

# lecture 6: differentiation

## deep learning for vision

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# outline

gradient descent

gradient computation

automatic differentiation: units

automatic differentiation: functions

# gradient descent

## gradient descent

- a **first-order** Taylor approximation of  $f : \mathbb{R}^d \rightarrow \mathbb{R}$  at  $\mathbf{x}_0$  is

$$f_{\mathbf{x}_0}^{(1)}(\mathbf{x}) := f(\mathbf{x}_0) + (\mathbf{x} - \mathbf{x}_0)^\top \nabla f(\mathbf{x}_0)$$

i.e. , the gradient points in the direction of the greatest increase rate

- a **second-order** approximation needs the Hessian matrix  $Hf$

$$f_{\mathbf{x}_0}^{(2)}(\mathbf{x}) := f_{\mathbf{x}_0}^{(1)}(\mathbf{x}) + \frac{1}{2}(\mathbf{x} - \mathbf{x}_0)^\top (Hf(\mathbf{x}_0))(\mathbf{x} - \mathbf{x}_0)$$

- assuming  $f$  is locally convex with isotropic  $Hf(\mathbf{x}_0) = \frac{1}{\epsilon}I$ , the gradient of  $f_{\mathbf{x}_0}^{(2)}$  is

$$\nabla f_{\mathbf{x}_0}^{(2)}(\mathbf{x}) = \nabla f(\mathbf{x}_0) + \frac{1}{\epsilon}(\mathbf{x} - \mathbf{x}_0)$$

- so if we were to minimize this approximation instead of  $f$ , we would let this gradient vanish and solve for  $\mathbf{x}$

$$\arg \min_{\mathbf{x}} f_{\mathbf{x}_0}^{(2)}(\mathbf{x}) = \mathbf{x}_0 - \epsilon \nabla f(\mathbf{x}_0)$$

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# gradient descent

- this yields the update rule

$$\mathbf{x}^{(\tau+1)} = \mathbf{x}^{(\tau)} - \epsilon \nabla f(\mathbf{x}^{(\tau)})$$

i.e. , we are moving in the direction of the greatest decrease rate such that locally (depending on  $\epsilon$ )

$$\begin{aligned}f_{\mathbf{x}^{(\tau)}}^{(1)}(\mathbf{x}^{(\tau+1)}) &= f(\mathbf{x}^{(\tau)}) + (\mathbf{x}^{(\tau+1)} - \mathbf{x}^{(\tau)})^\top \nabla f(\mathbf{x}^{(\tau)}) \\&= f(\mathbf{x}^{(\tau)}) - \epsilon \nabla f(\mathbf{x}^{(\tau)})^\top \nabla f(\mathbf{x}^{(\tau)}) \\&\leq f(\mathbf{x}^{(\tau)})\end{aligned}$$

- the step size  $\epsilon$  is inversely proportional to the curvature we assume for  $f$  at the local minimum

# gradient descent

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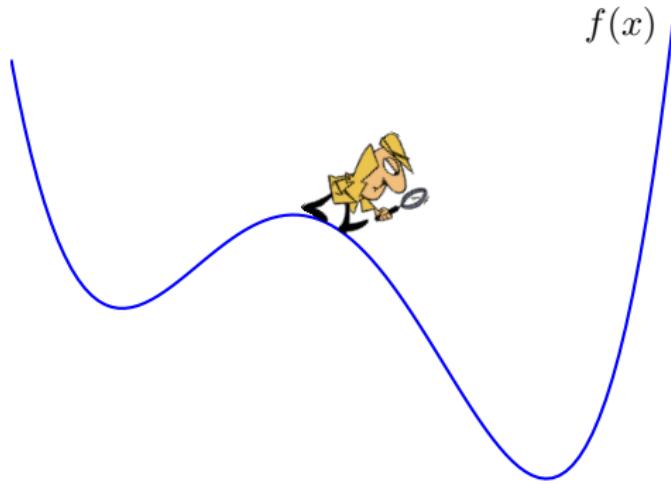
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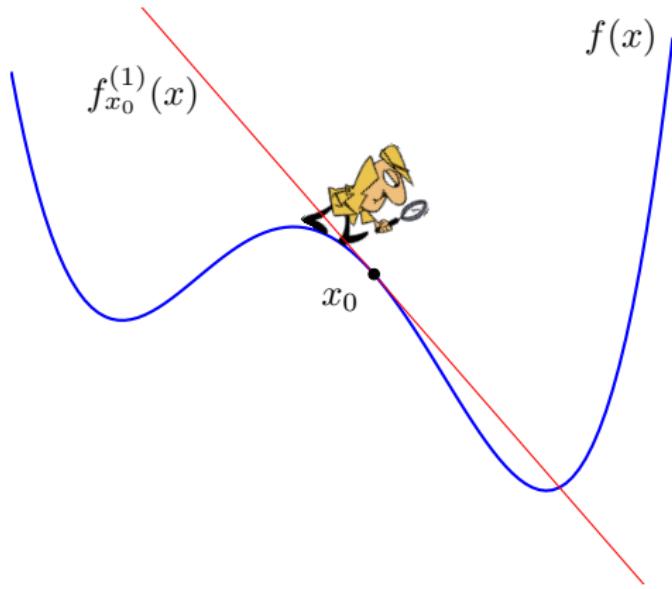
- the step size  $\epsilon$  is inversely proportional to the curvature we assume for  $f$  at the local minimum

# gradient descent in one dimension



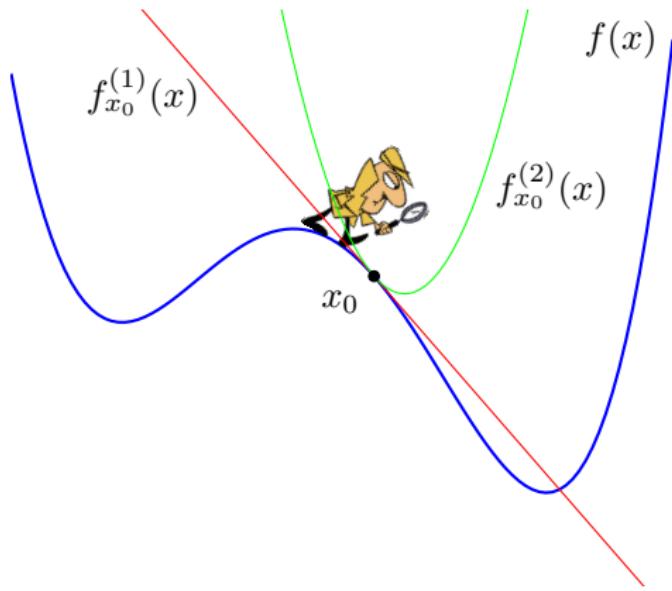
- $\epsilon = 0.05$ : converges to local minimum

# gradient descent in one dimension



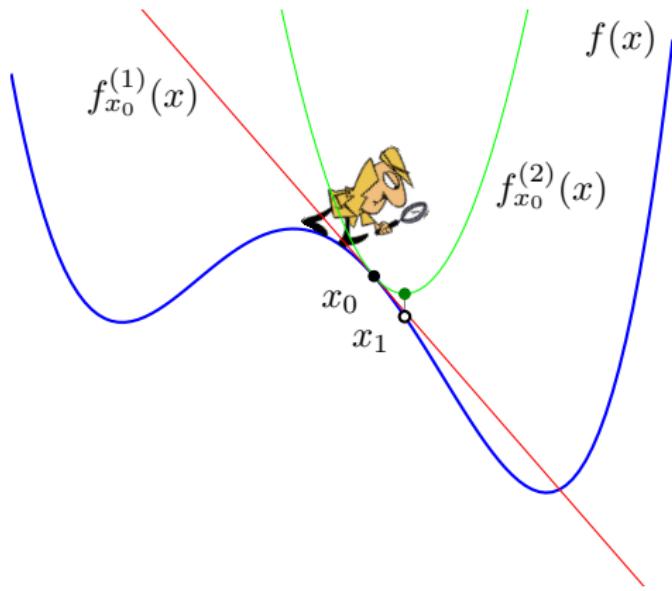
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# gradient descent in one dimension



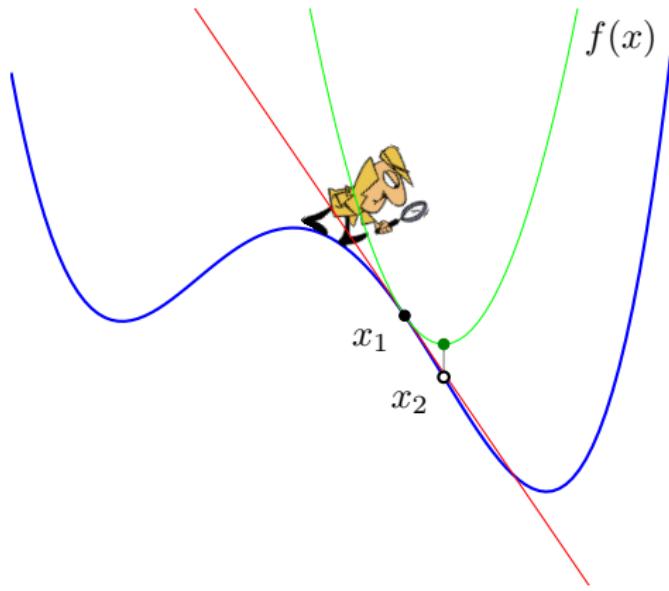
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# gradient descent in one dimension



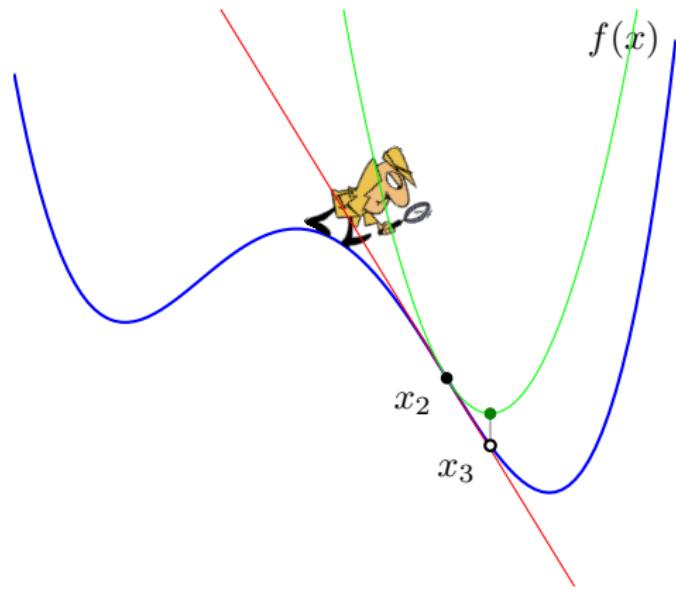
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# gradient descent in one dimension



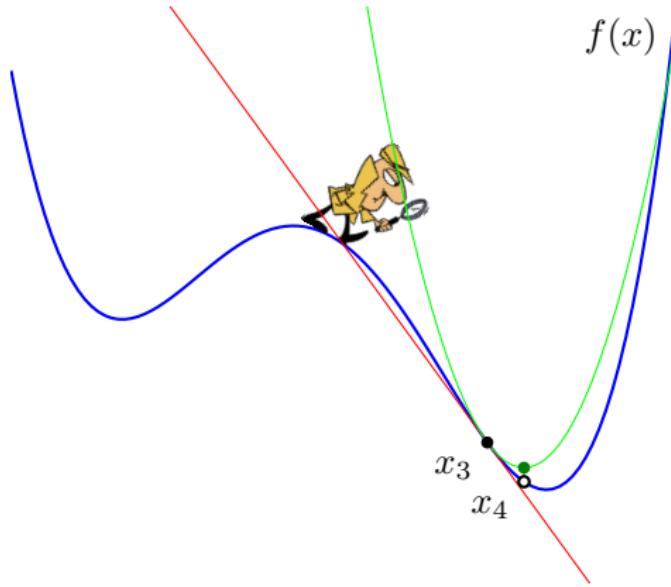
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# gradient descent in one dimension



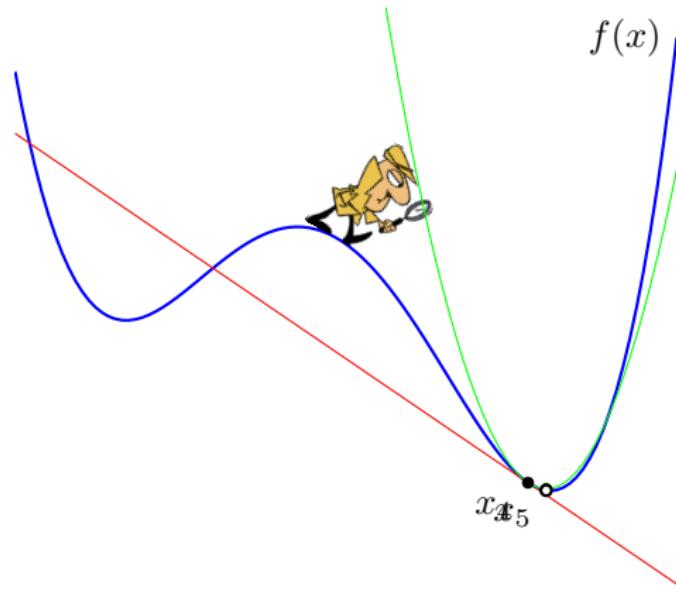
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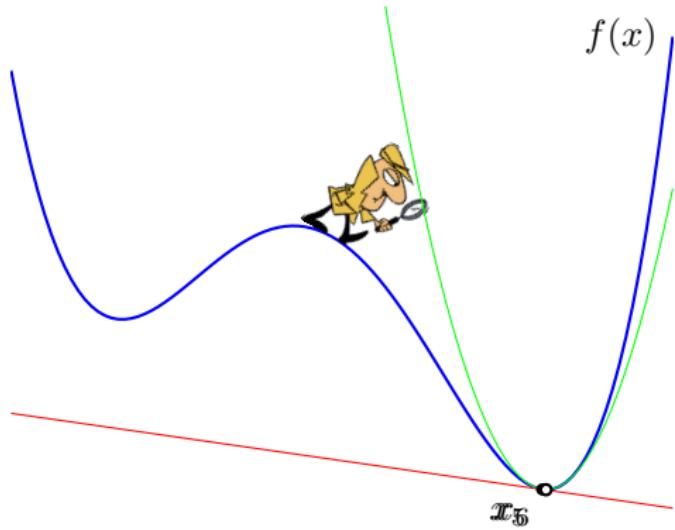
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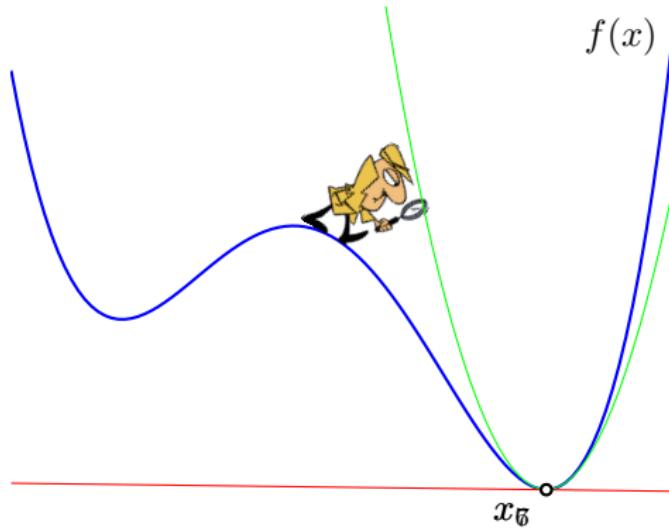
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# gradient descent in one dimension



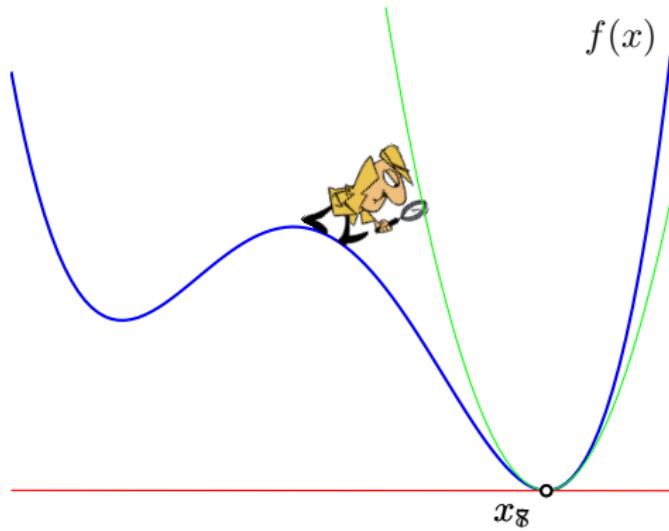
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# gradient descent in one dimension



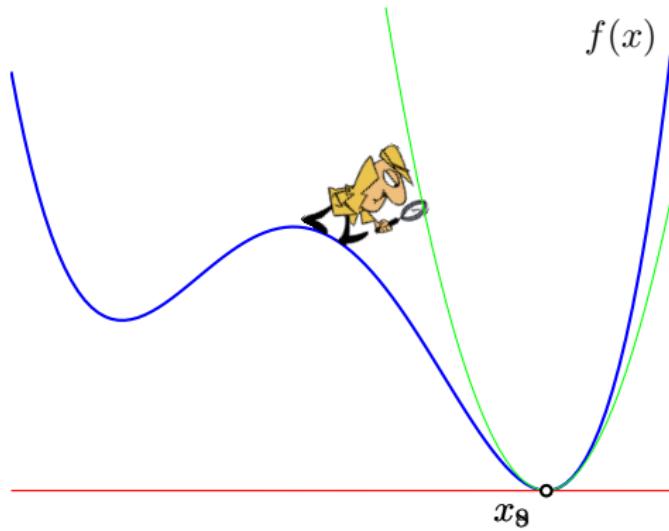
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# gradient descent in one dimension



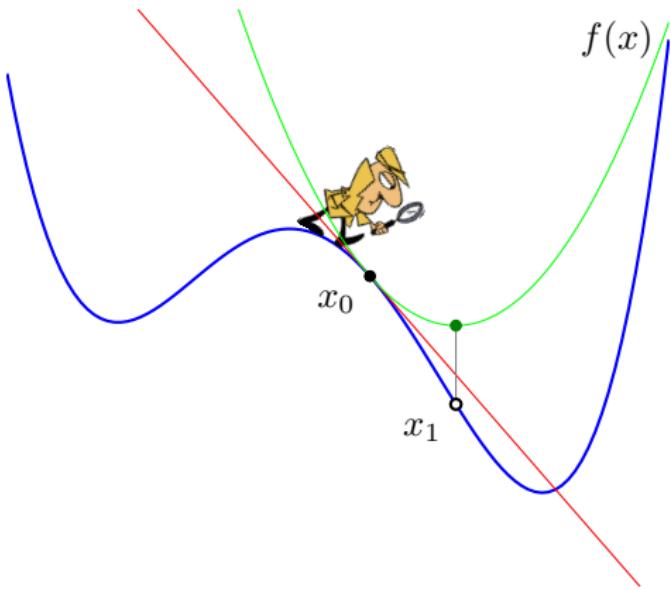
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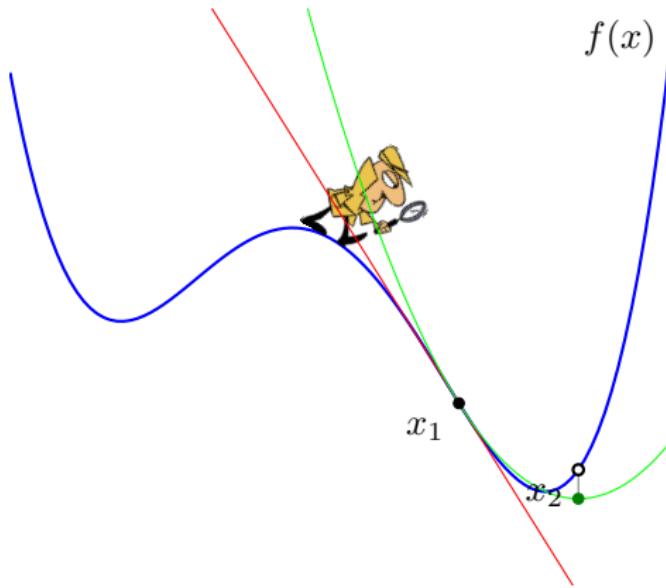
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# gradient descent in one dimension



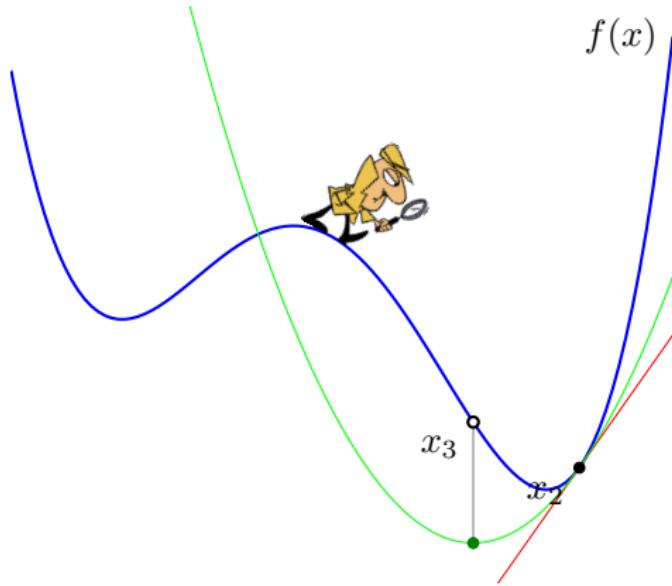
- $\epsilon = 0.14$ :  $1/\epsilon$  less than actual curvature, does not converge

# gradient descent in one dimension



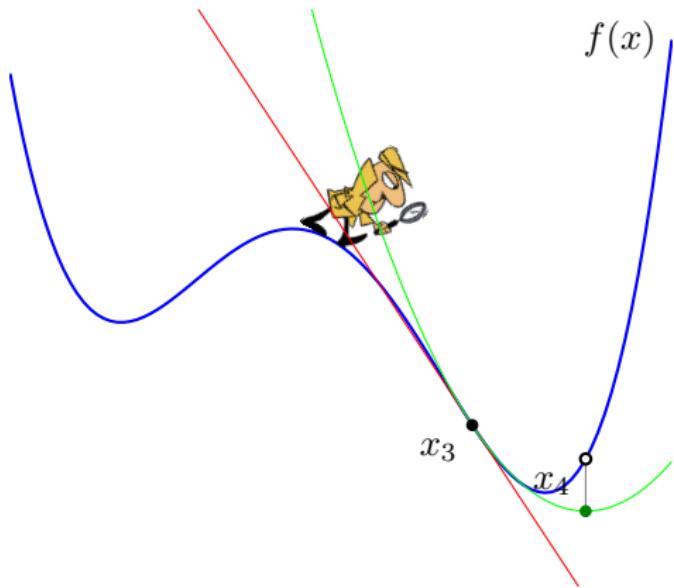
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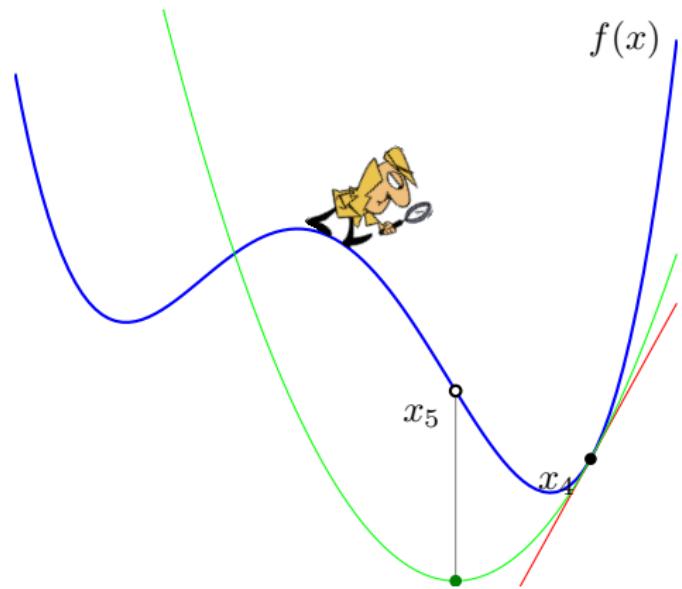
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# gradient descent in one dimension



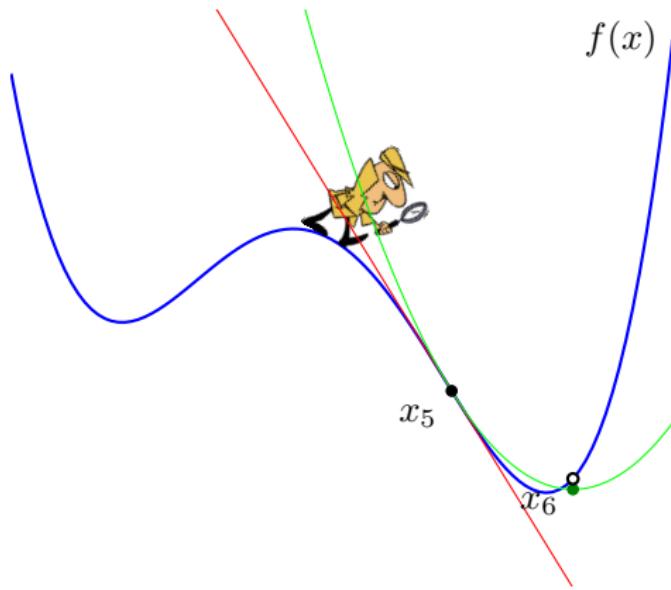
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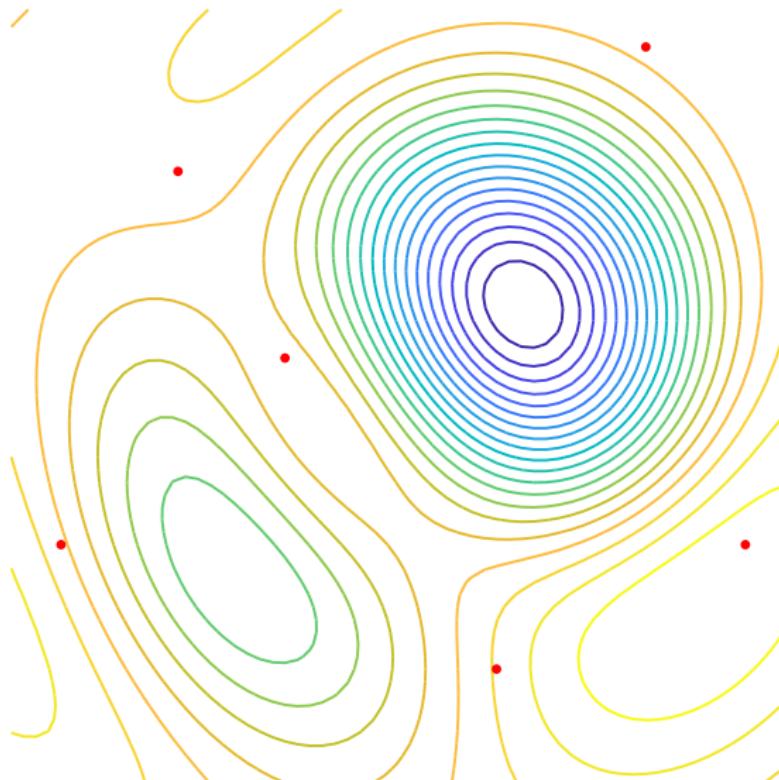
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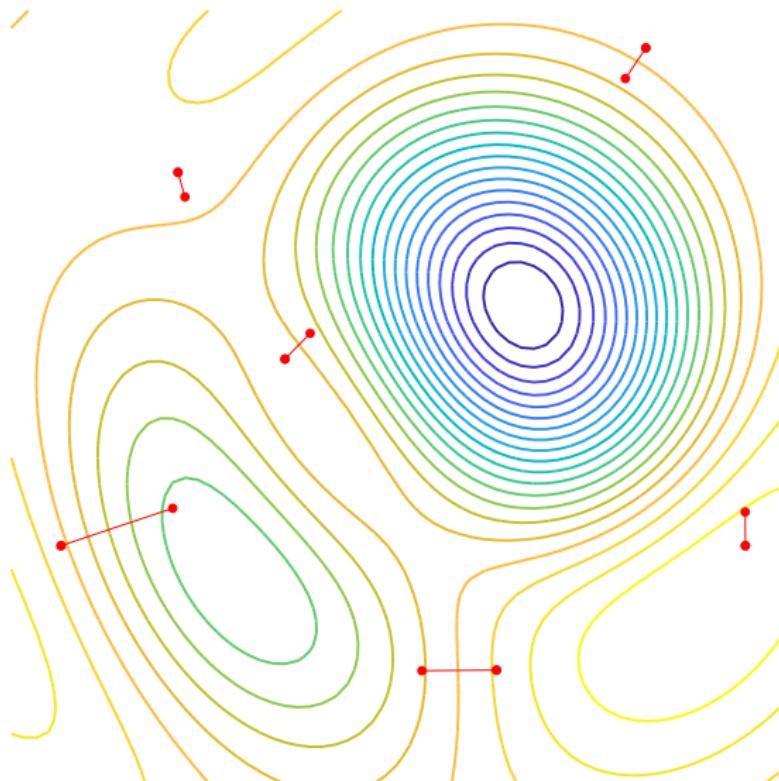
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# gradient descent in two dimensions



