

lecture 7: convolution and architectures

deep learning for vision

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Rennes, Nov. 2017 – Jan. 2018



outline

introduction

definition and properties

variants and their derivatives

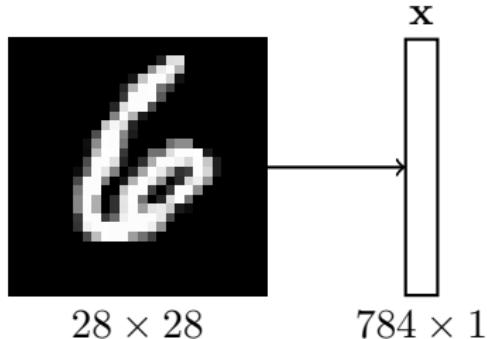
pooling

fun

network architectures

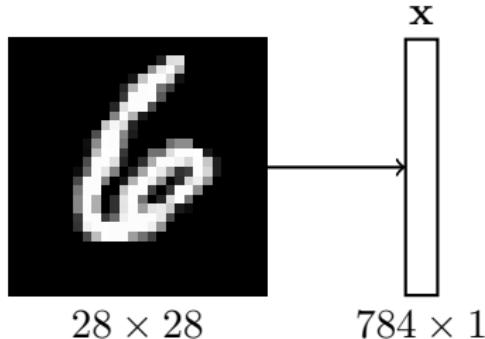
introduction

input image representation



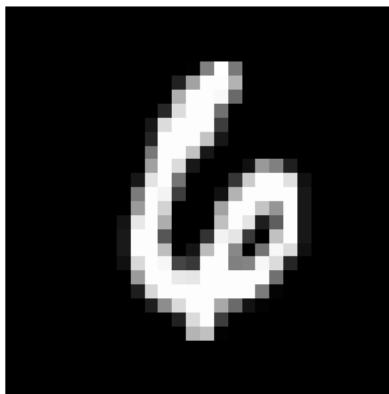
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- remember that for both MNIST and CIFAR10, we flattened images (1-channel or 3-channel) into vectors, and the **order** of the elements (pixels) plays no role in learning
- so what if we **permute** the elements in all images, both training and test set?

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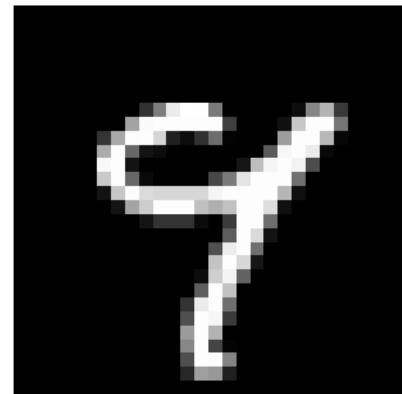
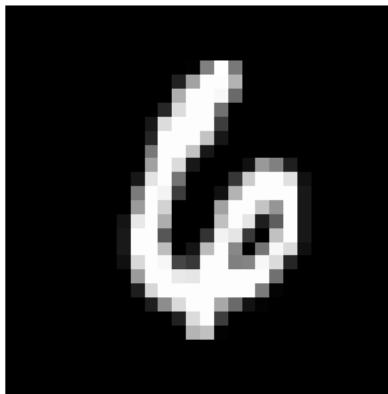
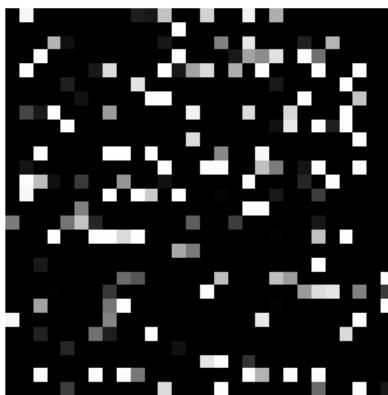
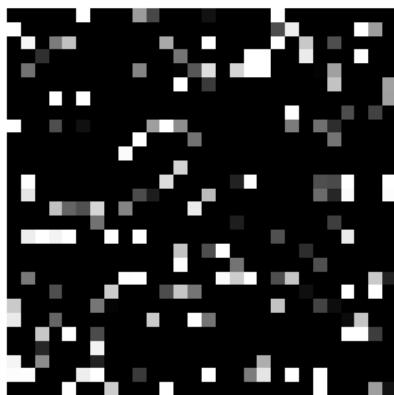


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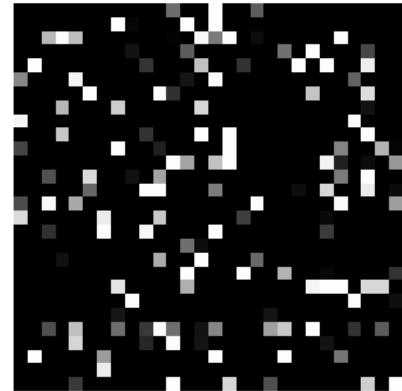
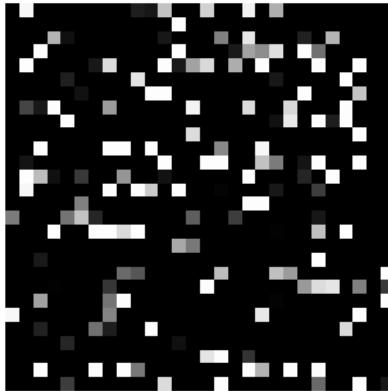
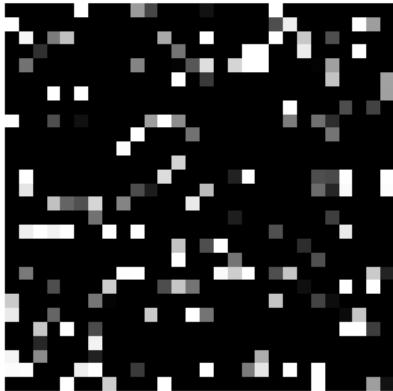
shuffling the dimensions



shuffling the dimensions

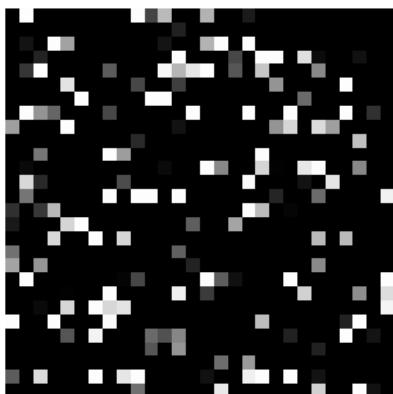
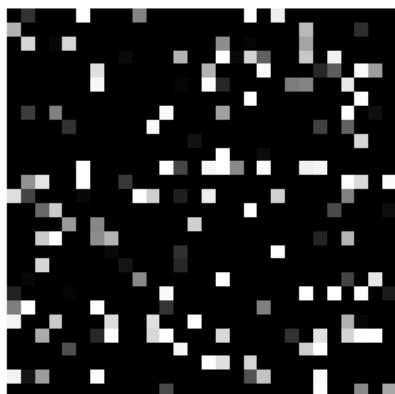
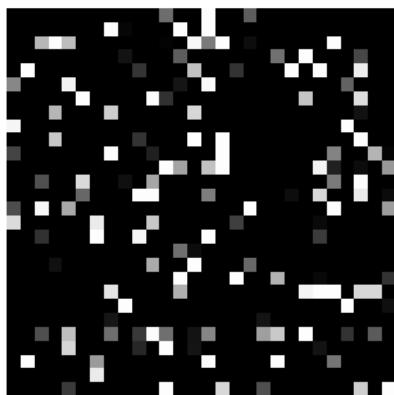
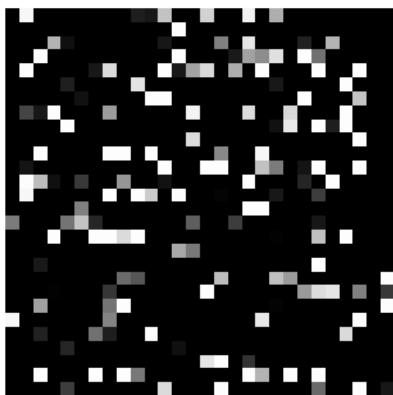
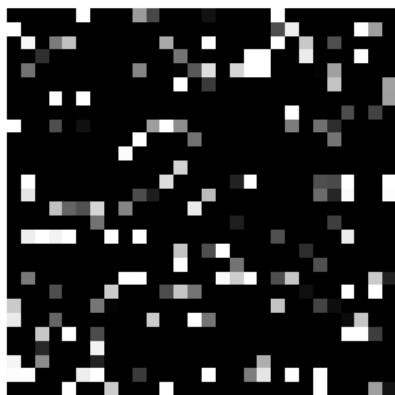


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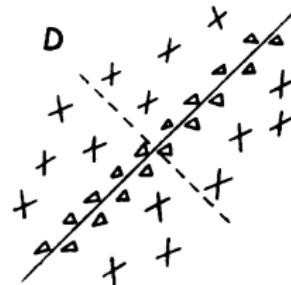
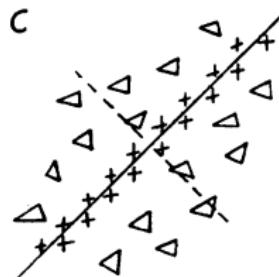


- this is what the computer sees
- it must make more sense when you start looking at more than one samples per class

shuffling the dimensions

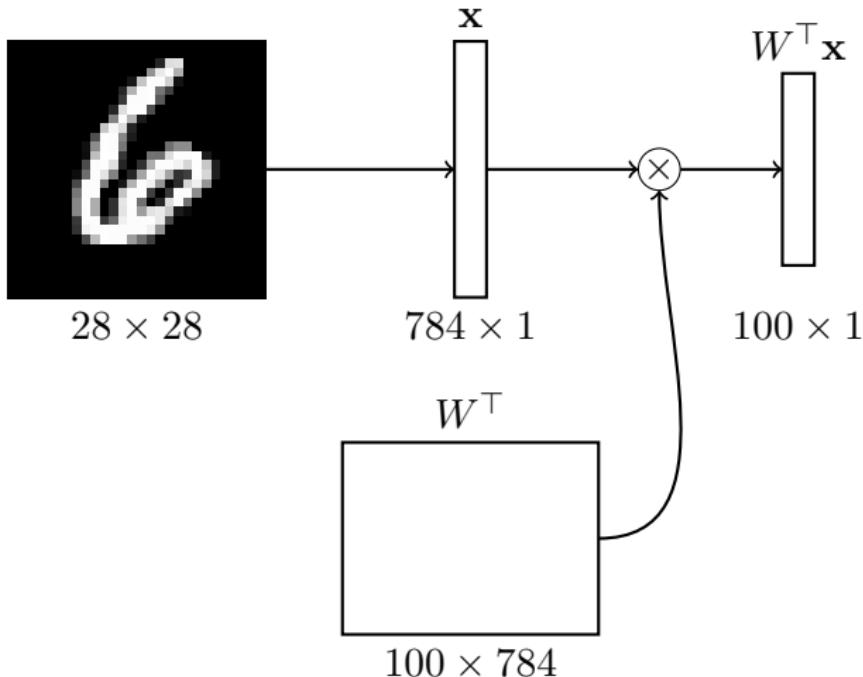


remember receptive fields?



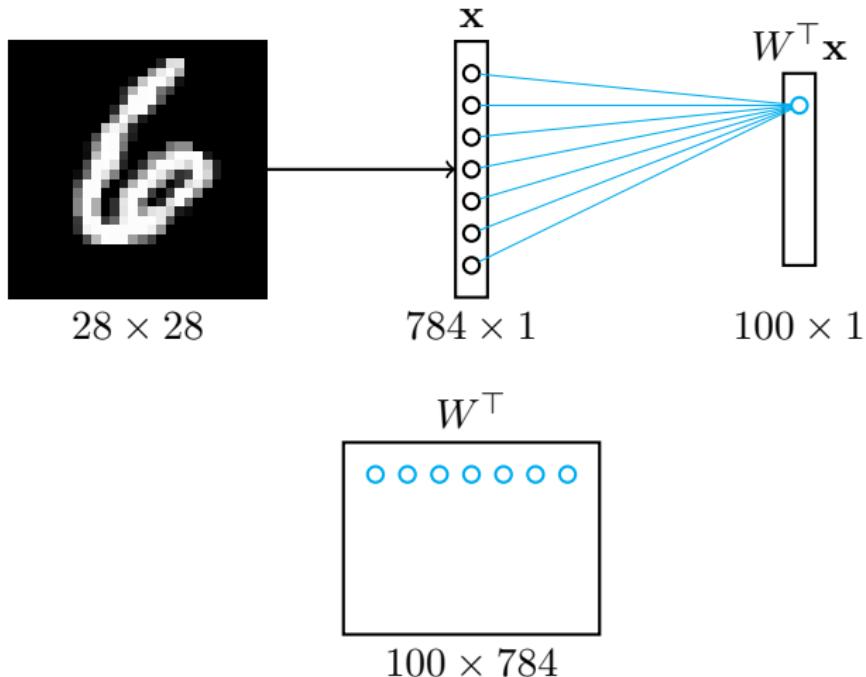
- A: 'on'-center LGN; B: 'off'-center LGN; C, D: simple cortical
- each cell only has a localized response over a **receptive field**
- **x**: excitatory ('on'), **△**: inhibitory ('off') responses
- **topographic mapping**: there is one cell with the same response pattern centered at each position

matrix multiplication



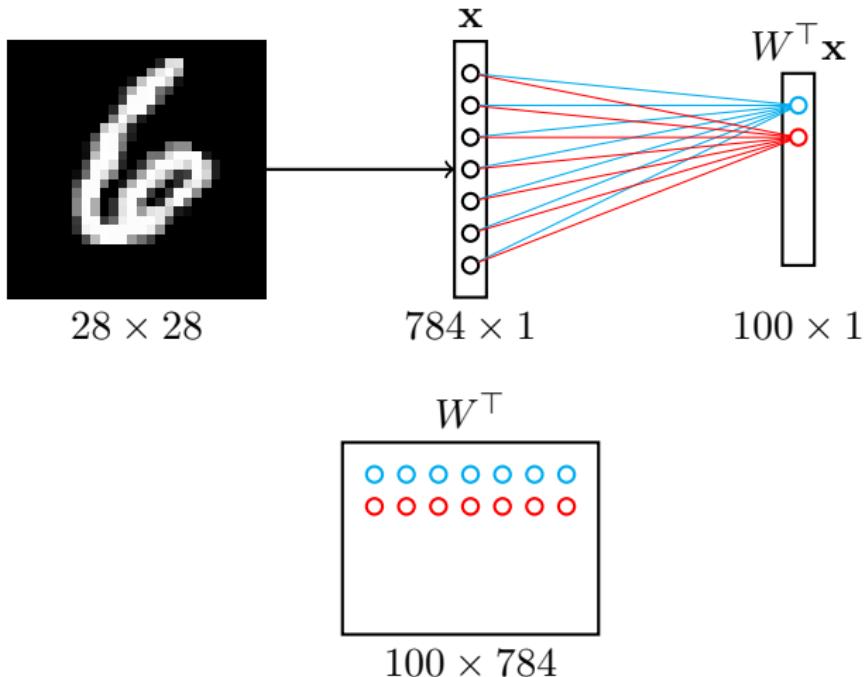
- inputs \mathbf{x} are mapped to activations $W^\top \mathbf{x}$
- columns/rows of W^\top correspond to input/activation elements

matrix multiplication → fully connected



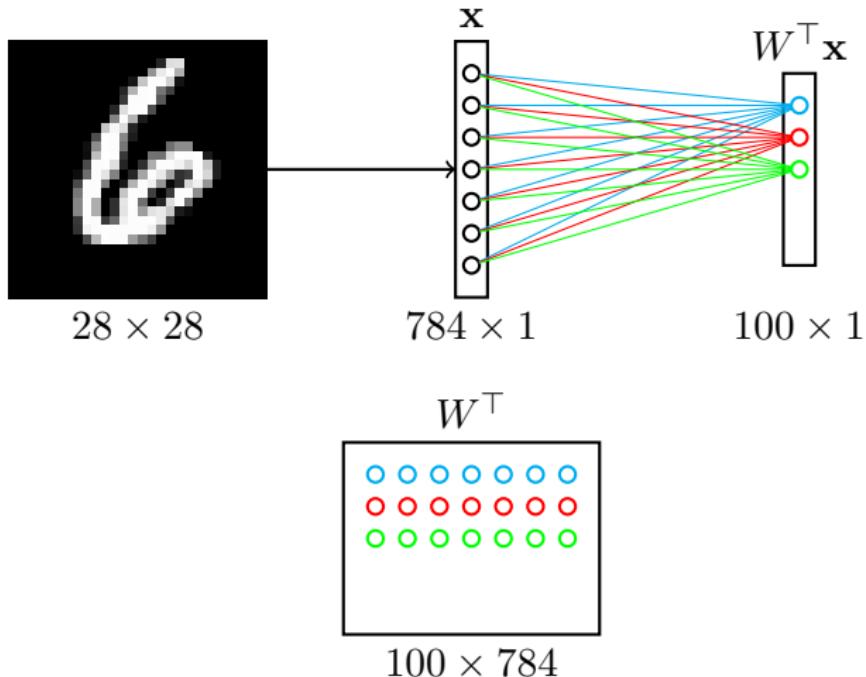
- each row of W^T yields one activation element (**cell**)
- each cell is **fully connected** to all input elements

matrix multiplication → fully connected



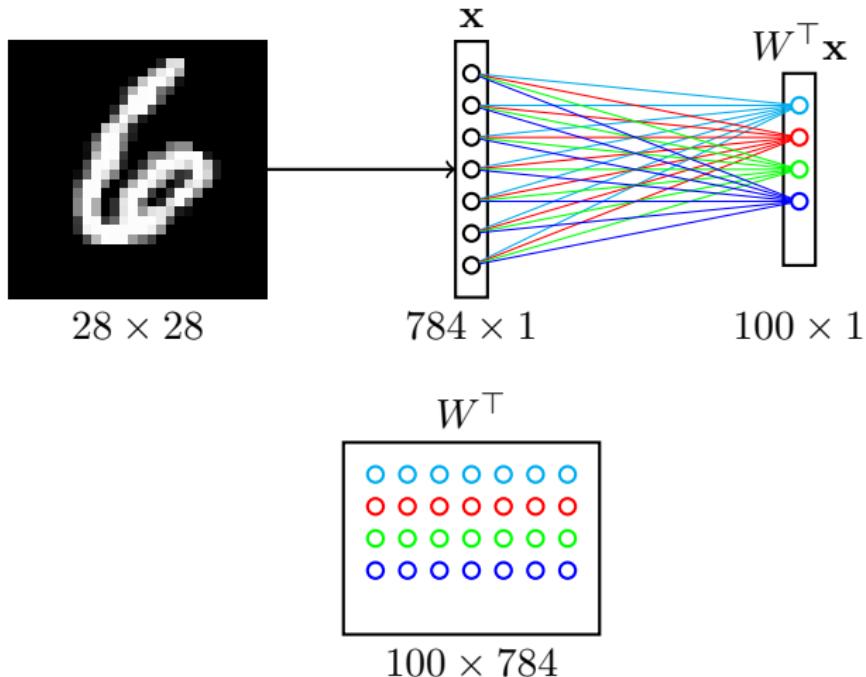
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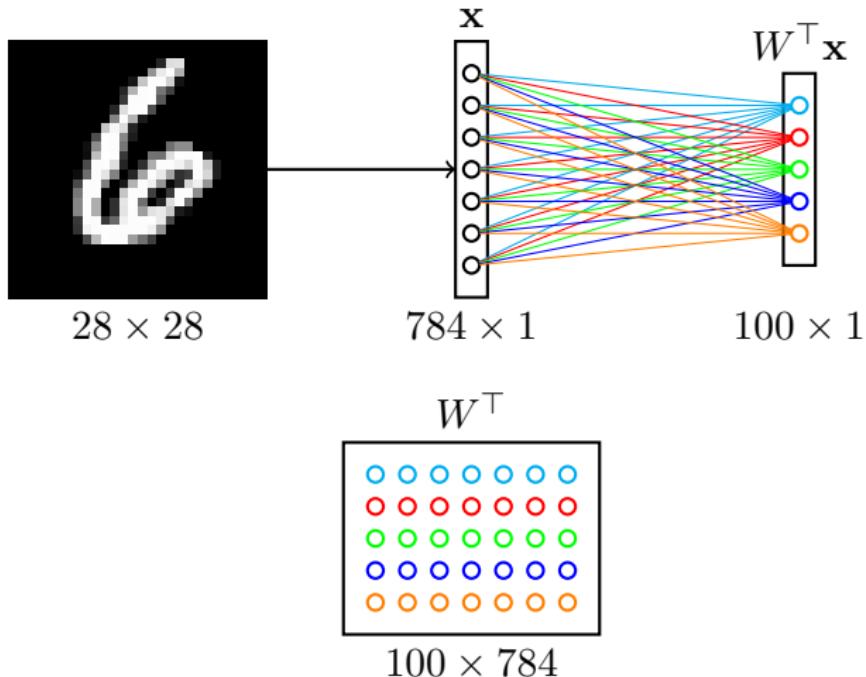
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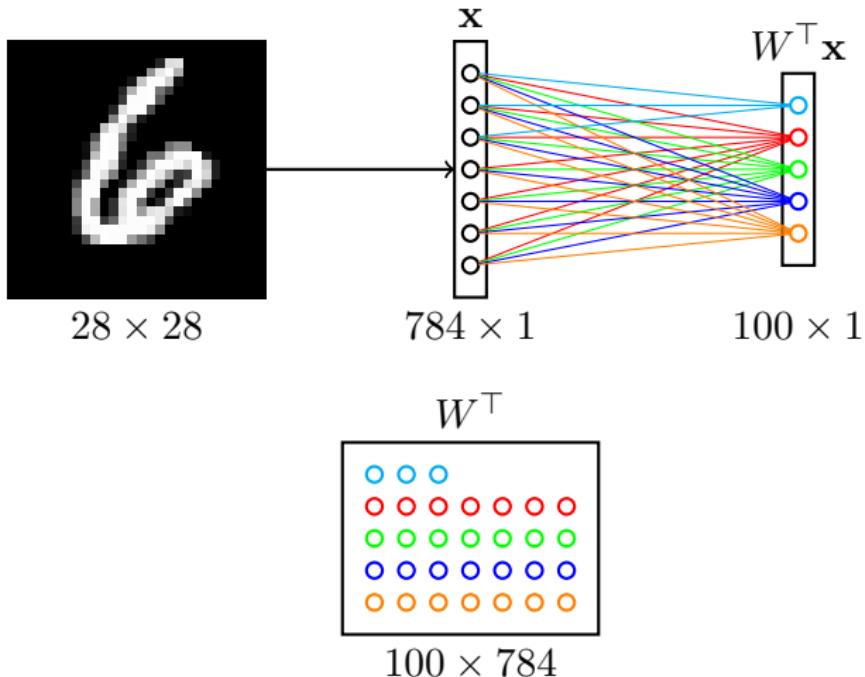
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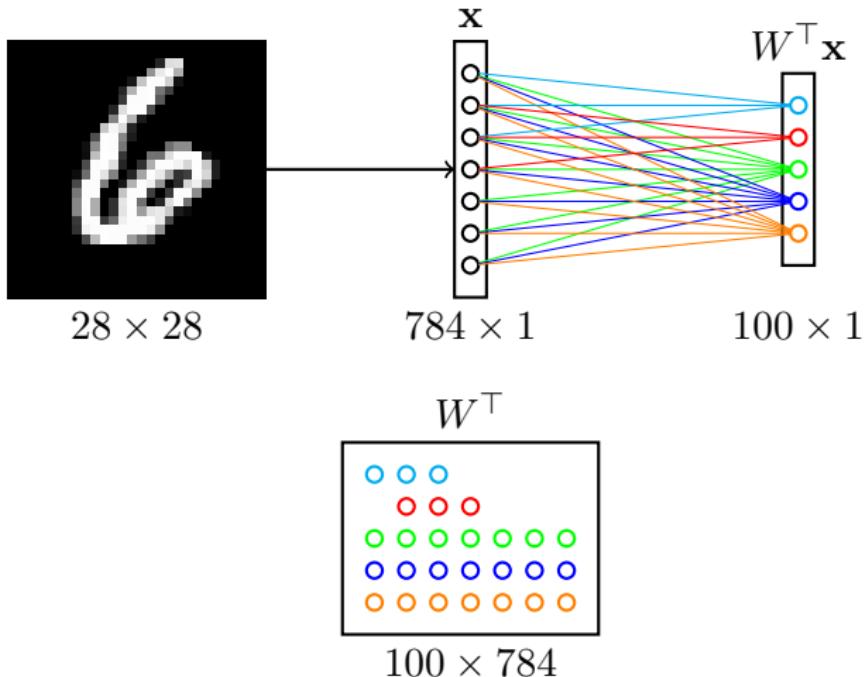
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sparse connections



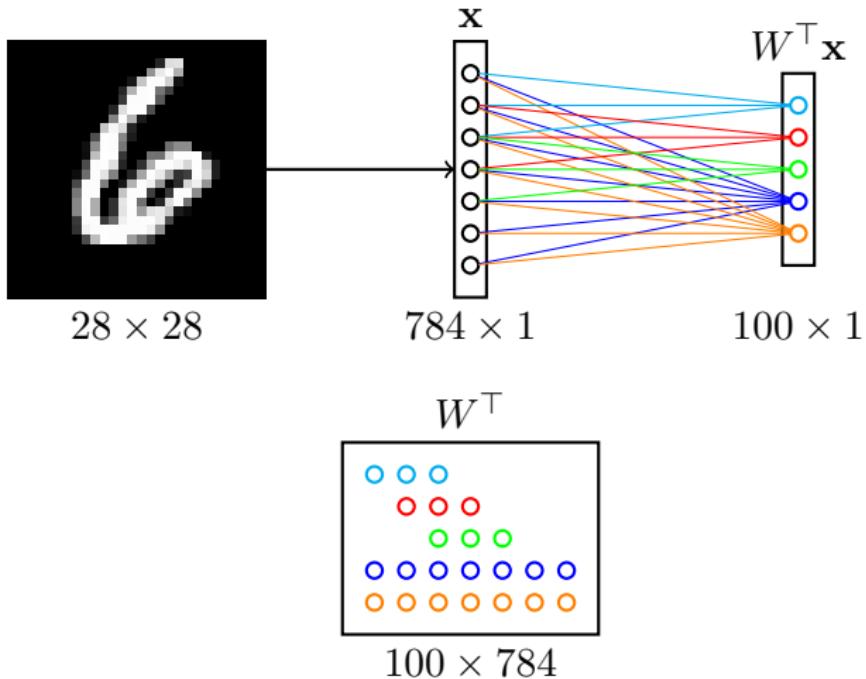
- now, we only keep a **sparse** set of connections
- and matrix W becomes sparse as well

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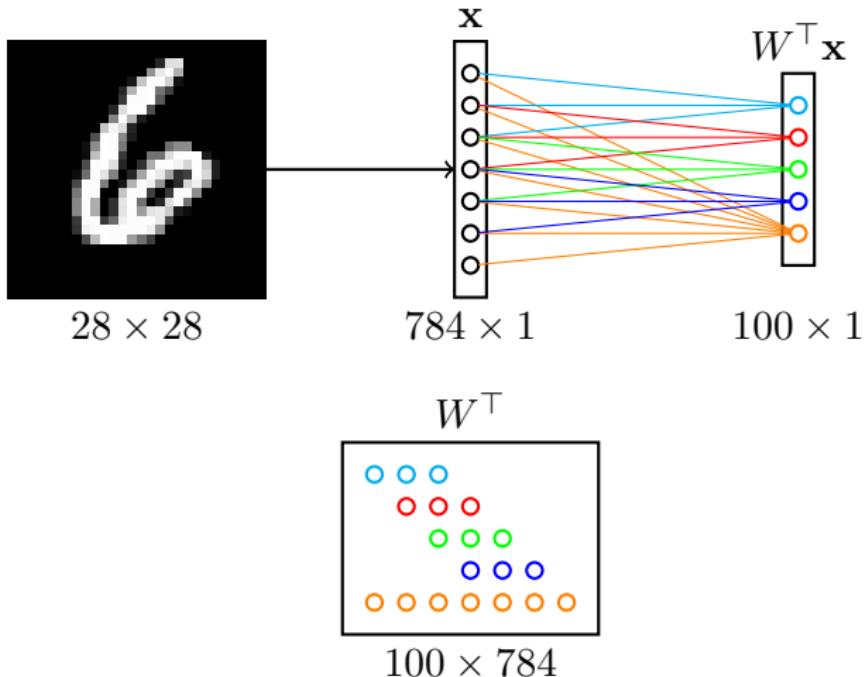
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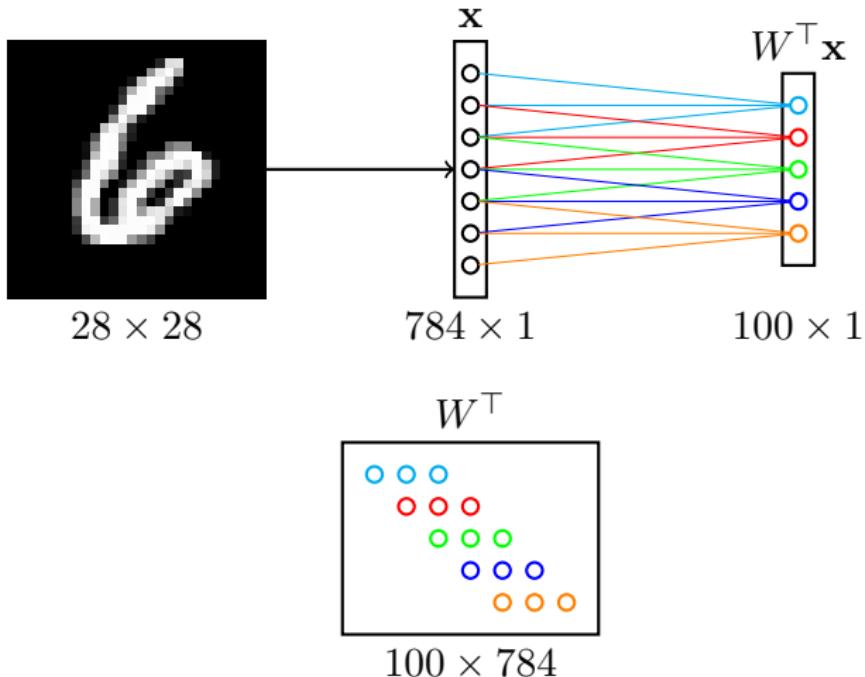
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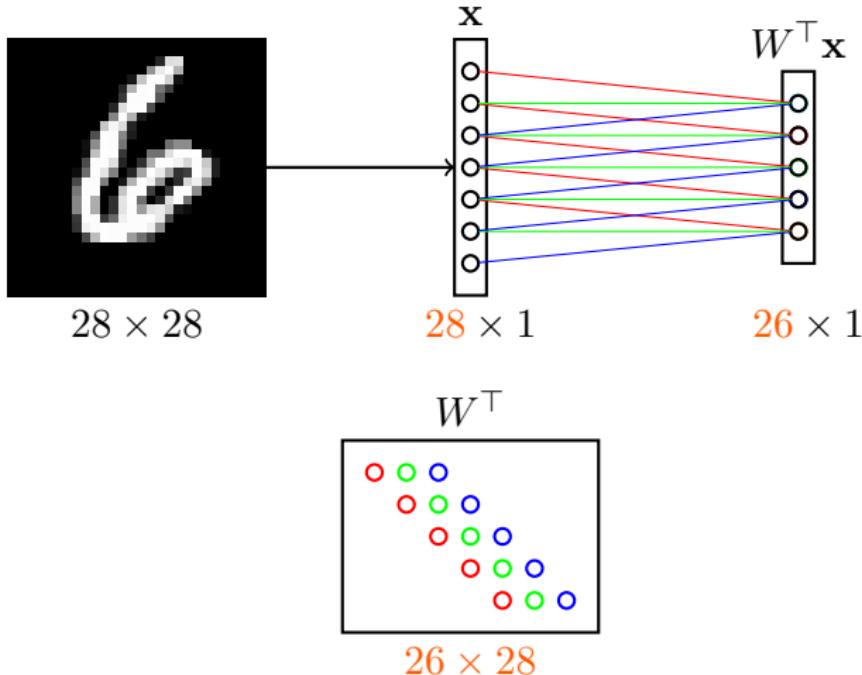
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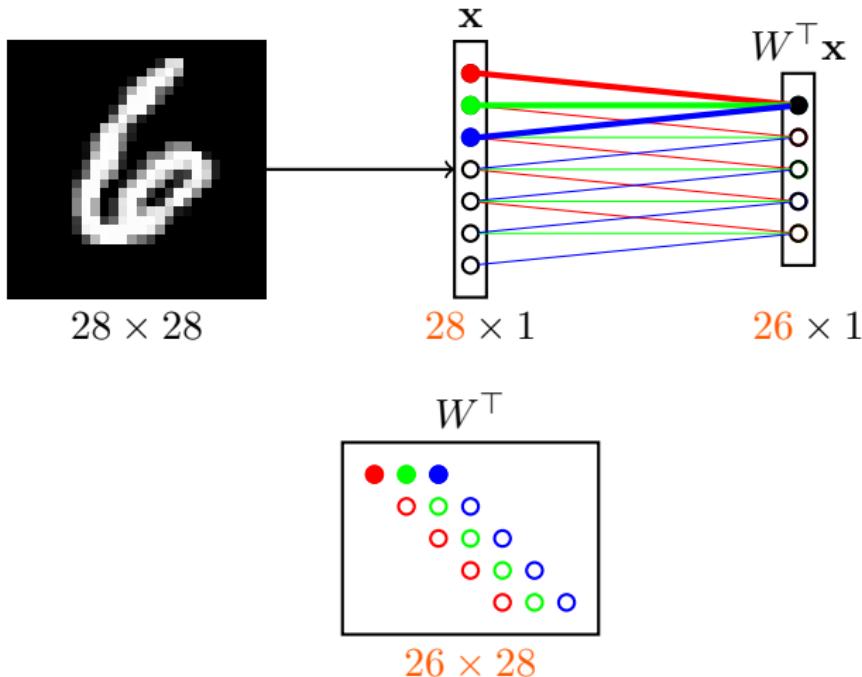
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Toeplitz matrix



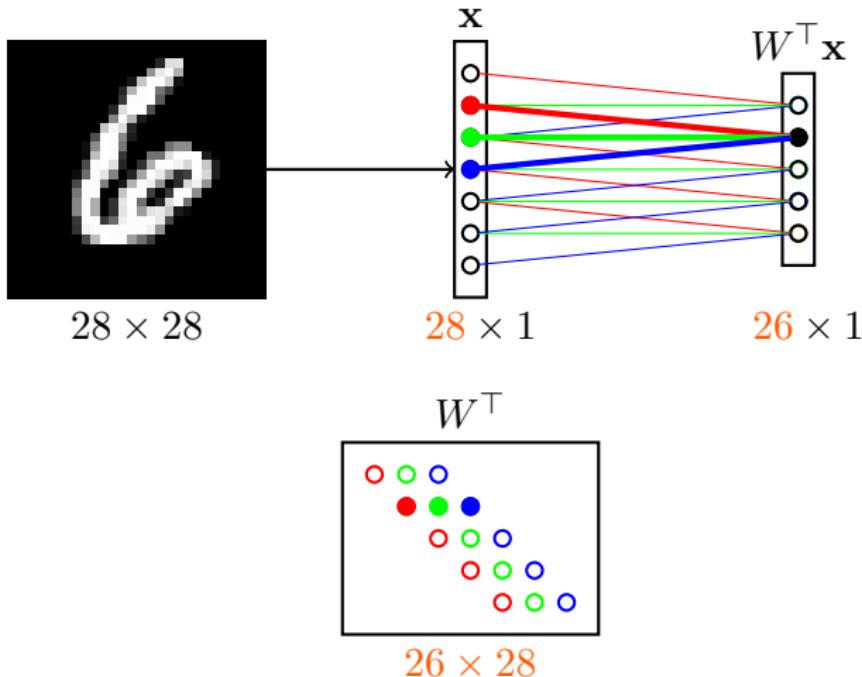
- now, we only refer to one input column; we will repeat
- and all weights having the same color are made equal (**shared**)

Toeplitz matrix \rightarrow convolution



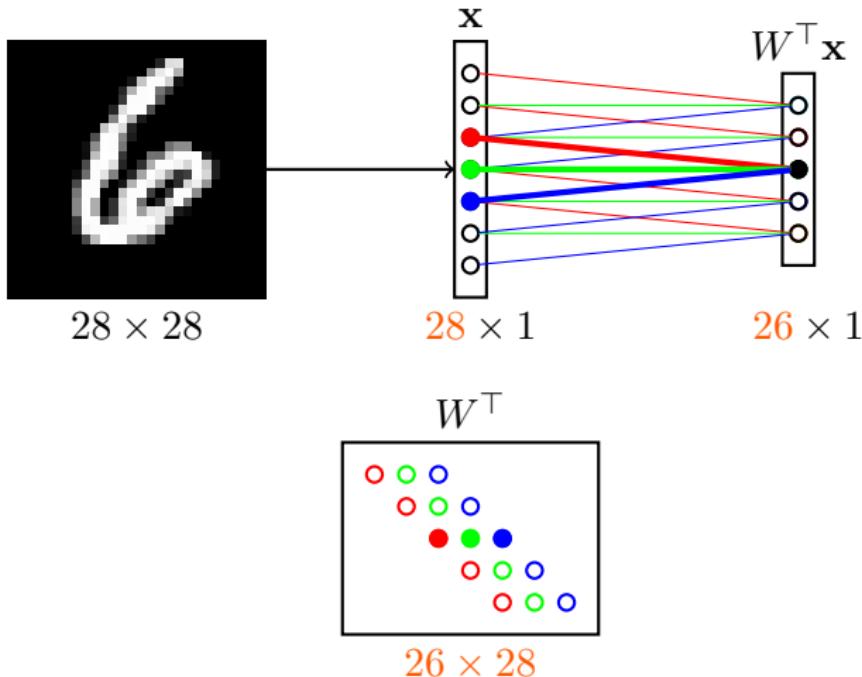
- this can be seen as **shifting** the same weight triplet (**kernel**)
- the set of inputs seen by each cell is its **receptive field**

