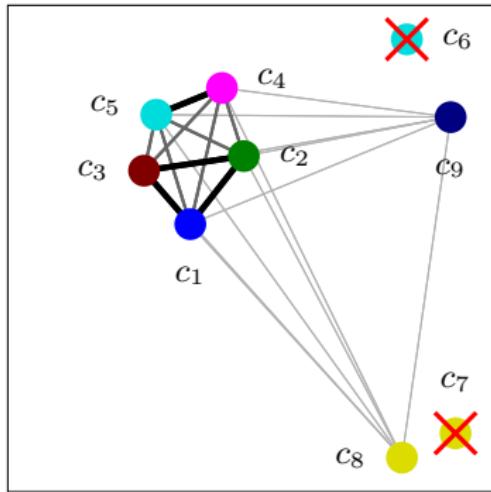


|       | $p$ | $q$ | similarity score                          |
|-------|-----|-----|---|
| $c_1$ |     |     | $(2 + \frac{1}{2}2 + \frac{1}{4}2)w(c_1)$ |
| $c_2$ |     |     | $(2 + \frac{1}{2}2 + \frac{1}{4}2)w(c_2)$ |
| $c_3$ |     |     | $(2 + \frac{1}{2}2 + \frac{1}{4}2)w(c_3)$ |
| $c_4$ |     |     | $(1 + \frac{1}{2}3 + \frac{1}{4}2)w(c_4)$ |
| $c_5$ |     |     | $(1 + \frac{1}{2}3 + \frac{1}{4}2)w(c_5)$ |
| $c_6$ |     |     | 0   |
| $c_7$ |     |     | 0   |
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| $c_9$ |     |     | $(\frac{1}{4}6)w(c_9)$                    |

- correspondence  $c$  contributes by  $w(c)$ , based e.g. on visual word
- conflicting correspondences in the same bin  $b$  are **erased**
- in a bin  $b$  with  $n_b$  correspondences, each groups with  $[n_b - 1]_+$  others
- level 1 weight  $\frac{1}{2}$