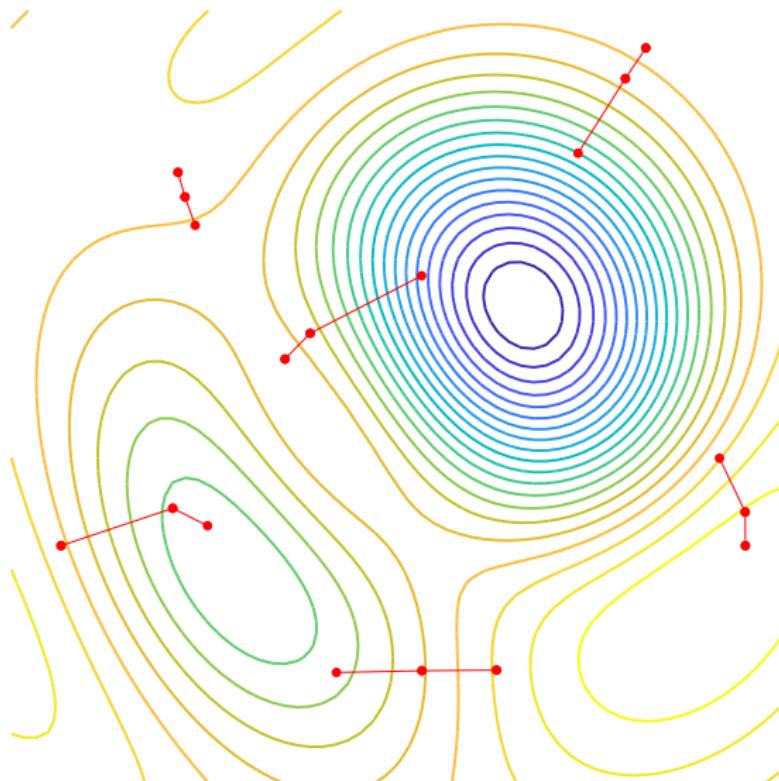
$\epsilon = 0.14$ , iteration 0

# gradient descent in two dimensions



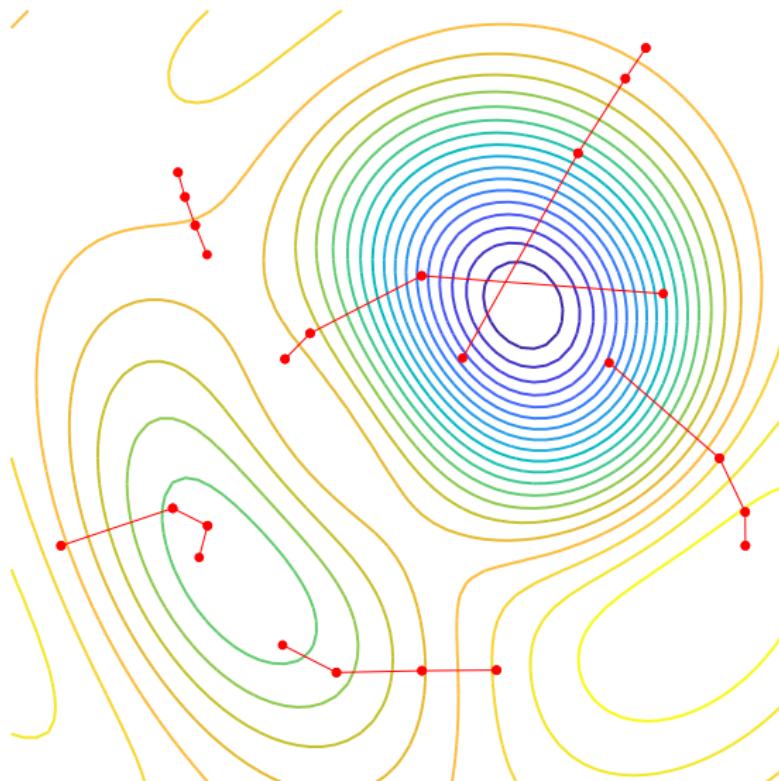
$\epsilon = 0.14$ , iteration 1

# gradient descent in two dimensions



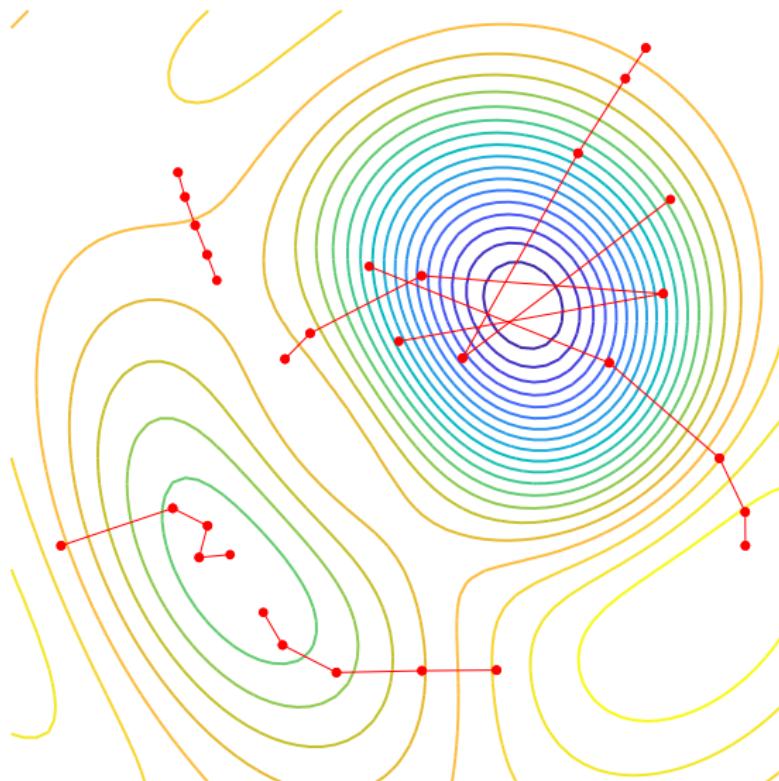
$\epsilon = 0.14$ , iteration 2

# gradient descent in two dimensions



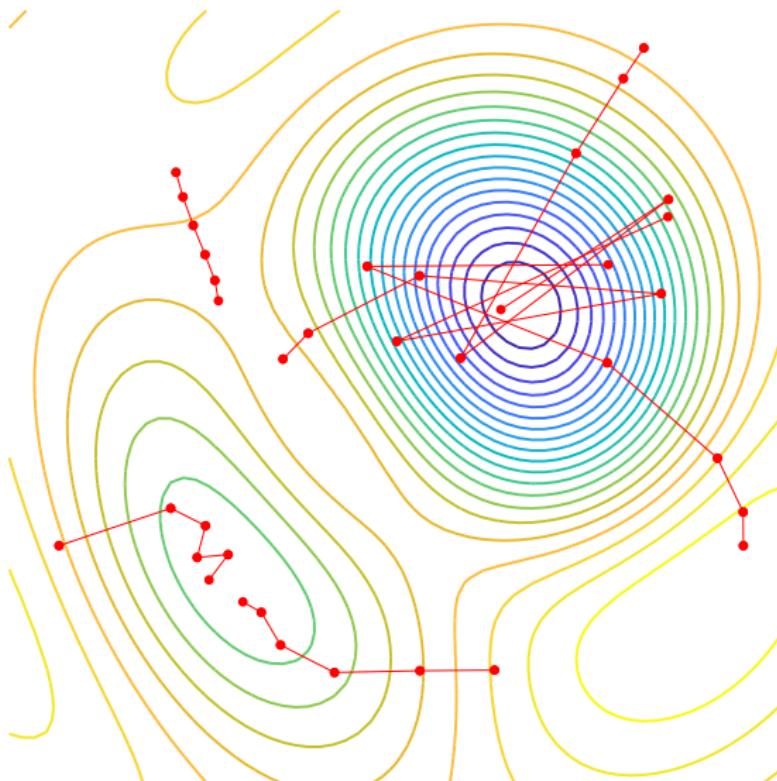
$\epsilon = 0.14$ , iteration 3

# gradient descent in two dimensions

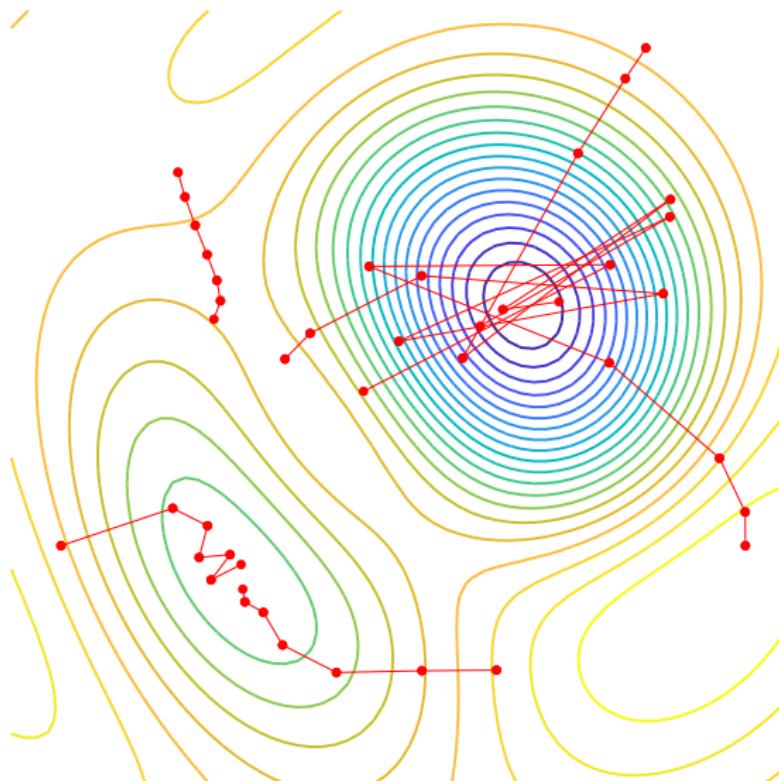


$\epsilon = 0.14$ , iteration 4

# gradient descent in two dimensions

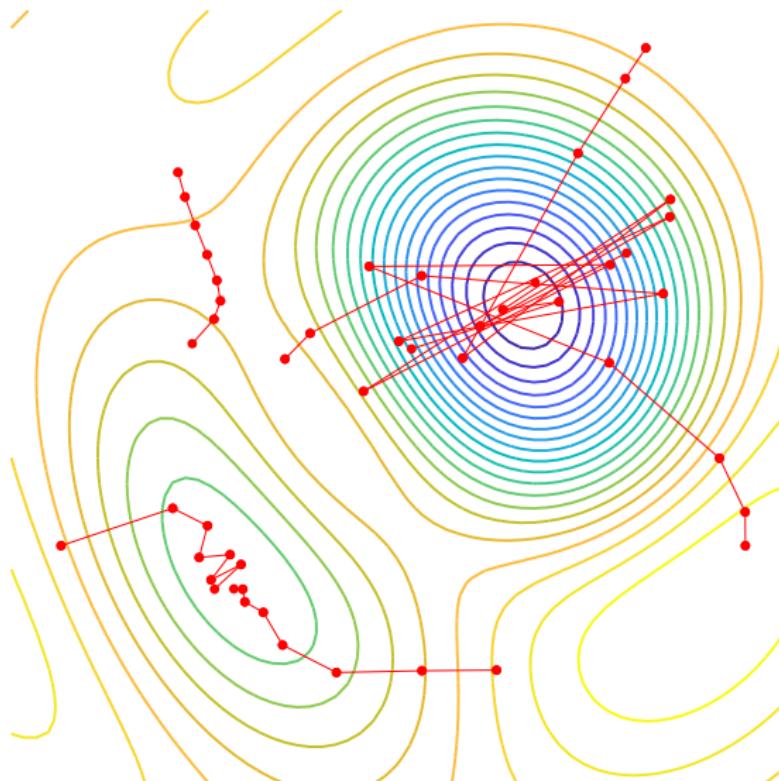


# gradient descent in two dimensions



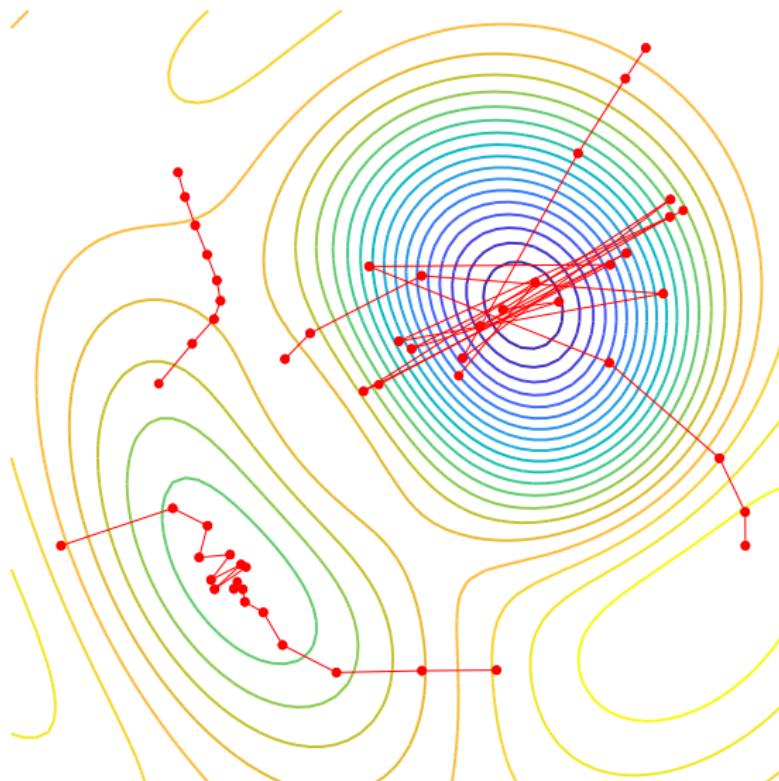
$\epsilon = 0.14$ , iteration 6

# gradient descent in two dimensions



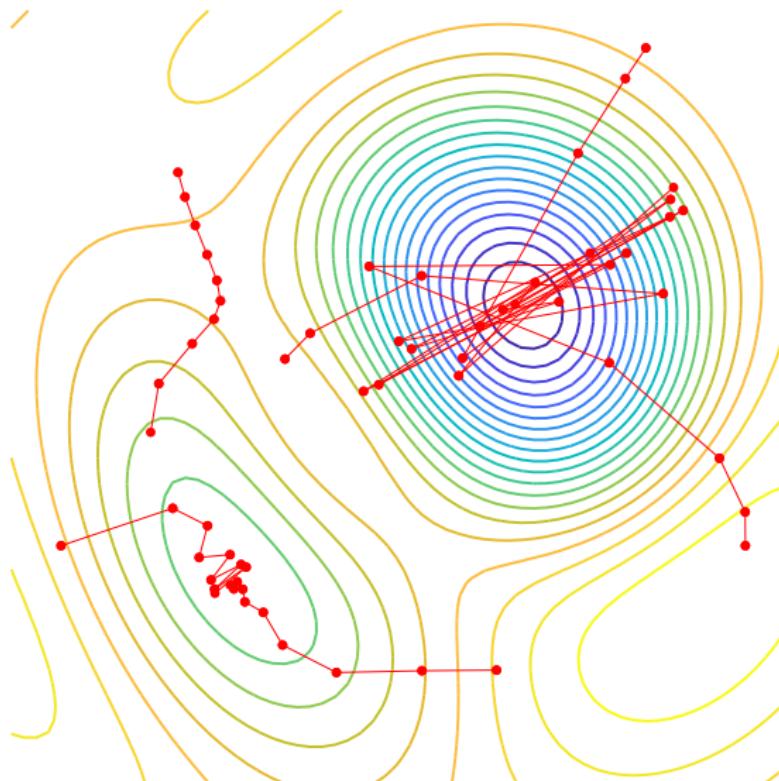
$\epsilon = 0.14$ , iteration 7

# gradient descent in two dimensions



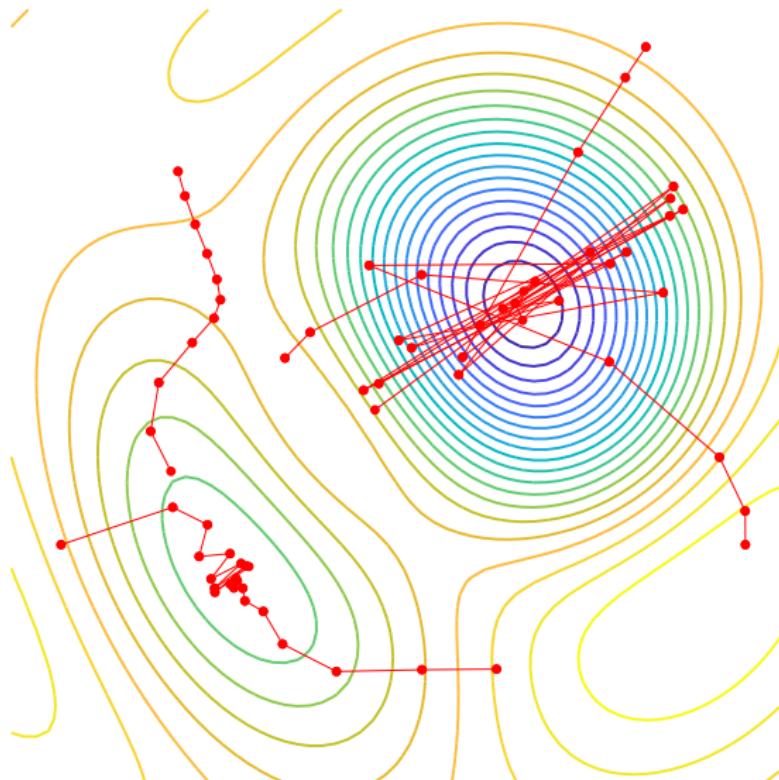
$\epsilon = 0.14$ , iteration 8

# gradient descent in two dimensions



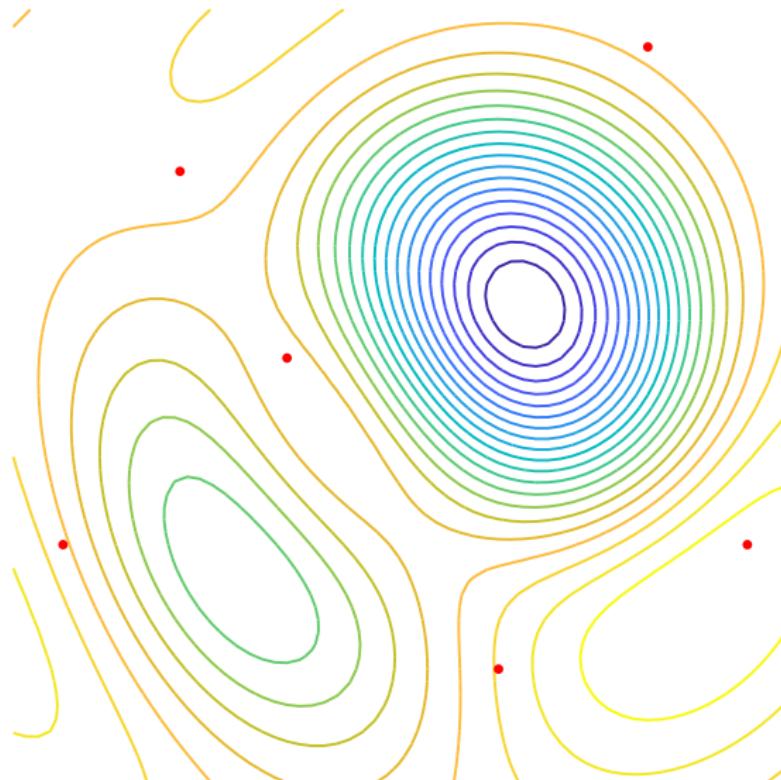
$\epsilon = 0.14$ , iteration 9

# gradient descent in two dimensions



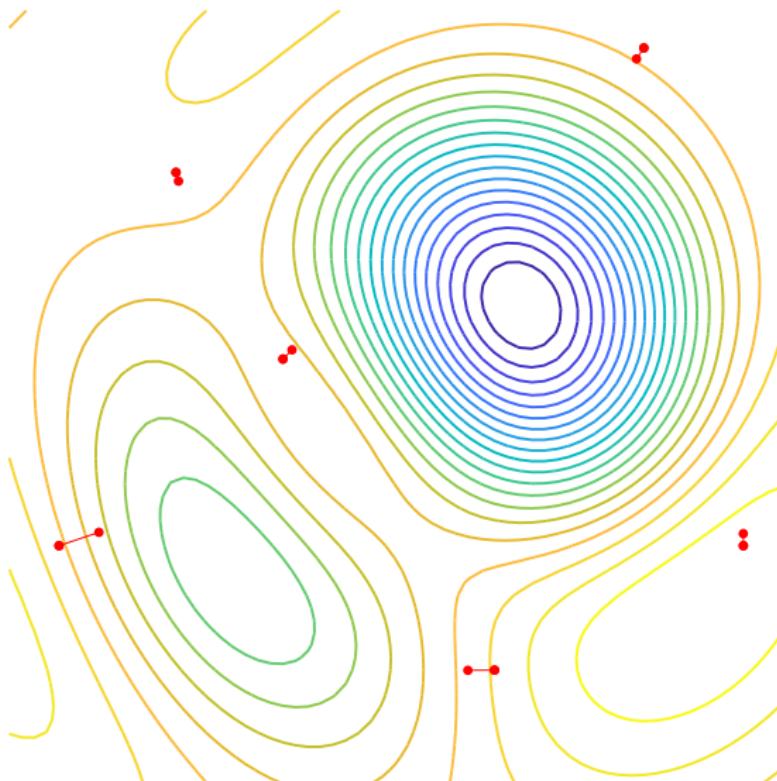
$\epsilon = 0.14$ , iteration 10

# gradient descent in two dimensions



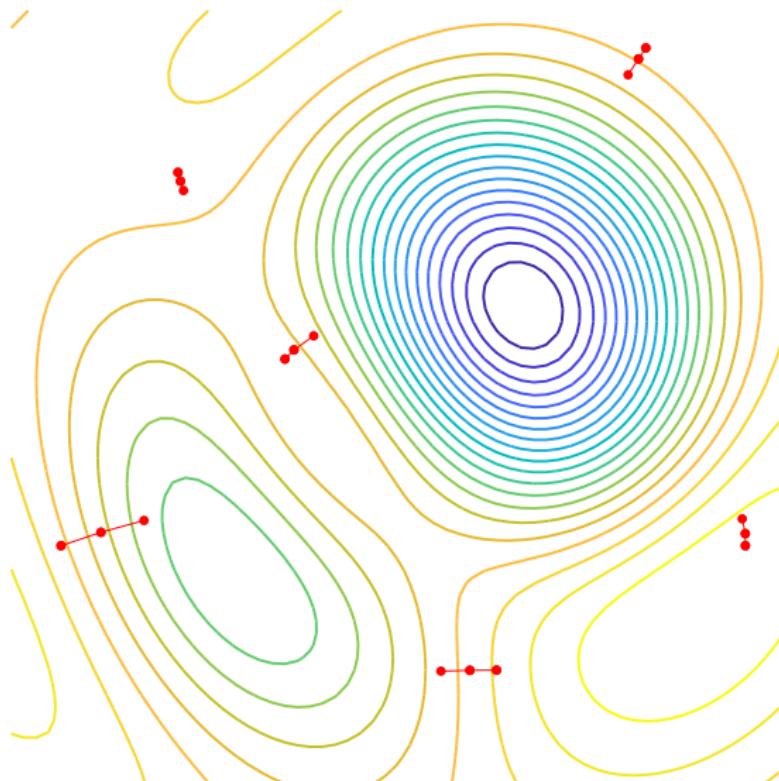
$\epsilon = 0.05$ , iteration 0

# gradient descent in two dimensions



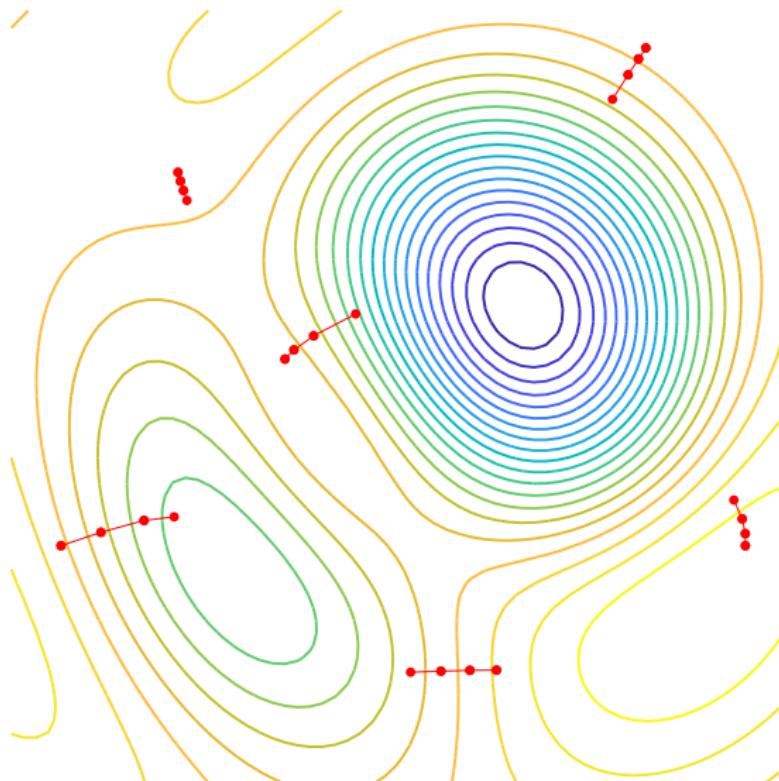
$\epsilon = 0.05$ , iteration 1

# gradient descent in two dimensions



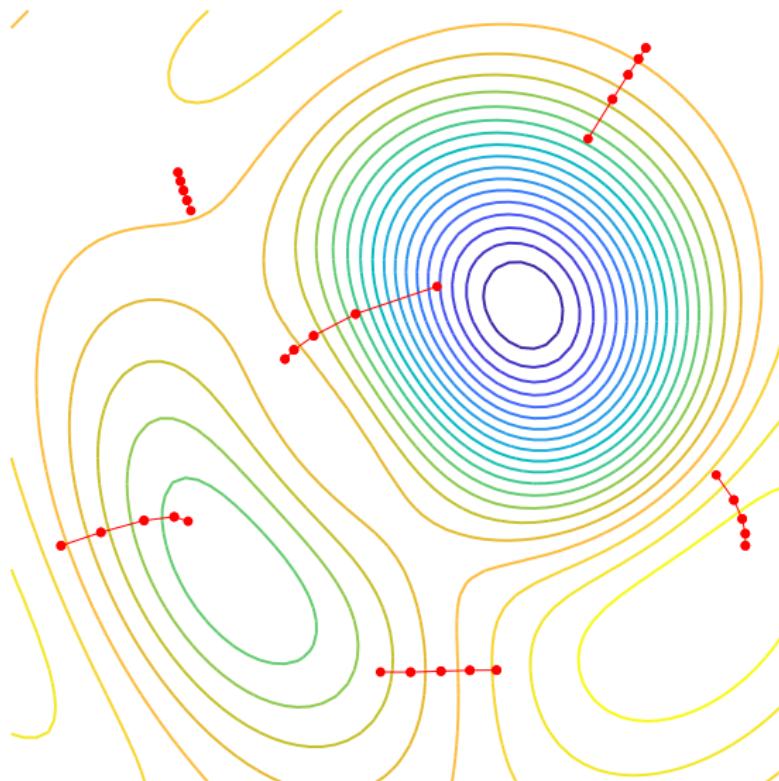
$\epsilon = 0.05$ , iteration 2

# gradient descent in two dimensions

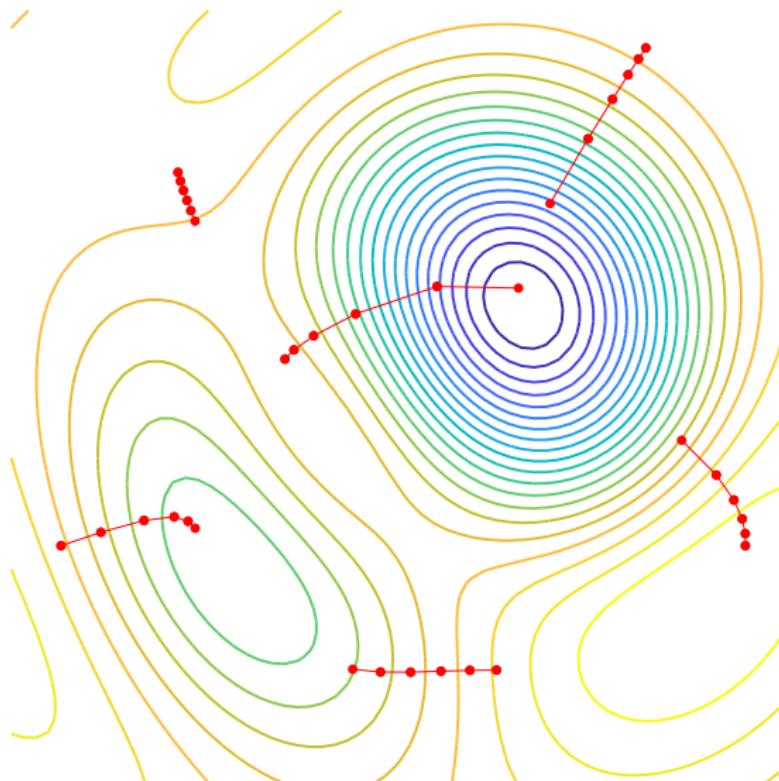


$\epsilon = 0.05$ , iteration 3

# gradient descent in two dimensions

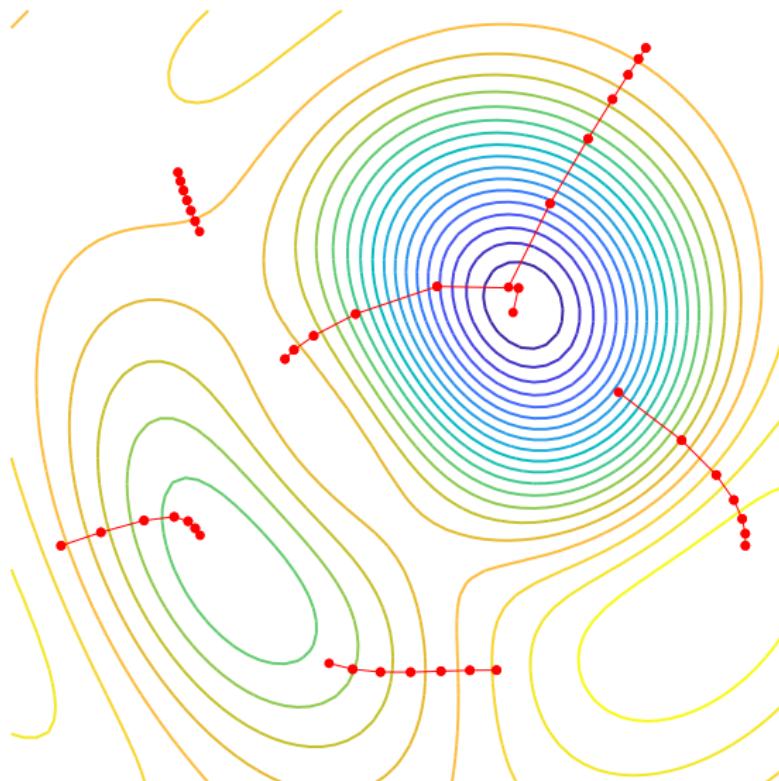


# gradient descent in two dimensions



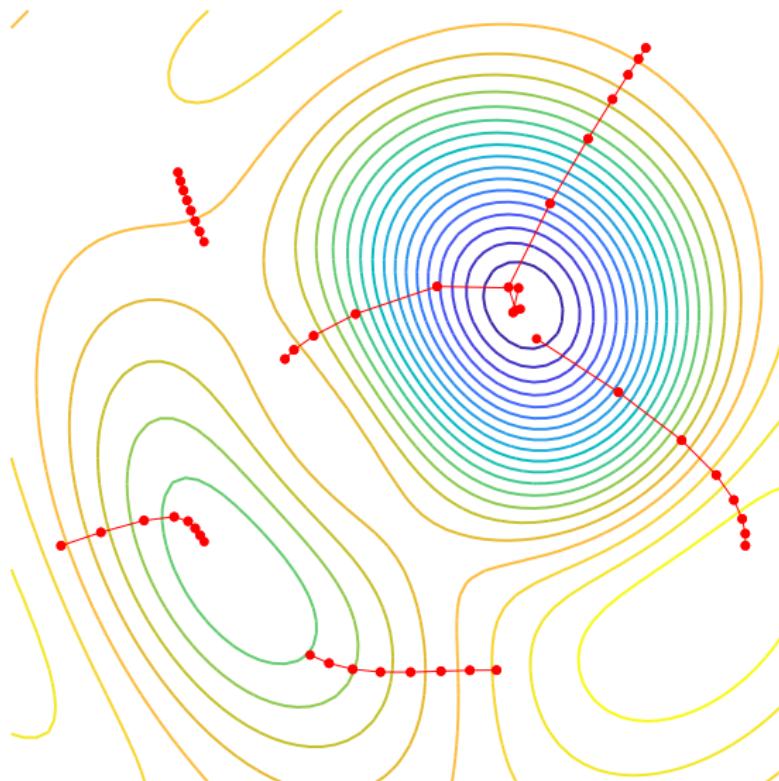
$\epsilon = 0.05$ , iteration 5

# gradient descent in two dimensions



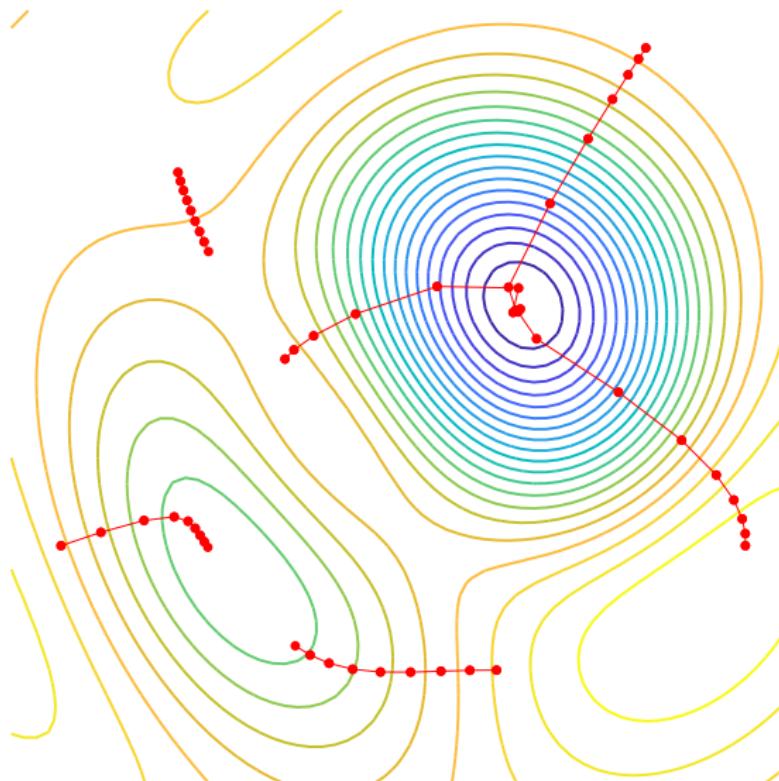
$\epsilon = 0.05$ , iteration 6

# gradient descent in two dimensions



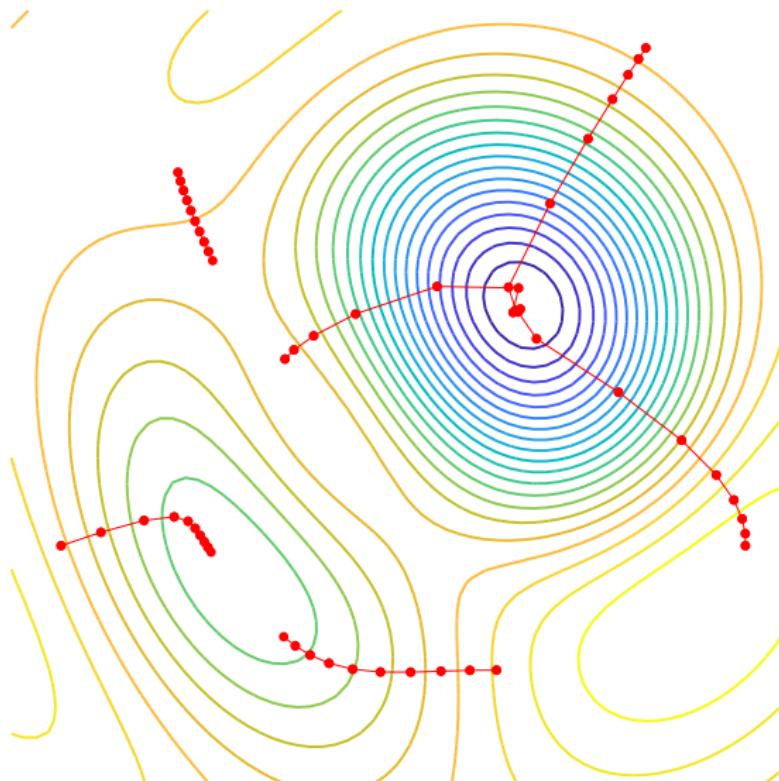
$\epsilon = 0.05$ , iteration 7

# gradient descent in two dimensions



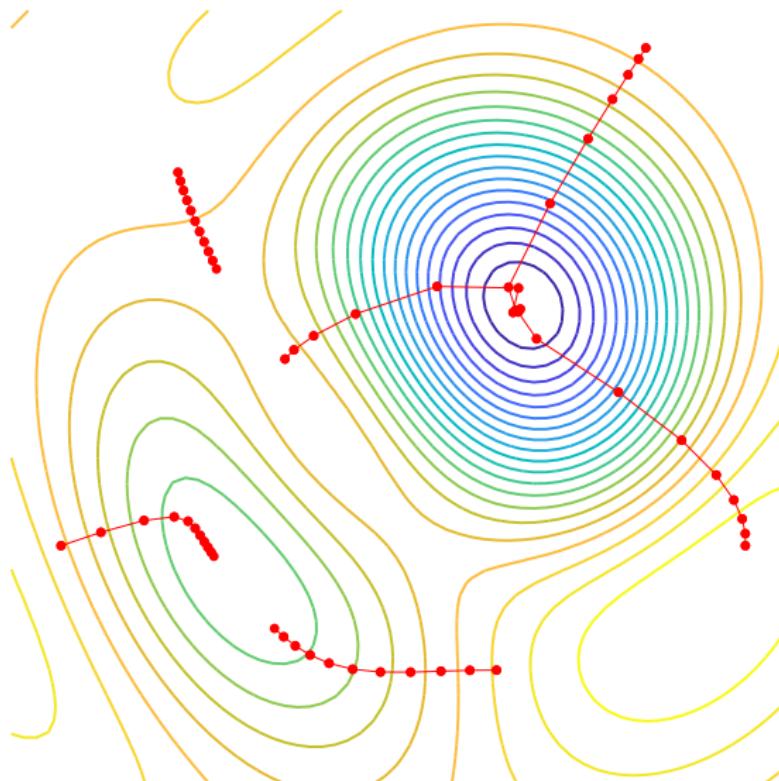
$\epsilon = 0.05$ , iteration 8

# gradient descent in two dimensions



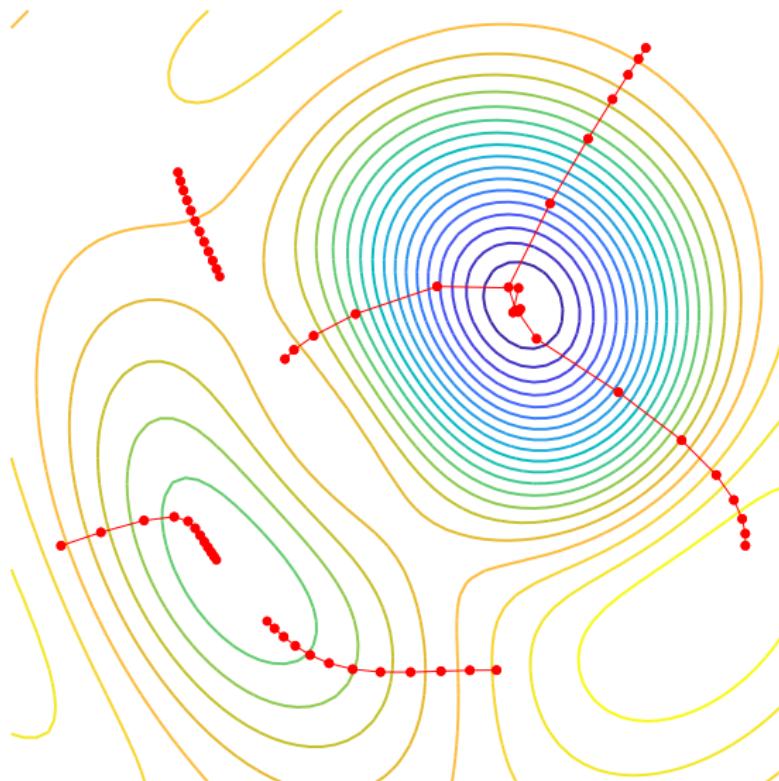
$\epsilon = 0.05$ , iteration 9

# gradient descent in two dimensions



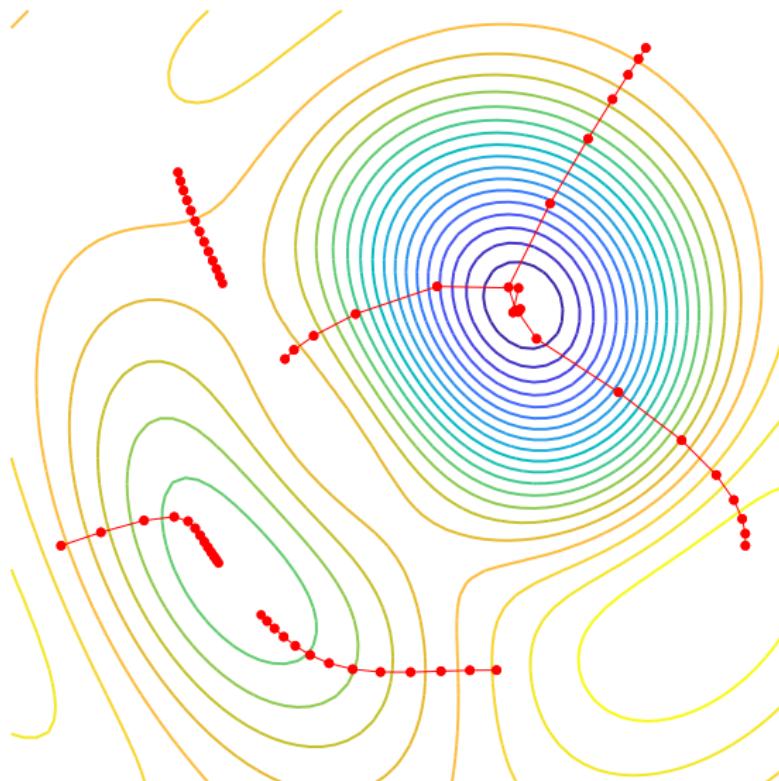
$\epsilon = 0.05$ , iteration 10

# gradient descent in two dimensions



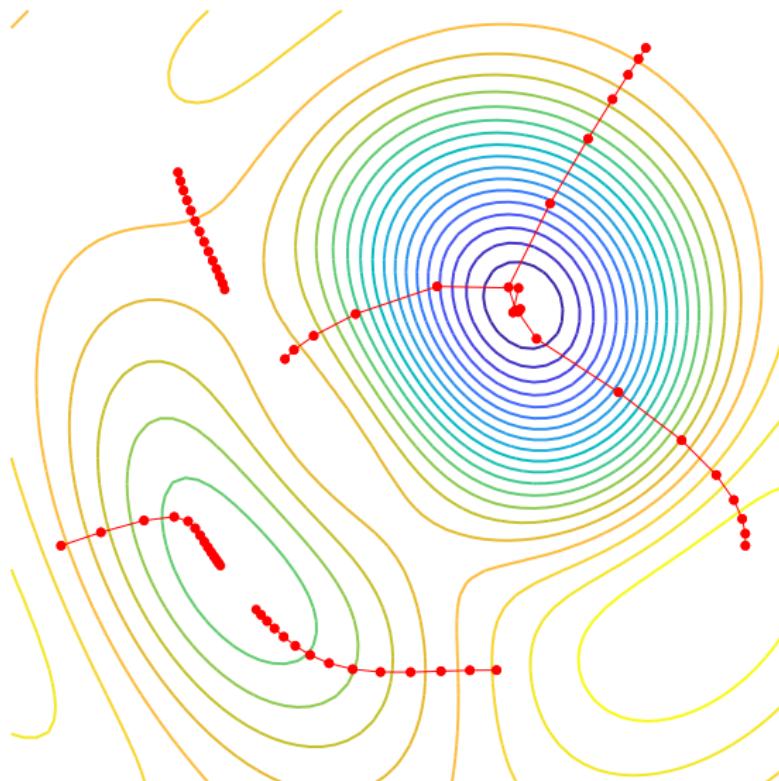
