Recurrent Problems

In recurrence, solution to a problem depends on the solution to smaller instance of the same problem.

Tower of Hanoi: We are given a tower of eight disks, initially stacked in decreasing size on one of three pegs. The objective is to transfer the entire tower to one of the other pegs, moving only one disk at a time and never moving a large one onto a smaller. ***(These Two are Constraints)***

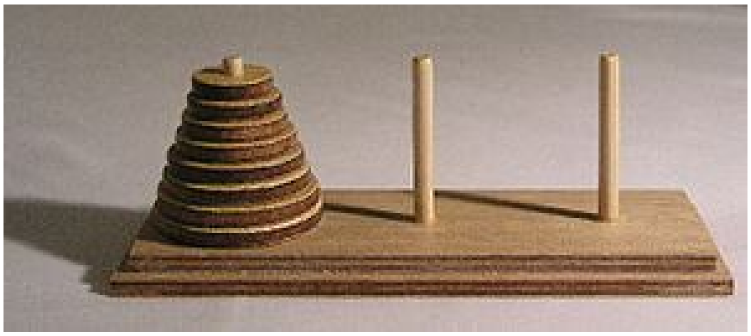

Fig: Tower of Hanoi with 8 disks

Consider small cases:

$T_0 = 0$, [0 disk requires 0 move]

$T_1 = 1$, [1 disk requires only 1 move]

$T_2 = 3$ [***Top disk from L to M; Bottom Disk from L to R; Finally, Top disk from M to R (see the Note below!)***]

$T_3 = 7$, [Transferring small 2 disks to middle peg requires 3 moves. Then moving the largest disk from left peg to right peg requires only 1 move. Finally, moving 2 smaller disks from middle peg to right peg requires another 3 moves. Thus, total 3+1+3=7 moves require for moving 3 disks from one peg to another]

Similarly, for n disks to transfer from one peg to another we require:

$T_n = T_{n-1} + 1 + T_{n-1} = 2T_{n-1} + 1$

Thus, the recurrence for Tower of Hanoi stands

$T_0 = 0$

$T_n = 2T_{n-1} + 1,$ for n > 0

We can find out the *closed form* of any recurrence to get quick result from the problem.

$$\begin{aligned}
T_n &= 2T_{n-1} + 1 \\
&= 2(2T_{n-2} + 1) + 1 \\
&= 2^2 T_{n-2} + 2 + 1 \\
&= 2^2(2T_{n-3} + 1) + 2 + 1 \\
&= 2^3 T_{n-3} + 2^2 + 2 + 1 \\
&\;\;\vdots \\
&= 2^n T_{n-n} + 2^{(n-1)} + 2^{(n-2)} + \cdots + 2^2 + 2 + 1 \\
&= 2^n T_0 + 2^{(n-1)} + 2^{(n-2)} + \cdots + 2^2 + 2^1 + 2^0
\end{aligned}$$

$T_n = 2^0 + 2^1 + 2^2 + \cdots + 2^{(n-2)} + 2^{(n-1)}$

$= \dfrac{2^n - 1}{2 - 1} \quad [\because a^0 + a^1 + a^2 + \cdots + a^n = \dfrac{a^{n+1}\mathbf{-1}}{a-1}]$

$= 2^n - 1$

$T_n = 2^n - 1$ is called the *closed form* solution of the "Tower of Hanoi" problem. *Mathematical induction* is a general way to prove any *closed form* solution. It has three parts.

- Basis – prove the formula for smallest possible value.
- Hypothesis – Consider that, the formula is true for first *n* values. ***(or, first n-1 values)***
- Induction – Try to prove the formula for (*n*+1)-th value. ***(or, n-th value, if hypothsis true for n-1 values)***

For example, we are going prove the *closed from* of Tower of Hanoi solution.

Basis: $T_0 = 2^0 - 1 = 1 - 1 = 0$ **(OK, because We know T0 is 0)**

Hypothesis: Let, $T_n = 2^n - 1$

Induction: $T_{n+1} = 2T_n + 1 = 2(2^n - 1) + 1 = 2^{n+1} - 2 + 1 = 2^{n+1} - 1$ (Proved)

☺Good Luck☺