Mathematical Analysis

CSE - 313

Chapter - 01

Recurrent Problem

Fibonacci equation,

$$f_n = n \; ; \; n = 0/1 \quad \text{(base condition)}$$

$$f_n = f_{n-1} + f_{n-2} \; ; \; n > 1 \quad \text{———— (RR)}$$

A set of equalities like that are called recurrence relation / recursion relation. It gives a boundary value and an equation for the general value in terms of earlier ones.

To find f_3, $f_n = f_{n-1} + f_{n-2}$ will be used,

$$f_3 = f_2 + f_1 \Rightarrow 1 + 1 \Rightarrow$$

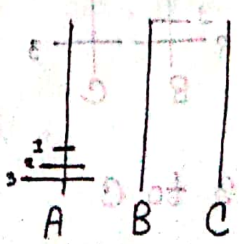
$$f_2 = f_1 + f_0 \Rightarrow 1 + 0 \Rightarrow 1$$

$$f_1 = 1 \; , \; f_0 = 0 \quad \text{[base condition]}$$

$\therefore$ fibonacci num = 0, 1, 1, 2

Tower of hanoi :

1. Solve by drawing :



Here,

T_n = minimum number of moves

n = no of disc

rule :

1. If the num of disc ~~of~~ is Even, 1st disc goes to B.

2. If the num of disc is Odd, 1st disc goes to C.

2. Base cond,

$T_0 = 0$; $n = 0$

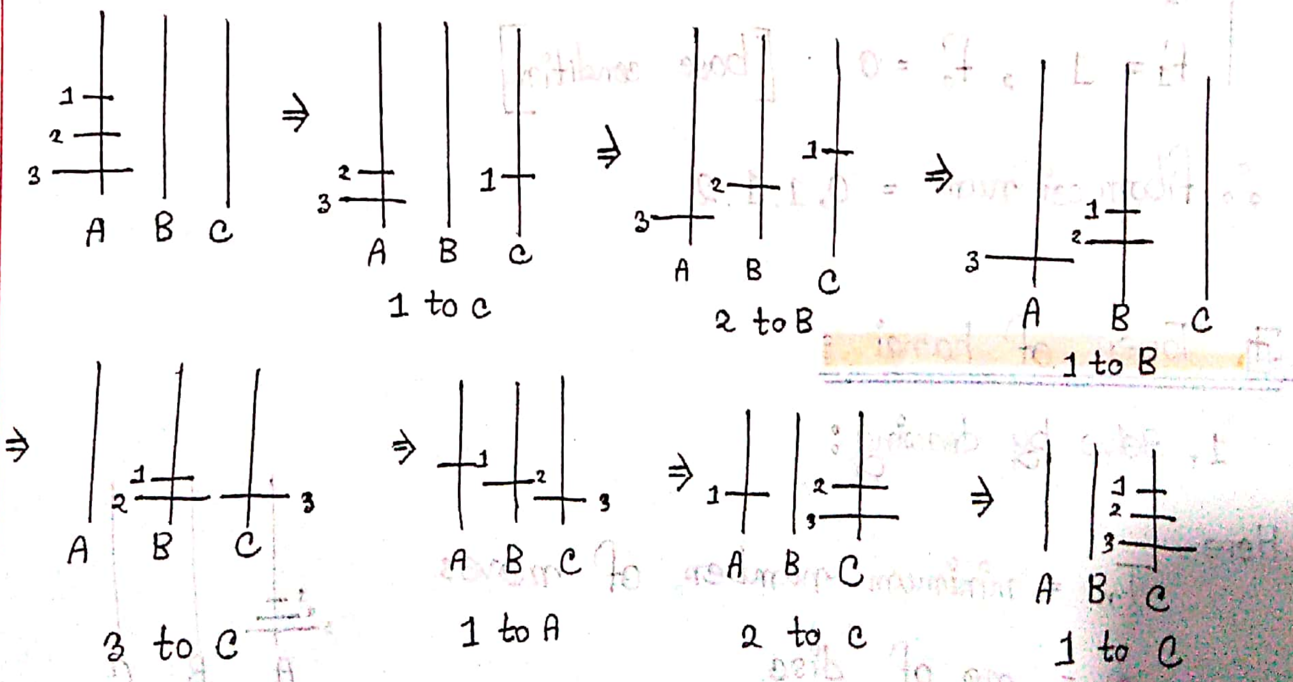
$T_1 = 1$; $n = 1$ (base case)

Recurrence relation,

$$T_n = 2T_{n-1} + 1$$

Q1 - Solve by drawing $T_3 = ?$; $n = 3$.

→ Since 3 is odd,



$\therefore T_3 = 7$ (Ans)

Q2 - Solve with recurrence relation, $T_{10} = ?$ when $n = 10$.

→ We know,

$$T_1 = 1 \; ; \; n = 1$$

$$T_n = 2T_{n-1} + 1 \; ; \; n > 1$$

$\therefore T_{10} = 2T_9 + 1 \Rightarrow 2 \times 511 + 1 \Rightarrow 1023$

$T_9 = 2T_8 + 1 \Rightarrow 2 \times 255 + 1 \Rightarrow 511$

$T_8 = 2T_7 + 1 \Rightarrow 2 \times 127 + 1 \Rightarrow 255$

$T_7 = 2T_6 + 1 \Rightarrow 2 \times 63 + 1 \Rightarrow 127$

$T_6 = 2T_5 + 1 \Rightarrow 2 \times 31 + 1 \Rightarrow 63$

$T_5 = 2T_4 + 1 \Rightarrow 2 \times 15 + 1 \Rightarrow 31$

$T_4 = 2T_3 + 1 \Rightarrow 2 \times 7 + 1 \Rightarrow 15$

$T_3 = 2T_2 + 1 \Rightarrow 2 \times 3 + 1 \Rightarrow 7$

$T_2 = 2T_1 + 1 \Rightarrow 2 \times 1 + 1 \Rightarrow 3$

$T_1 = 1$ (base case)

$\therefore T_{10} = 1023$

(Ans)

problem of recurrence problem is it repeat the same eqn too many times. So it is hard to solve bigger problem. And time consuming.

Close form eqution ~~sole~~ solves that problem.

3. Closed form eqn :

$$T_n = 2^n - 1$$

Q3 - $T_{50} = ?$ ~~we~~ when $n = 50$

$\rightarrow$ $T_n = 2^n - 1$

$\Rightarrow T_{50} = 2^{50} - 1$

$= 1125899906842623$

(Ans)

Closed form prove : Tower of hanoi

Base case $\rightarrow T_1 = 1$; $n = 1$

Recurrent relation $\rightarrow T_n = 2T_{n-1} + 1$; $n > 1$ —— (I)

iterative / back substituition method from (I),

$T_{n-1} = 2T_{n-2} + 1$ —— (II) $[n = n-1]$

$T_{n-2} = 2T_{n-3} + 1$ —— (III) $[n = n-2]$

from eqn (III) taking T_{n-2},

$T_{n-1} = 2(2T_{n-3} + 1) + 1$

$\Rightarrow T_{n-1} = 2^2 T_{n-3} + 2 + 1$ —— (IV)

from (I) we get,

$T_n = 2(2^2 T_{n-3} + 2 + 1) + 1$

$\Rightarrow T_n = 2^3 . T_{n-3} + 2^2 + 2 + 1$

$\Rightarrow T_n = 2^p T_{n-p} + [2^{p-1} + \ldots + 2 + 1]$ (p times)

Let, $n - p = 1$

$\Rightarrow p = n - 1$

$\therefore T_n = 2^{n-1} T_1 + 2^{n-2} + \ldots + 2 + 1$

$= 2^{n-1} + [2^{n-2} + \ldots + 2 + 1]$ $[\because T_1 = 1]$

$= \frac{2^n - 1}{2 - 1}$

$T_n = 2^n - 1$

Thus, closed form eqn of tower of hanoi.

[proved]

Mathematical induction :

Closed form eqn কে (0 থেকে n-1) true ধরে n এর জন্য CFE true prove করা।

$\rightarrow T_n = 2^n - 1$ ——— (I)

$T_{n-1} = 2^{n-1} - 1$ ——— (II)

taking value of T_{n-1} from (II) in recurrent relation,

$T_n = 2 T_{n-1} + 1$

$= 2(2^{n-1} - 1) + 1$

$= 2^n - 2 + 1$

$= 2^n - 1$

Hence, it's true.

Lines in the plain :

1 Straight line :

Rule -

1. must intersect all other line.
2. can not intersect in the same point.

n = no of lines

L_n = max no of region

$L_0 = 1$ 

, $L_1 = 2$ 

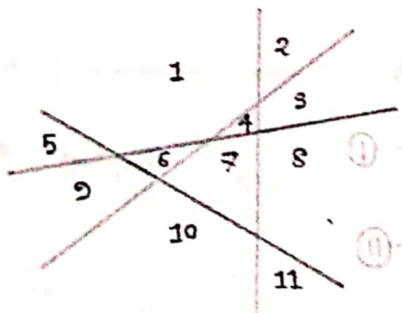
, $L_2 = 4$ 

finding region by drawing ,

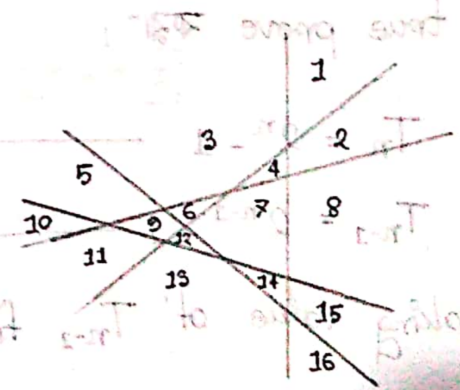
Q1 : $L_n = L_4$ and L_5 . So what are the number of region ?

→ $L_4 = 11$

$L_5 = 16$



$L_4 = 11$



$L_5 = 16$

Using recurrent relation :

base case → $L_n = 1$; $n = 0$

recurrent relation → $L_n = L_{n-1} + n$; $n > 0$

Q2 : find $L_{10} = ?$ using recurrent relation.

→ $L_{10} = L_9 + 10 = 46 + 10 = 56$

⇒ $L_9 = L_8 + 9 = 37 + 9 = 46$

⇒ $L_8 = L_7 + 8 = 29 + 8 = 37$

⇒ $L_7 = L_6 + 7 = 22 + 7 = 29$

⇒ $L_6 = L_5 + 6 = 16 + 6 = 22$

⇒ $L_5 = L_4 + 5 = 11 + 5 = 16$

⇒ $L_4 = L_3 + 4 = 7 + 4 = 11$

⇒ $L_3 = L_2 + 3 = 4 + 3 = 7$

⇒ $L_2 = L_1 + 2 = 2 + 2 = 4$

⇒ $L_1 = L_0 + 1 = 1 + 1 = 2$

⇒ $L_0 = 1$

∴ $L_{10} = 56$

(Ans)

Closed form eqn of lines in the plain :

$L_0 = 1 \; ; \; n=0$

$L_n = L_{n-1} + n \; ; \; n>0$ ——— (i)

replacing n with $n-1$ and $n-2$,

$L_{n-1} = L_{n-2} + (n-1)$ ——— (ii)

$L_{n-2} = L_{n-3} + (n-2)$ ——— (iii)

from (iii) in (ii),

$L_{n-1} = L_{n-3} + (n-2) + (n-1)$

from (i)

$L_n = L_{n-3} + (n-2) + (n-1) + n$

replacing n with 1,3,3 in (i),

$L_1 = L_0 + 1$ (from i)

$L_2 = L_1 + 2 = (L_0 + 1) + 2$

$L_3 = L_2 + 3 = (L_0 + 1 + 2) + 3$

⋮

$L_{n-3} = L_0 + 1 + 2 + 3 + \dots + (n-3)$ ——— (iv)

(ans)

from (iv),

$L_n = L_0 + 1 + 2 + 3 + \dots + (n-3) + (n-2) + (n-1) + n$

$\Rightarrow L_n = 1 + \frac{n(n+1)}{2}$ (Ans)

$$\boxed{\text{CFL}, \quad L_n = 1 + \frac{n(n+1)}{2}}$$

(iii) Solving with mathematical induction :

$$L_n = \frac{n(n+1)}{2} + 1$$

$$\Rightarrow L_{n-1} = \frac{(n-1)(n-1+1)}{2} + 1$$

$$= \frac{n(n-1)}{2} + 1$$

recurrence relation, $L_n = L_{n-1} + n$

$$\Rightarrow L_n = \left(\frac{n(n-1)}{2} + 1\right) + n$$

$$\Rightarrow L_n = \frac{n}{2}(n-1+2) + 1$$

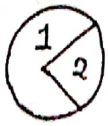
$$\Rightarrow L_n = \frac{n(n+1)}{2} + 1$$