$\epsilon = 0.05$ , iteration 11

# gradient descent in two dimensions



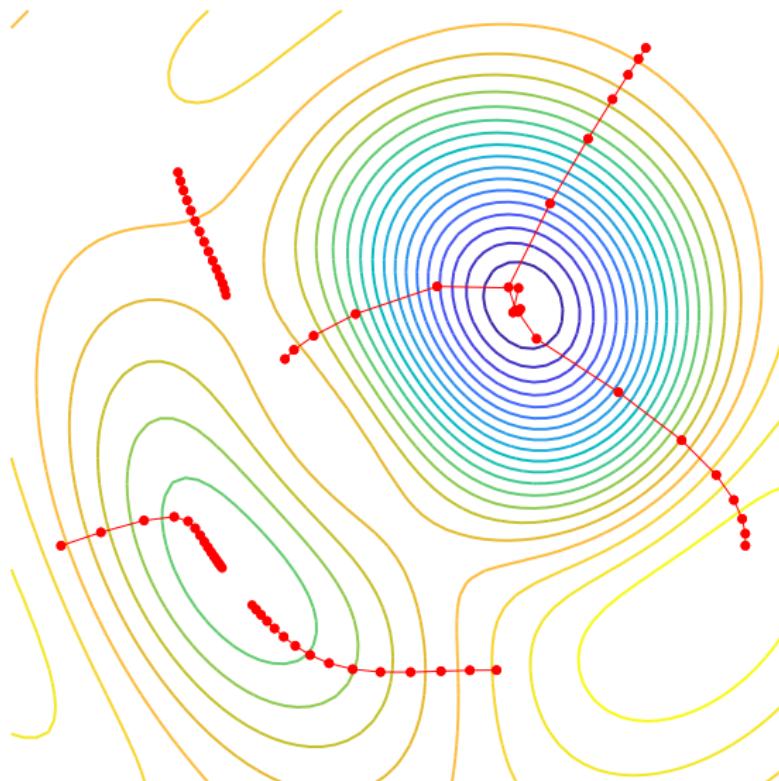
$\epsilon = 0.05$ , iteration 12

# gradient descent in two dimensions



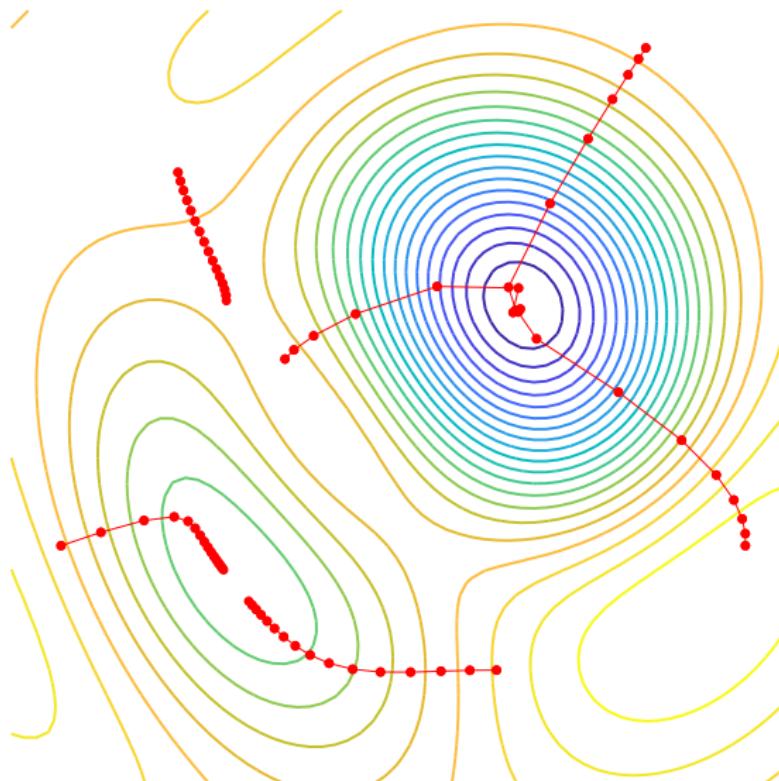
$\epsilon = 0.05$ , iteration 13

# gradient descent in two dimensions



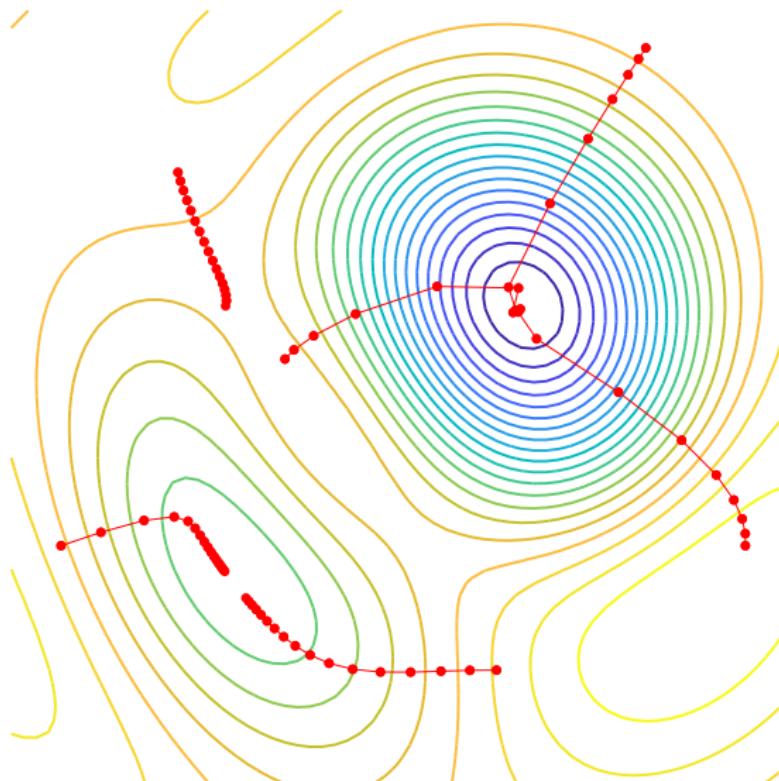
$\epsilon = 0.05$ , iteration 14

# gradient descent in two dimensions



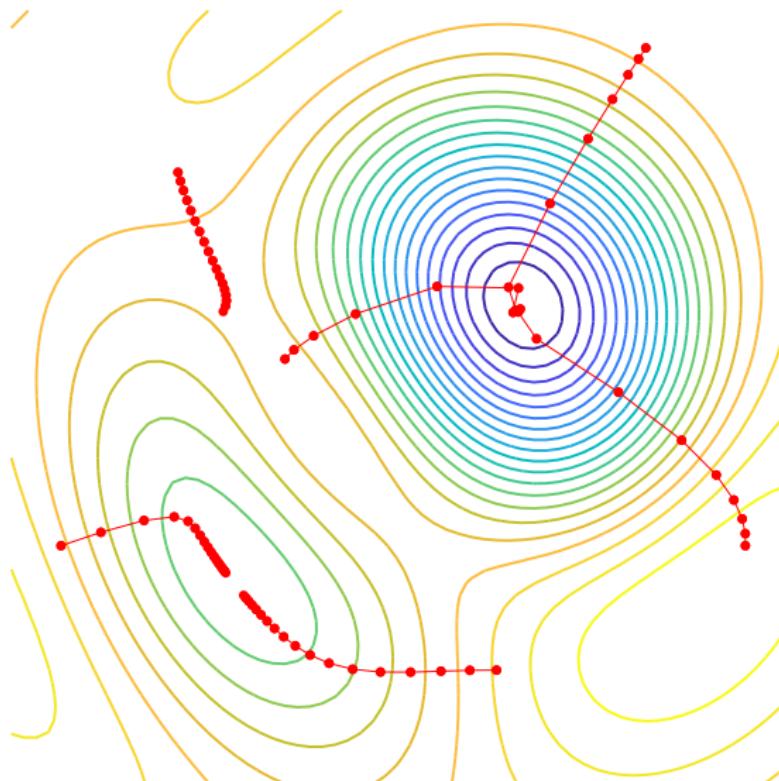
$\epsilon = 0.05$ , iteration 15

# gradient descent in two dimensions



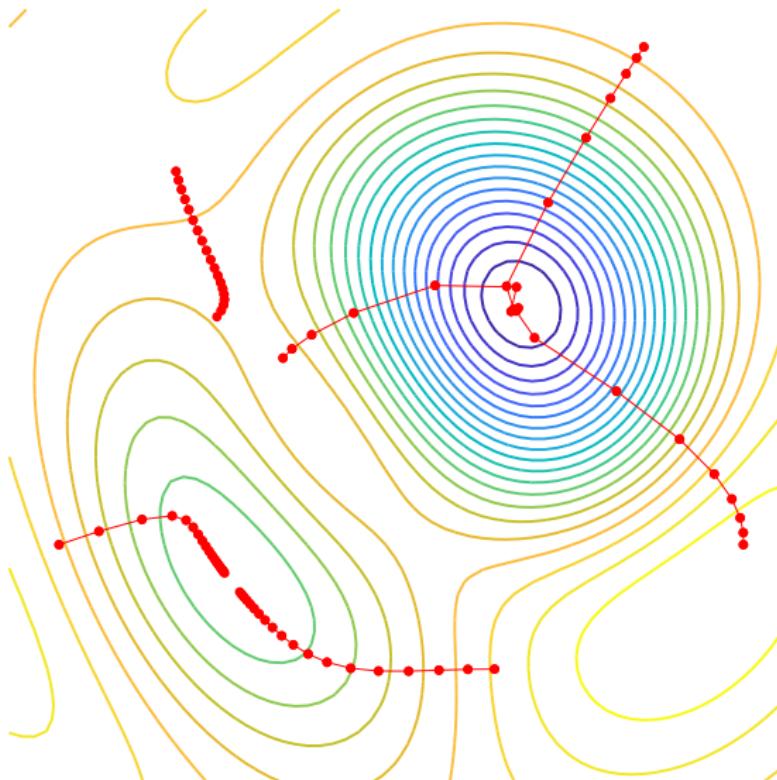
$\epsilon = 0.05$ , iteration 16

# gradient descent in two dimensions



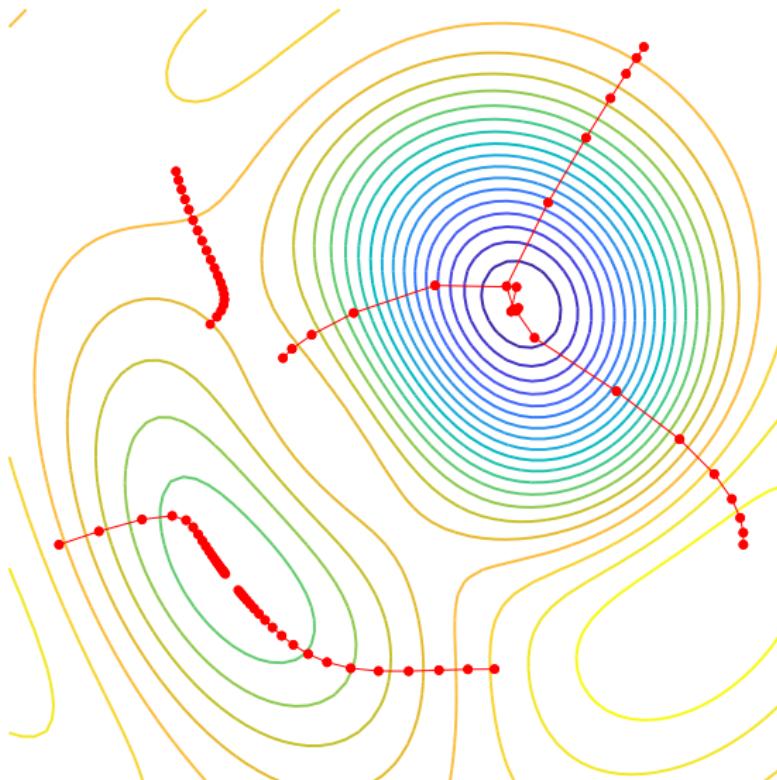
$\epsilon = 0.05$ , iteration 17

# gradient descent in two dimensions



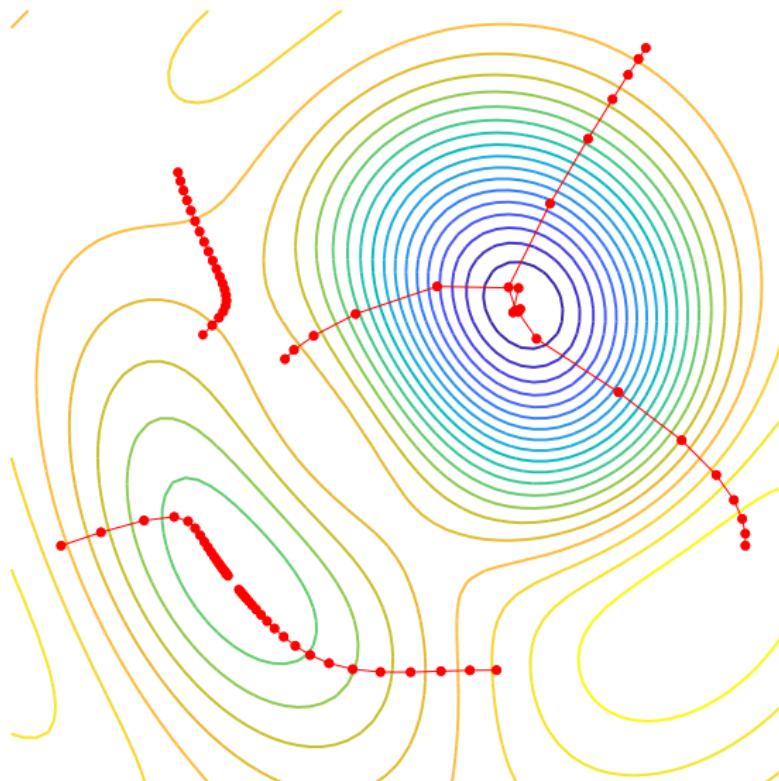
$\epsilon = 0.05$ , iteration 18

# gradient descent in two dimensions



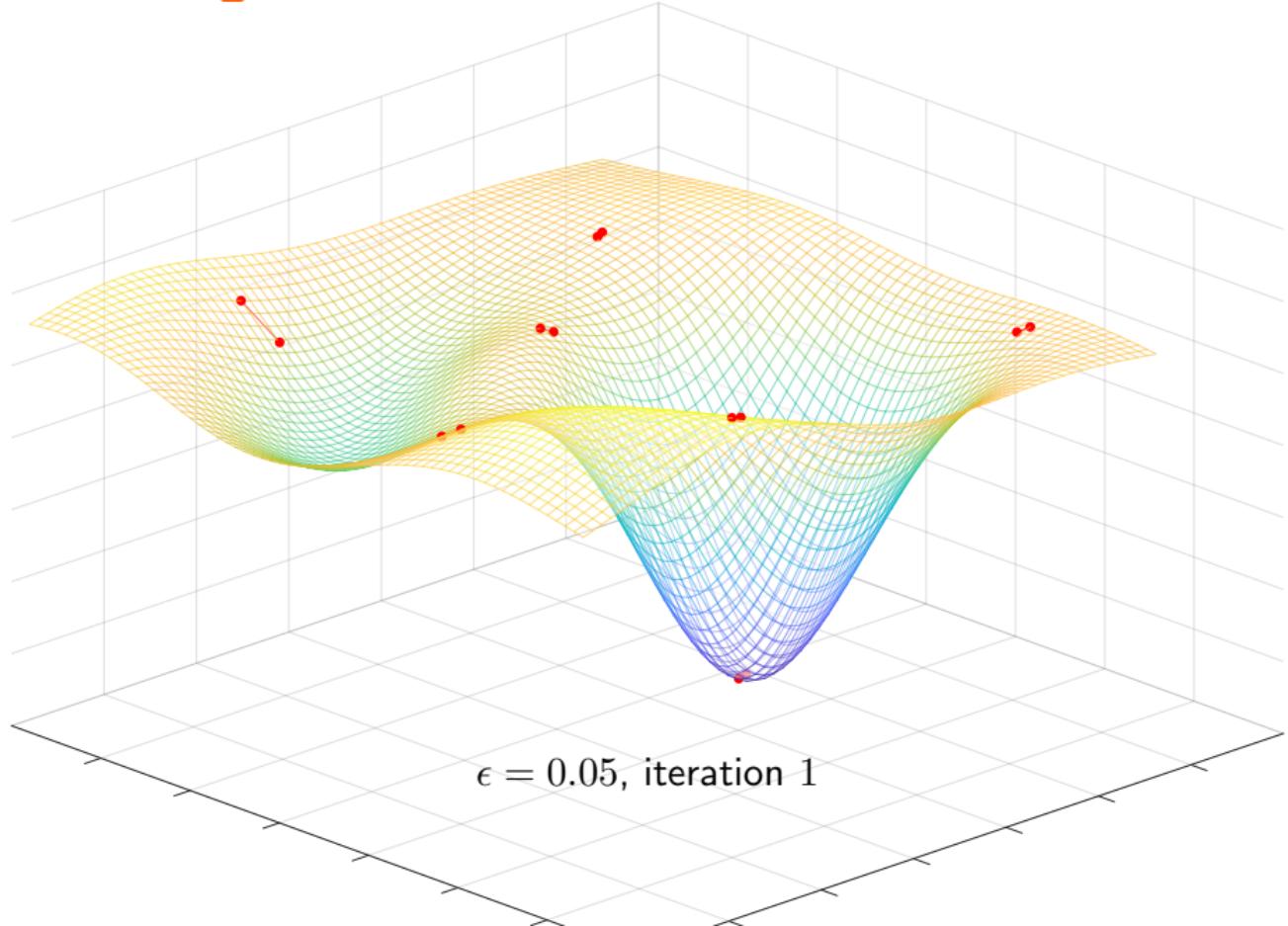
$\epsilon = 0.05$ , iteration 19

# gradient descent in two dimensions

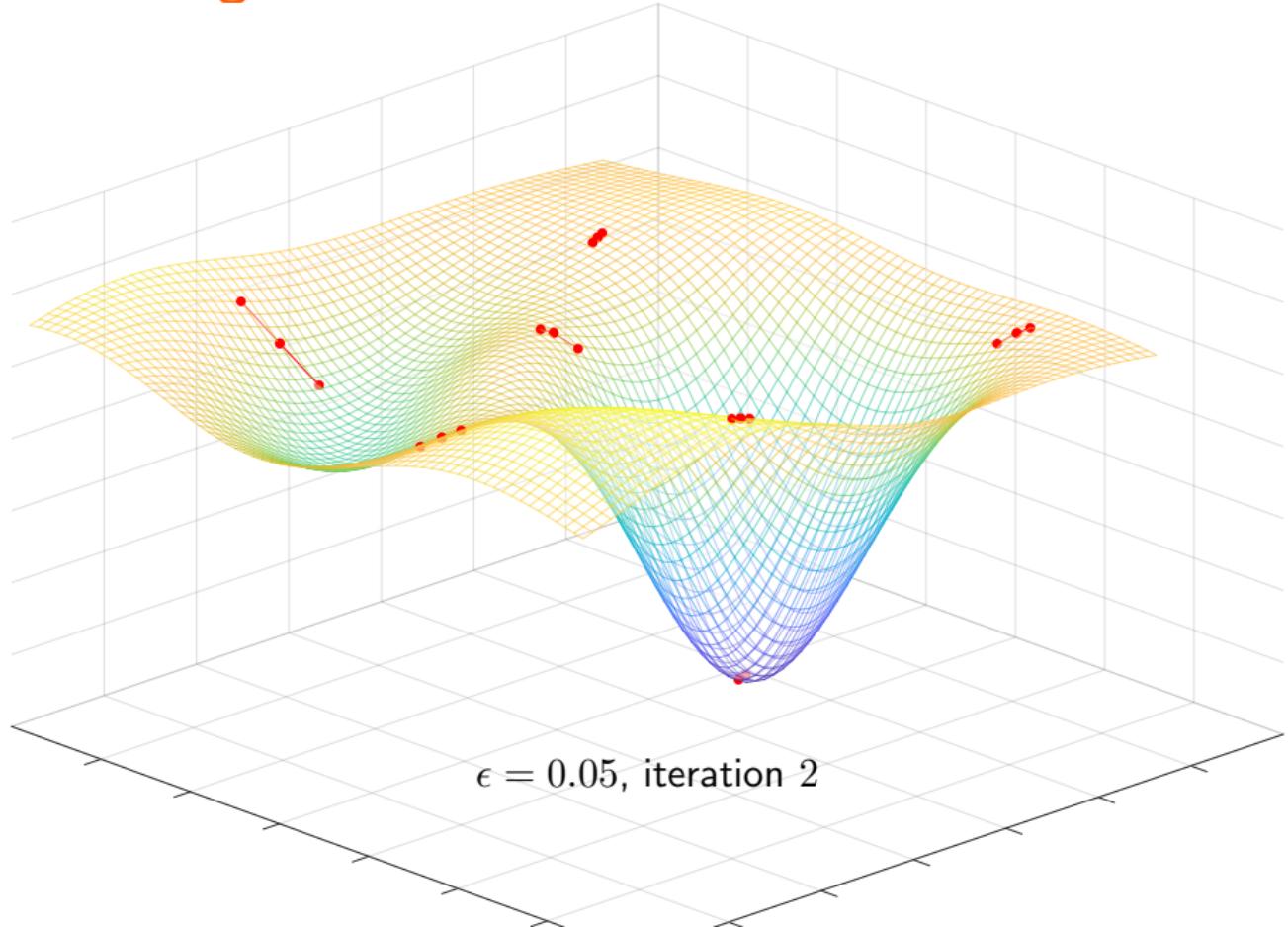


$\epsilon = 0.05$ , iteration 20

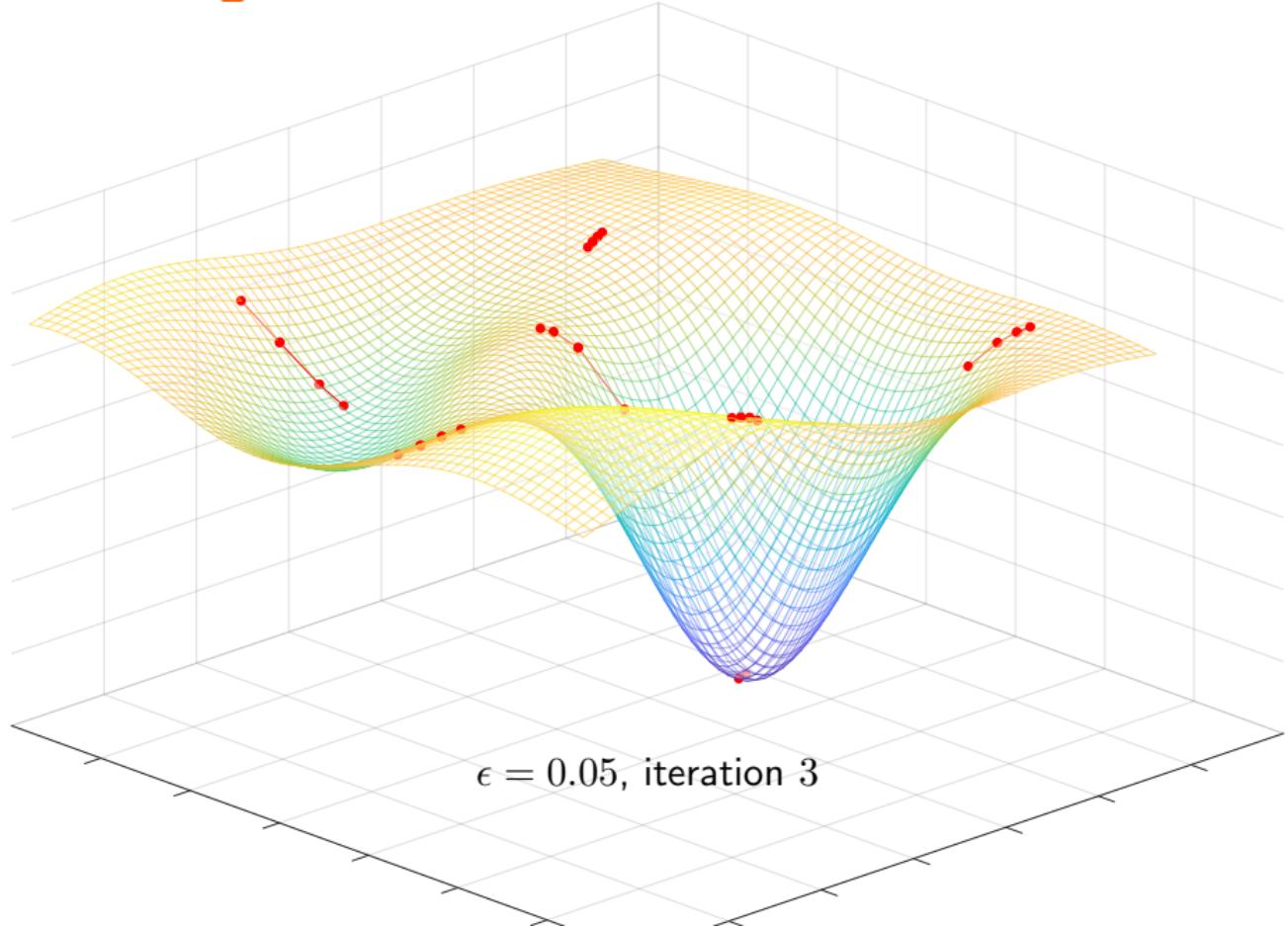
# gradient descent in two dimensions



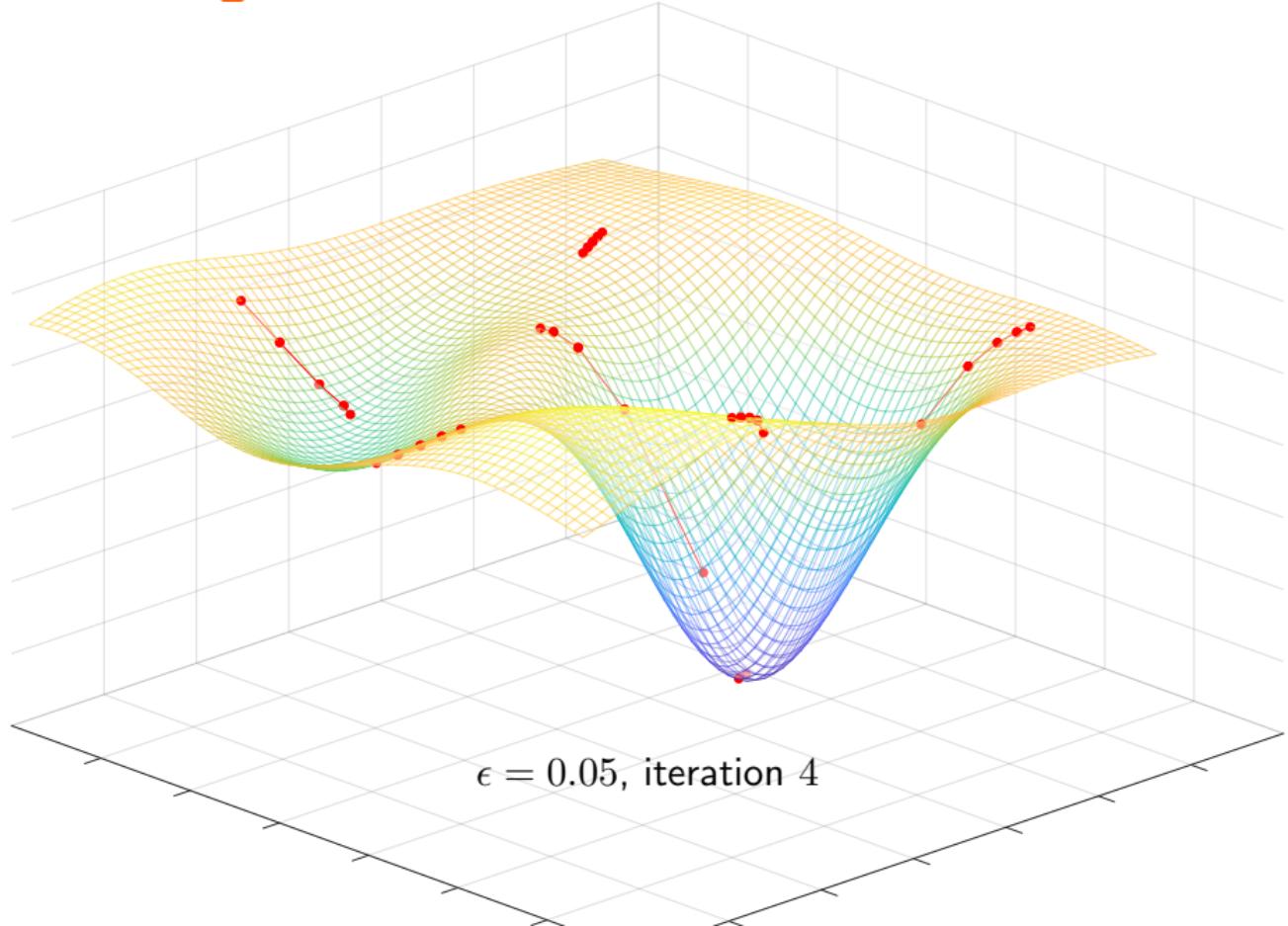
# gradient descent in two dimensions



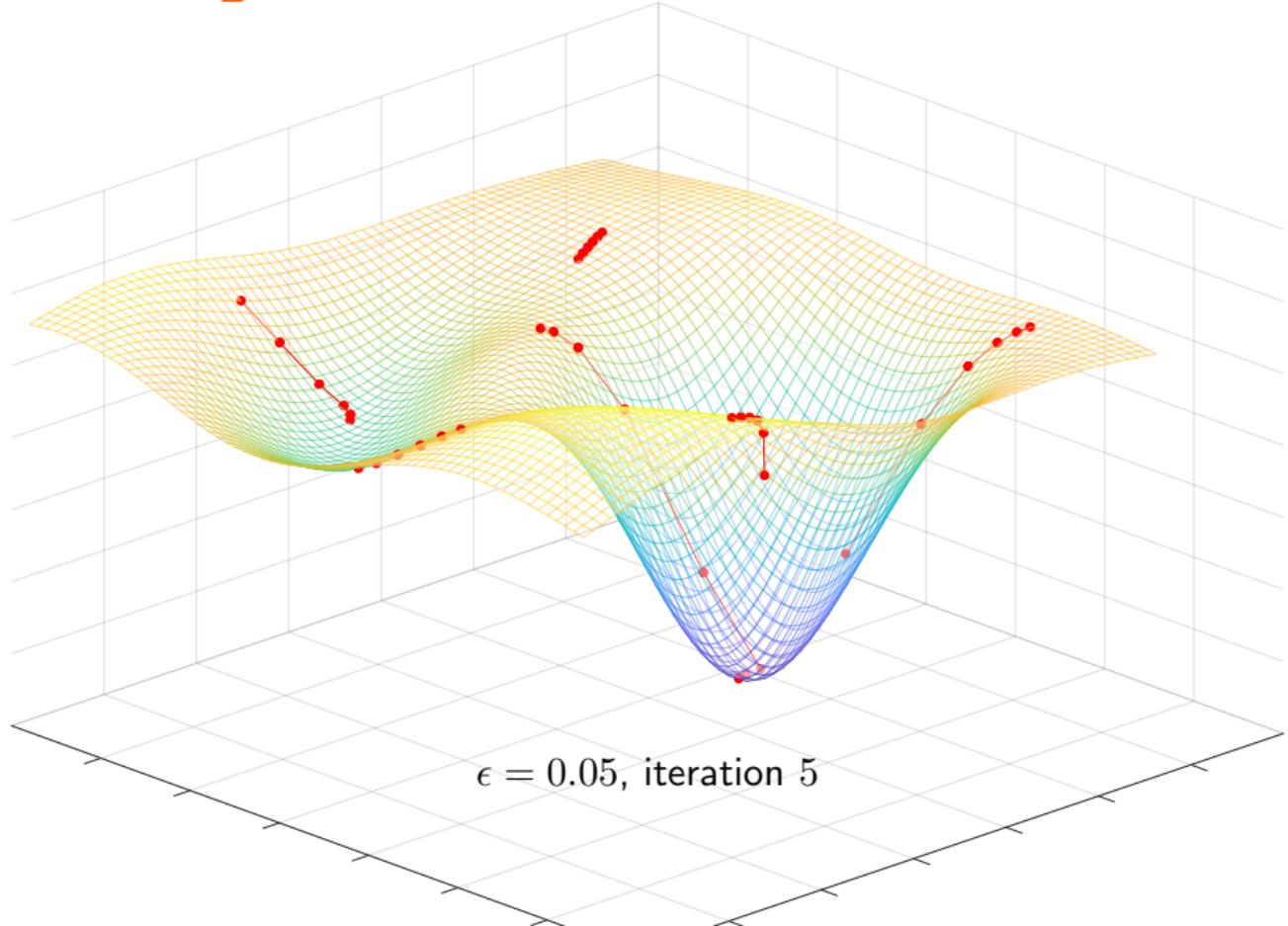
# gradient descent in two dimensions



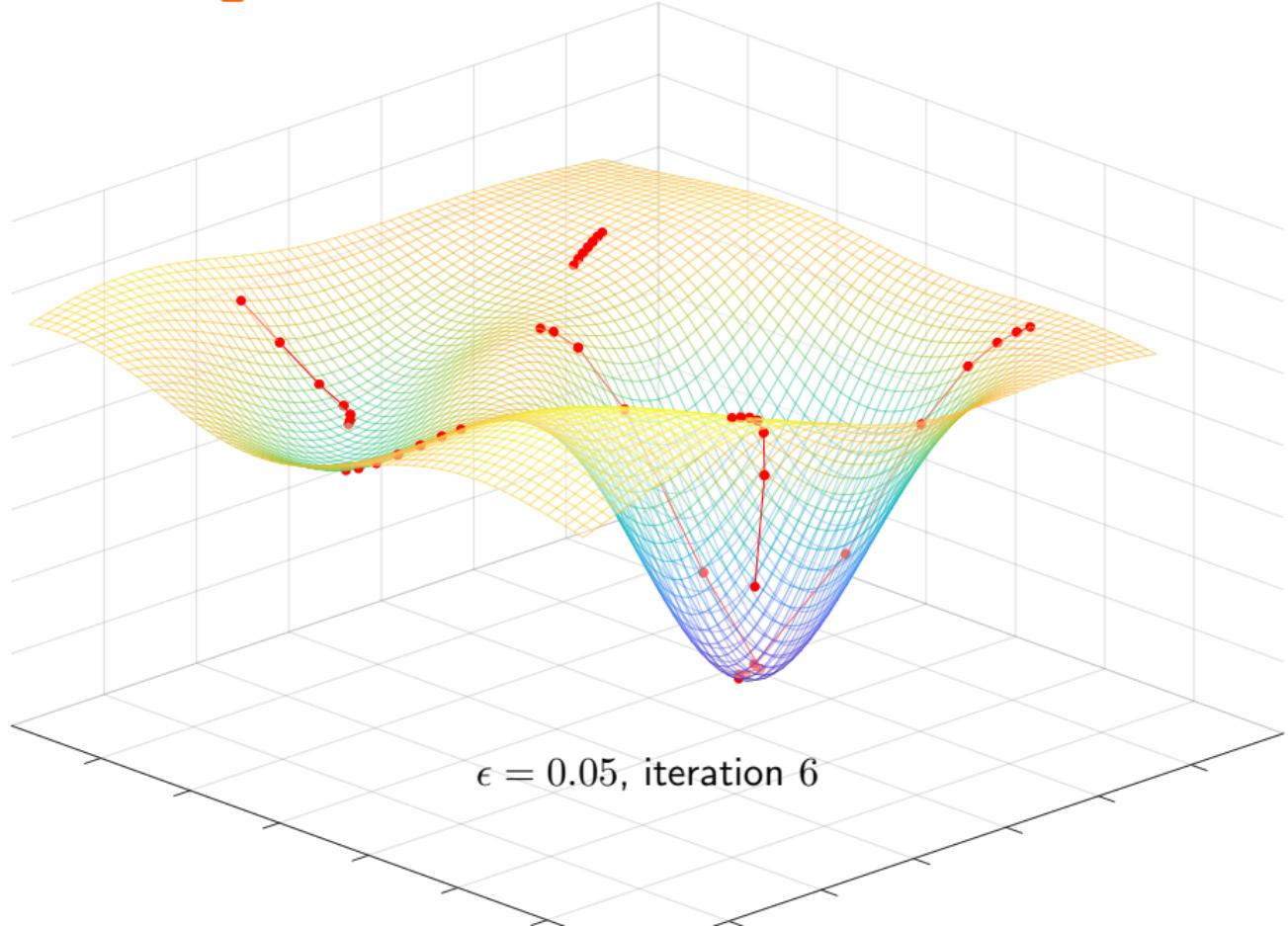
# gradient descent in two dimensions



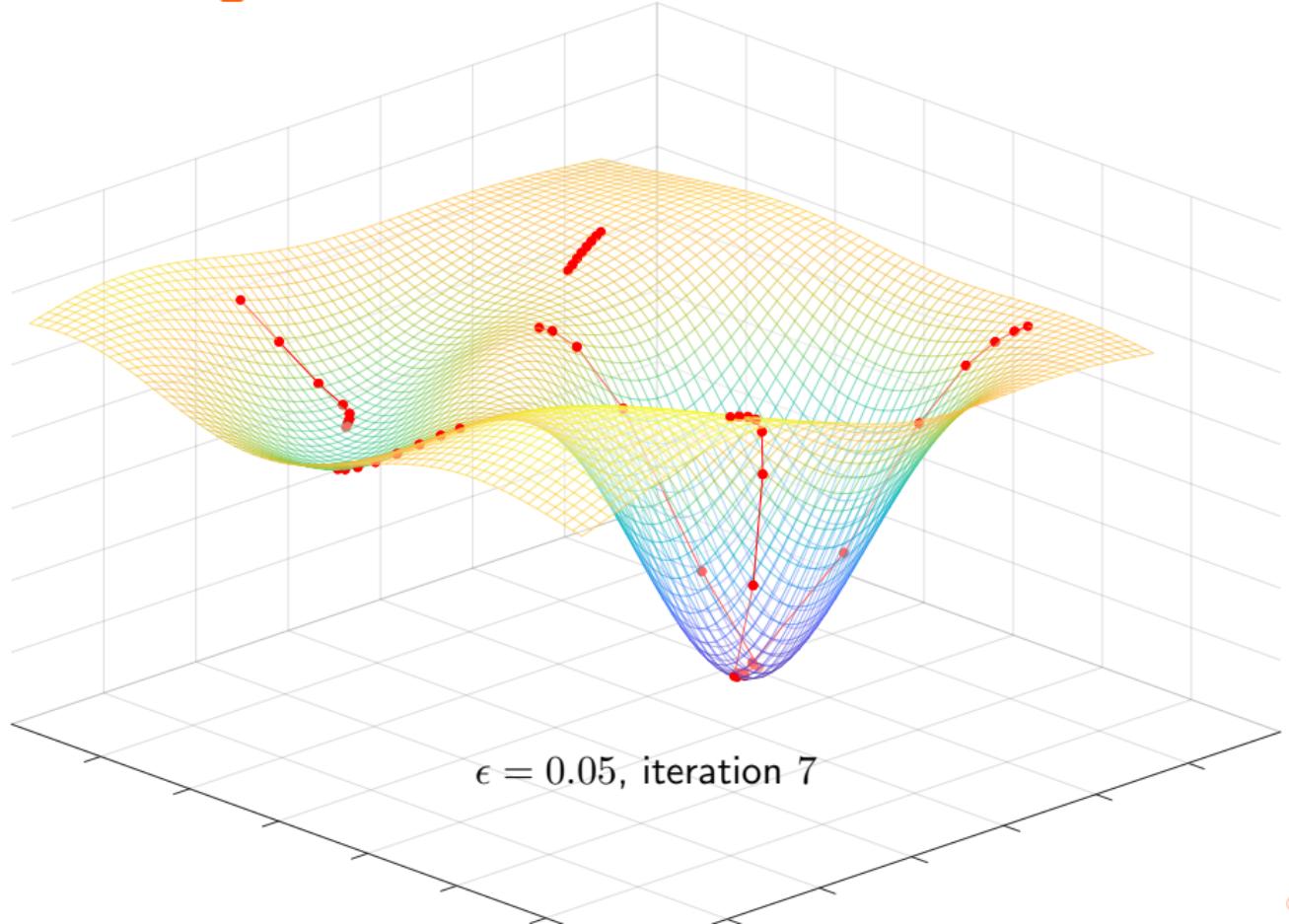
# gradient descent in two dimensions



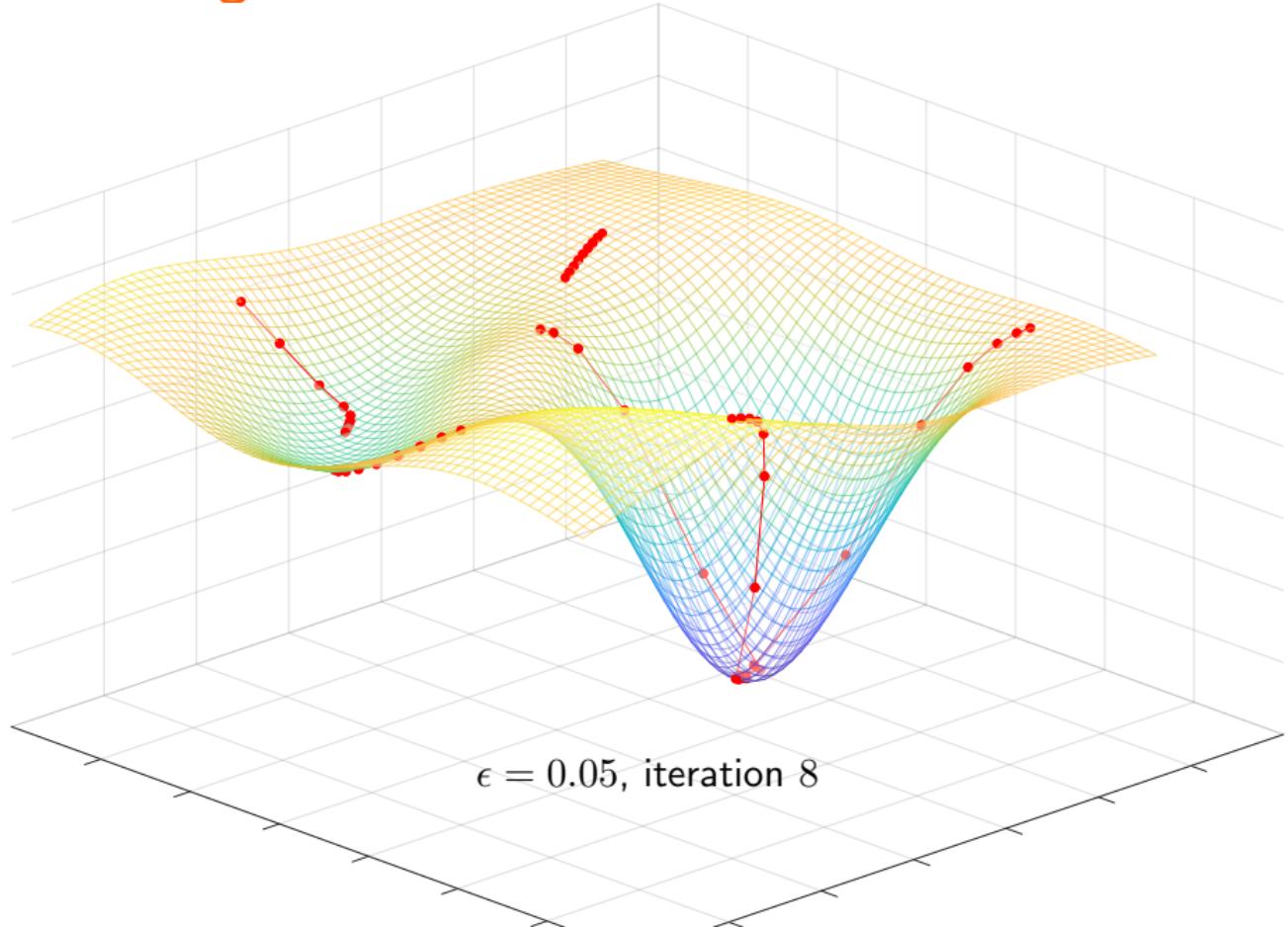
# gradient descent in two dimensions



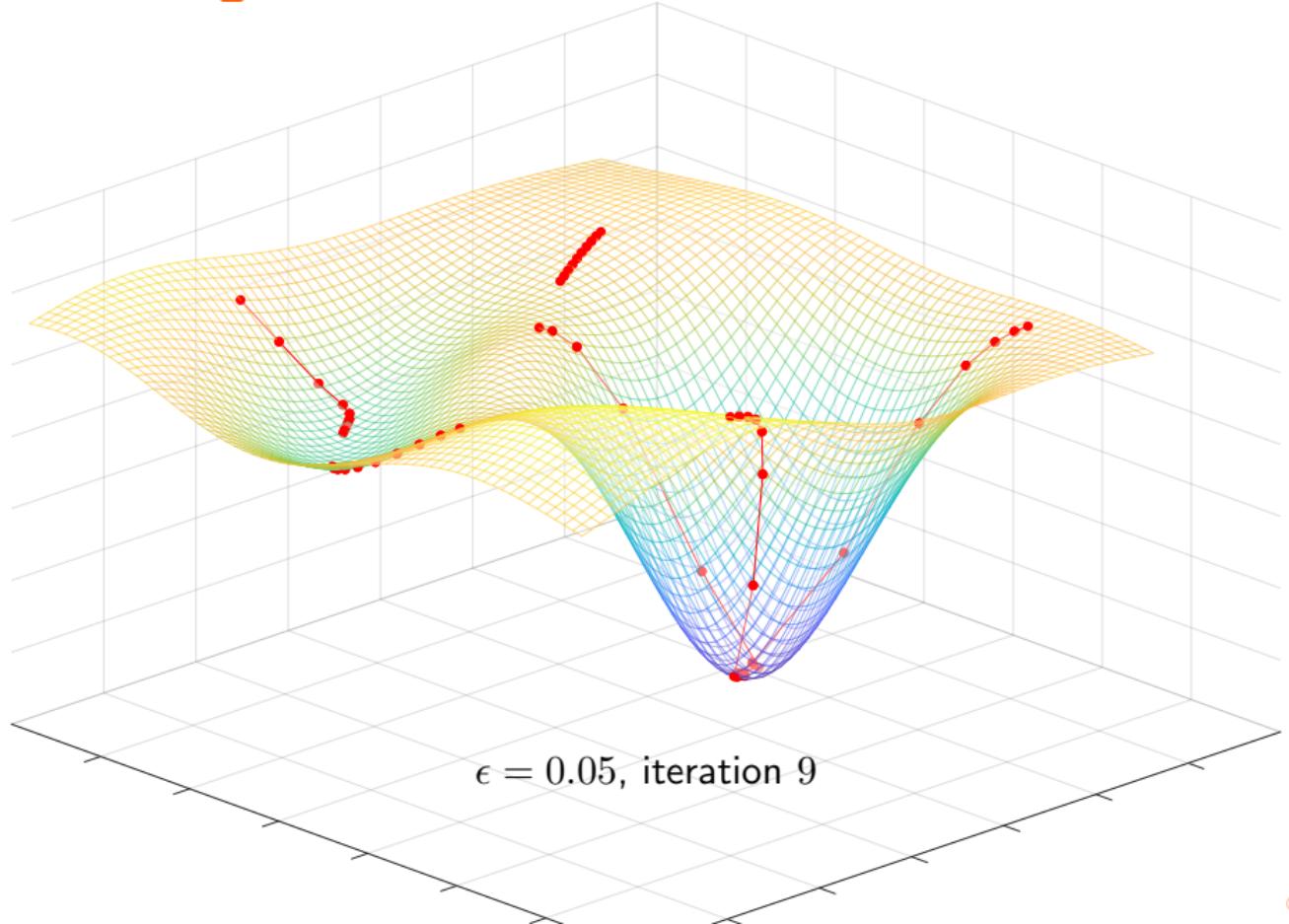
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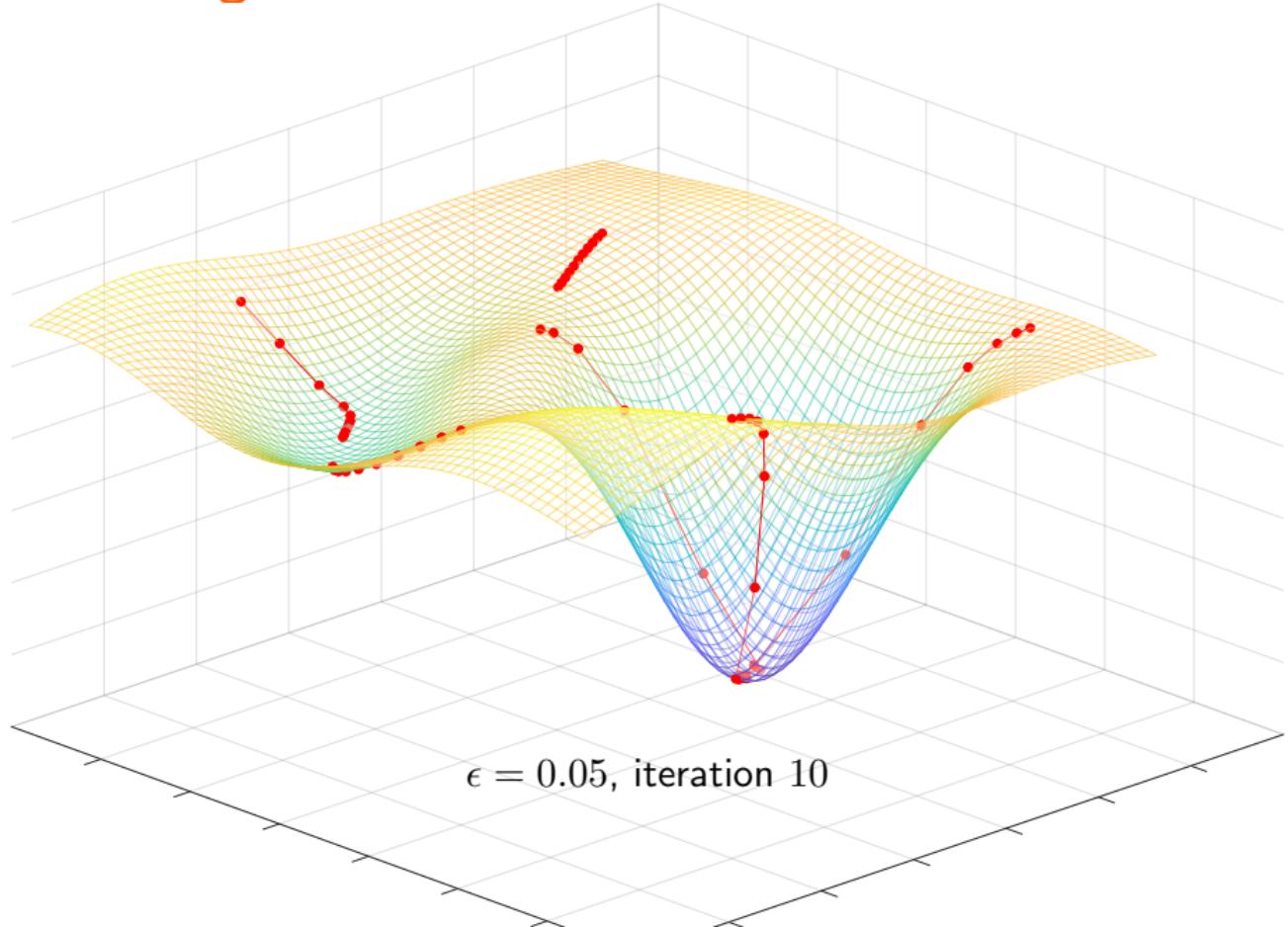
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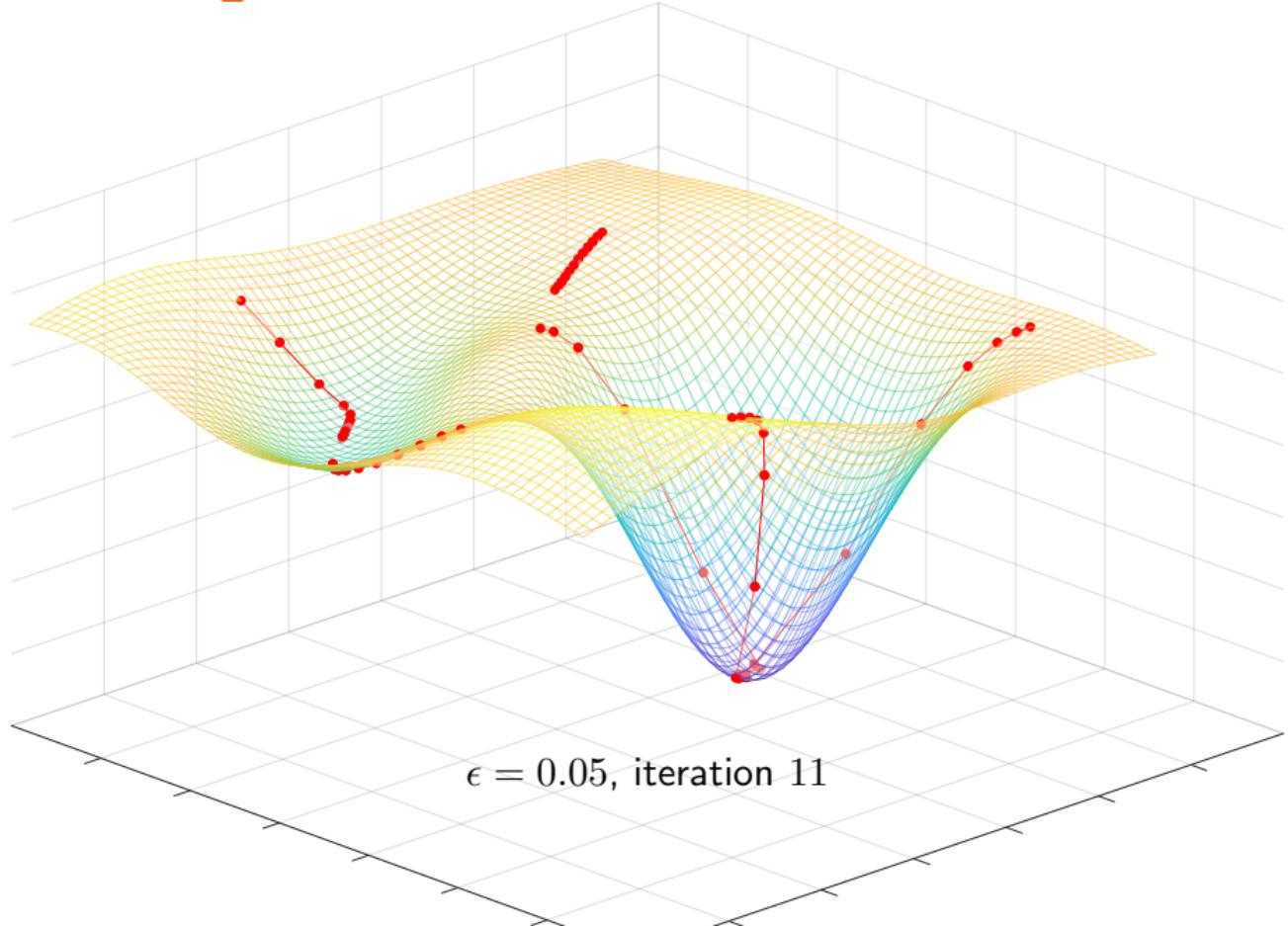
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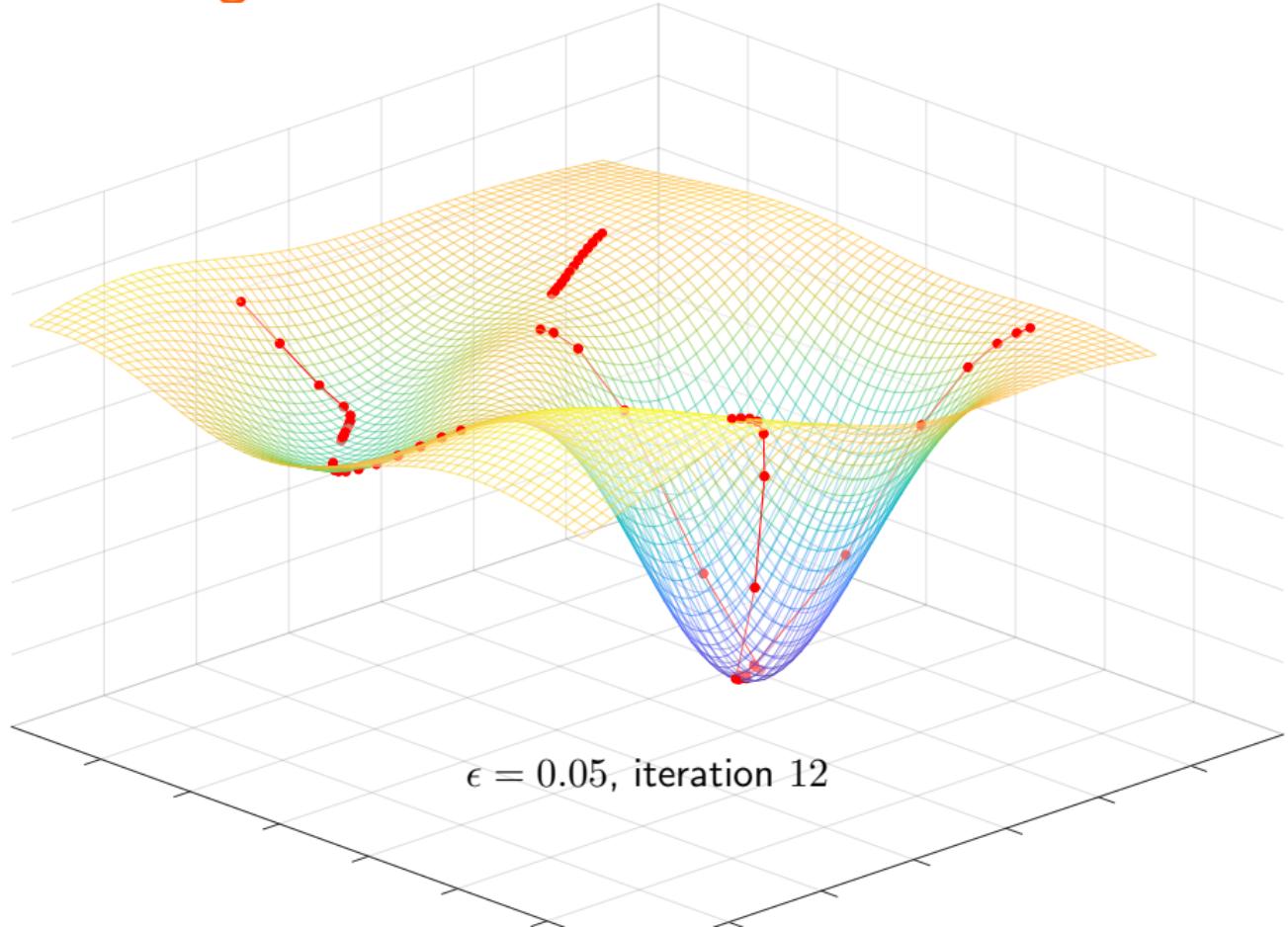
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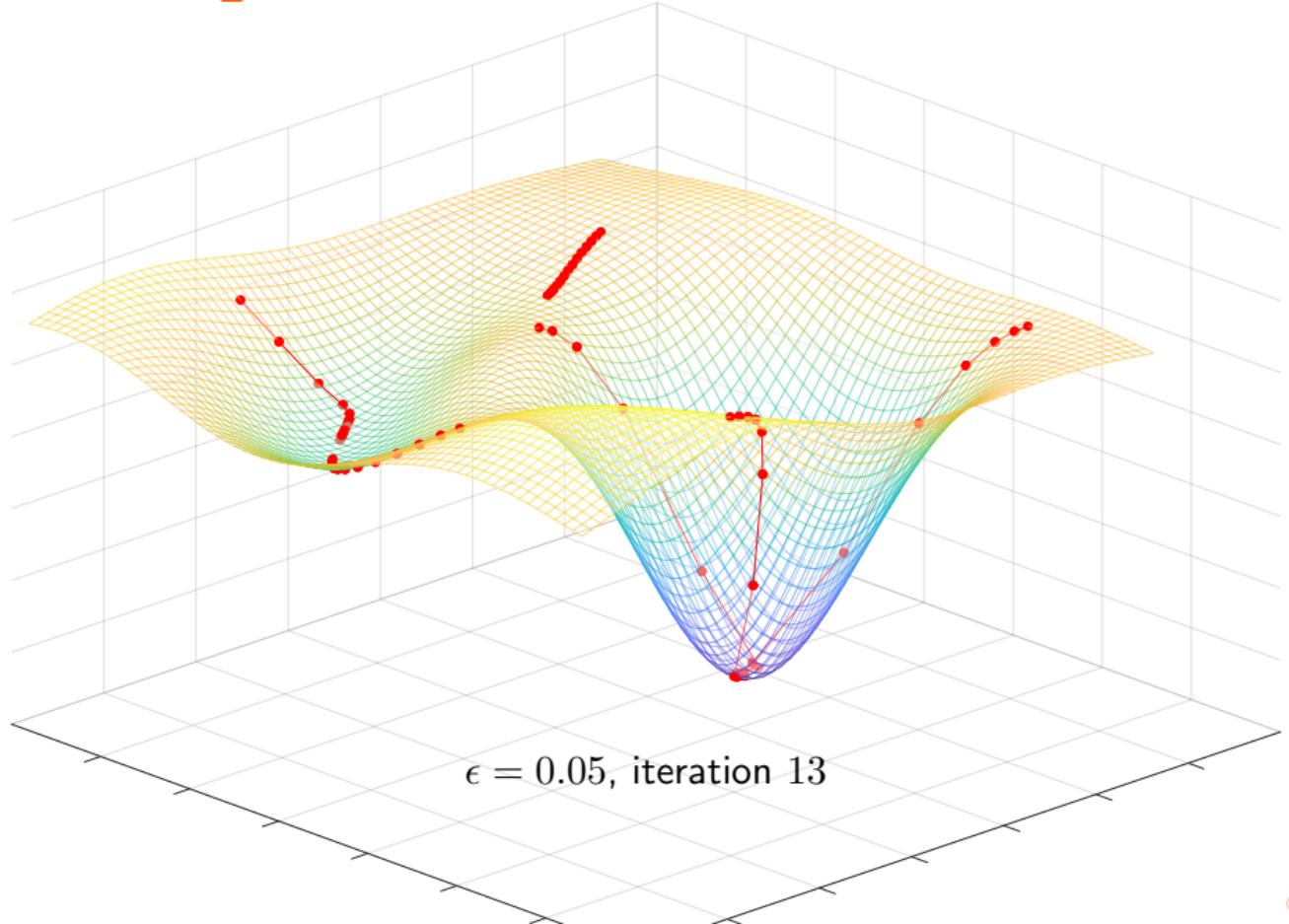
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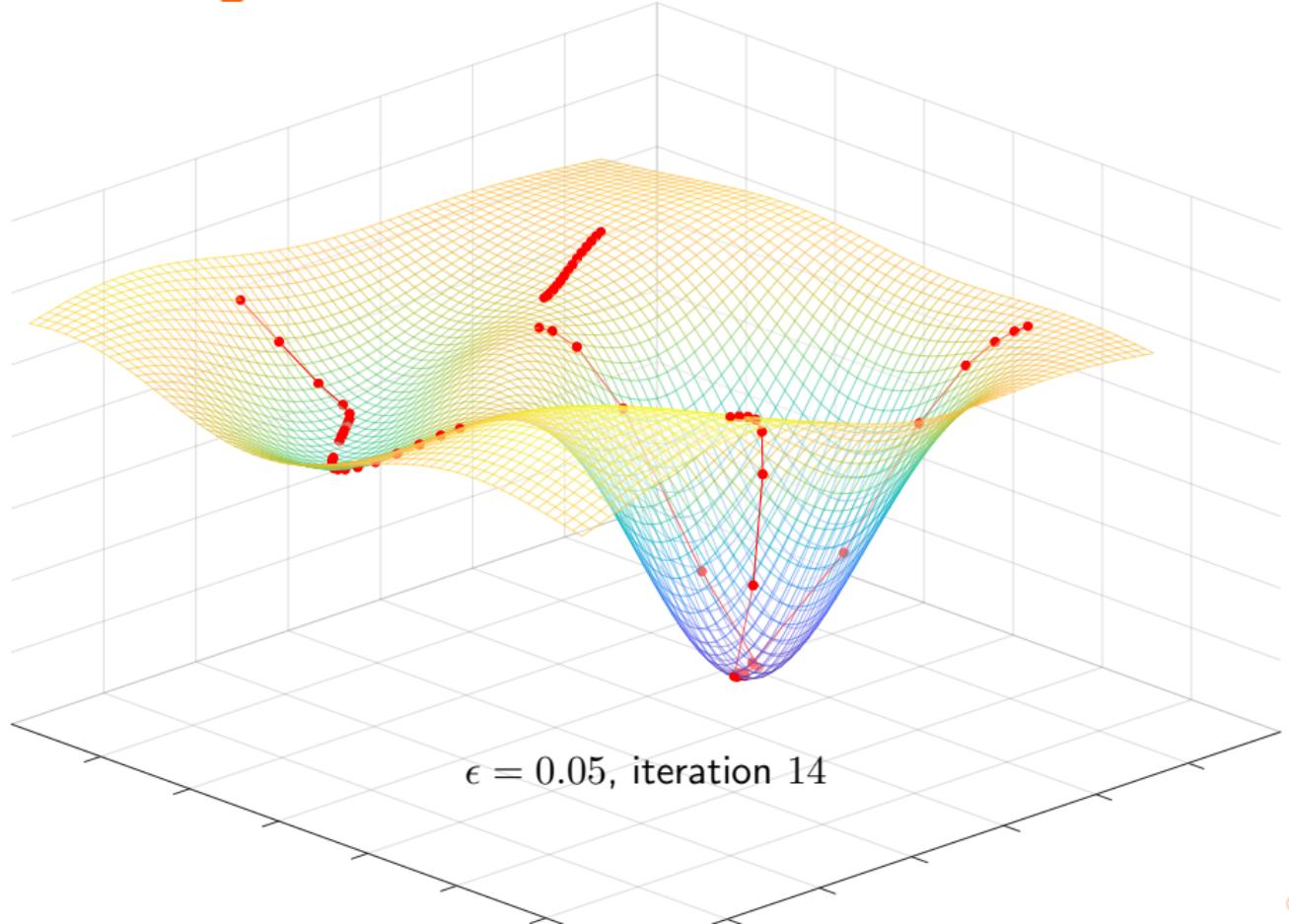
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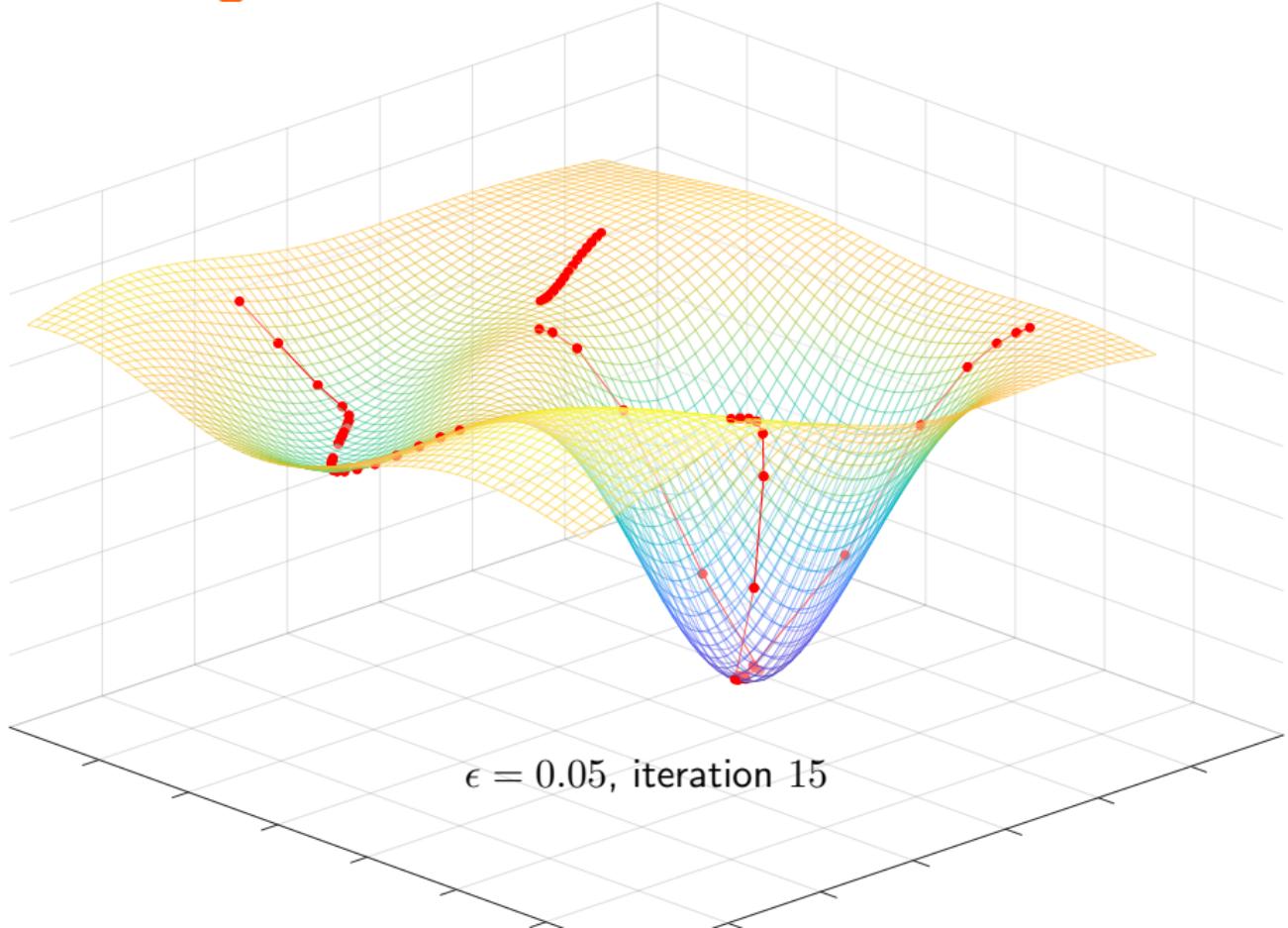
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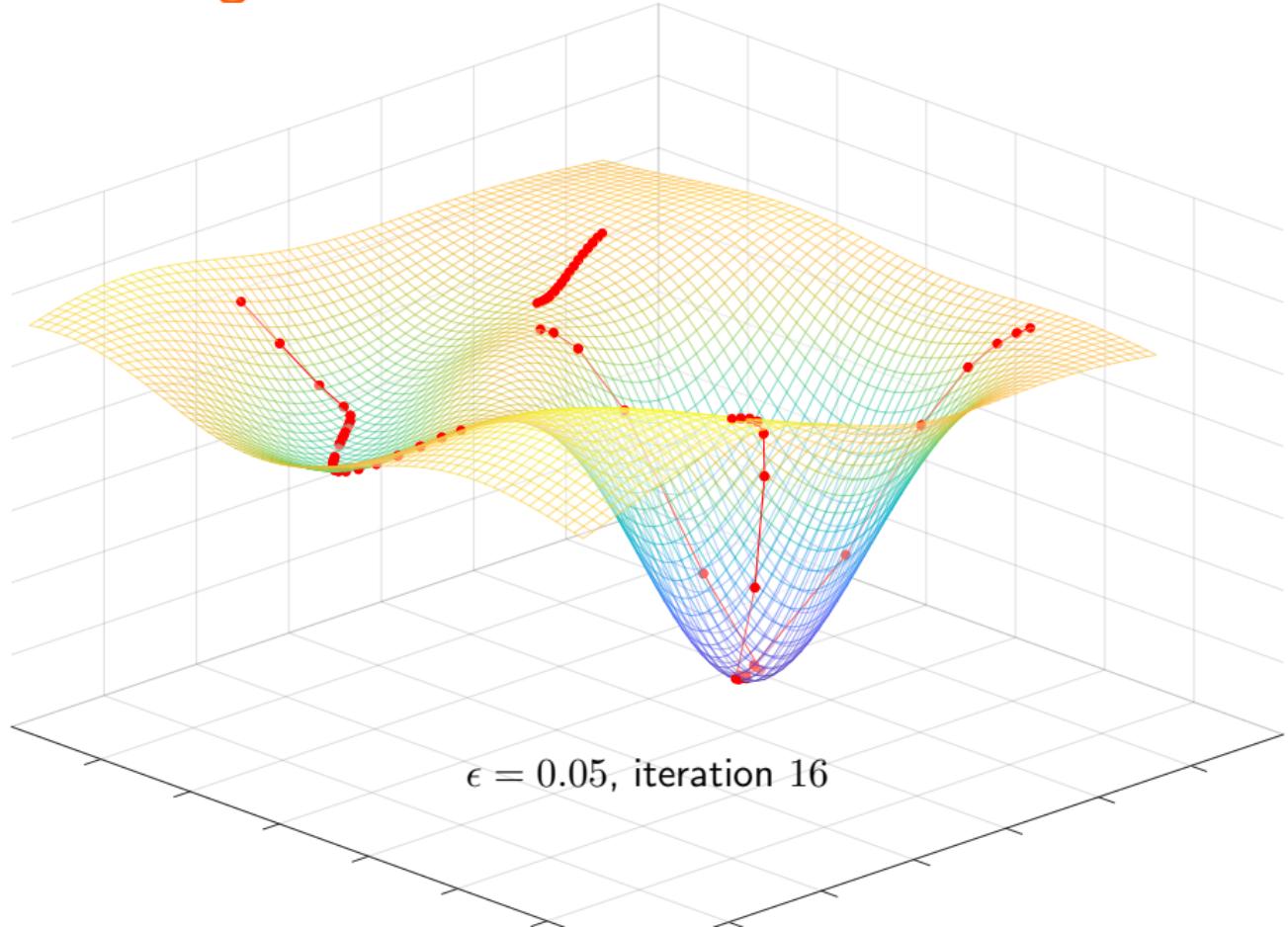
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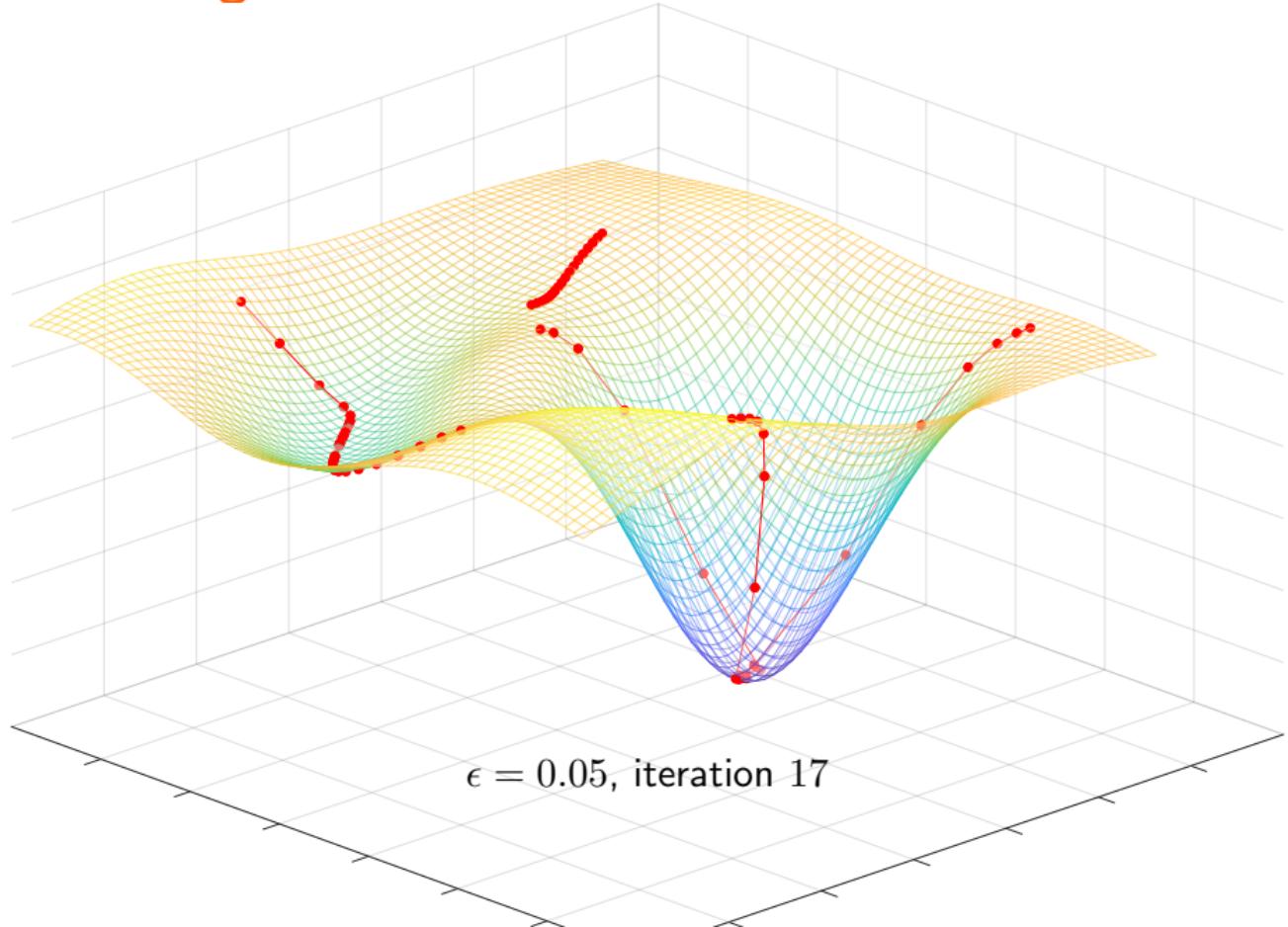
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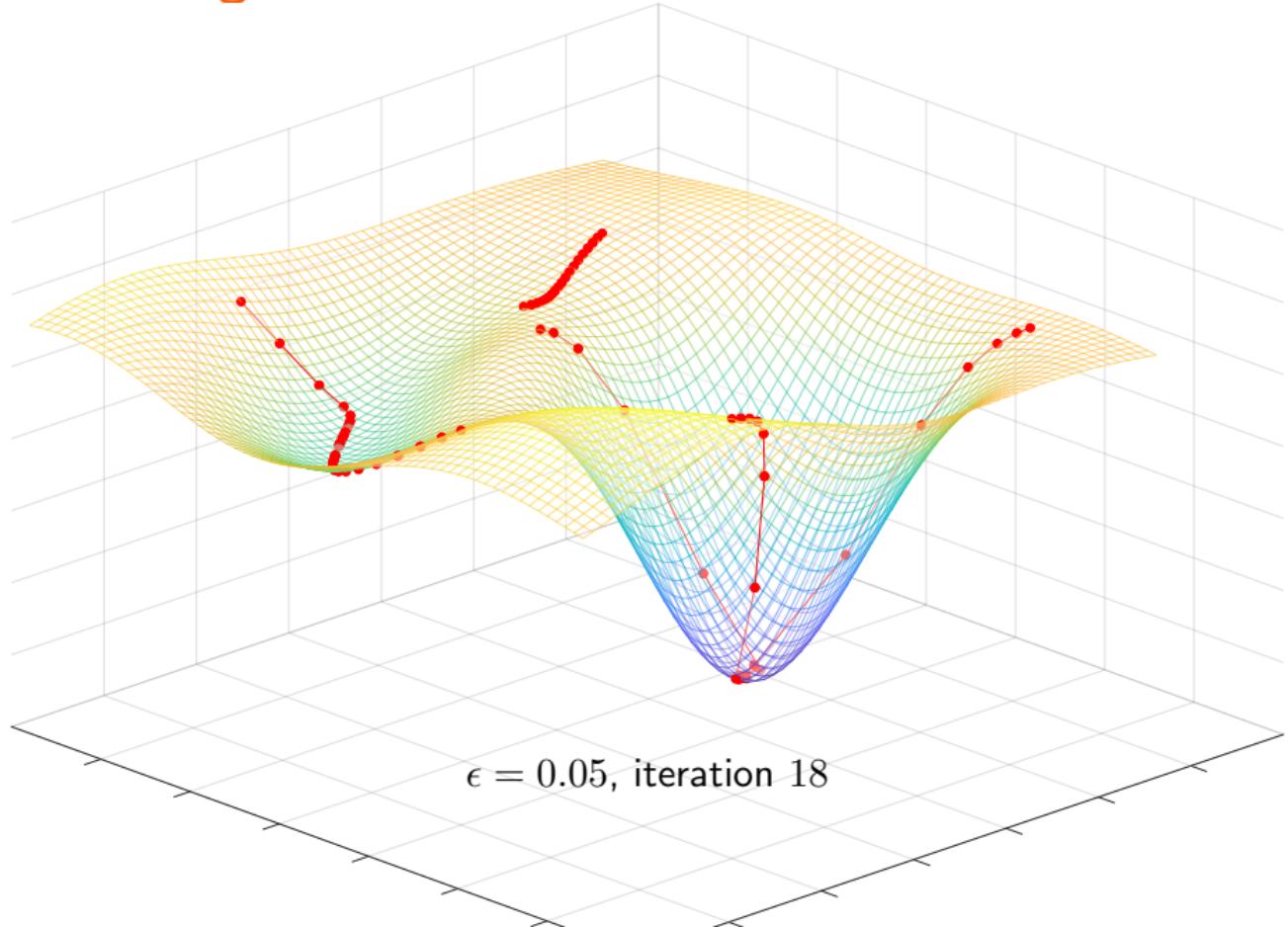
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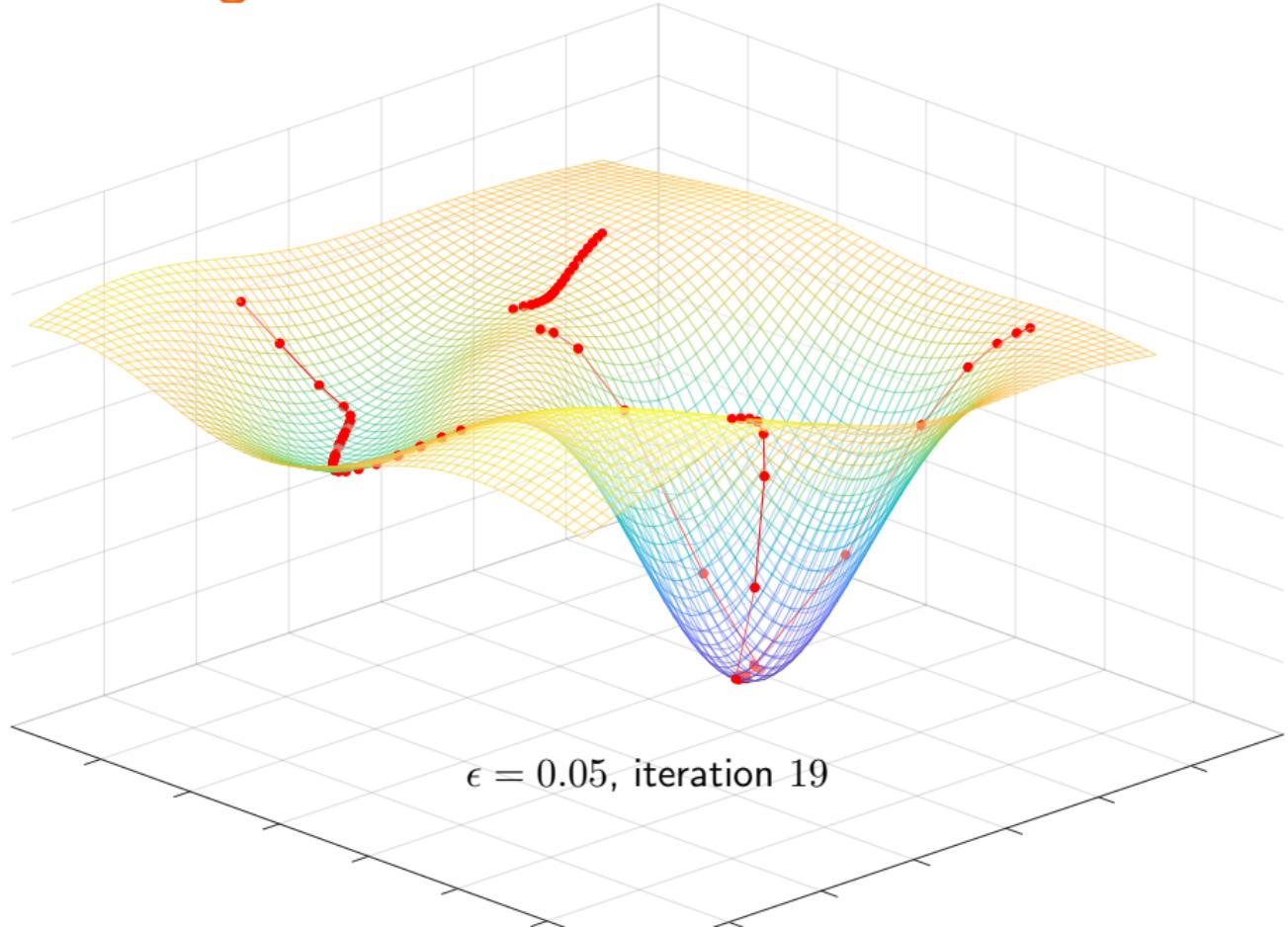
