Elementary Signals

Sinusoidal & Exponential Signals

- Sinusoids and exponentials are important in signal and system analysis because they arise naturally in the solutions of the differential equations.
- Sinusoidal Signals can expressed in either of two ways :

cyclic frequency *form*- $A \sin 2\Pi f_0 t = A \sin(2\Pi/T_0)t$

radian frequency form- $A \sin \omega_0 t$

$$\omega_0 = 2\Pi f_0 = 2\Pi/T_0$$

T_0 = Time Period of the Sinusoidal Wave

Sinusoidal & Exponential Signals Contd.

$$\left.\begin{aligned} x(t) &= A \sin (2\Pi f_0 t + \theta) \\ &= A \sin (\omega_0 t + \theta) \end{aligned}\right\} \text{Sinusoidal signal}$$

$x(t) = Ae^{at}$ Real Exponential

$= Ae^{j\omega_0 t} = A[\cos (\omega_0 t) + j \sin (\omega_0 t)]$ Complex Exponential

θ = Phase of sinusoidal wave

A = amplitude of a sinusoidal or exponential signal

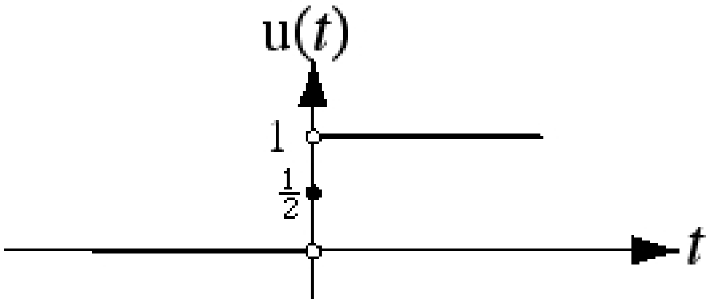
f_0 = fundamental cyclic frequency of sinusoidal signal

ω_0 = radian frequency

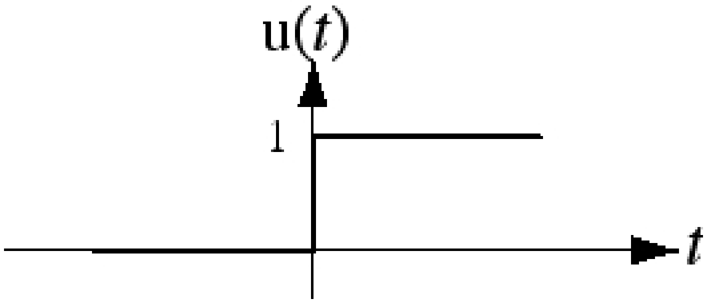
Unit Step Function

$$\mathrm{u}(t) = \begin{cases} 1 & , \quad t > 0 \\ 1/2 & , \quad t = 0 \\ 0 & , \quad t < 0 \end{cases}$$

Precise Graph



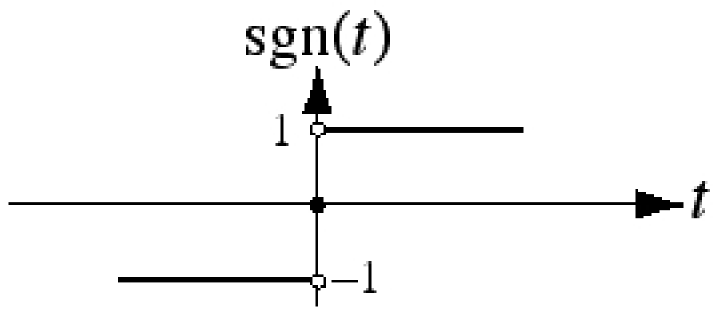
Commonly-Used Graph



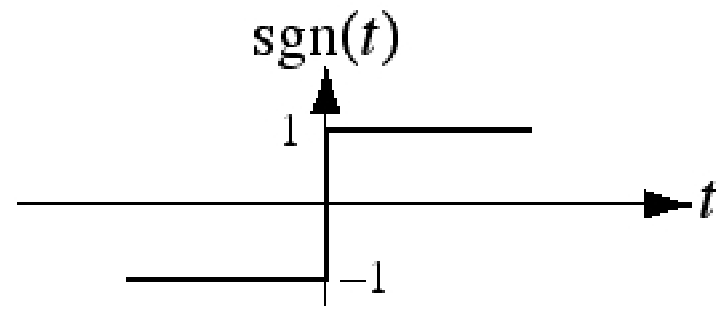
Signum Function

$$\operatorname{sgn}(t) = \begin{cases} 1 & , \ t > 0 \\ 0 & , \ t = 0 \\ -1 & , \ t < 0 \end{cases} = 2\mathrm{u}(t) - 1$$

Precise Graph

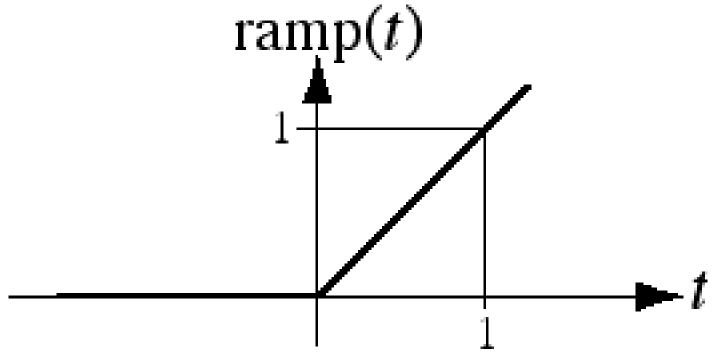


Commonly-Used Graph



The signum function, is closely related to the unit-step function.

Unit Ramp Function

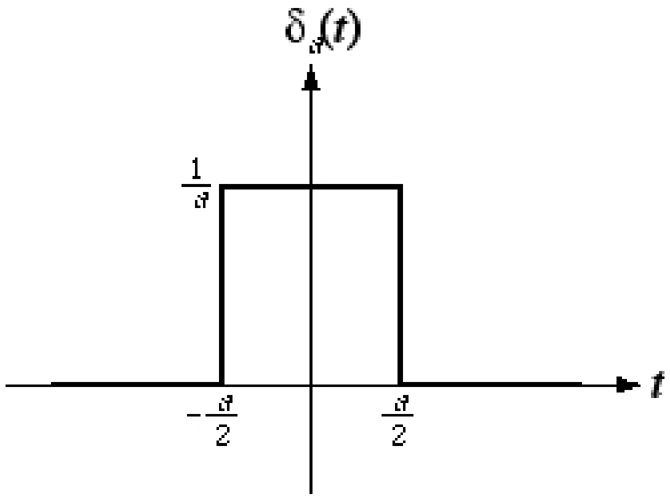


$$\text{ramp}(t) = \begin{Bmatrix} t & , & t > 0 \\ 0 & , & t \le 0 \end{Bmatrix} = \int_{-\infty}^{t} \text{u}(\lambda)\, d\lambda = t\,\text{u}(t)$$

- The unit ramp function is the integral of the unit step function.
- It is called the unit ramp function because for positive t, its slope is one amplitude unit per time.

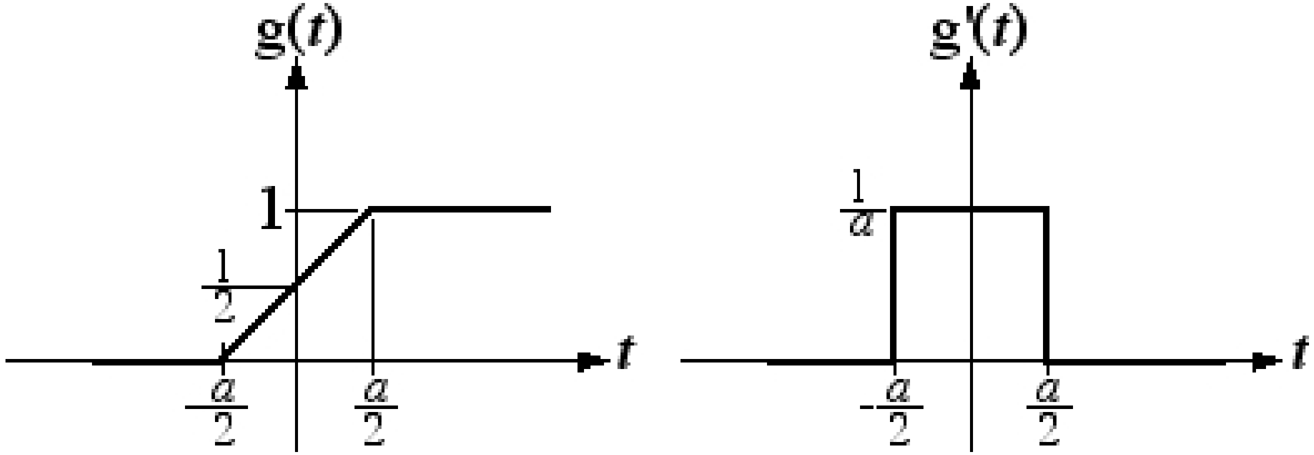
Rectangular Pulse or Gate Function

Rectangular pulse, $$\delta_a(t) = \begin{cases} 1/a & , \ |t| < a/2 \\ 0 & , \ |t| > a/2 \end{cases}$$



Unit Impulse Function

As a approaches zero, $\mathrm{g}(t)$ approaches a unit step and $\mathrm{g}'(t)$ approaches a unit impulse

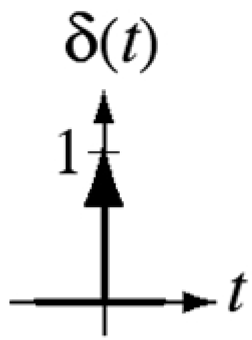


Functions that approach unit step and unit impulse

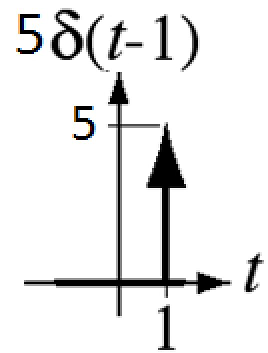
So unit impulse function is the **derivative** of the unit step function or unit step is the integral of the unit impulse function

Representation of Impulse Function

The area under an impulse is called its strength or weight. It is represented graphically by a vertical arrow. An impulse with a strength of one is called a unit impulse.