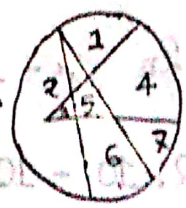
(proved)

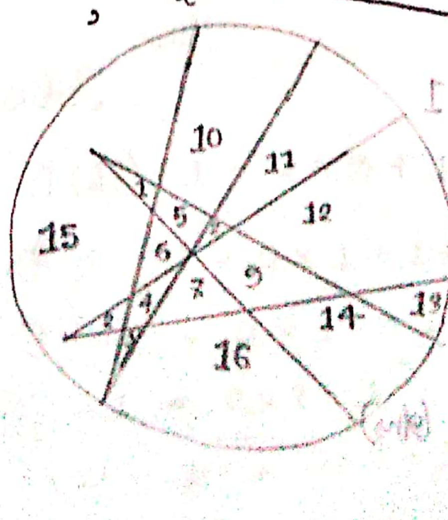
2. Zig line :

Z_n = max no of region, n = no of zig line


zig line

$Z_1 = 2 \longrightarrow$ 

, $Z_2 = 7 \longrightarrow$ 

$Z_3 = 16 \longrightarrow$ 

Closed form of zig line :

Zig line is the double of straight line.

from CFE of straight line,

$$L_n = 1 + \frac{n(n+1)}{2}$$

$$Z_n = L_{2n} - 2n \text{ ——— (i)}$$

if we simplify the eqn (i),

$$L_{2n} = 1 + \frac{2n(2n+1)}{2}$$ [n এর জায়গায় 2n দিলে]

$$\therefore Z_n = 1 + \frac{2n(2n+1)}{2} - 2n$$

$$= \frac{2 + 4n^2 + 2n - 4n}{2}$$

$$= \frac{4n^2 - 2n + 2}{2} = \frac{\cancel{2}(2n^2 - n + 1)}{\cancel{2}}$$

$$\boxed{= 2n^2 - n + 1}$$

(Ans)

Q1 : n = 10, find the no of region.

→ n = 10

$$Z_n = 2n^2 - n + 1$$

$$= 2.10^2 - 10 + 1$$

$$= 2.100 - 9$$

$$= 200 - 9$$

$$= 191$$

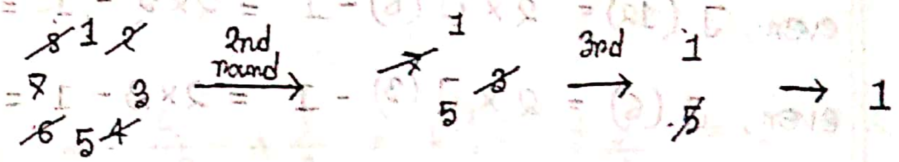
(Ans)

Josephus problem : who survives ?

n = no of person , $J(n)$ = Survivor number = (position)

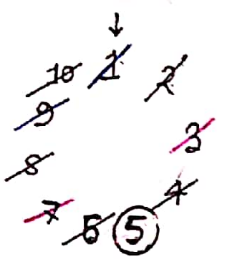
rule : 2nd person dies

$J(8) = 1$

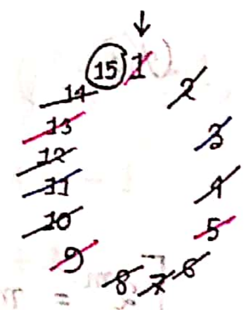


draw the answer,

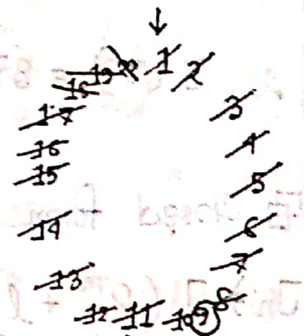
Q1 : find $J(10)$, $J(15)$ and $J(20)$.



$J(10) = 5$



$J(15) = 15$



$J(20) = 9$

Recurrence Relation :

Base – $J(1) = 1$, $n = 1$

Even $\rightarrow$ $J(2n) = J(n) = 2 \times J\left(\frac{n}{2}\right) - 1$; $n > 1$

Odd $\rightarrow$ $J(2n+1) = J(n) = 2 \times J\left(\frac{n}{2}\right) + 1$; $n > 1$

[n = 1st round এ যে ছিল]

Q2 : $J(35) = ?$

$\rightarrow$ Since 35 is odd,

$J(35) = 2 \times J(17) + 1 = 2 \times 3 + 1 = 7$

odd, $J(17) = 2 \times J(8) + 1 = 2 \times 1 + 1 = 3$

even, $J(8) = 2 \times J(4) - 1 = 2 \times 1 - 1 = 1$

even, $J(4) = 2 \times J(2) - 1 = 2 \times 1 - 1 = 1$

even, $J(2) = 2 \times J(1) - 1 = 2 \times 1 - 1 = 1$ (Ans)

$J(1) = 1$

Note : $J(35)$ থেকে 17 এসেছে $\frac{35-1}{2}$ করতে হবে।

Q3: $J(50) = ?$

$\rightarrow \quad J(50) = 2 \times J(25) - 1 = 2 \times 19 - 1 = 37$

odd, $J(25) = 2 \times J(12) + 1 = 2 \times 9 + 1 = 19$

even, $J(12) = 2 \times J(6) - 1 = 2 \times 5 - 1 = 9$

even, $J(6) = 2 \times J(3) - 1 = 2 \times 3 - 1 = 5$

odd, $J(3) = 2 \times J(1) + 1 = 2 \times 1 + 1 = 3$

$J(1) = 1$

$\therefore J(50) = 37$ position. (Ans)

Closed form eqn :

$J(n) \Rightarrow J(2^m + l) = 2l + 1$

$\Rightarrow J(50) = J(2^5 + 18) = 2 \times 18 + 1$

$= 37$

[2^m = max power of 2
l = যত plus করলে $J(n)$ এর সমান হবে]

n	1	2	3	4	5	6	7	8	9	10	11	12
	2^0	2^1		2^2				2^3				
$J(n)$	1	1	3	1	3	5	7	1	3	5	7	9
	1	2		4				8				

$\therefore J(12) \Rightarrow J(2^3 + 4)$

$= (2l + 1)$

$= 2 \times 4 + 1$

$= 9$

Mathematical induction :

Base case → $J(1) = 1$, $n=1$

Recurrent → $J(2n) = 2 \times J(n) - 1$ [even] ——— (i)

$J(2n+1) = 2 \times J(n) + 1$ [odd] ——— (ii)

Let, $2n = 2^m + l$

$\Rightarrow n = \dfrac{2^m + l}{2} = \dfrac{2^m}{2} + \dfrac{l}{2} = 2^{-1} \cdot 2^m + \dfrac{l}{2} = 2^{m-1} + \dfrac{l}{2}$

from eqn (i),

$J(2^m + l) = 2 \times J\left(2^{m-1} + \dfrac{l}{2}\right) - 1$ ...... $[2n = 2^m + l]$

$= 2(l+1) - 1$

$= 2l + 2 - 1$

$= 2l + 1$

$\left[J\left(2^{\overset{m}{m-1}} + \overset{l}{\dfrac{l}{2}}\right) = \left(2 \times \dfrac{l}{2}\right) + 1 = l + 1 \right.$

from CFE,

$J(2^m + l) = 2l + 1$

for odd,

Let, $2n+1 = 2^m + l$

$\Rightarrow 2n = 2^m + l - 1$

$\Rightarrow n = \dfrac{2^m + l - 1}{2} = 2^{m-1} + \dfrac{l}{2} - \dfrac{1}{2}$

from eqn (ii),

$J(2^m + l) = 2J\left(2^{\overset{m}{m-1}} + \overset{l}{\dfrac{l}{2}} - \dfrac{1}{2}\right)$

$= 2(l+1) - \dfrac{1}{2} - 1$

$= \dfrac{2 \cdot 2(l+1)}{2} - 1$

$= 2l + 2 - 1$

$= 2l + 1$ [proved]

Ch-2

Chapter - 2

1. Three dot notation : $a_1 + a_2 + \dots + a_n$

$2^0 + 2^1 + \dots 2^{n-1}$

2. Sigma ~~notion~~ notation : delimited form

$$\sum_{K=1}^{n} a_K$$

$n \rightarrow$ upper limit

$a_K \rightarrow$ summand

$K=1 \rightarrow$ lower limit

[serial maintain করে]

3. Genarelized Sigma notation (GSN) :

$$\sum_{\substack{1 \le K \le n \\ K = \text{odd}}} a_K$$

$\therefore$ sum $= a_1 + a_3 + a_5 + a_7 \dots$

Q1: (K+1) transform in sigma notation to 3 dot

$\rightarrow \sum_{K=0}^{n} a_{K+1}$

lower,

$K+1 = 1$

$\Rightarrow K = 1-1$

$= 0$

Upper

$K+1 = n$

$\Rightarrow K = n-1$

$\therefore$ sum $= a_0 + a_1 + a_2 + a_3 + \dots a_{n-1}$

Q2: (1-100) odd square of odd num. Convert in SN and GSN

$1^2+3^2+5^2+\dots+99^2$.

→ **GSN**

$\sum K^2$

$1 \le K \le 100$

K odd

SN

$\sum_{K=1}^{50} (2K-1)^2$ [serial wise যায় তাই]

or

$\sum_{K=0}^{49} (2K+1)^2$

Q3: (1-200) square of the even num

→ **GSN**

$\sum K^2$

$1 \le K \le 200$

K even

SN

$\sum_{K=1}^{100} (2K)^2$

or,

$\sum_{K=0}^{99} (2K+2)^2$

☆ # Reciprocal / inverse : prime number

Q4: 20 পর্যন্ত prime এর inverse

→ **GSN**

$\sum \frac{1}{K}$

$K \le 20$

K prime

SN

$\sum_{K=1}^{\pi(N)} \frac{1}{P_K}$ $(\pi(N) \le 20)$

$P_K = K$ position এর prime num

Prime num reciprocal sum : GSN

$$\sum_{\substack{P \le N \\ \text{P prime}}}^{N} \frac{1}{P} \Rightarrow \sum_{\substack{P \le 20 \\ \text{p prime}}} \frac{1}{P}$$

$$\Rightarrow \frac{1}{2}+\frac{1}{3}+\frac{1}{5}+\frac{1}{7}+\frac{1}{11}+\frac{1}{13}+\frac{1}{17}+\frac{1}{19} = 1.4554$$

Iverson conversion :

Condition true = 1 , false = 0

$$\to [\text{p prime}] = \begin{cases} 1, & \text{if p is prime.} \\ 0, & \text{if p is not prime} \end{cases}$$

for $P \le 20$,

$$\sum_{P} [\text{p prime}]\,[P \le 20] / P$$

$$= \frac{0}{1}+\frac{1}{2}+\frac{1}{3}+\frac{0}{4}+\frac{1}{5}+\frac{0}{6}+\frac{1}{7}+\frac{0}{8}+\frac{0}{9}+\frac{0}{10} + \frac{1}{11}+\frac{0}{12}+\frac{1}{13}+\frac{0}{14}+\frac{0}{15}+\frac{0}{16}+\frac{1}{17}+\frac{0}{18}+\frac{1}{19}+\frac{0}{20}$$

$= 1.4554$

for $P \le 30$

$$\sum_{P} [\text{p prime}]\,[P \le 30] / P$$

$$= \frac{0}{1}+\frac{1}{2}+\frac{1}{3}+\frac{0}{4}+\frac{1}{5}+\frac{0}{6}+\frac{1}{7}+\frac{0}{8}+\frac{0}{9}+\frac{0}{10}+\frac{1}{11}+\frac{0}{12}+\frac{1}{13} + \frac{0}{14}+\frac{0}{15}+\frac{0}{16}+\frac{1}{17}+\frac{0}{18}+\frac{1}{19}+\frac{0}{20}+\frac{0}{21}+\frac{0}{22}+\frac{1}{23}+\frac{0}{24}+\frac{0}{25} + \frac{0}{26}+\frac{0}{27}+\frac{0}{28}+\frac{1}{29}+\frac{0}{30} \Rightarrow 1.533$$

Advantage and disadvantage of sigma (SN) and generalized sigma (GNS)

1. Generalized sigma notation :

advantage :

1) More useful , simple , easy to see what is happened .
2) Less likely to make a mistake .
3) It can be manipulated more easily than delimited / SN .
4) It allows us to take sums over index - sets that aren't restricted to consecutive integers .
5) More clean and less cumbersome . Examle : $\sum_{\substack{1 \le K \le 50 \\ K \text{ odd}}} K^2$

dis advatage :

1) Takes time to write due to 8 symbols .
2) Takes multiple line to write .