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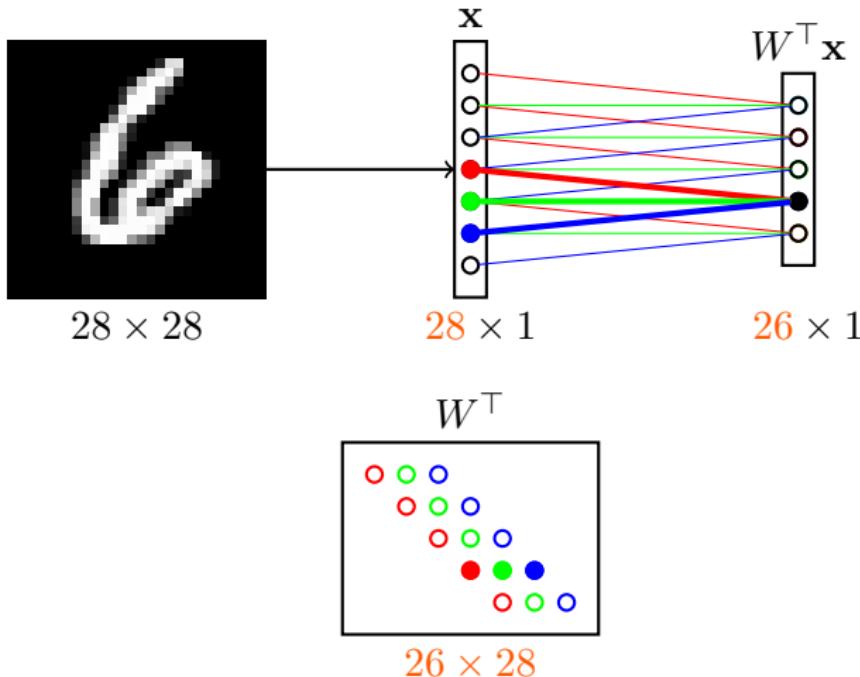
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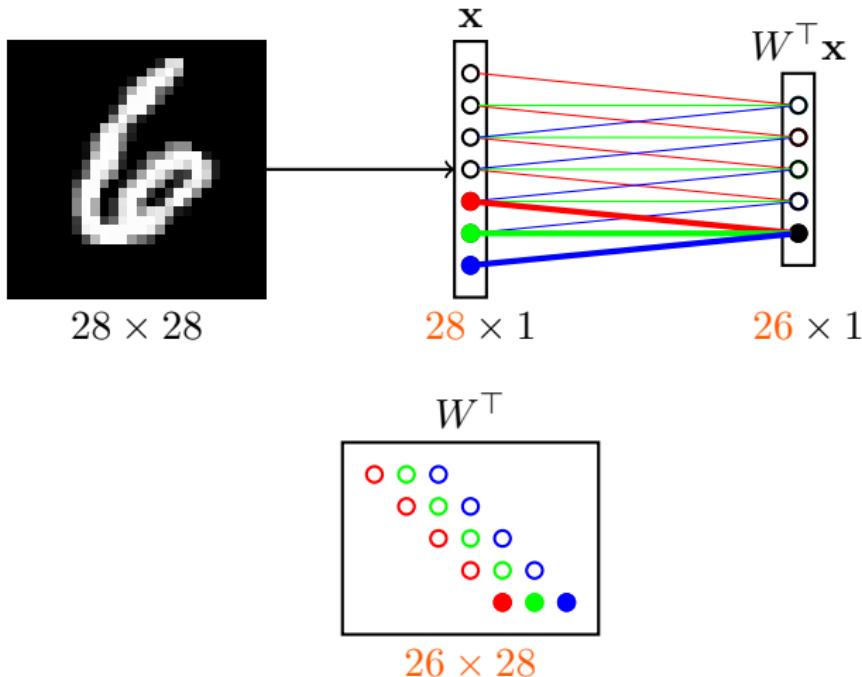
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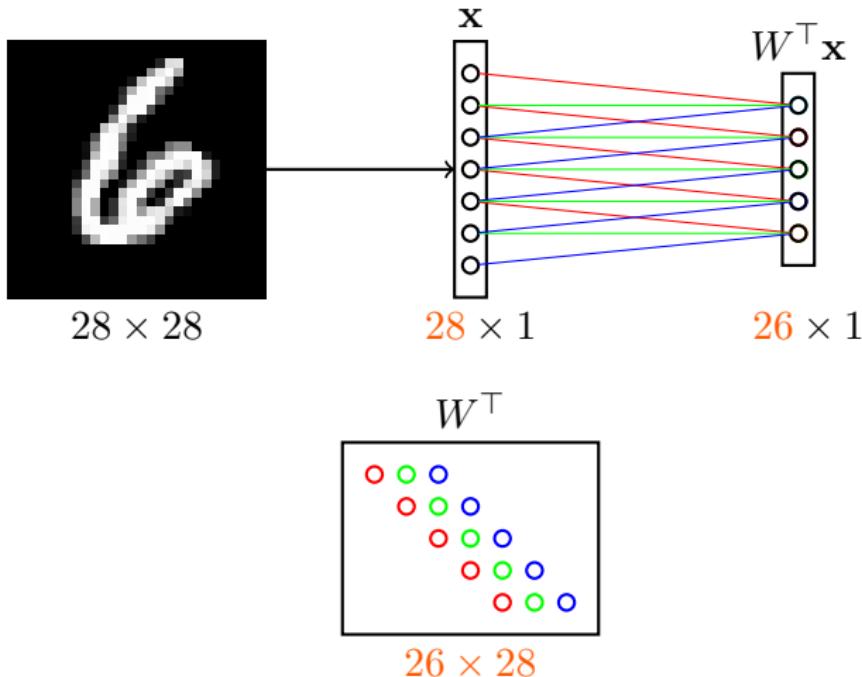
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- this is an 1d **convolution** and generalizes to 2d
- this new mapping is a **convolutional layer**

convolutional networks

convolutional layer

- 1 still linear, still matrix multiplication, just constrained
- 2 local receptive fields → sparse connections between units
- 3 translation equivariant → shared weights
- 4 sparse + shared → regularized: less parameters to learn

convolutional network

- a network of convolutional layers, optionally followed by fully-connected layers
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definition and properties

linear time-invariant (LTI) system

- discrete-time **signal**: $x[n]$, $n \in \mathbb{Z}$
- **system** (filter): $f(x)[n]$, $n \in \mathbb{Z}$
- **translation** (or shift, or delay): $s_k(x)[n] = x[n - k]$, $k \in \mathbb{Z}$
- **linear system**: commutes with linear combination

$$f\left(\sum_i a_i x_i\right) = \sum_i a_i f(x_i)$$

- **time-invariant** system: commutes with translation

$$f(s_k(x)) = s_k(f(x))$$

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LTI system \equiv convolution

- unit impulse $\delta[n] = \mathbb{1}[n = 0]$
- every signal x expressed as

$$x[n] = \sum_k x[k]\delta[n - k] = \sum_k x[k]s_k(\delta)[n]$$

- if f is LTI with impulse response $h = f(\delta)$, then $f(x) = x * h$:

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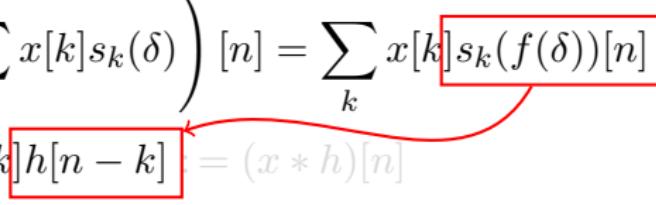
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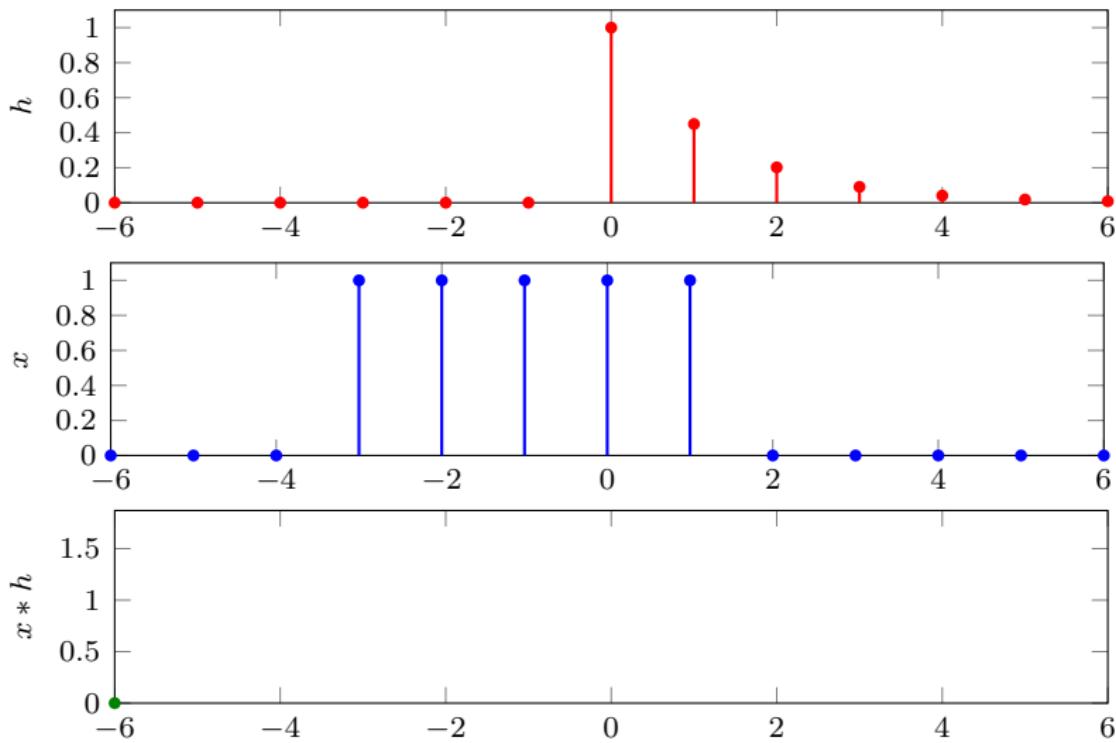
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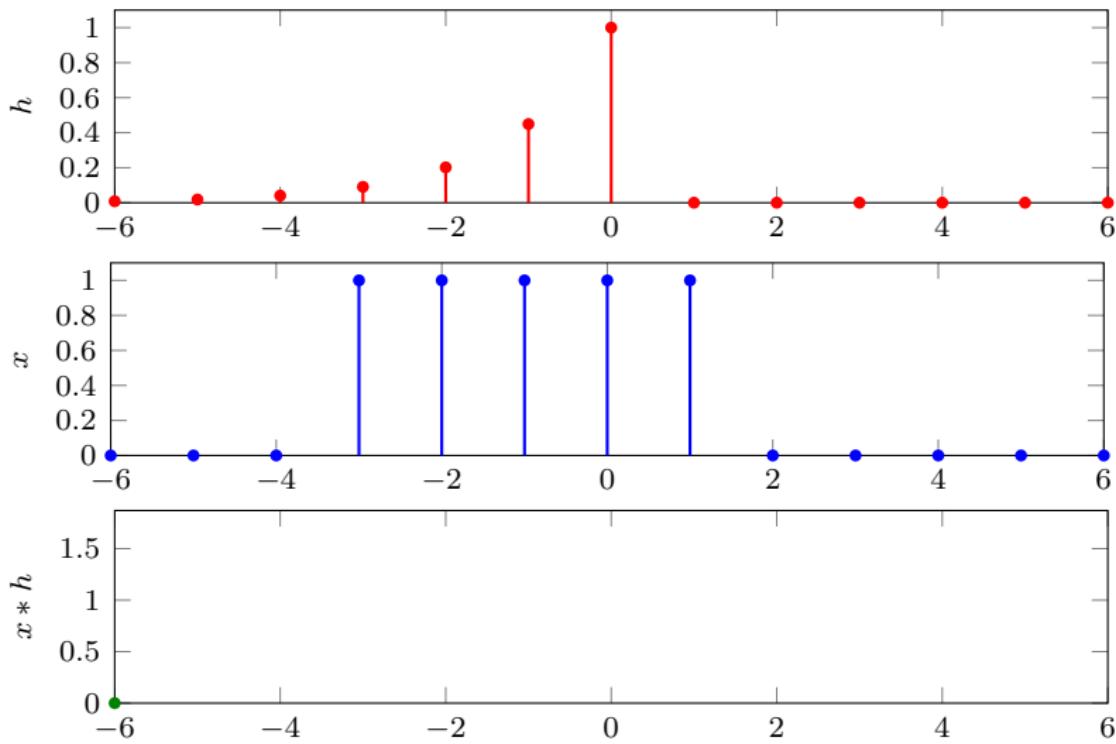
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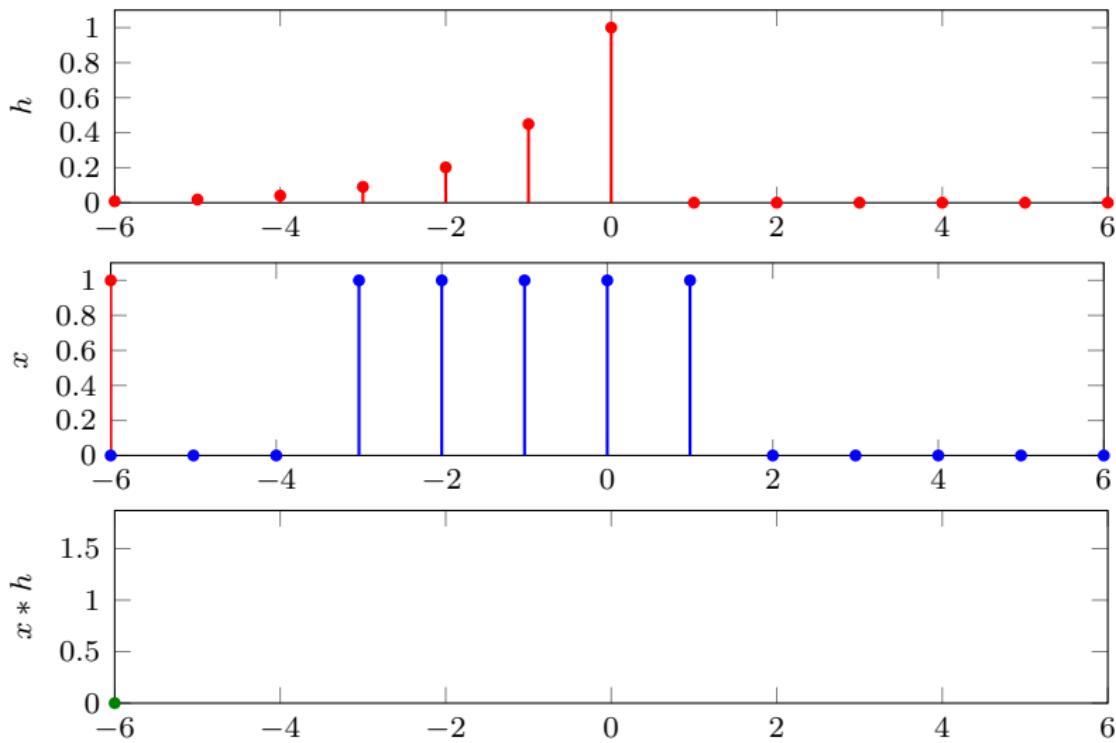
1d convolution



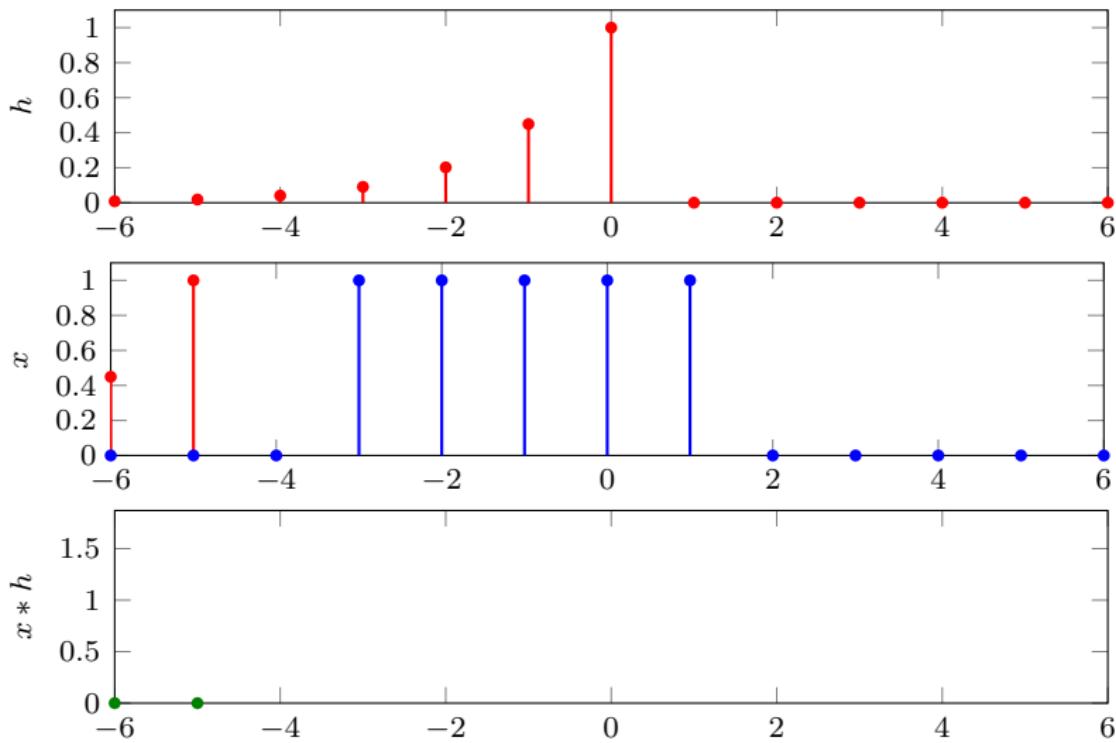
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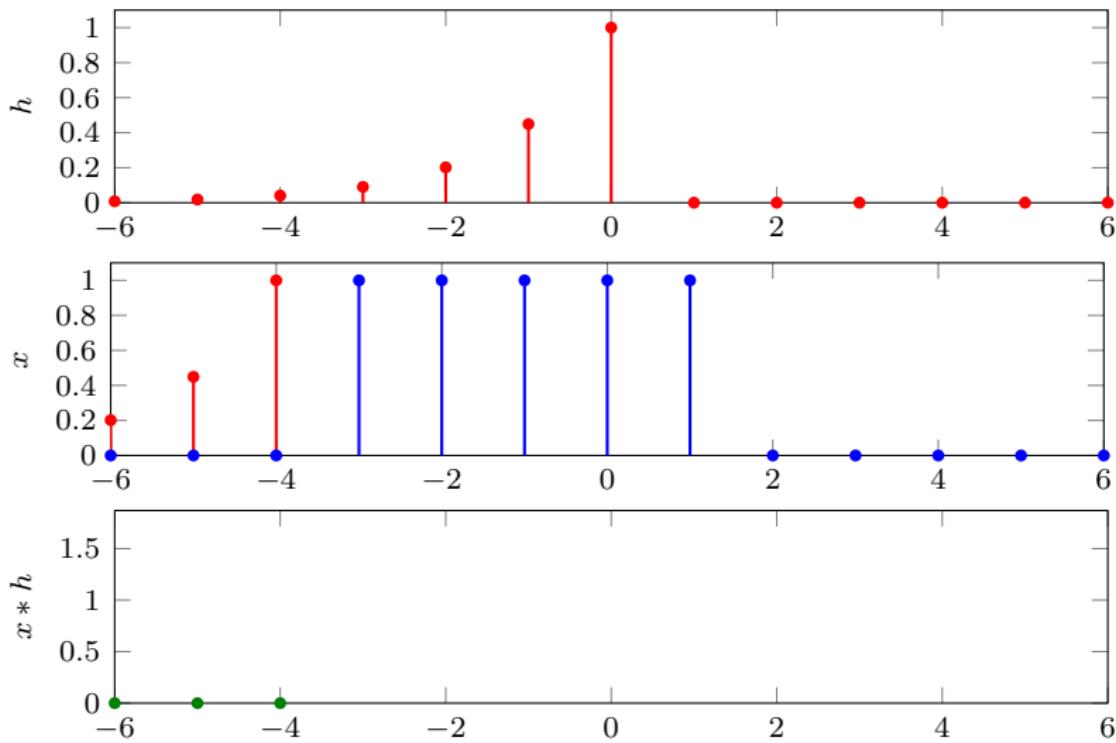
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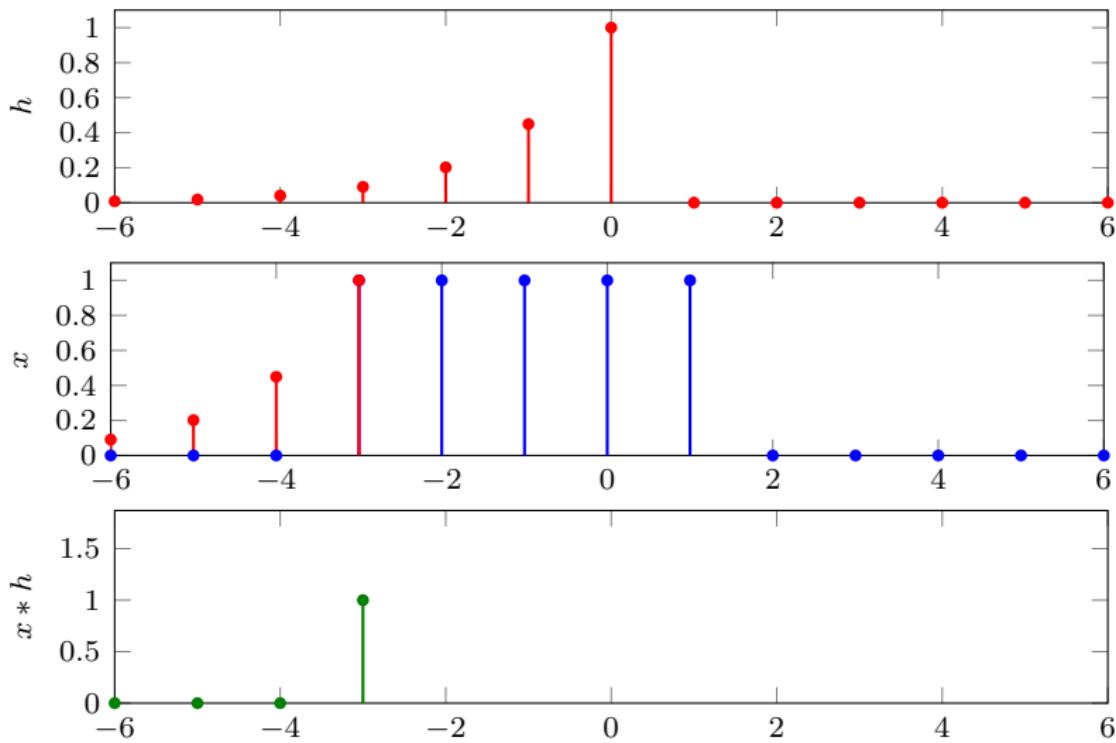
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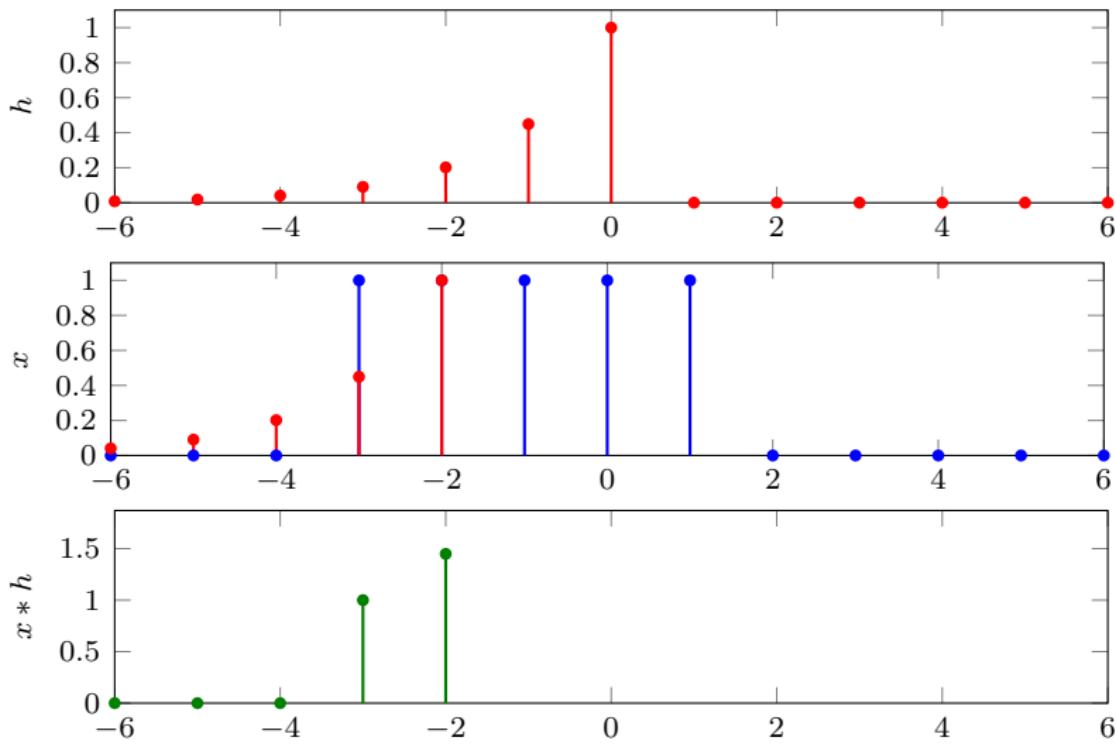
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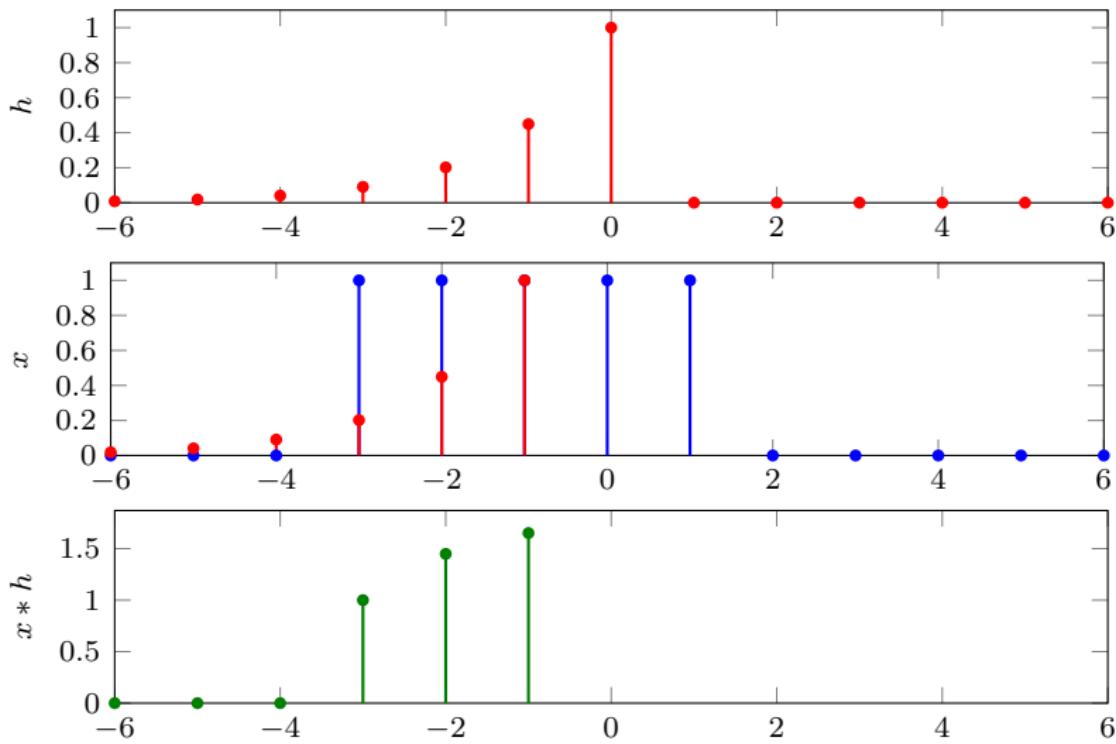
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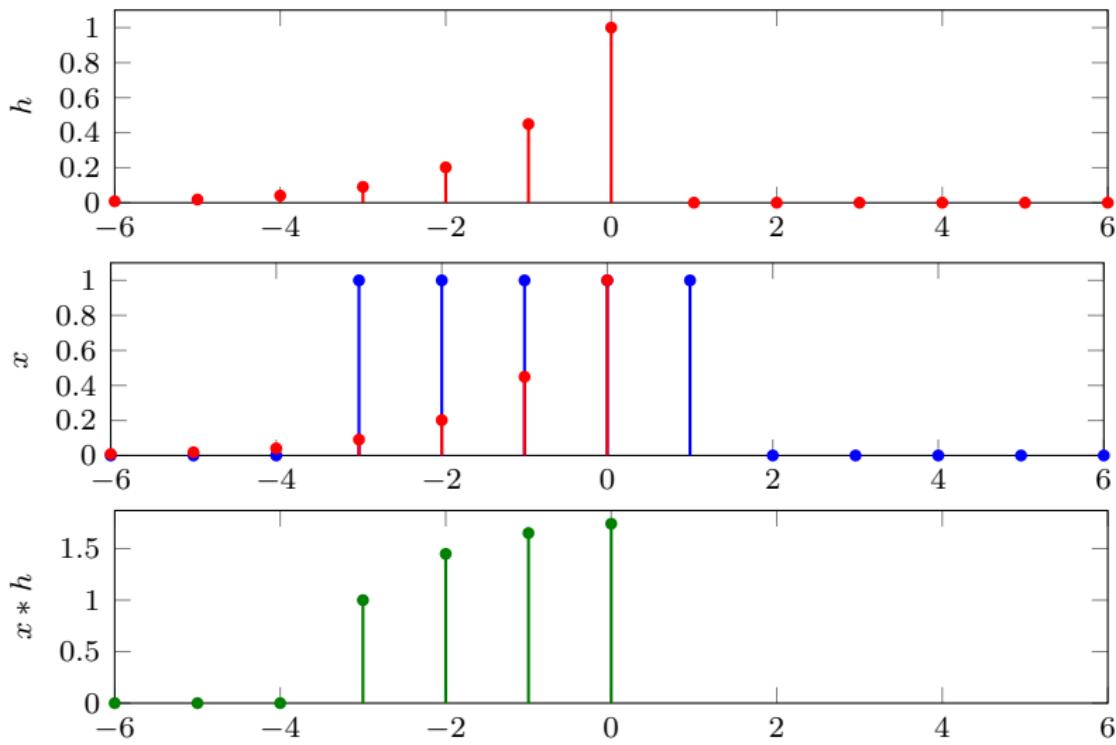
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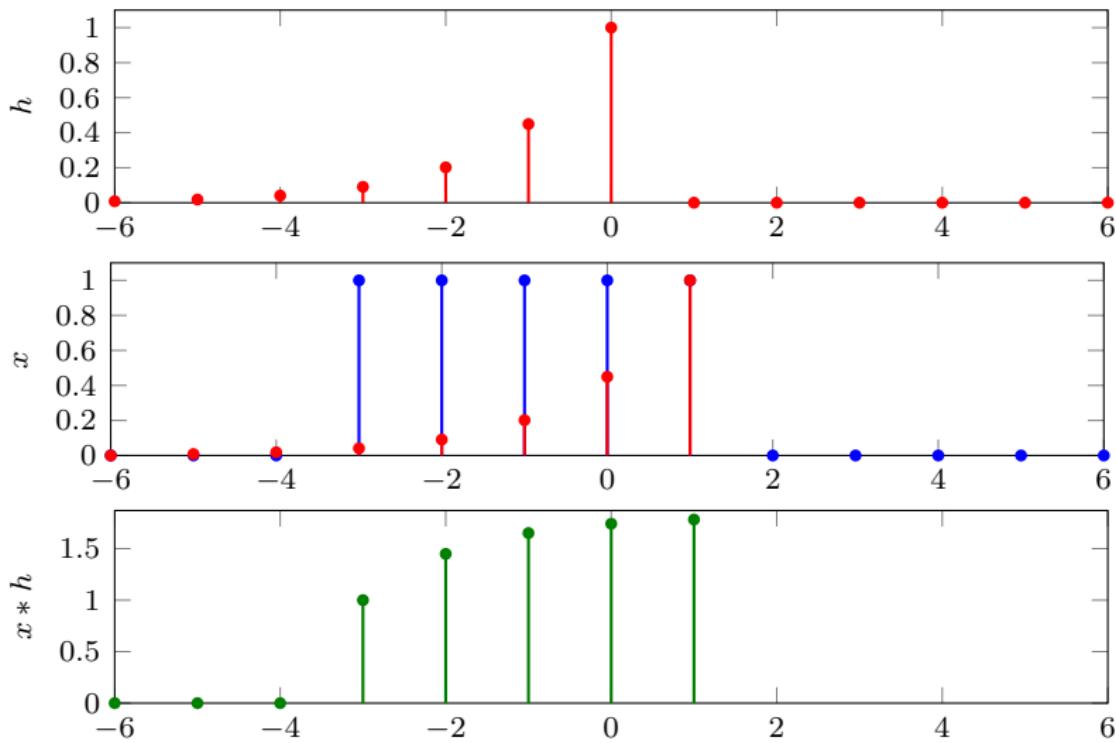
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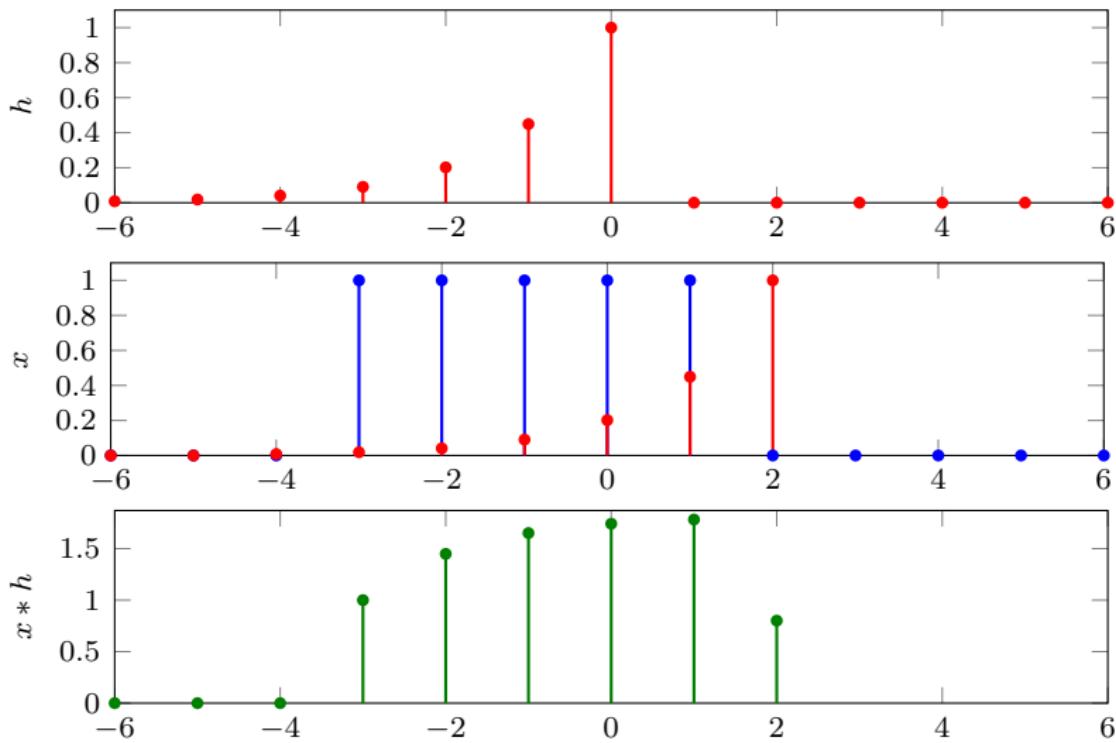
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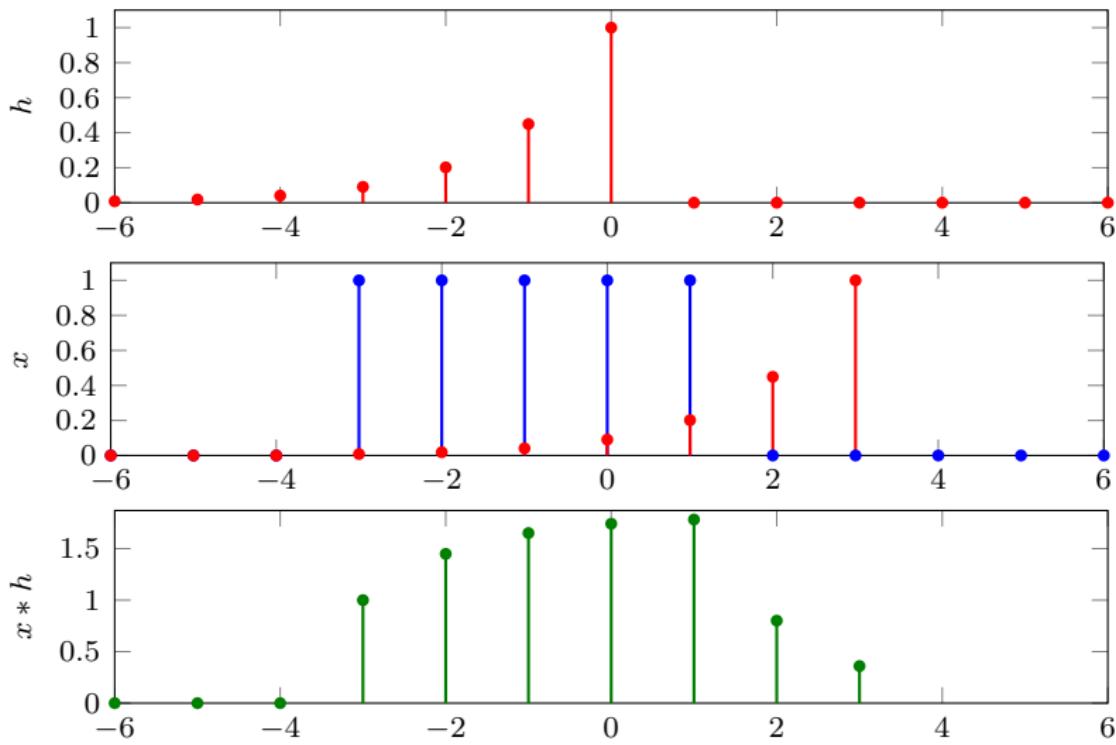
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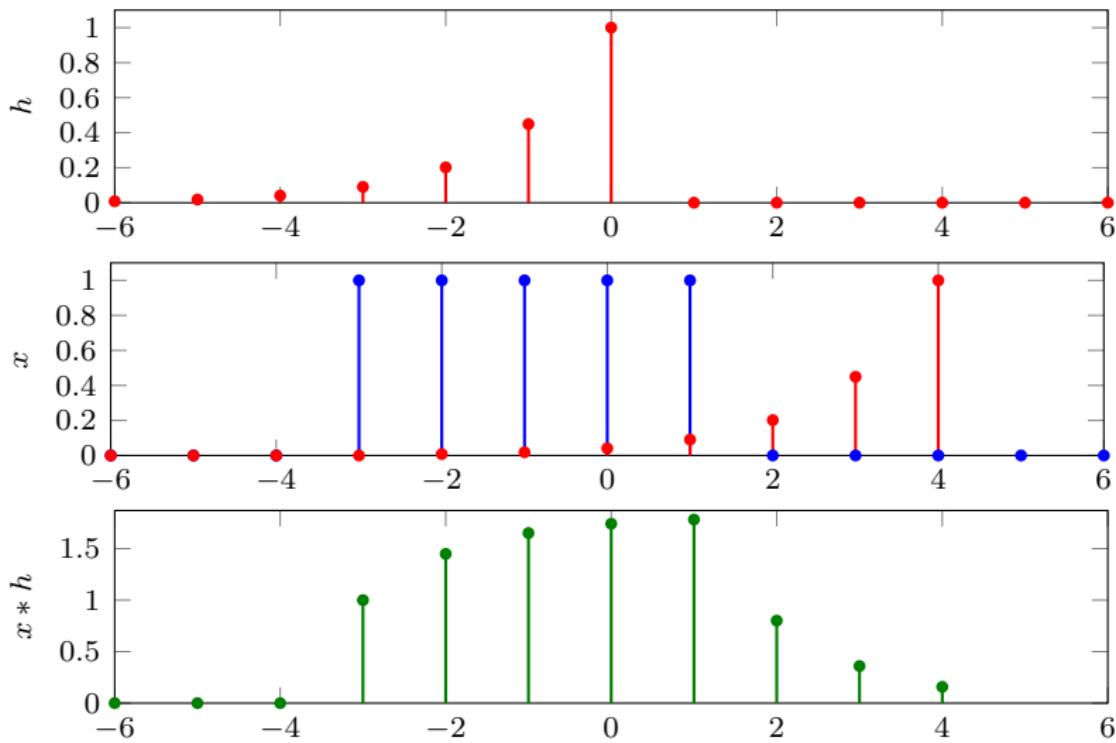
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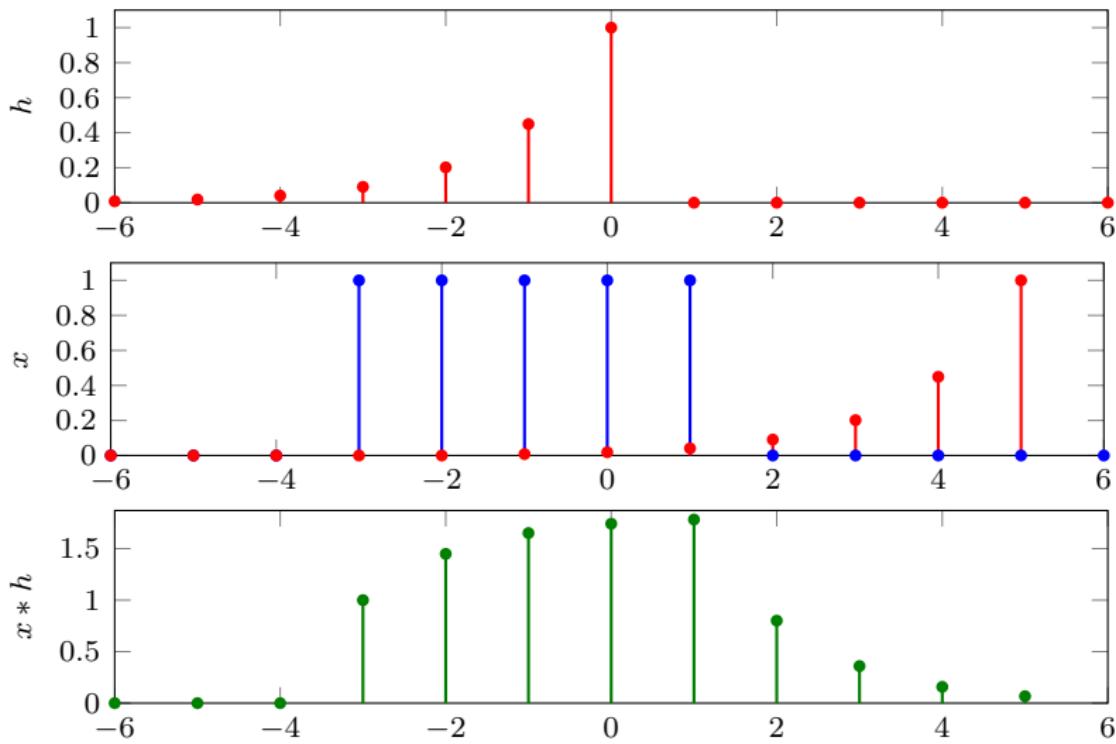
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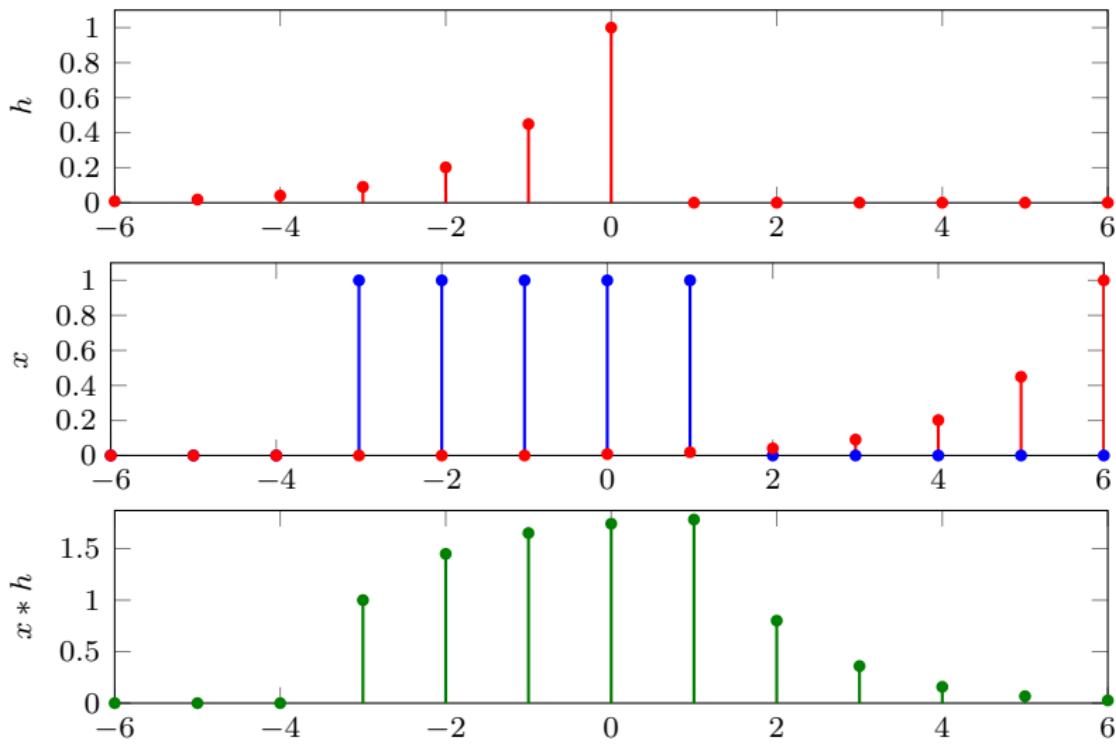
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invariance vs. equivariance