Representation of Unit Impulse



Shifted Impulse of Amplitude 5

Properties of the Impulse Function

The Sampling Property

$$\int_{-\infty}^{\infty} g(t)\,\delta(t-t_0)\,dt = g(t_0)$$

The Scaling Property

$$\delta\left(a\left(t-t_0\right)\right) = \frac{1}{|a|}\delta\left(t-t_0\right)$$

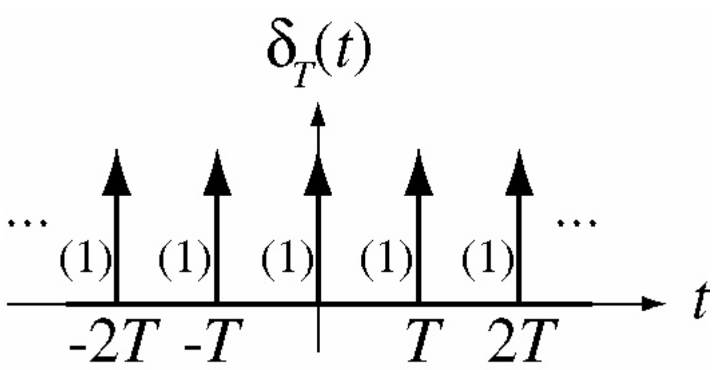
The Replication Property

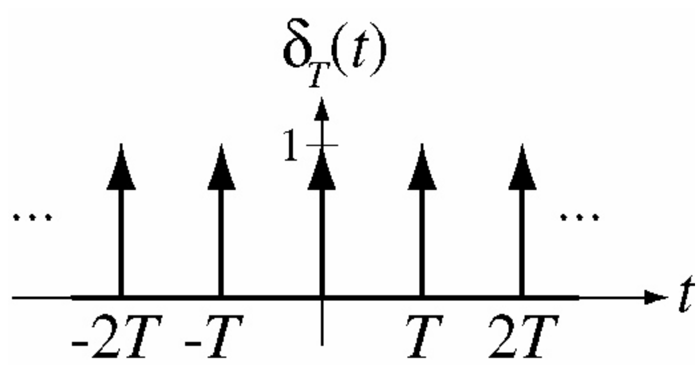
$$g(t) \otimes \delta(t) = g(t)$$

Unit Impulse Train

The unit impulse train is a sum of infinitely uniformly-spaced impulses and is given by

$$\delta_T(t) = \sum_{n=-\infty}^{\infty} \delta(t - nT) \quad , \quad n \text{ an integer}$$

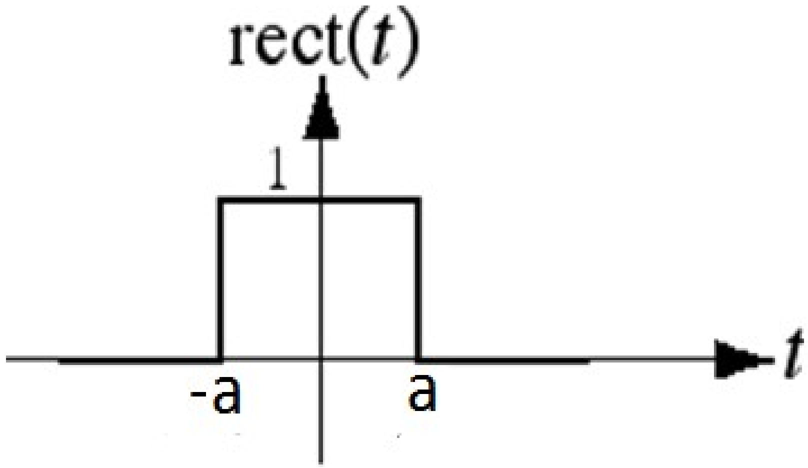




The Unit Rectangle Function

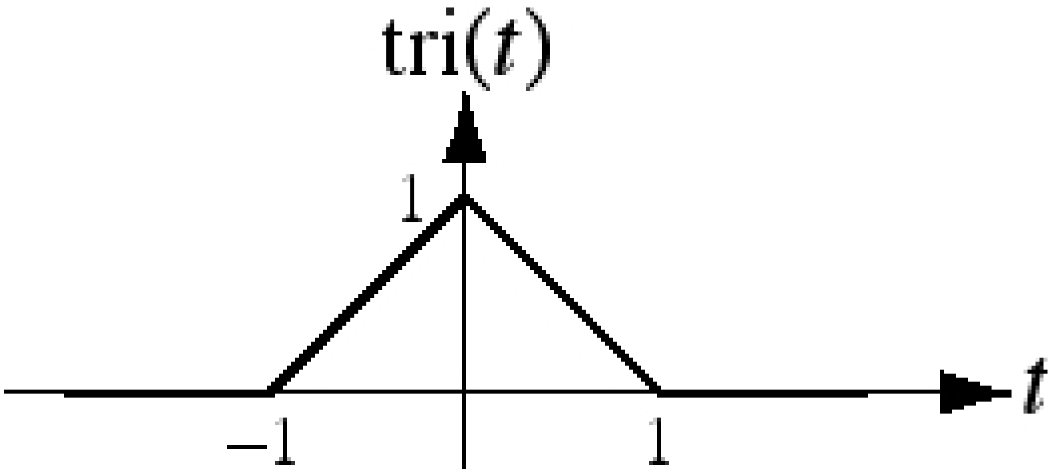
The unit rectangle or gate signal can be represented as combination of two shifted unit step signals as shown

$$\text{rect}(t) = u(t+a) - u(t-a)$$



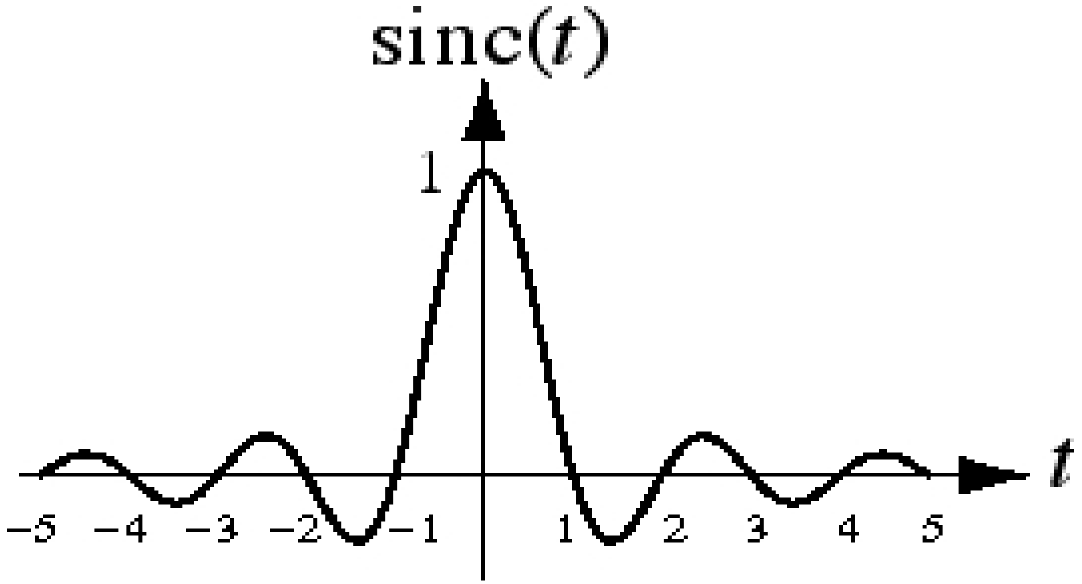
The Unit Triangle Function

A triangular pulse whose height and area are both one but its base width is not, is called unit triangle function. The unit triangle is related to the unit rectangle through an operation called **convolution**.



Sinc Function

$$\operatorname{sinc}(t) = \frac{\sin(\pi t)}{\pi t}$$



Discrete-Time Signals

- **Sampling** is the acquisition of the values of a continuous-time signal at discrete points in time
- $\mathrm{x}(t)$ is a continuous-time signal, $\mathrm{x}[n]$ is a discrete-time signal

$\mathrm{x}[n] = \mathrm{x}(nT_s)$ where T_s is the time between samples

Discrete Time Exponential and Sinusoidal Signals

- DT signals can be defined in a manner analogous to their continuous-time counter part