2. Sigma notation : delimited form

advantage :

1) It is convenient for writing in a single line .
2) It is compact and often suggestive of manipulation
3) Can be written quickly due to having 7 symbols .
4) Zero term are not generally harmful while working with SN .

disadvantage :

1) More cumbersum and less clear.

$\sum_{K=1}^{25} (2K-1)^2$

2) More likely to make mistake.

3) Harded to manipulate than GSN.

Repertory : Sum to recurrence relation :

$\sum_{K=0}^{n} a_K$

$S_n = \underbrace{\boxed{a_0 + a_1 + \dots + a_{n-1}}}_{S_{n-1}} + a_n$

$S_0 = a_0$

$S_n = S_{n-1} + a_n \; ; \; n>0$

$R_n = \boxed{a_0 + a_1 + \dots + a_{n-1}} + a_n$

→ $R_0 = a_0$

$R_n = R_{n-1} + a_n$

Let,

$a_n = \text{constant} + (\text{constant} * n)$

$a_0 = \alpha$

$\therefore a_n = \beta + \gamma n$

Hence,

$$\left.\begin{aligned} R_0 &= \alpha \\ R_n &= R_{(n-1)} + \underbrace{\beta + \gamma n}_{a_n} \end{aligned}\right\} \text{——— (1)}$$

$R_1 = R_0 + \beta + \gamma 1$

$R_2 = R_1 + \beta + 2\gamma$

$\vdots$

$R_n = A(n)\alpha + B(n)\beta + C(n)\gamma$ ———— (ii)

Let,

$R_n = 1$

$R_0 = 1 = \alpha$

[n does not matter]

$R_{n-1} = 1$

from (i),

$1 = 1 + \beta + \gamma n$

$\Rightarrow 0 = \beta + \gamma n$

$\Rightarrow \beta = 0, \gamma = 0$

from (ii),

$A(n) = R_n$

$\Rightarrow A(n) = 1$

$R_n = n$

$R_0 = 0 = \alpha$ [from (i)]

$R_{n-1} = n - 1$

from (i),

$n = n - 1 + \beta + \gamma n$

$\Rightarrow 0 = \cancel{n} - \cancel{n} - 1 + \beta + \gamma n$

$\Rightarrow 0 = (-1 + \beta) + \gamma n$

$\Rightarrow \beta = 1 \quad , \gamma = 0$

from (ii),

$R_n = A(n)\alpha + B(n)\beta + C(n)\gamma$

$R_n = B(n)\beta$

$\Rightarrow n = B(n)$

~~$\Rightarrow B(n) =$~~

$R_n = n^2$

$R_0 = 0 = \alpha$

$R(n-1) = (n-1)^2$

$= n^2 - 2n + 1$

from (i),

$n^2 = n^2 - 2n + 1 + \beta + \gamma n$

$\Rightarrow 0 = \cancel{n^2} - 2n + 1 + \beta + \gamma n - \cancel{n^2}$

$\Rightarrow 0 = (1 + \beta) + n(\gamma - 2)$

$\Rightarrow \beta = -1 \quad , \gamma = 2$

from (ii),

$n^2 = -B(n) + 2C(n)$

$\Rightarrow n^2 = -1n + 2C(n)$ [from $R_n = n$]

$\Rightarrow n^2 + n = 2C(n)$

$\Rightarrow C(n) = \frac{n^2 + n}{2}$

$= \frac{n(n+1)}{2}$

from (ii),

$R_n = \alpha + n\beta + \frac{n(n+1)}{2}\gamma$

Let,

$\alpha, \beta = a \quad , \gamma = b$

$$\boxed{\therefore R_n = a(n+1) + \frac{bn(n+1)}{2}} \longrightarrow \text{CFE}$$

Recurrence relation to sum:

→ 2 way

1. division → 2^n
2. multiplication → summation factor

1 division method:

$$\left.\begin{array}{l} T_0 = 0 \quad ; n=0 \\ T_n = 2T_{n-1}+1 \quad ; \quad n \ge 1 \end{array}\right\} \text{———— (I)} \quad [\text{Tower of hanoi}]$$

divide (I) by 2^n,

$$\frac{T_0}{2^n} = 0$$

$$\frac{T_n}{2^n} = \frac{2T_{n-1}+1}{2^n}$$

$$= \frac{T_{n-1}}{2^n \cdot 2^{-1}} + \frac{1}{2^n}$$

$$= \frac{T_{n-1}}{2^{n-1}} + \frac{1}{2^n}$$

$$= \frac{T_{n-1}}{2^{n-1}} + 2^{-n} \text{ ———— (II)}$$

Let,

$$S_n = \frac{T_n}{2n}$$

$$\Rightarrow S_0 = \frac{T_0}{2^0} \quad [\text{Since } n=0]$$

$$\therefore S_{n-1} = \frac{T_{n-1}}{2n-1} \longleftarrow \text{(III)}$$

Putting value of (III) in (II),

$$S_n = S_{n-1} + 2^{-n}$$

$\therefore SN = \sum_{K=1}^{n} 2^{-K}$ | $GSN = \sum_{1 \le K \le n} 2^{-K}$

$$\Rightarrow 2^{-1} + 2^{-2} + 2^{-3} + \dots + 2^{-n}$$

$$\Rightarrow \frac{1}{2} + \left(\frac{1}{2}\right)^2 + \left(\frac{1}{2}\right)^3 + \dots \left(\frac{1}{2}\right)^n$$

Mathematical induction, / CFE

$$S_n = \frac{T_n}{2^n} \Rightarrow T_n = S_n \cdot 2^n$$

$$\Rightarrow T_n = \left(1 - 2^{-n}\right) \cdot 2^n \quad \left[S_n = S_{n-1} + 2^{-n}\right]$$

$$= 2^n - 1$$

(Ans)

(2) multiplication method :

$T_0 = 0$

$T_n = 2T_{n-1} + 1$ —— (I)

from (I)

$a_n T_n = b_n T_{n-1} + c_n$ ——→ (II)

multiplying by summation factor f_n,

$f_n a_n T_n = f_n b_n T_{n-1} + f_n c_n$ —— (III)

$\Rightarrow \boxed{f_n b_n = f_{n-1} \cdot a_{n-1}}$ —— (V)

Let, $S_n = f_n a_n T_n$ —— (IV)

$S_{n-1} = f_{n-1} \cdot a_{n-1} \cdot T_{n-1}$

$\therefore\ S_{n-1} = \boxed{f_n b_n} . T_{n-1}$ —(v)

from (III),

$$S_n = S_{n-1} + f_n c_n$$

$$\Rightarrow S_n = \sum_{K=1}^{n} f_K C_K$$

from (IV), (for zero's term)

$$S_n = f_n a_n T_n$$

$$\Rightarrow S_0 = f_0 a_0 T_0$$

$$= f_1 b_1 T_0 \quad [f_1 b_1 = f_0 a_0] \text{ —(v)}$$

from (IV),

$$S_n = f_n a_n T_n$$

$$\Rightarrow T_n = \frac{1}{f_n a_n}\left(f_1 b_1 T_0 + \sum_{K=1}^{n} f_K C_K\right)$$

Manipulation of sum :

① distribution law :

$$\sum_{K\in K} c a_K = c \sum_{K=K} a_K$$

$\boxed{K = -1, 0, 1}$

$\boxed{P(K) = -K}$

prove → $\sum_{-1 \le K \le 1} c a_K = c a_{-1} + c a_0 + c a_1$

$= c(a_{-1} + a_0 + a_1)$

$= c \sum_{-1 \le K \le 1} a_K$

[proved]

② Associative law :

$$\sum_{K\in K}(a_K + b_K) = \sum_{K\in K} a_K + \sum_{K\in K} b_K.$$

→ $\sum_{-1 \le K \le 1} (a_K + b_K) = (a_{-1} + b_{-1}) + (a_0 + b_0) + (a_1 + b_1)$

$= (a_{-1} + a_0 + a_1) + (b_{-1} + b_0 + b_1)$

$= \sum_{-1 \le K \le 1} a_K + \sum_{-1 \le K \le 1} b_K$

(proved)

③ Commulative law :

$$\sum_{K\in K} a_K = \sum_{P(K)\in K} a_{P(K)} \rightarrow -1 \le K \le 1$$

→ L.H.S $= \sum_{-1 \le K \le 1} a_K$

$= a_{-1} + a_0 + a_1$

R.H.S $= \sum a_{P(K)} = \sum a_{-K}$

$= a_1 + a_0 + a_{-1}$

∴ L.H.S = R.H.S

[proved]

The genaral sum of an arithmatic program:

$$S = \sum_{0 \le k \le n} (a + bk) \qquad \text{——— (I)}$$

$$\Rightarrow S = \sum_{0 \le n-k \le n} (a + b(n-k))$$

$$\Rightarrow S = \sum_{0 \le k \le n} (a + bn - bk) \qquad \text{——— (II)}$$

(I) + (II)

$$2S = \sum_{0 \le k \le n} (a + \cancel{bk} + a + bn - \cancel{bk})$$

$$\Rightarrow 2S = \sum_{0 \le k \le n} (2a + bn)$$

$$\Rightarrow 2S = (2a + bn) \sum_{0 \le k \le n} 1 \qquad \text{[distributive law]}$$

$$\Rightarrow 2S = (2a + bn)(n+1)$$

$$\Rightarrow S = \frac{1}{2}(2a + bn)(n+1)$$

$$= \frac{1}{2} \cdot 2\left(a + \frac{bn}{2}\right)(n+1)$$

[0 থেকে n সংখ্যক 1 এর sum = n+1 সংখ্যক]

$$= \left(a + \frac{bn}{2}\right)(n+1)$$

☐ **Multiple sum:** The term of a sum might be specified by two or more indices, not just by one, that called multiple sum.

\# $\sum\limits_{1 \le j,k \le 3} a_j b_k$

$= \sum\limits_{j,k} a_j b_k \ [1 \le j,k \le 3]$

$= \sum\limits_{j,k} a_j b_k \ [1 \le j \le 3] \ [1 \le k \le 3]$

$= \sum\limits_{j} a_j \ [1 \le j \le 3] \sum\limits_{k} b_k \ [1 \le k \le 3]$

$= \left(\sum\limits_{j=1}^{3} a_j\right)\left(\sum\limits_{k=1}^{3} b_k\right)$

$= (a_1 + a_2 + a_3)(b_1 + b_2 + b_3)$

\# $\sum\limits_{1 \le j,k,l \le 4} a_j b_k c_l$

$0^2+1^2+2^2+3^2+\ldots+n^2$ to solve

$$\square_n = \sum_{K=0}^{n} K^2$$

① Guess the ans and prove by induction :

n	0	1	2	3
n^2	0	1	4	9
$\square_n$	0	1	5	14
$3\square_n$	0	3	15	42
$n(n+1)$	0	2	6	12
$\frac{3\square_n}{n(n+1)}$	–	1.5	2.5	3.5

$$\frac{3\square_n}{n(n+1)} = n+\frac{1}{2}$$

$$\Rightarrow 3\square_n = \left(n+\frac{1}{2}\right).\, n(n+1)$$

$$= n(n+1)\left(n+\frac{1}{2}\right)$$

Mathematical induction,

n কে $n-1$ ধরে,

$$3\square_{n-1} = (n-1)(n)\left(n-\frac{1}{2}\right) \text{ ——— } ①$$

Here

$$\square_n = \left[0^2+1^2+2^2+\ldots+(n-1)^2\right]+n^2$$

$$\Rightarrow \square_n = \square_{n-1}+n^2$$

$$\Rightarrow 3\square_n = 3\square_{n-1}+3n^2$$

$$\Rightarrow 3\Box_n = \left[n(n+1)\left(n-\frac{1}{2}\right)\right] + 3n^2$$

$$= n\left\{n^2 - \frac{1}{2}n - n + \frac{1}{2} + 3n\right\}$$

$$= n\left(n^2 + \frac{3}{2}n + \frac{1}{2}\right)$$

$$= n\left(n^2 + \frac{1}{2}n + n + \frac{1}{2}\right)$$

$$= n\left\{n\left(n+\frac{1}{2}\right) + 1\left(n+\frac{1}{2}\right)\right\}$$

$$= n(n+1)\left(n+\frac{1}{2}\right)$$

[proved]