- time invariance: invariance to **absolute** time (or position)
- translation (or shift) equivariance: equivariance to **relative** time (or position)
- despite confusion, both mean the same thing: **system commutes with translation**

$$f(s_k(x)) = s_k(f(x))$$

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finite impulse response (FIR)

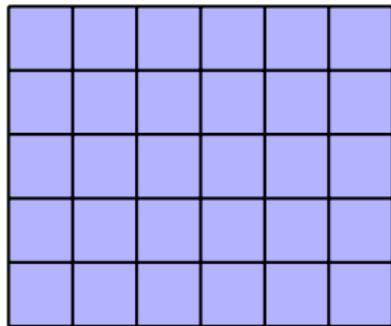
- an FIR system has impulse response h of finite duration (or spatial extent), because it settles to zero in finite time (extent) from the input impulse
- “sparse connections and local receptive fields” mean exactly that h is of finite duration (extent)
- we assume this in the following, starting with a 2d extension, where we write $x[\mathbf{n}]$, $\mathbf{n} \in \mathbb{Z}^2$

2d convolution

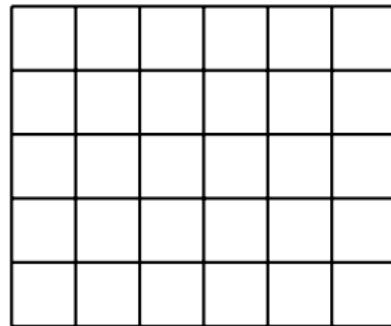
1	2	3
4	5	6
7	8	9

h

$$(x * h)[\mathbf{n}] = \sum_{\mathbf{k}} x[\mathbf{k}]h[\mathbf{n} - \mathbf{k}]$$
$$= \sum_{\mathbf{k}} h[\mathbf{k}]x[\mathbf{n} - \mathbf{k}]$$



x



$x * h$

2d convolution

1	2	3
4	5	6
7	8	9

h

$$(x * h)[\mathbf{n}] = \sum_{\mathbf{k}} x[\mathbf{k}]h[\mathbf{n} - \mathbf{k}]$$
$$= \sum_{\mathbf{k}} h[\mathbf{k}]x[\mathbf{n} - \mathbf{k}]$$

9	8	7				
6	5	4				
3	2	1				

x

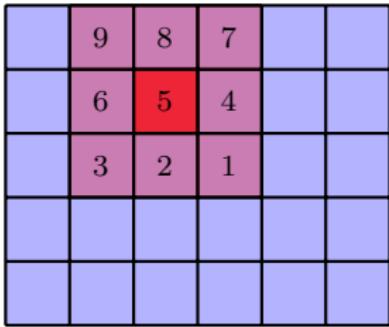
$x * h$

2d convolution

$$\begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{matrix}$$

h

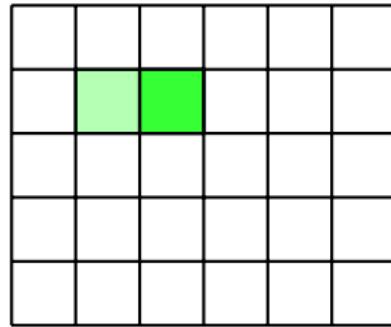
$$(x * h)[\mathbf{n}] = \sum_{\mathbf{k}} x[\mathbf{k}]h[\mathbf{n} - \mathbf{k}]$$
$$= \sum_{\mathbf{k}} h[\mathbf{k}]x[\mathbf{n} - \mathbf{k}]$$



A 5x5 grid of colored squares representing the input image x . The colors are blue, purple, and red. The values are arranged as follows:

6	9	8	7		
3	2	1			

x



A 5x5 grid of colored squares representing the convolved image $x * h$. The colors are white, light green, and dark green. The values are arranged as follows:

		1	2	3	
		4	5	6	
		7	8	9	

$x * h$

2d convolution

1	2	3
4	5	6
7	8	9

h

$$(x * h)[\mathbf{n}] = \sum_{\mathbf{k}} x[\mathbf{k}]h[\mathbf{n} - \mathbf{k}]$$
$$= \sum_{\mathbf{k}} h[\mathbf{k}]x[\mathbf{n} - \mathbf{k}]$$

		9	8	7	
		6	5	4	
		3	2	1	

x

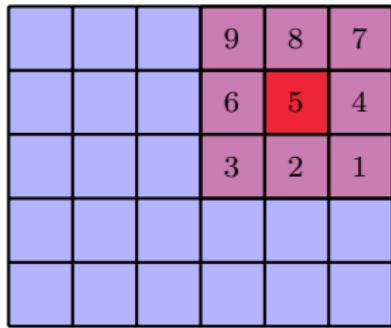
$x * h$

2d convolution

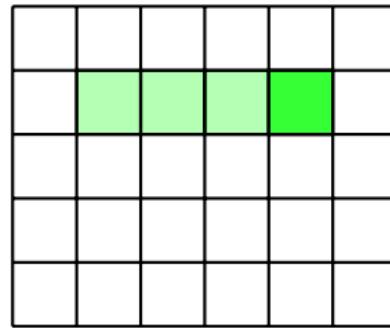
$$\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 & 8 & 9 \\ \hline \end{array}$$

h

$$(x * h)[\mathbf{n}] = \sum_{\mathbf{k}} x[\mathbf{k}]h[\mathbf{n} - \mathbf{k}]$$
$$= \sum_{\mathbf{k}} h[\mathbf{k}]x[\mathbf{n} - \mathbf{k}]$$



x



$x * h$

2d convolution

1	2	3
4	5	6
7	8	9

h

$$(x * h)[\mathbf{n}] = \sum_{\mathbf{k}} x[\mathbf{k}]h[\mathbf{n} - \mathbf{k}]$$
$$= \sum_{\mathbf{k}} h[\mathbf{k}]x[\mathbf{n} - \mathbf{k}]$$

9	8	7				
6	5	4				
3	2	1				

x

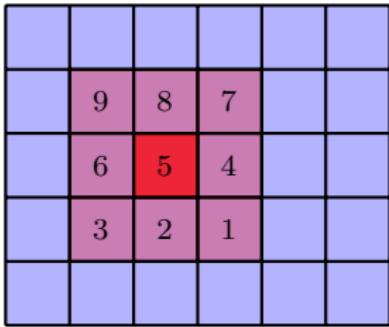
$x * h$

2d convolution

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 & 8 & 9 \\ \hline \end{array}$$

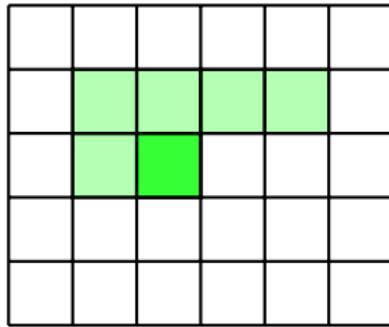
h

$$(x * h)[\mathbf{n}] = \sum_{\mathbf{k}} x[\mathbf{k}]h[\mathbf{n} - \mathbf{k}]$$
$$= \sum_{\mathbf{k}} h[\mathbf{k}]x[\mathbf{n} - \mathbf{k}]$$



A 5x5 grid of squares. The bottom-left 3x3 subgrid contains values 9, 8, 7; 6, 5, 4; and 3, 2, 1 respectively, all in red/pink. The rest of the grid is filled with light blue squares.

x



A 5x5 grid of squares. The bottom-left 3x3 subgrid contains values 9, 8, 7; 6, 5, 4; and 3, 2, 1 respectively, all in green. The rest of the grid is filled with white squares.

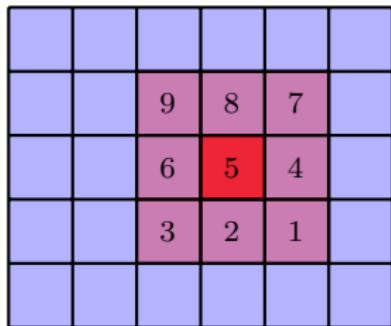
$x * h$

2d convolution

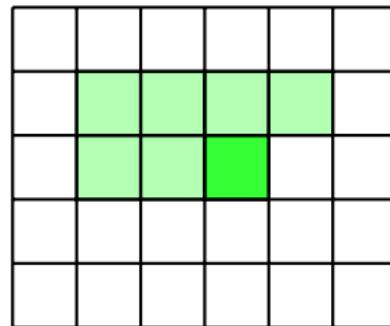
1	2	3
4	5	6
7	8	9

h

$$(x * h)[\mathbf{n}] = \sum_{\mathbf{k}} x[\mathbf{k}]h[\mathbf{n} - \mathbf{k}]$$
$$= \sum_{\mathbf{k}} h[\mathbf{k}]x[\mathbf{n} - \mathbf{k}]$$



x



$x * h$

2d convolution

1	2	3
4	5	6
7	8	9

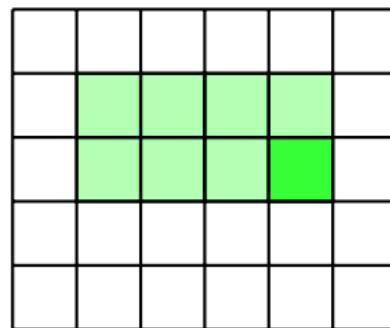
h

$$(x * h)[\mathbf{n}] = \sum_{\mathbf{k}} x[\mathbf{k}]h[\mathbf{n} - \mathbf{k}]$$

$$= \sum_{\mathbf{k}} h[\mathbf{k}]x[\mathbf{n} - \mathbf{k}]$$



x



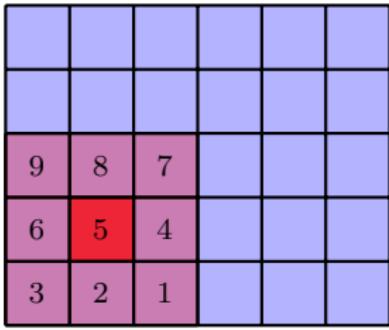
$x * h$

2d convolution

$$\begin{matrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{matrix}$$

h

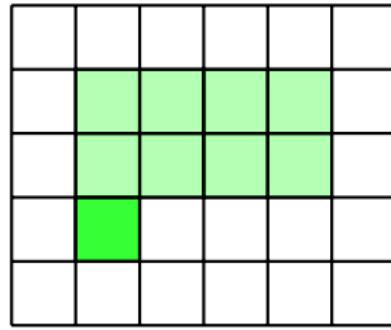
$$(x * h)[\mathbf{n}] = \sum_{\mathbf{k}} x[\mathbf{k}]h[\mathbf{n} - \mathbf{k}]$$
$$= \sum_{\mathbf{k}} h[\mathbf{k}]x[\mathbf{n} - \mathbf{k}]$$



A 3x3 input matrix x with values:

9	8	7
6	5	4
3	2	1

x



A 5x5 output matrix $x * h$ with values:

	green	green	green	green
	green	green	green	green

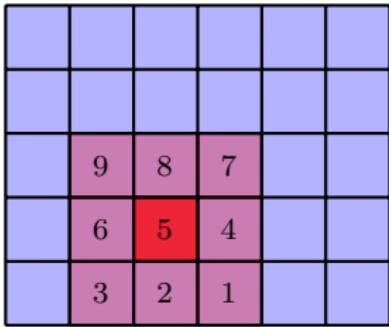
$x * h$

2d convolution

$$\begin{array}{|c|c|c|} \hline 1 & 2 & 3 \\ \hline 4 & 5 & 6 \\ \hline 7 & 8 & 9 \\ \hline \end{array}$$

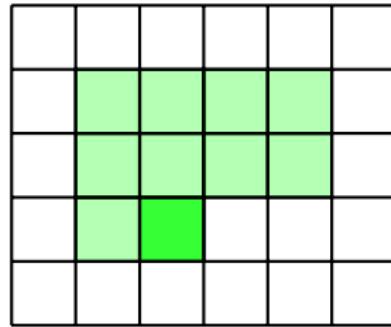
h

$$(x * h)[\mathbf{n}] = \sum_{\mathbf{k}} x[\mathbf{k}]h[\mathbf{n} - \mathbf{k}]$$
$$= \sum_{\mathbf{k}} h[\mathbf{k}]x[\mathbf{n} - \mathbf{k}]$$



The input image x is a 5x5 grid of blue squares. It contains a 3x3 subgrid of red squares in the center, labeled with values 9, 8, 7, 6, 5, 4, 3, 2, and 1 respectively.

x



The convolved image $x * h$ is a 5x5 grid. It has a 3x3 green subgrid in the center, with the bottom-right square colored bright green. All other squares are white.

$x * h$

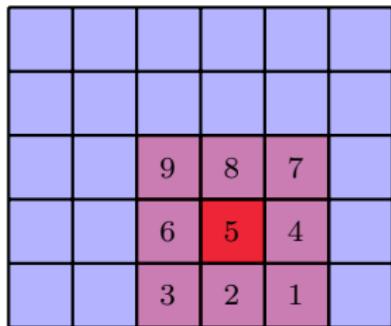
2d convolution

1	2	3
4	5	6
7	8	9

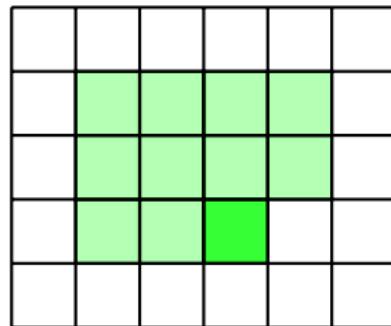
h

$$(x * h)[\mathbf{n}] = \sum_{\mathbf{k}} x[\mathbf{k}]h[\mathbf{n} - \mathbf{k}]$$

$$= \sum_{\mathbf{k}} h[\mathbf{k}]x[\mathbf{n} - \mathbf{k}]$$



x



$$x * h$$

2d convolution

1	2	3
4	5	6
7	8	9

h

$$(x * h)[\mathbf{n}] = \sum_{\mathbf{k}} x[\mathbf{k}]h[\mathbf{n} - \mathbf{k}]$$

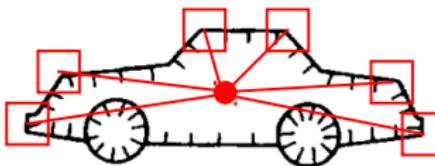
$$= \sum_{\mathbf{k}} h[\mathbf{k}]x[\mathbf{n} - \mathbf{k}]$$

			9	8	7	
			6	5	4	
			3	2	1	

x

$x * h$

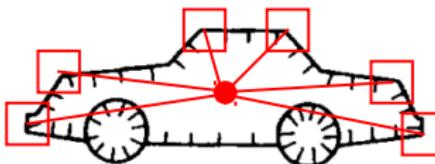
remember Hough in translation space?



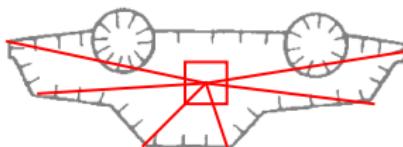
model image

- **model:** record coordinates relative to reference point
- **test:** votes for all possible coordinates of reference point
- impulse response is the **flipped** pattern to be detected

remember Hough in translation space?



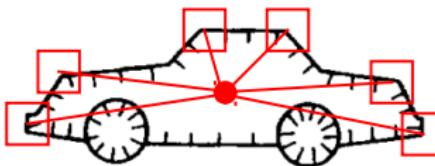
model image



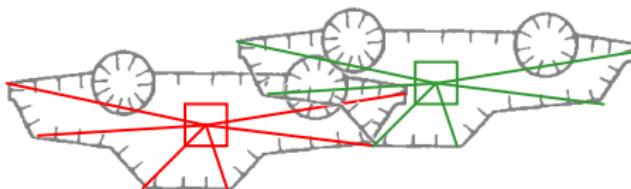
test image

- **model:** record coordinates relative to reference point
- **test:** votes for all possible coordinates of reference point
- impulse response is the **flipped** pattern to be detected

remember Hough in translation space?



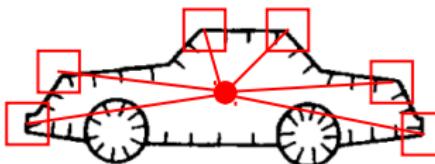
model image



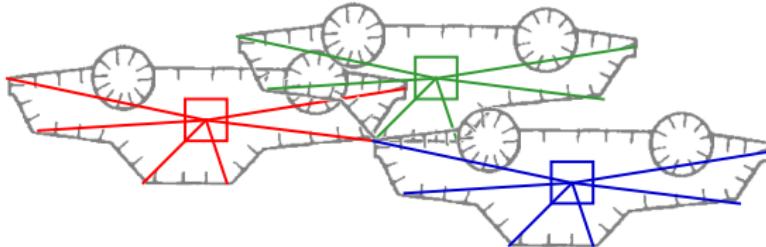
test image

- **model:** record coordinates relative to reference point
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- impulse response is the **flipped** pattern to be detected

remember Hough in translation space?



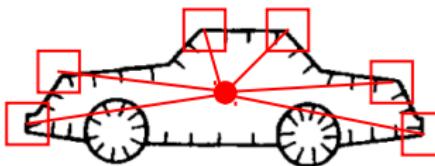
model image



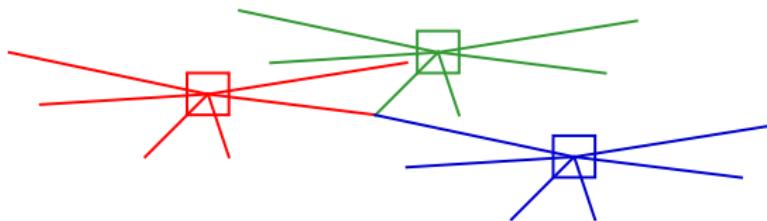
test image

- **model:** record coordinates relative to reference point
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- impulse response is the **flipped** pattern to be detected

remember Hough in translation space?



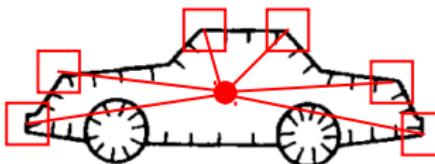
model image



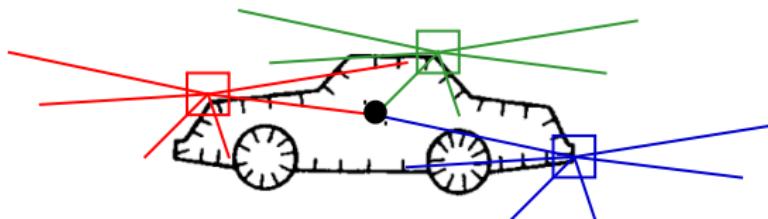
test image

- **model:** record coordinates relative to reference point
- **test:** votes for all possible coordinates of reference point
- impulse response is the **flipped** pattern to be detected

remember Hough in translation space?



model image



test image

- **model:** record coordinates relative to reference point
- **test:** votes for all possible coordinates of reference point
- impulse response is the **flipped** pattern to be detected

cross-correlation

- convolution is commutative

$$(x * h)[\mathbf{n}] := \sum_{\mathbf{k}} x[\mathbf{k}]h[\mathbf{n} - \mathbf{k}] = \sum_{\mathbf{k}} h[\mathbf{k}]x[\mathbf{n} - \mathbf{k}] = (h * x)[\mathbf{n}]$$

- cross-correlation is not

$$(h \star x)[\mathbf{n}] := \sum_{\mathbf{k}} h[\mathbf{k}]x[\mathbf{k} + \mathbf{n}] = \sum_{\mathbf{k}} x[\mathbf{k}]h[\mathbf{k} - \mathbf{n}] = (x \star h)[-\mathbf{n}]$$

- both are LTI; the only difference is that in cross-correlation, h refers to the **flipped** impulse response
- but if h is even ($h[n] = h[-n]$), then $h \star x = x * h = h * x$
- in the following, we use **cross-correlation** $w \star x$ or **convolution** $x * h$, where $h[n] = w[-n]$ is the impulse response
- we call w the **kernel** of the operation

cross-correlation

- convolution is commutative

$$(x * h)[\mathbf{n}] := \sum_{\mathbf{k}} x[\mathbf{k}] h[\mathbf{n} - \mathbf{k}] = \sum_{\mathbf{k}} h[\mathbf{k}] x[\mathbf{n} - \mathbf{k}] = (h * x)[\mathbf{n}]$$

- cross-correlation is not

$$(h \star x)[\mathbf{n}] := \sum_{\mathbf{k}} h[\mathbf{k}] x[\mathbf{k} + \mathbf{n}] = \sum_{\mathbf{k}} x[\mathbf{k}] h[\mathbf{k} - \mathbf{n}] = (x \star h)[-\mathbf{n}]$$

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cross-correlation

- convolution is commutative

$$(x * h)[\mathbf{n}] := \sum_{\mathbf{k}} x[\mathbf{k}] h[\mathbf{n} - \mathbf{k}] = \sum_{\mathbf{k}} h[\mathbf{k}] x[\mathbf{n} - \mathbf{k}] = (h * x)[\mathbf{n}]$$

- cross-correlation is not

$$(h \star x)[\mathbf{n}] := \sum_{\mathbf{k}} h[\mathbf{k}] x[\mathbf{k} + \mathbf{n}] = \sum_{\mathbf{k}} x[\mathbf{k}] h[\mathbf{k} - \mathbf{n}] = (x \star h)[-\mathbf{n}]$$

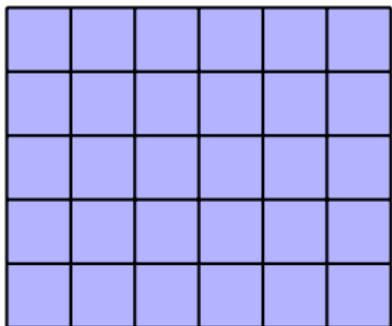
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- but if h is even ($h[n] = h[-n]$), then $h \star x = x * h = h * x$
- in the following, we use **cross-correlation** $w \star x$ or **convolution** $x * h$, where $h[n] = w[-n]$ is the impulse response
- we call w the **kernel** of the operation

2d convolution, again

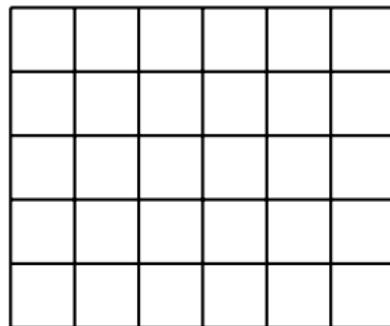
1	2	3
4	5	6
7	8	9

w

$$(w \star x)[\mathbf{n}] = \sum_{\mathbf{k}} w[\mathbf{k}]x[\mathbf{k} + \mathbf{n}]$$
$$= \sum_{\mathbf{k}} x[\mathbf{k}]w[\mathbf{k} - \mathbf{n}]$$



x



$w \star x$

2d convolution, again

1	2	3
4	5	6
7	8	9

w

$$(w \star x)[\mathbf{n}] = \sum_{\mathbf{k}} w[\mathbf{k}]x[\mathbf{k} + \mathbf{n}]$$
$$= \sum_{\mathbf{k}} x[\mathbf{k}]w[\mathbf{k} - \mathbf{n}]$$

1	2	3			
4	5	6			
7	8	9			

x

w \star x

2d convolution, again

1	2	3
4	5	6
7	8	9

w

$$(w \star x)[\mathbf{n}] = \sum_{\mathbf{k}} w[\mathbf{k}]x[\mathbf{k} + \mathbf{n}]$$
$$= \sum_{\mathbf{k}} x[\mathbf{k}]w[\mathbf{k} - \mathbf{n}]$$

	1	2	3		
	4	5	6		
	7	8	9		

x

w \star x

2d convolution, again

1	2	3
4	5	6
7	8	9

w

$$(w \star x)[\mathbf{n}] = \sum_{\mathbf{k}} w[\mathbf{k}]x[\mathbf{k} + \mathbf{n}]$$
$$= \sum_{\mathbf{k}} x[\mathbf{k}]w[\mathbf{k} - \mathbf{n}]$$

		1	2	3	
		4	5	6	
		7	8	9	

x

w \star x

2d convolution, again

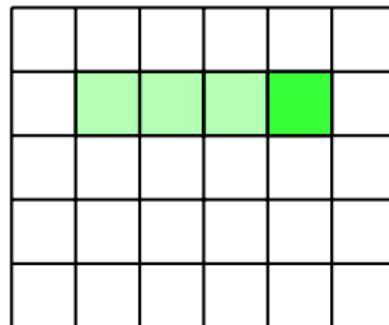
1	2	3
4	5	6
7	8	9

w

$$(w \star x)[\mathbf{n}] = \sum_{\mathbf{k}} w[\mathbf{k}]x[\mathbf{k} + \mathbf{n}]$$
$$= \sum_{\mathbf{k}} x[\mathbf{k}]w[\mathbf{k} - \mathbf{n}]$$



x



w \star *x*

2d convolution, again

1	2	3
4	5	6
7	8	9

w

$$(w \star x)[\mathbf{n}] = \sum_{\mathbf{k}} w[\mathbf{k}]x[\mathbf{k} + \mathbf{n}]$$
$$= \sum_{\mathbf{k}} x[\mathbf{k}]w[\mathbf{k} - \mathbf{n}]$$

1	2	3				
4	5	6				
7	8	9				

x

w \star x

2d convolution, again

1	2	3
4	5	6
7	8	9

w

$$(w \star x)[\mathbf{n}] = \sum_{\mathbf{k}} w[\mathbf{k}]x[\mathbf{k} + \mathbf{n}]$$
$$= \sum_{\mathbf{k}} x[\mathbf{k}]w[\mathbf{k} - \mathbf{n}]$$

1	2	3			
4	5	6			
7	8	9			

x

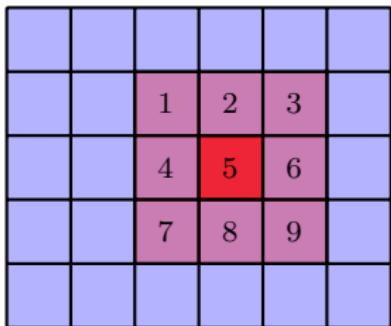
w \star x

2d convolution, again

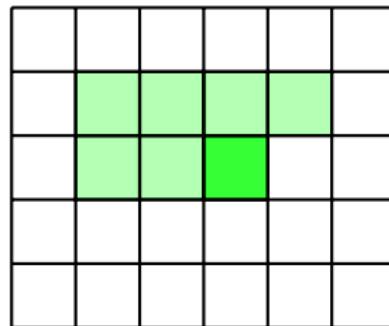
1	2	3
4	5	6
7	8	9

w

$$(w \star x)[\mathbf{n}] = \sum_{\mathbf{k}} w[\mathbf{k}]x[\mathbf{k} + \mathbf{n}]$$
$$= \sum_{\mathbf{k}} x[\mathbf{k}]w[\mathbf{k} - \mathbf{n}]$$



x



w \star *x*

2d convolution, again

1	2	3
4	5	6
7	8	9

w

$$(w \star x)[\mathbf{n}] = \sum_{\mathbf{k}} w[\mathbf{k}]x[\mathbf{k} + \mathbf{n}]$$
$$= \sum_{\mathbf{k}} x[\mathbf{k}]w[\mathbf{k} - \mathbf{n}]$$

			1	2	3
			4	5	6
			7	8	9

x

w \star x

2d convolution, again

1	2	3
4	5	6
7	8	9

w

$$(w \star x)[\mathbf{n}] = \sum_{\mathbf{k}} w[\mathbf{k}]x[\mathbf{k} + \mathbf{n}]$$
$$= \sum_{\mathbf{k}} x[\mathbf{k}]w[\mathbf{k} - \mathbf{n}]$$

1	2	3			
4	5	6			
7	8	9			

x

w \star x

2d convolution, again

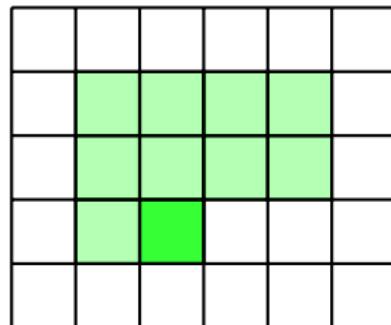
1	2	3
4	5	6
7	8	9

w

$$(w \star x)[\mathbf{n}] = \sum_{\mathbf{k}} w[\mathbf{k}]x[\mathbf{k} + \mathbf{n}]$$
$$= \sum_{\mathbf{k}} x[\mathbf{k}]w[\mathbf{k} - \mathbf{n}]$$



x



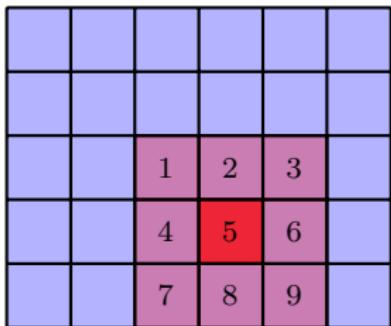
w \star *x*

2d convolution, again

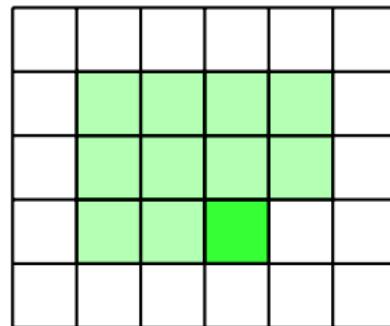
1	2	3
4	5	6
7	8	9

w

$$(w \star x)[\mathbf{n}] = \sum_{\mathbf{k}} w[\mathbf{k}]x[\mathbf{k} + \mathbf{n}]$$
$$= \sum_{\mathbf{k}} x[\mathbf{k}]w[\mathbf{k} - \mathbf{n}]$$



x



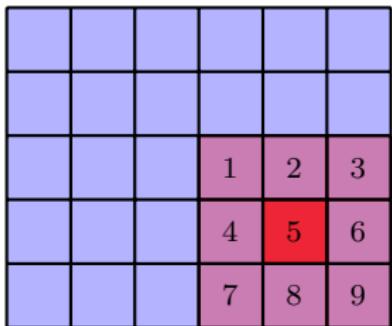
w \star x

2d convolution, again

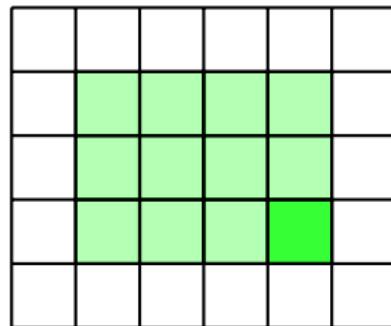
1	2	3
4	5	6
7	8	9

w

$$(w \star x)[\mathbf{n}] = \sum_{\mathbf{k}} w[\mathbf{k}]x[\mathbf{k} + \mathbf{n}]$$
$$= \sum_{\mathbf{k}} x[\mathbf{k}]w[\mathbf{k} - \mathbf{n}]$$



x



w \star *x*

features

- something is still missing: so far we had activations \mathbf{a} and outputs \mathbf{y} of the form

$$\mathbf{a} = W^\top \mathbf{x} + \mathbf{b}, \quad \mathbf{y} = h(\mathbf{a}) = h(W^\top \mathbf{x} + \mathbf{b})$$

where \mathbf{x} is the input, $W = (\mathbf{w}_1, \dots, \mathbf{w}_k)$ a weight matrix and \mathbf{b} a bias