$x[n] = A \sin(2\Pi n/N_0+\theta)$

$= A \sin(2\Pi F_0 n+ \theta)$

Discrete Time Sinusoidal Signal

Discrete Time Exponential Signal

$x[n] = a^n$

n = the discrete time

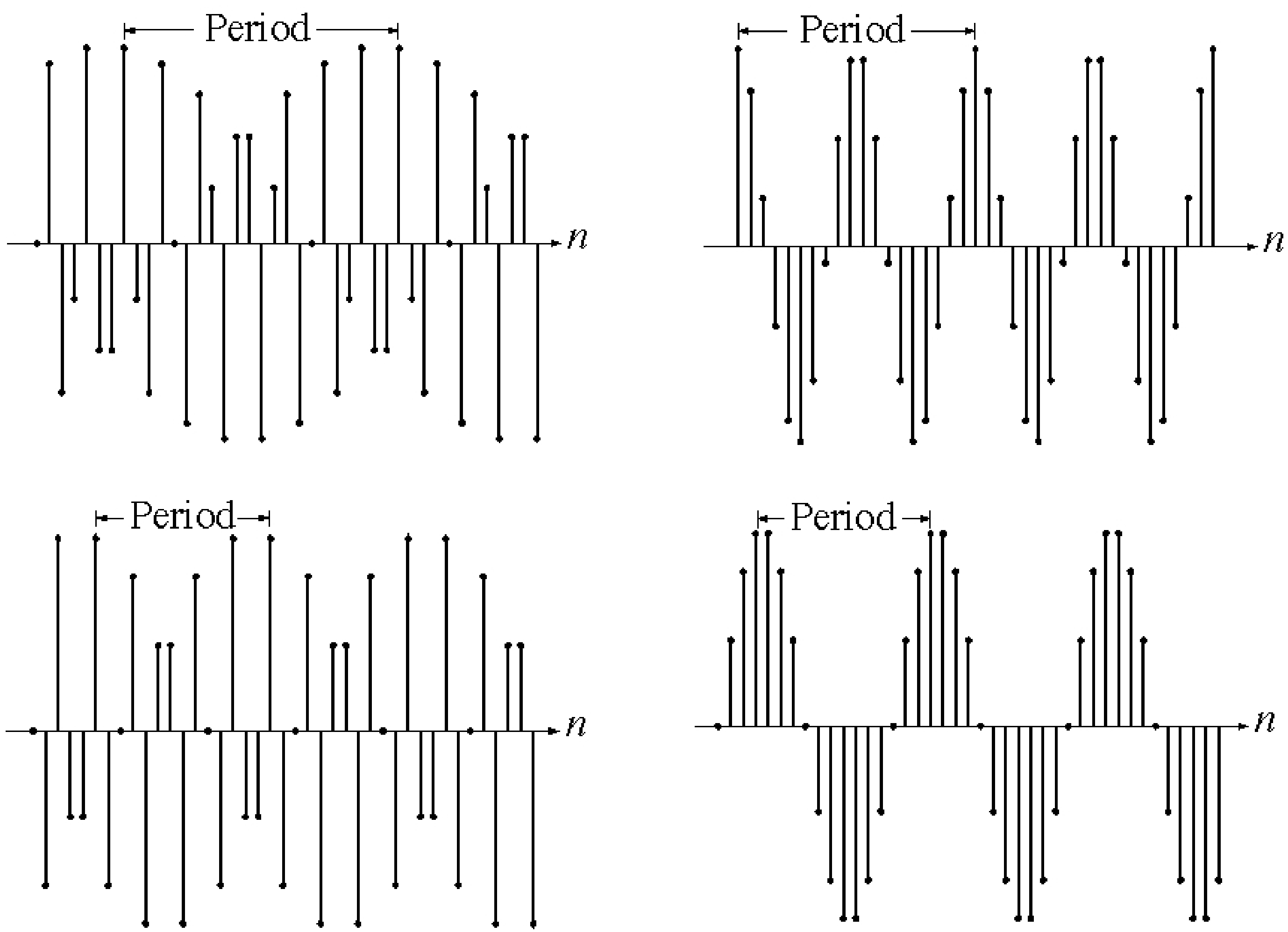
A = amplitude

θ = phase shifting radians,

N_0 = Discrete Period of the wave

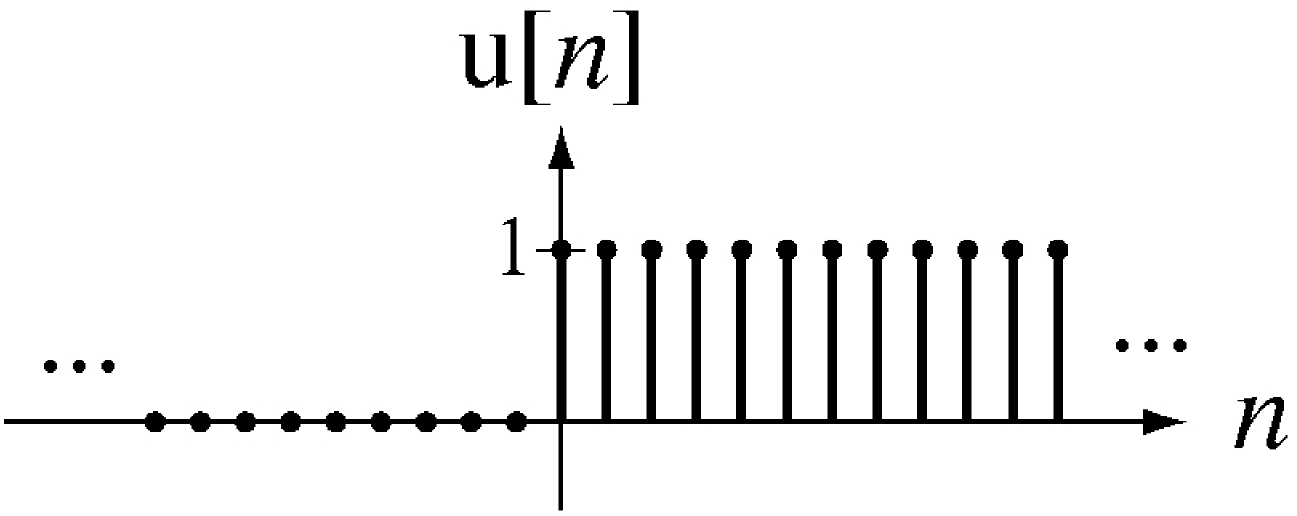
$1/N_0 = F_0 = \Omega_0/2\Pi$ = Discrete Frequency

Discrete Time Sinusoidal Signals



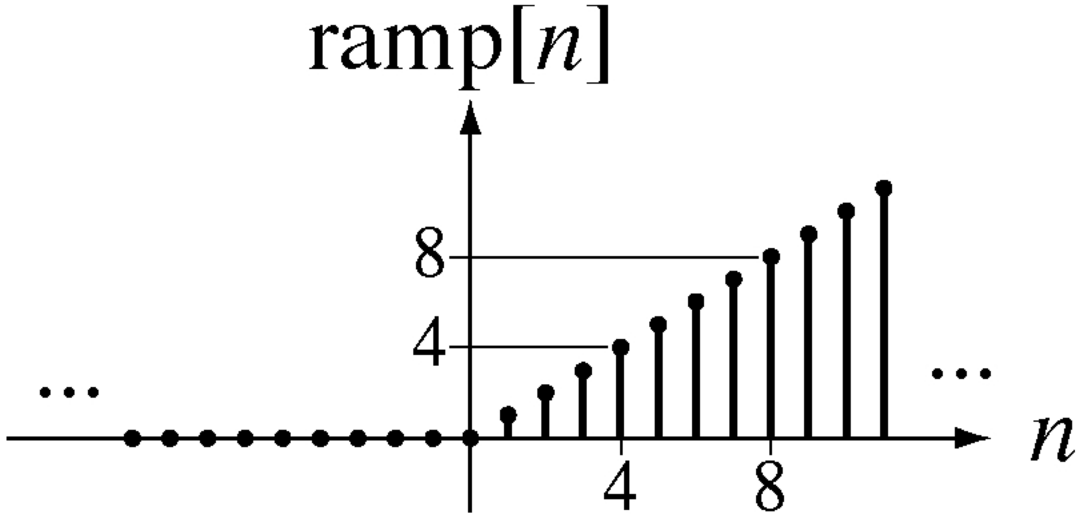
Discrete Time Unit Step Function or Unit Sequence Function

$$\mathrm{u}[n] = \begin{cases} 1 \ , & n \geq 0 \\ 0 \ , & n < 0 \end{cases}$$



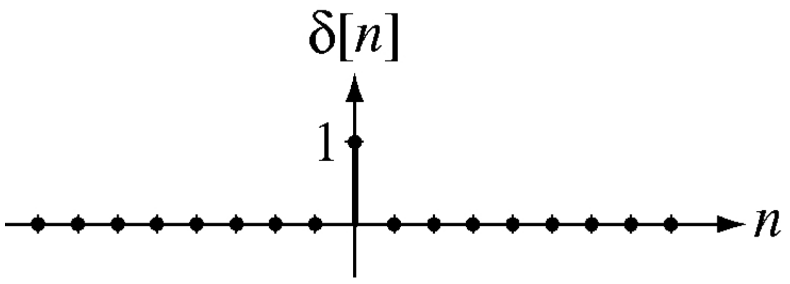
Discrete Time Unit Ramp Function

$$\mathrm{ramp}[n] = \begin{Bmatrix} n & , & n \geq 0 \\ 0 & , & n < 0 \end{Bmatrix} = \sum_{m=-\infty}^{n} \mathrm{u}[m-1]$$



Discrete Time Unit Impulse Function or Unit Pulse Sequence

$$\delta[n] = \begin{cases} 1 \ , & n = 0 \\ 0 \ , & n \neq 0 \end{cases}$$



$\delta[n] = \delta[an]$ for any non-zero, finite integer a.

Unit Pulse Sequence Contd.

- The discrete-time unit impulse is a function in the ordinary sense in contrast with the continuous-time unit impulse.
- It has a sampling property.
- It has no scaling property i.e.

  $\delta[n] = \delta[an]$ for any non-zero finite integer ‘a’

Operations of Signals

- Sometime a given mathematical function may completely describe a signal .
- Different operations are required for different purposes of arbitrary signals.
- The operations on signals can be

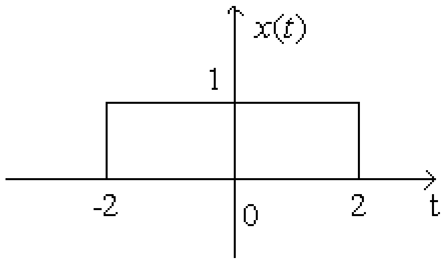
  Time Shifting

  Time Scaling

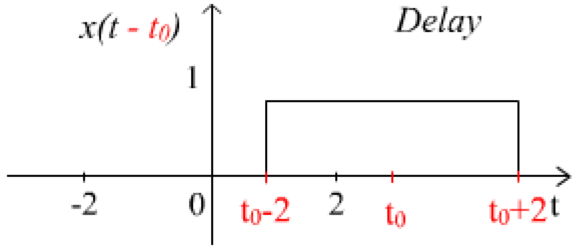
  Time Inversion or Time Folding

Time Shifting

- The original signal $x(t)$ is shifted by an amount t_o.