$\boxplus$ Perturb the sum: ⊞ to □

$$⊡_n = 1^3 + 2^3 + \ldots + n^3$$

$$\Rightarrow ⊡_n + (n+1)^3 = 1^3 + 2^3 + \ldots + n^3 + (n+1)^3$$ (mul by $n+1$)

$$\Rightarrow ⊡_n + (n+1)^3 = \sum_{0 \le K \le n} (K+1)^3$$

$$\Rightarrow ⊡_n + (n+1)^3 = \sum_{0 \le K \le n} (K^3 + 3K^2 + 3K + 1)$$ [$(a+b)^3$ formula]

$$\Rightarrow ⊡_n + (n+1)^3 = \sum_{0 \le K \le n} K^3 + \sum_{0 \le K \le n} 3K^2 + \sum_{0 \le K \le n} 3K + \sum_{0 \le K \le n} 1$$

$$\Rightarrow ⊡_n + (n+1)^3 = ⊡_n + 3\Box_n + 3\frac{n(n+1)}{2} + (n+1)$$

$$\Rightarrow (n+1)^3 = 3\Box_n + \frac{3n(n+1)}{2} + (n+1)$$

$$\Rightarrow 3\Box_n = (n+1)^3 - (n+1) - \frac{3n(n+1)}{2}$$

$$= (n+1)\left\{(n+1)^2 - 1 - \frac{3n}{2}\right\}$$

$$= (n+1)\left(n^2+2n+1-1-\frac{3n}{2}\right)$$

$$= (n+1)\, n \left(n+2-\frac{3}{2}\right)$$

$$= n(n+1)\left(n+\frac{1}{2}\right)$$

Assignment 1 : Repertoire method for n^2 (Ans), build $3\Box_n = n(n+1)\left(n+\frac{1}{2}\right)$

$\rightarrow R_0 = a_0$

$R_n = R_{n-1} + a_n$

Let,

$a_0 = \alpha$

$a_n = \beta + \gamma n + \delta n^2$

$\therefore R_n = R_{(n-1)} + \beta + \gamma n + \delta n^2$ ——— (1)

$R_0 = \alpha$

~~$\therefore R_1 = 0 + \beta + \gamma$~~

$\therefore R_n = A(n)\alpha + B(n)\beta + C(n)\gamma + D(n)\delta$ ——— (11)

Let,

$R_n = 1$

$R_0 = 1 = \alpha$

$\therefore R_{(n-1)} = 1$

from (i),

~~$1 = 1 + B(n)\beta + C(n)\gamma +$~~

$R_n = 1 + \beta + \gamma n + \delta n^2$

$\Rightarrow 1 = 1 + \beta + \gamma n + \delta n^2$

$\Rightarrow 0 = \beta + \gamma n + \delta n^2$

$\Rightarrow \beta = 0, \gamma = 0, \delta = 0$

from (ii),

$1 = A(n)\alpha$

$\Rightarrow A(n) = \alpha$

For,

$R_n = n$

$R_0 = 0 = \alpha$

$R_{(n-1)} = n-1$

from (i),

$n = n-1 + \beta + \gamma n + \delta n^2$

$\Rightarrow 0 = (-1+\beta) + \gamma n + \delta n^2$

$\Rightarrow \alpha = 0, \beta = 1, \gamma = 0, \delta = 0$

from (ii),

$n = B(n)\beta$

$\Rightarrow n = B(n).1$

$\Rightarrow B(n) = n$

For,

$R_n = n^2$

$R_0 = 0 = \alpha$

$R_{(n-1)} = (n-1)^2$

$= n^2 - 2n + 1$

From (i),

$R_n = n^2 - 2n + 1 + \beta + \gamma n + \delta n^2$

$\Rightarrow n^2 = n^2 - 2n + 1 + \beta + \gamma n + \delta n^2$

$\Rightarrow 0 = -2n + 1 + \beta + \gamma n + \delta n^2$

$\Rightarrow 0 = 1 + \beta + n(\gamma - 2) + \delta n^2$

$\Rightarrow \beta = -1, \gamma = 2, \delta = 0, \alpha = 0$

from (ii),

$\cancel{n^2 = 1 + n +}$

$n^2 = -B(n) + 2C(n)$

$\Rightarrow n^2 + n = 2C(n)$

$\Rightarrow C(n) = \frac{n(n+1)}{2}$

For,

$R_n = n^3$

$R_0 = 0 = \alpha$

$R_{n-1} = (n-1)^3$

$= n^3 - 3n^2 + 3n - 1$

form (i),

$n^3 = n^3 - 3n^2 + 3n - 1 + \beta + \gamma n + \delta n^2$

$\Rightarrow 0 = -3n^2 + 3n - 1 + \beta + \gamma n + \delta n^2$

$\Rightarrow 0 = \delta n^2 - 3n^2 + \gamma n + 3n - 1 + B$

$\Rightarrow 0 = n^2(\delta - 3) + n(\gamma + 3) - 1 + \beta$

$\Rightarrow \alpha = 0, \beta = 1, \gamma = -3, \delta = 3$

from (ii),

$$n^3 = B(n) - 3C(n) + 3D(n)$$

$$\Rightarrow n^3 - n + 3\left\{\frac{n(n+1)}{2}\right\} = 3D(n)$$

$$\Rightarrow 3D(n) = \frac{2n^3 - 2n + 3n^2 + 3n}{2}$$

$$\Rightarrow 3D(n) = \frac{2n^3 + 3n^2 + n}{2}$$

$$= n\left(\frac{2n^2 + 2n + n + 1}{2}\right)$$

$$= n\left(\frac{2n(n+1) + 1(n+1)}{2}\right)$$

$$= n\,\frac{(n+1)\ (2n+1)}{2} \qquad = n\ (n+1)\ \frac{(2n+1)}{2}$$

$$= n\ \left(n + \frac{1}{2}\right)\ (n+1)$$

$\therefore$

Chapter - 03

Floor = $\lfloor x \rfloor$ = the greatest int less or equal to x.

Celling = $\lceil x \rceil$ = the least int greater than or equal to x.

Formula

$\lfloor -x \rfloor = -\lceil x \rceil$

$\lceil -x \rceil = -\lfloor x \rfloor$

Property : তত point যত mark

1. $\lfloor x \rfloor \leq x$; $\lceil x \rceil \geq x$
2. $\lceil x \rceil = \lfloor x \rfloor = x$ (if integer)
3. $\lceil x \rceil - \lfloor x \rfloor = 1$ (none integer)
4. $x - 1 < \lfloor x \rfloor \leq x \leq \lceil x \rceil < x + 1$ $(1.5 < 2 \leq 2.5 \leq 3 < 3.5)$
5. $\lfloor -x \rfloor = -\lceil x \rceil$

   $\lceil -x \rceil = -\lfloor x \rfloor$

Swapping formula :

$$\left.\begin{array}{ll} \text{(i)} & \lfloor x \rfloor = n \Leftrightarrow n \leq x < n+1 \\ \text{(ii)} & x < n \Longleftrightarrow \lfloor x \rfloor < n \\ \text{(iii)} & n \leq x \Leftrightarrow n \leq \lfloor x \rfloor \end{array}\right\} \text{floor}$$

$$\text{Celling} \left\{\begin{array}{ll} \text{(i)} & \lceil x \rceil = n \Longleftrightarrow n-1 < x \leq n \\ \text{(ii)} & n < x \Longleftrightarrow n < \lceil x \rceil \\ \text{(iii)} & x \leq n \Longleftrightarrow \lceil x \rceil \leq n \end{array}\right.$$

For ceiling—

Formula,

$\lceil x \rceil = n \iff n-1 < x \leq n$ — (i)

$n < x \iff n < \lceil x \rceil$ — (ii)

$x \leq n \iff \lceil x \rceil \leq n$ — (iii)

proof—

$$\lceil \sqrt{\lceil x \rceil} \rceil = \lceil \sqrt{x} \rceil$$

Let,

$m = \lceil \sqrt{\lceil x \rceil} \rceil$

$\Rightarrow m-1 < \sqrt{\lceil x \rceil} \leq m$ [from i]

$\Rightarrow (m-1)^2 < \lceil x \rceil \leq m^2$ [square]

$\Rightarrow (m-1)^2 < x \leq m^2$ [if iii]

$\Rightarrow m-1 < \sqrt{x} \leq m$ ($\sqrt{}$ করে)

$\Rightarrow m = \lceil \sqrt{x} \rceil$ from (i) (Prove)

Proof: $\lfloor \sqrt{\lfloor x \rfloor} \rfloor = \lfloor \sqrt{x} \rfloor$

let, $m = \lfloor \sqrt{\lfloor x \rfloor} \rfloor$

$m \leq \sqrt{\lfloor x \rfloor} < m+1$ (equation 1 [illegible])

$\Rightarrow m^2 \leq \lfloor x \rfloor < (m+1)^2$ ([illegible])

$\Rightarrow m^2 \leq x < (m+1)^2$ I.

[∵ ii, iii]

$\Rightarrow m \leq \sqrt{x} < (m+1)$ (from 1 compare)

$\Rightarrow m = \lfloor \sqrt{x} \rfloor$ (from i)

formula-

(i) $\lfloor x \rfloor = n \Leftrightarrow n \leq x < n+1$

(ii) $n \leq x \Leftrightarrow n \leq \lfloor x \rfloor$

(iii) $x < n \Leftrightarrow \lfloor x \rfloor < n$

$\lfloor x \rfloor = n \longleftrightarrow n \leq x < n+1$

$x < n \longleftrightarrow \lfloor x \rfloor < n$

$n \leq x \Longleftrightarrow n \leq \lfloor x \rfloor$

Modules :

$$x \bmod y = x - \left(y \left\lfloor x/y \right\rfloor\right)$$

Ques :

1. $-77 \bmod -3$
2. $77 \bmod -3$
3. $77 \bmod 3$

→ 1. $-77 \bmod -3$

$= -77 - \left(-3 \left\lfloor \frac{-77}{-3} \right\rfloor\right)$

$= -77 - (-3 \times 25)$

$= -77 - (-75)$

$= -77 + 75 = -2$

→ 2. $77 \bmod -3$

$= 77 - \left(-3 \left\lfloor \frac{-77}{3} \right\rfloor\right)$

$= 77 - \left(-3 - \left\lceil \frac{77}{3} \right\rceil\right)$

$= 77 - (-3 \times -26)$

$= 77 - 78$

$= 1$

→ 3. $77 \bmod 3$

$= 77 - \left(3 \left\lfloor \frac{77}{3} \right\rfloor\right)$

$= 77 - 3 \times 25$

$= 77 - 75$

$= 2$ (Ans)

proof

$c(x \bmod y) = cx \bmod cy$

$\Rightarrow c\{x - (y\lfloor x/y \rfloor)\}$

$\Rightarrow cx - c(y\lfloor x/y \rfloor)$

$\Rightarrow cx - cy\lfloor cx/cy \rfloor$

$\Rightarrow cx \bmod cy = R.H.S$ (prove)

#

$n \rightarrow$ no of thing

$m \rightarrow$ no of grp

n	m	$\lceil n/m \rceil$
500	8	63
437	7	63
374	6	63
311	5	63
248	4	62
186	3	62
124	2	62
62	1	62

$n_1 - \lceil n/m \rceil_1 = n_2$

$m_1 - 1 = m_2$

Σ 500

Final Number Ch-04

Chapter - 4

Number Theory

Divisibility :

$$\frac{n}{m} = k \quad ; \quad m > 0$$

$$\Rightarrow n = mk$$

Prime number :

$12 = 2 \times 2 \times 3 = 2^2 . 3$ [HCF] aka

$50 = 2 \times 5 \times 5 = 2 . 5^2$

$$n = P_1 . P_2 \ldots . P_m = \prod_{K=1}^{m} P_K$$

$= \prod p^{np} \rightarrow$ P number power ; $np \geq 0$

$P \rightarrow$ all prime number

Prime exponent representation :

$50 = 2^1 . 3^0 . 5^2 . 7^0$ $[2 \times 5 \times 5]$

$= \langle 1, 0, 2, 0, \ldots \rangle$

$100 = 2^2 . 3^0 . 5^2 . 7^0$ $[2 \times 2 \times 5 \times 5]$

$= \langle 2, 0, 2, 0, \ldots \rangle$

$250 = 2^1 . 3^0 . 5^3 . 7^0$ $[5 \times 2 \times 5 \times 5]$

$= \langle 1, 0, 3, 0, \ldots \rangle$

$350 = 2^1 . 3^0 . 5^2 . 7^1$ $[5 \times 2 \times 5 \times 7]$

$= \langle 1, 0, 2, 1, 0 \ldots \rangle$

GCD : min number of common power

$\rightarrow$ $GCD(100, 250) = 2^{\min(2,1)} \cdot 5^{\min(2,3)}$

$= 2^1 \cdot 5^2 = 50$

Since,

$100 = 2^2 \cdot 5^2$

$250 = 2^1 \cdot 3^0 \cdot 5^3 \cdot 7^0$

LCM $(100, 250)$ = max num power

$= 2^{\max(2,1)} \cdot 5^{\max(2,3)}$

$= 2^2 \cdot 5^3$

$= 500$

Q2: GCD and LCM of $(2500, 175)$

$\rightarrow$ $2500 = 2\times 2\times 5\times 5\times 5\times 5 = 2^2 \cdot 5^4 \cdot 7^0$

$175 = 5\times 5\times 7 = 5^2 \cdot 7^1 \cdot 2^0$

$\therefore GCD(2500, 175) = 2^{\min(2,0)} \cdot 5^{\min(4,2)} \cdot 7^{\min(0,1)}$

$= 2^0 \cdot 5^2 \cdot 7^0$

$= 25$

$\therefore LCM(2500, 175) = 2^{\max(2,0)} \cdot 5^{\max(4,2)} \cdot 7^{\max(0,1)}$

$= 2^2 \cdot 5^4 \cdot 7^1$

$= 17500$

(Ans)

Q3: $(1500, 75)$ find GCD, LCM.