# Hough pyramid matching (HPM)\*

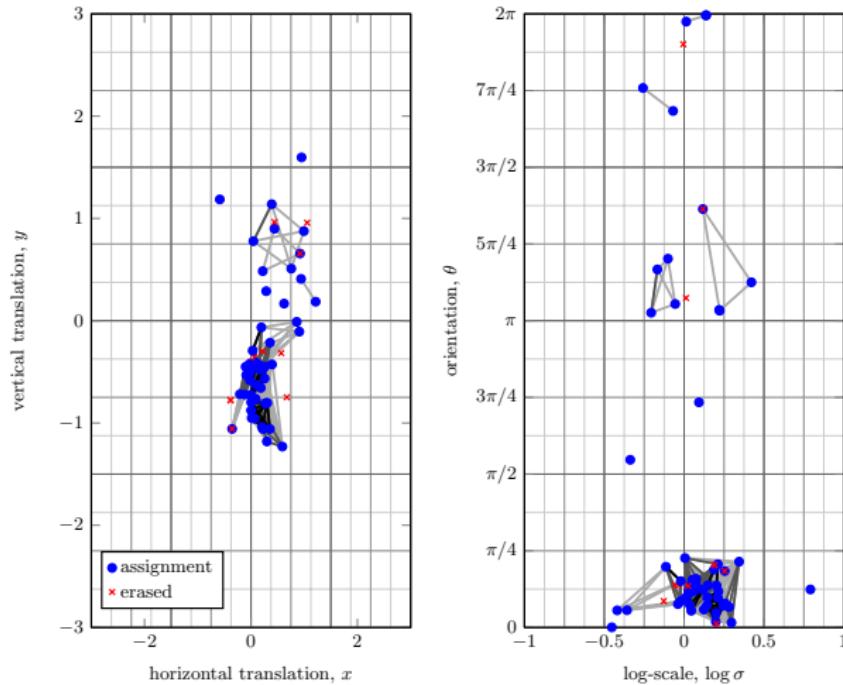
pyramid level 2



|       | $p$ | $q$   | similarity score                          |
|-------|-----|-------|---|
| $c_1$ |     | $c_6$ | $(2 + \frac{1}{2}2 + \frac{1}{4}2)w(c_1)$ |
| $c_2$ |     | $c_6$ | $(2 + \frac{1}{2}2 + \frac{1}{4}2)w(c_2)$ |
| $c_3$ |     | $c_6$ | $(2 + \frac{1}{2}2 + \frac{1}{4}2)w(c_3)$ |
| $c_4$ |     | $c_6$ | $(1 + \frac{1}{2}3 + \frac{1}{4}2)w(c_4)$ |
| $c_5$ |     | $c_6$ | $(1 + \frac{1}{2}3 + \frac{1}{4}2)w(c_5)$ |
| $c_6$ |     |       | 0   |
| $c_7$ |     |       | 0   |
| $c_8$ |     | $c_7$ | $(\frac{1}{4}6)w(c_8)$                    |
| $c_9$ |     |       | $(\frac{1}{4}6)w(c_9)$                    |

- correspondence  $c$  contributes by  $w(c)$ , based e.g. on visual word
- conflicting correspondences in the same bin  $b$  are **erased**
- in a bin  $b$  with  $n_b$  correspondences, each groups with  $[n_b - 1]_+$  others
- level 2 weight  $\frac{1}{4}$

# Hough pyramid matching (HPM)\*



- mode seeking: we are looking for regions where density is maximized in the transformation space

# Hough pyramid matching (HPM)\*

- linear in the number of correspondences; no need to count inliers
- robust to deformations and multiple matching surfaces, invariant to transformations
- only applies to same instance matching

## summary

- bag of words: treating geometry separately from appearance, and quantizing descriptors
- BoW for instance and class recognition: what is common, what is different
- $k$ -means, HKM\*, vocabulary tree\*, AKM\*, soft/multiple assignment, max pooling, burstiness
- beyond BoW—matching between sets of features/descriptors that cannot be expressed as dot product: HE, VLAD\*, ASMK\*
- design or learn embeddings: EMK, PMK, SPM, HPM\*?
- a sum of similarities is better than a sum of distances

# discussion

# representation

- convolution is linear + translation invariant (or equivariant) and is the **only** function having these properties
- Gabor filters or histograms of gradient orientations are more or less the same thing and are just the **first layer** of extracting a representation
- they record responses at every possible position, **scale and orientation**, resulting in a 4-dimensional representation; rotation and change of scale in the image behave like translation in the representation space
- convolution means that for every pixel we are looking at some **spatial neighborhood** (in the image domain), but the image has only **one channel** (grayscale)
- histograms can be expressed as two stages of **encoding + pooling**; then we can generalize these operations for the next layers

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# codebooks

- so, for the **second layer** we still have histograms of some kind but now they are over vectors (the filter responses of the first stage) rather than scalars (orientation and scale)
- to make a histogram we need a small number of such vectors, and this we obtain through **vector quantization** (or sampling) of the layer one responses of a given dataset
- so, the concept that such representations are “**hand-crafted**” is incorrect; codebooks are learned from data in an unsupervised fashion
- codebook size, parameters in the encoding and pooling stages etc. are just **hyperparameters** that will we learn through **cross-validation**
- in contrast to layer one, there is **no spatial neighborhood** here (with the exception of HMAX) but there is **depth**, i.e. a number of channels corresponding to the dimensions of these vectors; we will combine both, resulting in 3-dimensional filter kernels

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## local features

- depending on the task (e.g. stereopsis, motion estimation, instance recognition compared to class recognition), not all spatial regions are equally important
- classification works best with dense features, but still, through encoding, the responses to most “visual words” are zero; so there some **sparsity** in the representation, at least before pooling
- in order to make change of scale really behave like translation in the representation space, we also need **scale normalization** and a **logarithm**
- operators that detect local features can be expressed as convolution followed by some kind of competition, but they can require **more than one layers** with **nonlinearities** in between; we will follow this idea for more complex patterns
- given a sparse set of local features, matching becomes easier to formulate compared to e.g. continuous distributions

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# matching

- descriptors are really meant to be used for matching one image to another (e.g. for instance recognition) or one image to a pattern (for classification)
- we want to learn a descriptor such that dot product will be good enough for matching
- we can start by thinking about pairwise matching between two sets of descriptors and come up with (design or learn) a representation, maybe at a higher dimension, such that dot product will be approximating this pairwise matching process
- there should be some invariance to geometric transformations; whether this should be designed or learned is up to discussion

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