- the elements of \mathbf{x} , \mathbf{a} , \mathbf{b} and \mathbf{y} were representing **features** (or cells);
the elements of W were representing **connections**
- now we have x as a 2d array, w as a 2d kernel, but no features yet

feature maps

- now \mathbf{b} remains a vector but \mathbf{x} , \mathbf{a} , \mathbf{y} become **3d tensors** with input feature i and output feature j at spatial position \mathbf{n} represented by

$$x_i[\mathbf{n}], \quad a_j[\mathbf{n}], \quad b_j, \quad y_j[\mathbf{n}]$$

- x_i and y_j are 2d arrays we call **feature maps**, each corresponding to one feature; and a_j a 2d array we call **activation map**
- if x_i refers to the input image, there is just one feature that is the image intensity of a monochrome image, or three features corresponding to the three **channels** of a color image
- W becomes a **4d tensor** with a connection from input feature i to output feature j at spatial position \mathbf{k} represented by

$$w_{ij}[\mathbf{k}]$$

feature maps

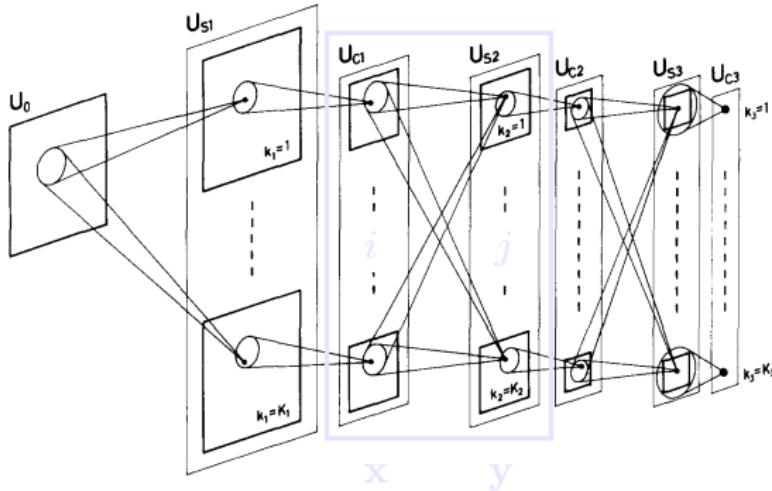
- now \mathbf{b} remains a vector but \mathbf{x} , \mathbf{a} , \mathbf{y} become **3d tensors** with input feature i and output feature j at spatial position \mathbf{n} represented by

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convolution on feature maps



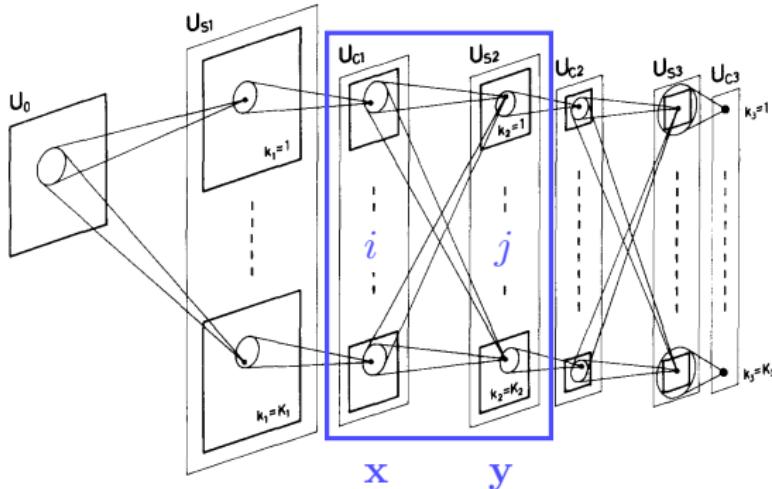
- matrix multiplication and convolution combined

$$\mathbf{a} = W^T \star \mathbf{x} + \mathbf{b}, \quad \mathbf{y} = h(\mathbf{a}) = h(W^T \star \mathbf{x} + \mathbf{b})$$

$$(W^T \star \mathbf{x})_j[n] = (\mathbf{w}_j^T \star \mathbf{x})[n] := \sum_i (w_{ij} \star x_i)[n] = \sum_k w_{ij}[k] x_i[k+n]$$

Fukushima. BC 1980. Neocognitron: A Self-Organizing Neural Network Model for a Mechanism of Pattern Recognition Unaffected By Shift in Position.

convolution on feature maps



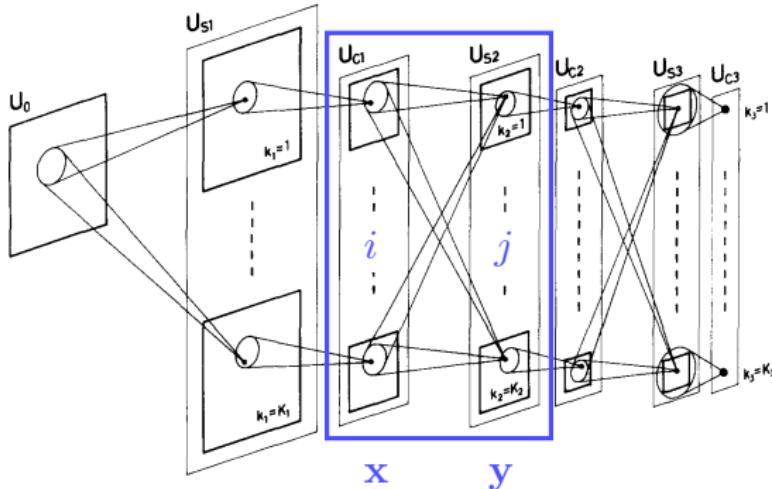
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convolution on feature maps



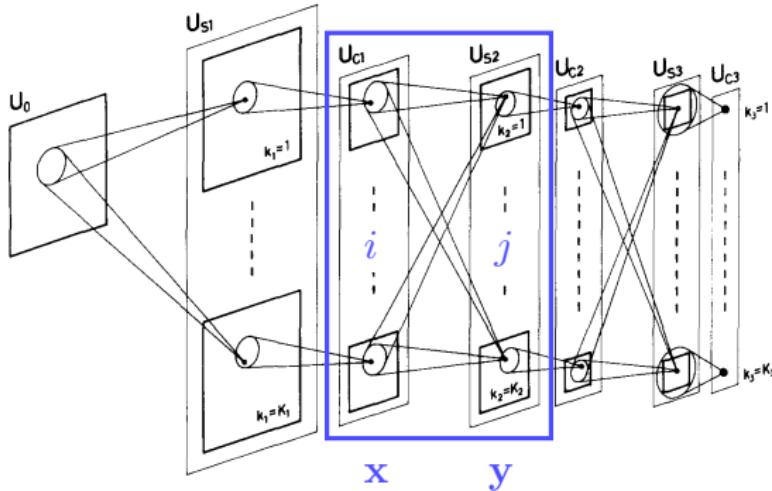
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convolution on feature maps



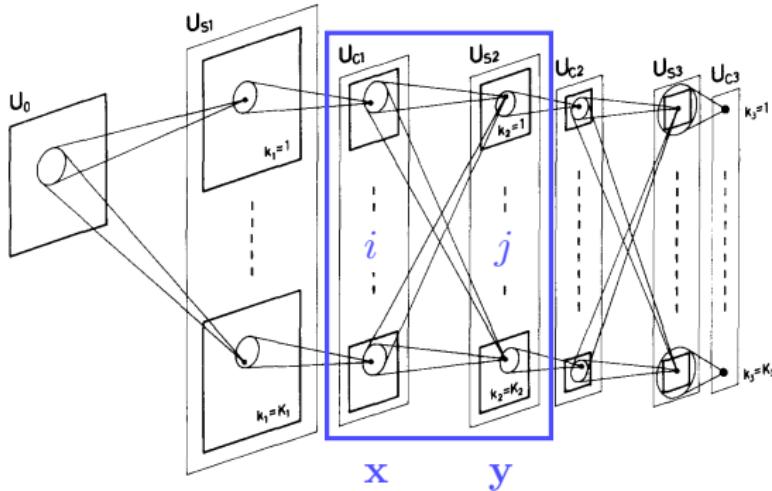
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convolution on feature maps



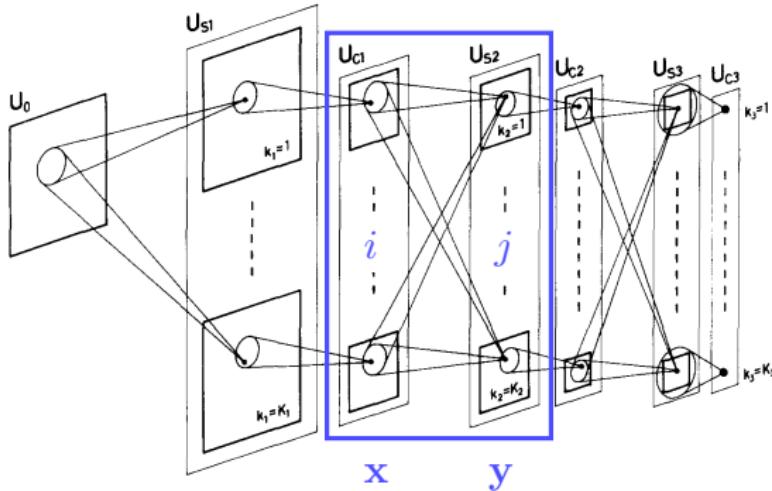
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Fukushima. BC 1980. Neocognitron: A Self-Organizing Neural Network Model for a Mechanism of Pattern Recognition Unaffected By Shift in Position.

convolution on feature maps

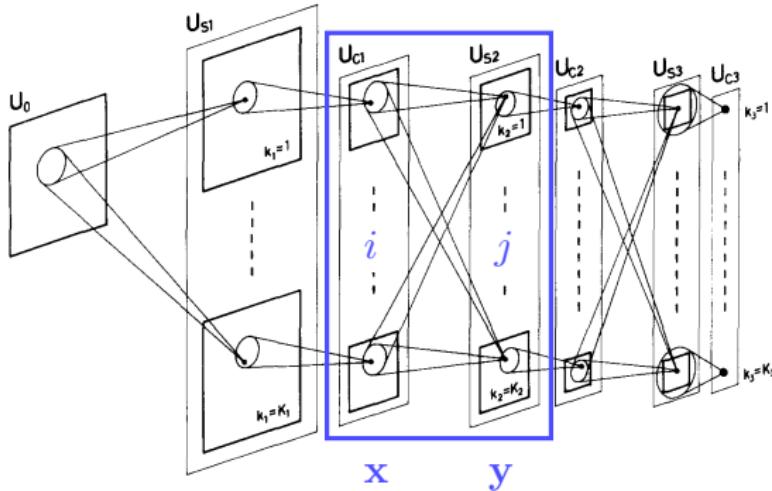


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convolution on feature maps

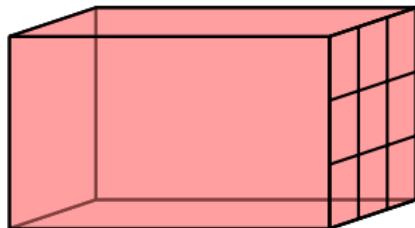


- matrix multiplication and convolution combined

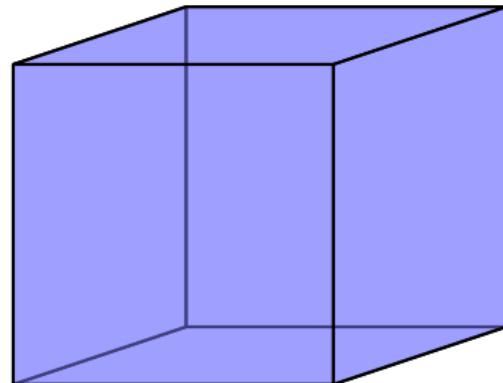
$$\mathbf{a} = W^\top \star \mathbf{x} + \mathbf{b}, \quad \mathbf{y} = h(\mathbf{a}) = h(W^\top \star \mathbf{x} + \mathbf{b})$$

$$(W^\top \star \mathbf{x})_j[\mathbf{n}] = (\mathbf{w}_j^\top \star \mathbf{x})[\mathbf{n}] := \sum_i (w_{ij} \star x_i)[\mathbf{n}] = \sum_{i,\mathbf{k}} w_{ij}[\mathbf{k}] x_i[\mathbf{k} + \mathbf{n}]$$

convolution on feature maps

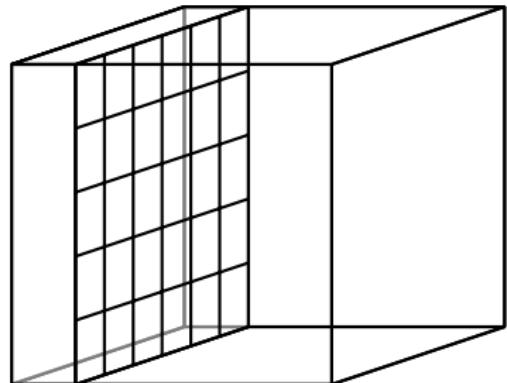


kernel w_1



input x

kernel weights shared
among all spatial positions

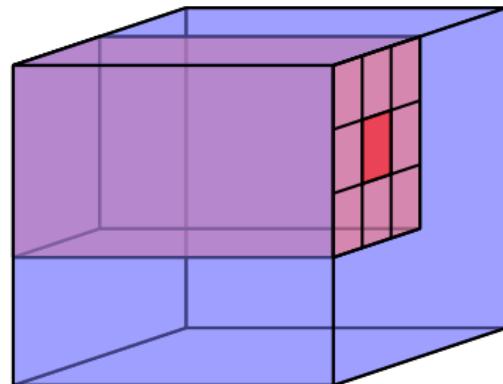


$$\text{output } y_1 = h(\mathbf{w}_1^\top \star \mathbf{x} + b_1)$$

convolution on feature maps

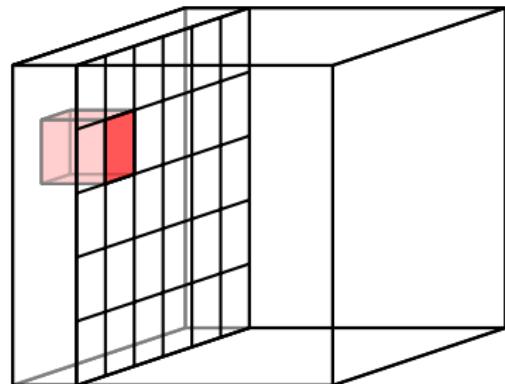


kernel w_1



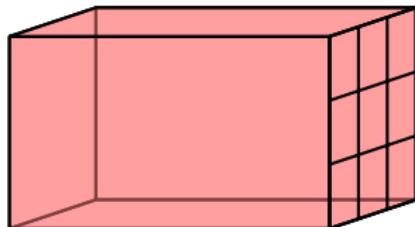
input x

kernel weights shared
among all spatial positions

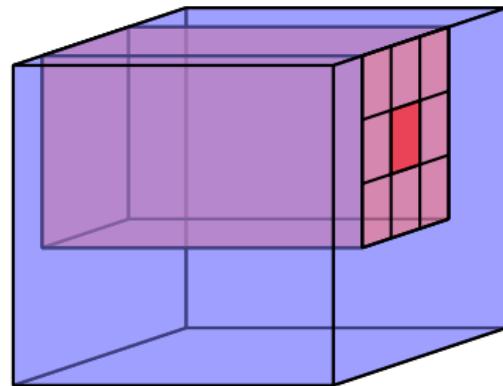


$$\text{output } y_1 = h(w_1^\top \star x + b_1)$$

convolution on feature maps

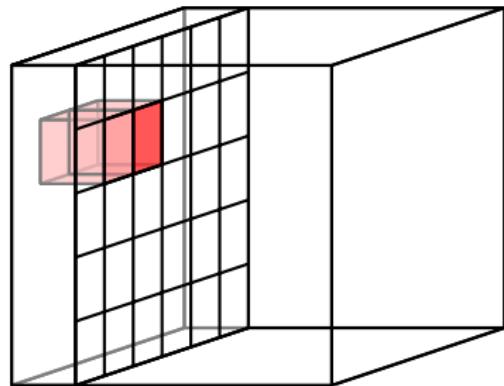


kernel w_1



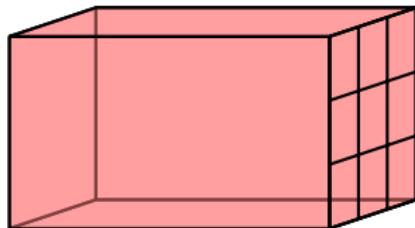
input x

kernel weights shared
among all spatial positions

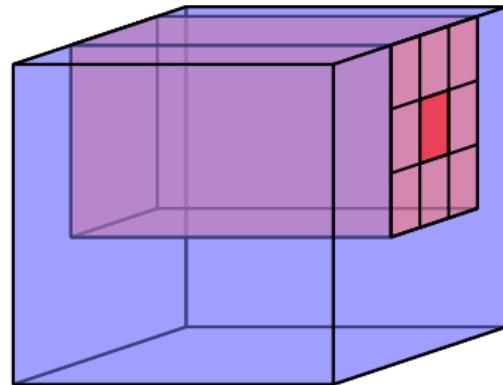


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convolution on feature maps

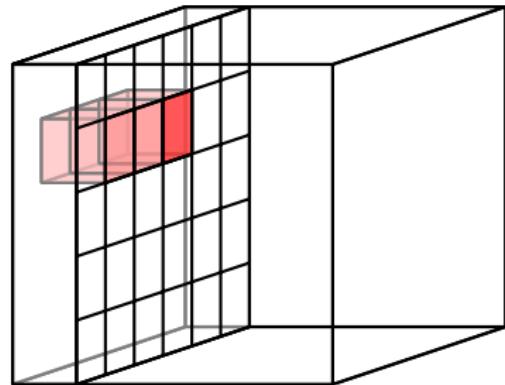


kernel w_1



input x

kernel weights shared
among all spatial positions

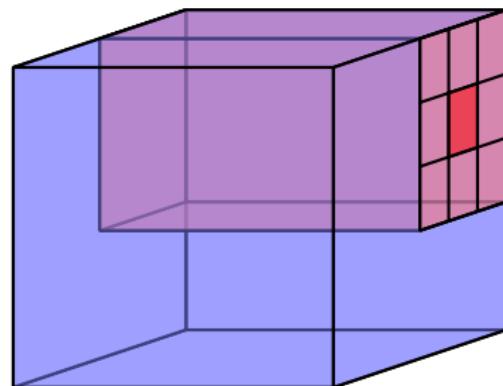


$$\text{output } y_1 = h(w_1^\top \star x + b_1)$$

convolution on feature maps

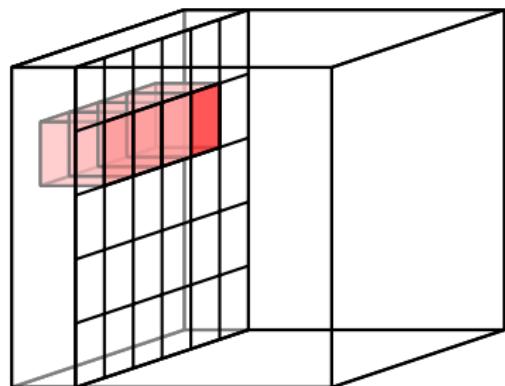


kernel w_1



input x

kernel weights shared
among all spatial positions

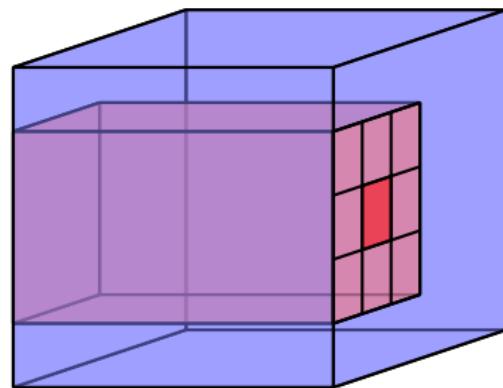


$$\text{output } y_1 = h(w_1^\top \star x + b_1)$$

convolution on feature maps

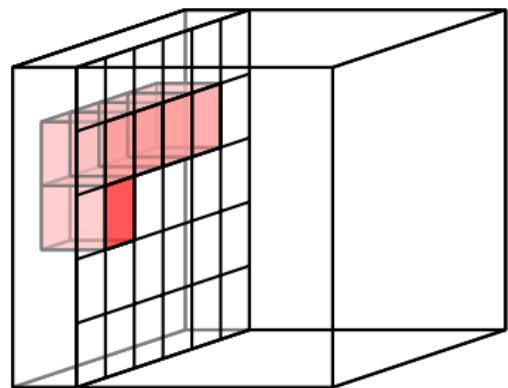


kernel w_1



input x

kernel weights shared
among all spatial positions

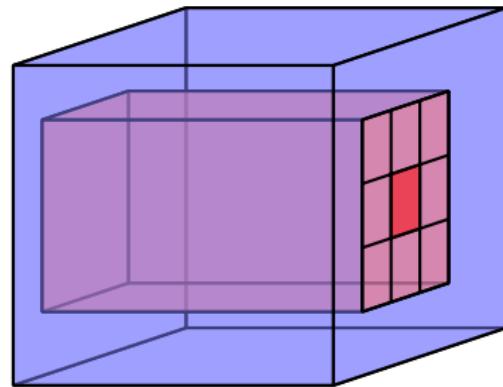


$$\text{output } y_1 = h(w_1^\top \star x + b_1)$$

convolution on feature maps

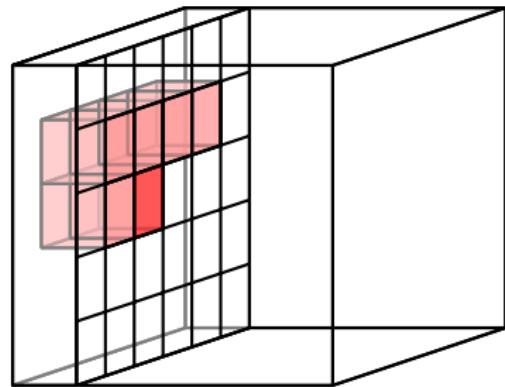


kernel w_1



input x

kernel weights shared
among all spatial positions

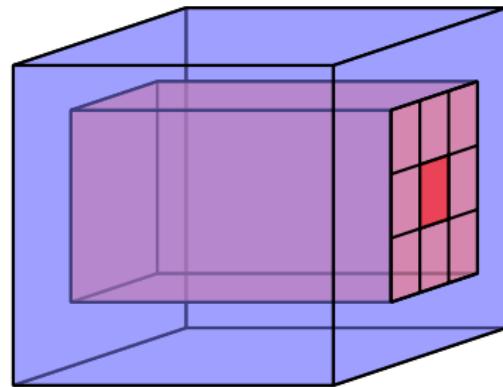


$$\text{output } y_1 = h(w_1^\top \star x + b_1)$$

convolution on feature maps

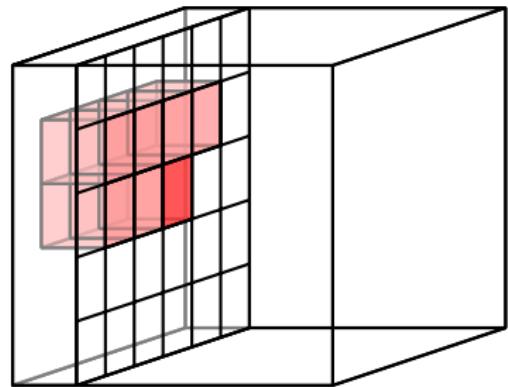


kernel w_1



input x

kernel weights shared
among all spatial positions

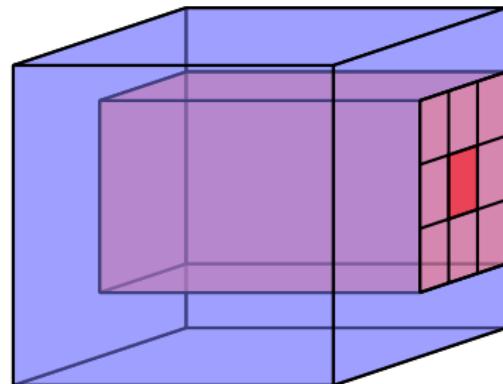


$$\text{output } y_1 = h(w_1^\top \star x + b_1)$$

convolution on feature maps

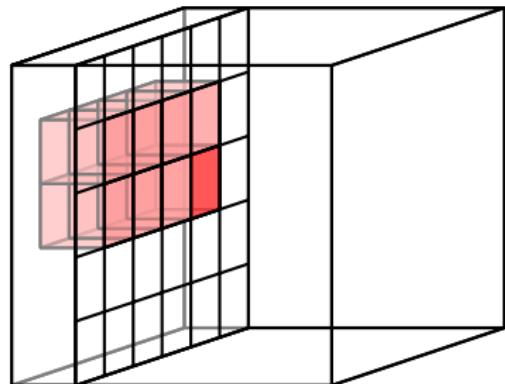


kernel w_1



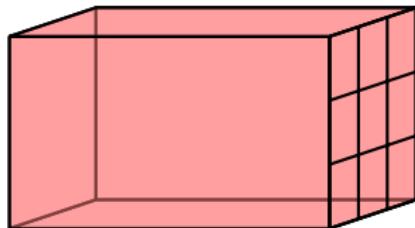
input x

kernel weights shared
among all spatial positions

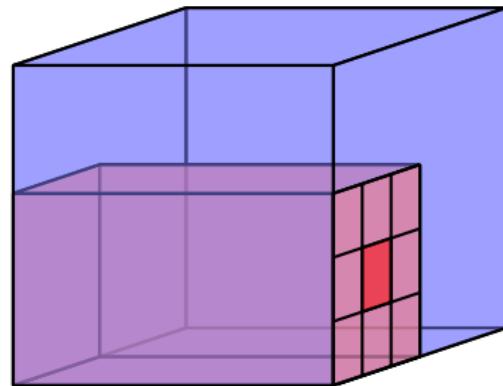


$$\text{output } y_1 = h(w_1^\top \star x + b_1)$$

convolution on feature maps

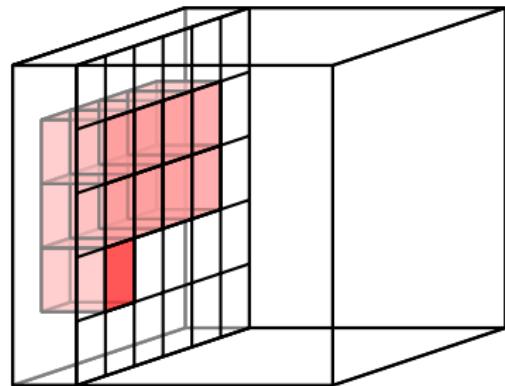


kernel w_1



input x

kernel weights shared
among all spatial positions

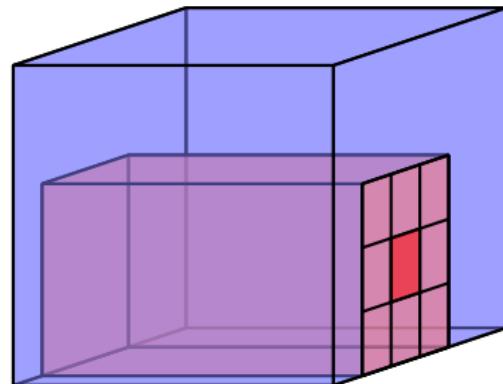


$$\text{output } y_1 = h(w_1^\top \star x + b_1)$$

convolution on feature maps

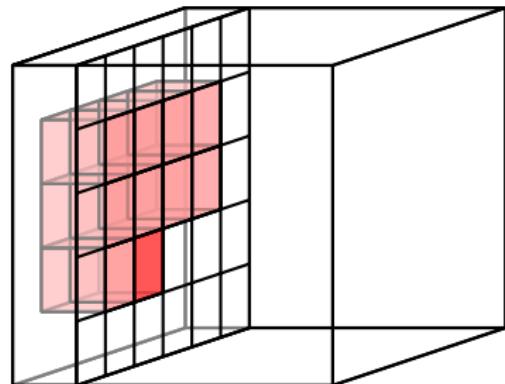


kernel w_1



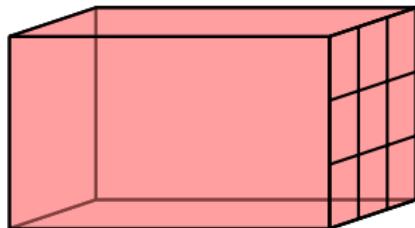
input x

kernel weights shared
among all spatial positions



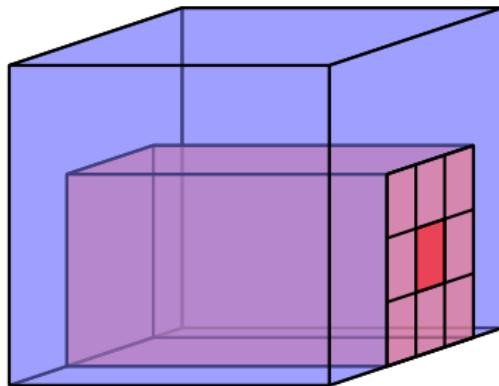
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convolution on feature maps

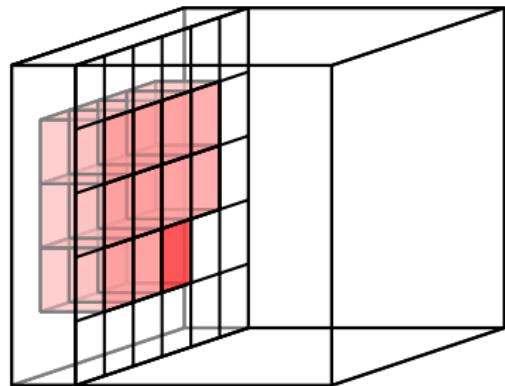


kernel w_1

kernel weights shared
among all spatial positions

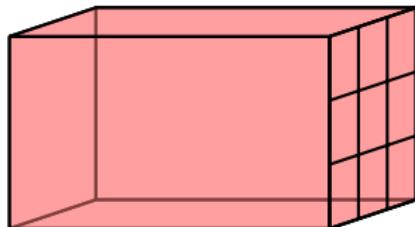


input x

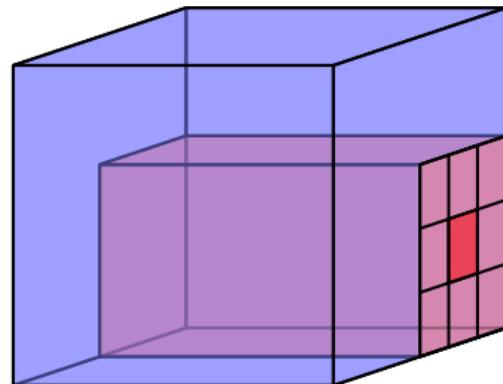


$$\text{output } y_1 = h(\mathbf{w}_1^\top \star \mathbf{x} + b_1)$$

convolution on feature maps

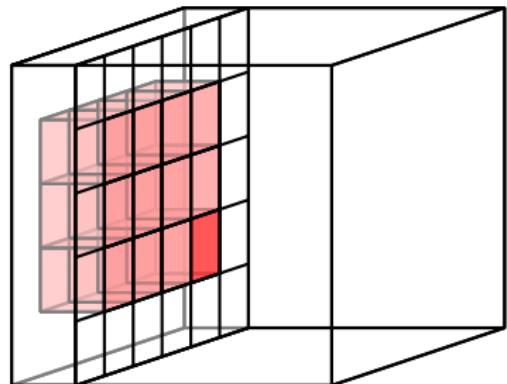


kernel w_1



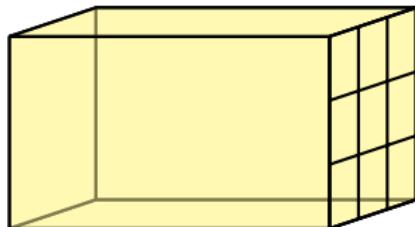
input x

kernel weights shared
among all spatial positions

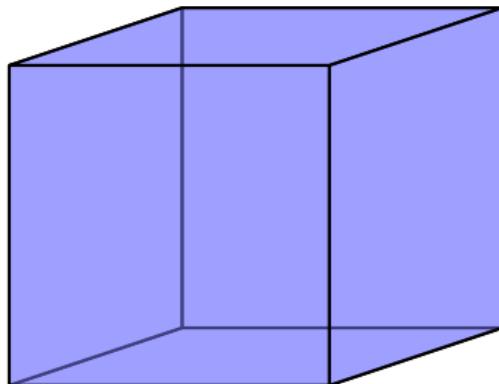


$$\text{output } y_1 = h(w_1^\top \star x + b_1)$$

convolution on feature maps

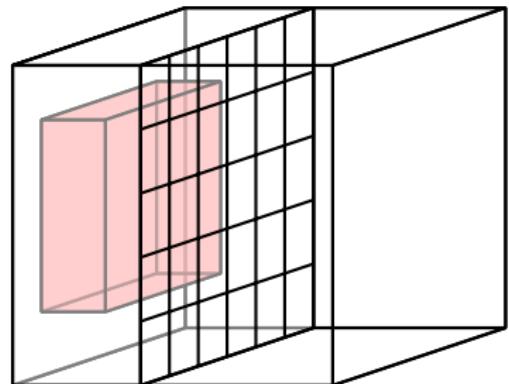


kernel w_2



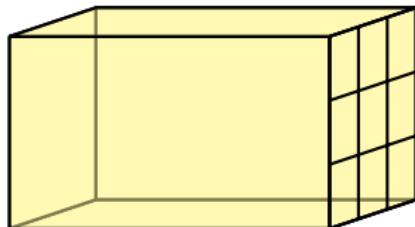
input x

new kernel, but still shared
among all spatial positions

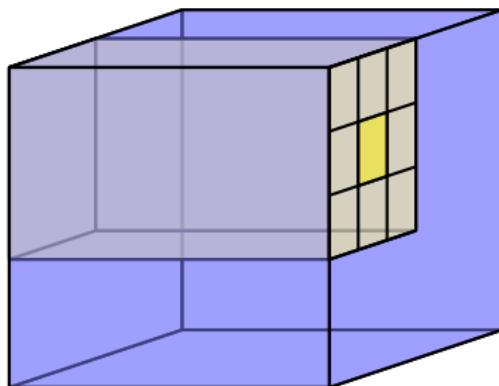


$$\text{output } y_2 = h(w_2^T \star x + b_2)$$

convolution on feature maps

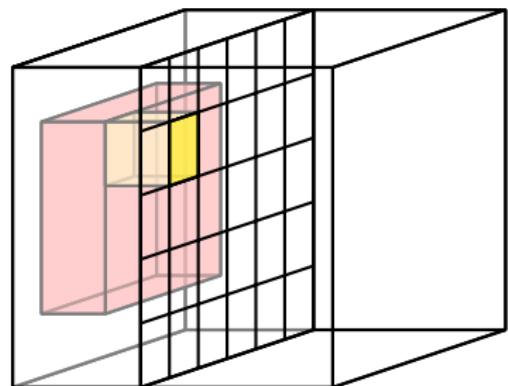


kernel w_2



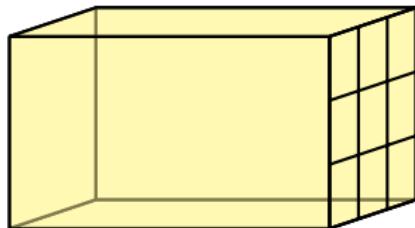
input x

new kernel, but still shared
among all spatial positions

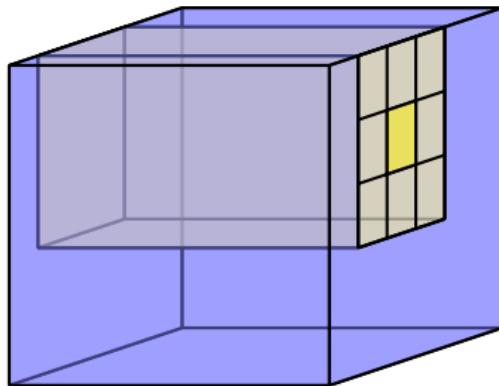


$$\text{output } y_2 = h(w_2^T \star x + b_2)$$

convolution on feature maps

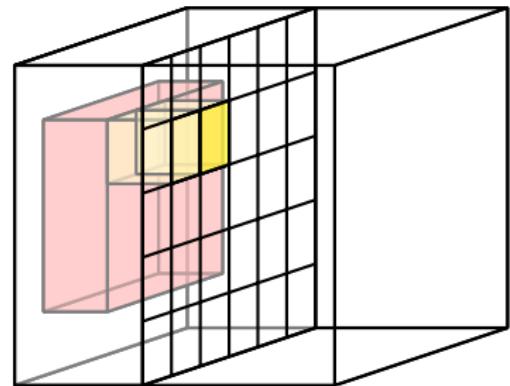


kernel w_2



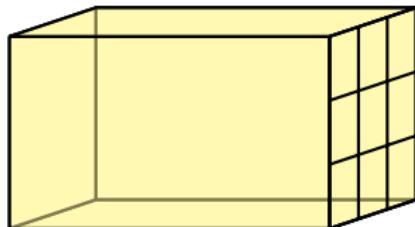
input x

new kernel, but still shared
among all spatial positions

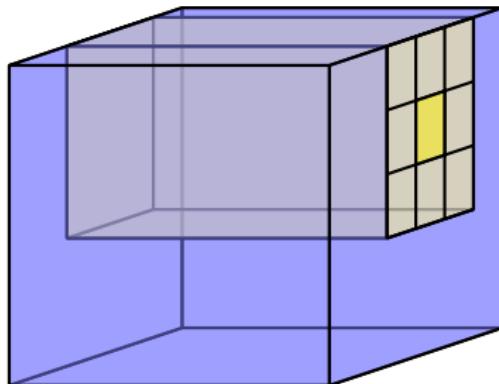


$$\text{output } y_2 = h(w_2^T \star x + b_2)$$

convolution on feature maps

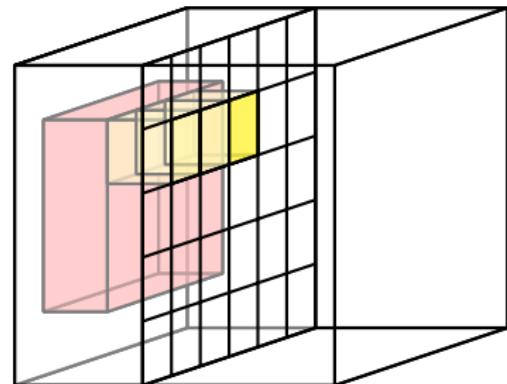


kernel w_2



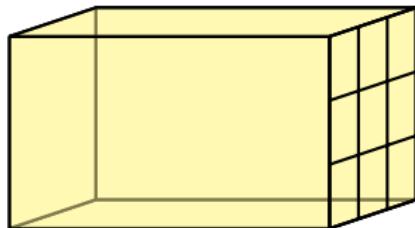
input x

new kernel, but still shared
among all spatial positions

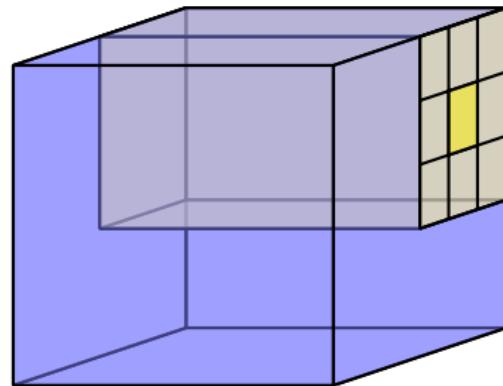


$$\text{output } y_2 = h(\mathbf{w}_2^\top \star \mathbf{x} + b_2)$$

convolution on feature maps

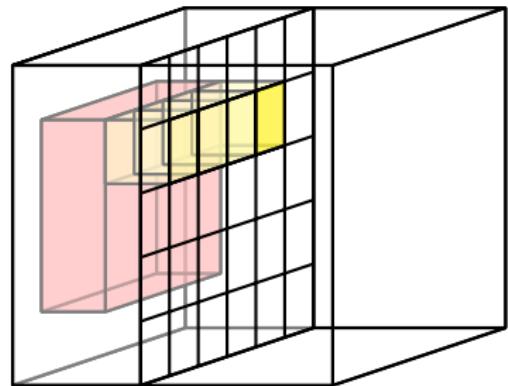


kernel w_2



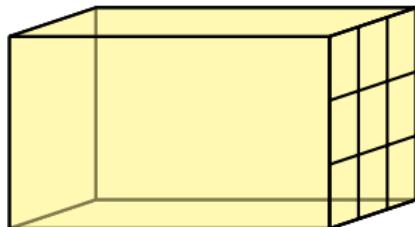
input x

new kernel, but still shared
among all spatial positions

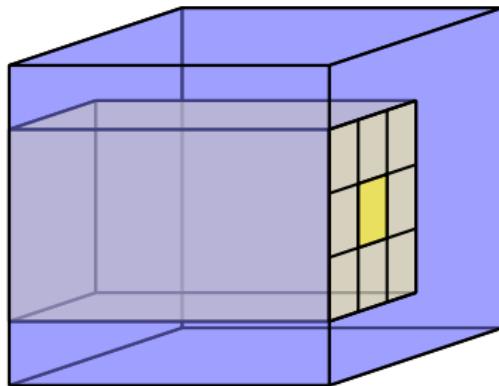


$$\text{output } y_2 = h(w_2^T \star x + b_2)$$

convolution on feature maps

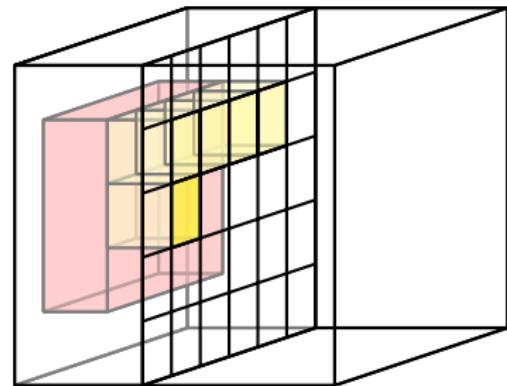


kernel w_2



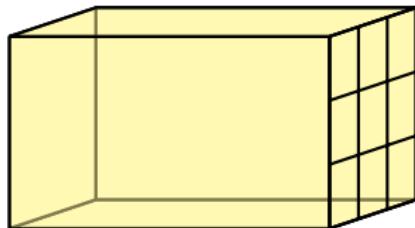
input x

new kernel, but still shared
among all spatial positions

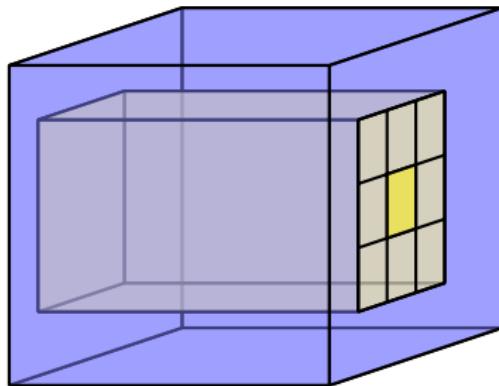


$$\text{output } y_2 = h(w_2^T \star x + b_2)$$

convolution on feature maps

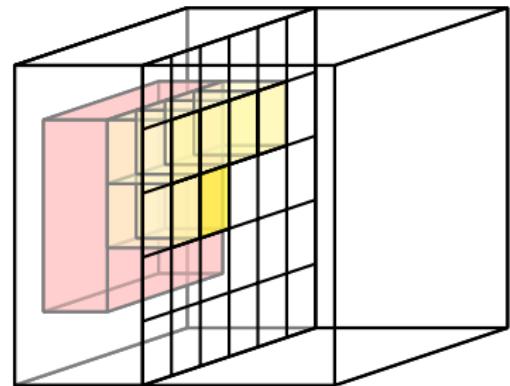


kernel w_2



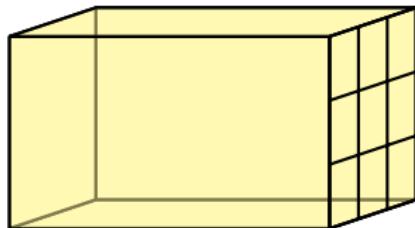
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new kernel, but still shared
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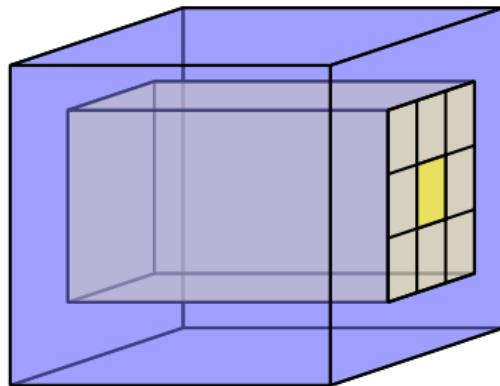