$\rightarrow$ $1500 = 2\times2\times3\times5\times5\times5 = 2^2 . 3^1 . 5^3$

$\langle 2, 1, 3, 0, \ldots \rangle$

$75 = 3\times5\times5 = 2^0 . 3^1 . 5^2$

$\langle 0, 1, 2, 0, \ldots \rangle$

$\therefore$ GCD $(1500, 75) = 2^{\min(2,0)} . 3^{m(1,1)} . 5^{\min(3,2)}$

$= 2^0 . 3^1 . 5^2$

$= 75$

$\therefore$ LCM $(1500, 75) = 2^{\max(2,0)} . 3^{\max(1,1)} . 5^{\max(3,2)}$

$= 2^2 . 3^1 . 5^3$

$= 1500$

(Ans)

Euclid number:

$e_n = e_1 . e_2 \ldots . e_{n-1} + 1 ; n \geq 1$

$e_0 = 1$

Hence,

$e_1 = e_0 + 1 = 1 + 1 = 2$

$e_2 = e_1 + 1 = 2 + 1 = 3$

$e_3 = e_1 . e_2 + 1 = 2 . 3 + 1 = 7$

$e_4 = e_1 . e_2 . e_3 + 1 = 2 . 3 . 7 + 1 = 43$

$e_5 = e_1 . e_2 . e_3 . e_4 + 1 = 2 . 3 . 7 . 43 + 1 = 1807$

$e_6 = e_1 . e_2 . e_3 . e_4 . e_5 + 1 = 2 . 3 . 7 . 43 . 1807 + 1 = 3263443$

euclid number → relative prime (২টি num কে 1 ছাড়া ভাগ করা যায় না)

$\therefore \gcd(\text{euclid num}) = 1$

$\Rightarrow \gcd(e_m, e_n) = 1 \quad ; m \neq n$

Mersenne number :

$\boxed{2^P - 1} \quad ; P = \text{prime num}$

$2^2 - 1 = 4 - 1 = 3$

$2^3 - 1 = 8 - 1 = 7$

$2^5 - 1 = 32 - 1 = 31$

$2^7 - 1 = 128 - 1 = 127$

$2^{11} - 1 = 2048 - 1 = 2047$

MOD : The congruence relation ($\equiv$)

$a \equiv b \pmod m$ if, $[a \bmod m = b \bmod m]$

Q1: $9 \equiv -16 \pmod 5$

LHS $= \cancel{16}\, 9 \bmod 5$ $\quad [x \bmod y = x - y\lfloor x/y \rfloor]$

$= 4$

RHS $= -16 - 5 \cdot (\lfloor -16/5 \rfloor)$

$= -16 - 5 \cdot (\lceil 16/5 \rceil) \quad [\lfloor -x \rfloor = -\lceil x \rceil]$

$= -16 - 5 \times -4$

$= -16 + 20$

$= 4$ (proved)

$\therefore 9 \equiv -16 \pmod 5$

Q2: $200 \equiv -16 \pmod 4$

$\rightarrow$ LHS = 200 mod 4

= 0

RHS = $-16 \pmod 4$

$= -16 - 4\left(\left\lfloor \frac{-16}{4} \right\rfloor\right)$

$= -16 - 4 \times -4$

$= -16+16 = 0$ = RHS [proved]

$\therefore$ congruent.

Q3: $284 \equiv 12 \pmod 2$

LHS = 284 mod 2

= 0

RHS = $12 \pmod 2$

$= 12 - 2 \times \left(\left\lfloor \frac{12}{2} \right\rfloor\right)$

$= 12 - 2 \times 6$

= 0

$\therefore 284 \equiv 12 \pmod 2$ (Ans)

Congruent is an equivalence relation:

(I) Reflexive law, $a \equiv a$

(II) Symmatric law, $a \equiv b \Rightarrow b \equiv a$

(III) Transitive law, $a \equiv b \equiv c \Rightarrow a \equiv c$

Euclidean :

Alg → $\gcd(a,b)$; $a>b$

while $(b>0)$

{ $r = a \bmod b$ $[a - b \times (\lfloor a/b \rfloor)]$

$a = b$

$b = r$ }

return (a) and [$\gcd = a$ if $b = r = 0$]

Q1: $\gcd(2322, 650)$

→ $r = a \bmod b$

$= a - b(\lfloor a/b \rfloor)$

$= 2322 - 650 \times (\lfloor 2322/650 \rfloor)$

$= 2322 - 650 \times 3$

$= 2322 - 1950 = 372$

$a = 650$, $b = 372$

$r = 650 - 372(\lfloor 650/372 \rfloor)$

$= 278$

$a = 372$, $b = 278$

$r = 372 - 278(\lfloor 372/278 \rfloor)$

$= 94$

$a = 278$, $b = 94$

$r = 278 - 94(\lfloor 278/94 \rfloor)$

$= 90$

$a = 94$, $b = 90$

$r = 94 - 90(\lfloor 94/90 \rfloor)$

$= 4$

$r = 90 - 4(\lfloor 90/4 \rfloor)$

$= 90 - 4 \times 22$

$= 90 - 88 = 2$

$a = 4$, $b = 2$

$r = 4 - 2(\lfloor 4/2 \rfloor)$

$= 4 - 4$

$= 0$

∴ $\gcd = 2$

(Ans)

Q2: gcd (1050, 20) , $a = 1050$, $b = 20$

$\rightarrow r = 1050 - 20 \left(\lfloor \frac{1050}{20} \rfloor\right)$ $\quad$ [$a \bmod b = a - b \left(\lfloor \frac{a}{b} \rfloor\right)$

$= 1050 - 1040$

$= 10$

$a = 20$, $b = 10$

$r = 20 - 10 \left(\lfloor \frac{20}{10} \rfloor\right)$

$= 20 - 20$

$= 0$; $a = 10$, $b = 0$

$\therefore$ gcd $= 10$ (Ans)

Q3: gcd (1128, 46)

$\rightarrow r = 1128 - 46 \left(\lfloor \frac{1128}{46} \rfloor\right)$

$= 1128 - 1104$

$= 24$

$a = 46$, $b = 24$

$r = 46 - 24 \left(\lfloor \frac{46}{24} \rfloor\right)$

$= 46 - 24$

$= 22$

$a = 24$, $b = 22$

$r = 24 - 22 \left(\lfloor \frac{24}{22} \rfloor\right)$

$= 24 - 22$

$= 2$

$a = 22$, $b = 2$

$r = 22 - 2 \left(\lfloor \frac{22}{2} \rfloor\right)$

$= 22 - 22$

$= 0$

$a = 2$, $b = 0$

$\therefore$ gcd $= 2$ (Ans)

law - 01 :

prove. $\gcd(na, nb) = n \cdot \gcd(a,b)$

Let,

$$\gcd(a,b) = P_1^{\min(\alpha_1, \beta_1)} \cdots\cdots P_k^{\min(\alpha_k \beta_k)}$$

$$n = P_1^{n_1} \cdots P_k^{n_k}$$

~~gcd~~ $$a = P_1^{\alpha_1} \cdots P_k^{\alpha_k}$$

$$b = P_1^{\beta_1} \cdots P_k^{\beta_k}$$

$$na = P_1^{(\alpha_1 + n_1)} \cdots\cdots P_k^{(\alpha_k + n_k)}$$

$$nb = P_1^{(\beta_1 + n_1)} \cdots\cdots P_k^{(\beta_k + n_k)}$$

$$\therefore LHS = \gcd(na, nb) = P_1^{\min(\alpha_1 + n_1, \beta_1 + n_1)} \cdots\cdots P_k^{(\alpha_k + n_k, \beta_k + n_k)}$$

$$= P_1^{\min(\alpha_1, \beta_1)} \cdot P_1^{n_1} \cdots\cdots P_k^{\min(\alpha_k, \beta_k)} \cdot P_k^{n_k}$$

$$= \left(P_1^{n_1} \cdots\cdots P_k^{n_k}\right)\left(P_1^{\min(\alpha_1 \beta_1)} \cdots\cdots P_k^{\min(\alpha_k, \beta_k)}\right)$$

$$= n \cdot \gcd(a,b)$$

$$= R.H.S$$

$\therefore$ LHS = RHS (proved)

Ques 1: Prove if $a = 500$, $b = 12$, $n = 5$

$\gcd(na, nb) = n \cdot \gcd(a, b)$

→

$$L.H.S = \gcd(500 \times 5, 12 \times 5)$$

$$= \gcd(2500, 60)$$

$$= 20$$

$$RHS = 5 \gcd(500, 12)$$

$$= 5 \times 4$$

$$= 20$$

$\therefore$ LHS = RHS

→

$2500 = 2 \times 2 \times 5 \times 5 \times 5 \times 5 = 2^2 \times 3^0 \times 5^4$

$60 = 2 \times 2 \times 3 \times 5 = 2^2 \times 3^1 \times 5^1$

$\therefore \gcd = \min(2^2 \times 3^0 \times 5^1)$

$= 20$

→

$500 = 2 \times 2 \times 5 \times 5 \times 5 = 2^2 \times 3^0 \times 5^3$

$12 = 2 \times 2 \times 3 = 2^2 \times 3^1 \times 5^0$

$\therefore \gcd = 2^2 \times 3^0 \times 5^0$

$= 4$

Law - 02 :

prove, $\gcd(a,b) \cdot \text{lcm}(a,b) = ab$

Let,

$$a = p_1^{\alpha_1} \cdots p_k^{\alpha_k}$$

$$b = p_1^{\beta_1} \cdots p_k^{\beta_k}$$

$$\gcd(a,b) = p_1^{\min(\alpha_1,\beta_1)} \cdots p_k^{\min(\alpha_k,\beta_k)}$$

$$\text{lcm}(a,b) = p_1^{\max(\alpha_1,\beta_1)} \cdots p_k^{\max(\alpha_k,\beta_k)}$$

$\therefore$ LHS $= \gcd(a,b) \cdot \text{lcm}(a,b)$

$$= p_1^{\min(\alpha_1,\beta_1)} \cdots p_k^{(\alpha_k,\beta_k)} \cdot p_1^{\max(\alpha_1,\beta_1)} \cdots p_k^{\max(\alpha_k,\beta_k)}$$

$$= p_1^{\min(\alpha_1,\beta_1)+\max(\alpha_1,\beta_1)} \cdots p_k^{\min(\alpha_1,\beta_1)+\max(\alpha_1,\beta_1)}$$

$$= p_1^{(\alpha_1,\beta_1)} \cdots p_k^{(\alpha_k,\beta_k)} \quad [\text{min nullfies max}]$$

$$= (p_1^{\alpha_1} \cdots p_k^{\alpha_k}) \cdot (p_1^{\beta_1} \cdots p_k^{\beta_k})$$

$$= a \cdot b = \text{LHS}$$

$\therefore$ LHS = RHS

Ques: $a = 200$, $b = 14$ then prove law 2.

