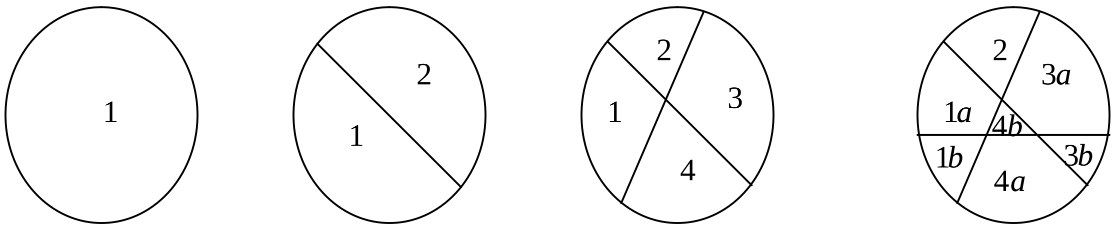
Recurrent Problems

Lines in the Plane: We have to find out how many slices of pizza can a person obtain by making n straight cuts with a knife. Academically, what is the maximum number of regions, L_n defined by n lines in the plane? We can start by looking at small cases.



$L_0 = 1$ $\qquad$ $L_1 = 2$, $L_1 = L_0 + 1$ $\qquad$ $L_2 = 4$, $L_2 = L_1 + 2$ $\qquad$ $L_3 = 4 + 3 = 7$, $L_3 = L_2 + 3$

If there exists $n-1$ lines in the plane, then n-th line have to cut previous $n-1$ lines to produce maximum number of new region in the plane. The $n-1$ intersection points create n new region in the plane. Thus, the recurrence for line in the plane therefore

$L_0 = 1$

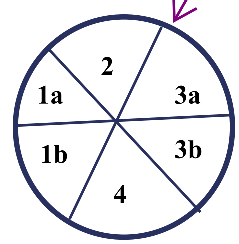
$L_n = L_{n-1} + n$, $\qquad$ for $n > 0$

But, if the newly added line goes thru any of the previous intersection points, then there will be less than "n new regions". For example:

We can find out the *closed form* of the recurrence through unfolding it to the end.

$$
\begin{aligned}
\boxed{L_n = L_{n-1} + n} \\
&= L_{n-2} + (n-1) + n \\
&= L_{n-3} + (n-2) + (n-1) + n \\
&\quad\vdots \\
&= L_{n-n} + (n-n+1) + \cdots + (n-2) + (n-1) + n \\
&= L_0 + 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n \\
&= S_n + 1, \quad \text{where } S_n = 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n
\end{aligned}
$$

(Also, if the new line is parallel to any previous line then, we will get less regions. Check it yourself!!!)



6 in stead of 7 regions

S_n is called the triangular number because it is number of bowling pins in an n-row triangular array. For example the usual four-row array has $S_4 = 10$ pins.

n	1	2	3	4	5	6	7	8	9	10
S_n	1	3	6	10	15	21	28	36	45	55

We can evaluate S_n using the following trick:

$$
\begin{array}{rl}
S_n = & 1 + 2 + 3 + \cdots + (n-2) + (n-1) + n \\
+S_n = & n + (n-1) + (n-2) + \cdots + 3 + 2 + 1 \\
\hline
2S_n = & (n+1) + (n+1) + (n+1) + \cdots + (n+1) + (n+1) + (n+1)
\end{array}
$$

$S_n = \dfrac{n(n+1)}{2}$, $\qquad$ for $n \ge 0$

Now, we have our *closed form* for lines in the plane problem.

$L_n = \dfrac{n(n+1)}{2} + 1$, $\qquad$ for $n \ge 0$

Suppose instead of straight lines we use bent lines, each containing one "zig". We have to find out the maximum number of regions, Z_n created by n such bent lines in the plane.

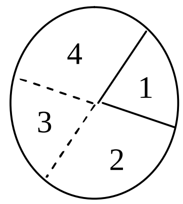


Fig: Bent lines in the plane

From small cases, we realize that a bent line is like two straight lines except that region merge when the "two" lines don't extent past their intersection point. Region 2, 3, 4 which would be distinct with two lines, become single region when there is a bent line, we lose two region. Thus

$$
\begin{aligned}
Z_n &= L_{2n} - 2n \\
&= \frac{2n(2n+1)}{2} + 1 - 2n \\
&= 2n^2 + n + 1 - 2n \\
&= 2n^2 - n + 1, \qquad \text{for } n \geq 0
\end{aligned}
$$

Zn = Maximum no. of regions obtained from n intersecting Zig shapes

Comparing *closed forms* of straight and bent lines in the plane, we find that for large *n*,

$$L_n \sim \frac{1}{2} n^2$$

$$Z_n \sim 2n^2$$

So, we get about four times as many regions with bent lines as with straight lines.

☺Good Luck☺

Also Find,
ZZn = Maximum no. of regions obtained from n intersecting ZigZag shapes
Wn = Maximum no. of regions obtained from n intersecting W shapes