$$\text{output } y_2 = h(\mathbf{w}_2^\top \star \mathbf{x} + b_2)$$

convolution on feature maps

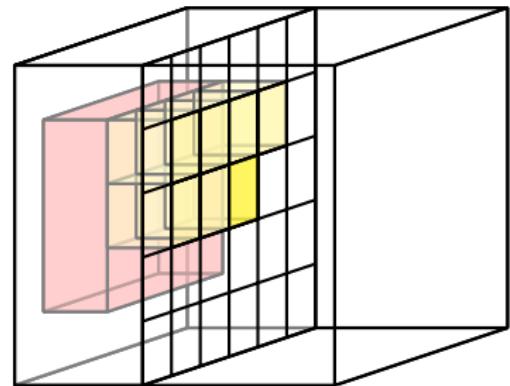


kernel w_2



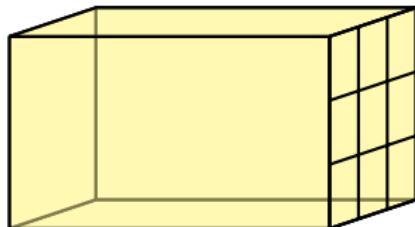
input x

new kernel, but still shared
among all spatial positions

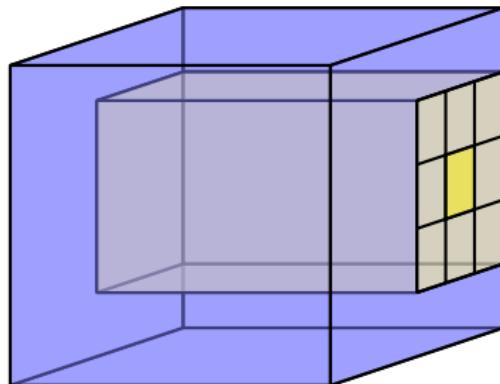


$$\text{output } y_2 = h(w_2^T \star x + b_2)$$

convolution on feature maps

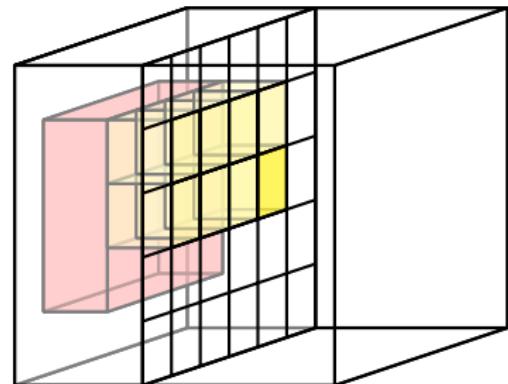


kernel w_2



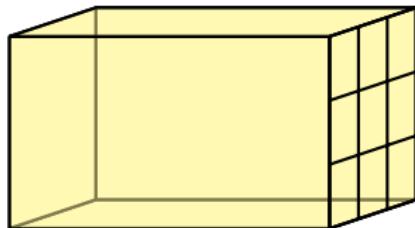
input x

new kernel, but still shared
among all spatial positions

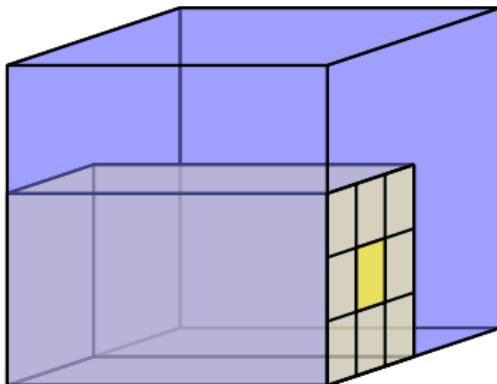


$$\text{output } y_2 = h(w_2^\top \star x + b_2)$$

convolution on feature maps

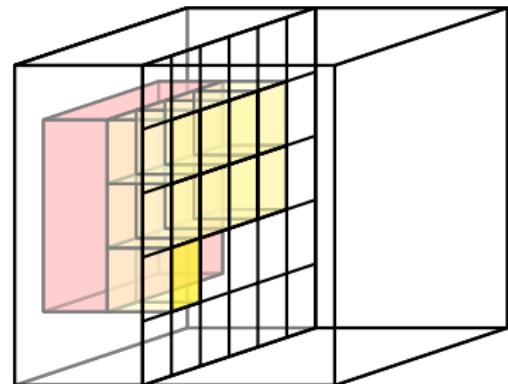


kernel w_2



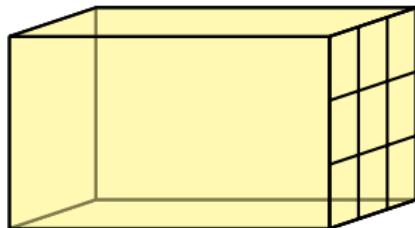
input x

new kernel, but still shared
among all spatial positions

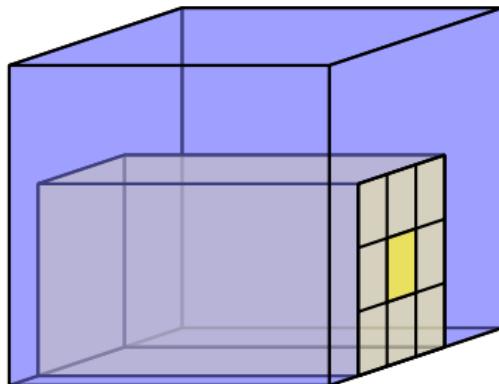


$$\text{output } y_2 = h(\mathbf{w}_2^\top \star \mathbf{x} + b_2)$$

convolution on feature maps

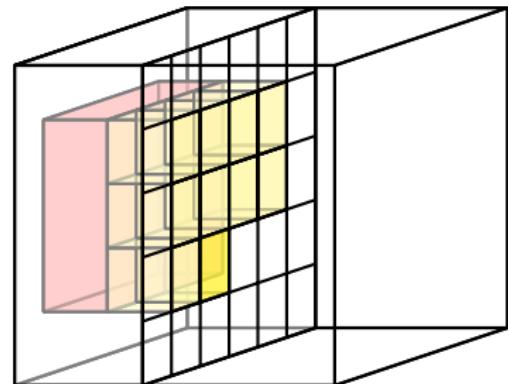


kernel w_2



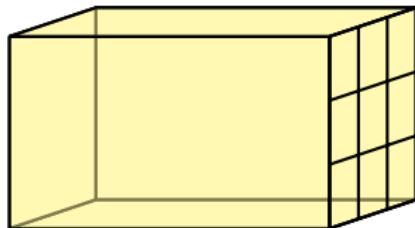
input x

new kernel, but still shared
among all spatial positions

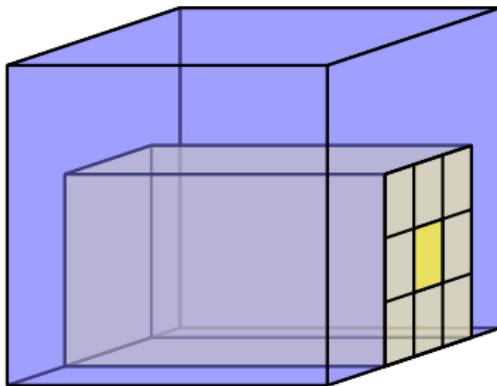


$$\text{output } y_2 = h(w_2^T \star x + b_2)$$

convolution on feature maps

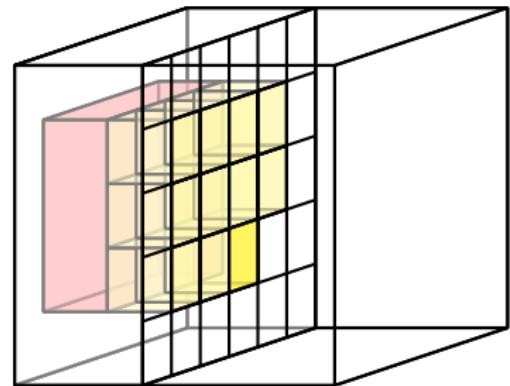


kernel w_2



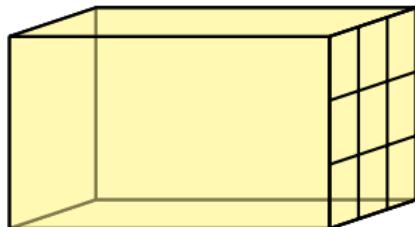
input x

new kernel, but still shared
among all spatial positions

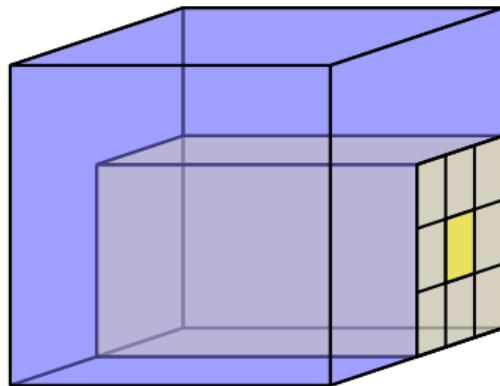


$$\text{output } y_2 = h(w_2^T \star x + b_2)$$

convolution on feature maps

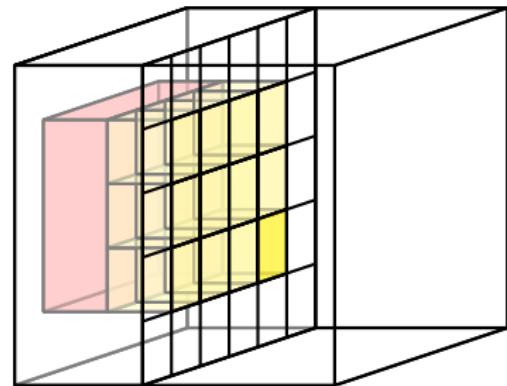


kernel w_2



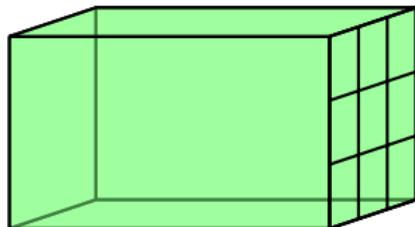
input x

new kernel, but still shared
among all spatial positions

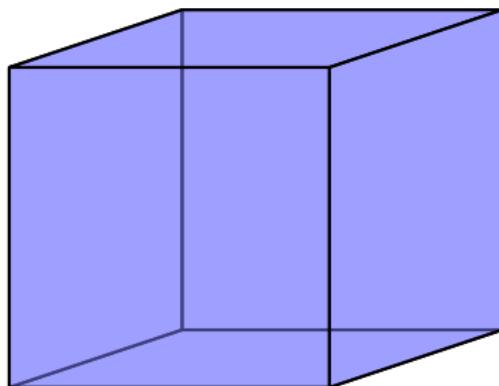


$$\text{output } y_2 = h(w_2^T \star x + b_2)$$

convolution on feature maps

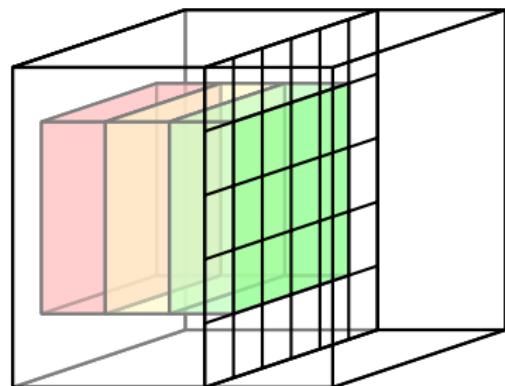


kernel w_3



input x

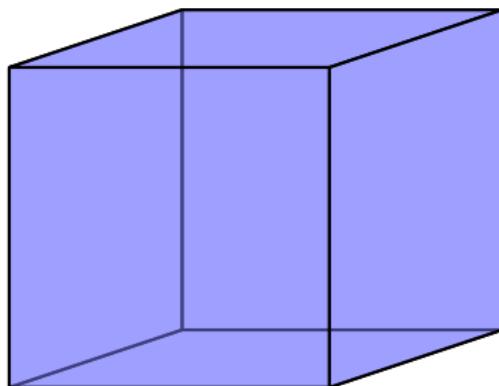
different kernel for
each output dimension



convolution on feature maps

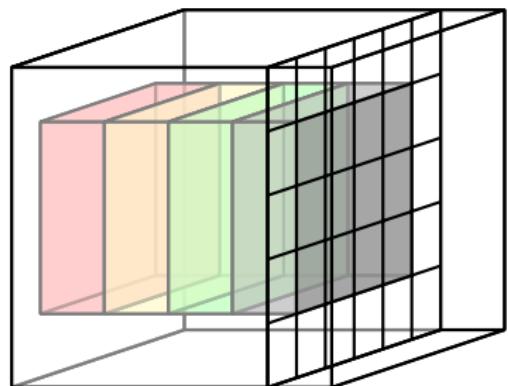


kernel w_4



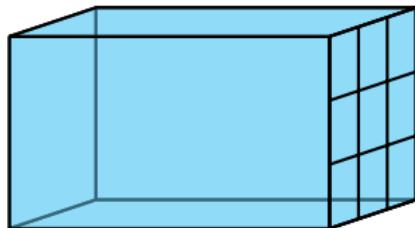
input x

different kernel for
each output dimension

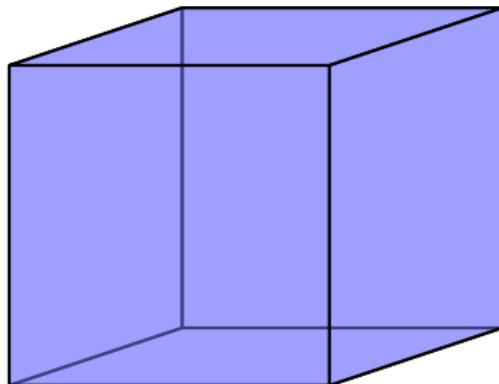


$$\text{output } y_4 = h(\mathbf{w}_4^\top \star \mathbf{x} + b_4)$$

convolution on feature maps

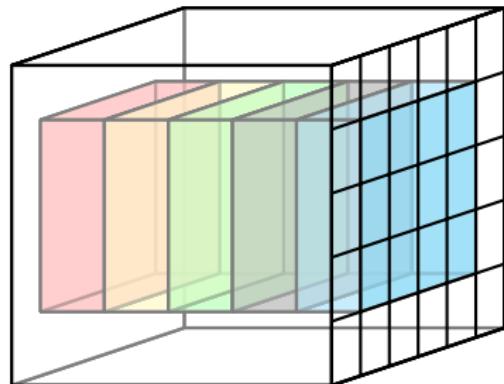


kernel w_5



input x

different kernel for
each output dimension



$$\text{output } y_5 = h(w_5^T \star x + b_5)$$

1×1 convolution

- if W has no spatial extent, it becomes a **2d matrix** again

$$\begin{aligned} (\mathbf{w}_j^\top \star \mathbf{x})[\mathbf{n}] &:= \sum_i (w_{ij} \star x_i)[\mathbf{n}] = \sum_{i,\mathbf{k}} w_{ij}[\mathbf{k}] x_i[\mathbf{k} + \mathbf{n}] \\ &= \sum_i w_{ij} x_i[\mathbf{n}] = \mathbf{w}_j^\top \mathbf{x}[\mathbf{n}] \end{aligned}$$

- the operation becomes a **matrix multiplication** just as in fully-connected layers, but now it is performed independently at each spatial location

$$(W^\top \star \mathbf{x})[\mathbf{n}] = W^\top \mathbf{x}[\mathbf{n}]$$

$$W^\top \star \mathbf{x} = W^\top \mathbf{x}$$

1×1 convolution

- if W has no spatial extent, it becomes a **2d matrix** again

$$\begin{aligned} (\mathbf{w}_j^\top \star \mathbf{x})[\mathbf{n}] &:= \sum_i (w_{ij} \star x_i)[\mathbf{n}] = \sum_{i,\mathbf{k}} w_{ij}[\mathbf{k}] x_i[\mathbf{k} + \mathbf{n}] \\ &= \sum_i w_{ij} x_i[\mathbf{n}] = \mathbf{w}_j^\top \mathbf{x}[\mathbf{n}] \end{aligned}$$

- the operation becomes a **matrix multiplication** just as in fully-connected layers, but now it is performed independently at each spatial location

$$(W^\top \star \mathbf{x})[\mathbf{n}] = W^\top \mathbf{x}[\mathbf{n}]$$

$$W^\top \star \mathbf{x} = W^\top \mathbf{x}$$

1×1 convolution

- if W has no spatial extent, it becomes a **2d matrix** again

$$\begin{aligned} (\mathbf{w}_j^\top \star \mathbf{x})[\mathbf{n}] &:= \sum_i (w_{ij} \star x_i)[\mathbf{n}] = \sum_{i,\mathbf{k}} w_{ij}[\mathbf{k}] x_i[\mathbf{k} + \mathbf{n}] \\ &= \sum_i w_{ij} x_i[\mathbf{n}] = \mathbf{w}_j^\top \mathbf{x}[\mathbf{n}] \end{aligned}$$

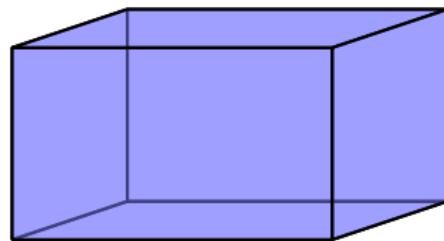
- the operation becomes a **matrix multiplication** just as in fully-connected layers, but now it is performed independently at each spatial location

$$\begin{aligned} (W^\top \star \mathbf{x})[\mathbf{n}] &= W^\top \mathbf{x}[\mathbf{n}] \\ W^\top \star \mathbf{x} &= W^\top \mathbf{x} \end{aligned}$$

1×1 convolution

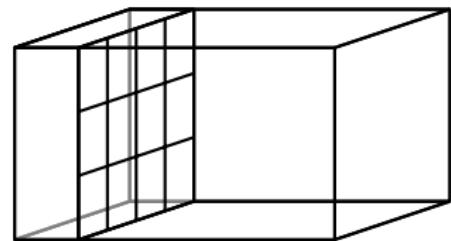


kernel w_1



input x

kernel weights shared
among all spatial positions

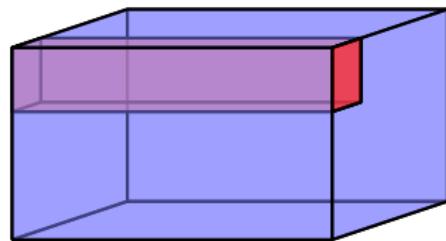


$$\text{output } y_1 = h(\mathbf{w}_1^\top \star \mathbf{x} + b_1)$$

1×1 convolution

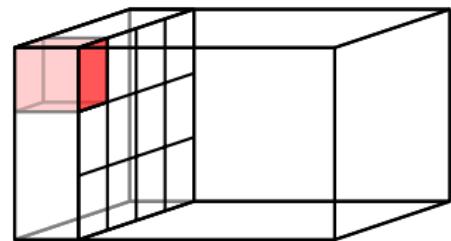


kernel w_1



input x

kernel weights shared
among all spatial positions

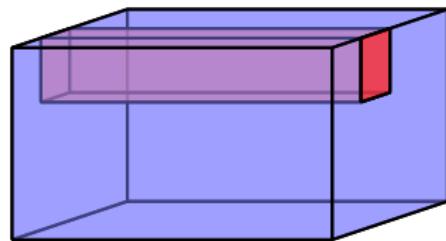


$$\text{output } y_1 = h(\mathbf{w}_1^\top \star \mathbf{x} + b_1)$$

1×1 convolution

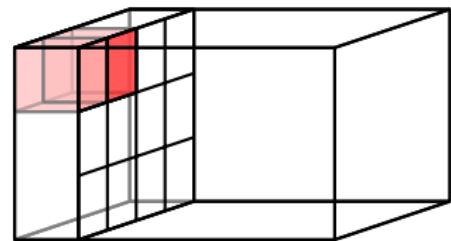


kernel w_1



input x

kernel weights shared
among all spatial positions

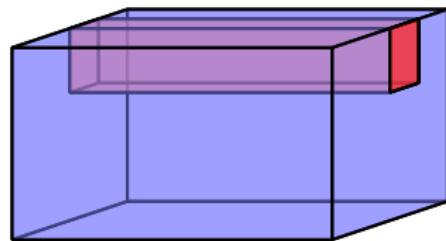


$$\text{output } y_1 = h(\mathbf{w}_1^\top \star \mathbf{x} + b_1)$$

1×1 convolution

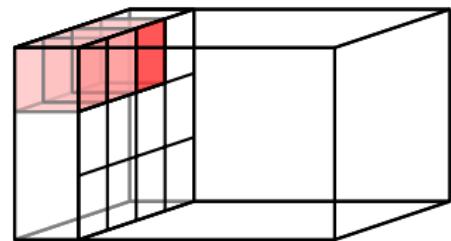


kernel w_1



input x

kernel weights shared
among all spatial positions

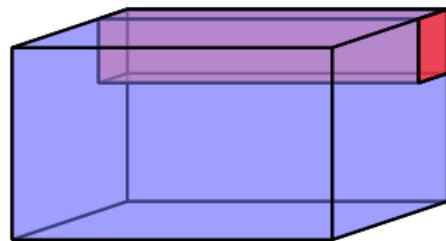


$$\text{output } y_1 = h(\mathbf{w}_1^\top \star \mathbf{x} + b_1)$$

1×1 convolution

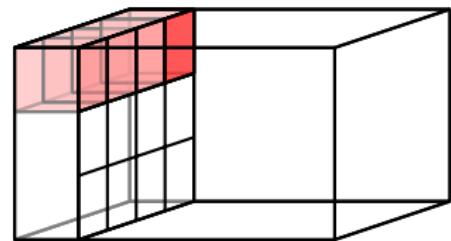


kernel w_1



input x

kernel weights shared
among all spatial positions

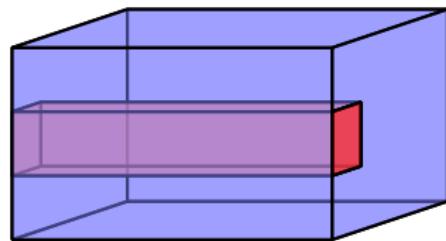


$$\text{output } y_1 = h(\mathbf{w}_1^\top \star \mathbf{x} + b_1)$$

1×1 convolution

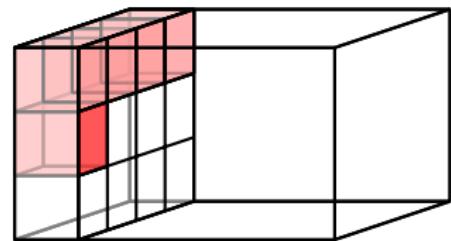


kernel w_1



input x

kernel weights shared
among all spatial positions

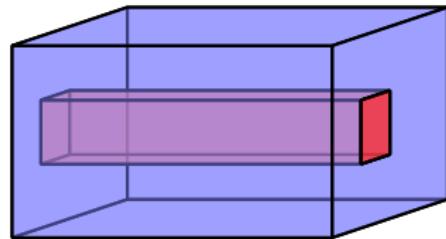


$$\text{output } y_1 = h(\mathbf{w}_1^\top \star \mathbf{x} + b_1)$$

1×1 convolution

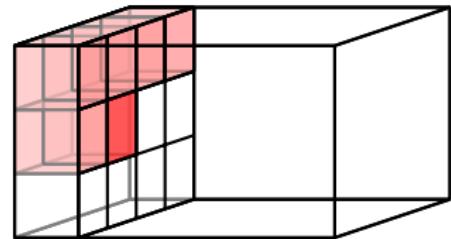


kernel w_1



input x

kernel weights shared
among all spatial positions

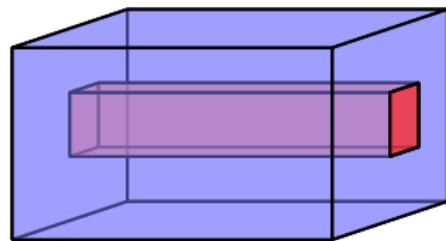


$$\text{output } y_1 = h(\mathbf{w}_1^\top \star \mathbf{x} + b_1)$$

1×1 convolution

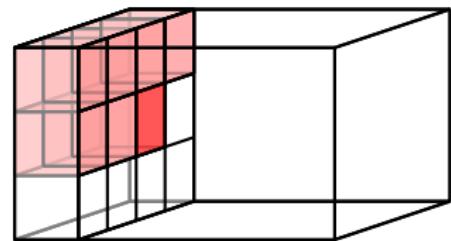


kernel w_1



input x

kernel weights shared
among all spatial positions

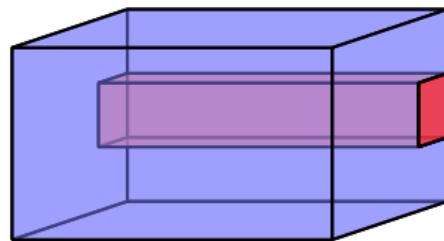


$$\text{output } y_1 = h(\mathbf{w}_1^\top \star \mathbf{x} + b_1)$$

1×1 convolution

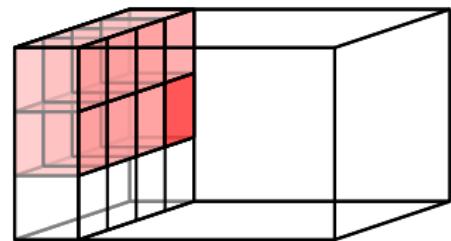


kernel w_1



input x

kernel weights shared
among all spatial positions

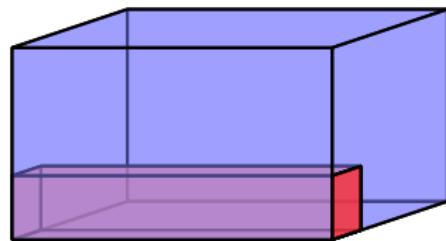


$$\text{output } y_1 = h(w_1^\top * x + b_1)$$

1×1 convolution

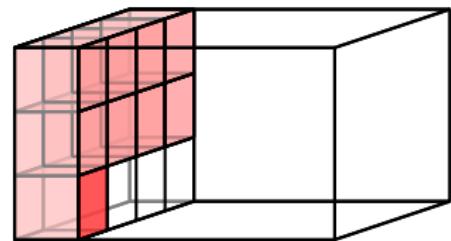


kernel w_1



input x

kernel weights shared
among all spatial positions

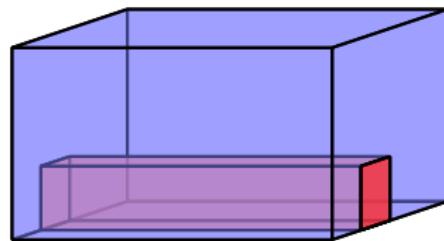


$$\text{output } y_1 = h(\mathbf{w}_1^\top \star \mathbf{x} + b_1)$$

1×1 convolution

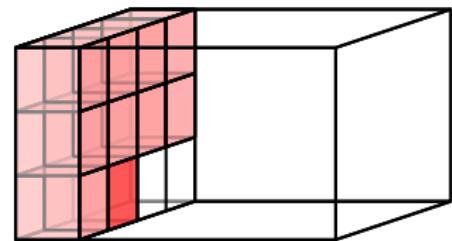


kernel w_1



input x

kernel weights shared
among all spatial positions

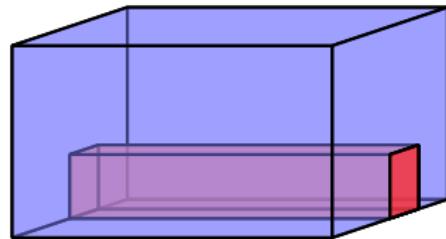


$$\text{output } y_1 = h(\mathbf{w}_1^\top \star \mathbf{x} + b_1)$$

1×1 convolution

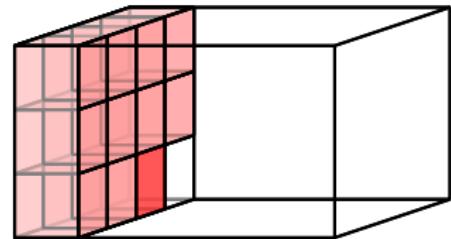


kernel w_1



input x

kernel weights shared
among all spatial positions

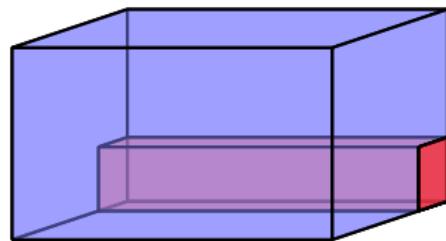


$$\text{output } y_1 = h(\mathbf{w}_1^\top \star \mathbf{x} + b_1)$$

1×1 convolution

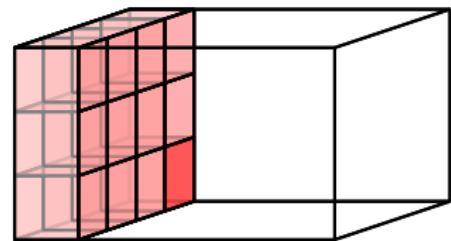


kernel w_1



input x

kernel weights shared
among all spatial positions

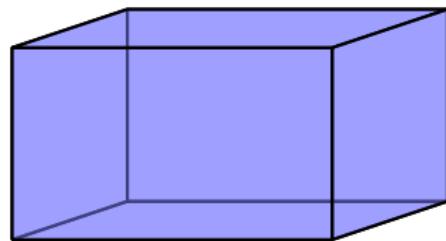


$$\text{output } y_1 = h(w_1^\top * x + b_1)$$

1×1 convolution

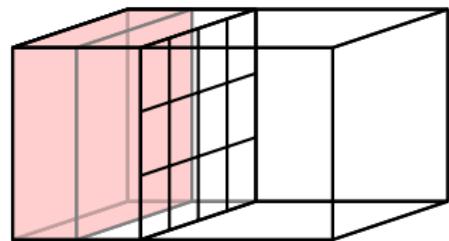


kernel w_2



input x

new kernel, but still shared
among all spatial positions

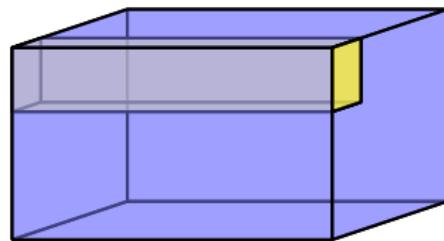


$$\text{output } y_2 = h(w_2^\top * x + b_2)$$

1×1 convolution

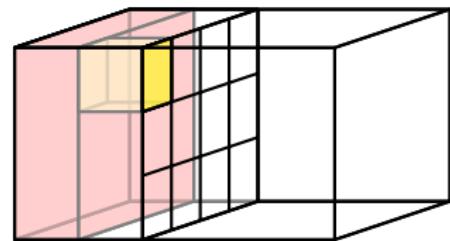


kernel w_2



input x

new kernel, but still shared
among all spatial positions

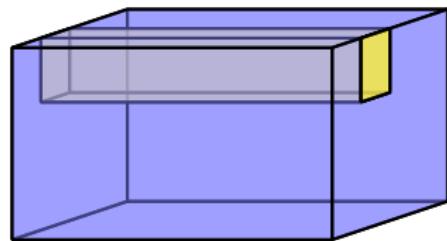


$$\text{output } y_2 = h(w_2^\top * x + b_2)$$

1×1 convolution

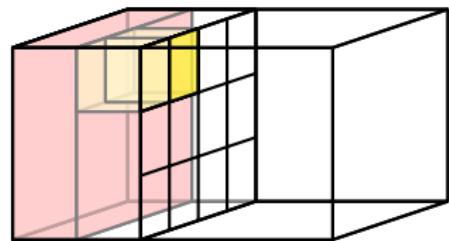


kernel w_2



input x

new kernel, but still shared
among all spatial positions

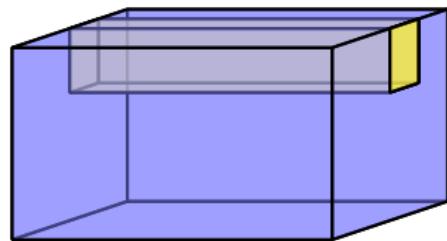


$$\text{output } y_2 = h(\mathbf{w}_2^\top \star \mathbf{x} + b_2)$$

1×1 convolution

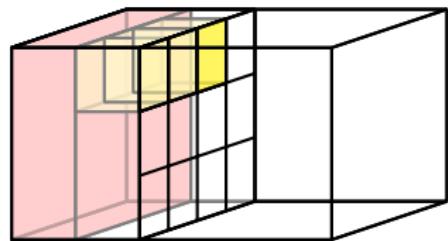


kernel w_2



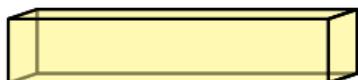
input x

new kernel, but still shared
among all spatial positions

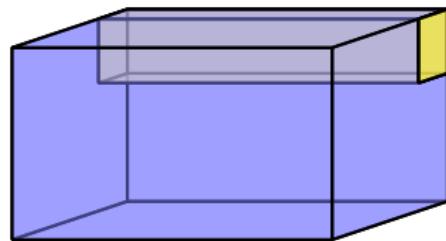


$$\text{output } y_2 = h(w_2^\top * x + b_2)$$

1×1 convolution

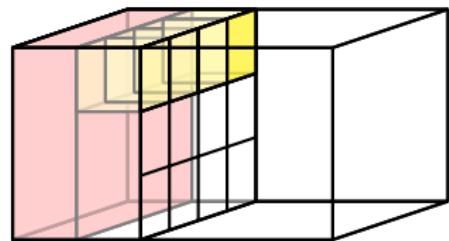


kernel w_2



input x

new kernel, but still shared
among all spatial positions

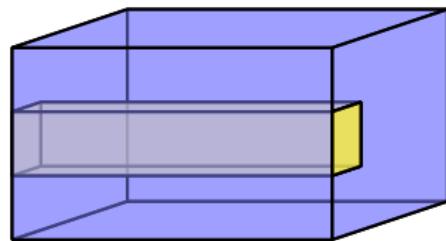


$$\text{output } y_2 = h(w_2^\top * x + b_2)$$

1×1 convolution

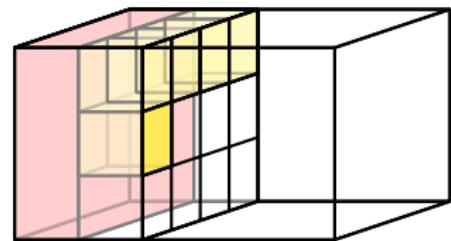


kernel w_2



input x

new kernel, but still shared
among all spatial positions

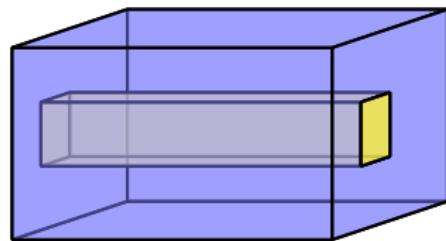


$$\text{output } y_2 = h(\mathbf{w}_2^\top \star \mathbf{x} + b_2)$$

1×1 convolution

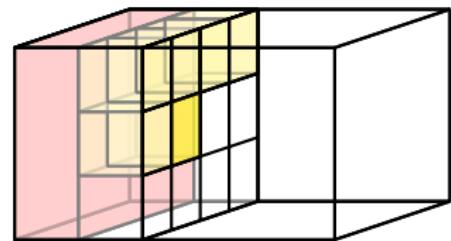


kernel w_2



input x

new kernel, but still shared
among all spatial positions

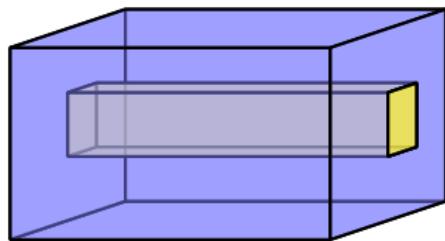


$$\text{output } y_2 = h(w_2^\top * x + b_2)$$

1×1 convolution

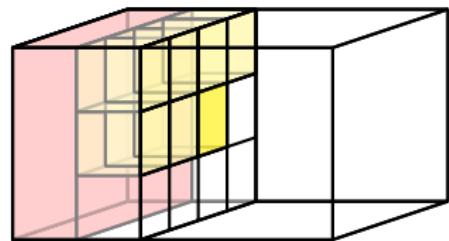


kernel w_2



input x

new kernel, but still shared
among all spatial positions

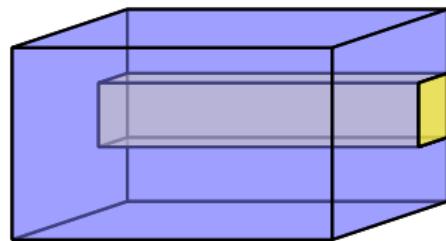


$$\text{output } y_2 = h(\mathbf{w}_2^\top \star \mathbf{x} + b_2)$$

1×1 convolution

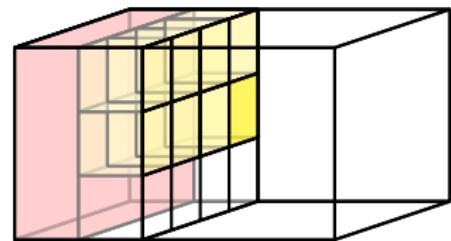


kernel w_2



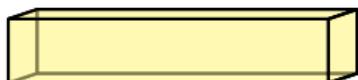
input x

new kernel, but still shared
among all spatial positions

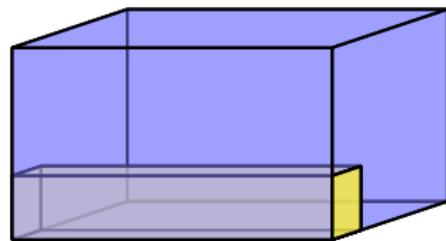


$$\text{output } y_2 = h(w_2^\top * x + b_2)$$

1×1 convolution

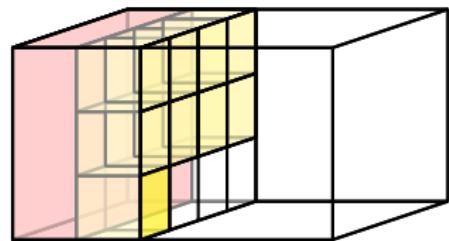


kernel w_2



input x

new kernel, but still shared
among all spatial positions

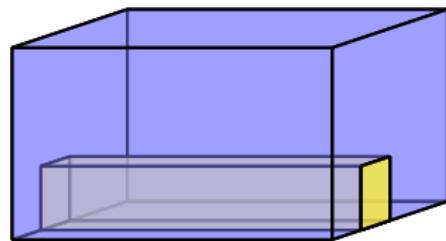


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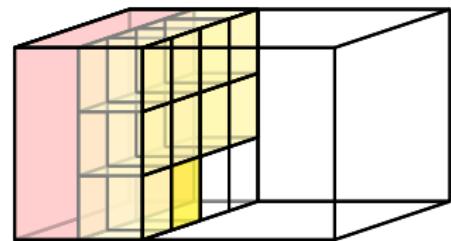