$\rightarrow$ LHS $= \gcd(200,14) \cdot \text{lcm}(200,14)$

$= 2 \cdot 1400$

$= 2800$

RHS $= 200 \cdot 14$

$= 2800$

$\therefore$ LHS = RHS (proved)

$200 = 2 \times 2 \times 2 \times 5 \times 5 = 2^3 \times 3^0 \times 5^2 \times 7^0$

$14 = 2 \times 7 = 2^1 \times 3^0 \times 5^0 \times 7^1$

$\therefore \gcd = 2^{\min} \cdot 5^{\min} \cdot 7^{\min}$

$= 2^1 \cdot 5^0 \cdot 7^0$

$= 2$

$\therefore \text{lcm} = 2^{\max} \cdot 5^{\max} \cdot 7^{\max}$

$= 2^3 \times 5^2 \times 7^1$

$= 1400$

Ross Probability

Chapter - 1

☐ **Sample space :** The set of all events, denoted by S.

$$S = \{HH, HT, TH, TT\}$$

Car এর lifetime এর $S = \{0, \alpha\}$ since end is undecided

☐ **Event :**

$E = \{H\}$

Ex → 1st and 3rd coin is head → $E = \{HTH, HHH\}$ [write the sample space before]

Ex-2 : 2 dice are rolled. Find the probability of sum = 7.

→ $S = \{(1,1), (1,2), (1,3), (1,4), (1,5), (1,6)$
$(2,1), (2,2), (2,3), (2,4), (2,5), (2,6)$
$(3,1), (3,2), (3,3), (3,4), (3,5), (3,6)$
$(4,1), (4,2), (4,3), (4,4), (4,5), (4,6)$
$(5,1), (5,2), (5,3), (5,4), (5,5), (5,6)$
$(6,1), (6,2), (6,3), (6,4), (6,5), (6,6)\} = 36$

$\therefore E = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\} = 6$

$\therefore P = \frac{6}{36} = \frac{1}{6}$ (Ans)

for, all even, $E = \{(2,2), (2,4), (2,6), (4,2), (4,4), (4,6),$
$(6,2), (6,4), (6,6)\} = 9$

$\therefore P = \frac{\cancel{9}^{3}}{\cancel{36}_{12}} = \frac{3}{12} = \frac{1}{4}$ (Ans)

Final
Ch-01

(#) Car lasts for 2 to 5 year = $E = \{2, 3, 4, 5\}$

Probability :

$P(S) = 1$

$P(E) = 0 \leq P(E) \leq 1$

$P(E \cup F) = P(E) + P(F) - P(E.F)$ $[P(E.F) = P(E \cap F)]$

If E and F mutually exclusive,

$P(E \cup F) = P(E) + P(F)$ [events are not dependent / একটা থেকে অন্যটা আলাদা]

$P(E) = \dfrac{\text{number of event}}{\text{total event}}$

Question: A person has 3 children. Find the P_1 of 1st and 3rd boy. And 2nd girl find P_2.

→ $S = \{ggg, ggb, gbb, gbg, bgg, bgb, bbg, bbb\}$

$E_1 = \{bgb, bbb\}$

$\therefore P_1 = \dfrac{2}{8} = \dfrac{1}{4}$ (Ans)

$E_2 = \{ggg, ggb, bgg, bgb\}$

$\therefore P_2 = \dfrac{4}{8} = \dfrac{1}{2}$

(Ans)

Q-3: 2 dice, sum = 12. What is P?

→ $S = \{(1,1) \ldots (6,6)\}$

$E = \{(6,6)\}$

$\therefore P = \dfrac{1}{36}$ (Ans)

[$P(E) = P(EF)/P(E \cap F)$]

A-A-A

Conditional probability :

$$P(E/F) = \frac{P(EF)}{P(F)}$$

[E = যে event এর P বের করবে]
[F = conditional event , F হলে E এর prob]

Suppose, given event $F = \{(4,1), (4,2), (4,3), (4,4), (4,5), (4,6)\} = \frac{1}{6}$

P of sum = 7 = $E = \{(4,3)\}$, $\therefore$ $(EF) = E \cup F = \{4,3\} = \frac{1}{36}$

$$\therefore P(E/F) = \frac{P(EF)}{P(F)}$$

$$= \frac{\frac{1}{36}}{\frac{1}{6}} = \frac{1}{\cancel{36}_6} \times \frac{\cancel{6}}{1} = \frac{1}{6}$$

(Ans).

Q1.5 : Family has 2 kid. Condition one of them is boy. Find P of 2 boys :

→ $S = \{gg, gb, bg, bb\} = 4$

$F = \{gb, bg, bb\}$ $\therefore P(F) = \frac{3}{4}$

$E = \{bb\}$

$\therefore EF = E = \{bb\}$ $\therefore P(EF) = \frac{1}{4}$

$$\therefore P(E/F) = \frac{P(EF)}{P(F)} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{\cancel{4}} \times \frac{\cancel{4}}{3} = \frac{1}{3}$$

(Ans)

Q2 : Has 2 kid. Condition atmost 1 boy. Find P of 2 girls.

→ $S = \{gg, gb, bg, bb\} = 4$

$F = \{gg, gb, bg\} = \frac{3}{4}$

$E = EF = \{gg\} = \frac{1}{4} = P(EF)$

$$\therefore P(E/F) = \frac{P(EF)}{P(F)} = \frac{\frac{1}{4}}{\frac{3}{4}} = \frac{1}{3}$$

(Ans)

Q3: 10 cards, Pick one card that is atleast 5. Find P of 10

→ $S = \{1, 2, 3 \ldots 9, 10\} = 10$

$F = \{5, 6, 7, 8, 9, 10\}$, $P(F) = \frac{6}{10}$

$E = EF = \{10\}$, $P(EF) = \frac{1}{10}$

$\therefore P(E/F) = \frac{P(EF)}{P(F)} = \frac{\frac{1}{10}}{\frac{6}{10}} = \frac{1}{6}$ (Ans)

Q4:

Q5: 3 coin. Atmost 2 coin head so find P 1st and 3rd coin being head.

→ $S = \{hhh, hht, hth, htt, thh, tht, tth, ttt\} = 8$

$F = \{ttt, hht, hth, htt, thh, thh, tth\} - 7$

$\therefore P(F) = \frac{7}{8}$

$E = EF = \{ hth, \}$

$\therefore P(EF) = \frac{1}{8}$

$\therefore P(E/F) = \frac{\frac{1}{8}}{\frac{7}{8}} = \frac{1}{\cancel{8}} \times \frac{\cancel{8}}{7} = \frac{1}{7}$

(Ans)

Q 1.7 : Ball = 12 in a urn. 7 black and 5 white. Pick 2 balls without replacing. P of both balls being black = ?

→ $S = 7+5 = 12$

1st ball being black $= P(F) = \frac{7}{12}$ [condition, 1st ball needs to be black]

After 1st ball $S = 6+5 = 11$

∴ 2nd ball will be black to $P(E/F) = \frac{6}{11}$

So, both ball being black, $P(EF) = P(E/F) \cdot P(F)$

$$= \frac{6}{11} \times \frac{7}{12}$$

$$= \frac{7}{22} \quad \text{(Ans)}$$

Independent event :

$P(EF) = P(E) \cdot P(F)$ [$P(EF) = P(E \cap F)$]

Ques : In a case of 2 dice only the sum 7 can make these independent. 1st dice is 3 suppose.

→ $E = \{(1,6), (2,5), (3,4), (4,3), (5,2), (6,1)\}$, $P(E) = \frac{6}{36} = \frac{1}{6}$

$F = \{(3,1), (3,2), (3,3), (3,4), (3,5), (3,6)\}$, $P(F) = \frac{6}{36} = \frac{1}{6}$

$EF/E \cap F = \{(3,4)\}$, $P(E \cap F) = \frac{1}{36}$

∴ $P(E) \cdot P(F) = \frac{1}{6} \times \frac{1}{6}$

$$= \frac{1}{36}$$

$$= P(EF) \quad \text{(proved)}$$

Pair-wise independent event that are not independent :

Suppose,

$S = \{1,2,3,4\}$

$E = \{1,2\} = \frac{1}{2}$

$F = \{1,3\} = \frac{1}{2}$

$G = \{1,4\} = \frac{1}{2}$

But,

$P(EFG) = \{1\}$

$P(EFG) = P(E).P(F).P(G)$

$\Rightarrow \frac{1}{4} = \frac{1}{2} . \frac{1}{2} . \frac{1}{2}$

$= \frac{1}{6}$

$\therefore$ LHS $\neq$ RHS

$\therefore$ Individually not independent.

pairwise,

$P(EF) = P(E).P(F)$ $[(EF) = \{1\}]$

$\Rightarrow \frac{1}{4} = \frac{1}{2} . \frac{1}{2}$

$= \frac{1}{4} =$ LHS

$P(FG) = P(F).P(G)$ $[(FG) = \{1\}]$

$\Rightarrow \frac{1}{4} = \frac{1}{2} . \frac{1}{2}$

$= \frac{1}{4} =$ LHS

$P(GE) = P(G).P(E)$ $[(GE) = \{1\}]$

$\Rightarrow \frac{1}{4} = \frac{1}{2} . \frac{1}{2}$

$= \frac{1}{4} =$ L.H.S

Hence, pair-wise independent.

Bayes' formula :

$$\boxed{E = \underbrace{EF \cup EF^c}_{\text{mutually exclusive}}}$$

$= E(F \cup F^c)$

$= E.1$ $[F^c = 1-F]$

$= E$

(proved)

$\therefore P(E) = P(EF \cup EF^c)$
$= P(EF) + P(EF^c)$
$= P(E/F).P(F) + P(E/F^c).P(F^c)$

$\Rightarrow P(E/F) = \frac{P(EF)}{P(F)}$
$\Rightarrow P(E \cup F) = P(E) + P(E) - P(EF)$
$\Rightarrow P(F^c) = 1 - P(F)$

$$\boxed{P(E) = P(E/F).P(F) + P(E/F^c).P(F^c)}$$

[Output দেওয়া থাকে input বের করতে হবে]

Must

Q 1.12 : Urn 1 → 2w + 7B balls = 9
Urn 2 → 5w + 6B ball = 11

Coin tossed, if H then 1st urn, if T then 2nd ~~হ~~ থেকে ball pick করবো।
Find the P of ball picked w and H on coin.

→ H = Head w = white
H^c = Tail

$$\therefore P(H/w) = \frac{P(HW)}{P(W)}$$

$$= \frac{P(w/H).P(H)}{P(w)}$$

$$= \frac{P(w/H).P(H)}{P(w/H).P(H) + P(w/H^c).P(H^c)}$$

$$= \frac{\frac{2}{9} \cdot \frac{1}{2}}{\left(\frac{2}{9} \cdot \frac{1}{2}\right) + \left(\frac{5}{11} \cdot \frac{1}{2}\right)}$$

$$= \frac{11}{67} = 0.164$$

$$P(w/H) = \frac{P(HW)}{P(H)}$$

$\Rightarrow P(HW) = P(w/H).P(H)$

from bayes's,
$P(w) = P(W/H).P(H) + P(w/H^c).P(H^c)$

$\Rightarrow P(w/H) = \frac{2}{9}$, $P(H) = \frac{1}{2}$ $\therefore P(H^c) = \frac{1}{2}$

$\Rightarrow P(w/H^c) = \frac{5}{11}$

Q : (1.14) :- 95% time ~~accurate~~ right result. 1% falls Positive 0.5% has actually covid. Your test result positive. What is the P of you actually having covid ?