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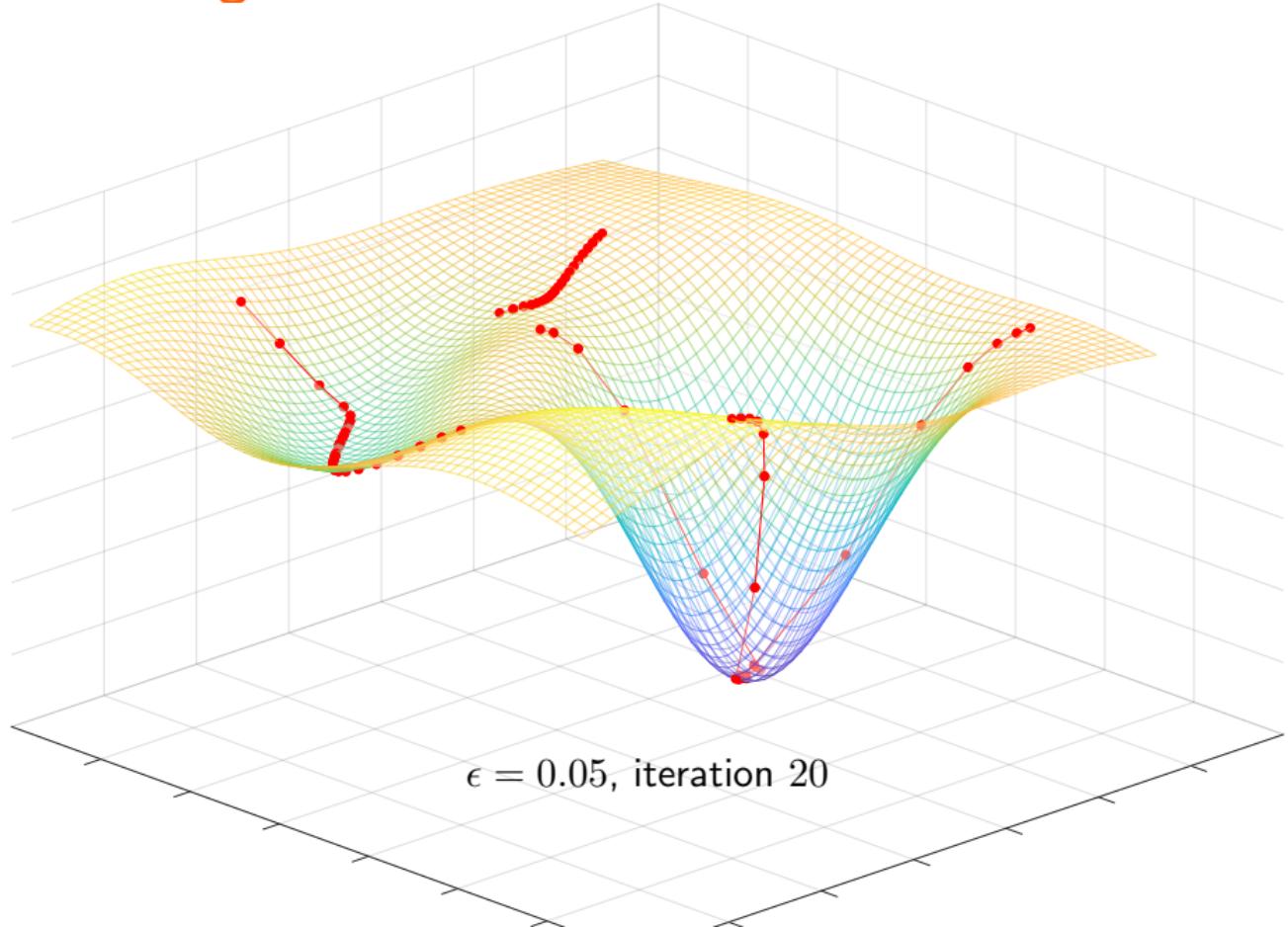
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# problems

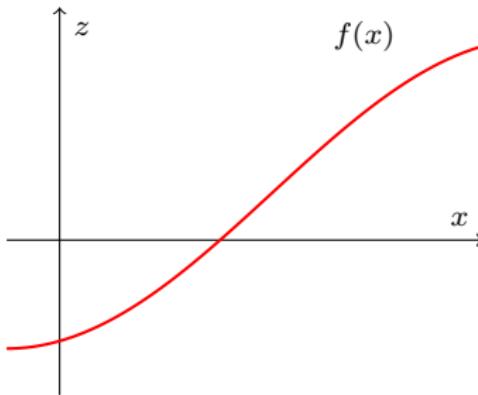
- $f$  non-convex: local minima
- $d \times d$  Hessian matrix too expensive ( $d$  can be millions): unknown curvature
- high condition number: elongated regions
- plateaus, saddle points: no progress
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# sequential estimation

[Robbins and Monro 1951]



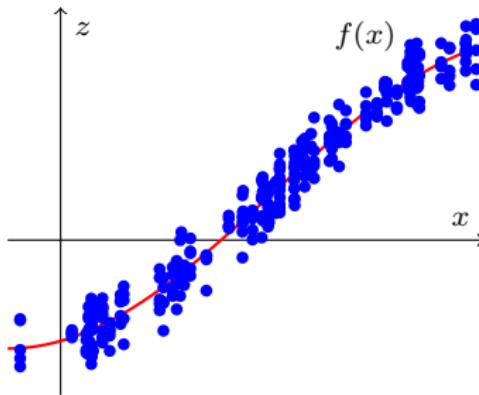
- suppose  $f^*$  is the expectation of random variable  $z$  conditional on  $x$ , and  $f$  is its empirical estimate on  $n$  samples

$$f^*(x) := \mathbb{E}[z|x] \quad f(x) := \frac{1}{n} \sum_{i=1}^n f_i(x)$$

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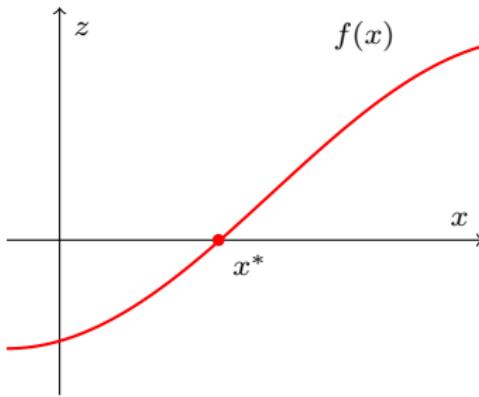
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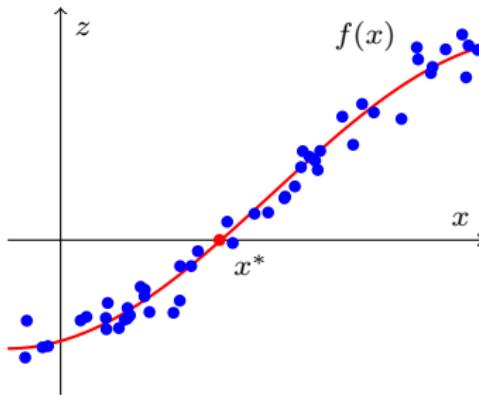
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# sequential estimation

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- then we can estimate  $x^*$  sequentially

$$x^{(\tau+1)} = x^{(\tau)} - \epsilon_\tau z(x^{(\tau)}) = x^{(\tau)} - \epsilon_\tau f_i(x^{(\tau)})$$

where  $z(x^{(\tau)})$  is an observation of  $z$  when  $x = x^{(\tau)}$  and  $i$  is a random index in  $\{1, \dots, n\}$

# sufficient conditions for convergence

- successive corrections decrease in magnitude

$$\lim_{\tau \rightarrow \infty} \epsilon_\tau = 0$$

- the algorithm does not converge short of the root

$$\sum_{\tau=1}^{\infty} \epsilon_\tau = \infty$$

- the accumulated “noise” has finite variance

$$\sum_{\tau=1}^{\infty} \epsilon_\tau^2 < \infty$$

## online gradient descent

- now, replace  $x$  by the parameters  $\theta$  of our model, and  $f$  by  $\nabla E$ , the gradient of our empirical risk
- the update rule becomes

$$\theta^{(\tau+1)} \leftarrow \theta^{(\tau)} - \epsilon_\tau \nabla E_i(\theta^{(\tau)})$$

- and, under the same conditions, it converges to a root of

$$\nabla E(\theta) = \frac{1}{n} \sum_{i=1}^n \nabla E_i(\theta) = \frac{1}{n} \sum_{i=1}^n \nabla L(f(\mathbf{x}_i; \theta), t_i)$$

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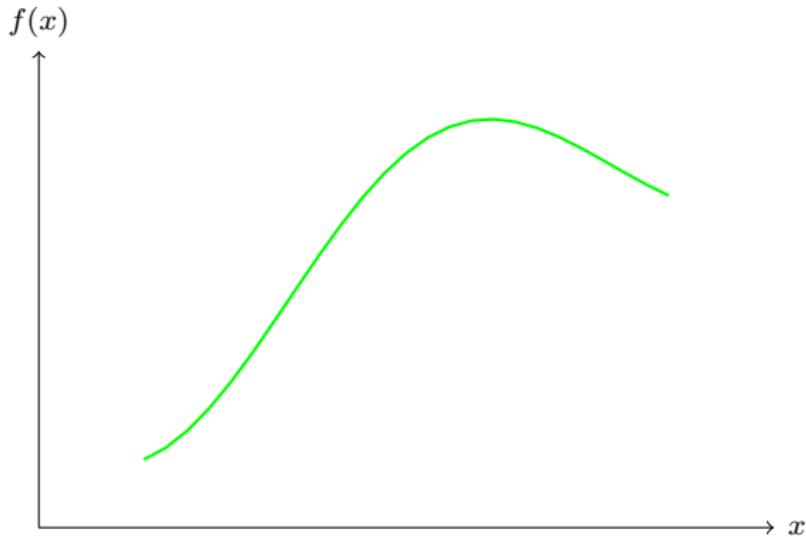
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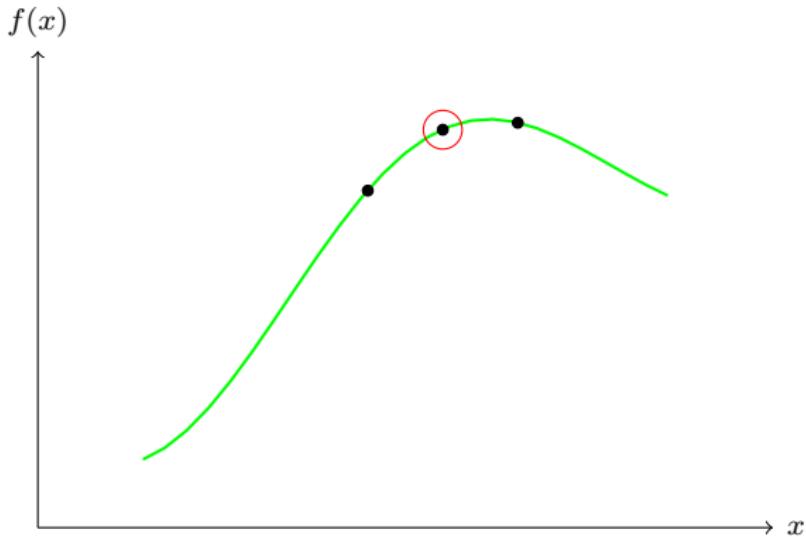
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# gradient computation