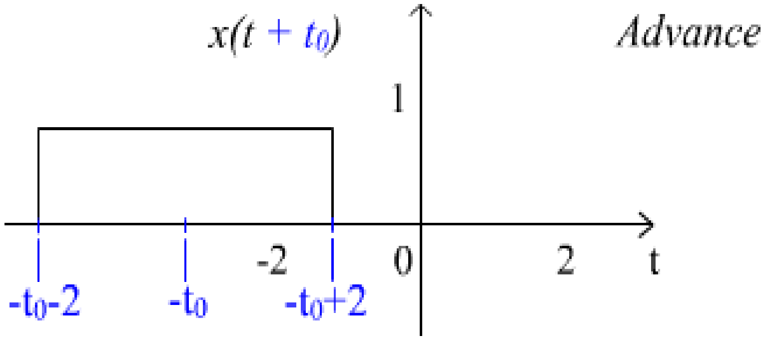


- X(t)→X(t-to) →Signal Delayed→ Shift to the right



Time Shifting Contd.

- $X(t) \rightarrow X(t+to)$ $\rightarrow$ Signal Advanced $\rightarrow$ Shift to the left

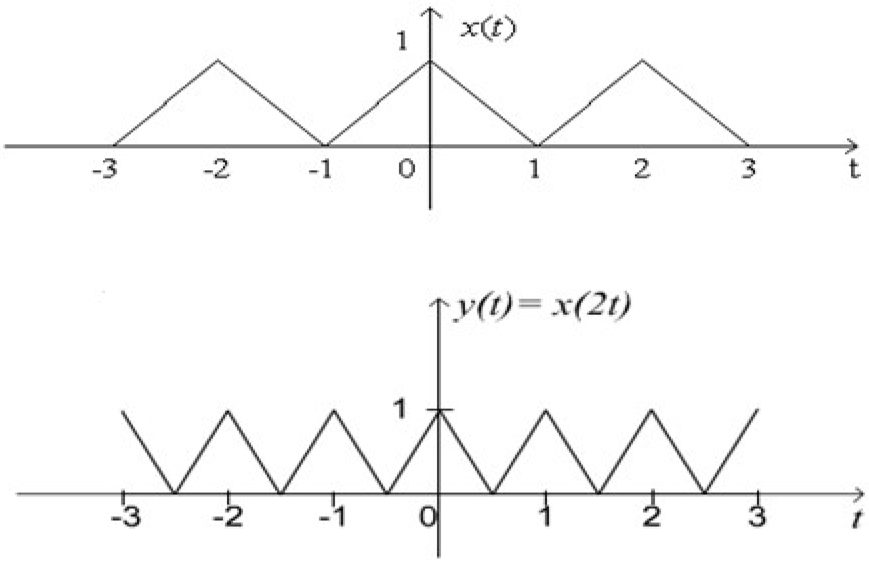


Time Scaling

- For the given function x(t), x(at) is the time scaled version of x(t)
- For a > 1,period of function x(t) reduces and function speeds up. Graph of the function shrinks.
- For a < 1, the period of the x(t) increases and the function slows down. Graph of the function expands.

Time scaling Contd.

Example: Given $x(t)$ and we are to find $y(t) = x(2t)$.

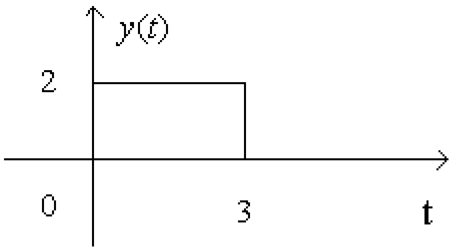


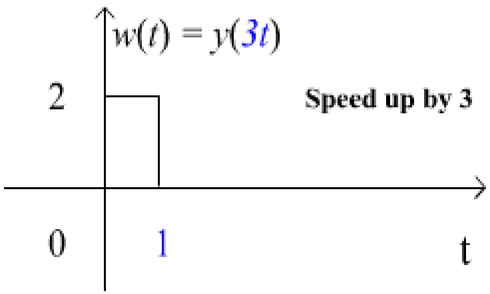
The period of $x(t)$ is 2 and the period of $y(t)$ is 1,

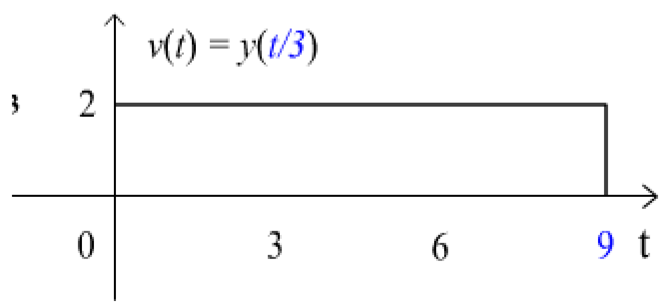
Time scaling Contd.

- Given $y(t)$,
  - find $w(t) = y(3t)$

  $and\ v(t) = y(t/3)$.





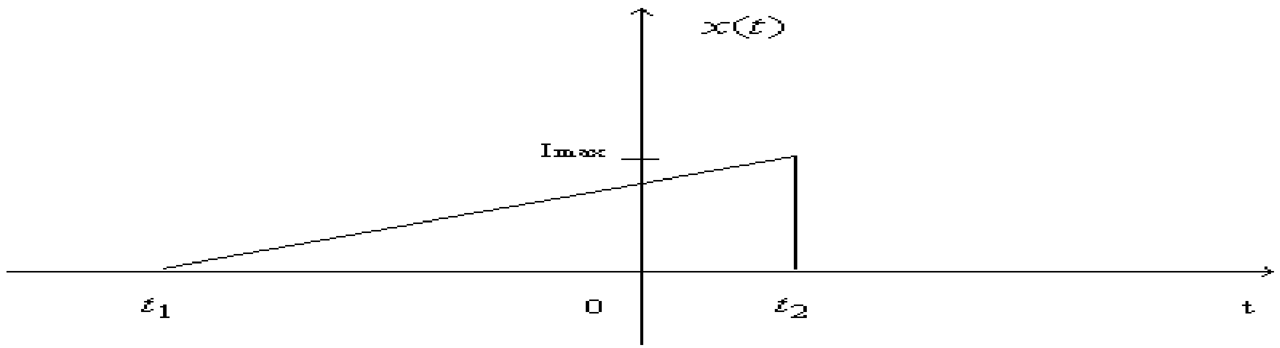


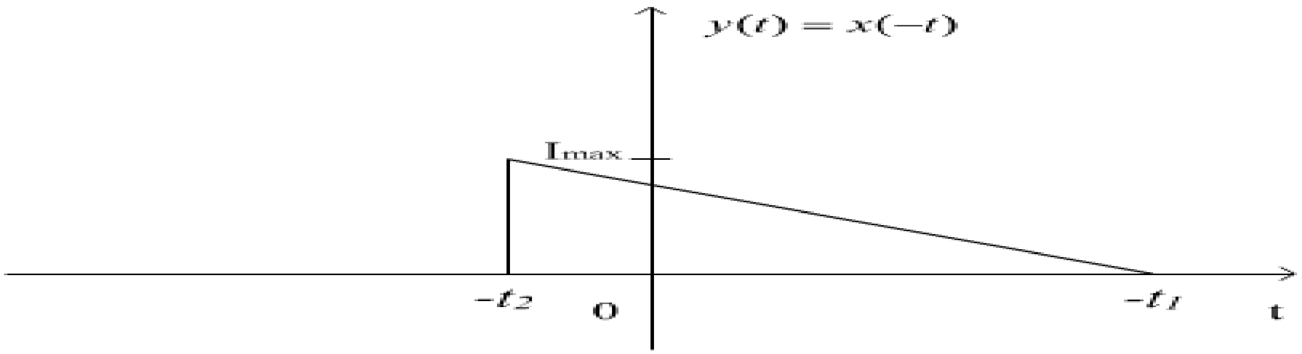
Time Reversal

- Time reversal is also called time folding
- In Time reversal signal is reversed with respect to time i.e.

  $y(t) = x(-t)$ is obtained for the given function

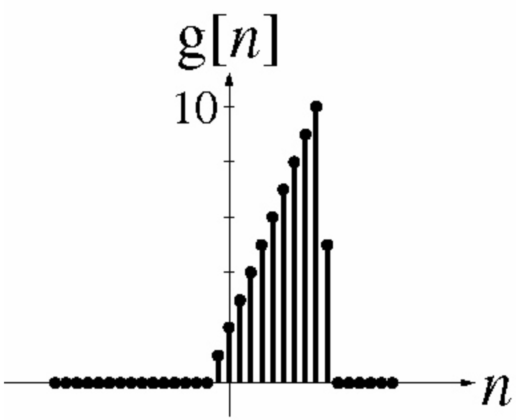
Time reversal Contd.