→ D → you have covid

E → test result positive

$$P(E/D) = \frac{P(ED)}{P(D)}$$

$$P(D/E) = \frac{P(ED)}{P(D)}$$

$$\Rightarrow P(E|D).P(D) = P(ED)$$

$$= \frac{P(ED)}{P(E)}$$

$$= \frac{P(E/D).P(D)}{P(E/D).P(D) + P(E/D^c)P(D^c)}$$

$$= \frac{(0.95 \times 0.005) \times 0.005}{(0.95) \times (0.005) + (0.01) \times (1 - 0.005)}$$

$$= 0.323$$

Note:

Chapter - 2

Random variable

.1 : Random variable

find $P(x)$ where $X \to$ sum of two dice

$\to \quad x = \{2-12\}$ since

$\therefore P(x=2) = P\{(1,1)\} = \frac{1}{36}$

$P(x=3) = P\{(1,2),(2,1)\} = \frac{2}{36} = \frac{1}{18}$

$P(x=4) = P\{(1,3),(2,2),(3,1)\} = \frac{3}{36} = \frac{1}{12}$

$P(x=5) = P\{(1,4),(2,3),(3,2),(4,1)\} = \frac{4}{36} = \frac{1}{9}$

$P(x=6) = P\{(1,5),(2,4),(3,3),(4,2),(5,1)\} = \frac{5}{36}$

$P(x=7) = P\{(1,6),(2,5),(3,4),(4,3),(5,2),(6,1)\} = \frac{6}{36} = \frac{1}{6}$

$P(x=8) = P\{(2,6),(3,5),(4,4),(5,3),(6,2)\} = \frac{5}{36}$

$P(x=9) = P\{(3,6),(4,5),(5,4),(6,3)\} = \frac{4}{36} = \frac{1}{9}$

$P(x=10) = P\{(4,6),(5,5),(6,4)\} = \frac{3}{36} = \frac{1}{12}$

$P(x=11) = P\{(5,6),(6,5)\} = \frac{2}{36} = \frac{1}{18}$

$P(x=12) = P\{(6,6)\} = \frac{1}{36}$

$\therefore \sum P = \frac{1}{36} + \frac{1}{18} + \frac{1}{12} + \frac{1}{9} + \frac{5}{36} + \frac{1}{6} + \frac{5}{36} + \frac{1}{9} + \frac{1}{12} + \frac{1}{18} + \frac{1}{36}$

$= 1$

(Ans)

Q: 2 coin toss. Y is a random variable.
$\therefore P(Y) = ?$

$P(Y) = \{0, -2\}$

$P(Y=1) = P\{HT, TH\} = \frac{2}{4}$

$P(Y=2) = P\{HH\} = \frac{1}{4}$

$P(Y=0) = P(TT) = \frac{1}{4}$

Total P =

$\therefore \frac{2}{4} + \frac{1}{4} + \frac{1}{4} = 1$

8×2

H.W.

Q2: 4 coin toss. x is random variable that represents num of tail. P of (x) ?

$\rightarrow P(x) = \{0 - 4\}$

$P(x=0) = P\{(HHHH)\} = \frac{1}{16}$

$P(x=1) = P\{(HHHT), (HHTH), (HTHH), (THHH)\} = \frac{4}{16}$

$P(x=2) = P\{(HHTT), (HTTH), (TTHH), (HTHT), (THTH), (THHT)\} = \frac{6}{16}$

Note:

$P(x=3) = P\{(HTTT), (TTTH), (THTT), (TTHT)\} = \frac{4}{16}$

$P(x=4) = P\{(TTTT)\} = \frac{1}{16}$

$\therefore \frac{1}{16} + \frac{4}{16} + \frac{6}{16} + \frac{4}{16} + \frac{1}{16} = 1$ (dom)

□ Random variable → Discrete (1,2,3...) → Bernoulli
→ Binominal
→ Continous

□ Probability mass function : Pmf

$$Pmf(a) = P(x = a)$$

↓ random variable value

□ Discrete variable, $Pmf(x_i) > 0$ if $i = 1, 2, 3 \ldots \alpha$

$Pmf(x) = 0$ for other value

$$Sum = \sum_{i=1}^{\alpha} Pmf(x_i) = 1$$

A random variable that can take ~~on the most~~ at most a countable value of number is said to be discrete random variable.

□ Bernoulli random variable : A random variable that can only ~~the~~ take 2 possible value which are success and fail. 1 = success and 0 = fail

1 → P, 0 → 1 - P

$\therefore Pmf(0) = P(x=0) = 1-P$

$\therefore Pmf(1) = P(x=1) = P$

Binomial Random variable: (n time)

n = total num of trial

$S = P$

$F = 1-P$

$\therefore P_{mf}(i) = \binom{n}{i} \cdot p^i \cdot (1-P)^{n-i}$; $i = 0, 1, 2$

$\Rightarrow {}^nC_i$

Ques 1: independently toss a coin 4 times. P of (hh), (tt) = ?

$\rightarrow$ $n = 4$

$i = 2$ (possible outcome of a coin) (hh or tt = 2)

h = success $= P = \frac{1}{2}$

t = fail $= 1-P = \frac{1}{2}$

$\therefore P(x=2) = {}^4C_2 \cdot p^i \cdot (1-P)^{n-i}$

$= {}^4C_2 \cdot \left(\frac{1}{2}\right)^2 \cdot \left(\frac{1}{2}\right)^{4-2}$

$= \frac{3}{8} = 0.375$ (Ans)

Ques 2: 8 times toss coin. P of 5 tail = ? when outcome is independent!

$\rightarrow$ $n = 8$ $\quad P = \frac{1}{2}$, since a coin has 2 possibility

$i = 5$ (ttttt) $\quad \therefore (1-P) = 1 - \frac{1}{2} = \frac{1}{2}$

$\therefore P(x=5) = {}^nC_i \cdot p^i \cdot (1-P)^{n-i}$

$= {}^8C_5 \cdot \left(\frac{1}{2}\right)^5 \cdot \left(\frac{1}{2}\right)^{8-5}$

$= 0.21875$ (Ans)

Ques 3: ~~Att~~ · Atleast 5 tail · 8 toss · (Same Q) · $P = 0.5$

→ $n = 8$

~~$P_{mf}(i) = P(x = [5-8])$~~

$x = (5-8)$

$\therefore (x = 5) = {}^8C_5 \cdot (0.5)^5 \cdot (0.5)^3 = 0.2187$

$(x = 6) = {}^8C_6 \cdot (0.5)^6 \cdot (0.5)^2 = 0.1093$

$(x = 7) = {}^8C_7 \cdot (0.5)^7 \cdot (0.5)^1 = 0.0312$

$(x = 8) = {}^8C_8 \cdot (0.5)^8 \cdot (0.5)^0 = 0.0039$

$\therefore \sum P = 0.2187 + 0.1093 + 0.0312 + 0.0039 = 0.3632$

(Ans)

Ques 4: defective item $P = 0.1$, out of 3 item almost 1 defective. Find $P(x) = ?$

→ $n = 3$

$x = \cancel{(1,2,3)}\ (0,1)$

$P = 0.1$

$\therefore 1-P = 1 - 0.1 = 0.9$

$\therefore (x = 0) = {}^3C_0 \cdot (0.1)^0 \cdot (0.9)^{3-0} = 0.729$

$\therefore (x = 1) = {}^3C_1 \cdot (0.1)^1 \cdot (0.9)^{3-1} = 0.243$

$\therefore P(x) = 0.729 + 0.243 = 0.972$

(Ans)

Q-4: alleast 1 defective.

→ $n = 3$

$x = (1-3)$

$p = 0.1$

$(1-p) = 0.9$

$\therefore (x=1) = {}^{3}C_{1} \cdot (0.1)^{1} \cdot (0.9)^{3-1} = 0.243$

$(x=2) = {}^{3}C_{2} \cdot (0.1)^{2} \cdot (0.9)^{3-2} = 0.027$

$(x=3) = {}^{3}C_{3} \cdot (0.1)^{3} \cdot (0.9)^{3-3} = 0.001$

$\therefore P(x) = 0.243 + 0.027 + 0.001$

$= 0.271$ (Ans)

Ques-5: 2.8 : 2 airplane. 1 has 4 another has 2 engine. attest 50% operate হলে airplane will fly. For what value of P=? for 4 engine is preferable.

→ $n = 4$

$i = 2, 3, 4$ (তিন থাকলে চলবে)

$p = ?$

$P(x=2) = {}^{4}C_{2} \cdot p^{2} \cdot (1-p)^{2} = 6p^{2} \cdot (1-p)^{2}$

$P(x=3) = {}^{4}C_{3} \cdot p^{3} \cdot (1-p)^{1} = 4p^{3} \cdot (1-p)$

$P(x=4) = {}^{4}C_{4} \cdot p^{4} \cdot (1-p)^{0} = p^{4}$

$\therefore \sum P_{4} = 6p^{2}(1-p)^{2} + 4p^{3}(1-p) + p^{4}$

for, $n = 2$

$i = 2, 1$ (since 50% operates)

$\therefore P(x=2) = {}^2C_2 \cdot P^2 (1-P)^{2-2} = P^2$

$P(x=1) = {}^2C_1 \cdot P^1 \cdot (1-P)^{2-1} = 2P(1-P)$

Hence,

$6p^2 (1-p)^2 + 4p^3 (1-p) + p^4 \geq 2p(1-p) + p^2$

$\cancel{p^2 (6p (1-p)^2 + 4p}$

$\Rightarrow p(6p(1-p)^2 + 4p^2(1-p) + p^3) \geq p(2(1-p) + p)$

$\Rightarrow 6p(1-p)^2 + 4p^2(1-p) + p^3 \geq 2 - 2p + p$

$\Rightarrow 6p(1-p)^2 + 4p^2(1-p) + p^3 \geq 2 - p$

$\Rightarrow 6p(1 - 2p + p^2) + 4p^2 - 4p^3 + p^3 \geq 2 - p$

$\Rightarrow 6p - 12p^2 + 6p^3 + 4p^2 - 4p^3 + p^3 + p - 2 \geq 0$

$\Rightarrow 3p^3 - 8p^2 + 7p - 2 \geq 0$

$\Rightarrow 3p^3 - 3p^2 - 5p^2 + 5p + 2p - 2 \geq 0$

$\Rightarrow 3p^2(p-1) - 5p(p-1) + 2(p-1) \geq 0$

$\Rightarrow (p-1)(3p^2 - 5p + 2) \geq 0$

$\Rightarrow (p-1)(3p^2 - 2p - 3p + 2) \geq 0$

$\Rightarrow (p-1)(p-1)(3p-2) \geq 0$

since, $p \geq 1$ not possible. $p \geq \frac{2}{3}$ (Ans)

$\boxplus$ Poisson Random : (n specified na)

$$x = 0, 1, 2, \dots \alpha$$

$\boxplus$ Probability mass function :

$$P_{mf}(i) = P(x=i) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

$$\sum_{i=0}^{\alpha} P_{mf}(i) = \sum_{i=0}^{\alpha} e^{-\lambda} \cdot \frac{\lambda^i}{i!}$$

$$= e^{-\lambda} \sum_{i=0}^{\alpha} \frac{\lambda^i}{i!}$$

$$= e^{-\lambda} \left(\lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \dots \right)$$

$$= e^{-\lambda} \cdot e^{\lambda}$$

$$= 1$$

Tipographical error $\lambda = 1$

~~attest~~ (atleast 1 error)

$i = (1, 2, 3 \dots \alpha)$, $\lambda =$ কত (x)

$$P(x \geq 1) = 1 - P(x=0)$$

$$= 1 - e^{-1} \cdot \frac{1^0}{1!}$$

$$= 0.63$$

Q-1: num of acc atmost 3 হওয়ার P = ?

$\rightarrow \lambda = 1$ (parameter)

$i = (1, 2, 3)$.

atmost

$\therefore P(x=0) = e^{-1} = 0.367$

$P(x=1) = e^{-1} \cdot \frac{1^1}{1!} = 0.367$

$P(x=2) = e^{-1} \cdot \frac{1^2}{2!} = 0.1839$

$P(x=3) = e^{-1} \frac{1^3}{3!} = 0.0613$

$\therefore \sum P = 0.367 + 0.367 + 0.1839 + 0.0613$

$= 0.9792$

Q 2.11 : $\lambda = 3$

$i = (0, 1, 2 \ldots \alpha)$

$x = 0$

$\therefore P(x=0) = e^{-\lambda} \cdot \frac{\lambda^i}{i!}$

$= e^{-3} \cdot \frac{3^0}{0!}$

$= 0.049$ (Ans)

$\boxed{\square}$ Continous random variable :

$x \in \{-\alpha, \alpha\}$

$\int_{-\alpha}^{\alpha} f(x) = 1$

$\therefore P\{x \in (-\alpha, \alpha)\} = \int_{-\alpha}^{\alpha} f(x) = 1$

Uniform random $\rightarrow$ $f(x) = 1$ if $0 < x < 1$

$f(x) = 0$ if otherwise

Normal random variable :

$$P_{df}(x) = \frac{1}{\sqrt{2\pi} \cdot \sigma} e^{-(x-\mu)^2/2\sigma^2}$$

$\boxed{\square}$ Expectation of RV :

$E[x] = \sum x \cdot P(x)$

2.15 : $E(x) = ?$, X is a dice .