kernel w_2



input x

new kernel, but still shared
among all spatial positions

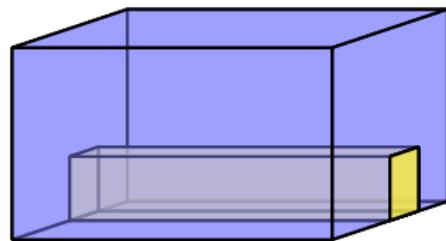


$$\text{output } y_2 = h(w_2^\top * x + b_2)$$

1×1 convolution

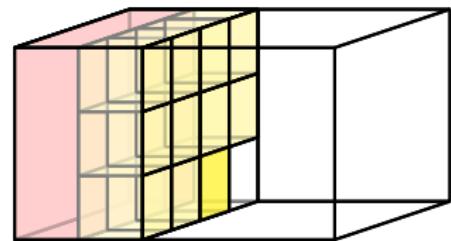


kernel w_2



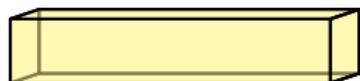
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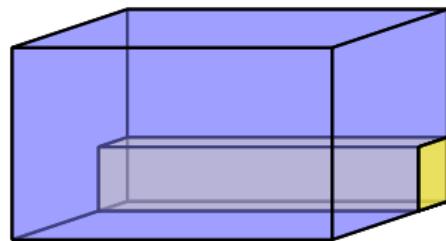


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1×1 convolution

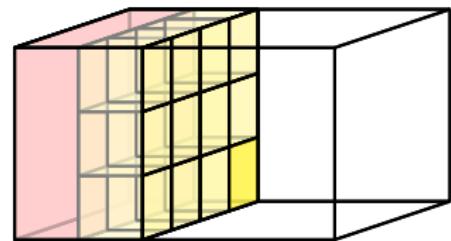


kernel w_2



input x

new kernel, but still shared
among all spatial positions

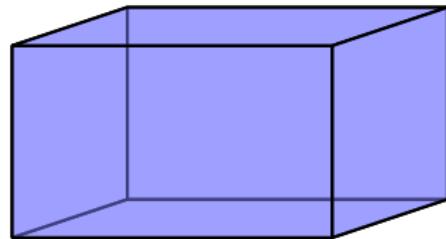


$$\text{output } y_2 = h(w_2^\top * x + b_2)$$

1×1 convolution

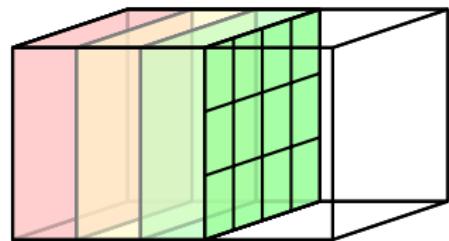


kernel w_3



input x

different kernel for
each output dimension

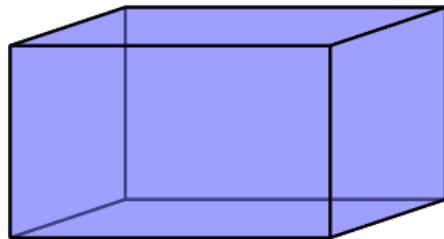


$$\text{output } y_3 = h(\mathbf{w}_3^\top \star \mathbf{x} + b_3)$$

1×1 convolution

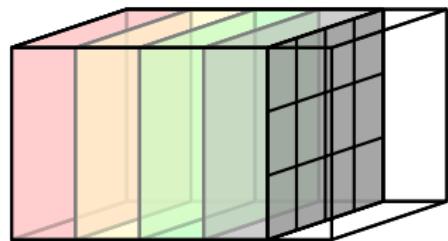


kernel w_4



input x

different kernel for
each output dimension

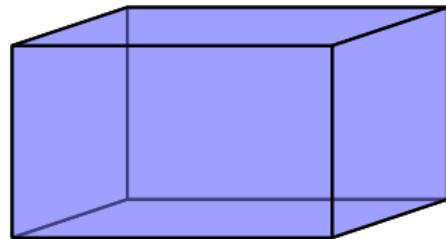


$$\text{output } y_4 = h(\mathbf{w}_4^\top \star \mathbf{x} + b_4)$$

1×1 convolution

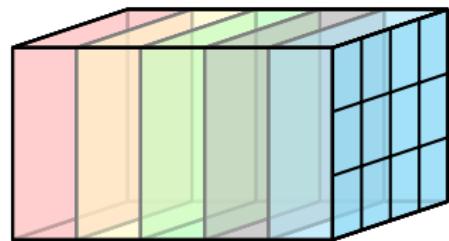


kernel w_5



input x

different kernel for
each output dimension



$$\text{output } y_5 = h(w_5^\top * x + b_5)$$

convolution as regularization

- suppose a fully connected layer is given by

$$\mathbf{a} = \begin{pmatrix} w_1 & w_2 & w_3 \\ w_4 & w_5 & w_6 \end{pmatrix} \mathbf{x}$$

- now if we add the following term to our error function

$$\frac{\lambda}{2} ((w_6 - w_2)^2 + (w_5 - w_1)^2 + w_3^2 + w_4^2)$$

then, as $\lambda \rightarrow \infty$, the weight matrix tends to the constrained **Toeplitz** form

$$\begin{pmatrix} w_1 & w_2 & 0 \\ 0 & w_1 & w_2 \end{pmatrix}$$

and the layer becomes **convolutional**

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convolution as Gaussian mixture prior

- remember, weight decay is equivalent to a zero-centered Gaussian prior if the weight vector/matrix is considered a random variable
- in this analogy, error term

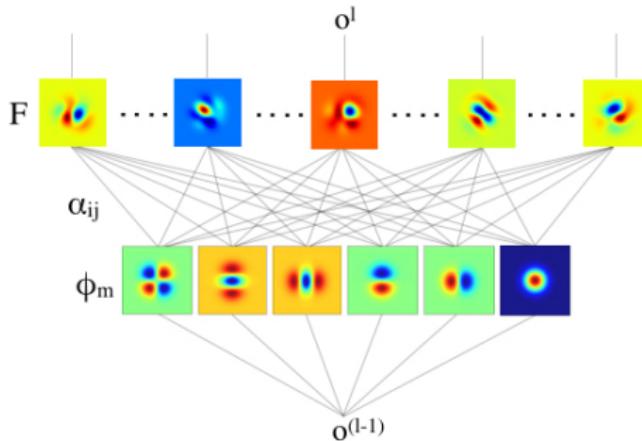
$$\frac{\lambda}{2} ((w_6 - w_2)^2 + (w_5 - w_1)^2 + w_3^2 + w_4^2)$$

corresponds to two Gaussian priors centered at w_1 , w_2 for w_5 , w_6 and one zero-centered Gaussian for w_3 , w_4

- that is, a Gaussian mixture prior

structured convolution

[Jacobsen et al. 2016]



- we can constrain parameters even more by considering a fixed basis of **steerable filters** consisting of **separable Gaussian derivatives**
 - the network then only learns the parameters needed to construct a filter as a linear combination of the basis filters
 - this applies to all layers

variants and their derivatives

convolution variants

- we will examine a number of variants of convolution, each only in one dimension
- this leaves an extension to one more spatial dimension (convolution), and one more feature dimension (matrix multiplication)
- in each case, we will write convolution as matrix multiplication, where the matrix has some special structure: derivatives are then straightforward

standard convolution

- input size n , kernel size r , output size n'

$$x \quad \begin{array}{|c|c|c|c|c|c|c|} \hline \textcolor{blue}{\square} & \textcolor{blue}{\square} & \textcolor{blue}{\square} & \textcolor{blue}{\square} & \textcolor{blue}{\square} & \textcolor{blue}{\square} & \textcolor{blue}{\square} \\ \hline \end{array} \quad n = 7, r = 3$$

$$a = w \star x \quad \begin{array}{|c|c|c|c|c|} \hline & & & & \\ \hline \end{array} \quad n' = n - r + 1 = 5$$

- written as matrix multiplication

$$\mathbf{a} = W^\top \cdot \mathbf{x}$$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} = \begin{pmatrix} w_1 & w_2 & w_3 \\ & w_1 & w_2 & w_3 \\ & & w_1 & w_2 & w_3 \\ & & & w_1 & w_2 & w_3 \\ & & & & w_1 & w_2 & w_3 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix}$$

standard convolution

- input size n , kernel size r , output size n'

$$x \quad \begin{array}{cccccc} 1 & 2 & 3 & \text{blue} & \text{blue} & \text{blue} \end{array} \quad n = 7, r = 3$$

$$a = w \star x \quad \begin{array}{ccccc} \text{green} & \text{white} & \text{white} & \text{white} & \text{white} \end{array} \quad n' = n - r + 1 = 5$$

- written as matrix multiplication

$$\mathbf{a} = W^\top \cdot \mathbf{x}$$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \end{pmatrix} = \begin{pmatrix} w_1 & w_2 & w_3 \\ & w_1 & w_2 & w_3 \\ & & w_1 & w_2 & w_3 \\ & & & w_1 & w_2 & w_3 \\ & & & & w_1 & w_2 & w_3 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix}$$

standard convolution

- input size n , kernel size r , output size n'

$$x \quad \begin{array}{ccccccc} \text{blue} & 1 & \text{red} & 3 & \text{blue} & \text{blue} & \text{blue} \end{array} \quad n = 7, r = 3$$

$$a = w \star x \quad \begin{array}{ccccc} \text{green} & \text{green} & \text{white} & \text{white} & \text{white} \end{array} \quad n' = n - r + 1 = 5$$

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standard convolution

- input size n , kernel size r , output size n'

$$x \quad \begin{array}{c} \text{blue} \\ \text{blue} \\ \text{purple} \\ | \\ 1 \quad 2 \quad 3 \\ \text{purple} \\ \text{blue} \end{array} \quad n = 7, r = 3$$

$$a = w \star x \quad \begin{array}{c} \text{green} \\ \text{green} \\ \text{green} \\ \text{white} \\ \text{white} \end{array} \quad n' = n - r + 1 = 5$$

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standard convolution

- input size n , kernel size r , output size n'

$$x \quad \begin{array}{ccccccccc} \text{blue} & \text{blue} & \text{blue} & 1 & 2 & 3 & \text{purple} & \text{blue} \\ \hline \end{array} \quad n = 7, r = 3$$

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standard convolution

- input size n , kernel size r , output size n'

$$x \quad \begin{array}{ccccccccc} \text{blue} & \text{blue} & \text{blue} & \text{blue} & \text{purple} & 1 & 2 & 3 & \text{purple} \end{array} \quad n = 7, r = 3$$

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standard convolution: input derivative

- in general, $C = AB \rightarrow dA = (dC)B^\top, dB = A^\top dC$
- here, $\mathbf{a} = W^\top \mathbf{x}$: derivative with respect to input \mathbf{x}

$$d\mathbf{x} = W \cdot d\mathbf{a}$$

$$d \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix} = \begin{pmatrix} w_1 & & & & & & \\ w_2 & w_1 & & & & & \\ w_3 & w_2 & w_1 & & & & \\ & w_3 & w_2 & w_1 & & & \\ & & w_3 & w_2 & w_1 & & \\ & & & w_3 & w_2 & w_1 & \\ & & & & w_3 & w_2 & w_1 \end{pmatrix} \cdot d \begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{pmatrix}$$

standard convolution: weight derivative

- in general, $C = AB \rightarrow dA = (dC)B^\top, dB = A^\top dC$
- here, $\mathbf{a} = W^\top \mathbf{x}$: derivative with respect to weights W

$$dW = \mathbf{x} \cdot d\mathbf{a}^\top$$

$$dW = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix} \cdot d(a_1 \quad a_2 \quad a_3 \quad a_4 \quad a_5)$$

- this is not convenient: we really want $d\mathbf{w} = (dw_1, dw_2, dw_3)$
- if $da_i = \mathbb{1}[i = 4]$, then $d\mathbf{w} = (x_4, x_5, x_6)$: we learn the pattern that generated the activation

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$$dw = da \star x$$

$$d \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = d \begin{pmatrix} a_1 & a_2 & a_3 & a_4 & a_5 \\ & a_1 & a_2 & a_3 & a_4 & a_5 \\ & & a_1 & a_2 & a_3 & a_4 & a_5 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix}$$

- sharing in forward \equiv adding in backward
- if $da_i = \mathbb{1}[i = 4]$, then $dw = (x_4, x_5, x_6)$: we learn the pattern that generated the activation

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$$d \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = d \begin{pmatrix} a_1 & a_2 & a_3 & \textcolor{red}{a}_4 & a_5 \\ & a_1 & a_2 & a_3 & \textcolor{red}{a}_4 & a_5 \\ & & a_1 & a_2 & a_3 & \textcolor{red}{a}_4 & a_5 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ \textcolor{red}{x}_4 \\ \textcolor{red}{x}_5 \\ x_6 \\ x_7 \end{pmatrix}$$

- sharing in forward \equiv adding in backward
- if $da_i = \mathbb{1}[i = 4]$, then $d\mathbf{w} = (x_4, x_5, x_6)$: we learn the pattern that generated the activation

padded convolution

- input size n , kernel size r , padding p , padded input $\mathbf{x}_{(p)} = (\mathbf{0}_p; \mathbf{x}; \mathbf{0}_p)$, output size n'

$$\mathbf{x}_{(p)} \quad \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline & \text{white} & \text{blue} & \text{blue} & \text{blue} & \text{blue} & \text{blue} & \text{blue} & \text{white} \\ \hline \end{array} \quad n = 7, r = 3, p = 1$$

$$a = w \star x_{(p)} \quad \begin{array}{|c|c|c|c|c|c|c|c|c|} \hline & \text{white} \\ \hline \end{array} \quad n' = (n + 2p) - r + 1 = 7$$

- written as matrix multiplication

$$\mathbf{a} = W^\top \cdot \mathbf{x}$$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{pmatrix} = \begin{pmatrix} w_2 & w_3 \\ w_1 & w_2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix}$$

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$$\mathbf{x}_{(p)} \quad \begin{array}{ccccccc} 1 & 2 & 3 & & & & \\ \text{red} & \text{red} & \text{purple} & \text{blue} & \text{blue} & \text{blue} & \text{white} \end{array} \quad n = 7, r = 3, p = 1$$

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$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{pmatrix} = \begin{pmatrix} w_2 & w_3 \\ w_1 & w_2 & w_3 \\ & w_1 & w_2 & w_3 \\ & & w_1 & w_2 & w_3 \\ & & & w_1 & w_2 & w_3 \\ & & & & w_1 & w_2 & w_3 \\ & & & & & w_1 & w_2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix}$$

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- written as matrix multiplication

$$\mathbf{a} = W^\top \cdot \mathbf{x}$$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{pmatrix} = \begin{pmatrix} w_2 & w_3 \\ w_1 & w_2 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix}$$

padded convolution

- input size n , kernel size r , padding p , padded input $\mathbf{x}_{(p)} = (\mathbf{0}_p; \mathbf{x}; \mathbf{0}_p)$, output size n'

$$a = w \star x_{(p)} \quad \begin{array}{|c|c|c|c|c|c|c|} \hline \textcolor{lightgreen}{\square} & \textcolor{lightgreen}{\square} & \textcolor{lightgreen}{\square} & \textcolor{lightgreen}{\square} & \textcolor{lightgreen}{\square} & \textcolor{red}{\square} & \textcolor{lightgreen}{\square} \\ \hline \end{array} \quad n' = (n + 2p) - r + 1 = 7$$

- written as matrix multiplication

$$\mathbf{a} = W^\top \cdot \mathbf{x}$$

padded convolution

- input size n , kernel size r , padding p , padded input $\mathbf{x}_{(p)} = (\mathbf{0}_p; \mathbf{x}; \mathbf{0}_p)$, output size n'

$$\mathbf{x}_{(p)} \quad \begin{array}{ccccccccc} \text{white} & \text{blue} & \text{blue} & \text{blue} & \text{blue} & \text{blue} & \text{purple} & \text{red} & \text{red} \\ \hline \end{array} \quad n = 7, r = 3, p = 1$$

$$a = w \star x_{(p)} \quad \begin{array}{ccccccccc} \text{light green} & \text{bright green} \\ \hline \end{array} \quad n' = (n + 2p) - r + 1 = 7$$

- written as matrix multiplication

$$\mathbf{a} = W^\top \cdot \mathbf{x}$$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \\ a_4 \\ a_5 \\ a_6 \\ a_7 \end{pmatrix} = \begin{pmatrix} \textcolor{orange}{w_2} & \textcolor{orange}{w_3} & & & & & \\ w_1 & w_2 & w_3 & & & & \\ & w_1 & w_2 & w_3 & & & \\ & & w_1 & w_2 & w_3 & & \\ & & & w_1 & w_2 & w_3 & \\ & & & & w_1 & w_2 & w_3 \\ & & & & & \textcolor{orange}{w_1} & \textcolor{orange}{w_2} \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix}$$

padding preserves size

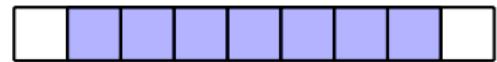
- if kernel size $r = 2\ell + 1$ and $p = \ell$, then $n' = n + 2p - r + 1 = n$ and the size is preserved
- over several layers:

$$p = 0$$

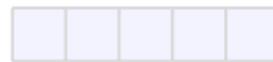
L_1



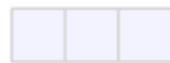
$$p = 1$$



L_2



L_3

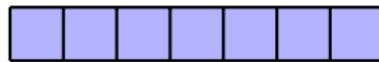


padding preserves size

- if kernel size $r = 2\ell + 1$ and $p = \ell$, then $n' = n + 2p - r + 1 = n$ and the size is preserved
- over several layers:

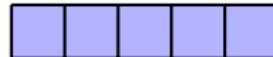
$$p = 0$$

L_1

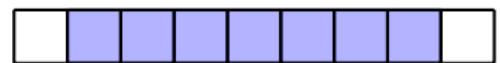
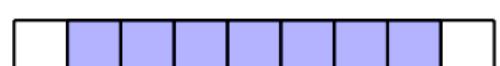


$$p = 1$$

L_2

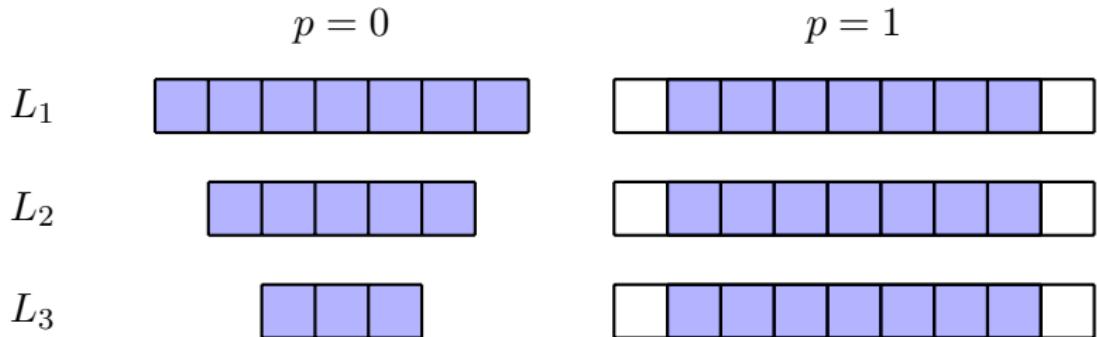


L_3



padding preserves size

- if kernel size $r = 2\ell + 1$ and $p = \ell$, then $n' = n + 2p - r + 1 = n$ and the size is preserved
- over several layers:



strided convolution (down-sampling)

- input size n , kernel size r , stride s , output size n'

$$x \quad \begin{array}{|c|c|c|c|c|c|c|} \hline \textcolor{blue}{\square} & \textcolor{blue}{\square} & \textcolor{blue}{\square} & \textcolor{blue}{\square} & \textcolor{blue}{\square} & \textcolor{blue}{\square} & \textcolor{blue}{\square} \\ \hline \end{array} \quad n = 7, r = 3, s = 2$$

$$a = (w \star x) \downarrow_s \quad \begin{array}{|c|c|c|} \hline \textcolor{white}{\square} & \textcolor{white}{\square} & \textcolor{white}{\square} \\ \hline \end{array} \quad n' = \lfloor (n - r)/s \rfloor + 1 = 3$$

- like standard convolution followed by **down-sampling**, but efficient
- written as matrix multiplication (rows sub-sampled)

$$\mathbf{a} = W^\top \cdot \mathbf{x}$$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} w_1 & w_2 & w_3 & & & \\ & w_1 & w_2 & w_3 & & \\ & & w_1 & w_2 & w_3 & \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix}$$

strided convolution (down-sampling)

- input size n , kernel size r , stride s , output size n'

$$x \quad \begin{array}{cccccc} 1 & 2 & 3 & \text{blue} & \text{blue} & \text{blue} \end{array} \quad n = 7, r = 3, s = 2$$

$$a = (w \star x) \downarrow_s \quad \begin{array}{c|c|c} \text{green} & \text{white} & \text{white} \end{array} \quad n' = \lfloor (n - r)/s \rfloor + 1 = 3$$

- like standard convolution followed by **down-sampling**, but efficient
- written as matrix multiplication (rows sub-sampled)

$$\mathbf{a} = W^\top \cdot \mathbf{x}$$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} w_1 & w_2 & w_3 & & & \\ & w_1 & w_2 & w_3 & & \\ & & w_1 & w_2 & w_3 & \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix}$$

strided convolution (down-sampling)

- input size n , kernel size r , stride s , output size n'

$$x \quad \begin{array}{ccccccccc} \text{blue} & \text{blue} & | & 1 & 2 & 3 & | & \text{blue} & \text{blue} \end{array} \quad n = 7, r = 3, s = 2$$

$$a = (w \star x) \downarrow_s \quad \begin{array}{ccc} \text{light green} & \text{green} & | & \text{white} \end{array} \quad n' = \lfloor (n - r)/s \rfloor + 1 = 3$$

- like standard convolution followed by **down-sampling**, but efficient
- written as matrix multiplication (rows sub-sampled)

$$\mathbf{a} = W^\top \cdot \mathbf{x}$$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} w_1 & w_2 & w_3 & & & \\ & w_1 & w_2 & w_3 & & \\ & & w_1 & w_2 & w_3 & \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix}$$

strided convolution (down-sampling)

- input size n , kernel size r , stride s , output size n'

$$x \quad \begin{array}{ccccccccc} & \text{blue} & & \text{blue} & & \text{blue} & & \text{red} & \text{red} \\ & 1 & & 2 & & 3 & & & \end{array} \quad n = 7, r = 3, s = 2$$

$$a = (w \star x) \downarrow_s \quad \begin{array}{ccc} & \text{green} & \text{green} & \text{green} \\ & 1 & 2 & 3 \end{array} \quad n' = \lfloor (n - r)/s \rfloor + 1 = 3$$

- like standard convolution followed by **down-sampling**, but efficient
- written as matrix multiplication (rows sub-sampled)

$$\mathbf{a} = W^\top \cdot \mathbf{x}$$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} w_1 & w_2 & w_3 & & & \\ & w_1 & w_2 & w_3 & & \\ & & w_1 & w_2 & w_3 & \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix}$$

strided convolution: input derivative

- in general, $C = AB \rightarrow dA = (dC)B^\top, dB = A^\top dC$
- here, $\mathbf{a} = W^\top \mathbf{x}$: derivative with respect to input \mathbf{x}

$$d\mathbf{x} = W \cdot d\mathbf{a}$$

$$d \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix} = \begin{pmatrix} w_1 & & & & \\ w_2 & w_1 & & & \\ w_3 & w_2 & w_1 & & \\ & w_3 & w_2 & w_1 & \\ & & & & w_3 \end{pmatrix} \cdot d \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix}$$

strided convolution: weight derivative

- in general, $C = AB \rightarrow dA = (dC)B^\top, dB = A^\top dC$
- here, $\mathbf{a} = W^\top \mathbf{x}$: derivative with respect to weights W

$$dW = \mathbf{x} \cdot d\mathbf{a}^\top$$

$$d \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = d \begin{pmatrix} a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \\ a_1 & a_2 & a_3 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix}$$

- again e.g. by writing W as a function of $\mathbf{w} = (w_1, w_2, w_3)$ and applying the chain rule, or **by just observing the moving pattern**

dilated convolution (up-sampling)

- input size n , kernel size r , dilation factor t , effective kernel size $\hat{r} = r + (r - 1)(t - 1)$, output size n'

$$x \quad \begin{array}{|c|c|c|c|c|c|c|}\hline \textcolor{blue}{\square} & \textcolor{blue}{\square} & \textcolor{blue}{\square} & \textcolor{blue}{\square} & \textcolor{blue}{\square} & \textcolor{blue}{\square} & \textcolor{blue}{\square} \\ \hline \end{array} \quad n = 7, r = 3, t = 2$$

$$a = w \uparrow^t \star x \quad \begin{array}{|c|c|c|}\hline \textcolor{blue}{\square} & \textcolor{blue}{\square} & \textcolor{blue}{\square} \\ \hline \end{array} \quad n' = n - \hat{r} + 1 = 3$$

- written as matrix multiplication (like strided backward!)

$$\mathbf{a} = W^\top \cdot \mathbf{x}$$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} w_1 & w_2 & w_3 \\ & w_1 & w_2 & w_3 \\ & & w_1 & w_2 & w_3 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix}$$

dilated convolution (up-sampling)

- input size n , kernel size r , dilation factor t , effective kernel size $\hat{r} = r + (r - 1)(t - 1)$, output size n'

$$x \quad \begin{array}{c} 1 \\ \text{purple} \end{array} \quad \begin{array}{c} 2 \\ \text{red} \end{array} \quad \begin{array}{c} 3 \\ \text{purple} \end{array} \quad n = 7, r = 3, t = 2$$

$$a = w \uparrow^t \star x \quad \begin{array}{c} \text{green} \\ \text{white} \\ \text{white} \end{array} \quad n' = n - \hat{r} + 1 = 3$$

- written as matrix multiplication (like strided backward!)

$$\mathbf{a} = W^\top \cdot \mathbf{x}$$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} w_1 & w_2 & w_3 \\ w_1 & w_2 & w_3 \\ w_1 & w_2 & w_3 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix}$$

dilated convolution (up-sampling)

- input size n , kernel size r , dilation factor t , effective kernel size $\hat{r} = r + (r - 1)(t - 1)$, output size n'

$$x \quad \begin{array}{ccccccccc} \text{blue} & | & \text{purple} & | & \text{blue} & | & \text{red} & | & \text{blue} & | & \text{purple} & | & \text{blue} \\ & 1 & & & & 2 & & & & 3 & & & & \end{array} \quad n = 7, r = 3, t = 2$$

$$a = w \uparrow^t \star x \quad \begin{array}{ccc} \text{light green} & | & \text{green} & | & \text{white} \end{array} \quad n' = n - \hat{r} + 1 = 3$$

- written as matrix multiplication (like strided backward!)

$$\mathbf{a} = W^\top \cdot \mathbf{x}$$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} w_1 & & w_2 & & w_3 & & \\ & w_1 & & w_2 & & w_3 & \\ & & w_1 & & w_2 & & w_3 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix}$$

dilated convolution (up-sampling)

- input size n , kernel size r , dilation factor t , effective kernel size $\hat{r} = r + (r - 1)(t - 1)$, output size n'

$$x \quad \begin{array}{c} \text{blue} \\ \text{blue} \\ 1 \\ \text{blue} \\ \text{red} \\ \text{blue} \\ 3 \\ \text{purple} \end{array} \quad n = 7, r = 3, t = 2$$

$$a = w \uparrow^t \star x \quad \begin{array}{c} \text{light green} \\ \text{light green} \\ \text{green} \end{array} \quad n' = n - \hat{r} + 1 = 3$$

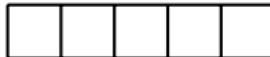
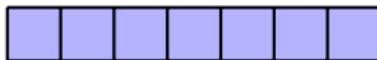
- written as matrix multiplication (like strided backward!)

$$\mathbf{a} = W^\top \cdot \mathbf{x}$$

$$\begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} w_1 & w_2 & w_3 \\ & w_1 & w_2 & w_3 \\ & & w_1 & w_2 & w_3 \end{pmatrix} \cdot \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \\ x_7 \end{pmatrix}$$

dilated convolution (up-sampling)

- suppose a filter has been trained at a given resolution

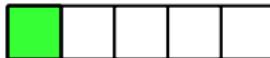


- à trous algorithm: given an input at twice the resolution, apply the same filter dilated by a factor of 2



dilated convolution (up-sampling)

- suppose a filter has been trained at a given resolution



- à trous algorithm: given an input at twice the resolution, apply the same filter dilated by a factor of 2



dilated convolution (up-sampling)

- suppose a filter has been trained at a given resolution

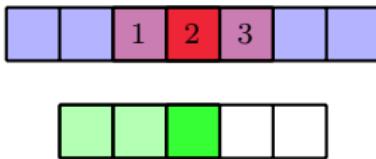


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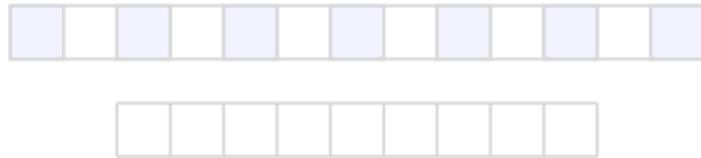


dilated convolution (up-sampling)

- suppose a filter has been trained at a given resolution

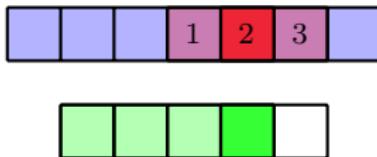


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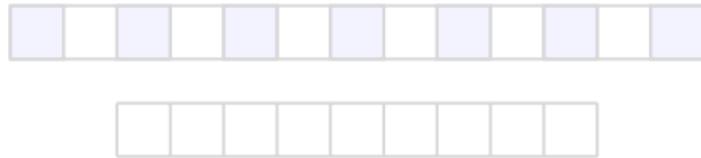


dilated convolution (up-sampling)

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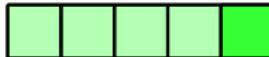
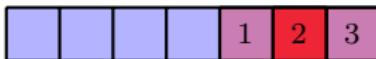


- à trous algorithm: given an input at twice the resolution, apply the same filter dilated by a factor of 2

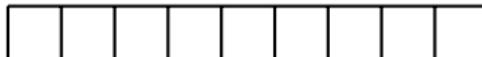


dilated convolution (up-sampling)

- suppose a filter has been trained at a given resolution

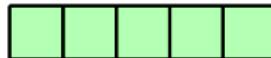
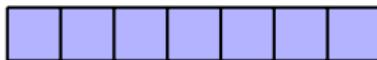


- **à trous algorithm:** given an input at twice the resolution, apply the same filter dilated by a factor of 2



dilated convolution (up-sampling)

- suppose a filter has been trained at a given resolution

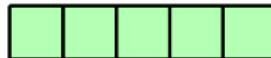
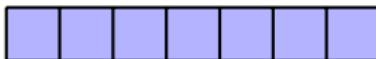


- **à trous algorithm:** given an input at twice the resolution, apply the same filter dilated by a factor of 2

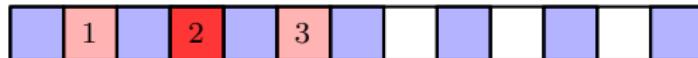


dilated convolution (up-sampling)

- suppose a filter has been trained at a given resolution

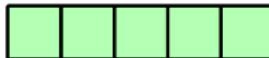
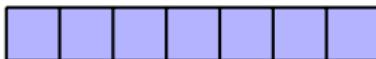


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dilated convolution (up-sampling)

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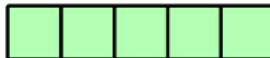
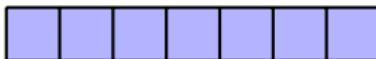


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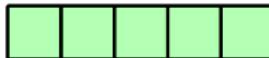
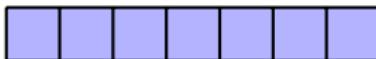


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dilated convolution (up-sampling)

- suppose a filter has been trained at a given resolution

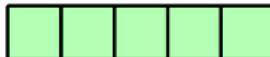
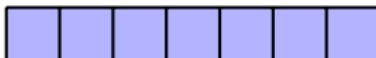


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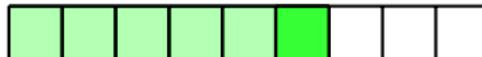
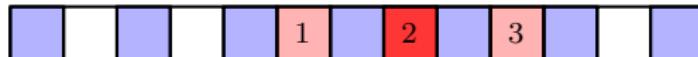


dilated convolution (up-sampling)

- suppose a filter has been trained at a given resolution

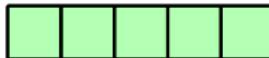
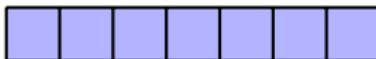


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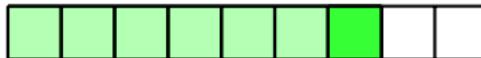
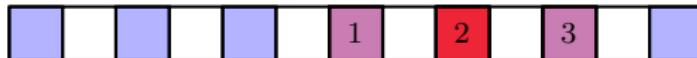


dilated convolution (up-sampling)

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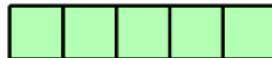
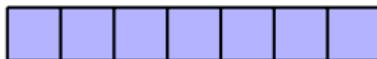


- **à trous algorithm:** given an input at twice the resolution, apply the same filter dilated by a factor of 2

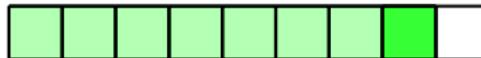


dilated convolution (up-sampling)

- suppose a filter has been trained at a given resolution

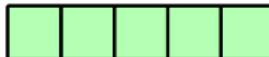
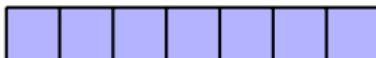


- **à trous algorithm:** given an input at twice the resolution, apply the same filter dilated by a factor of 2

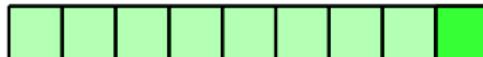


dilated convolution (up-sampling)