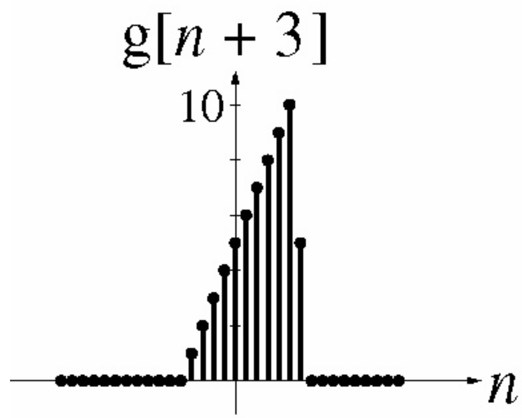


Operations of Discrete Time Functions

Time shifting $n \rightarrow n + n_0$, n_0 an integer



n	$g[n]$
–1	1
0	2
1	3
2	4
3	5
4	6
5	7
6	8
7	9
8	10
9	5
10	0

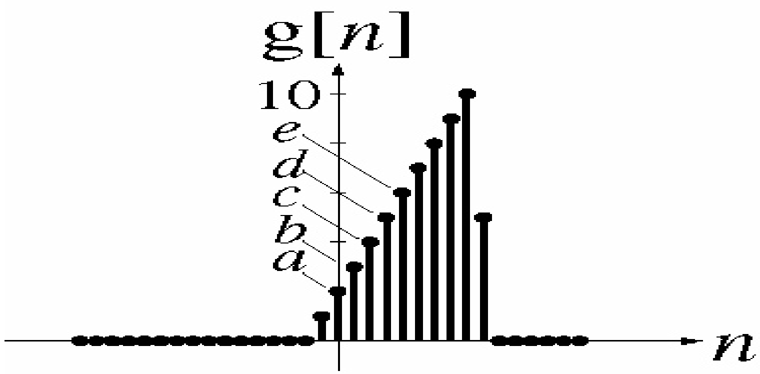


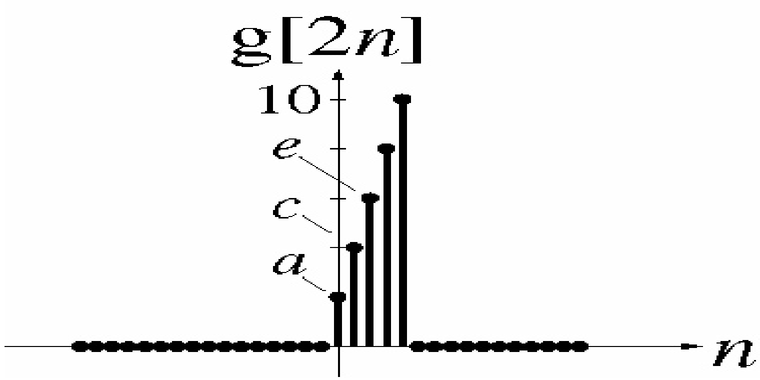
n	$n+3$	$g[n+3]$
–4	–1	1
–3	0	2
–2	1	3
–1	2	4
0	3	5
1	4	6
2	5	7
3	6	8
4	7	9
5	8	10
6	9	5
7	10	0

Operations of Discrete Functions Contd.

Scaling; Signal Compression

$n \to Kn$ $\quad$ K an integer > 1





n	$2n$	$g[2n]$
0	0	2
1	2	4
2	4	6
3	6	8
4	8	10

Classification of Signals

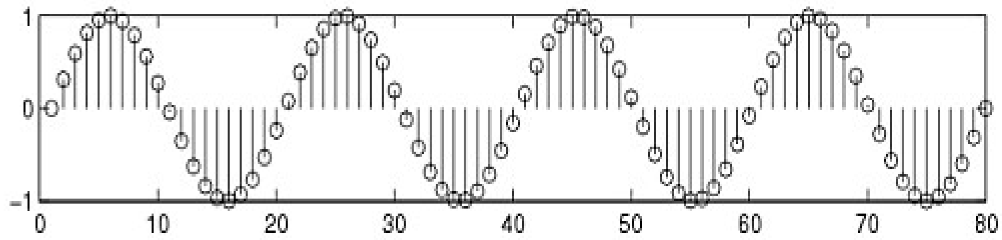
- Deterministic & Non Deterministic Signals
- Periodic & A periodic Signals
- Even & Odd Signals
- Energy & Power Signals

Deterministic & Non Deterministic Signals

Deterministic signals

- Behavior of these signals is predictable w.r.t time
- There is no uncertainty with respect to its value at any time.
- These signals can be expressed mathematically.

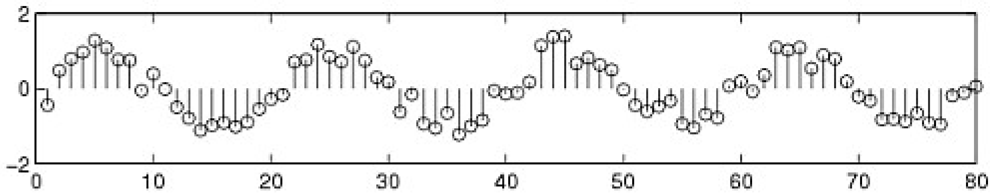
For example $x(t) = \sin(3t)$ is deterministic signal.



Deterministic & Non Deterministic Signals Contd.

Non Deterministic or Random signals

- Behavior of these signals is random i.e. not predictable w.r.t time.
- There is an uncertainty with respect to its value at any time.
- These signals can't be expressed mathematically.
- For example Thermal Noise generated is non deterministic signal.



Periodic and Non-periodic Signals

- Given $x(t)$ is a continuous-time signal
- $x(t)$ is periodic iff $x(t) = x(t+T_o)$ for any T and any integer n
- Example
  - $x(t) = A\cos(\omega t)$
  - $x(t+T_o) = A\cos[\omega(t+T_o)] = A\cos(\omega t+\omega T_o) = A\cos(\omega t+2\pi) = A\cos(\omega t)$
  - Note: $T_o = 1/f_o$; $\omega = 2\pi f_o$

Periodic and Non-periodic Signals Contd.

- For non-periodic signals

$$x(t) \neq x(t+T_o)$$

- A non-periodic signal is assumed to have a period $T = \infty$
- Example of non periodic signal is an exponential signal

Important Condition of Periodicity for Discrete Time Signals

- A discrete time signal is periodic if

$$x(n) = x(n+N)$$

- For satisfying the above condition the frequency of the discrete time signal should be ratio of two integers

$$\text{i.e. } f_o = k/N$$

Sum of periodic Signals

- $X(t) = x1(t) + X2(t)$
- $X(t+T) = x1(t+m_1T_1) + X2(t+m_2T_2)$
- $m_1T_1=m_2T_2 = T_o$ = Fundamental period
- Example: $\cos(t\pi/3)+\sin(t\pi/4)$
  - $T1=(2\pi)/(\pi/3)=6$; $T2 =(2\pi)/(\pi/4)=8$;
  - $T1/T2=6/8 = 3/4$ = (rational number) = $m2/m1$
  - $m_1T_1=m_2T_2 \rightarrow$ Find m1 and m2 $\rightarrow$
  - $6.4 = 3.8 = 24 = T_o$

Sum of periodic Signals – may not always be periodic!

$$x(t) = x_1(t) + x_2(t) = \cos t + \sin \sqrt{2t}$$

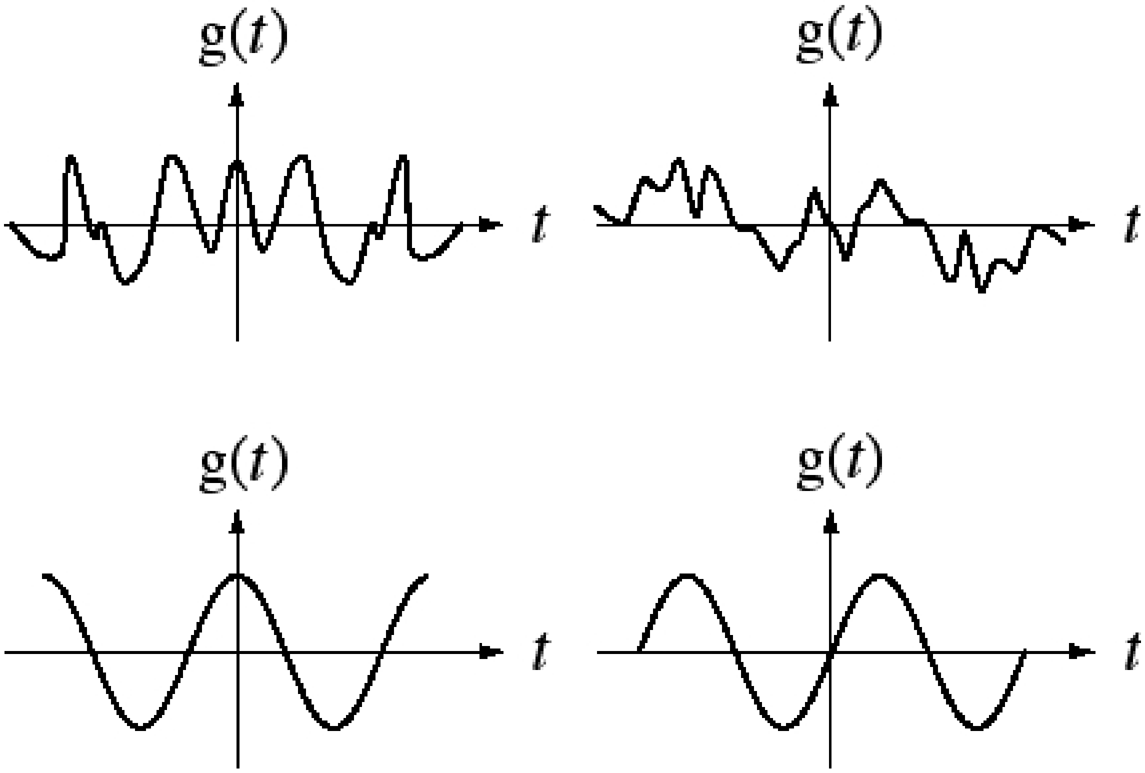
T1=(2π)/(1)= 2π;   T2 =(2π)/(sqrt(2));

T1/T2= sqrt(2);