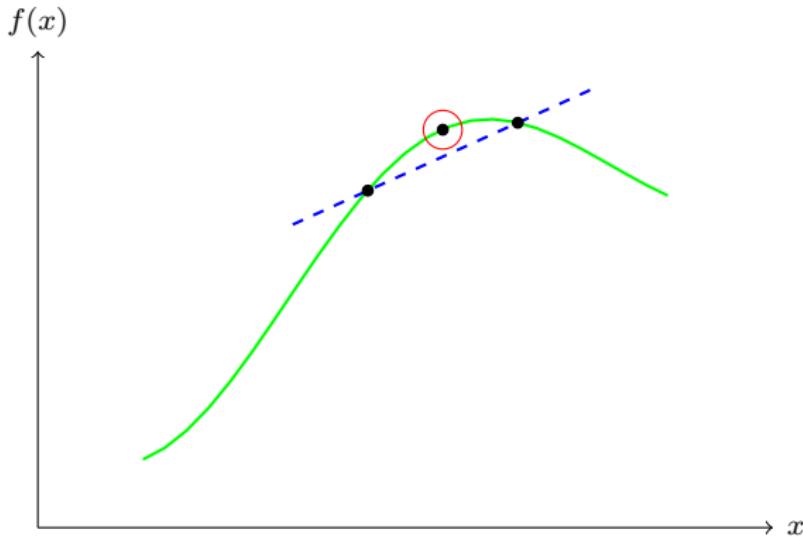
# numerical approximation



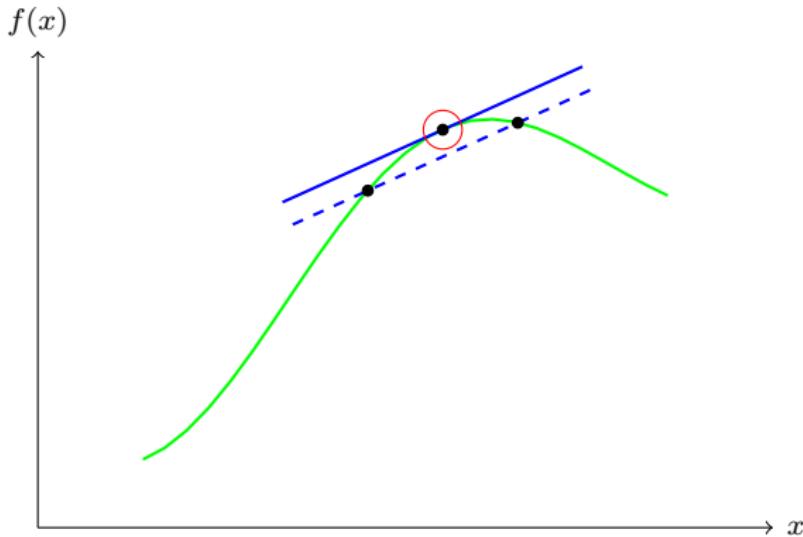
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# numerical approximation



$$\frac{df}{dx}(x) \approx \frac{f(x + \delta) - f(x - \delta)}{2\delta}$$

# numerical approximation

- given  $f : \mathbb{R}^p \rightarrow \mathbb{R}$ , its gradient is the vector function

$$\nabla f := \left( \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_p} \right)$$

- each partial derivative  $\frac{\partial f}{\partial x_i}$  can be approximated at  $\mathbf{x}$  by the **symmetric difference** formula

$$\Delta_i f(\mathbf{x}; \delta) := \frac{f(\mathbf{x} + \delta \mathbf{e}_i) - f(\mathbf{x} - \delta \mathbf{e}_i)}{2\delta}$$

for small  $\delta > 0$ , where  $\mathbf{e}_i$  is the  $i$  standard basis vector of  $\mathbb{R}^m$

- in practice, the smallest  $\delta$  should be used that does not cause numerical issues, e.g.  $\delta \in [10^{-10}, 10^{-5}]$  for double-precision arithmetic

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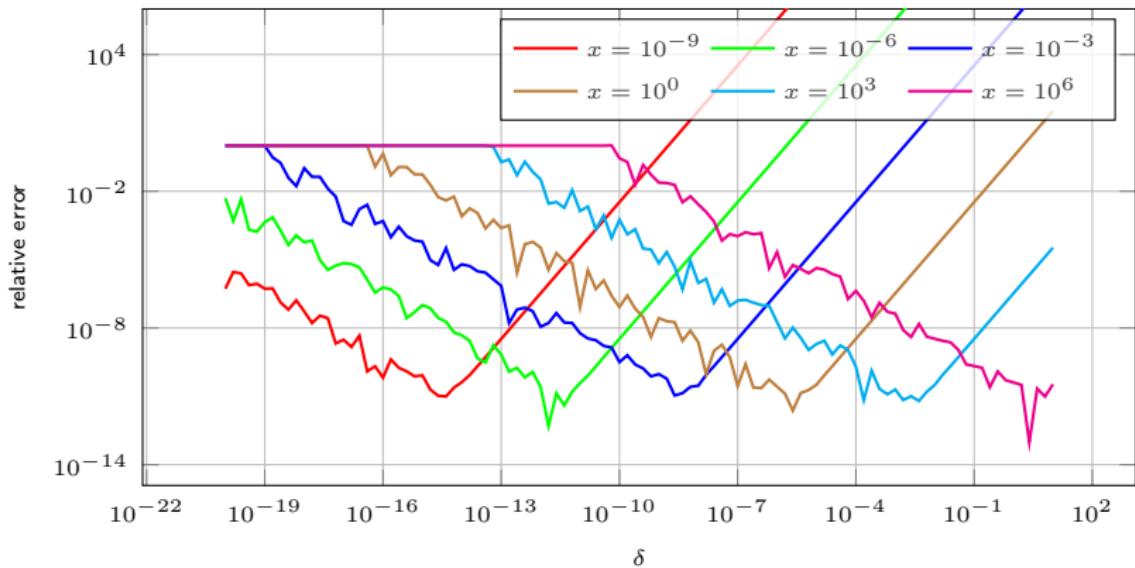
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# example



- relative error for  $f(x) = x^3$ ,  $\nabla f(x) = 3x^2$

$$\frac{|\Delta f(x; \delta) - \nabla f(x)|}{\nabla f(x)}$$

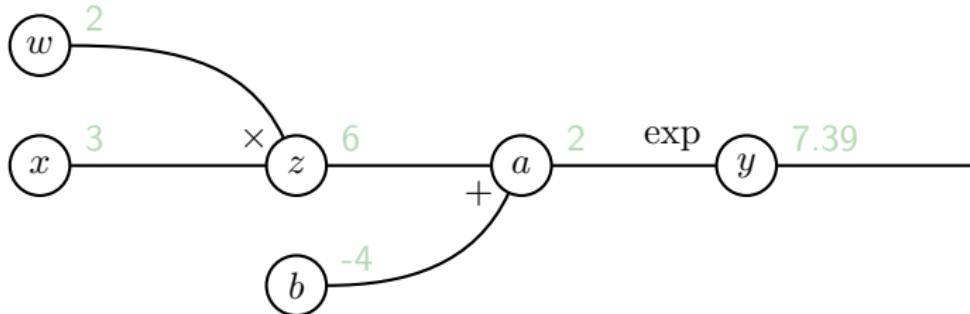
## numerical vs. analytical

- apart from accuracy issues, the numerical approximation is impractical in high dimensions: one evaluation of  $\Delta f$  requires  $2p$  evaluations of  $f$ , and dimension  $p$  is easily in the order of millions
- we turn to analytical computation of the gradient, which costs roughly as much as one evaluation of  $f$
- but the numerical approximation always remains useful for double-checking

# analytical computation

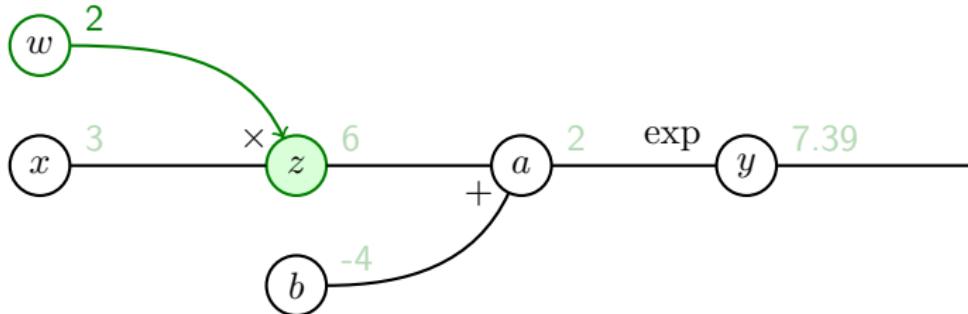
- all derivatives we care about are the derivatives of the error function with respect to the model parameters: the error function is **scalar** and we need its **gradient**
- we are going to write the error function as a composition of simpler functions, and use the **chain rule** to compute the gradient efficiently
- the error function can be as complex as a program with **control flow** statements
- each component function, called a **unit**, is assumed to be at least piecewise differentiable with a known formula for its derivative
- a unit may be a vector function, so we need **Jacobian matrices** in general, not just gradients

## example



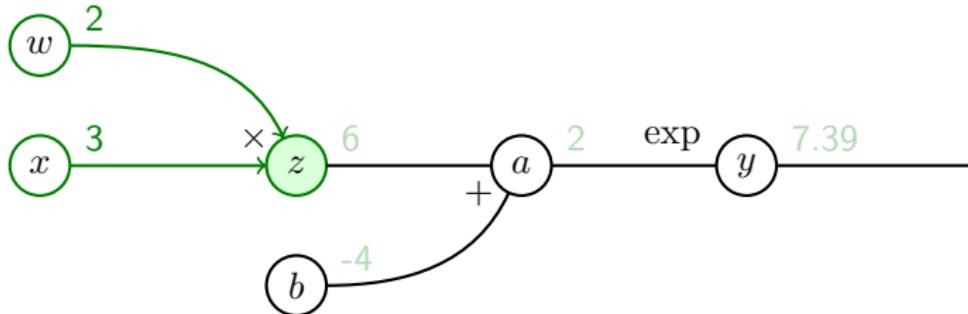
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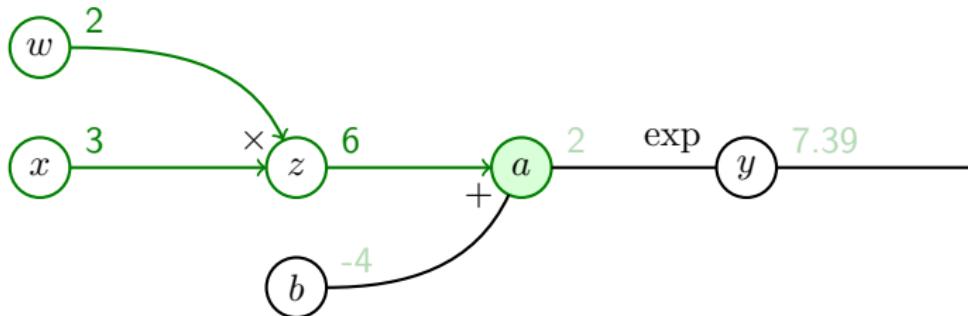
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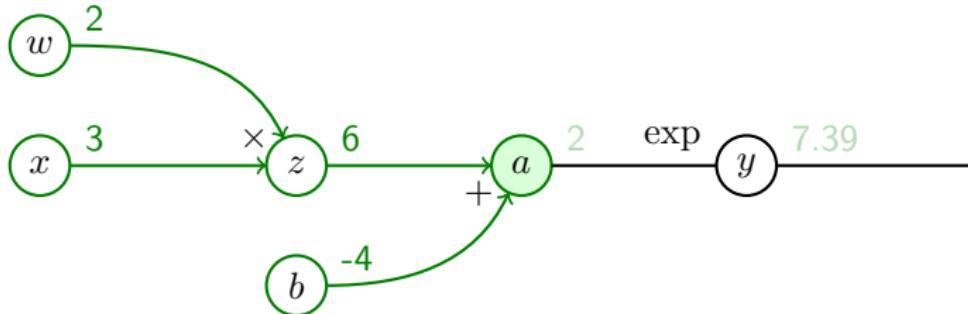
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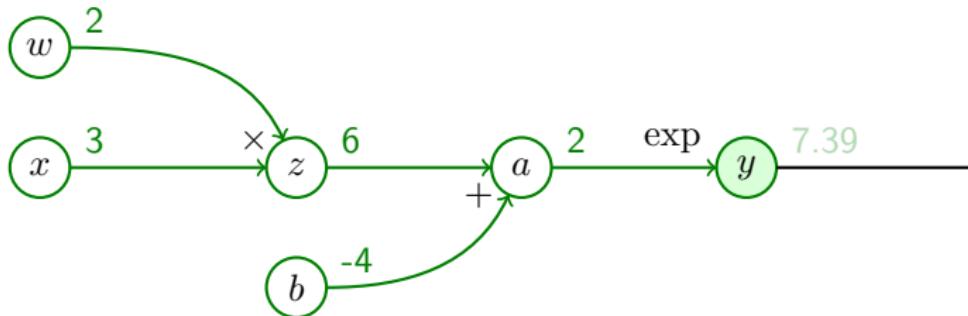
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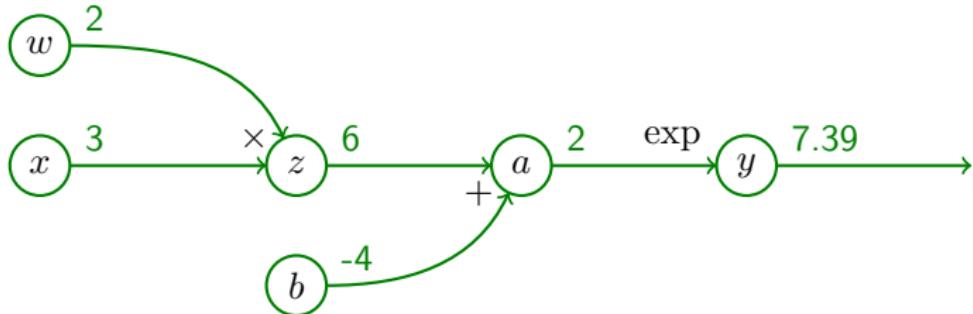
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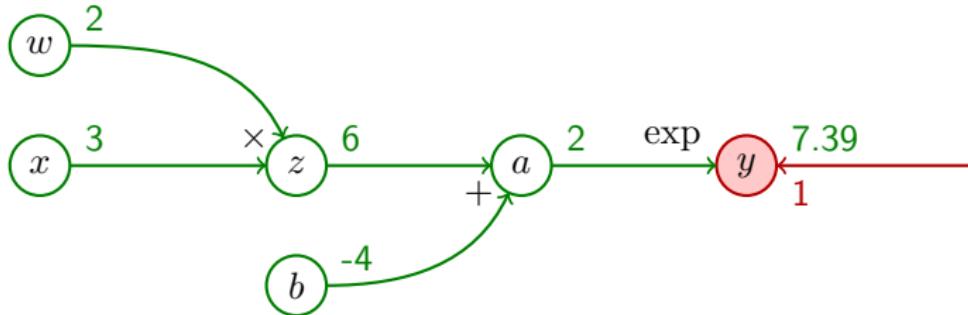
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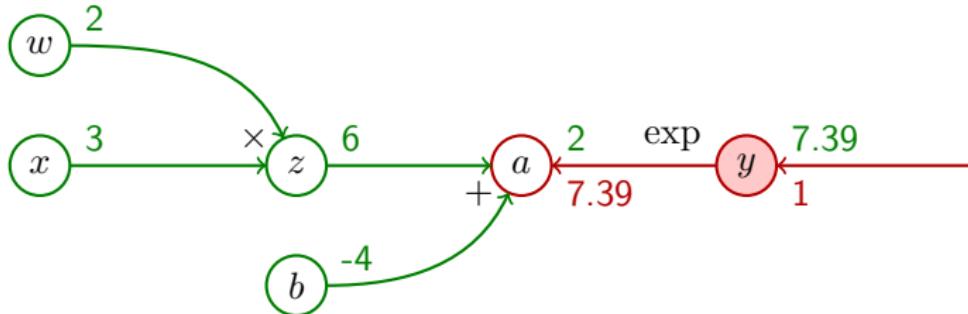
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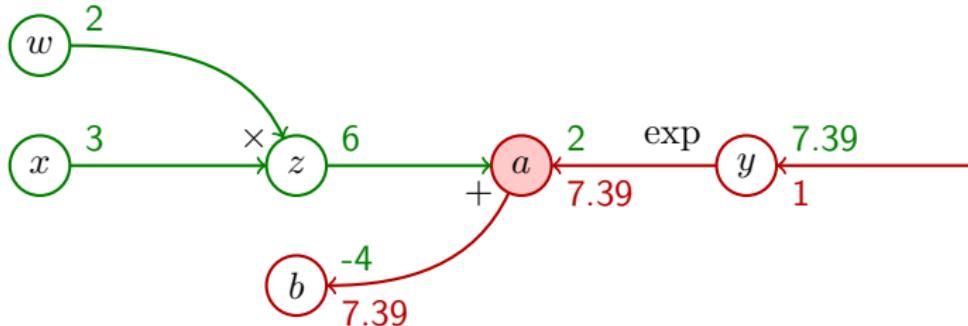
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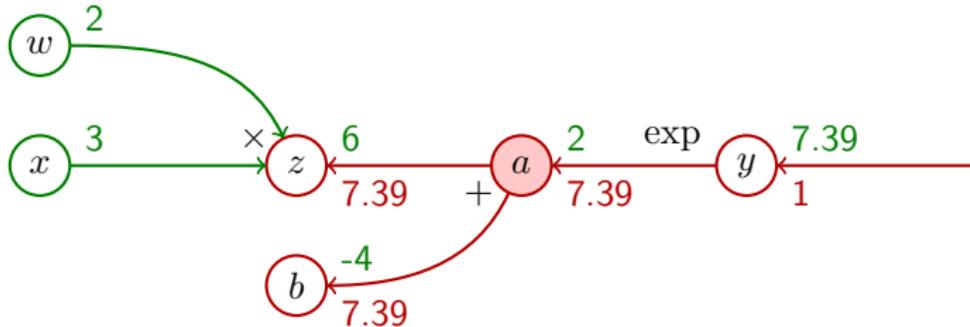
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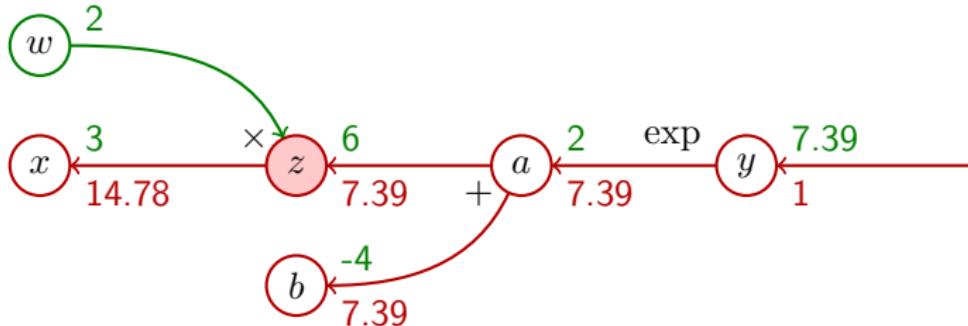
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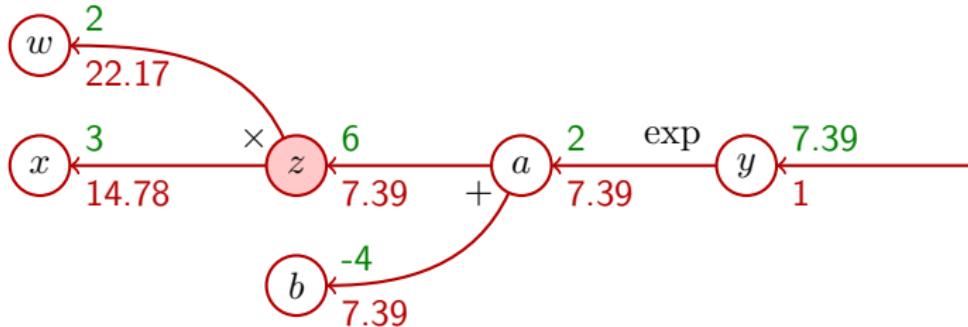
- $y = e^{wx+b}$  is broken down into  $y = e^a$ ,  $a = z + b$ ,  $z = wx$
- we seek  $\frac{\partial y}{\partial w}$ ,  $\frac{\partial y}{\partial x}$ ,  $\frac{\partial y}{\partial b}$  for  $(w, x, b) = (2, 3, -4)$
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## vector functions: derivative

- a function  $f : \mathbb{R}^p \rightarrow \mathbb{R}^q$  is **differentiable** at  $\mathbf{x}$  if there is a  $q \times p$  matrix  $A$  such that

$$\frac{f(\mathbf{x} + \mathbf{h}) - f(\mathbf{x}) - A \cdot \mathbf{h}}{|\mathbf{h}|} \rightarrow \mathbf{0}$$

as  $\mathbf{h} \rightarrow \mathbf{0}$ ; matrix  $A$  is the **derivative** of  $f$  at  $\mathbf{x}$ , denoted as  $Df(\mathbf{x})$

- if

$$f(\mathbf{x}) = A\mathbf{x}$$

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## vector functions: derivative vs. Jacobian

- given  $f = (f_1, \dots, f_q) : \mathbb{R}^p \rightarrow \mathbb{R}^q$  whose its partial derivatives exist at  $\mathbf{x}$ , and  $\mathbf{y} = f(\mathbf{x})$ , its **Jacobian matrix** at  $\mathbf{x}$  can be written as

$$\frac{\partial \mathbf{y}}{\partial \mathbf{x}} = \frac{\partial f}{\partial \mathbf{x}} := \begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_p} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_q}{\partial x_1} & \cdots & \frac{\partial f_q}{\partial x_p} \end{pmatrix}$$

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## scalar functions: derivative vs. gradient

- the gradient of a scalar  $f : \mathbb{R}^p \rightarrow \mathbb{R}$  with respect to an input vector  $\mathbf{x}$  is a **column** vector in  $\mathbb{R}^p$ , the same size as  $\mathbf{x}$

$$\nabla f := \left( \frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_p} \right)$$

- in contrast, the derivative is an  $1 \times p$  **row** vector

$$Df(\mathbf{x}) := \begin{pmatrix} \frac{\partial f}{\partial x_1} & \dots & \frac{\partial f}{\partial x_p} \end{pmatrix} = (\nabla f)^\top$$

- the following analysis uses derivatives/Jacobians, so we will **transpose** them to make them compatible with  $\mathbf{x}$

## chain rule

- if  $f : \mathbb{R}^p \rightarrow \mathbb{R}^q$  is differentiable at  $\mathbf{x}$  and  $g : \mathbb{R}^q \rightarrow \mathbb{R}^r$  is differentiable at  $\mathbf{y} = f(\mathbf{x})$ , then  $g \circ f : \mathbb{R}^p \rightarrow \mathbb{R}^r$  is differentiable at  $\mathbf{x}$  and

$$D(g \circ f)(\mathbf{x}) = Dg(\mathbf{y}) \cdot Df(\mathbf{x})$$

where  $\cdot$  denotes matrix multiplication

- how to use it:

$$\frac{\partial z}{\partial x_1} = \frac{\partial z}{\partial x_2} \cdot \frac{\partial x_2}{\partial x_1}$$



## chain rule

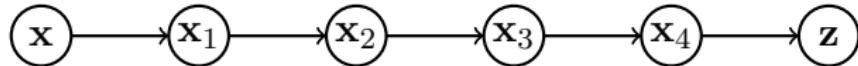
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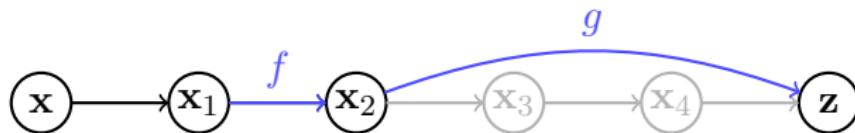
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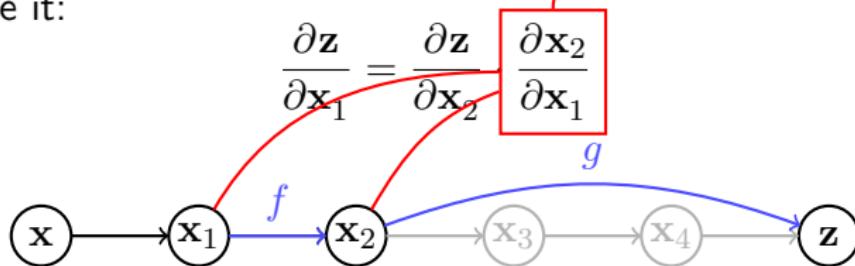
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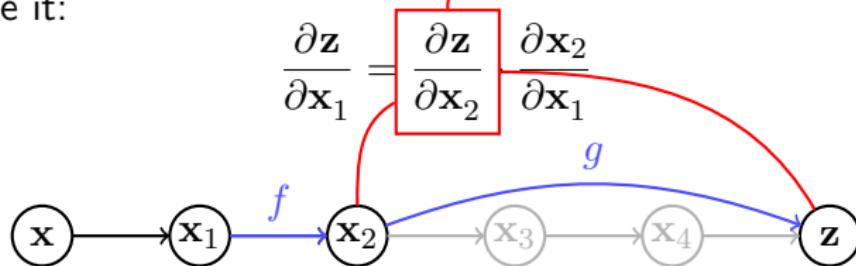
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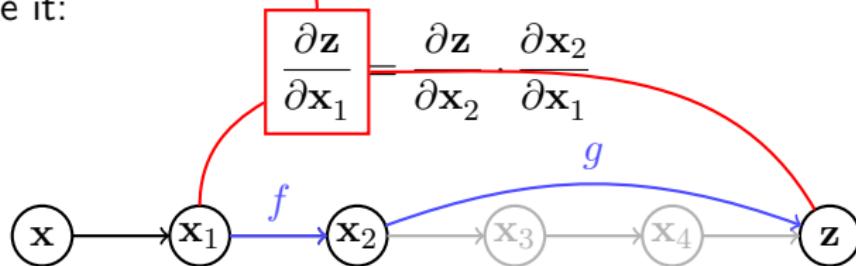
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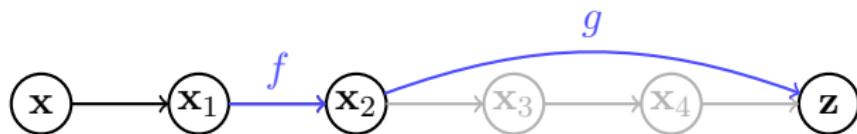
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- now, for all  $i$ , let us call the partial derivatives

$$d\mathbf{x}_i^\top := \frac{\partial \mathbf{z}}{\partial \mathbf{x}_i}$$

## chain rule

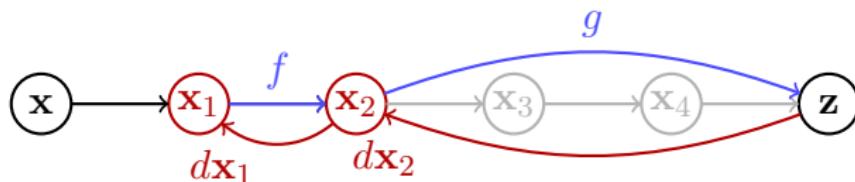
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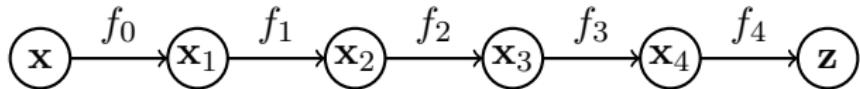


- then, we are **back-propagating** from  $d\mathbf{x}_2$  to  $d\mathbf{x}_1$

$$d\mathbf{x}_1^\top = d\mathbf{x}_2^\top \cdot Df(\mathbf{x}_1)$$

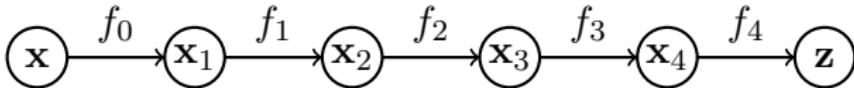
# chaining

- let  $f = f_4 \circ f_3 \circ f_2 \circ f_1 \circ f_0$  and  $\mathbf{z} = f(\mathbf{x})$



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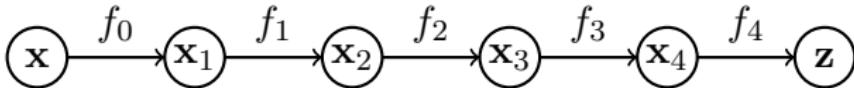


- we apply the chain rule

$$\begin{aligned}\frac{\partial \mathbf{z}}{\partial \mathbf{x}} &= Df(\mathbf{x}) = D(f_4 \circ f_3 \circ f_2 \circ f_1)(\mathbf{x}_1) \cdot Df_0(\mathbf{x}) \\ &= D(f_4 \circ f_3 \circ f_2)(\mathbf{x}_2) \cdot Df_1(\mathbf{x}_1) \cdot Df_0(\mathbf{x}) \\ &= D(f_4 \circ f_3)(\mathbf{x}_3) \cdot Df_2(\mathbf{x}_2) \cdot Df_1(\mathbf{x}_1) \cdot Df_0(\mathbf{x}) \\ &= Df_4(\mathbf{x}_4) \cdot Df_3(\mathbf{x}_3) \cdot Df_2(\mathbf{x}_2) \cdot Df_1(\mathbf{x}_1) \cdot Df_0(\mathbf{x}) \\ &= d\mathbf{x}_4^\top \cdot Df_3(\mathbf{x}_3) \cdot Df_2(\mathbf{x}_2) \cdot Df_1(\mathbf{x}_1) \cdot Df_0(\mathbf{x}) \\ &= d\mathbf{x}_3^\top \cdot Df_2(\mathbf{x}_2) \cdot Df_1(\mathbf{x}_1) \cdot Df_0(\mathbf{x}) \\ &= d\mathbf{x}_2^\top \cdot Df_1(\mathbf{x}_1) \cdot Df_0(\mathbf{x}) \\ &= d\mathbf{x}_1^\top \cdot Df_0(\mathbf{x}) \\ &= d\mathbf{x}^\top\end{aligned}$$

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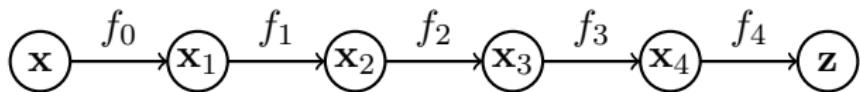
- let  $f = f_4 \circ f_3 \circ f_2 \circ f_1 \circ f_0$  and  $\mathbf{z} = f(\mathbf{x})$



- we apply the chain rule, then collect back into factors  $d\mathbf{x}_i$

$$\begin{aligned}\frac{\partial \mathbf{z}}{\partial \mathbf{x}} &= Df(\mathbf{x}) = D(f_4 \circ f_3 \circ f_2 \circ f_1)(\mathbf{x}_1) \cdot Df_0(\mathbf{x}) \\&= D(f_4 \circ f_3 \circ f_2)(\mathbf{x}_2) \cdot Df_1(\mathbf{x}_1) \cdot Df_0(\mathbf{x}) \\&= D(f_4 \circ f_3)(\mathbf{x}_3) \cdot Df_2(\mathbf{x}_2) \cdot Df_1(\mathbf{x}_1) \cdot Df_0(\mathbf{x}) \\&= Df_4(\mathbf{x}_4) \cdot Df_3(\mathbf{x}_3) \cdot Df_2(\mathbf{x}_2) \cdot Df_1(\mathbf{x}_1) \cdot Df_0(\mathbf{x}) \\&= d\mathbf{x}_4^\top \cdot Df_3(\mathbf{x}_3) \cdot Df_2(\mathbf{x}_2) \cdot Df_1(\mathbf{x}_1) \cdot Df_0(\mathbf{x}) \\&= d\mathbf{x}_3^\top \cdot Df_2(\mathbf{x}_2) \cdot Df_1(\mathbf{x}_1) \cdot Df_0(\mathbf{x}) \\&= d\mathbf{x}_2^\top \cdot Df_1(\mathbf{x}_1) \cdot Df_0(\mathbf{x}) \\&= d\mathbf{x}_1^\top \cdot Df_0(\mathbf{x}) \\&= d\mathbf{x}^\top\end{aligned}$$

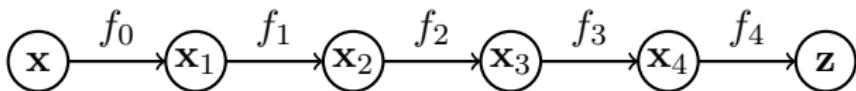
# back-propagation



forward pass

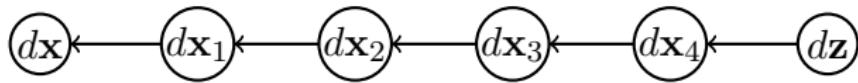
$$\mathbf{x}_1 = f_0(\mathbf{x}) \quad \mathbf{x}_2 = f_1(\mathbf{x}_1) \quad \mathbf{x}_3 = f_2(\mathbf{x}_2) \quad \mathbf{x}_4 = f_3(\mathbf{x}_3) \quad \mathbf{z} = f_4(\mathbf{x}_4)$$

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backward pass

$$d\mathbf{z}^\top = I \qquad d\mathbf{x}_4^\top = d\mathbf{z}^\top \cdot Df_4(\mathbf{x}_4) \quad d\mathbf{x}_3^\top = d\mathbf{x}_4^\top \cdot Df_3(\mathbf{x}_3)$$
$$d\mathbf{x}_2^\top = d\mathbf{x}_3^\top \cdot Df_2(\mathbf{x}_2) \quad d\mathbf{x}_1^\top = d\mathbf{x}_2^\top \cdot Df_1(\mathbf{x}_1) \quad d\mathbf{x}^\top = d\mathbf{x}_1^\top \cdot Df_0(\mathbf{x})$$

# back-propagation is dynamic programming

- we need to store all the  $x_i$  that we compute in the forward pass before the backward pass begins
- the  $dx_i$  can be computed on the fly in reverse order on a chain, but may need to be all stored on a general network structure
- that's exactly what we do in **dynamic programming**: break the problem down into a collection of smaller, overlapping subproblems, store their solutions and save computation time at the expense of a (hopefully) modest expenditure in storage space
- as in all dynamic programming problems, there is a **bottom-up** approach that we have just described, and a **top-down** approach coming out of the **recursive** formulation through **memoization**; this can be useful if we are looking for the derivative with respect to only few parameters

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# partial derivatives

- in the following, for any vector  $\mathbf{x}$  appearing in our function, we will use the symbol

$$d\mathbf{x}^\top := \frac{\partial}{\partial \mathbf{x}}$$

for the partial derivative operator of any quantity with respect to  $\mathbf{x}$

- in practice, we will apply this to the quantity we want to optimize, i.e. the error
- the error gradient will consist of the partial derivatives with respect to the model parameters, but we still need to compute partial derivatives with respect to all variables appearing in back-propagation

# partial derivatives

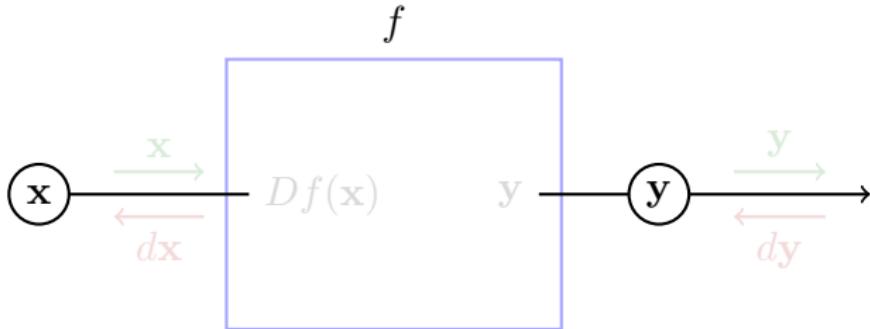
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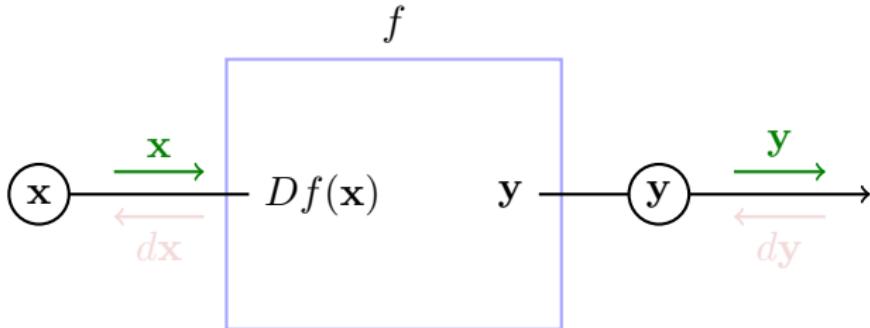
# nodes



- to every variable  $y$  is associated a node with the function  $f$  that produces it, from input variable  $x$
- given  $x$ , derivative  $Df(x)$  is “stored”, and output  $y$  is computed and flows forward
- given  $dy$ , partial derivative  $dx$  is computed and flows backward

$$d\mathbf{x}^\top = d\mathbf{y}^\top \cdot Df(\mathbf{x}) \quad \text{or} \quad \frac{\partial}{\partial \mathbf{x}} = \frac{\partial}{\partial \mathbf{y}} \cdot \frac{\partial \mathbf{y}}{\partial \mathbf{x}}$$

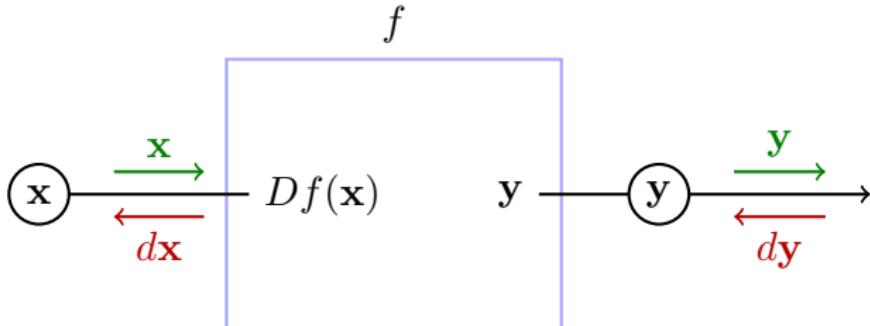
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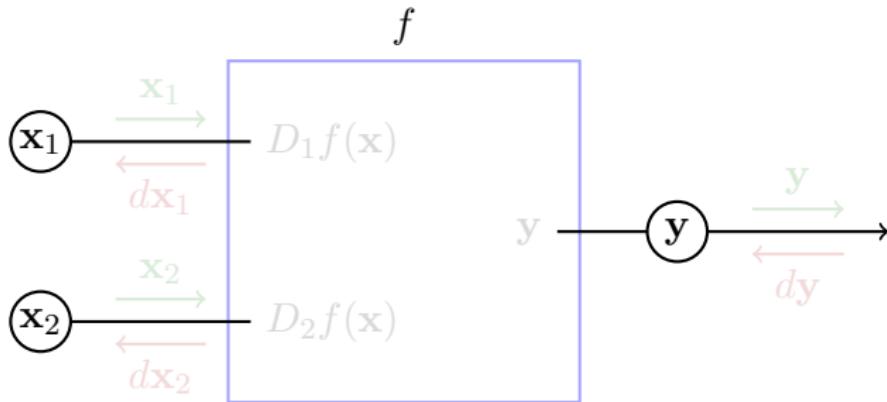
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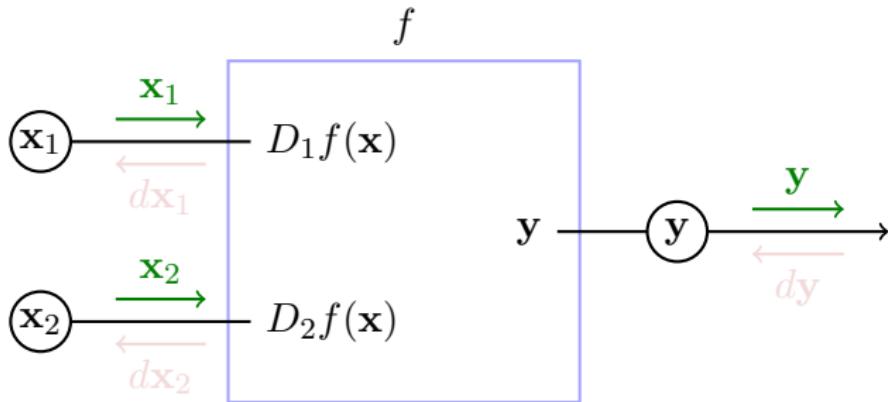
$$d\mathbf{x}^\top = d\mathbf{y}^\top \cdot Df(\mathbf{x}) \quad \text{or} \quad \frac{\partial}{\partial \mathbf{x}} = \frac{\partial}{\partial \mathbf{y}} \cdot \frac{\partial \mathbf{y}}{\partial \mathbf{x}}$$

# splitting the input



- we split input vector  $\mathbf{x}$  into subvectors as  $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2)$
- then, the derivative consists of blocks stacked horizontally
$$Df(\mathbf{x}) = (D_1 f \ D_2 f)(\mathbf{x}) \quad \text{or} \quad \frac{\partial \mathbf{y}}{\partial \mathbf{x}} = \begin{pmatrix} \frac{\partial \mathbf{y}}{\partial \mathbf{x}_1} & \frac{\partial \mathbf{y}}{\partial \mathbf{x}_2} \end{pmatrix}$$
- $d\mathbf{x}$  is also split as  $d\mathbf{x} = (d\mathbf{x}_1, d\mathbf{x}_2)$  and  $d\mathbf{x}^\top = d\mathbf{y}^\top \cdot Df(\mathbf{x})$  becomes
$$d\mathbf{x}_i^\top = d\mathbf{y}^\top \cdot D_i f(\mathbf{x}) \quad \text{or} \quad \frac{\partial}{\partial \mathbf{x}_i} = \frac{\partial}{\partial \mathbf{y}} \cdot \frac{\partial \mathbf{y}}{\partial \mathbf{x}_i}$$