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- **à trous algorithm:** given an input at twice the resolution, apply the same filter dilated by a factor of 2



convolutional layer arithmetic

- **input volume** $v = w \times h \times k$
- **hyperparameters** k' filters, kernel size r , padding p , stride s , dilation factor t
- effective kernel size $\hat{r} = r + (r - 1)(t - 1)$
- **output volume** $v' = w' \times h' \times k'$ with

$$w' = \lfloor (w + 2p - \hat{r})/s \rfloor + 1$$
$$h' = \lfloor (h + 2p - \hat{r})/s \rfloor + 1$$

- r^2kk' weights, k' biases, $(r^2k + 1)k'$ **parameters** in total
- $(r^2k + 1)v' = (r^2k + 1)k' \times w' \times h'$ **operations** in total

convolutional layer arithmetic

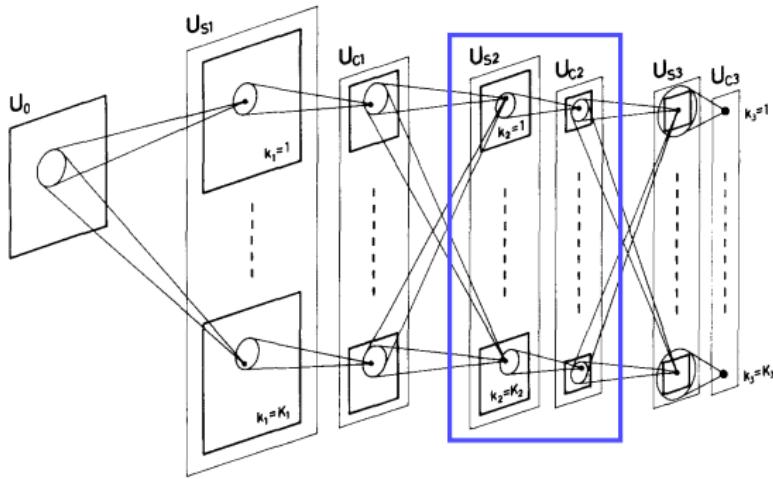
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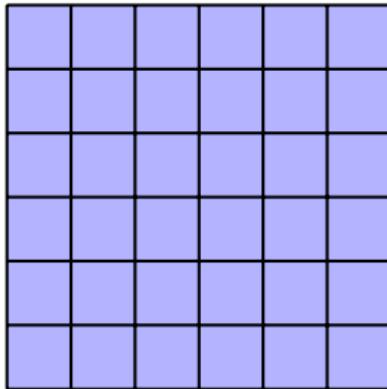
pooling

spatial pooling

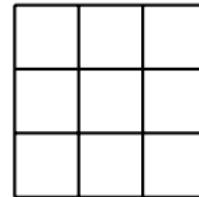


- the deeper a layer is, the larger becomes the **receptive field** of each cell and the **density** of cells decreases accordingly
- gradually introduces translation and deformation **invariance**
- pooling is **independent** per feature map and connections are **fixed**

spatial pooling



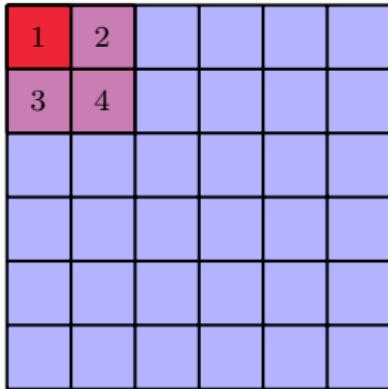
$$n = 6, r = 2, s = 2$$



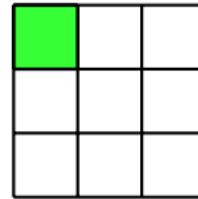
$$n' = \lfloor n/s \rfloor = 3$$

- same “sliding window” as in convolution, only has **no parameters** and performs orderless pooling rather than dot product per neighborhood, e.g. average or max
- no padding but usually stride $s > 1$
- typically, $r = s$ such that $n' = \lfloor (n - r)/s \rfloor + 1 = \lfloor n/s \rfloor$

spatial pooling



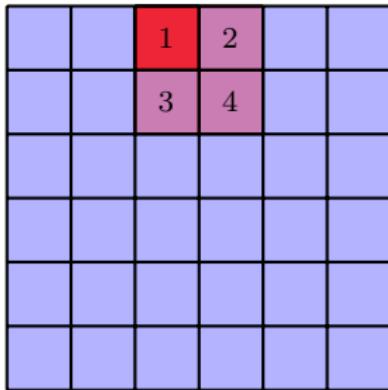
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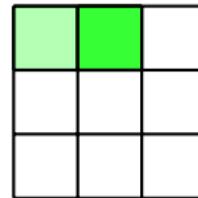
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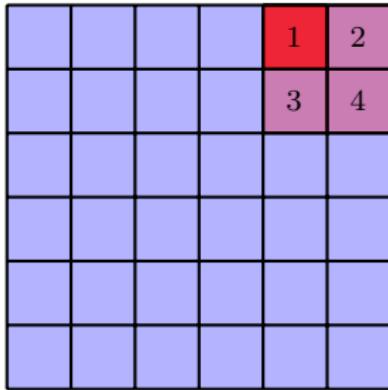
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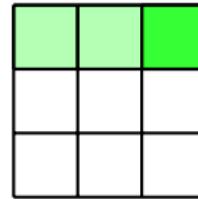
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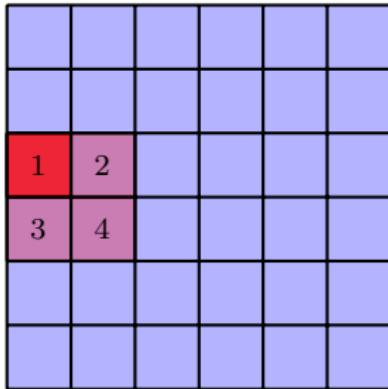
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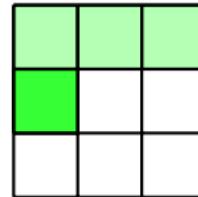
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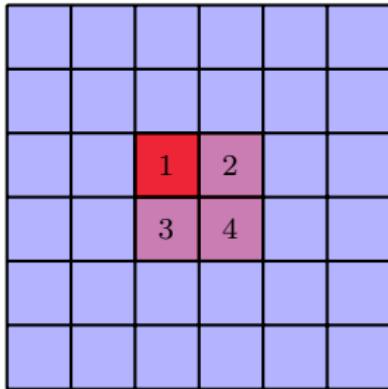
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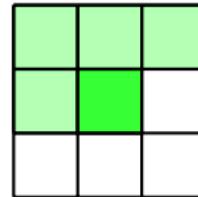
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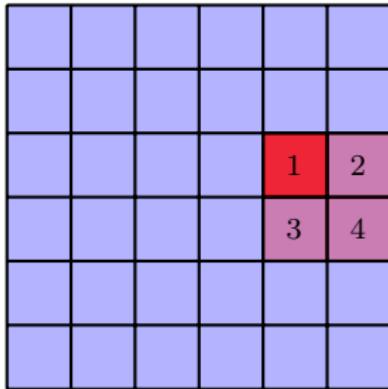
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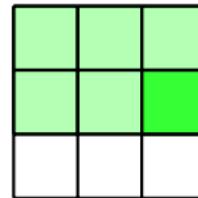
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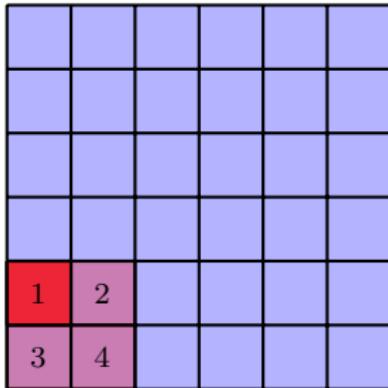
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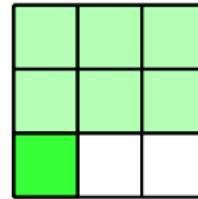
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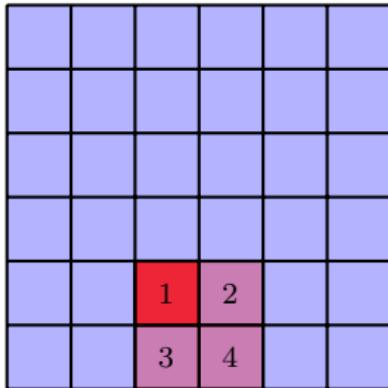
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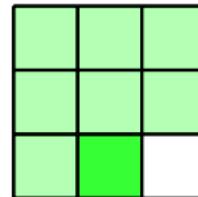
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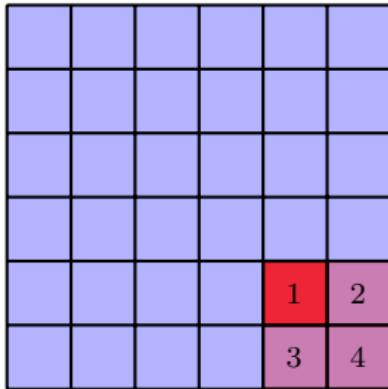
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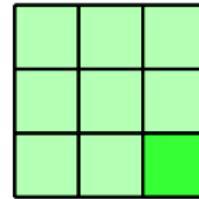
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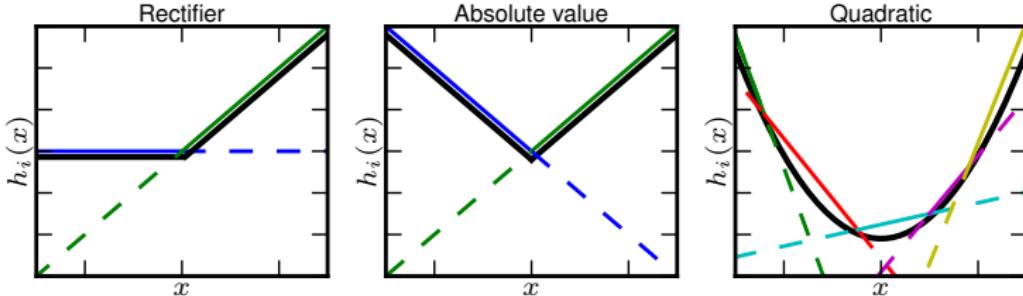
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feature pooling e.g. maxout

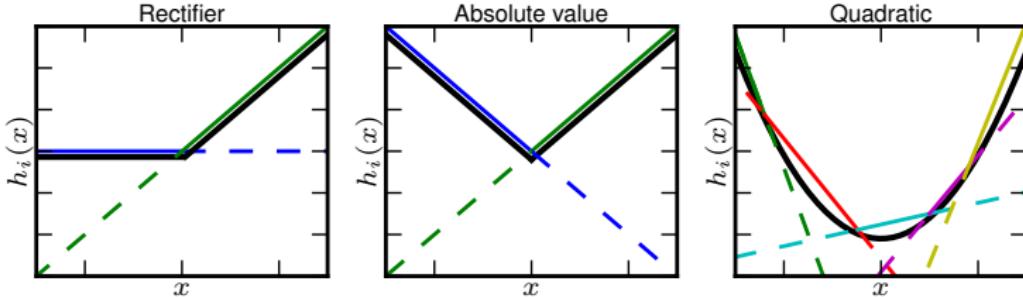


- unlike most activation functions that are element-wise, maxout groups several (e.g. k) activations together and takes their maximum

$$a = \max_j \mathbf{w}_j^\top \mathbf{x} + b_j$$

- does not saturate or “die”, but increases the cost by k
- can approximate any convex function
- two such units can approximate any smooth function!

feature pooling e.g. maxout

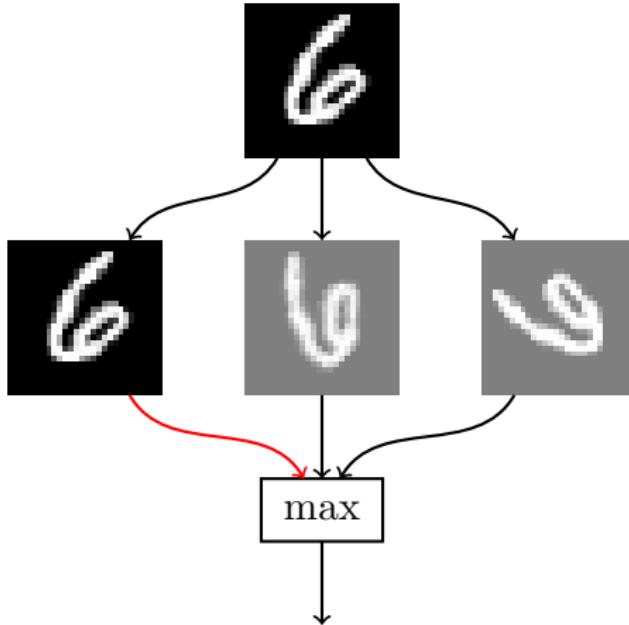


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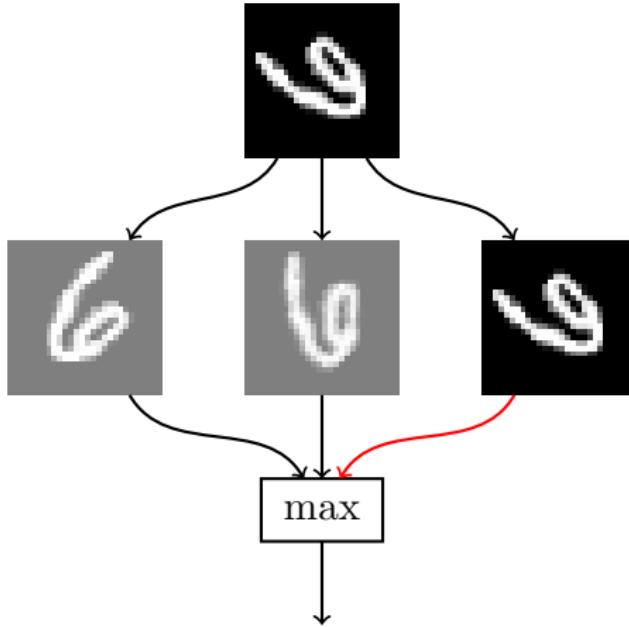
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feature pooling: pose invariance



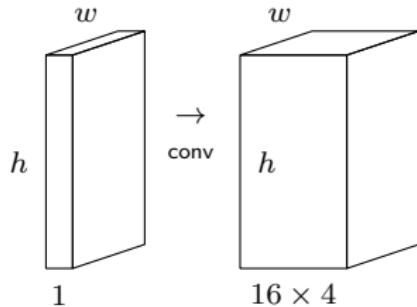
- if each activation responds to a different pose or view, maxout will respond to any

feature pooling: pose invariance



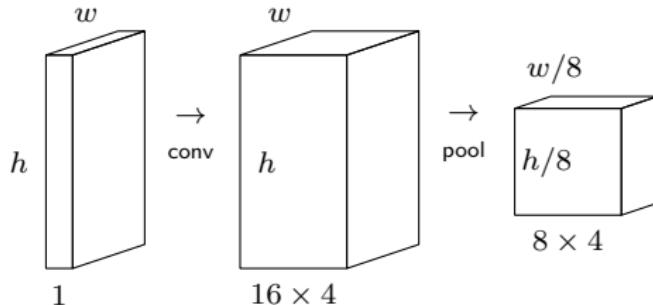
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remember HMAX?



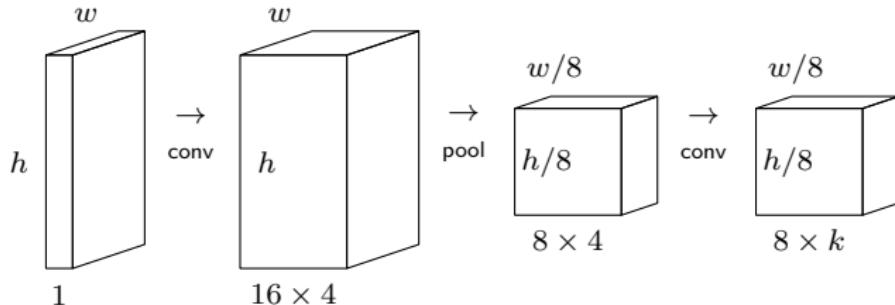
- apply filters at 16 scales \times 4 orientations
- max-pooling over 8×8 spatial cells and over 2 scales
- convolutional RBF matching of input patches X to $k = 4072$ prototypes P (4×4 patches at 4 orientations) extracted at random during learning: activation $Y = \exp(-\gamma \|X - P\|^2)$
- global max pooling over positions and scales
- orientations are not pooled

remember HMAX?



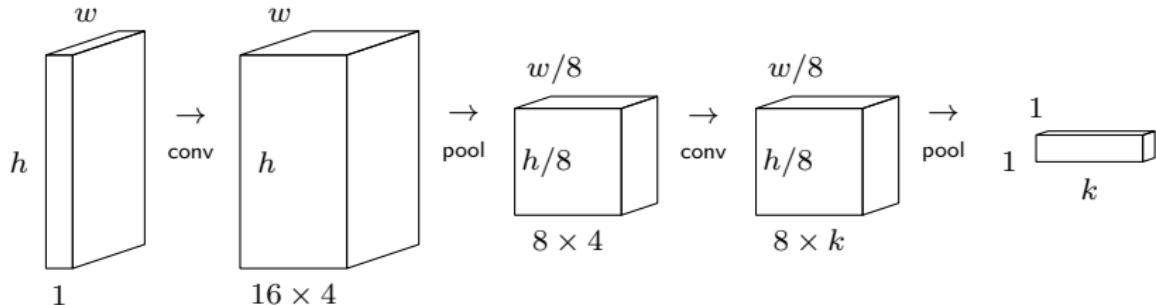
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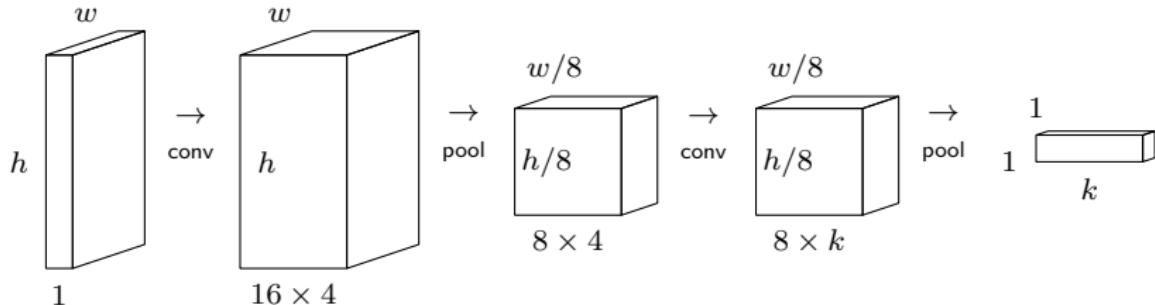
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fun

convolutional network

			MNIST		CIFAR10		
		param	ops	volume	param	ops	volume
x =	input	0	0	$28 \times 28 \times 1$	0	0	$32 \times 32 \times 3$
z ₁ =	conv(5, 32) (x)	832	479232	$24 \times 24 \times 32$	2432	1906688	$28 \times 28 \times 32$
p ₁ =	pool(2) (z ₁)	0	18432	$12 \times 12 \times 32$	0	25088	$14 \times 14 \times 32$
z ₂ =	conv(5, 64) (p ₁)	51264	3280896	$8 \times 8 \times 64$	51264	5126400	$10 \times 10 \times 64$
p ₂ =	pool(2) (z ₂)	0	4096	$4 \times 4 \times 64$	0	6400	$5 \times 5 \times 64$
z ₃ =	fc(100) (p ₂)	102500	102500	100	160100	160100	100
a ₄ =	fc(10) (z ₃)	1010	1010	10	1010	1010	10
y =	softmax (a ₄)	0	0	10	0	0	10

- most **parameters** in first fully connected layer
- most **operations** in second convolutional layer
- most **memory** in first convolutional layer

`conv(r, k'[, p = 0][, s = 1]); (max)- pool(r[, s = r][, p = 0]);`

convolutional network

	MNIST		CIFAR10			
	param	ops	param	ops		
input	0	$28 \times 28 \times 1$	0	$32 \times 32 \times 3$		
conv(5, 32)	832	479232	$24 \times 24 \times 32$	2432	1906688	$28 \times 28 \times 32$
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fc(100)	102500	102500	100	160100	160100	100
fc(10)	1010	1010	10	1010	1010	10
softmax	0	0	10	0	0	10

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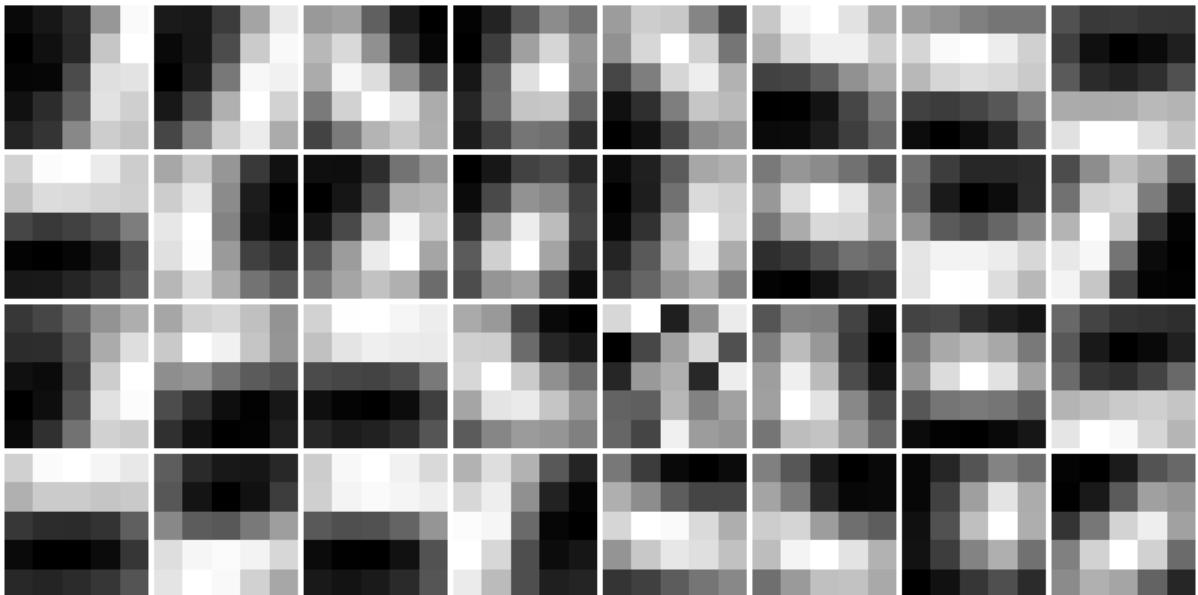
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MNIST layer 1 filters



- mini-batch $m = 128$, learning rate $\epsilon = 10^{-2}$, regularization strength $\lambda = 10^{-2}$, Gaussian initialization $\sigma = 0.1$
- test error: 1.2%

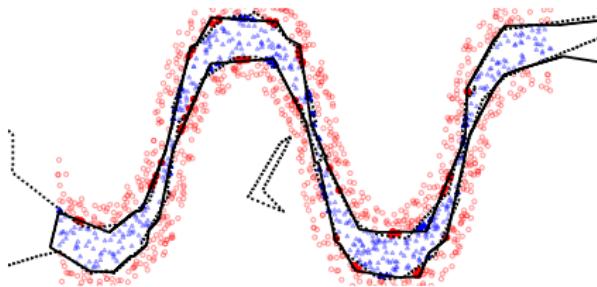
CIFAR10 layer 1 filters



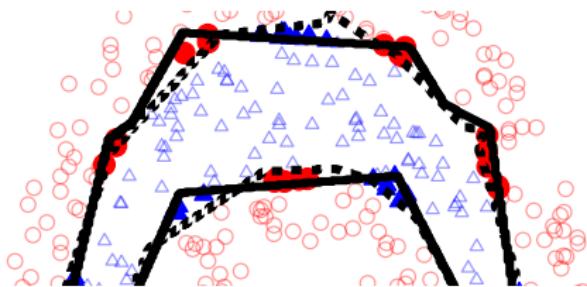
- mini-batch $m = 128$, learning rate $\epsilon = 10^{-2}$, regularization strength $\lambda = 10^{-2}$, Gaussian initialization $\sigma = 0.1$
- test error: 28%

towards deeper networks

[Montufar et al. 2014]



2-layer: solid; 3-layer: dashed
(20 hidden units each)



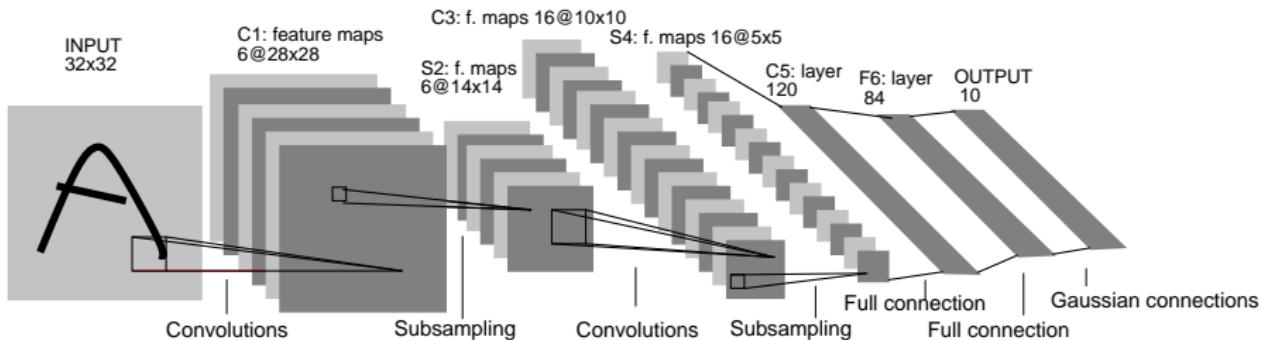
close-up

- “deep networks are able to separate their input space into **exponentially** more linear response regions than their shallow counterparts, despite using the same number of computational units”

network architectures

LeNet-5

[LeCun et al. 1998]



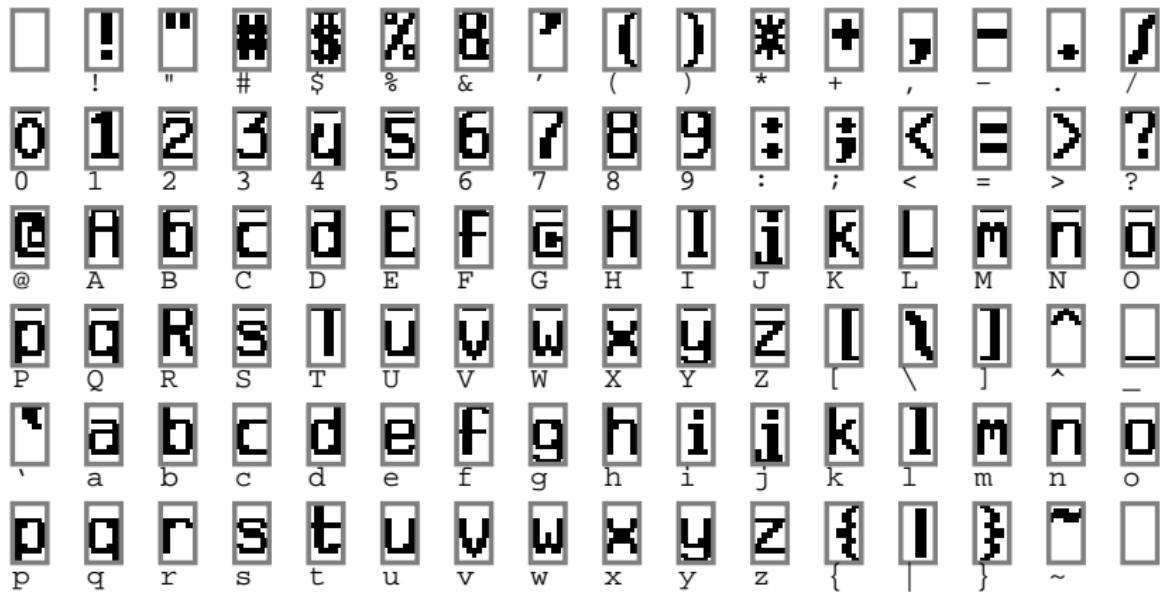
- first convolutional neural network to use back-propagation
- applied to character recognition

LeNet-5

	parameters	operations	volume
input(32, 1)	0	0	$32 \times 32 \times 1$
conv(5, 6)	156	122,304	$28 \times 28 \times 6$
avg(2)	0	4,704	$14 \times 14 \times 6$
conv(5, 16)	2,416	241,600	$10 \times 10 \times 16$
avg(2)	0	1,600	$5 \times 5 \times 16$
conv(5, 120)	48,120	48,120	$1 \times 1 \times 120$
fc(84)	10,164	10,164	84
rbf(10)	850	850	10
softmax	0	10	10

- subsampling by average pooling with learnable global weight and bias
- scaled tanh function after first pooling layer and FC layer
- last convolutional layer allows variable-sized input
- output RBF units: Euclidean distance to 7×12 distributed codes
- softmax-like loss function

LeNet-5 distributed codes



- 7 × 12 character bitmaps
- chosen by hand to initialize the FC-RBF connections
- structured output

LeNet-5 connections between convolutional layers

	0	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15
0	X			X	X	X			X	X	X	X		X	X	
1	X	X			X	X	X			X	X	X	X		X	
2	X	X	X			X	X	X			X		X	X	X	
3		X	X	X		X	X	X	X			X		X	X	
4			X	X	X		X	X	X	X		X	X		X	
5				X	X	X			X	X	X	X		X	X	

- number of connections limited
- forces break of symmetry

ImageNet

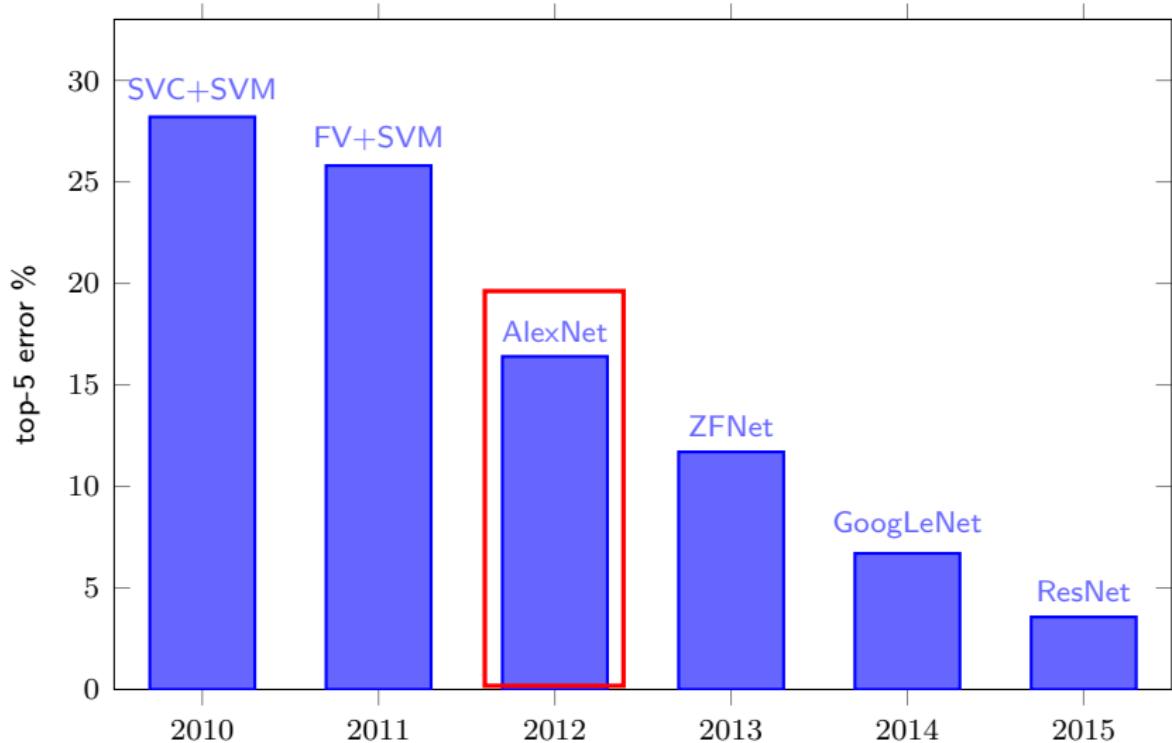
[Russakovsky et al. 2014]



- 22k classes, 15M samples
- ImageNet Large-Scale Visual Recognition Challenge (ILSVRC): 1000 classes, 1.2M training images, 50k validation images, 150k test images

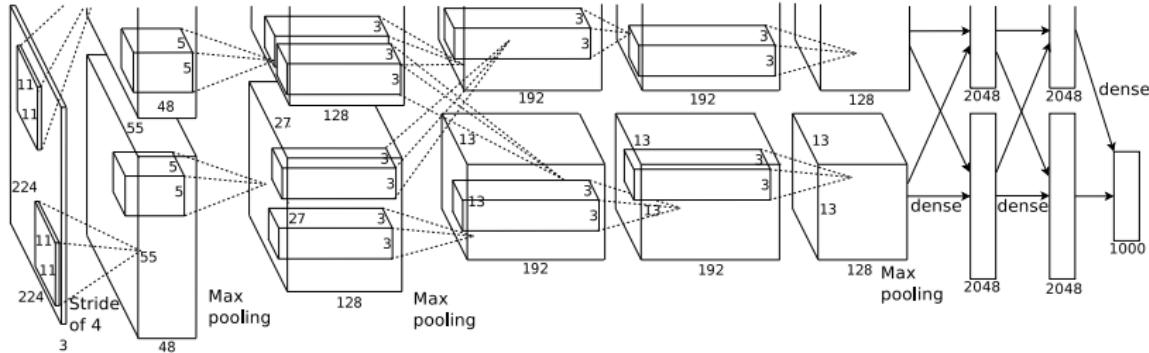
Russakovsky, Deng, Su, Krause, et al. 2014. Imagenet Large Scale Visual Recognition Challenge.