- Note: T1/T2 = sqrt(2) is an irrational number
- X(t) is aperiodic

Even and Odd Signals

Even Functions   Odd Functions



Even and Odd Parts of Functions

The **even part** of a function is $g_e(t) = \dfrac{g(t) + g(-t)}{2}$

The **odd part** of a function is $g_o(t) = \dfrac{g(t) - g(-t)}{2}$

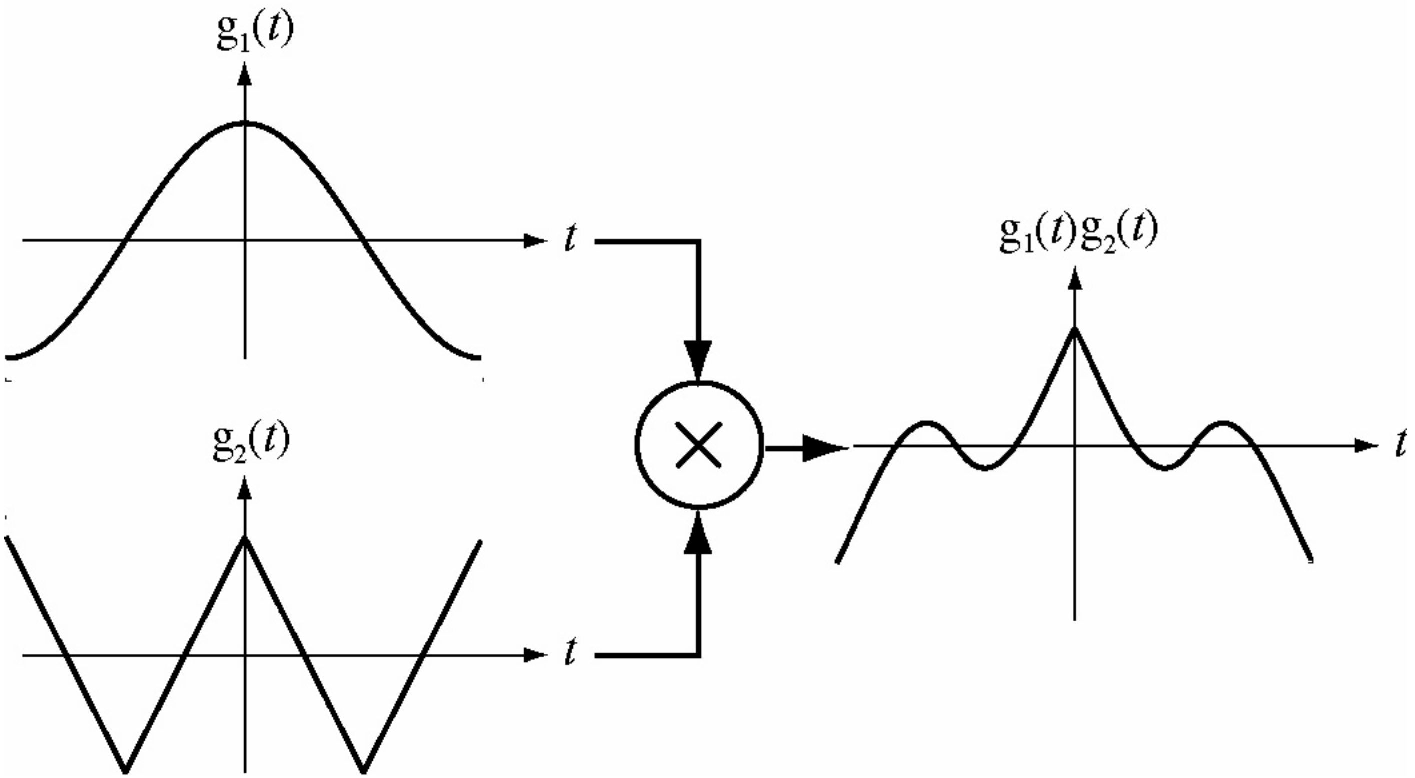
A function whose even part is zero, is odd and a function whose odd part is zero, is even.

Various Combinations of even and odd functions

Function type	Sum	Difference	Product	Quotient
Both even	Even	Even	Even	Even
Both odd	Odd	Odd	Even	Even
Even and odd	Neither	Neither	Odd	Odd

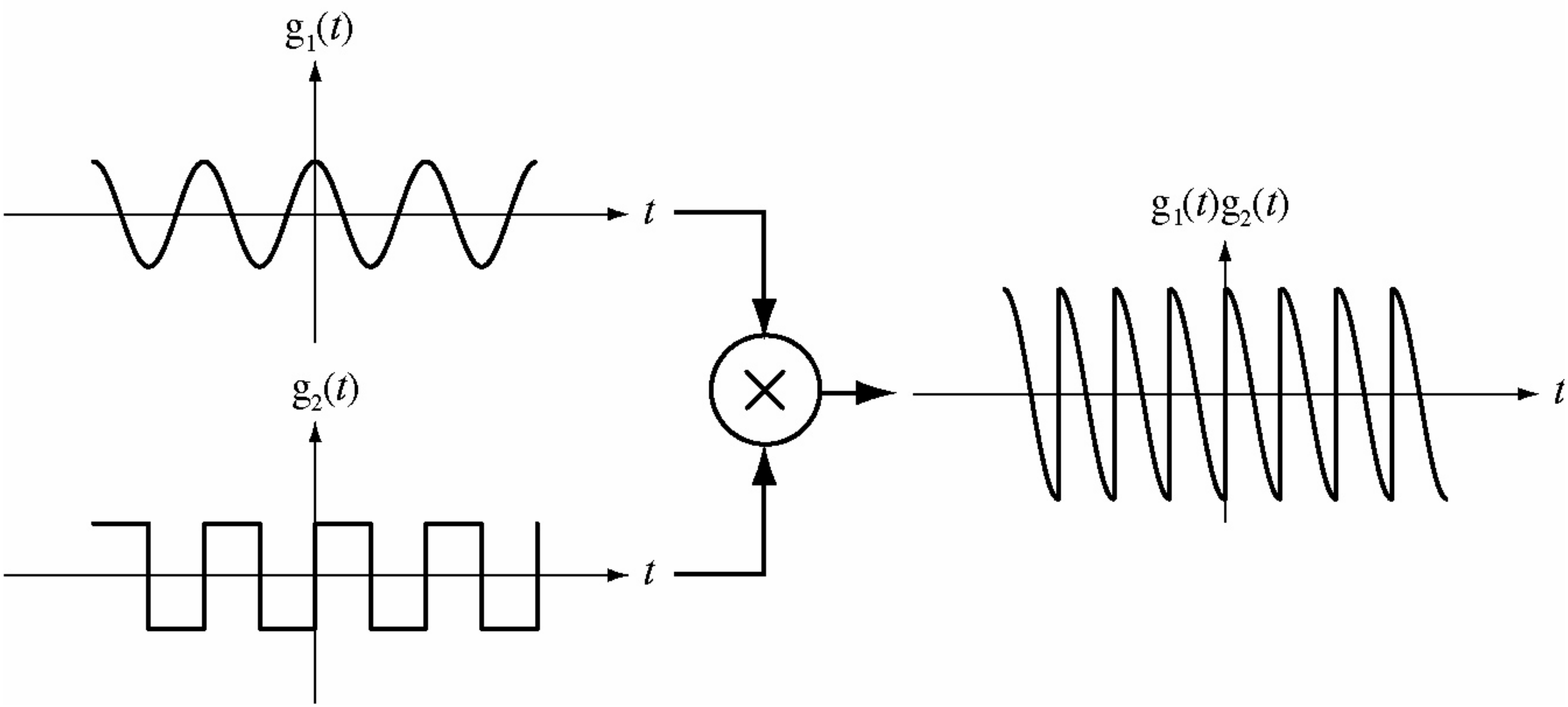
Product of Even and Odd Functions

Product of Two Even Functions



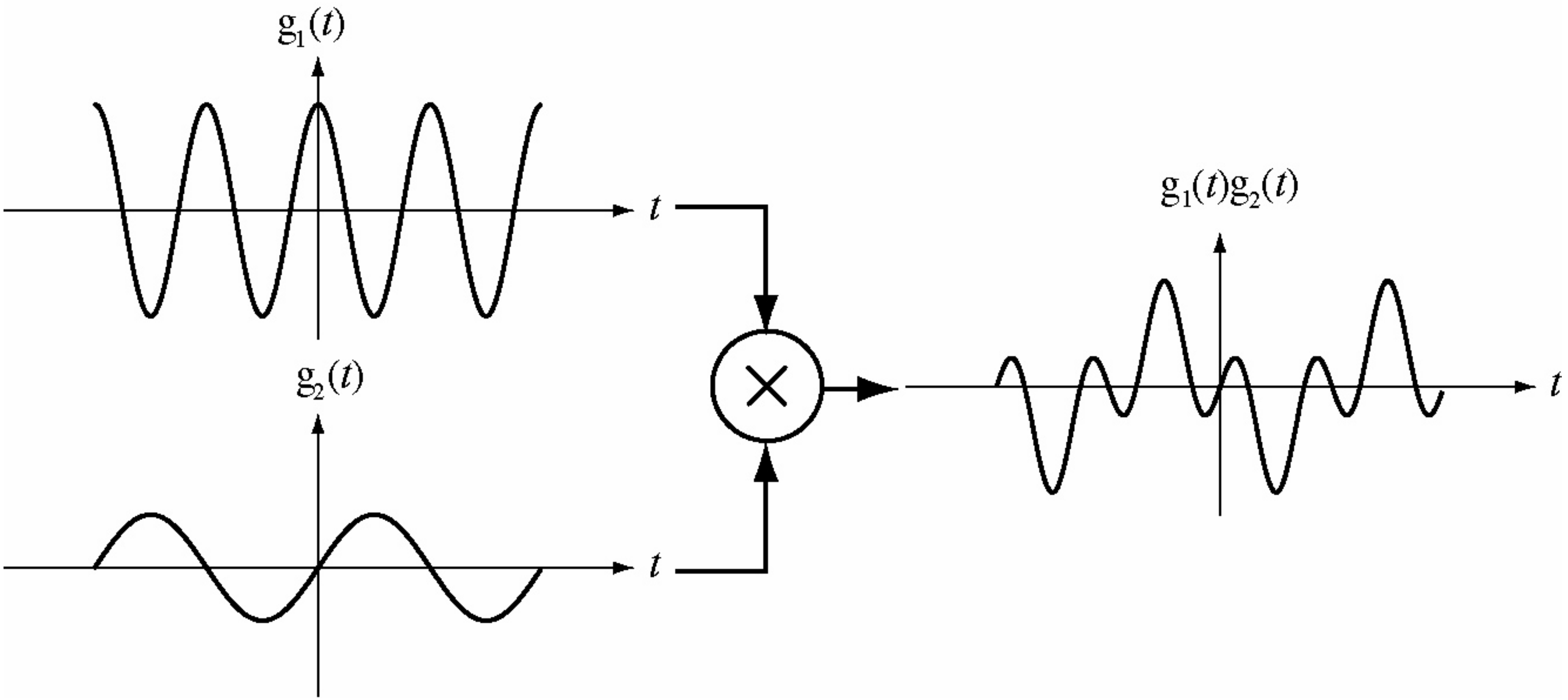
Product of Even and Odd Functions Contd.