# splitting the input



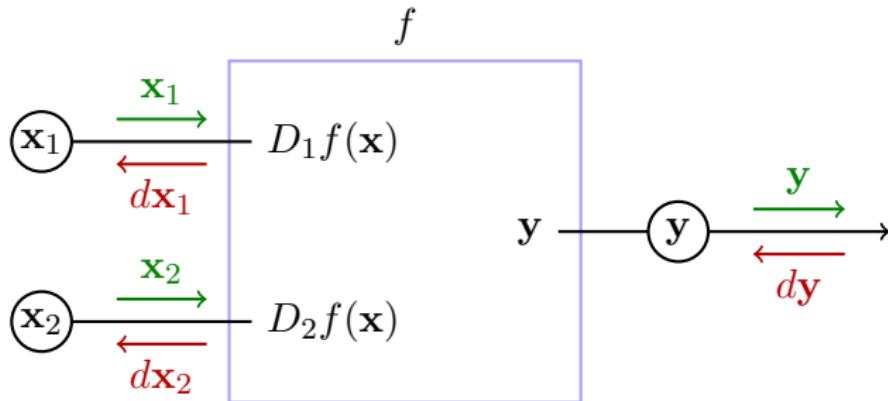
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- $d\mathbf{x}$  is also split as  $d\mathbf{x} = (dx_1, dx_2)$  and  $d\mathbf{x}^\top = dy^\top \cdot Df(\mathbf{x})$  becomes

$$d\mathbf{x}_i^\top = dy^\top \cdot D_i f(\mathbf{x}) \quad \text{or} \quad \frac{\partial}{\partial \mathbf{x}_i} = \frac{\partial}{\partial \mathbf{y}} \cdot \frac{\partial \mathbf{y}}{\partial \mathbf{x}_i}$$

## splitting the input



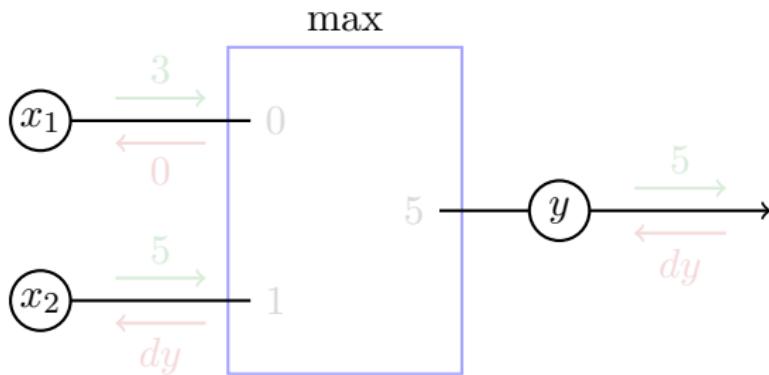
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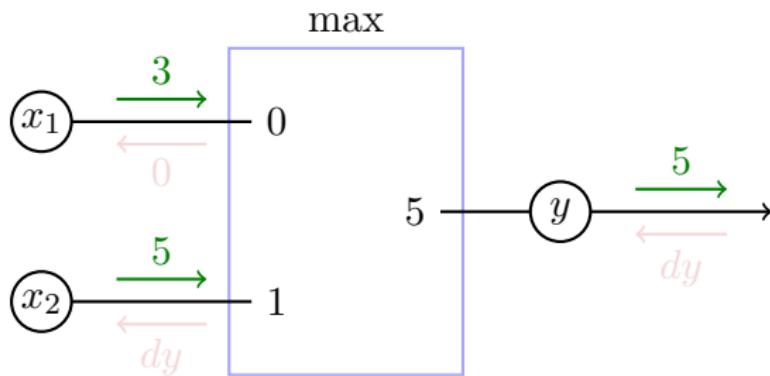
$$d\mathbf{x}_i^\top = d\mathbf{y}^\top \cdot D_i f(\mathbf{x}) \quad \text{or} \quad \frac{\partial}{\partial \mathbf{x}_i} = \frac{\partial}{\partial \mathbf{y}} \cdot \frac{\partial \mathbf{y}}{\partial \mathbf{x}_i}$$

## example: maximum



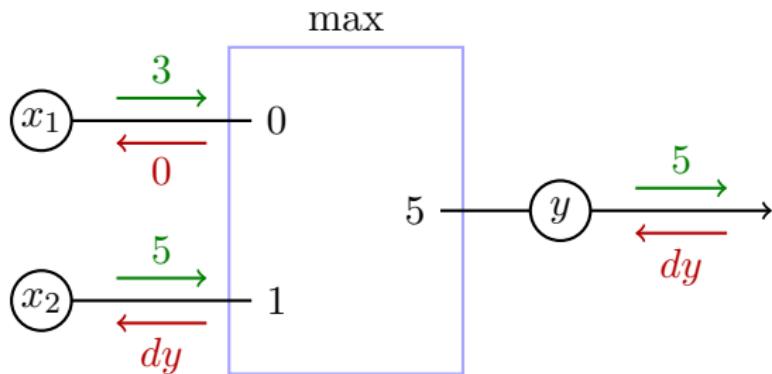
- if  $f(x_1, x_2) = \max(x_1, x_2)$ , then  $D_i f(x_1, x_2) = \mathbb{1}[x_i = \max(x_1, x_2)]$
- and  $dy$  is **routed** into the branch of the maximum input

## example: maximum



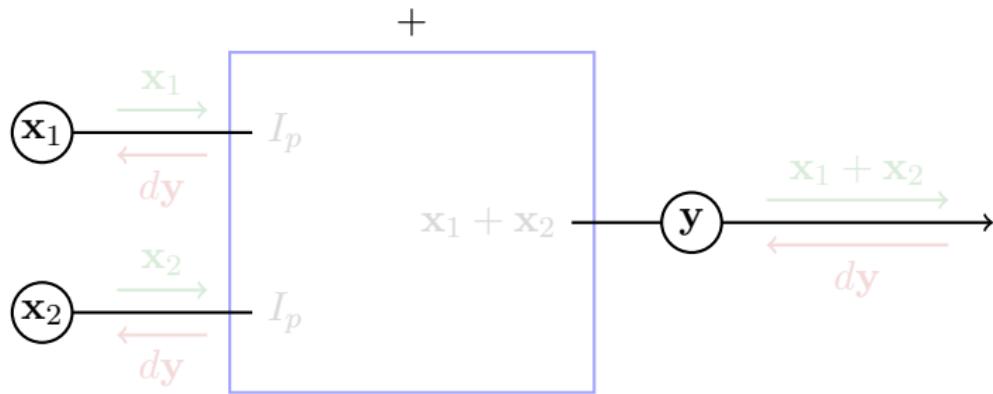
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- and  $dy$  is **routed** into the branch of the maximum input

## example: maximum



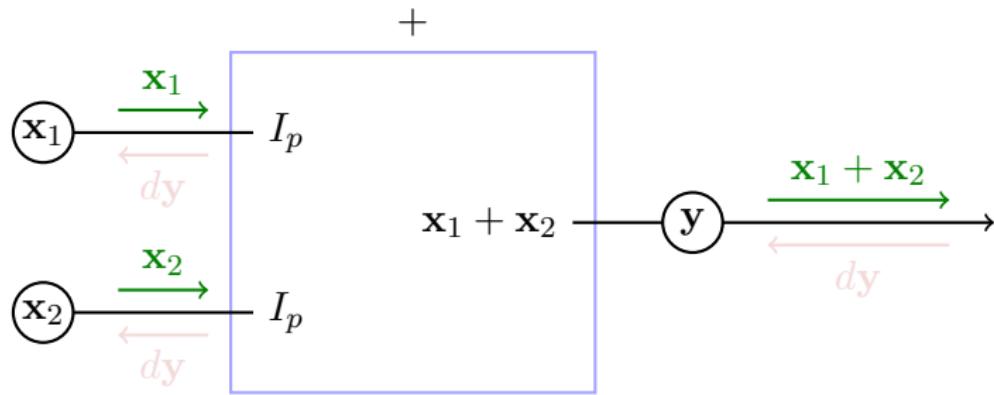
- if  $f(x_1, x_2) = \max(x_1, x_2)$ , then  $D_i f(x_1, x_2) = \mathbb{1}[x_i = \max(x_1, x_2)]$
- and  $dy$  is **routed** into the branch of the maximum input

## example: sum



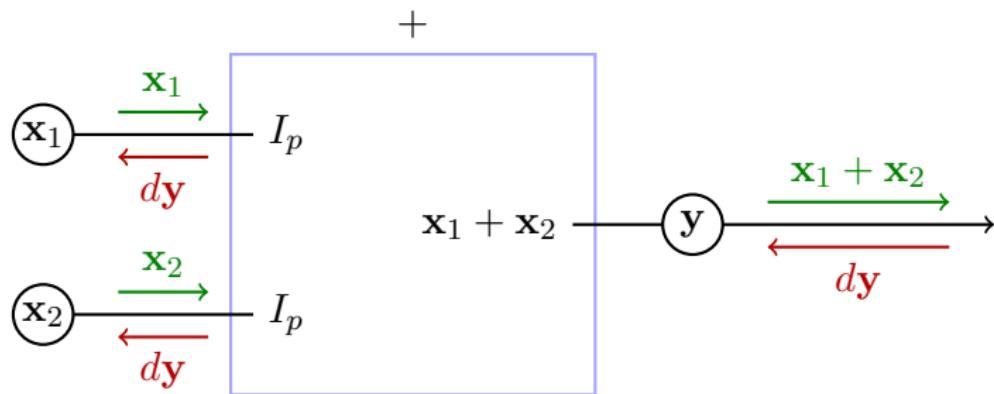
- if  $f(\mathbf{x}_1, \mathbf{x}_2) = \mathbf{x}_1 + \mathbf{x}_2$  and  $\mathbf{x}_i \in \mathbb{R}^p$ , then  $D_i f(\mathbf{x}_1, \mathbf{x}_2) = I_p$
- and  $d\mathbf{y}$  is **distributed** to both branches

## example: sum



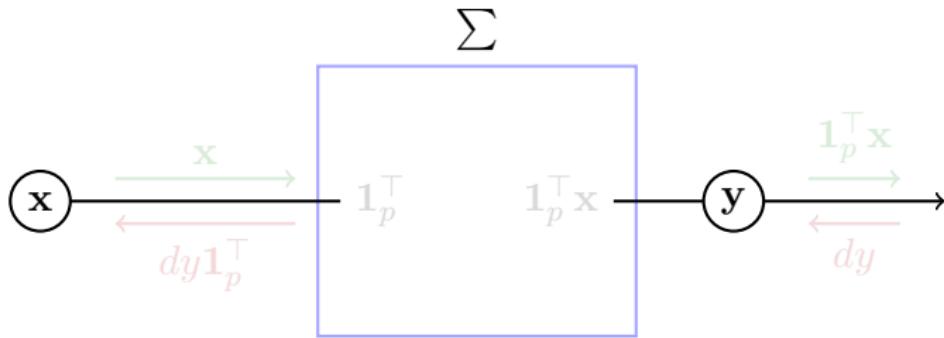
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## example: sum



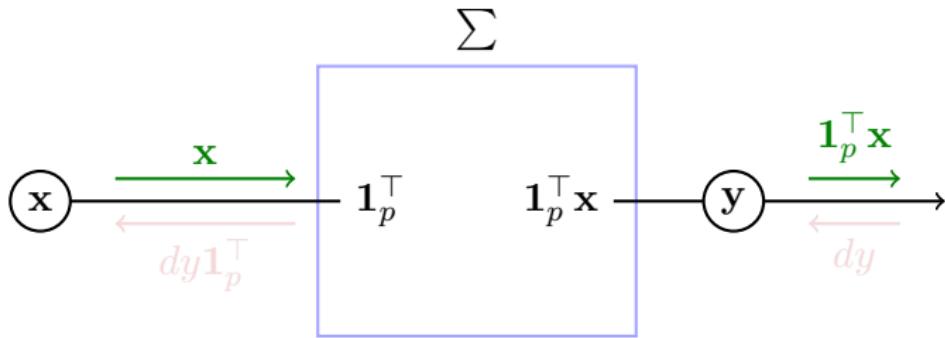
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- and  $d\mathbf{y}$  is distributed to both branches

## example: vector sum\*



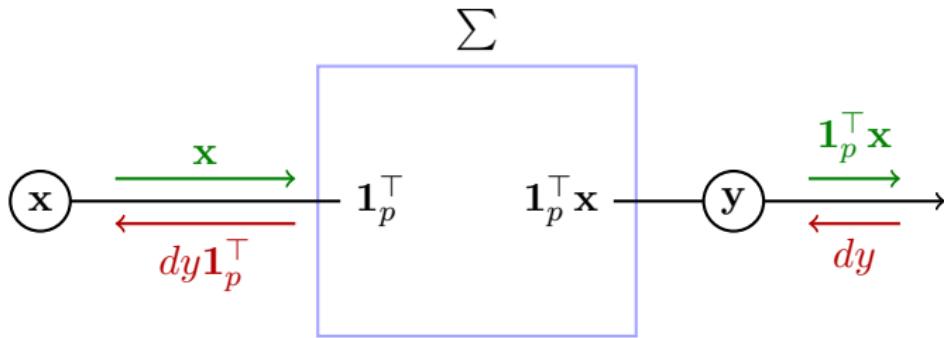
- if  $f(\mathbf{x}) = \mathbf{1}_p^\top \mathbf{x} = \sum_{i=1}^p x_i$  and  $\mathbf{x} \in \mathbb{R}^p$ , then  $Df(\mathbf{x}) = \mathbf{1}_p^\top$
- and  $d\mathbf{y}$  is **distributed** to every element

## example: vector sum\*



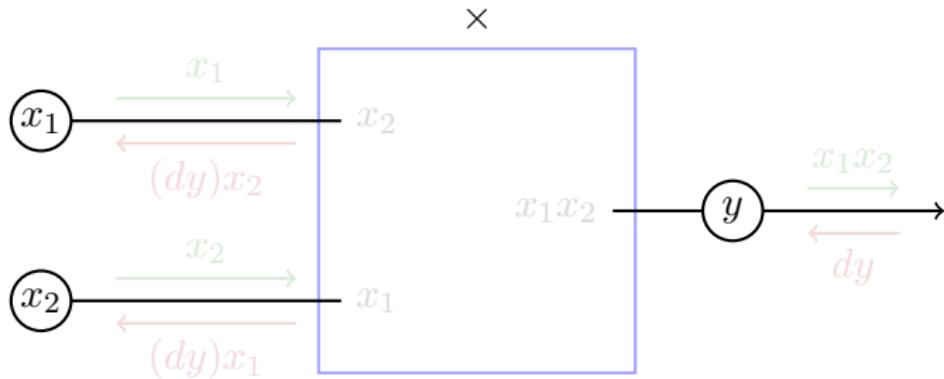
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- and  $d\mathbf{y}$  is distributed to every element

## example: vector sum\*



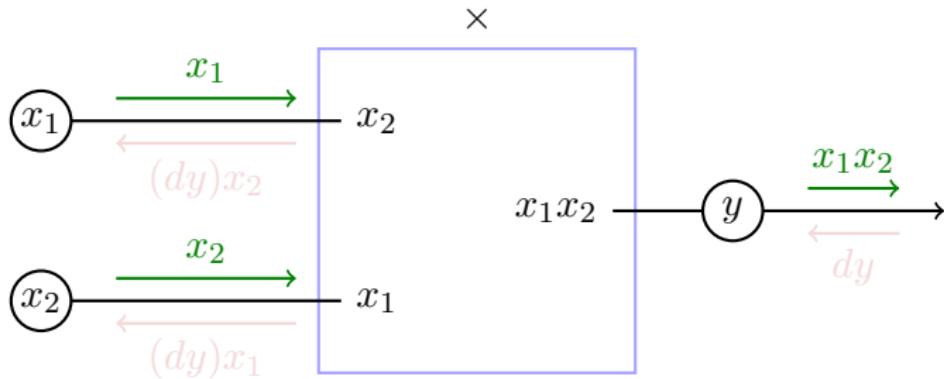
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- and  $d\mathbf{y}$  is **distributed** to every element

## example: product\*



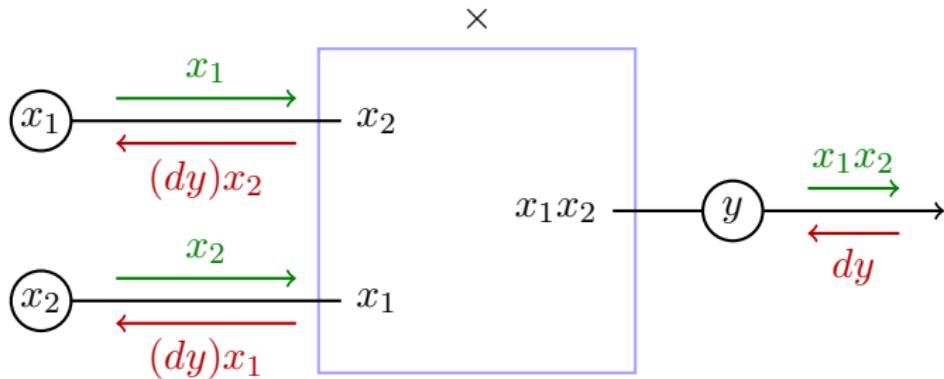
- if  $f(x_1, x_2) = x_1x_2$ , then  $D_1f(x_1, x_2) = x_2$  and  $D_2f(x_1, x_2) = x_1$
- the derivative on each branch is multiplied by the input of the other

## example: product\*



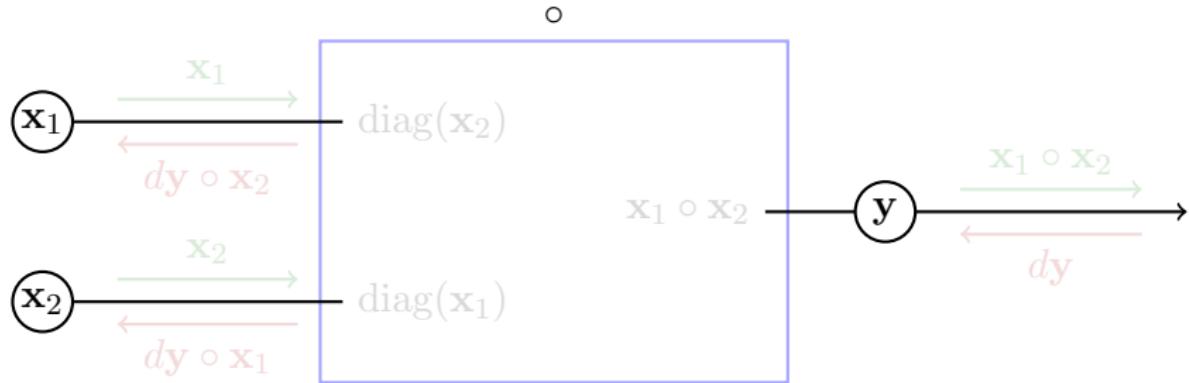
- if  $f(x_1, x_2) = x_1x_2$ , then  $D_1f(x_1, x_2) = x_2$  and  $D_2f(x_1, x_2) = x_1$
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## example: product\*



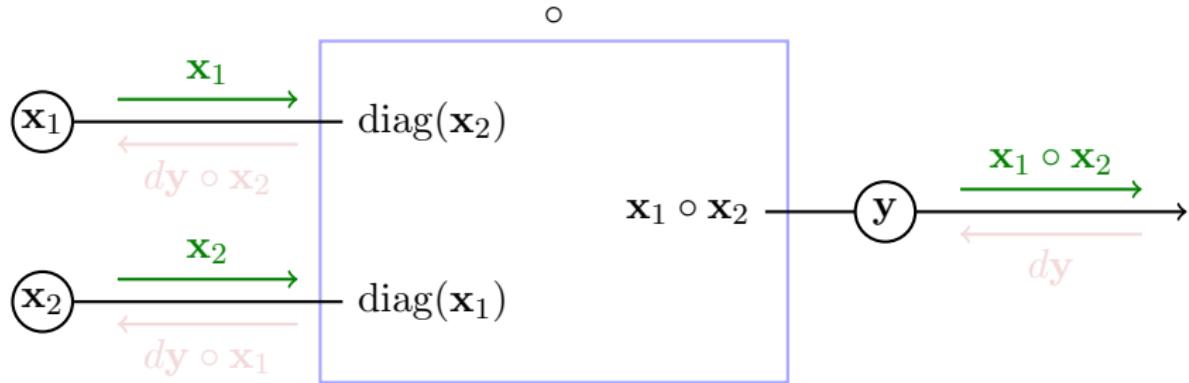
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## example: Hadamard (element-wise) product\*



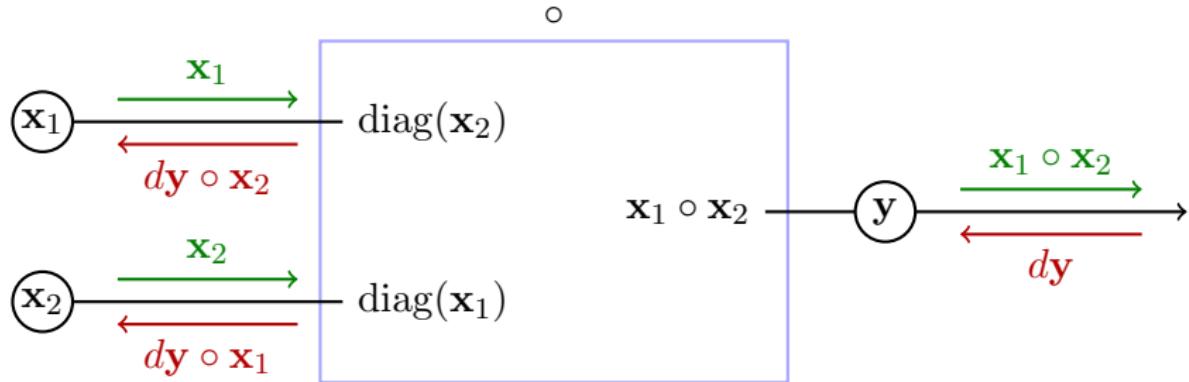
- if  $f(x_1, x_2) = x_1 \circ x_2$ , then  $D_1 f(x_1, x_2) = \text{diag}(x_2)$  and  $D_2 f(x_1, x_2) = \text{diag}(x_1)$
- the derivative on each branch is element-wise multiplied by the input of the other

## example: Hadamard (element-wise) product\*



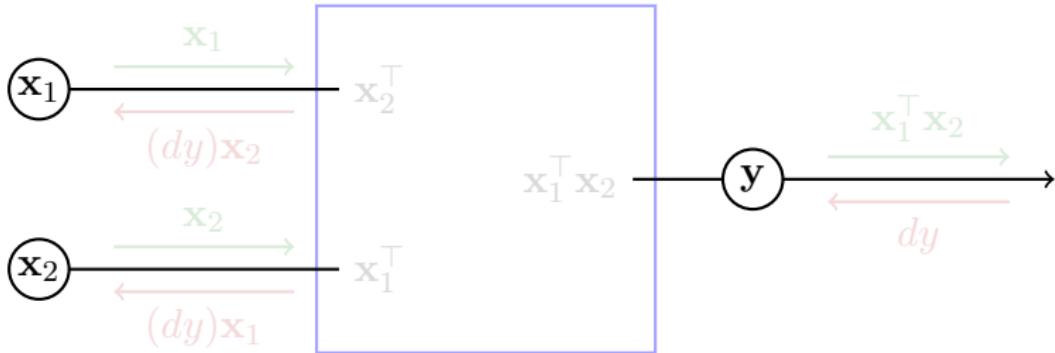
- if  $f(\mathbf{x}_1, \mathbf{x}_2) = \mathbf{x}_1 \circ \mathbf{x}_2$ , then  $D_1 f(\mathbf{x}_1, \mathbf{x}_2) = \text{diag}(\mathbf{x}_2)$  and  $D_2 f(\mathbf{x}_1, \mathbf{x}_2) = \text{diag}(\mathbf{x}_1)$
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## example: Hadamard (element-wise) product\*



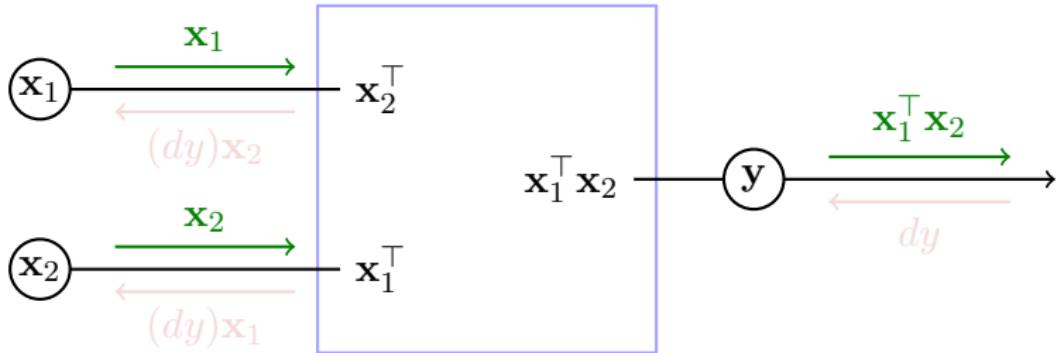
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- the derivative on each branch is element-wise multiplied by the input of the other

## example: dot product\*



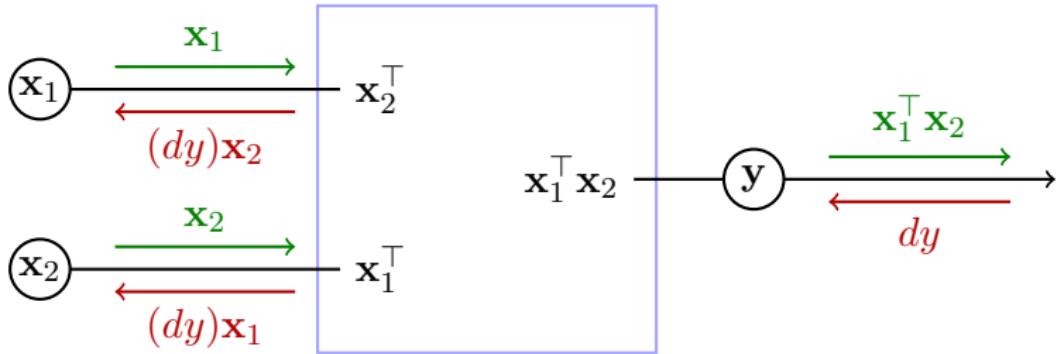
- if  $f(\mathbf{x}_1, \mathbf{x}_2) = \mathbf{x}_1 \cdot \mathbf{x}_2 = \mathbf{x}_1^\top \mathbf{x}_2$ , then  $D_1 f(\mathbf{x}_1, \mathbf{x}_2) = \mathbf{x}_2$  and  $D_2 f(\mathbf{x}_1, \mathbf{x}_2) = \mathbf{x}_1$
- the derivative on each branch is multiplied by the input of the other; this can be seen by composing an element-wise product with a vector sum

## example: dot product\*



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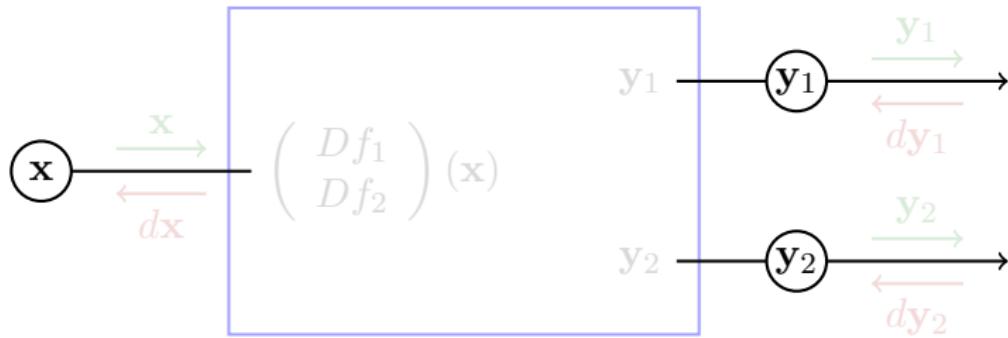
## example: dot product\*



- if  $f(\mathbf{x}_1, \mathbf{x}_2) = \mathbf{x}_1 \cdot \mathbf{x}_2 = \mathbf{x}_1^\top \mathbf{x}_2$ , then  $D_1 f(\mathbf{x}_1, \mathbf{x}_2) = \mathbf{x}_2$  and  $D_2 f(\mathbf{x}_1, \mathbf{x}_2) = \mathbf{x}_1$
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# splitting the output

$$f = (f_1, f_2)$$



- we split output  $\mathbf{y}$  into subvectors as  $\mathbf{y} = (\mathbf{y}_1, \mathbf{y}_2) = (f_1(\mathbf{x}), f_2(\mathbf{x}))$
- then, the derivative consists of blocks stacked vertically

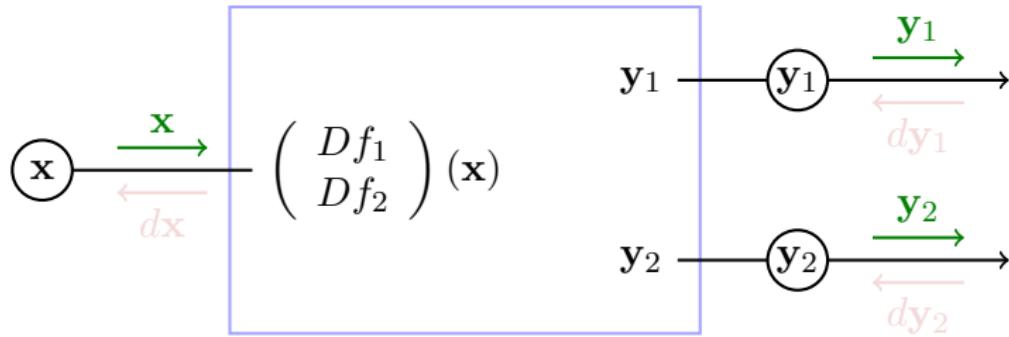
$$Df(\mathbf{x}) = (Df_1; Df_2)(\mathbf{x}) \quad \text{or} \quad \frac{\partial \mathbf{y}}{\partial \mathbf{x}} = \left( \frac{\partial \mathbf{y}_1}{\partial \mathbf{x}}; \frac{\partial \mathbf{y}_2}{\partial \mathbf{x}} \right)$$

- $d\mathbf{y}$  is also split as  $d\mathbf{y} = (d\mathbf{y}_1, d\mathbf{y}_2)$  and  $d\mathbf{x}^\top = d\mathbf{y}^\top \cdot Df(\mathbf{x})$  becomes

$$d\mathbf{x}^\top = \sum_i d\mathbf{y}_i^\top \cdot Df_i(\mathbf{x}) \quad \text{or} \quad \frac{\partial}{\partial \mathbf{x}} = \sum_i \frac{\partial}{\partial \mathbf{y}_i} \cdot \frac{\partial \mathbf{y}_i}{\partial \mathbf{x}}$$

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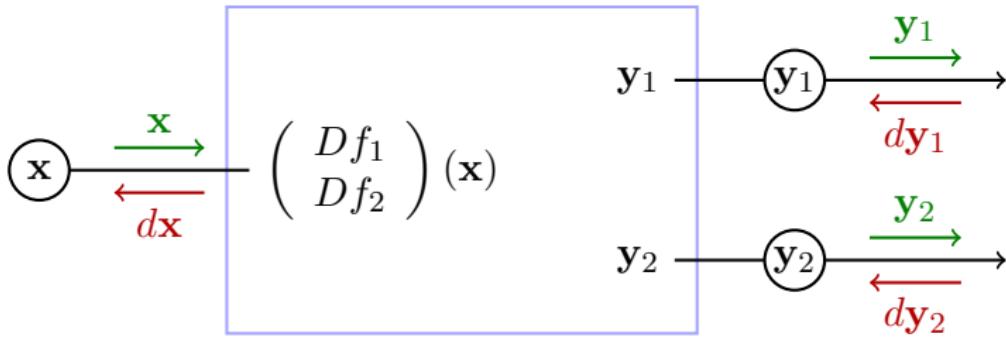
$$Df(\mathbf{x}) = (Df_1; Df_2)(\mathbf{x}) \quad \text{or} \quad \frac{\partial \mathbf{y}}{\partial \mathbf{x}} = \left( \frac{\partial \mathbf{y}_1}{\partial \mathbf{x}}; \frac{\partial \mathbf{y}_2}{\partial \mathbf{x}} \right)$$

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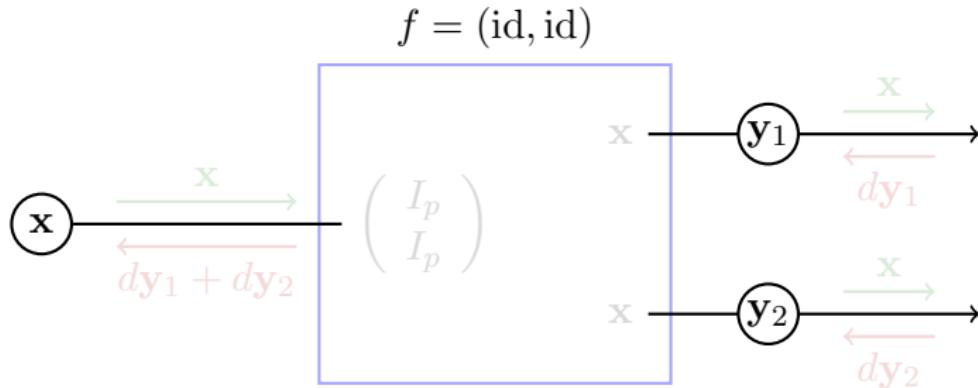
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## example: splitter (sharing)

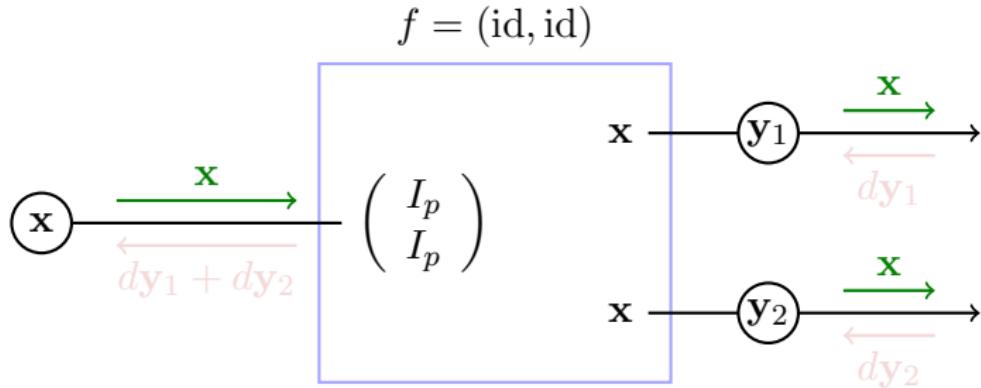


- if  $f(\mathbf{x}) = (\mathbf{x}, \mathbf{x})$  and  $\mathbf{x} \in \mathbb{R}^p$ , then  $Df(\mathbf{x}) = (I_p; I_p)$
- and the node behaves like **sum** backwards

$$d\mathbf{x} = d\mathbf{y}_1 + d\mathbf{y}_2 \quad \text{or} \quad \frac{\partial}{\partial \mathbf{x}} = \frac{\partial}{\partial \mathbf{y}_1} + \frac{\partial}{\partial \mathbf{y}_2}$$

- whenever a variable is shared (used more than once), we need to sum the gradients flowing from all paths where it appears

## example: splitter (sharing)

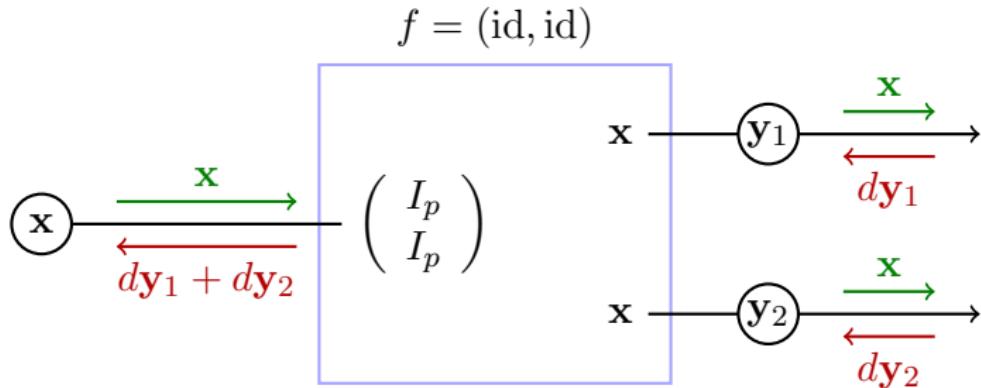


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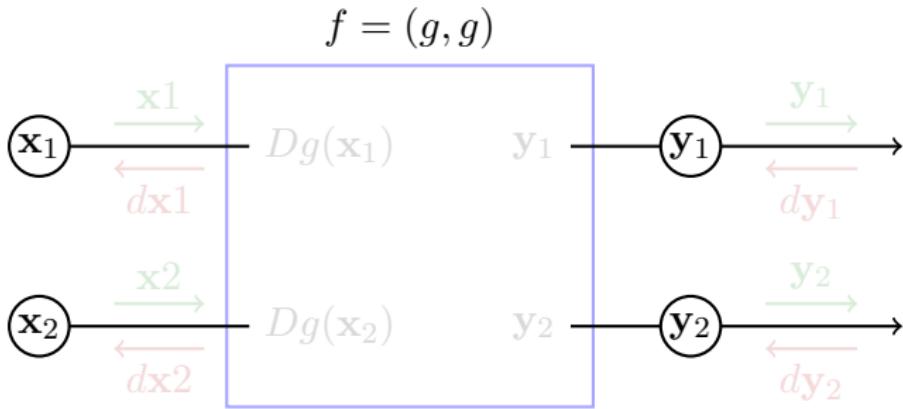


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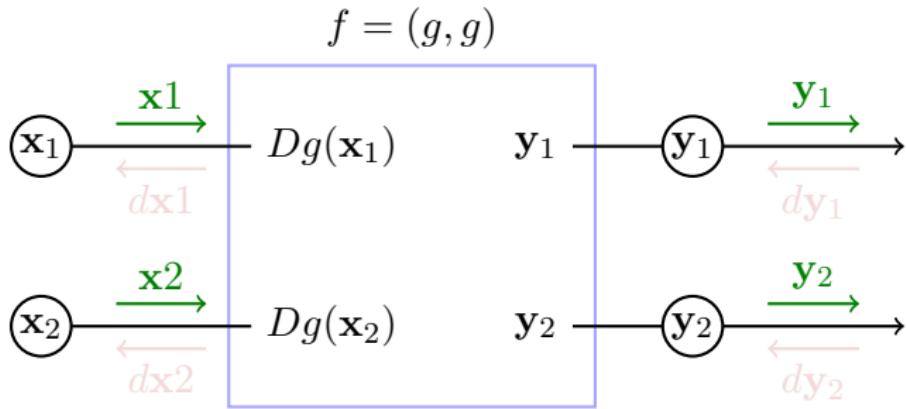
## example: tuples\*



- if  $\mathbf{x} = (\mathbf{x}_1, \mathbf{x}_2)$ ,  $\mathbf{y} = (\mathbf{y}_1, \mathbf{y}_2)$  and  $f = (g, g)$ , then  $Df(\mathbf{x})$  is block-wise diagonal:  $\text{diag}(Dg(\mathbf{x}_1), Dg(\mathbf{x}_2))$
- and the backward paths flow independently like the forward

$$d\mathbf{x}_i^\top = d\mathbf{y}_i^\top \cdot Dg(\mathbf{x}_i) \quad \text{or} \quad \frac{\partial}{\partial \mathbf{x}_i} = \frac{\partial}{\partial \mathbf{y}_i} \cdot \frac{\partial \mathbf{y}_i}{\partial \mathbf{x}_i}$$

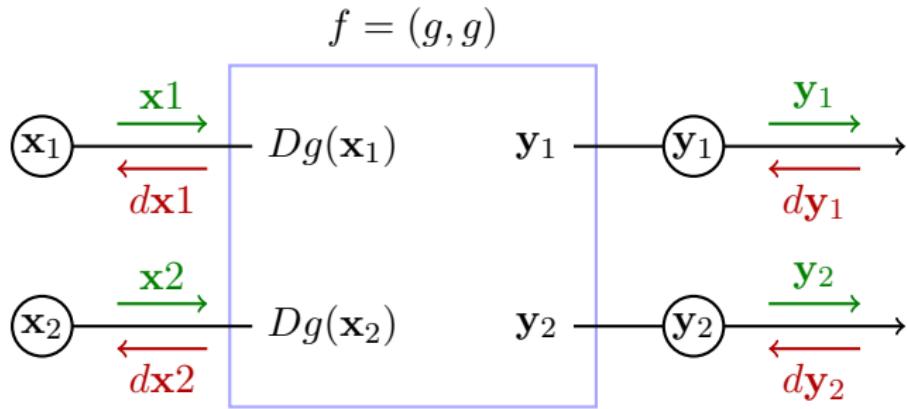
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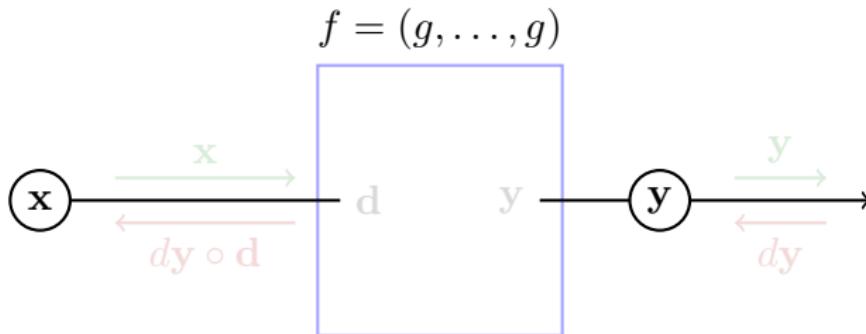
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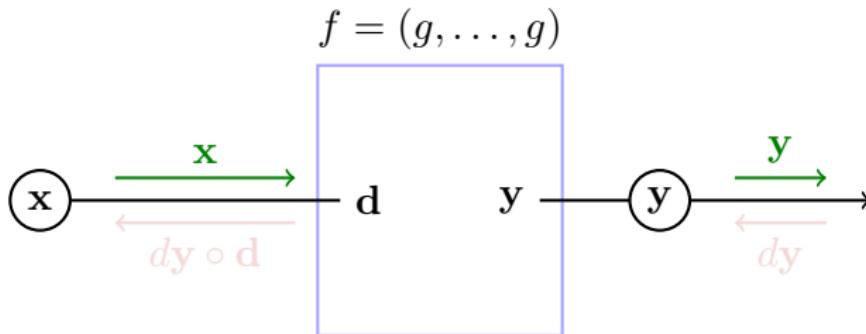
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## example: element-wise functions



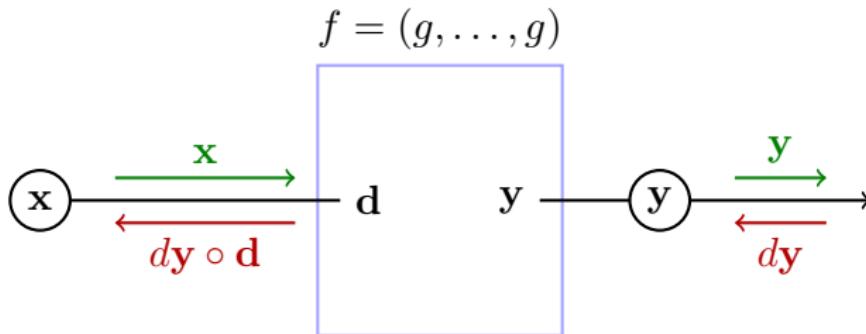
- if  $\mathbf{x} \in \mathbb{R}^p$  and  $f$  is element-wise with  $f(\mathbf{x}) = (g(x_1), \dots, g(x_p))$  where  $g : \mathbb{R} \rightarrow \mathbb{R}$ , then  $Df(\mathbf{x}) = \text{diag } \mathbf{d}$  is diagonal, where  $\mathbf{d} = (Dg(x_1), \dots, Dg(x_p))$
- and the partial derivatives are element-wise multiplied