$\rightarrow x = \cancel{\{1-6\}} \; \{1-6\}$

$\therefore E[x] = 1 \times \frac{1}{6} + 2 \times \frac{1}{2} + 3 \times \frac{1}{6} + 4 \times \frac{1}{6} + 5 \times \frac{1}{6} + 6 \times \frac{1}{6}$

$= 3.5$

Continous Random variable :

$x \in \{-\alpha, \alpha\}$ [Integration]

$$\int_{-\alpha}^{\alpha} f(x) = 1$$

$$\therefore P\{x \in (-\alpha, \alpha)\} = \int_{-\alpha}^{\alpha} f(x) = 1$$

Types : ① Uniform random variable :

$$f(x) = \begin{cases} 1, & 0 < x < 1 \\ 0, & \text{otherwise} \end{cases}$$

→ 0 - 1 এর মধ্যে হলে $f(x) = 1$

Pdf (prob density function)

→ otherwise -0

☐ Sample space এর probability :

$$\int_0^1 1\,dx$$

$$[x]_0^1$$

$$= 1 - 0 = 1$$

Note:

② exponential random variable :

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0 \\ 0, & \text{if } x < 0 \end{cases}$$

$$\int_0^1 \lambda e^{-\lambda x} dx = \lambda . \int_0^1 e^{-\lambda x} dx$$

SKIP

Day:

Date :

$$? = \lambda \cdot \left[\frac{e^{-\lambda x}}{-\lambda}\right]_0^{\alpha}$$

$$= \left[-e^{-\lambda x}\right]_0^{\alpha} = -e^{-\lambda\alpha} + e^0$$

$$= -\frac{1}{e^{\alpha}} + e^0$$

$$= -0 + 1$$

$$= 1$$

$$?=$$

$$\int_0^{\alpha} \lambda e^{\lambda x} dx$$

$$= \lambda \left[\frac{e^{-\lambda x}}{-\lambda}\right]_0^{\alpha}$$

$$= \left[-e^{-\lambda x}\right]_0^{\alpha}$$

$$= -e^{\alpha} + e^0$$

$$= -\frac{1}{e^{\alpha}} + e^0$$

?

$$= -\frac{1}{e^{\alpha}} + e^0 = -0 + 1 = 1$$

$$E[x] = \sum x \cdot P(x)$$

Note:

SKIP

(F-8)

Normal Random variable: (Continous)

range $x = \{-\alpha, \alpha\}$

$$\text{Pdf}(x) = \frac{1}{\sqrt{2\pi}\cdot\sigma} e^{-\left((x-\mu)^2/2\sigma^2\right)} \; ; \; x = \{-\alpha, \alpha\}$$

Expectation of a random variable: (ERV) [Discrete]

$$E[x] = \sum_{x : P(x) > 0} x \cdot P(x) \rightarrow \text{probability of } x$$

$P(1) = \frac{1}{2} = P(2)$ ~~P(2)~~

Q: একটি ছক্কা x = random variable. ERV OR $E[x] = ?$

$x = \{1, 2, 3, 4, 5, 6\}$

$\rightarrow 1 \times \frac{1}{6} + 2 \times \frac{1}{6} + 3 \times \frac{1}{6} + 4 \times \frac{1}{6} + 5 \times \frac{1}{6} + 6 \times \frac{1}{6}$

$= \cancel{\frac{21}{6}} \frac{7}{2} = \cancel{15.167}\ 3.5 = E[x]$

Q: 2 dice. x = sum of 2 dice (12). $E[x] = ?$

$\rightarrow x = \{2 - 12\}$

$P(x=2) = \frac{1}{36}$

$P(x=3) = \{(1,2)\ (2,1)\} = \frac{2}{36}$

$P(x=4) = \frac{3}{36}$

Note: $P(x=5) = \frac{4}{36}$

$P(x=6) = \frac{5}{36}$

$P(x=7) = \frac{6}{36}$

$P(x=8) = \frac{5}{36}$

$P(x=9) = \frac{4}{36}$

$P(x=10) = \frac{3}{36}$

$P(x=11) = \frac{2}{36}$

$P(x=12) = \frac{1}{36}$

Day: s s m t w t f

Date :

$$= 2\times\frac{1}{36} + 3\times\frac{2}{36} + 4\times\frac{3}{36} + 5\times\frac{4}{36} + 6\times\frac{5}{36} + 7\times\frac{6}{36}$$
$$+ 8\times\frac{5}{36} + 9\times\frac{4}{36} + 10\times\frac{3}{36} + 11\times\frac{2}{36} + 12\times\frac{1}{36}$$

$$= 7$$

Ques: 3 coin toss, x random variables denotes num of tail

$S = \{hhh, hht, hth, htt, thh, tht, tth, ttt\}$, $E[x] = ?$

$x = \{0, 1, 2, 3\}$

$P(x=0) = \frac{1}{8}$ $\quad$ $P(x=2) = \frac{3}{8}$

$P(x=1) = \frac{3}{8}$ $\quad$ $P(x=3) = \frac{1}{8}$

$$E[x] = 0\times\frac{1}{8} + 1\times\frac{3}{8} + 2\times\frac{3}{8} + 3\times\frac{1}{8}$$
$$= 0 + \frac{3}{8} + \frac{6}{8} + \frac{3}{8} = 1.5$$

(Ans)

~~Bernoli~~ Bernouli : [failure or success]

$P(0) = 1-P \rightarrow$ ~~success~~ fail

$P(1) = P \rightarrow$ success

Note:

$$\therefore E[x] = 0\times(1-P) + (1\times P)$$
$$= P$$

Ques: Bernouli random var এর $E[x] = ?$ or $E[x] = P$ prove কর।

Binomial Random : $P_{mp}(i) = \binom{n}{i} p^i (1-p)^{n-i}, \quad i = 0, 1, 2, \ldots n$

$$E(x) = \sum_{i=0}^{n} i \cdot P_{mp}(i)$$

$$= \sum_{i=0}^{n} i \cdot \binom{n}{i} \cdot p^i \cdot (1-p)^{n-i} \qquad [i=0 \text{ ignore করে}]$$

$$= \sum_{i=1}^{n} i \cdot \frac{n!}{i!\,(n-i)!} \times p^i \times (1-p)^{n-i}$$

$$= \sum_{i=1}^{n} \cancel{i} \cdot \frac{n!}{\cancel{i}\,(i-1)!\,(n-i)!} \times p^i \times (1-p)^{n-i}$$

$$= \sum_{i=1}^{n} \frac{n\,(n-1)!}{(i-1)!\,(n-i)!} \cdot p^1 \cdot p^i \cdot (1-p)^{n-i}$$

$$= \sum_{i=1}^{n} \frac{(n-1)!}{(i-1)!\,(n-i)!} \cdot p^{i-1} \cdot (1-p)^{n-i}$$

$$= np \sum_{k=0}^{n-1} \frac{(n-1)!}{k!\,(n-k-1)!} p^k (1-p)^{n-k-1}$$

Let,
$i - 1 = k$
$k = n - 1$

$$= np \boxed{\sum_{k=0}^{n-1} \binom{n-1}{k} p^k (1-p)^{n-k-1}}$$

$$= np\,(p + (1-p))^{n-1}$$

$$= np$$

Side note: $\boxed{(x+y)^n = \binom{n}{0} x^n y^0 + \binom{n}{1} x^{n-1} y^1 + \ldots + \binom{n}{n} x^0 y^n}$

Note:

☐ Poisson Random variable Expectation:

$$P_{mf}(i) = P(x=i) = e^{-\lambda}\frac{\lambda^i}{i!} \; ; \; i=0,1,2 \ldots \alpha$$

$$E(x) = \sum_{i=0}^{\alpha} i \cdot e^{-\lambda}\frac{\lambda^i}{i!}$$

$$= \sum_{i=1}^{\alpha} i \cdot e^{-\lambda}\frac{\lambda^i}{\cancel{i}(i-1)!}$$

$$= e^{-\lambda}\sum_{i=1}^{\alpha}\frac{\lambda \cdot \lambda^{i-1}}{(i-1)!}$$

$$= \lambda e^{-\lambda}\sum_{i=1}^{\alpha}\frac{\lambda^{i-1}}{(i-1)!}$$

$$= \lambda e^{-\lambda}\left(1+\lambda+\frac{\lambda^2}{2!}+\frac{\lambda^3}{3!}+\ldots\right)$$

$$= \lambda e^{-\lambda} \cdot e^{\lambda} = \lambda$$

Chapter - 04

Markov chains → Stochastic process (Ch-2, 2.9) (collection of random variable)

☐ What is stochastic proce?

Note: $X(t)$, $t \in T$

t = time

n-0 time period $= i$

$n+1 = j$

probability of predicting future state from present and past.

☐ $P(x_{n+1} = j \mid X_n = i, X_{n-1} = i_{n-1}, \ldots X_1 = i_1; X_0 = i_0) = P_{ij}$

Transition probability matrix : (TP/TPM)

TPM for i to j state =

$$\begin{array}{c|cccc} & 0 & 1 & & j \\ 0 & P_{00} & P_{01} & \cdots & P_{0j} \\ 1 & P_{10} & P_{11} & \cdots & P_{1j} \\ \vdots & \vdots & & & \\ i & P_{i0} & P_{i1} & \cdots & P_{ij} \end{array}$$

→ States

yes no

4.1 আগামীকাল বৃষ্টি হবে কী না তার P। আজ বৃষ্টি হলে কাল হওয়ার $P(\alpha)$ কিন্তু আজ না হলে কাল হওয়ার $P(\beta)$। Write TPM and draw TP diagram.

→ state 0 → when it rains

state 1 → " " doesnt rain

today → tommorow → day

$$TPM = \begin{array}{c|cc} & 0 & 1 \\ 0 & \alpha & (1-\alpha) \\ 1 & \beta & (1-\beta) \end{array}$$

do/dont

row = হবে / হবে না

colum = today

$\alpha = 0.7$

$\beta = 0.4$

Retension = same state এ থাকা

$$= \begin{bmatrix} 0.7 & 0.3 \\ 0.4 & 0.6 \end{bmatrix}$$

Retension and gain*

Retension and loss*

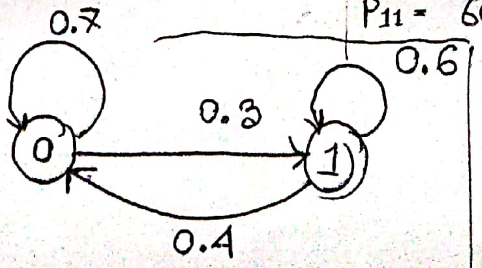
$P_{00} = 0.7 = 70\% =$ Retesion of state 0

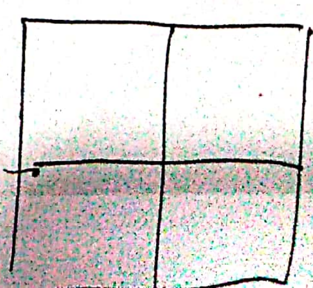
$P_{01} = 30\% =$ loss of s0, gain of s1

$P_{10} = 40\% =$ loss of s1, gain of s0

$P_{11} = 60\% =$ Retension of s1

Note: diagram :





→ table

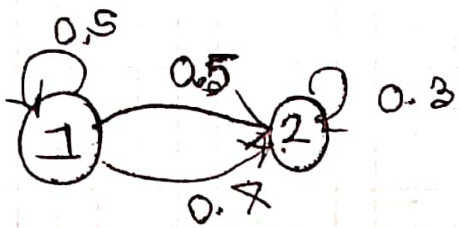
Day: s s m t w t f   Date :

Ch-4 (Markov chain) → 3 type থাকতে পারে
- intro, def
- 4.1, 4.2, 4.3

$\boxed{\text{রবিবার}}$ - TPM and initial prob দেওয়া থাকে

☐ TPM diagram

$$\begin{array}{c|cc} & 1 & 2 \\ \hline 1 & 0.5 & 0.5 \\ 2 & 0.7 & 0.3 \end{array}$$



BI/303

$\rightarrow P(x_n = a) = q_n(a)$, $\left(\therefore q_n = q_0 P^n \right)$

↓
TPM
Right থেকে count

$\boxed{\begin{array}{l} n = \text{time} \\ a = \text{state} \end{array}}$

OS