Product of an Even Function and an Odd Function



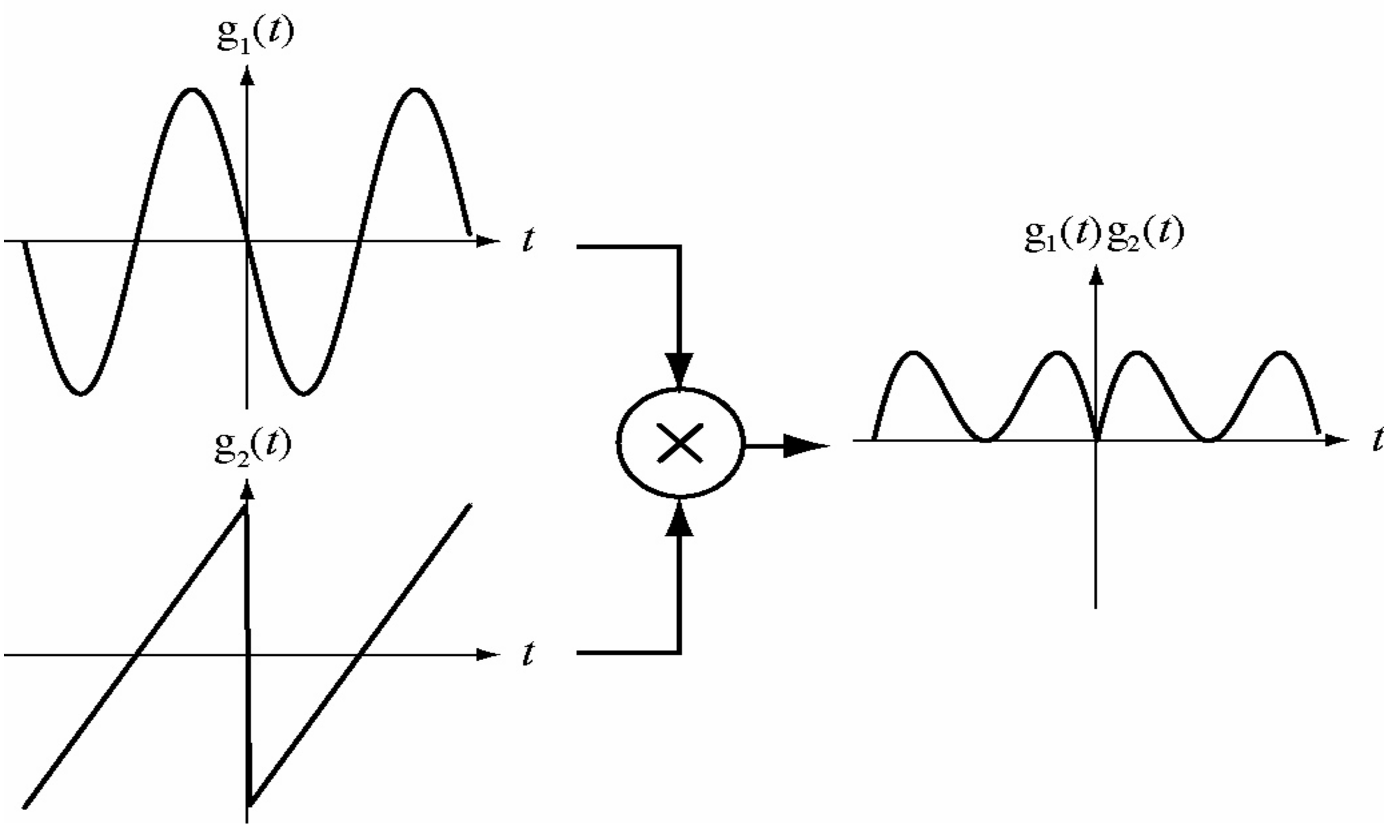
Product of Even and Odd Functions Contd.

Product of an Even Function and an Odd Function



Product of Even and Odd Functions Contd.

Product of Two Odd Functions



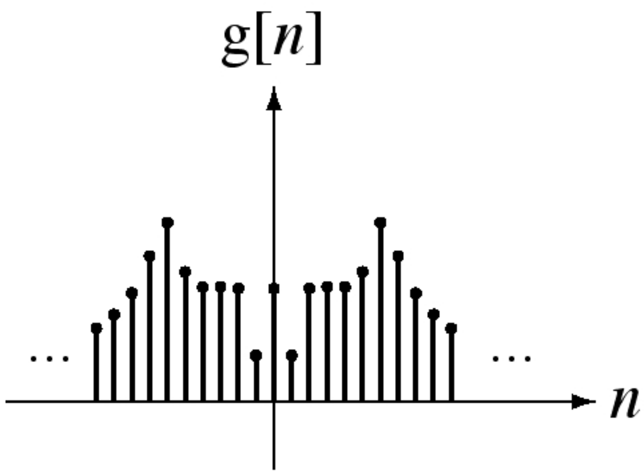
Derivatives and Integrals of Functions

Function type	Derivative	Integral
Even	Odd	Odd + constant
Odd	Even	Even

Discrete Time Even and Odd Signals

$$g[n] = g[-n]$$

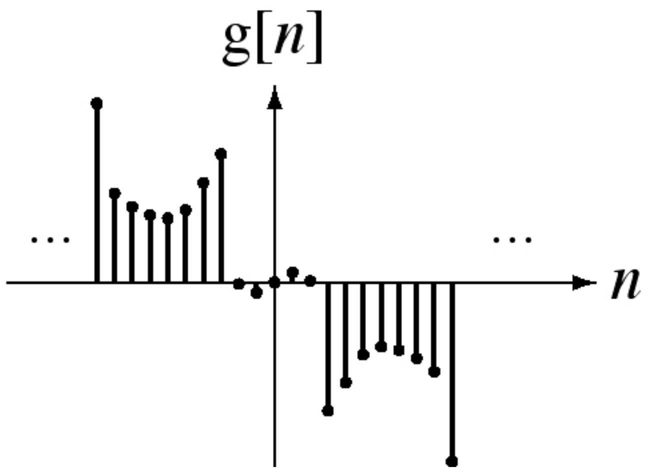
Even Function



$$g_e[n] = \frac{g[n] + g[-n]}{2}$$

$$g[n] = -g[-n]$$

Odd Function



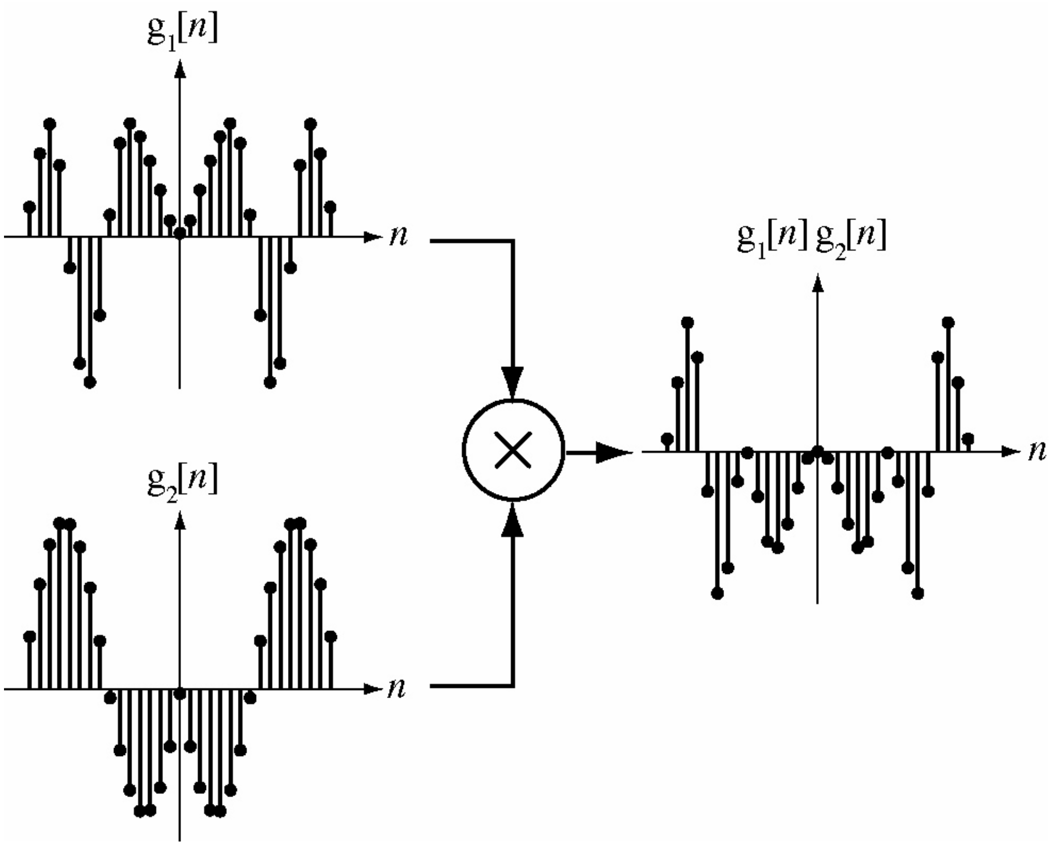
$$g_o[n] = \frac{g[n] - g[-n]}{2}$$

Combination of even and odd function for DT Signals

Function type	Sum	Difference	Product	Quotient
Both even	Even	Even	Even	Even
Both odd	Odd	Odd	Even	Even
Even and odd	Even or Odd	Even or odd	Odd	Odd