## example: element-wise functions



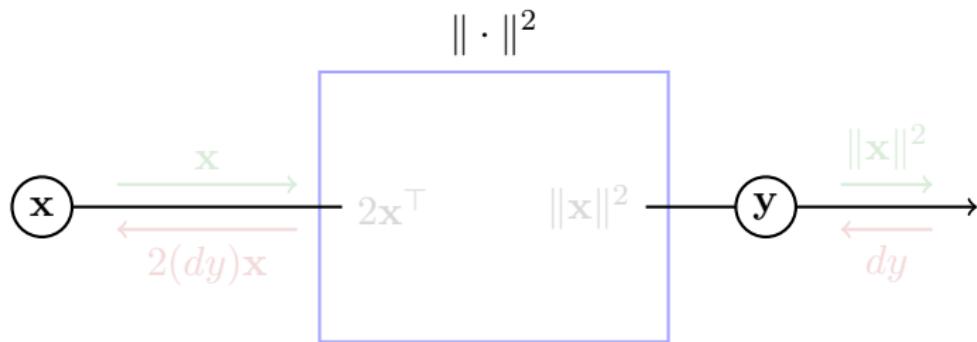
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## example: element-wise functions



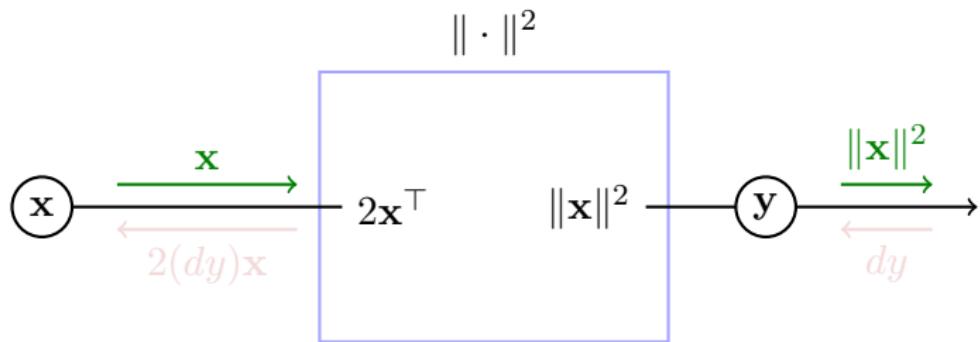
- if  $\mathbf{x} \in \mathbb{R}^p$  and  $f$  is element-wise with  $f(\mathbf{x}) = (g(x_1), \dots, g(x_p))$  where  $g : \mathbb{R} \rightarrow \mathbb{R}$ , then  $Df(\mathbf{x}) = \text{diag } \mathbf{d}$  is diagonal, where  $\mathbf{d} = (Dg(x_1), \dots, Dg(x_p))$
- and the partial derivatives are element-wise multiplied

## example: squared norm\*



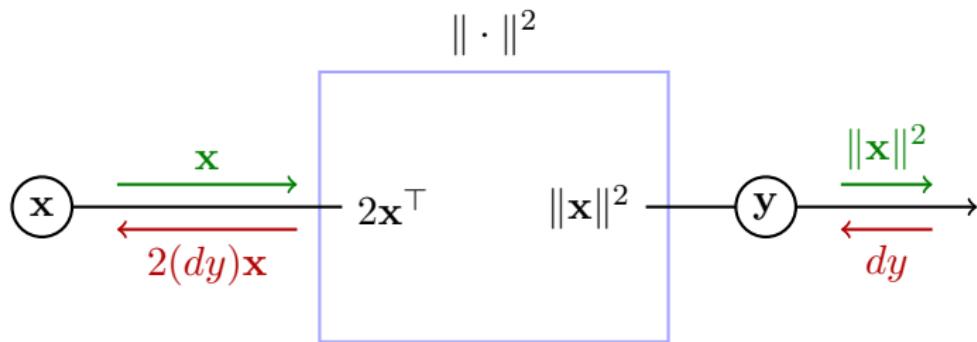
- if  $f(x) = \|x\|^2$  then  $Df(x) = 2x^\top$
- and  $dy$  is multiplied by  $2x^\top$ ; this can be seen by composing a splitter (factor 2) with a dot product (factor  $x^\top$ )

## example: squared norm\*



- if  $f(\mathbf{x}) = \|\mathbf{x}\|^2$  then  $Df(\mathbf{x}) = 2\mathbf{x}^\top$
- and  $d\mathbf{y}$  is multiplied by  $2\mathbf{x}^\top$ ; this can be seen by composing a splitter (factor 2) with a dot product (factor  $\mathbf{x}^\top$ )

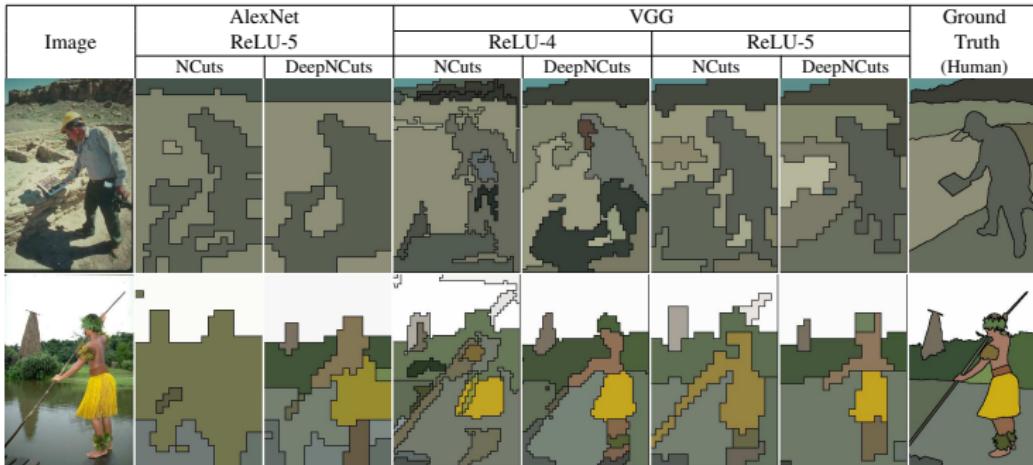
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# matrix derivatives\*

[Ionescu et al. 2015]



- derivatives for
  - SVD decomposition  $A = U\Sigma V^\top$
  - eigenvalue decomposition  $A = U\Sigma U^\top$
  - nonlinear matrix functions  $f(A) = Uf(\Sigma)U^\top$
- application to spectral methods for image segmentation

# matrix calculus\*

- results like these, and many more

$$\frac{\partial A\mathbf{x}}{\partial \mathbf{x}} = A$$

$$\frac{\partial \mathbf{x}^\top A\mathbf{x}}{\partial \mathbf{x}} = \mathbf{x}^\top (A + A^\top)$$

$$\frac{\partial \text{vec}(\mathbf{x}^\top A\mathbf{x})}{\partial \text{vec } A} = \mathbf{x}^\top \otimes \mathbf{x}^\top$$

$$\frac{\partial AXB}{\partial X} = B^\top \otimes A$$

$$\frac{dA^{-1}}{dA} = -(A^{-\top} \otimes A^{-1})$$

$$\frac{d \ln |A|}{dA} = \text{vec}(A^{-\top})^\top$$

$$\frac{\partial \text{tr}(AX)}{\partial X} = \text{vec}(A^\top)^\top$$

## in general

- apparently, we do not need to store the Jacobian matrix  $Df(\mathbf{x})$ , which may be huge, but only what is needed to compute the partial derivatives in the backward pass
- our function can be decomposed into a **directed acyclic graph** (DAG) of nodes, called a **computational graph**
- each time we call the function in the forward pass, a new graph may be constructed if our program contains control flow statements like conditionals and loops; methods supporting this operation are called **dynamic**

# automatic differentiation

[Wengert 1964]

- is the more general set of methods used to automatically evaluate the derivative of a given function at a given input; it is **not numerical** and **not symbolic**
- what we call back-propagation here is known as the **reverse accumulation** mode in this context and makes sense because we compute the gradient of a single scalar quantity with respect to maybe millions of parameters
- **forward accumulation** makes sense when we need the derivative of many variables with respect to few parameters
- we will use the term **automatic differentiation** to refer to the process of generating a computer program for the derivatives given the program for the original function and the input variables

## aside: higher-order derivatives\*

- the Hessian was assumed fixed and isotropic in gradient descent; if we knew it, we could use the **Newton method** instead and solve all curvature-related problems
- given  $f : \mathbb{R}^p \rightarrow \mathbb{R}$ , its **Hessian matrix** at  $\mathbf{x}$  is

$$Hf(\mathbf{x}) := \begin{pmatrix} \frac{\partial^2 f}{\partial x_1^2} & \cdots & \frac{\partial^2 f}{\partial x_1 \partial x_p} \\ \vdots & \ddots & \vdots \\ \frac{\partial^2 f}{\partial x_p \partial x_1} & \cdots & \frac{\partial^2 f}{\partial x_p^2} \end{pmatrix}(\mathbf{x}) = \nabla(Df)(\mathbf{x})$$

- unfortunately, this is a  $p \times p$  matrix and with  $p$  in the order of millions, it is impractical even to store it, let alone compute it

## aside: multiplication by Hessian\*

[Pearlmutter 1994]

- fortunately, in many cases what we need is only the **product** of the Hessian with a given vector  $\mathbf{v}$ , which is just a vector in  $\mathbb{R}^p$

$$\mathbf{v}^\top \cdot Hf(\mathbf{x}) = \mathbf{v}^\top \cdot \nabla(Df)(\mathbf{x}) = \nabla_{\mathbf{v}}(Df)(\mathbf{x})$$

- here  $\nabla_{\mathbf{v}}$  is the **directional derivative** operator

$$\nabla_{\mathbf{v}}(f) := \mathbf{v}^\top \cdot \nabla f$$

- remember that in back-propagation, for each variable  $\mathbf{x}$ , we defined a vector  $d\mathbf{x}$ , which was computed in the backward pass
- so all we need to do is allocate another vector  $\nabla_{\mathbf{v}}(\mathbf{x})$  for the forward pass and another  $\nabla_{\mathbf{v}}(d\mathbf{x})$  for the backward, and compute them by applying the chain rule in both passes!

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# automatic differentiation: units

# automatic differentiation

## forward

- evaluation is carried out by **units**, one calling another
- when invoked, each unit generates a **node** object
- each node holds the **gradient** with respect to its unit's inputs, including parameters
- it also holds any information needed for the backward pass

## backward

- all gradients are set to **zero**, except for the gradient with respect to the scalar quantity that is to be optimized (the error), which is set to **one**
- the **back()** method is invoked on the node of this quantity
- this, in turn, triggers the same method on all units that have participated in the forward pass

# automatic differentiation

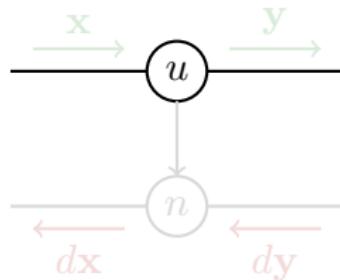
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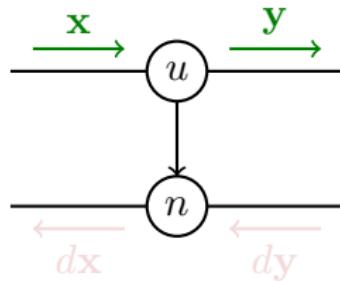
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# units and nodes



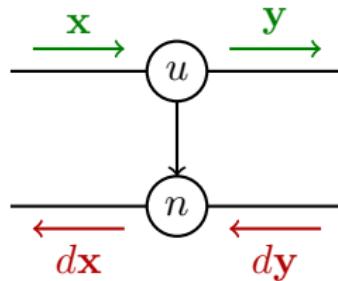
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# units and nodes



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- given a function  $f$  with derivative  $Df$ , a *unit* is a function of the form

```
def forward(x1, ..., xn):  
    y = f(x1, ..., xn)  
    def back(dy, dx1, ..., dxn):  
        dx1T += dyT · D1f(x1, ..., xn)  
        ...  
        dxnT += dyT · Dnf(x1, ..., xn)  
    return node(y, back)
```

- a *node* object:
  - holds  $y$  and an associated derivative  $dy$  of the same shape
  - exposes a method  $back(x_1, \dots, x_n)$  where  $x_i$  can be *nodes*
  - automatically adds its own  $dy$  as first argument
  - if an input  $x_i$  is a *node*, extracts the derivative part  $dx_i$
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# the affine unit

- input vectors are represented as rows of  $m \times p$  **input matrix**  $X$  where  $m$  is the **mini-batch size** and  $p$  the input dimension
- parameters are represented by  $p \times q$  **weight matrix**  $W$  and  $1 \times q$  **bias vector**  $\mathbf{b}$  where  $q$  is the output dimension
- the unit is defined as

```
def affine(X, (W, b)):  
    A = dot(X, W) + b  
def back(dA, dX, (dW, db)):  
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```

# the affine unit in math\*

## forward

- input  $X \in \mathbb{R}^{m \times p}$ ,  $W \in \mathbb{R}^{p \times q}$ ,  $\mathbf{b} \in \mathbb{R}^q$ , output  $A \in \mathbb{R}^{m \times q}$

$$A = f(X; W, \mathbf{b}) := XW + \mathbf{1}_m \mathbf{b}^\top$$

observe that in the code, addition of  $\mathbf{b}$  is handled by **broadcasting**

## backward

- if  $\mathbf{a}_i$ ,  $\mathbf{w}_i$  is the  $i$ -th column of  $A$ ,  $W$ ,

$$\frac{\partial \mathbf{a}_i}{\partial \mathbf{w}_i} = \frac{\partial (X\mathbf{w}_i)}{\partial \mathbf{w}_i} = X$$

and there are no other dependencies, so by the chain rule

$$d\mathbf{w}_i^\top := \frac{\partial}{\partial \mathbf{w}_i} = \frac{\partial}{\partial \mathbf{a}_i} \cdot \frac{\partial \mathbf{a}_i}{\partial \mathbf{w}_i} = d\mathbf{a}_i^\top \cdot X$$

- finally, the partial derivative with respect to  $W$

$$dW = (dA^\top X)^\top = X^\top dA$$

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$$dX = (dA)W^\top$$

- again, by replacing  $X$  and  $W$  by  $\mathbf{1}_m$  and  $\mathbf{b}^\top$  respectively in the previous result for  $dW$ ,

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# the logistic unit

- the input is an  $m \times q$  activation matrix  $A$  and the  $m \times k$  one-of- $k$  encoded target matrix, where  $k$  is the number of classes
- there are no parameters
- the unit integrates softmax with average cross-entropy loss

```
def logistic(A, T):
    E = exp(A)
    Y = E/sum_1(E)
    C = -sum_1(T * log(Y))
    D = sum_0(C)/m
    def back(dD, dA, _):
        dA += dD * (Y - T)/m
    return node(D, back)
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# the logistic unit in math\*

## forward

- $E$  is given element-wise as  $e_{ij} = \exp(a_{ij})$ , and  $m \times q$  matrix  $Y$  is row-normalized as

$$Y = (\text{diag}(E\mathbf{1}_k))^{-1}E$$

- the  $i$ -th row of  $Y$  is the softmax output of the  $i$ -th input sample representing the  $k$  posterior class probabilities
- $C$  is actually a  $m \times 1$  column vector and its  $i$ -th element represents the cross-entropy loss of the  $i$ -th input sample

$$c_i = - \sum_{j=1}^k t_{ij} \log(y_{ij})$$

- finally,  $D = \frac{1}{m} \sum_{i=1}^m c_i$  is a scalar and represents the average cross-entropy (data) error over the mini-batch

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# the logistic unit in math\*

## backward

- if  $\mathbf{a}_i^\top, \mathbf{y}_i^\top, \mathbf{t}_i^\top$  is the  $i$ -th row of  $A, Y, T$ , the derivative of the cross-entropy loss is, according to what we have seen,

$$\frac{\partial c_i}{\partial \mathbf{a}_i}(\mathbf{a}_i, \mathbf{t}_i) = (\sigma(\mathbf{a}_i) - \mathbf{t}_i)^\top = (\mathbf{y}_i - \mathbf{t}_i)^\top$$

- since  $D$  is the **average** of the individual sample losses  $c_i$ , the derivative of the total error, which is 1 by default, is **distributed** over the samples with a factor of  $\frac{1}{m}$

$$dA^\top = \frac{1}{m}(Y - T) \cdot dD$$

# why integrate softmax with cross-entropy?

- the simplified formula is faster compared to blind application of back-propagation at the level of elementary functions
- if this is not convincing, try evaluating the binary cross-entropy loss

$$\ell(x) := \ln(1 + e^{-x})$$

- $\ell(-1) = 1.3133$
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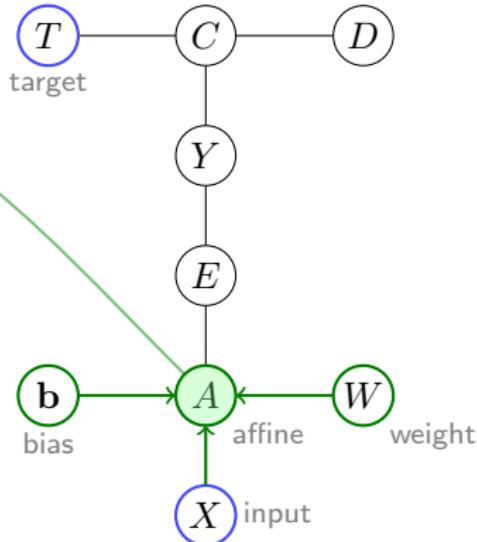
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# back-propagation

forward

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# back-propagation

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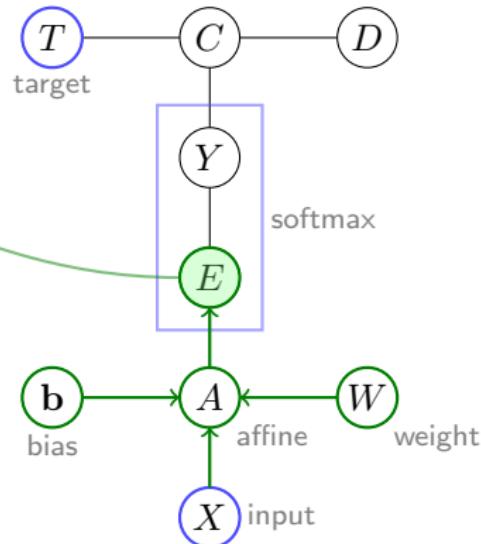
$$A = \text{dot}(X, W) + b$$

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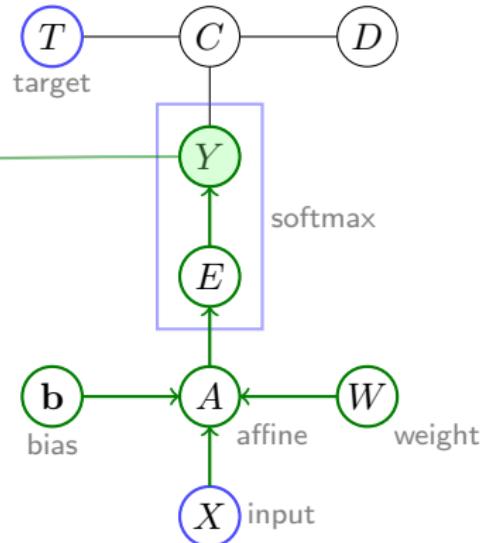
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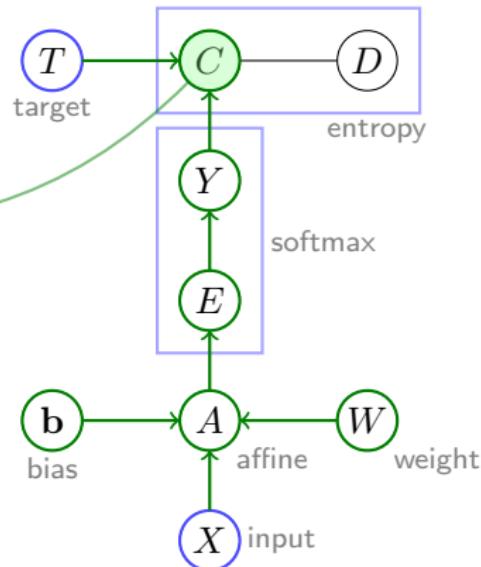
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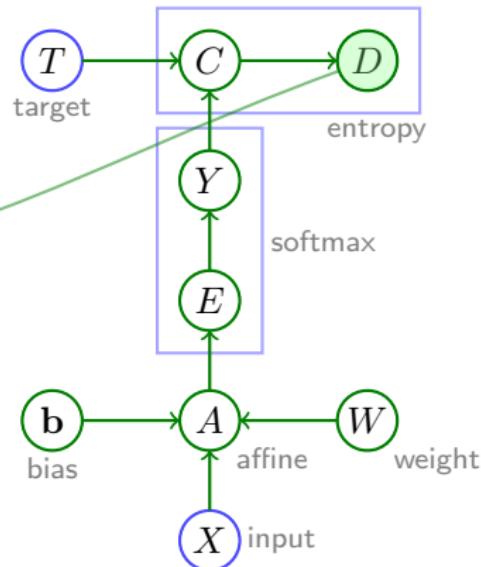
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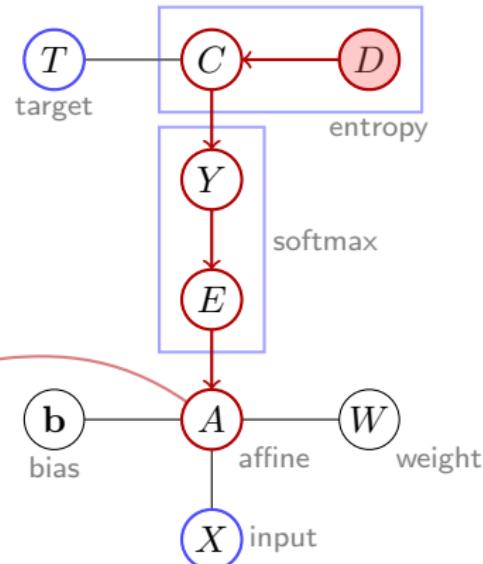
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## backward

$$dA = dD * (Y - T)/m$$



# back-propagation

## forward

$$A = \text{dot}(X, W) + b$$

$$E = \exp(A)$$

$$Y = E / \text{sum}_1(E)$$

$$C = -\text{sum}_1(T * \log(Y))$$

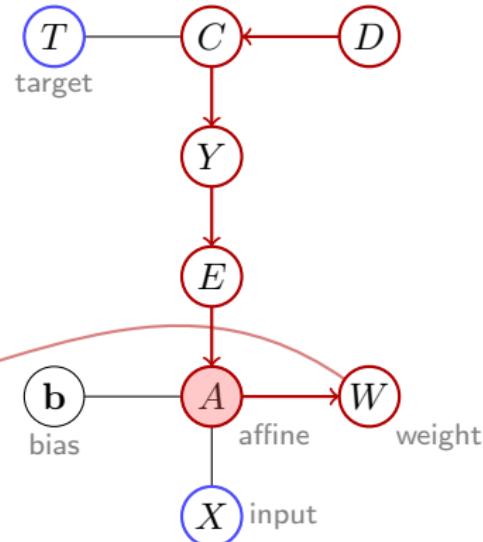
$$D = \text{sum}_0(C)/m$$

## backward

$$dA = dD * (Y - T)/m$$

$$dW += \text{dot}(X^\top, dA)$$

$$db = \text{sum}_0(dA)$$



# back-propagation

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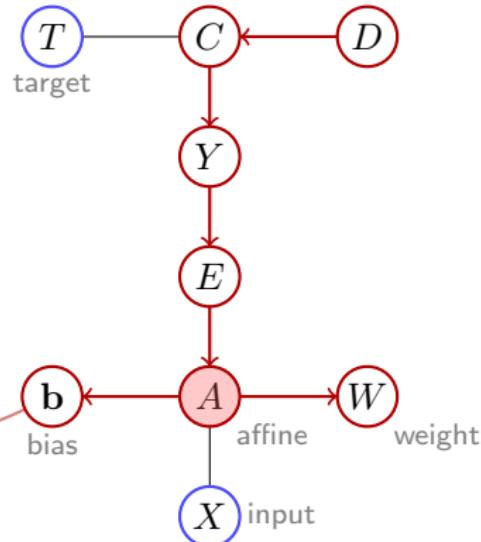
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# automatic differentiation

## forward

$$A = \text{dot}(X, W) + \mathbf{b}$$

$$E = \exp(A)$$

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$$D = \text{sum}_0(C)/m$$

## backward

$$dA = dD * (Y - T)/m$$

$$dW += \text{dot}(X^\top, dA)$$

$$d\mathbf{b} = \text{sum}_0(dA)$$

now we organize **forward** and **backward** code into units

# automatic differentiation

## forward

```
A = dot(X, W) + b  
E = exp(A)  
Y = E/sum1(E)  
C = -sum1(T * log(Y))  
D = sum0(C)/m
```

## backward

```
dA = dD * (Y - T)/m  
dW += dot(X⊤, dA)  
db = sum0(dA)
```

```
def affine(X, (W, b)):  
    A = dot(X, W) + b  
    def back(dA, dX, (dW, db)):  
        dW += dot(X⊤, dA)  
        db += sum0(dA)  
        dX += dot(dA, W⊤)  
    return node(A, back)
```

# automatic differentiation

## forward

```
A = affine(X, (W, b))  
E = exp(A)  
Y = E / sum1(E)  
C = -sum1(T * log(Y))  
D = sum0(C)/m
```

## backward

```
dA = dD * (Y - T)/m  
A.back(X, (W, b))
```

```
def affine(X, (W, b)):  
    A = dot(X, W) + b  
  
def back(dA, dX, (dW, db)):  
    dW += dot(X⊤, dA)  
    db += sum0(dA)  
    dX += dot(dA, W⊤)  
  
return node(A, back)
```

# automatic differentiation

## forward

```
A = affine(X, (W, b))
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```
E = exp(A)
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```
Y = E/sum1(E)
```

```
C = -sum1(T * log(Y))
```

```
D = sum0(C)/m
```

## backward

```
dA = dD * (Y - T)/m
```

```
A.back(X, (W, b))
```

```
def logistic(A, T):
```

```
E = exp(A)
```

```
Y = E/sum1(E)
```

```
C = -sum1(T * log(Y))
```

```
D = sum0(C)/m
```

```
def back(dD, dA, _):
```

```
dA += dD * (Y - T)/m
```

```
return node(D, back)
```

# automatic differentiation

## forward

```
A = affine(X, (W, b))
```

```
D = entropy(A, T)
```

```
def logistic(A, T):  
    E = exp(A)  
    Y = E/sum1(E)  
    C = -sum1(T * log(Y))  
    D = sum0(C)/m
```

## backward

```
D.back(A, T)  
A.back(X, (W, b))
```

```
def back(dD, dA, _):  
    dA += dD * (Y - T)/m  
return node(D, back)
```

# automatic differentiation: functions

# the relu unit\*

- relu is an element-wise activation function; its input is **activation matrix**  $A$  and returns matrix  $Z$  of the same size
- its backward pass behaves like a **switch**

```
def relu(A):  
    Z = max(0, A)  
    def back(dZ, dA):  
        dA += dZ * (Z > 0)  
    return node(Z, back)
```

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def relu(A):
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```

## the decay unit\*

- it takes as input a tuple or list  $W$  of weight matrices of any size and returns the weight decay error term  $\frac{\lambda}{2}\|w\|^2$  for each  $w \in W$ , where  $\|\cdot\|_F$  is the Frobenius norm
- the backward derivative is proportional to  $w$ , as for the  $\ell_2$  norm

```
def decay(W):
    R =  $\frac{\lambda}{2} * \text{sum}(\|w\|_F^2 \text{ for } w \text{ in } W)$ 
    def back(dR, dW):
        for (w, dw) in zip(W, dW):
            dw += dR *  $\lambda * w$ 
    return node(R, back)
```

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        for (w, dw) in zip(W, dW):
            dw += dR *  $\lambda * w$ 
    return node(R, back)
```

## the add unit\*

- it takes as input a tuple or list  $X$  of matrices (or vectors, or scalars) of the same size and returns their sum
- its backward pass **distributes** the derivative to all input branches

```
def add(X):
    S = sum(X)
    def back(dS, dX):
        for dx in dX:
            dx += dS
        return node(S, back)
```

- operator  $+$  is overloaded for *nodes* such that  $A + B$  means  $\text{add}((A, B))$

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- operator `+` is overloaded for *nodes* such that  $A + B$  means `add((A, B))`

# the loss function

- it takes as input the activation matrix  $A$ , the target matrix  $T$  and the weight matrix list  $W$
- it calls the logistic unit on  $(A, T)$  and the decay unit on  $W$ , and returns the sum of the two scalar terms

```
def loss(A, T, W):
    L = logistic(A, T) + decay(W)
    return block(L)
```

- addition is handled by add and the error derivative flows backward to both branches

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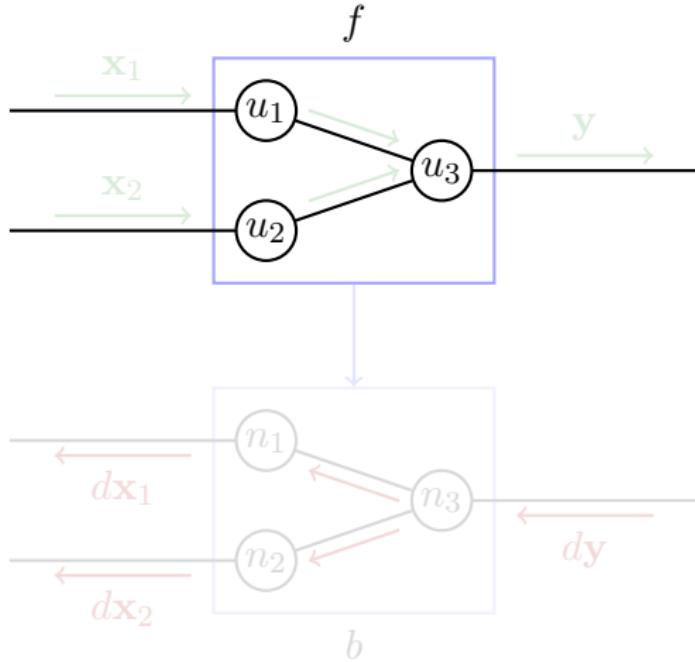
- addition is handled by add and the error derivative flows backward to both branches

## the model function

- this is a two-layer network model where an affine layer is followed by a relu activation function and another affine layer
- the parameter tuple  $U_i = (W_i, \mathbf{b}_i)$  for layer  $i$  contains a weight matrix  $W_i$  and a bias vector  $\mathbf{b}_i$
- unit calls are nested like every other function

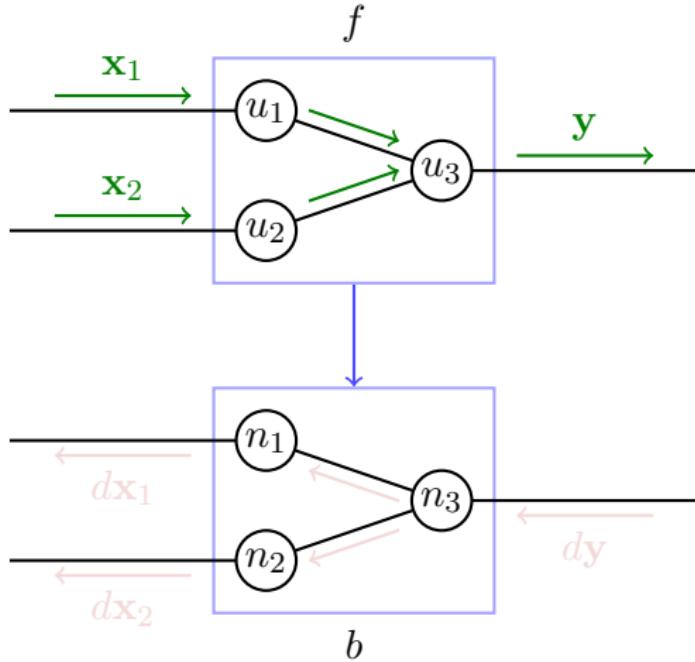
```
def model(X, (U1, U2)):  
    A = affine(relu(affine(X, U1)), U2)  
    return block(A)
```

# functions and blocks



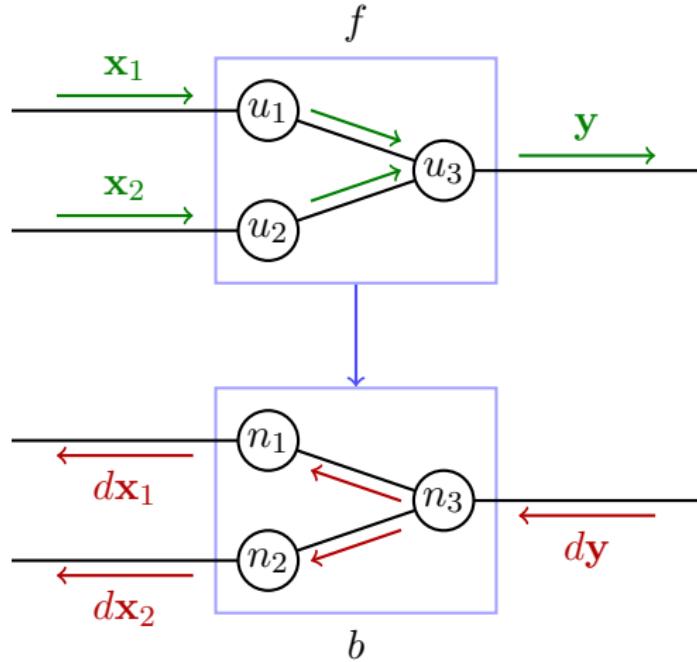
- function  $f$  containing units  $u_1, u_2, u_3$
- $f$  dynamically generates block  $b$  containing nodes  $n_1, n_2, n_3$ , manually generated by  $u_1, u_2, u_3$  respectively

# functions and blocks



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# functions and blocks

- a *function* is a function of the following form, where code is arbitrary but computation takes place through calls to *units* or *functions*