ImageNet classification performance



AlexNet

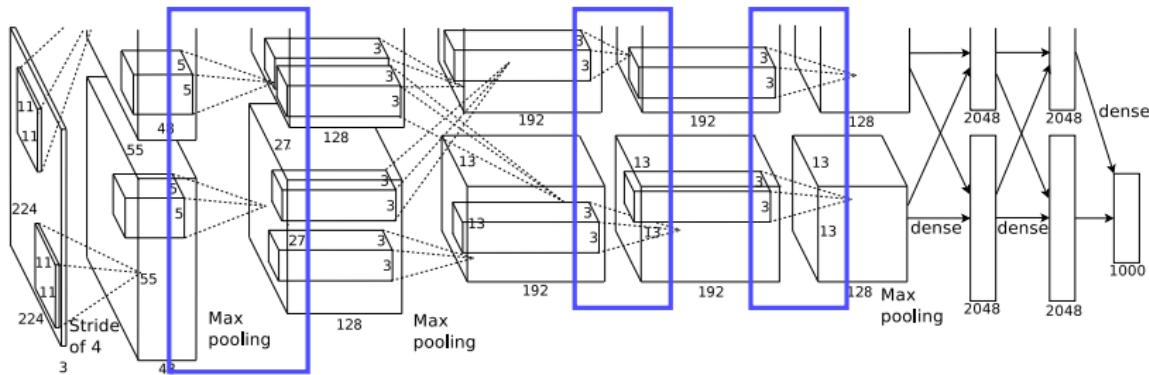
[Krizhevsky et al. 2012]



- 16.4% top-5 error on ILSVRC'12, outperformed all by 10%
- 8 layers
- ReLU, local response normalization, **data augmentation**, **dropout**
- stochastic gradient descent with **momentum**
- implementation on two GPUs; connectivity between the two subnetworks is limited

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[Krizhevsky et al. 2012]



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learned layer 1 kernels



- 96 kernels of size $11 \times 11 \times 3$
- top: 48 GPU 1 kernels; bottom: 48 GPU 2 kernels

AlexNet (CaffeNet)

	parameters	operations	volume
input(227, 3)	0	0	$227 \times 227 \times 3$
conv(11, 96, s4)	34,944	105,705,600	$55 \times 55 \times 96$
pool(3, 2)	0	290,400	$27 \times 27 \times 96$
norm	0	69,984	$27 \times 27 \times 96$
conv(5, 256, p2)	614,656	448,084,224	$27 \times 27 \times 256$
pool(3, 2)	0	186,624	$13 \times 13 \times 256$
norm	0	43,264	$13 \times 13 \times 256$
conv(3, 384, p1)	885,120	149,585,280	$13 \times 13 \times 384$
conv(3, 384, p1)	1,327,488	224,345,472	$13 \times 13 \times 384$
conv(3, 256, p1)	884,992	149,563,648	$13 \times 13 \times 256$
pool(3, 2)	0	43,264	$6 \times 6 \times 256$
fc(4096)	37,752,832	37,752,832	4,096
fc(4096)	16,781,312	16,781,312	4,096
fc(1000)	4,097,000	4,097,000	1,000
softmax	0	1,000	1,000

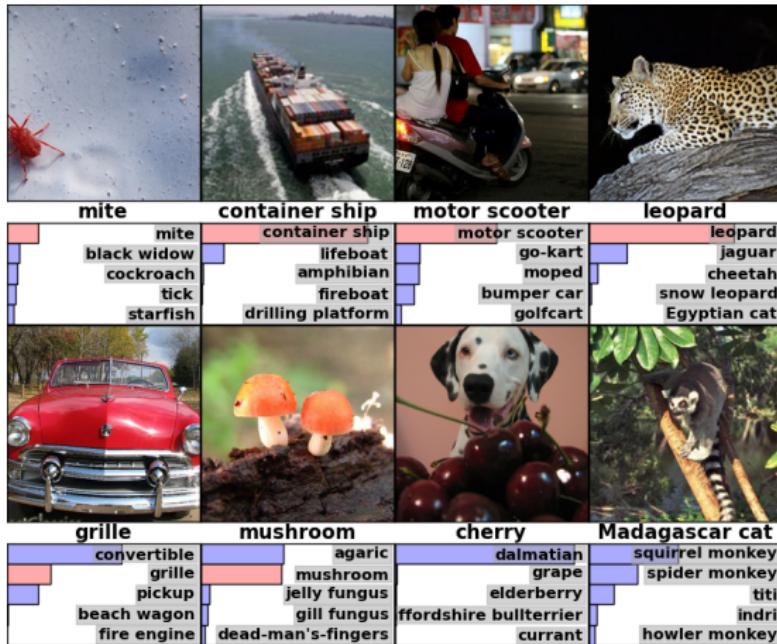
- ReLU follows each convolutional and fully connected layer
- CaffeNet: input size modified from 224×224 , pool/norm switched

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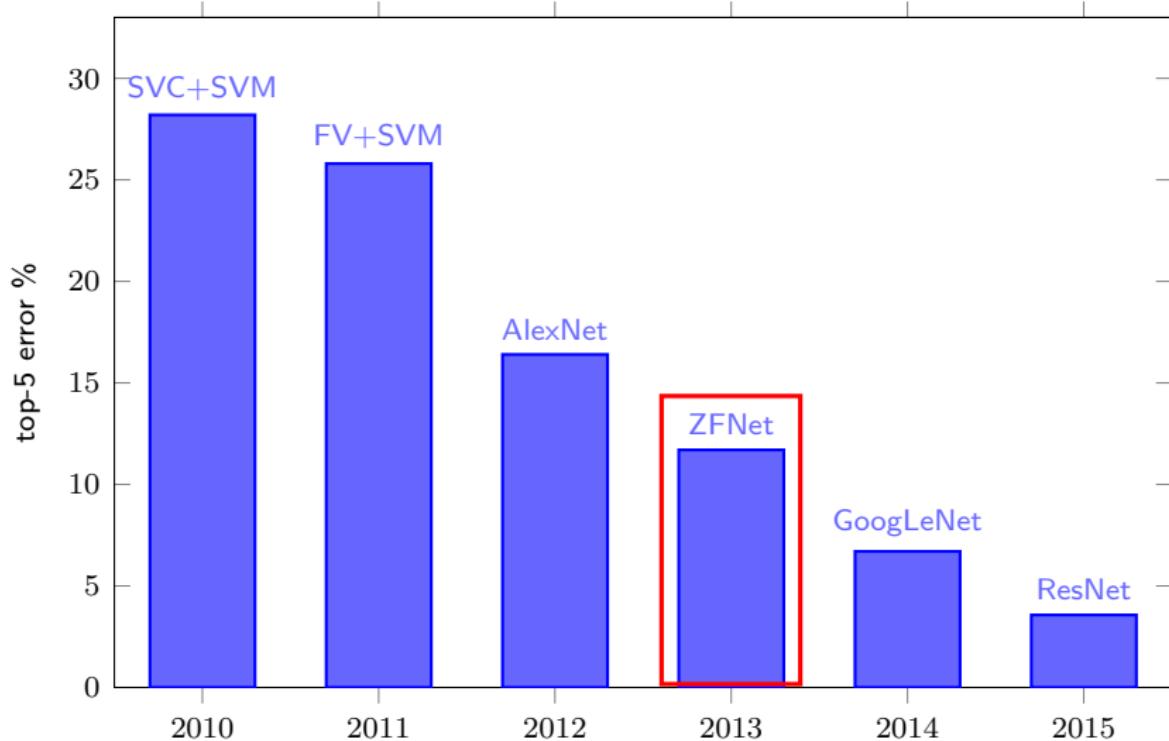
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AlexNet: classification examples

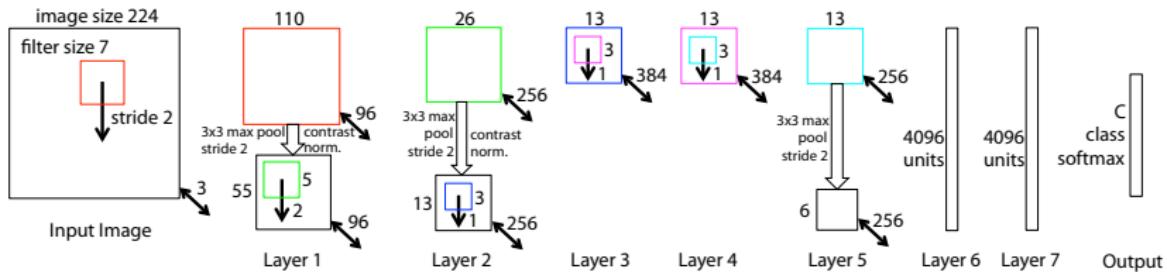


- correct label on top; its predicted probability with red if visible

ImageNet classification performance



ZFNet



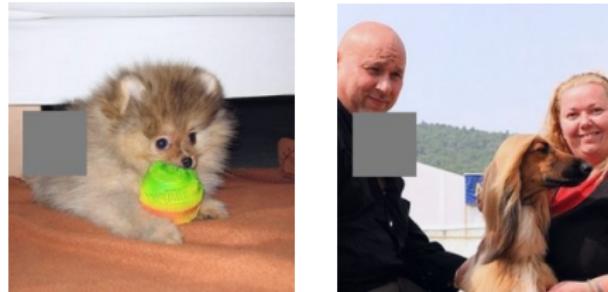
- 11.7% top-5 error on ILSVRC'13
 - 8 layers, refinement of AlexNet
 - layer 1 kernel size (stride) reduced from 11(4) to 7(2) to reduce aliasing artifacts
 - conv3,4,5 width increased to 512, 1024, 512

ZFNet

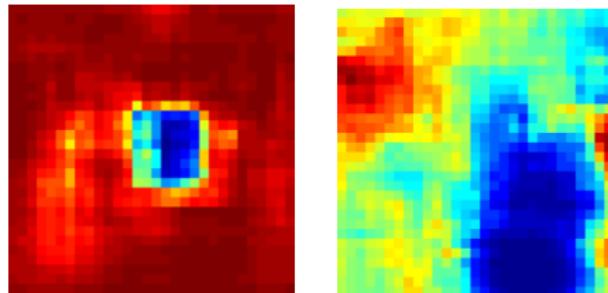
	parameters	operations	volume
input(224, 3)	0	0	$224 \times 224 \times 3$
conv(7, 96, s2, p1)	14,208	171,916,800	$110 \times 110 \times 96$
pool(3, 2, p1)	0	1,161,600	$55 \times 55 \times 96$
norm	0	290,400	$55 \times 55 \times 96$
conv(5, 256, s2)	614,656	415,507,456	$26 \times 26 \times 256$
pool(3, 2, p1)	0	173,056	$13 \times 13 \times 256$
norm	0	43,264	$13 \times 13 \times 256$
conv(3, 512, p1)	1,180,160	199,447,040	$13 \times 13 \times 512$
conv(3, 1024, p1)	4,719,616	797,615,104	$13 \times 13 \times 1024$
conv(3, 512, p1)	4,719,104	797,528,576	$13 \times 13 \times 512$
pool(3, 2)	0	86,528	$6 \times 6 \times 512$
fc(4096)	75,501,568	75,501,568	4,096
fc(4096)	16,781,312	16,781,312	4,096
fc(1000)	4,097,000	4,097,000	1,000
softmax	0	1,000	1,000

- layer widths adjusted by cross-validation; **depth matters**

ZFNet: occlusion sensitivity



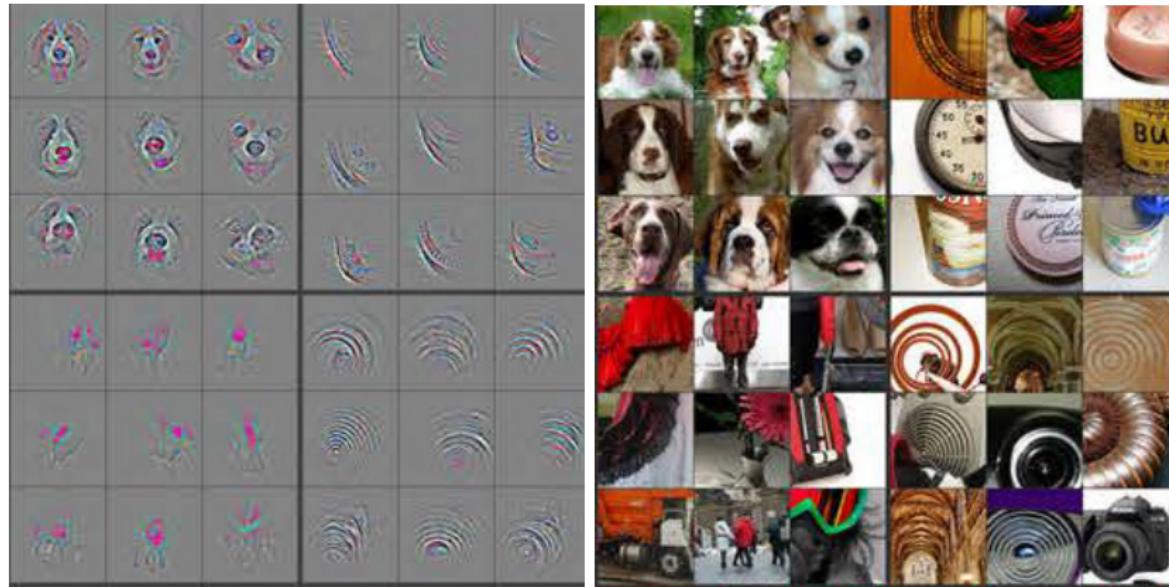
image



correct class probability

- image occluded by gray square; correct class probability shown as a function of the position of the square

ZFNet: visualizing intermediate layers



- reconstructed patterns from top 9 activations of selected features of layer 4 and corresponding image patches

VGG

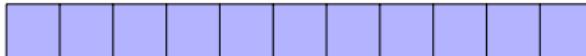
[Simonyan and Zisserman 2014]

ConvNet Configuration					
A	A-LRN	B	C	D	E
11 weight layers	11 weight layers	13 weight layers	16 weight layers	16 weight layers	19 weight layers
input (224 × 224 RGB image)					
conv3-64	conv3-64 LRN	conv3-64 conv3-64	conv3-64 conv3-64	conv3-64 conv3-64	conv3-64 conv3-64
maxpool					
conv3-128	conv3-128	conv3-128 conv3-128	conv3-128 conv3-128	conv3-128	conv3-128 conv3-128
maxpool					
conv3-256 conv3-256	conv3-256 conv3-256	conv3-256 conv3-256	conv3-256 conv3-256 conv1-256	conv3-256 conv3-256 conv3-256	conv3-256 conv3-256 conv3-256
maxpool					
conv3-512 conv3-512	conv3-512 conv3-512	conv3-512 conv3-512	conv3-512 conv3-512 conv1-512	conv3-512 conv3-512 conv3-512	conv3-512 conv3-512 conv3-512
maxpool					
conv3-512 conv3-512	conv3-512 conv3-512	conv3-512 conv3-512	conv3-512 conv3-512 conv1-512	conv3-512 conv3-512 conv3-512	conv3-512 conv3-512 conv3-512
maxpool					

- 7.3% top-5 error on ILSVRC'14
- depth increased up to 19 layers, kernel sizes (strides) reduced to 3(1)
- local response normalization doesn't do anything
- top/bottom layers of deep models pre-initialized by trained model A

effective receptive field

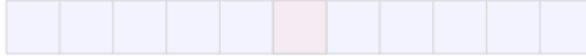
L_0



L_1



L_2

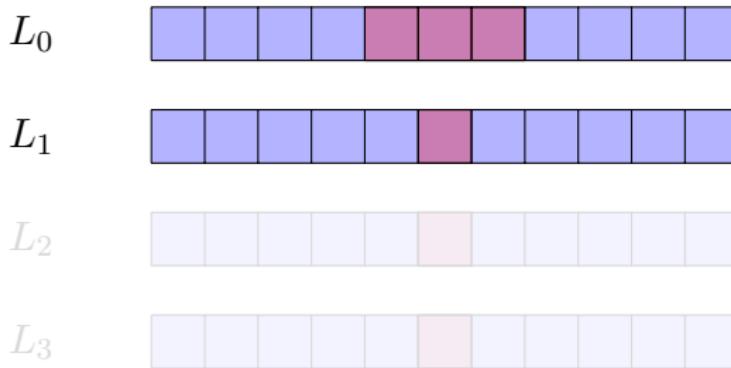


L_3



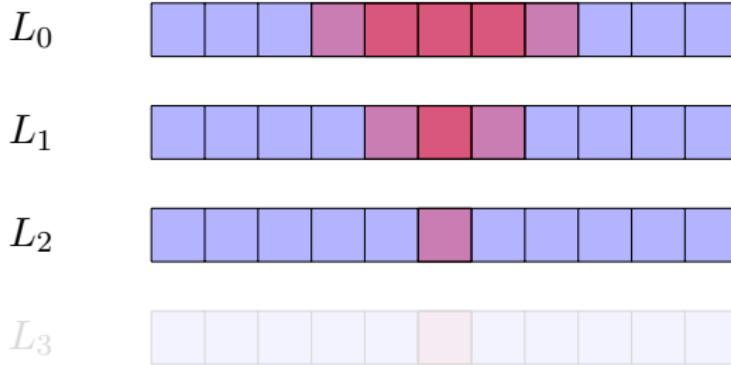
- is the part of the visual input that affects a given cell indirectly through previous layers
- grows linearly with depth
- stack of three 3×3 kernels of stride 1 has the same effective receptive field as a single 7×7 kernel, but fewer parameters

effective receptive field



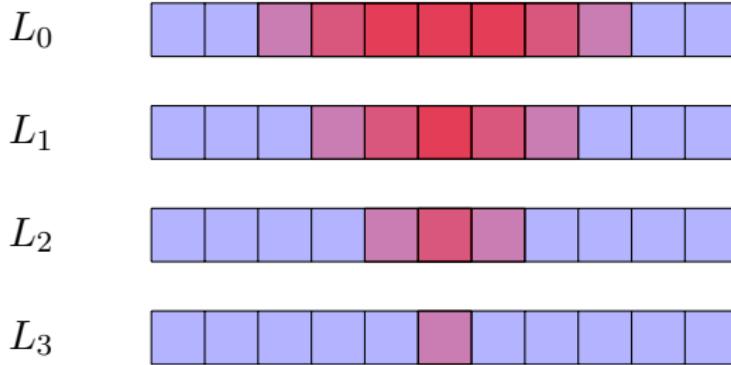
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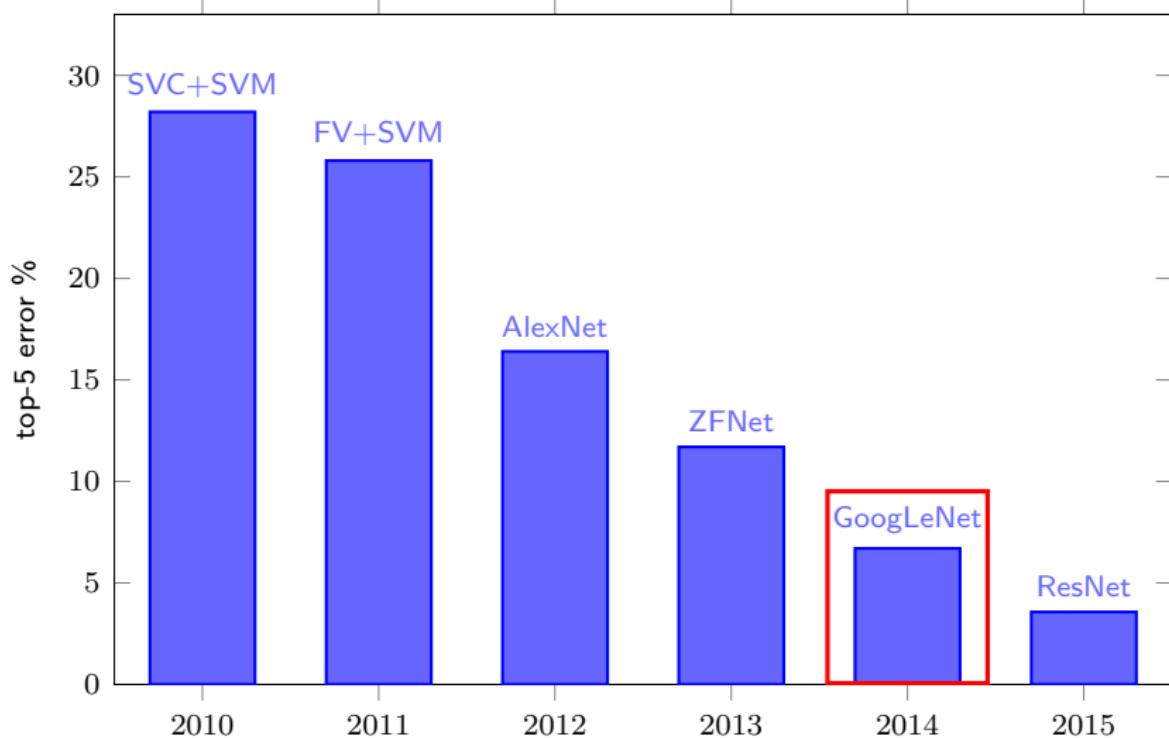
VGG-16

	parameters	operations	volume
input(224, 3)	0	0	$224 \times 224 \times 3$
conv(3, 64, p1)	1,792	89,915,390	$224 \times 224 \times 64$
conv(3, 64, p1)	36,928	1,852,899,328	$224 \times 224 \times 64$
pool(2)	0	3,211,264	$112 \times 112 \times 64$
conv(3, 128, p1)	73,856	926,449,664	$112 \times 112 \times 128$
conv(3, 128, p1)	147,584	1,851,293,696	$112 \times 112 \times 128$
pool(2)	0	1,605,632	$56 \times 56 \times 128$
conv(3, 256, p1)	295,168	925,646,848	$56 \times 56 \times 256$
conv(3, 256, p1)	590,080	1,850,490,880	$56 \times 56 \times 256$
conv(3, 256, p1)	590,080	1,850,490,880	$56 \times 56 \times 256$
pool(2)	0	802,816	$28 \times 28 \times 256$
conv(3, 512, p1)	1,180,160	925,245,440	$28 \times 28 \times 512$
conv(3, 512, p1)	2,359,808	1,850,089,472	$28 \times 28 \times 512$
conv(3, 512, p1)	2,359,808	1,850,089,472	$28 \times 28 \times 512$
pool(2)	0	401,408	$14 \times 14 \times 512$
conv(3, 512, p1)	2,359,808	462,522,368	$14 \times 14 \times 512$
conv(3, 512, p1)	2,359,808	462,522,368	$14 \times 14 \times 512$
conv(3, 512, p1)	2,359,808	462,522,368	$14 \times 14 \times 512$
pool(2)	0	100,352	$7 \times 7 \times 512$
fc(4096)	102,764,544	102,764,544	4,096
fc(4096)	16,781,312	16,781,312	4,096
fc(1000)	4,097,000	4,097,000	1,000
softmax	0	1,000	1,000

WE NEED TO GO

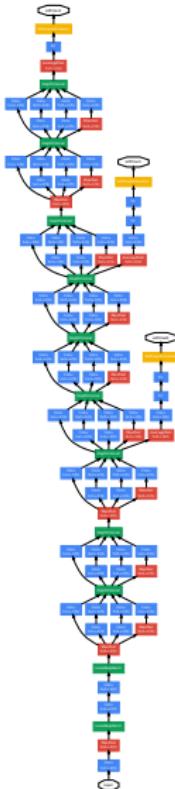
DEEPER

ImageNet classification performance



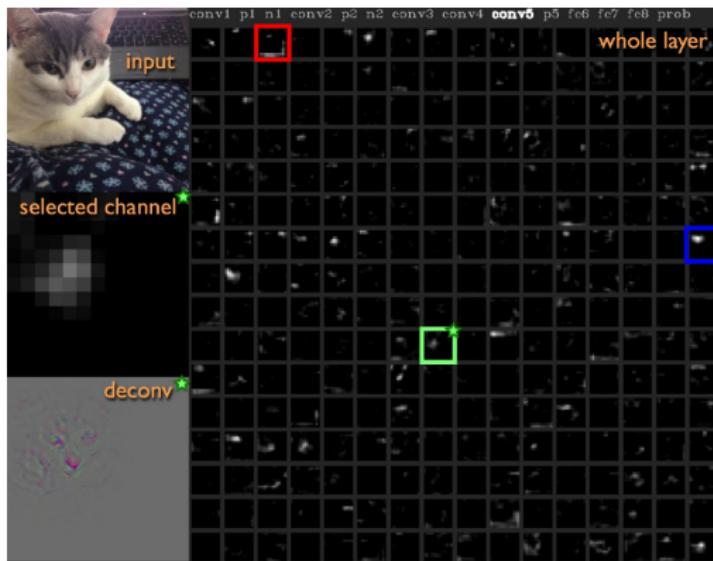
GoogLeNet

[Szegedy et al. 2015]



- 6.7% top-5 error on ILSVRC'14
- depth increased to 22 layers, kernel sizes 1×1 to 5×5
- inception module repeated 9 times
- 1×1 kernels used as “bottleneck” layers (dimensionality reduction)
- 25 times less parameters and faster than AlexNet
- auxiliary classifiers

convolutional features are sparse



- remember, features play the role of codebooks, and bag-of-words representations can be sparse even **after pooling**
- with relu, each feature represents a “**detector**” that fires when the activation is positive

convolutional features are sparse

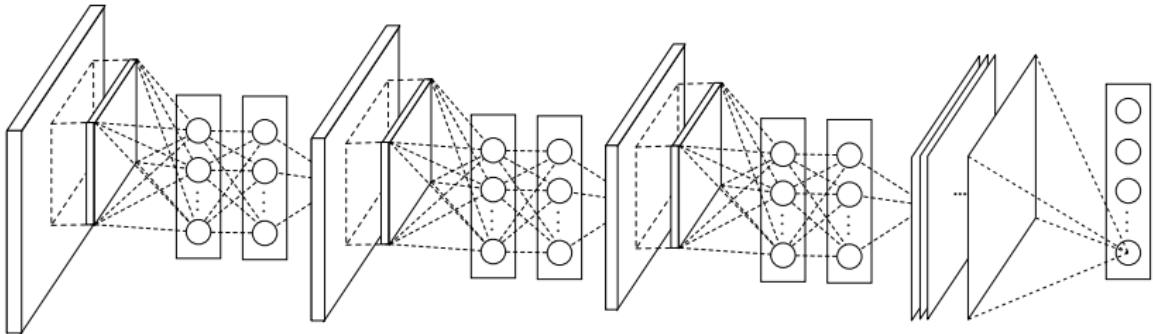
- deep layers have more features (e.g. 1024) and lower resolutions (e.g. 7×7)
 - detected patterns in many cases are as small as 5×5 , 3×3 or even 1×1
 - the convolution operation resembles more (sparse) matrix multiplication than convolution
- sparse matrix multiplication
 - is not as efficient as dense on parallel hardware
 - modern methods use coarse **partitioning** of nonzero elements
 - this is aligned with the Hebbian rule “**cells that fire together wire together**”

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network in network (NiN)

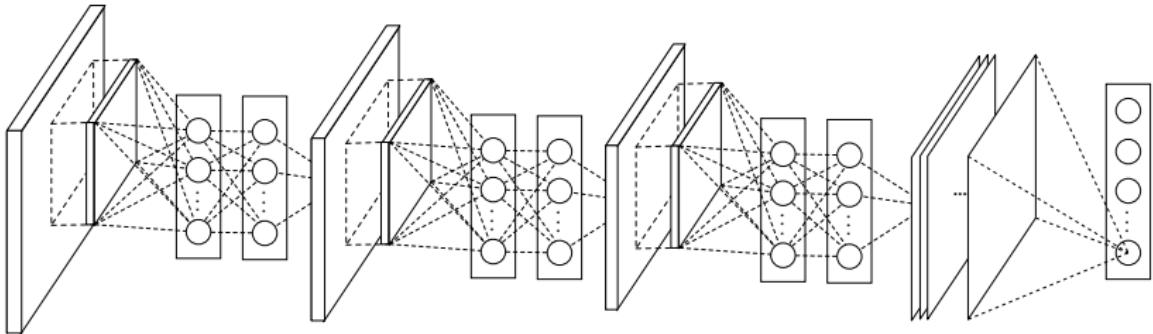
[Lin et al. 2013]



- activation functions are usually element-wise for simplicity; but here an entire **2-layer network** is used as activation function
- fully connected layers are simply replaced by **global average pooling**
- but this is nothing but convolution followed by two 1×1 convolutions
- 1×1 convolutions are just like matrix multiplications and can be used for **dimension reduction**

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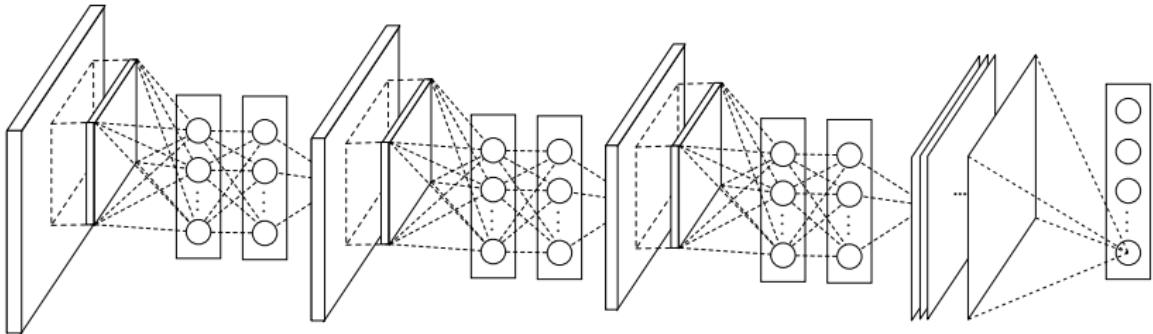
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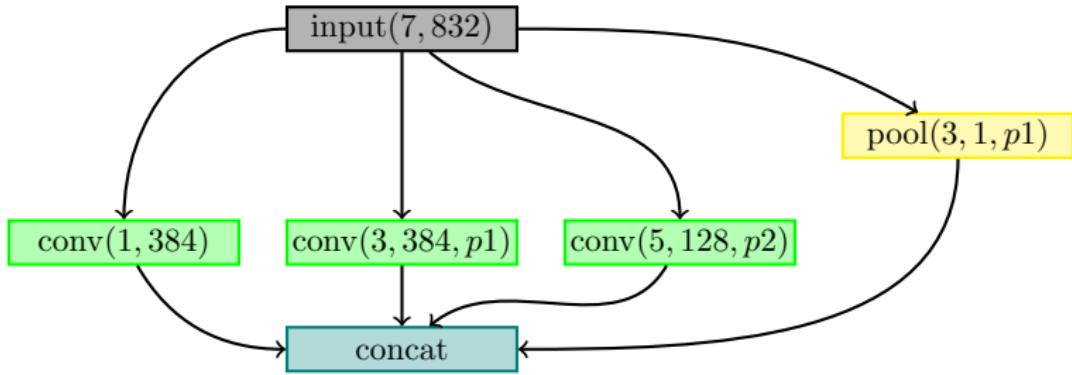
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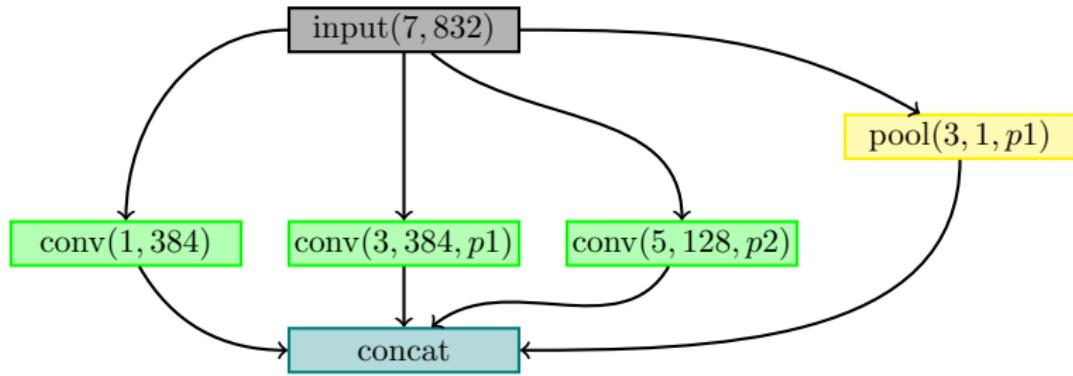
inception module



- **naive** inception module simply concatenates (feature-wise) three convolutions and one max-pooling
 - but this expensive and dimension keeps increasing
 - add **dimension reduction** to control cost, dimensions, and sparsity
 - this is referred to as **inception module**

inception module

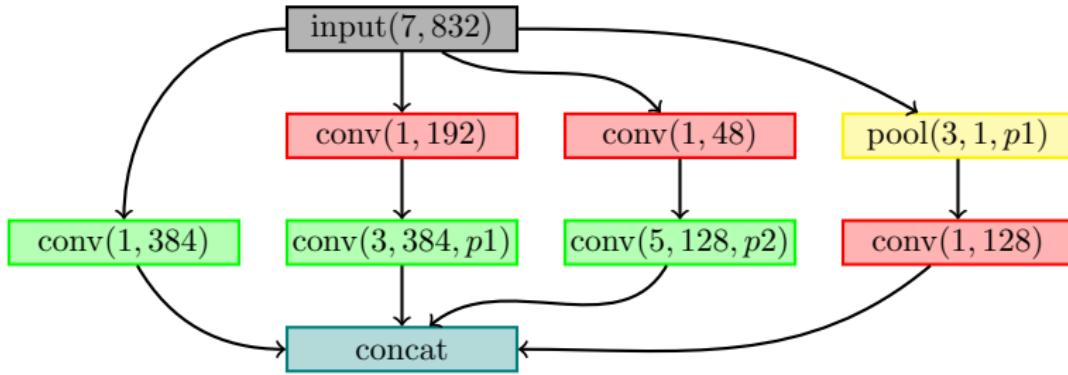
271,418,048 operations



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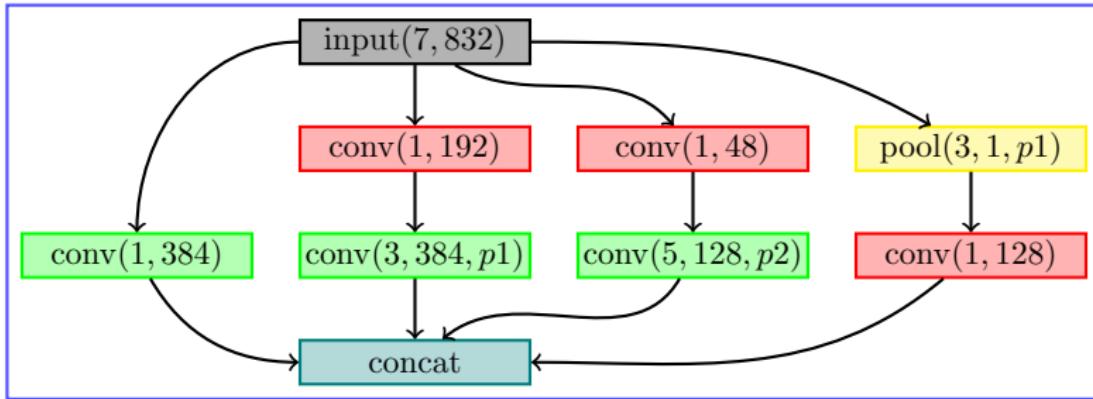
70,800,688 operations



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inception module

70,800,688 operations



$\text{inc}(384, (192, 384), (48, 128), 128)$

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alternatively: low-rank decomposition

$$Y = h \begin{pmatrix} W & X \end{pmatrix}$$

- X (Y): input (output) features (columns = spatial positions)
- W : weights; h : activation function
- low-rank approximation $W \approx UV^\top$; V is 1×1 spatially
- X was sparse; $V^\top X$ is not
- (in fact, V also includes a non-linearity)

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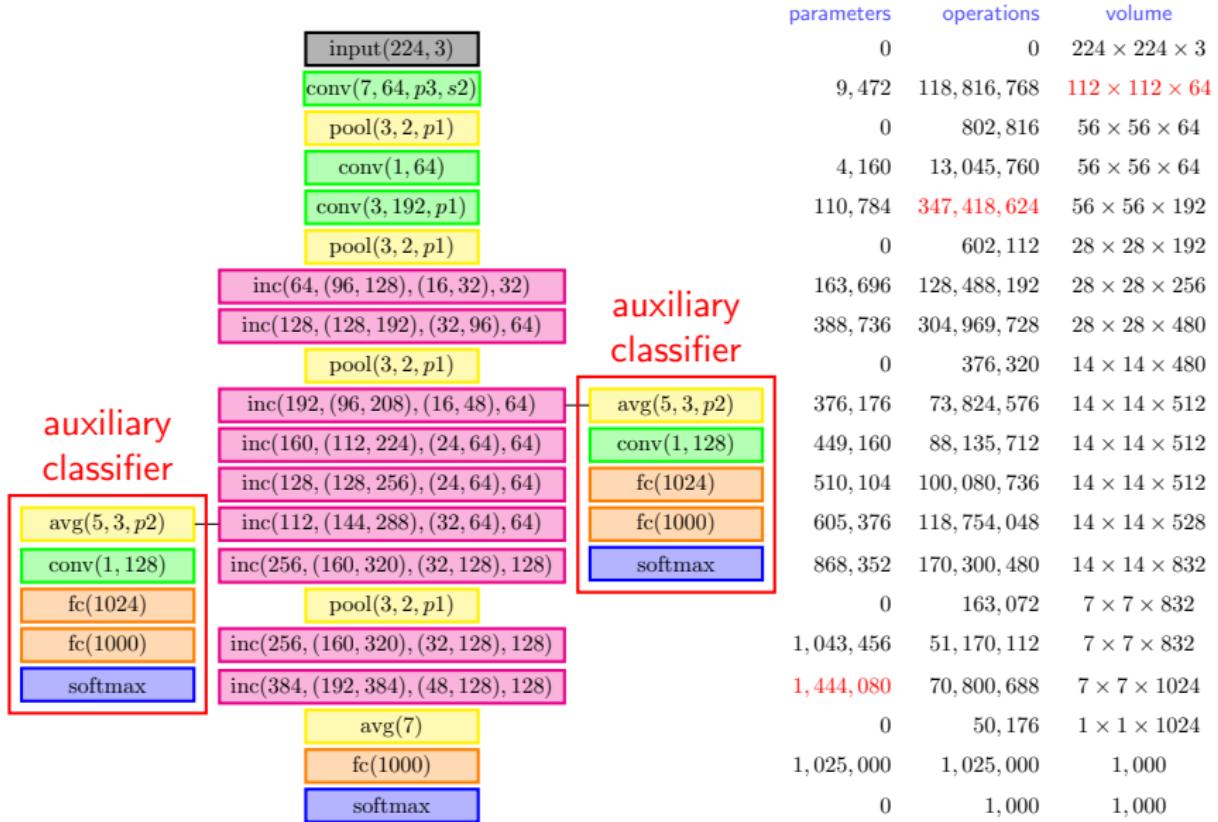
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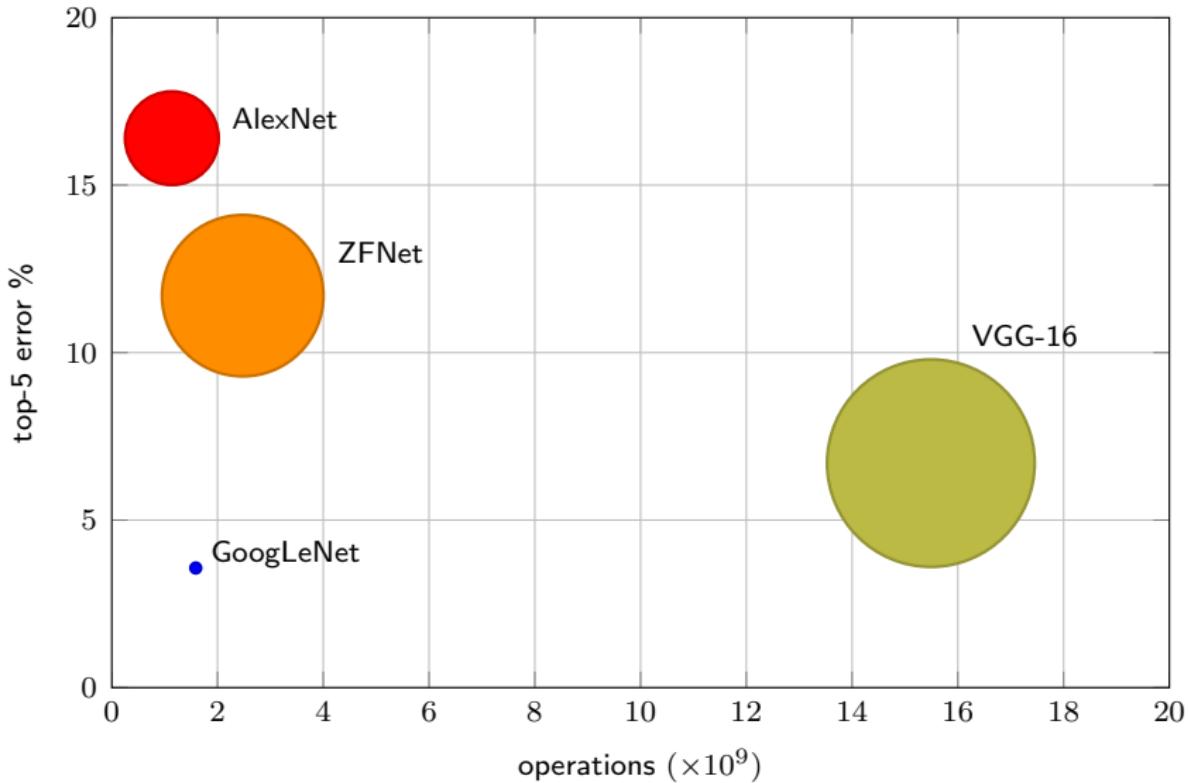
GoogLeNet

		parameters	operations	volume
	input(224, 3)	0	0	224 × 224 × 3
	conv(7, 64, p3, s2)	9,472	118,816,768	112 × 112 × 64
	pool(3, 2, p1)	0	802,816	56 × 56 × 64
	conv(1, 64)	4,160	13,045,760	56 × 56 × 64
	conv(3, 192, p1)	110,784	347,418,624	56 × 56 × 192
	pool(3, 2, p1)	0	602,112	28 × 28 × 192
	inc(64, (96, 128), (16, 32), 32)	163,696	128,488,192	28 × 28 × 256
	inc(128, (128, 192), (32, 96), 64)	388,736	304,969,728	28 × 28 × 480
	pool(3, 2, p1)	0	376,320	14 × 14 × 480
	inc(192, (96, 208), (16, 48), 64)	376,176	73,824,576	14 × 14 × 512
	inc(160, (112, 224), (24, 64), 64)	449,160	88,135,712	14 × 14 × 512
	inc(128, (128, 256), (24, 64), 64)	510,104	100,080,736	14 × 14 × 512
avg(5, 3, p2)	inc(112, (144, 288), (32, 64), 64)	605,376	118,754,048	14 × 14 × 528
conv(1, 128)	inc(256, (160, 320), (32, 128), 128)	868,352	170,300,480	14 × 14 × 832
fc(1024)	pool(3, 2, p1)	0	163,072	7 × 7 × 832
fc(1000)	inc(256, (160, 320), (32, 128), 128)	1,043,456	51,170,112	7 × 7 × 832
softmax	inc(384, (192, 384), (48, 128), 128)	1,444,080	70,800,688	7 × 7 × 1024
	avg(7)	0	50,176	1 × 1 × 1024
	fc(1000)	1,025,000	1,025,000	1,000
	softmax	0	1,000	1,000

GoogLeNet



network performance



summary

- convolution \equiv linearity, translation equivariance
- sparse connections, weight sharing: fully connected \rightarrow convolution
- cross-correlation
- feature maps: matrix multiplication & convolution combined
- 1×1 convolution
- convolution as regularization, structured convolution
- standard, padded, strided, dilated; and their derivatives
- pooling and invariance
- deeper networks
- LeNet-5, AlexNet, ZFNet, VGG-16, NiN, GoogLeNet