```
def name(x1, ..., xn):  
    <code generating the following>  
    r1 = call1(a1, ..., an1)  
    :  
    rs = calls(a1, ..., ans)  
    return block(rs)
```

- all calls are recorded as a list of *units* or *functions* by *call order*, each associated with a list of arguments
- a *block* object is a *node*, but
  - its method `back()` does not add its own derivative in the arguments
  - its method `back()` is *automatically generated* and its body calls the recorded functions with the same arguments in *reverse order*

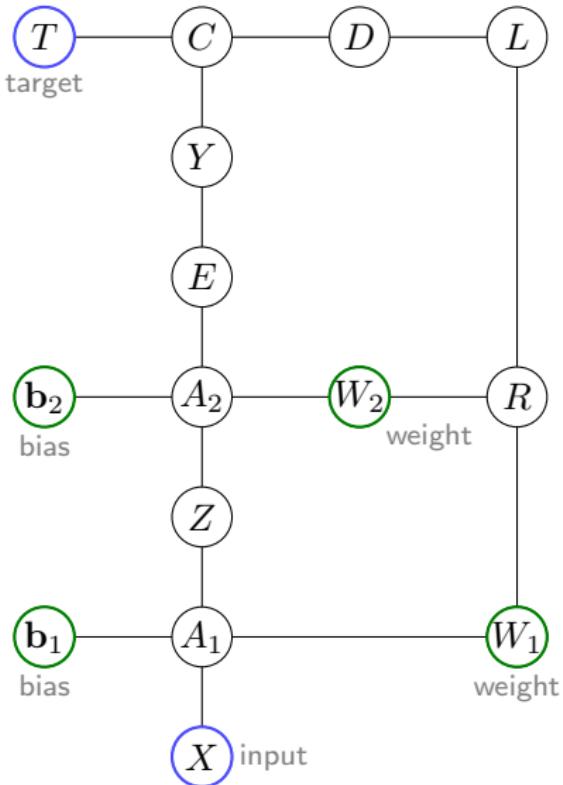
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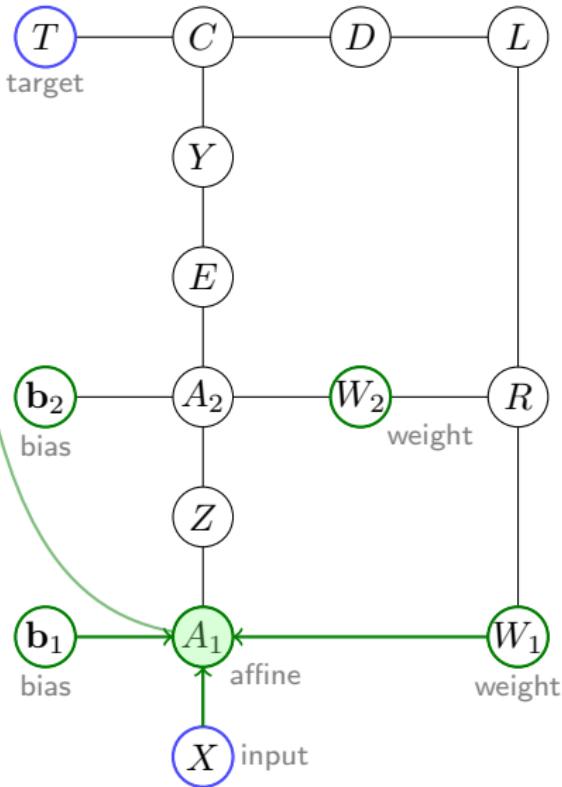
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# back-propagation



# back-propagation

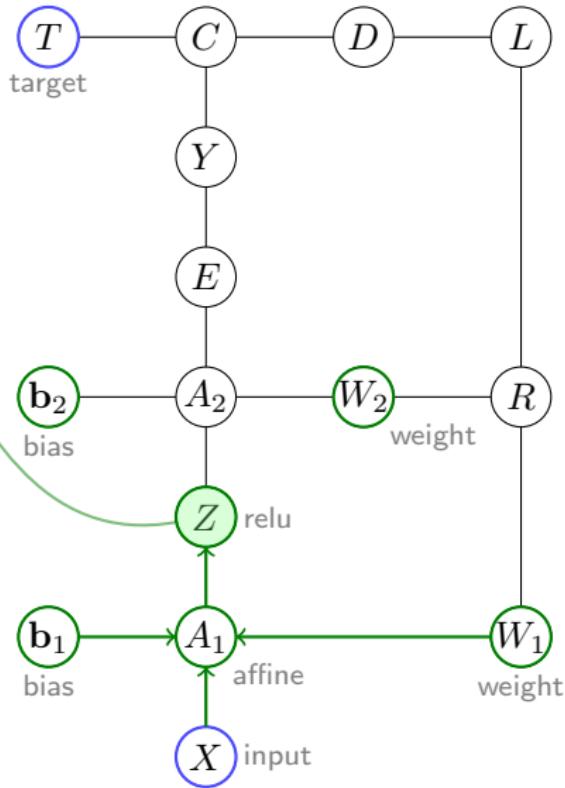
$$A_1 = \text{dot}(X, W_1) + \mathbf{b}_1$$



# back-propagation

$$A_1 = \text{dot}(X, W_1) + b_1$$

$$Z = \max(0, A_1)$$

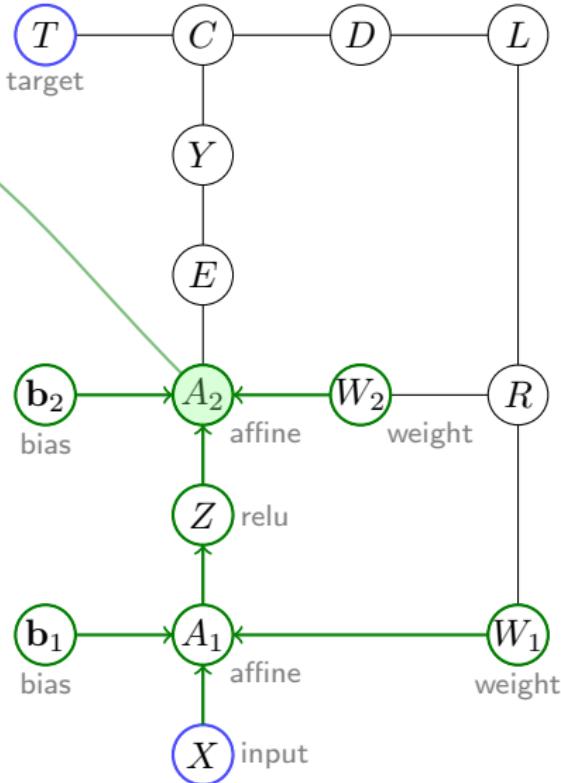


# back-propagation

$$A_1 = \text{dot}(X, W_1) + b_1$$

$$Z = \max(0, A_1)$$

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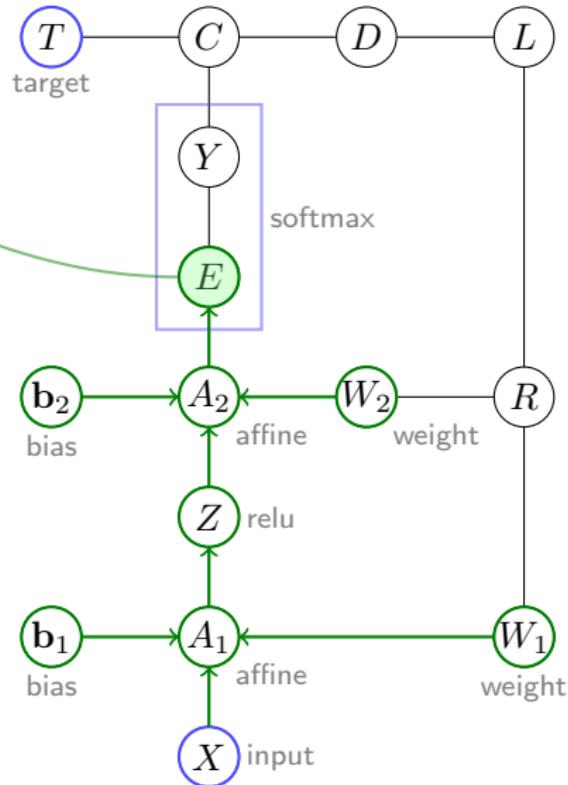
$$A_2 = \text{dot}(Z, W_2) + b_2$$

$$E = \exp(A_2)$$

$$Y = E / \text{sum}_1(E)$$

$$C = -\text{sum}_1(T * \log(Y))$$

$$D = \text{sum}_0(C)/m$$



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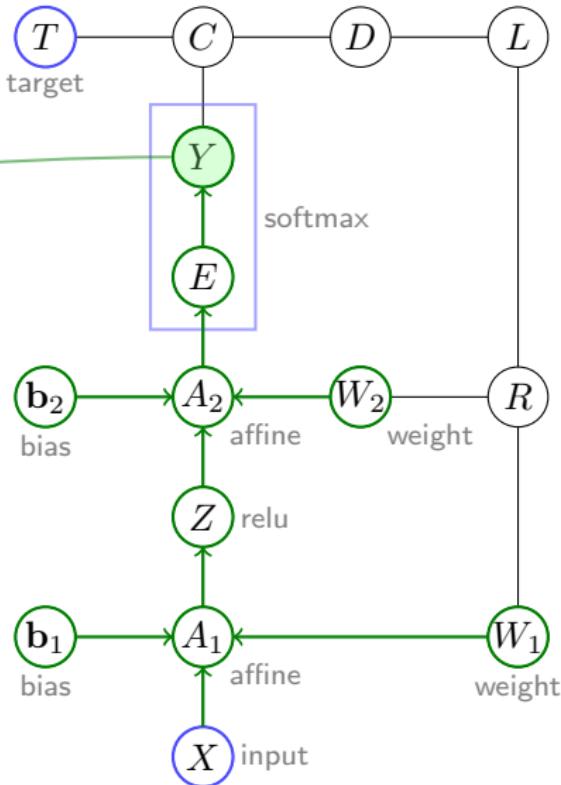
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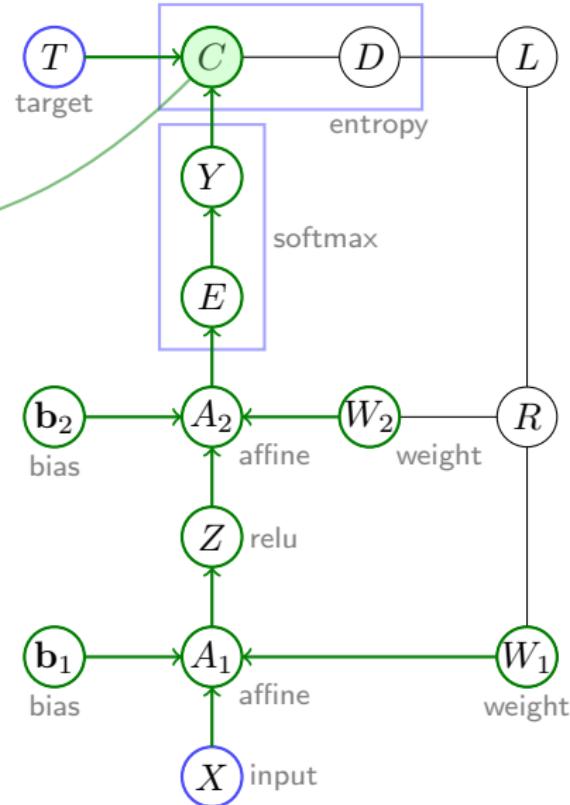
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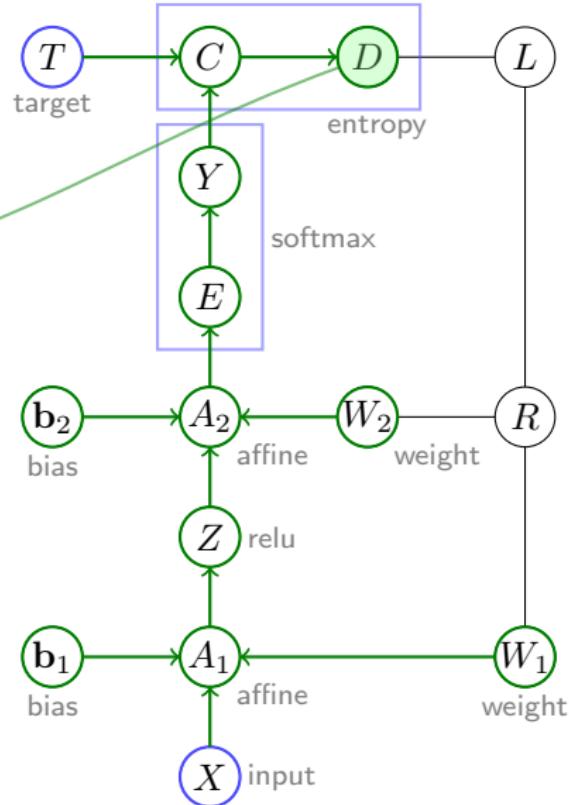
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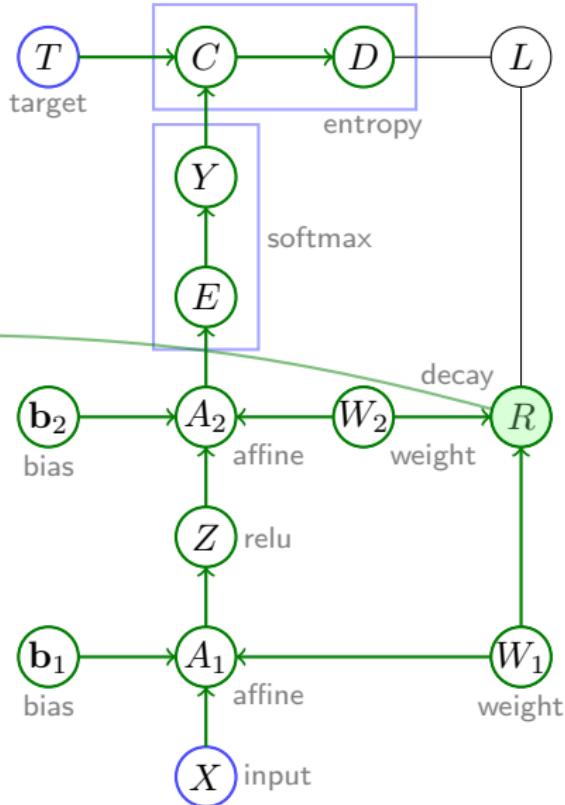
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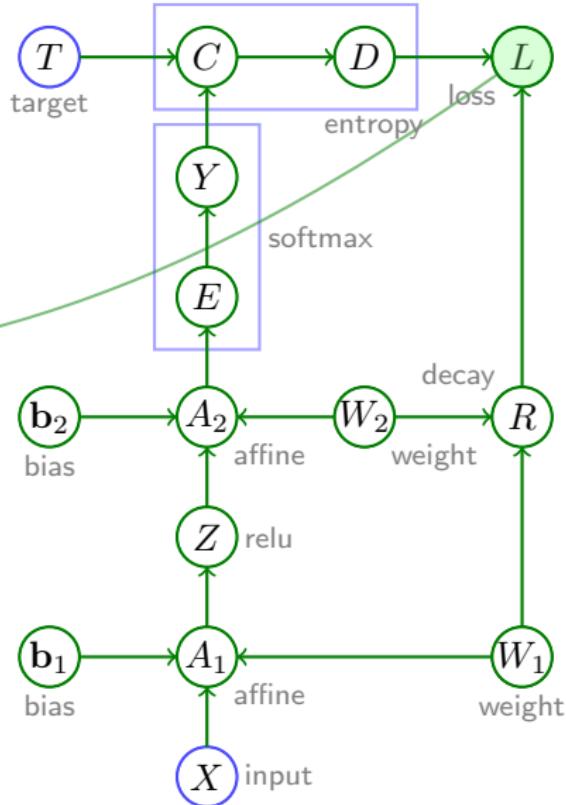
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$$L = D + R$$



# back-propagation

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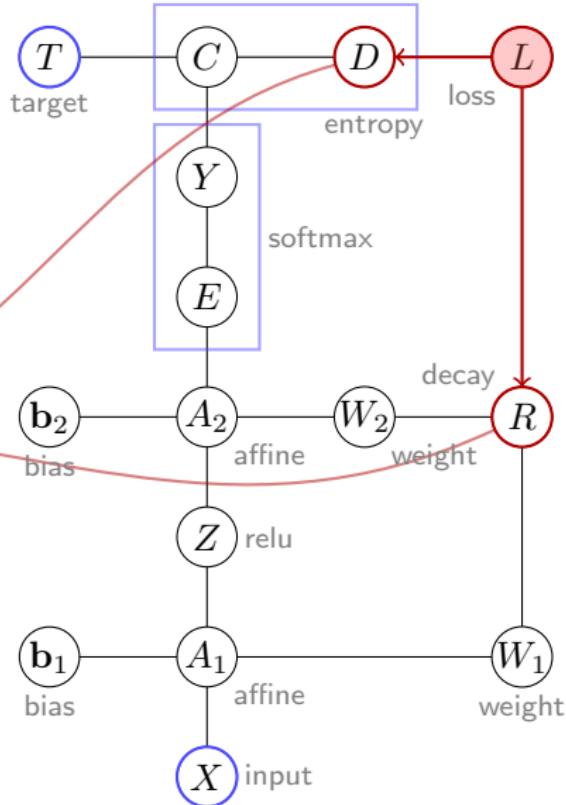
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$$L = D + R$$

$$(dD, dR) = (dL, dL)$$



# back-propagation

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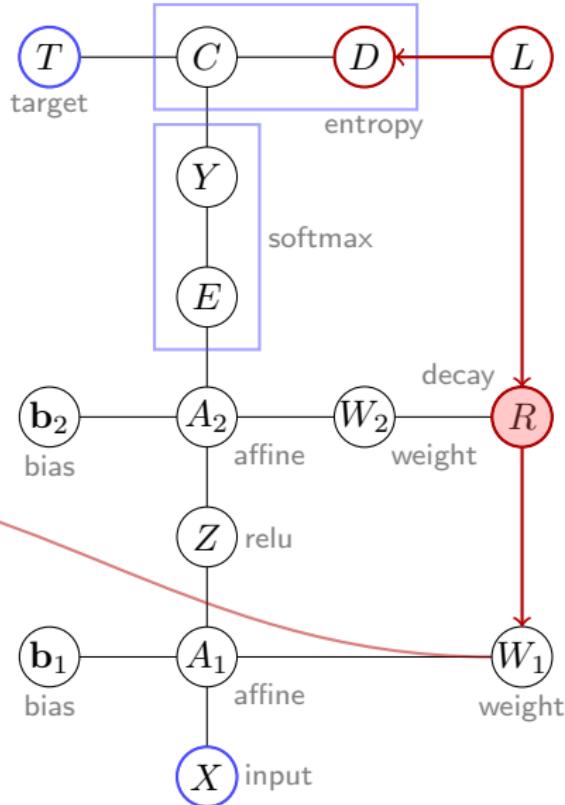
$$R = \frac{\lambda}{2} * (\|W_1\|_F^2 + \|W_2\|_F^2)$$

$$L = D + R$$

$$(dD, dR) = (dL, dL)$$

$$dW_1 = dR * \lambda * W_1$$

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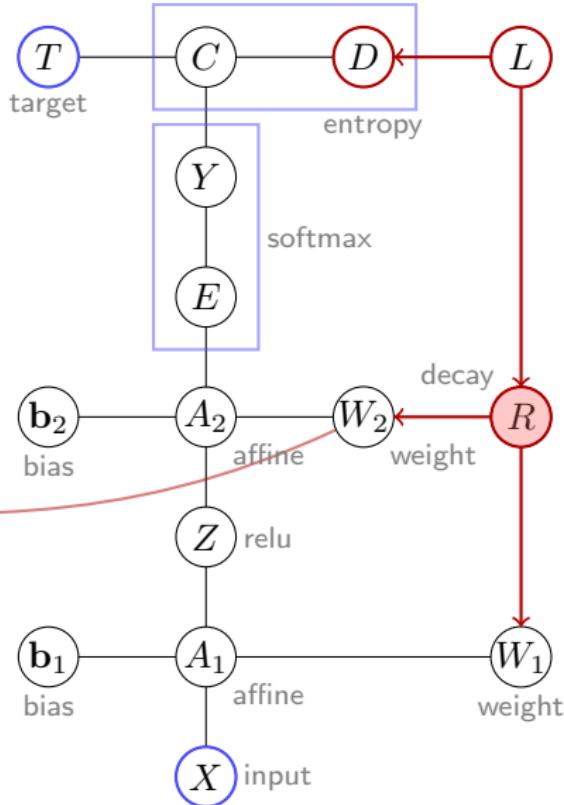
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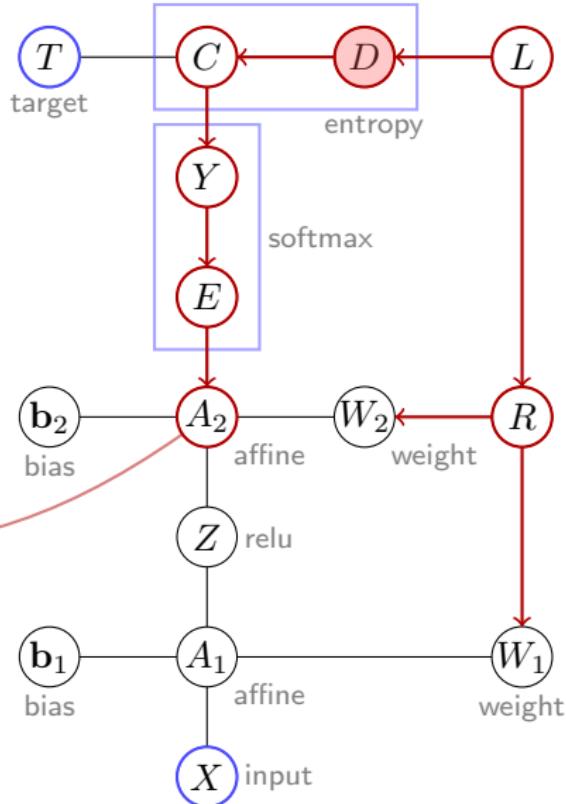
$$L = D + R$$

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$$dW_1 = dR * \lambda * W_1$$

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$$\boxed{dA_2 = dD * (Y - T)/m}$$



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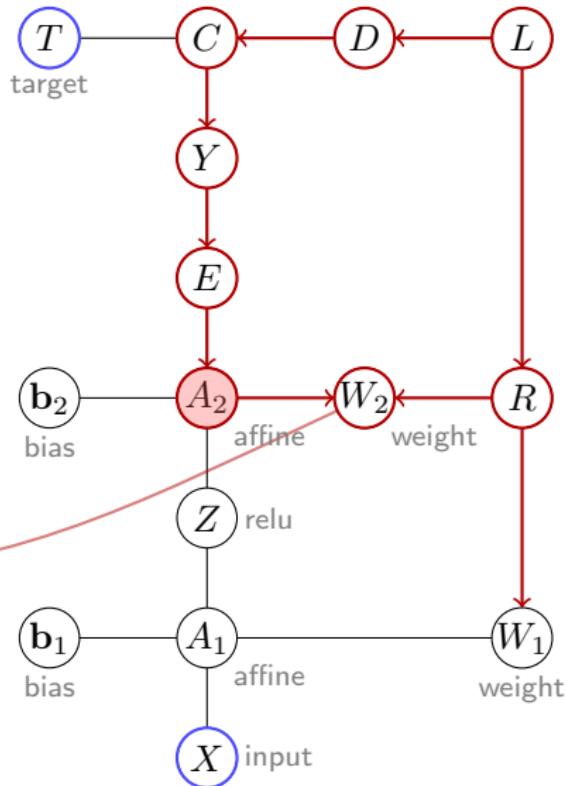
$$dW_2 = dR * \lambda * W_2$$

$$dA_2 = dD * (Y - T)/m$$

$$dW_2 += \text{dot}(Z^\top, dA_2)$$

$$d\mathbf{b}_2 = \text{sum}_0(dA_2)$$

$$dZ = \text{dot}(dA_2, W_2^\top)$$



# back-propagation

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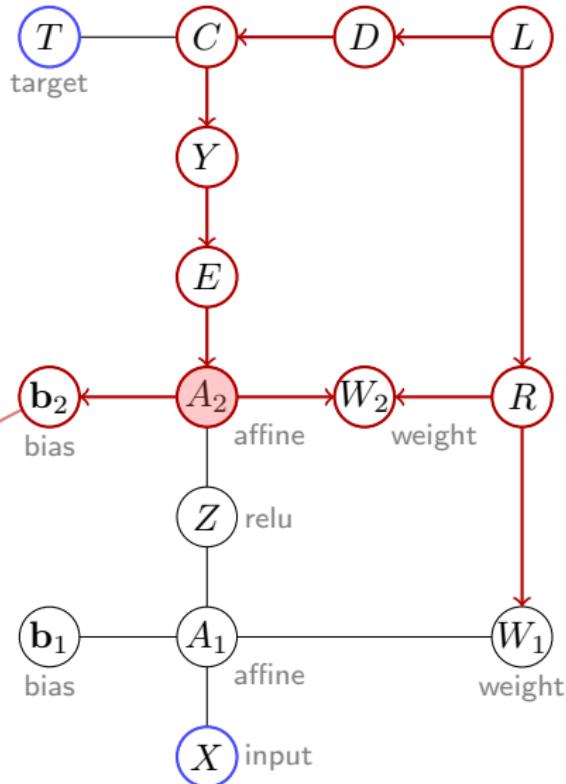
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$$\boxed{db_2 = \text{sum}_0(dA_2)}$$

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$$(dD, dR) = (dL, dL)$$

$$dW_1 = dR * \lambda * W_1$$

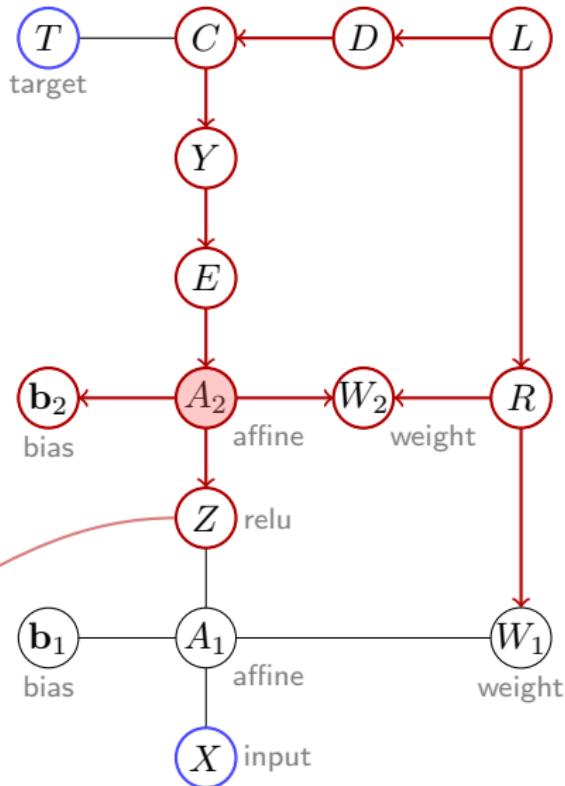
$$dW_2 = dR * \lambda * W_2$$

$$dA_2 = dD * (Y - T)/m$$

$$dW_2 += \text{dot}(Z^\top, dA_2)$$

$$\mathbf{d}\mathbf{b}_2 = \text{sum}_0(dA_2)$$

$$dZ = \text{dot}(dA_2, W_2^\top)$$



# back-propagation

$$A_1 = \text{dot}(X, W_1) + \mathbf{b}_1$$

$$Z = \max(0, A_1)$$

$$A_2 = \text{dot}(Z, W_2) + \mathbf{b}_2$$

$$E = \exp(A_2)$$

$$Y = E / \text{sum}_1(E)$$

$$C = -\text{sum}_1(T * \log(Y))$$

$$D = \text{sum}_0(C)/m$$

$$R = \frac{\lambda}{2} * (\|W_1\|_F^2 + \|W_2\|_F^2)$$

$$L = D + R$$

$$(dD, dR) = (dL, dL)$$

$$dW_1 = dR * \lambda * W_1$$

$$dW_2 = dR * \lambda * W_2$$

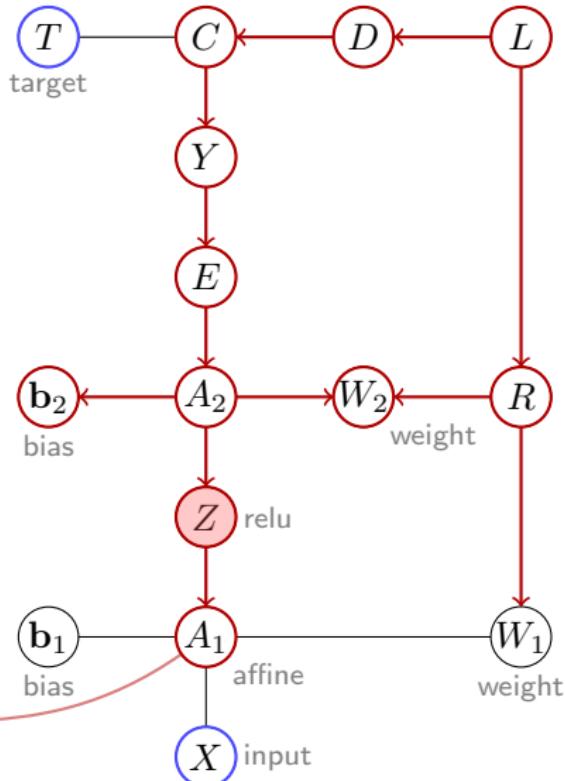
$$dA_2 = dD * (Y - T)/m$$

$$dW_2 += \text{dot}(Z^\top, dA_2)$$

$$d\mathbf{b}_2 = \text{sum}_0(dA_2)$$

$$dZ = \text{dot}(dA_2, W_2^\top)$$

$$dA_1 = dZ * (Z > 0)$$



# back-propagation

$$A_1 = \text{dot}(X, W_1) + \mathbf{b}_1$$

$$Z = \max(0, A_1)$$

$$A_2 = \text{dot}(Z, W_2) + \mathbf{b}_2$$

$$E = \exp(A_2)$$

$$Y = E / \text{sum}_1(E)$$

$$C = -\text{sum}_1(T * \log(Y))$$

$$D = \text{sum}_0(C)/m$$

$$R = \frac{\lambda}{2} * (\|W_1\|_F^2 + \|W_2\|_F^2)$$

$$L = D + R$$

$$(dD, dR) = (dL, dL)$$

$$dW_1 = dR * \lambda * W_1$$

$$dW_2 = dR * \lambda * W_2$$

$$dA_2 = dD * (Y - T)/m$$

$$dW_2 += \text{dot}(Z^\top, dA_2)$$

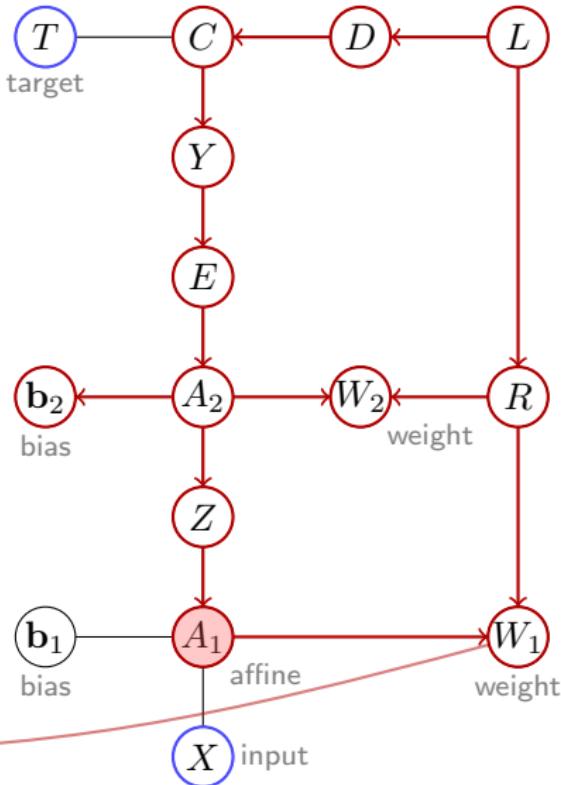
$$d\mathbf{b}_2 = \text{sum}_0(dA_2)$$

$$dZ = \text{dot}(dA_2, W_2^\top)$$

$$dA_1 = dZ * (Z > 0)$$

$$dW_1 += \text{dot}(X^\top, dA_1)$$

$$d\mathbf{b}_1 = \text{sum}_0(dA_1)$$



# back-propagation

$$A_1 = \text{dot}(X, W_1) + b_1$$

$$Z = \max(0, A_1)$$

$$A_2 = \text{dot}(Z, W_2) + b_2$$

$$E = \exp(A_2)$$

$$Y = E / \text{sum}_1(E)$$

$$C = -\text{sum}_1(T * \log(Y))$$

$$D = \text{sum}_0(C) / m$$

$$R = \frac{\lambda}{2} * (\|W_1\|_F^2 + \|W_2\|_F^2)$$

$$L = D + R$$

$$(dD, dR) = (dL, dL)$$

$$dW_1 = dR * \lambda * W_1$$

$$dW_2 = dR * \lambda * W_2$$

$$dA_2 = dD * (Y - T) / m$$

$$dW_2 += \text{dot}(Z^\top, dA_2)$$

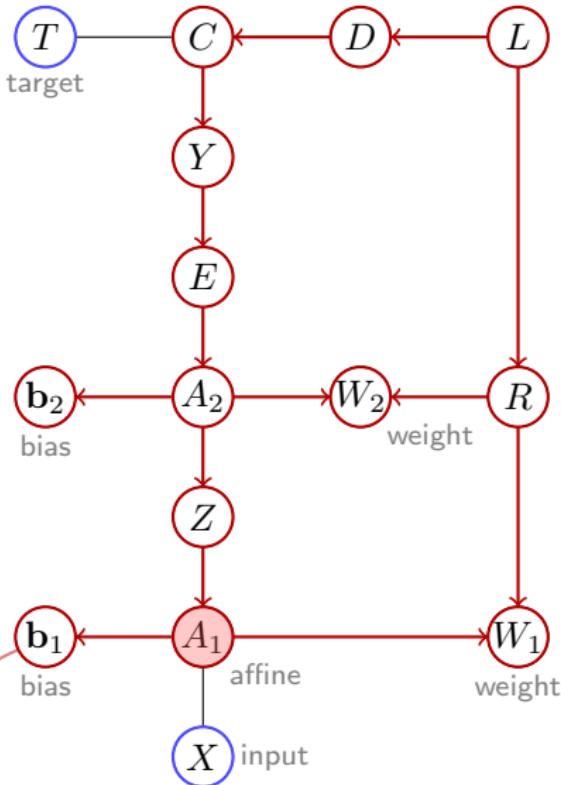
$$db_2 = \text{sum}_0(dA_2)$$

$$dZ = \text{dot}(dA_2, W_2^\top)$$

$$dA_1 = dZ * (Z > 0)$$

$$dW_1 += \text{dot}(X^\top, dA_1)$$

$$db_1 = \text{sum}_0(dA_1)$$



# automatic differentiation

```
A1 = dot(X, W1) + b1
Z = max(0, A1)
A2 = dot(Z, W2) + b2
E = exp(A2)
Y = E / sum1(E)
C = -sum1(T * log(Y))
D = sum0(C) / m
R = λ / 2 * (||W1||F2 + ||W2||F2)
L = D + R
```

```
(dD, dR) = (dL, dL)
dW1 = dR * λ * W1
dW2 = dR * λ * W2
dA2 = dD * (Y - T) / m
dW2 += dot(ZT, dA2)
db2 = sum0(dA2)
dZ = dot(dA2, W2T)
dA1 = dZ * (Z > 0)
dW1 += dot(XT, dA1)
db1 = sum0(dA1)
```

now we organize **forward** and **backward** code into units and functions

# automatic differentiation

```
A1 = dot(X, W1) + b1
Z = max(0, A1)
A2 = dot(Z, W2) + b2
E = exp(A2)
Y = E/sum1(E)
C = -sum1(T * log(Y))
D = sum0(C)/m
R = λ/2 * (||W1||F2 + ||W2||F2)
L = D + R
(dD, dR) = (dL, dL)
dW1 = dR * λ * W1
dW2 = dR * λ * W2
dA2 = dD * (Y - T)/m
dW2 += dot(ZT, dA2)
db2 = sum0(dA2)
dZ = dot(dA2, W2T)
dA1 = dZ * (Z > 0)
dW1 += dot(XT, dA1)
db1 = sum0(dA1)
```

```
def relu(A):
    Z = max(0, A)
    def back(dZ, dA):
        dA += dZ * (Z > 0)
    return node(Z, back)
```

# automatic differentiation

$$A_1 = \text{dot}(X, W_1) + \mathbf{b}_1$$

$$Z = \text{relu}(A_1)$$

$$A_2 = \text{dot}(Z, W_2) + \mathbf{b}_2$$

$$E = \exp(A_2)$$

$$Y = E / \text{sum}_1(E)$$

$$C = -\text{sum}_1(T * \log(Y))$$

$$D = \text{sum}_0(C) / m$$

$$R = \frac{\lambda}{2} * (\|W_1\|_F^2 + \|W_2\|_F^2)$$

$$L = D + R$$

$$(dD, dR) = (dL, dL)$$

$$dW_1 = dR * \lambda * W_1$$

$$dW_2 = dR * \lambda * W_2$$

$$dA_2 = dD * (Y - T) / m$$

$$dW_2 += \text{dot}(Z^\top, dA_2)$$

$$d\mathbf{b}_2 = \text{sum}_0(dA_2)$$

$$dZ = \text{dot}(dA_2, W_2^\top)$$

$$Z.\text{back}(A_1)$$

$$dW_1 += \text{dot}(X^\top, dA_1)$$

$$d\mathbf{b}_1 = \text{sum}_0(dA_1)$$

```
def relu(A):
```

```
    Z = max(0, A)
```

```
def back(dZ, dA):
```

```
    dA += dZ * (Z > 0)
```

```
return node(Z, back)
```

# automatic differentiation

```
A1 = dot(X, W1) + b1
Z = relu(A1)
A2 = dot(Z, W2) + b2
E = exp(A2)
Y = E / sum1(E)
C = -sum1(T * log(Y))
D = sum0(C) / m
R =  $\frac{\lambda}{2} * (\|W_1\|_F^2 + \|W_2\|_F^2)$ 
L = D + R
(dD, dR) = (dL, dL)
dW1 = dR * λ * W1
dW2 = dR * λ * W2
dA2 = dD * (Y - T) / m
dW2 += dot(Z⊤, dA2)
db2 = sum0(dA2)
dZ = dot(dA2, W2⊤)
Z.back(A1)
dW1 += dot(X⊤, dA1)
db1 = sum0(dA1)
```

```
def affine(X, (W, b)):
    A = dot(X, W) + b
    def back(dA, dX, (dW, db)):
        dW += dot(X⊤, dA)
        db += sum0(dA)
        dX += dot(dA, W⊤)
    return node(A, back)
```

# automatic differentiation

```
A1 = affine(X, (W1, b1))  
Z = relu(A1)  
A2 = affine(Z, (W2, b2))  
E = exp(A2)  
Y = E / sum1(E)  
C = -sum1(T * log(Y))  
D = sum0(C) / m  
R =  $\frac{\lambda}{2} * (\|W_1\|_F^2 + \|W_2\|_F^2)$   
L = D + R  
(dD, dR) = (dL, dL)  
dW1 = dR * λ * W1  
dW2 = dR * λ * W2  
dA2 = dD * (Y - T) / m  
A2.back(Z, (W2, b2))
```

```
def affine(X, (W, b)):  
    A = dot(X, W) + b  
  
def back(dA, dX, (dW, db)):  
    dW += dot(X⊤, dA)  
    db += sum0(dA)  
    dX += dot(dA, W⊤)  
  
return node(A, back)
```

Z.back(A<sub>1</sub>)

A<sub>1</sub>.back(X, (W<sub>1</sub>, b<sub>1</sub>))

# automatic differentiation

$A_1 = \text{affine}(X, (W_1, \mathbf{b}_1))$

$Z = \text{relu}(A_1)$

$A_2 = \text{affine}(Z, (W_2, \mathbf{b}_2))$

$E = \exp(A_2)$

$Y = E / \text{sum}_1(E)$

$C = -\text{sum}_1(T * \log(Y))$

$D = \text{sum}_0(C) / m$

$R = \frac{\lambda}{2} * (\|W_1\|_F^2 + \|W_2\|_F^2)$

$L = D + R$

$(dD, dR) = (dL, dL)$

$dW_1 = dR * \lambda * W_1$

$dW_2 = dR * \lambda * W_2$

$dA_2 = dD * (Y - T) / m$

$A_2.\text{back}(Z, (W_2, \mathbf{b}_2))$

**def** logistic( $A, T$ ):

$E = \exp(A)$

$Y = E / \text{sum}_1(E)$

$C = -\text{sum}_1(T * \log(Y))$

$D = \text{sum}_0(C) / m$

**def** back( $dD, dA, \_\_$ ):

$dA += dD * (Y - T) / m$

**return** node( $D, \text{back}$ )

$Z.\text{back}(A_1)$

$A_1.\text{back}(X, (W_1, \mathbf{b}_1))$

# automatic differentiation

$A_1 = \text{affine}(X, (W_1, \mathbf{b}_1))$

$Z = \text{relu}(A_1)$

$A_2 = \text{affine}(Z, (W_2, \mathbf{b}_2))$

$D = \text{logistic}(A_2, T)$

$$R = \frac{\lambda}{2} * (\|W_1\|_F^2 + \|W_2\|_F^2)$$

$$L = D + R$$

$$(dD, dR) = (dL, dL)$$

$$dW_1 = dR * \lambda * W_1$$

$$dW_2 = dR * \lambda * W_2$$

$D.\text{back}(A_2, T)$   
 $A_2.\text{back}(Z, (W_2, \mathbf{b}_2))$

```
def logistic(A, T):
    E = exp(A)
    Y = E/sum_1(E)
    C = -sum_1(T * log(Y))
    D = sum_0(C)/m

def back(dD, dA, _):
    dA += dD * (Y - T)/m
    return node(D, back)
```

$Z.\text{back}(A_1)$   
 $A_1.\text{back}(X, (W_1, \mathbf{b}_1))$

# automatic differentiation

$A_1 = \text{affine}(X, (W_1, \mathbf{b}_1))$

$Z = \text{relu}(A_1)$

$A_2 = \text{affine}(Z, (W_2, \mathbf{b}_2))$

$D = \text{logistic}(A_2, T)$

$$R = \frac{\lambda}{2} * (\|W_1\|_F^2 + \|W_2\|_F^2)$$

$$L = D + R$$

$$(dD, dR) = (dL, dL)$$

$$dW_1 = dR * \lambda * W_1$$

$$dW_2 = dR * \lambda * W_2$$

$$D.\text{back}(A_2, T)$$

$$A_2.\text{back}(Z, (W_2, \mathbf{b}_2))$$

**def** decay( $W$ ):

$$R = \frac{\lambda}{2} * \text{sum}(\|w\|_F^2 \text{ for } w \text{ in } W)$$

**def** back( $dR, dW$ ):

**for** ( $w, dw$ ) **in** `zip`( $W, dW$ ):  
 $dw += dR * \lambda * w$

**return** `node`( $R$ , back)

$Z.\text{back}(A_1)$

$A_1.\text{back}(X, (W_1, \mathbf{b}_1))$

# automatic differentiation

$A_1 = \text{affine}(X, (W_1, \mathbf{b}_1))$

$Z = \text{relu}(A_1)$

$A_2 = \text{affine}(Z, (W_2, \mathbf{b}_2))$

$D = \text{logistic}(A_2, T)$

$R = \text{decay}((W_1, W_2))$

$L = D + R$

$(dD, dR) = (dL, dL)$

$R.\text{back}((W_1, W_2))$

$D.\text{back}(A_2, T)$

$A_2.\text{back}(Z, (W_2, \mathbf{b}_2))$

$Z.\text{back}(A_1)$

$A_1.\text{back}(X, (W_1, \mathbf{b}_1))$

```
def decay(W):
    R =  $\frac{\lambda}{2} * \text{sum}(\|w\|_F^2)$  for w in W
    def back(dR, dW):
        for (w, dw) in zip(W, dW):
            dw += dR *  $\lambda * w$ 
    return node(R, back)
```

# automatic differentiation

$A_1 = \text{affine}(X, (W_1, \mathbf{b}_1))$

$Z = \text{relu}(A_1)$

$A_2 = \text{affine}(Z, (W_2, \mathbf{b}_2))$

$D = \text{logistic}(A_2, T)$

$R = \text{decay}((W_1, W_2))$

$L = D + R$

$(dD, dR) = (dL, dL)$

$R.\text{back}((W_1, W_2))$

$D.\text{back}(A_2, T)$

$A_2.\text{back}(Z, (W_2, \mathbf{b}_2))$

$Z.\text{back}(A_1)$

$A_1.\text{back}(X, (W_1, \mathbf{b}_1))$

```
def add(X):
    S = sum(X)
    def back(dS, dX):
        for dx in dX:
            dx += dS
    return node(S, back)
```

# automatic differentiation

$A_1 = \text{affine}(X, (W_1, \mathbf{b}_1))$

$Z = \text{relu}(A_1)$

$A_2 = \text{affine}(Z, (W_2, \mathbf{b}_2))$

$D = \text{logistic}(A_2, T)$

$R = \text{decay}((W_1, W_2))$

$L = \text{add}((D, R))$

$L.\text{back}((D, R))$

$R.\text{back}((W_1, W_2))$

$D.\text{back}(A_2, T)$

$A_2.\text{back}(Z, (W_2, \mathbf{b}_2))$

$Z.\text{back}(A_1)$

$A_1.\text{back}(X, (W_1, \mathbf{b}_1))$

**def** add( $X$ ):

$S = \text{sum}(X)$

**def** back( $dS, dX$ ):

**for**  $dx$  **in**  $dX$ :

$dx += dS$

**return** node( $S$ , back)

# automatic differentiation

$A_1 = \text{affine}(X, (W_1, \mathbf{b}_1))$

$Z = \text{relu}(A_1)$

$A_2 = \text{affine}(Z, (W_2, \mathbf{b}_2))$

$D = \text{logistic}(A_2, T)$

$R = \text{decay}((W_1, W_2))$

$L = \text{add}((D, R))$

$L.\text{back}((D, R))$

$R.\text{back}((W_1, W_2))$

$D.\text{back}(A_2, T)$

$A_2.\text{back}(Z, (W_2, \mathbf{b}_2))$

$Z.\text{back}(A_1)$

$A_1.\text{back}(X, (W_1, \mathbf{b}_1))$

```
def loss(A, T, W):
    D = logistic(A, T)
    R = decay(W)
    L = add((D, R))
    def back(A, T, W):
        L.back((D, R))
        R.back(W)
        D.back(A, T)
    return block(L, back)
```

# automatic differentiation

$A_1 = \text{affine}(X, (W_1, \mathbf{b}_1))$

$Z = \text{relu}(A_1)$

$A_2 = \text{affine}(Z, (W_2, \mathbf{b}_2))$

$L = \text{loss}(A_2, T, (W_1, W_2))$

$L.\text{back}(A_2, T, (W_1, W_2))$

$A_2.\text{back}(Z, (W_2, \mathbf{b}_2))$

$Z.\text{back}(A_1)$

$A_1.\text{back}(X, (W_1, \mathbf{b}_1))$

```
def loss(A, T, W):
    D = logistic(A, T)
    R = decay(W)
    L = add((D, R))

def back(A, T, W):
    L.back((D, R))
    R.back(W)
    D.back(A, T)

return block(L, back)
```

# automatic differentiation

$A_1 = \text{affine}(X, (W_1, \mathbf{b}_1))$

$Z = \text{relu}(A_1)$

$A_2 = \text{affine}(Z, (W_2, \mathbf{b}_2))$

$L = \text{loss}(A_2, T, (W_1, W_2))$

$L.\text{back}(A_2, T, (W_1, W_2))$

$A_2.\text{back}(Z, (W_2, \mathbf{b}_2))$

$Z.\text{back}(A_1)$

$A_1.\text{back}(X, (W_1, \mathbf{b}_1))$

```
def loss(A, T, W):
    L = logistic(A, T) + decay(W)
    return block(L)
```

```
def loss(A, T, W):
    D = logistic(A, T)
    R = decay(W)
    L = add((D, R))
```

```
def back(A, T, W):
    L.back((D, R))
    R.back(W)
    D.back(A, T)
    return block(L, back)
```

# automatic differentiation

```
A1 = affine(X, (W1, b1))  
Z = relu(A1)  
A2 = affine(Z, (W2, b2))  
L = loss(A2, T, (W1, W2))
```

$L.$ .back( $A_2, T, (W_1, W_2)$ )

$A_2.$ .back( $Z, (W_2, b_2)$ )

$Z.$ .back( $A_1$ )  
 $A_1.$ .back( $X, (W_1, b_1)$ )

**def** model( $X, (U_1, U_2)$ ):

```
A1 = affine(X, U1)  
Z = relu(A)  
A2 = affine(Z, U2)
```

**def** back( $X, (U_1, U_2)$ ):

```
A2.back(Z, U2)  
Z.back(A)  
A1.back(X, U1)
```

**return** block( $A_2$ , back)

# automatic differentiation

```
A2 = model(X, ((W1, b1), (W2, b2)))
```

```
L = loss(A2, T, (W1, W2))
```

```
L.back(A2, T, (W1, W2))
```

```
A2.back(X, ((W1, b1), (W2, b2)))
```

```
def model(X, (U1, U2)):  
    A1 = affine(X, U1)  
    Z = relu(A)  
    A2 = affine(Z, U2)  
  
def back(X, (U1, U2)):  
    A2.back(Z, U2)  
    Z.back(A)  
    A1.back(X, U1)  
    return block(A2, back)
```

# automatic differentiation

$A_2 = \text{model}(X, ((W_1, \mathbf{b}_1), (W_2, \mathbf{b}_2)))$

$L = \text{loss}(A_2, T, (W_1, W_2))$

$L.\text{back}(A_2, T, (W_1, W_2))$

$A_2.\text{back}(X, ((W_1, \mathbf{b}_1), (W_2, \mathbf{b}_2)))$

```
def model(X, (U1, U2)):  
    A = affine(relu(affine(X, U1)), U2)  
    return block(A)
```

```
def model(X, (U1, U2)):  
    A1 = affine(X, U1)  
    Z = relu(A)  
    A2 = affine(Z, U2)
```

```
def back(X, (U1, U2)):  
    A2.back(Z, U2)  
    Z.back(A)  
    A1.back(X, U1)  
    return block(A2, back)
```

# pynet

code available at <https://github.com/iaavr/pynet>

# deep learning software

Caffe



Caffe<sup>2</sup>



PYTORCH



TensorFlow

theano



dmlc  
**mxnet**

- automatically build computational graphs and compute derivatives
- run on GPU, multiple GPU, distributed
- component (unit, layer) libraries
- pre-trained models
- community

## summary

- stochastic gradient descent and its limitations
- numerical gradient approximation
- analytical computation by decomposing and applying the chain rule
- back-propagation as dynamic programming
- chaining, splitting and sharing
- common patterns between forward and backward flow
- decomposition into units (forward) and nodes (backward)
- grouping into functions (forward) and blocks (backward)