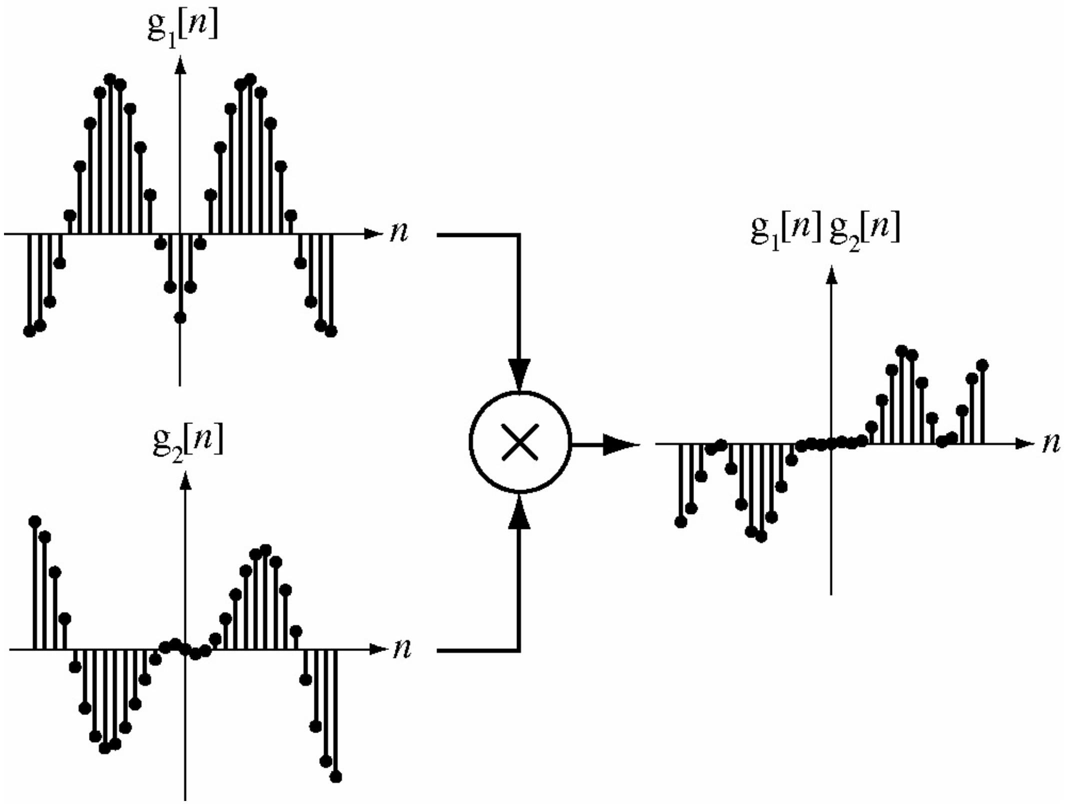
Products of DT Even and Odd Functions

Two Even Functions



Products of DT Even and Odd Functions Contd.

An Even Function and an Odd Function



Proof Examples

- Prove that product of two even signals is even.

Change t→ -t

$$x(t) = x_1(t) \times x_2(t) \rightarrow$$
$$x(-t) = x_1(-t) \times x_2(-t) =$$
$$x_1(t) \times x_2(t) = x(t)$$

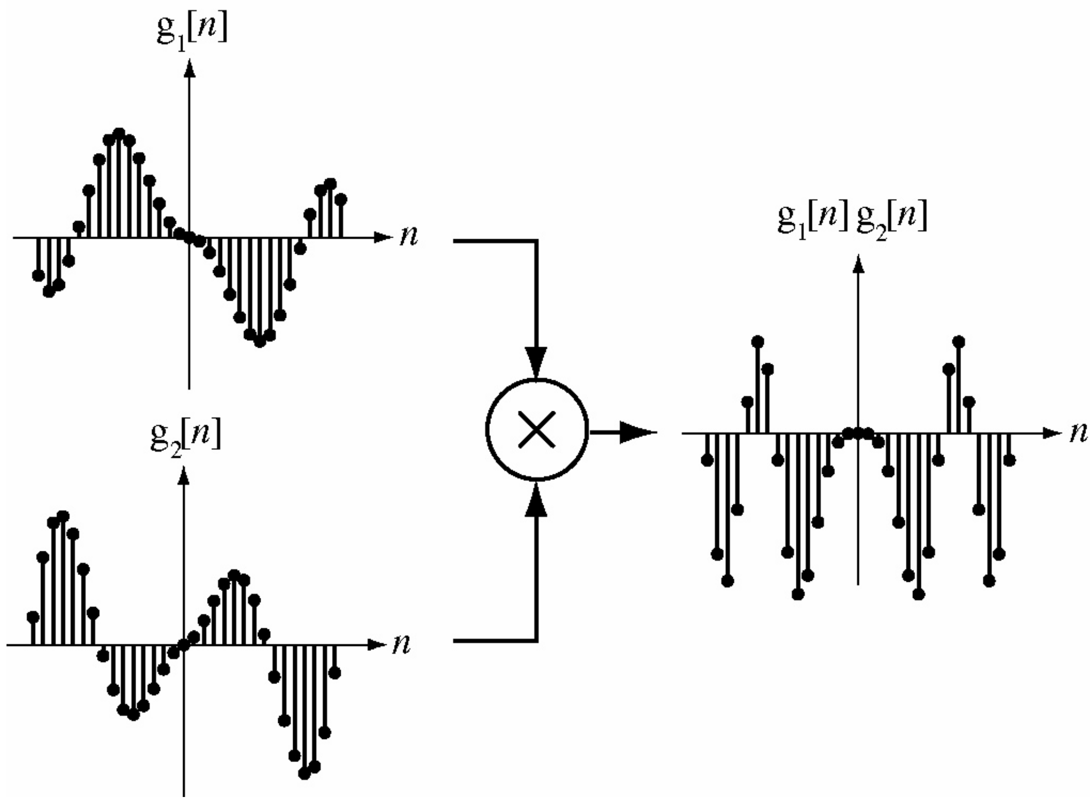
- Prove that product of two odd signals is odd.

- What is the product of an even signal and an odd signal? Prove it!

$$x(t) = x_1(t) \times x_2(t) \rightarrow$$
$$x(-t) = x_1(-t) \times x_2(-t) =$$
$$x_1(t) \times -x_2(t) = -x(t) =$$
$$x(-t) \leftarrow Even$$

Products of DT Even and Odd Functions Contd.

Two Odd Functions



Energy and Power Signals

Energy Signal

- A signal with finite energy and zero power is called Energy Signal i.e.for energy signal

$$0<E<\infty \text{ and } P=0$$

- Signal energy of a signal is defined as the *area under the square of the magnitude of the signal*.

$$E_x = \int_{-\infty}^{\infty} \left| x(t) \right|^2 dt$$

- The units of signal energy depends on the unit of the signal.

Energy and Power Signals Contd.

Power Signal

- Some signals have infinite signal energy. In that caseit is more convenient to deal with **average signal power.**
- For power signals

$$0<P<\infty \text{ and } E = \infty$$

- Average power of the signal is given by

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} \left|x(t)\right|^2 dt$$

Energy and Power Signals Contd.

- For a periodic signal x(t) the average signal power is

$$P_{\mathrm{x}} = \frac{1}{T}\int_{T} \left|\mathrm{x}(t)\right|^2 dt$$

- T is any period of the signal.
- Periodic signals are generally power signals.

Signal Energy and Power for DT Signal

- A discrtet time signal with finite energy and zero power is called Energy Signal i.e.for energy signal

$$0<E<\infty \text{ and } P=0$$

- The **signal energy** of a for a discrete time signal x[*n*] is

$$E_x = \sum_{n=-\infty}^{\infty} \left| x[n] \right|^2$$

Signal Energy and Power for DT Signal Contd.

The average signal power of a discrete time power signal x[n] is

$$P_x = \lim_{N \to \infty} \frac{1}{2N} \sum_{n=-N}^{N-1} \left| x[n] \right|^2$$

For a periodic signal x[n] the average signal power is

$$P_x = \frac{1}{N} \sum_{n=\langle N \rangle} \left| x[n] \right|^2$$

$\left(\begin{array}{l}\text{The notation } \sum_{n=\langle N \rangle} \text{ means the sum over any set of} \\ \text{consecutive } n\text{'s exactly } N \text{ in length.}\end{array}\right)$

What is System?

- Systems process input signals to produce output signals
- A system is combination of elements that manipulates one or more signals to accomplish a function and produces some output.

Examples of Systems

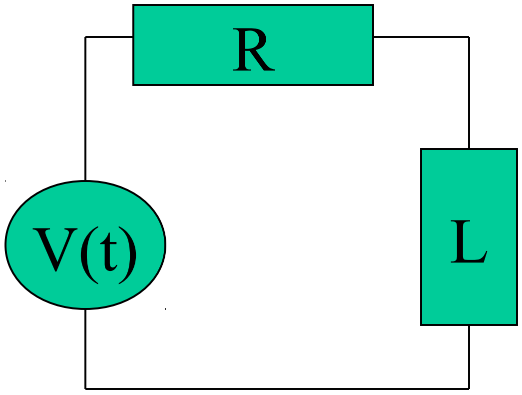
- A circuit involving a capacitor can be viewed as a system that transforms the source voltage (signal) to the voltage (signal) across the capacitor
- A communication system is generally composed of three sub-systems, the transmitter, the channel and the receiver. The channel typically attenuates and adds noise to the transmitted signal which must be processed by the receiver
- Biomedical system resulting in biomedical signal processing
- Control systems

System - Example

- Consider an RL series circuit
  - Using a first order equation:

$$V_L(t) = L\frac{di(t)}{dt}$$

$$V(t) = V_R + V_L(t) = i(t)\cdot R + L\frac{di(t)}{dt}$$



Mathematical Modeling of Continuous Systems

Most continuous time systems represent how continuous signals are transformed via **differential equations**.

E.g. RC circuit

$$\frac{dv_c(t)}{dt} + \frac{1}{RC} v_c(t) = \frac{1}{RC} v_s(t)$$

System indicating car velocity

$$m \frac{dv(t)}{dt} + \rho v(t) = f(t)$$

Mathematical Modeling of Discrete Time Systems

Most discrete time systems represent how discrete signals are transformed via **difference equations**

e.g. bank account, discrete car velocity system

$$y[n] = 1.01 y[n-1] + x[n]$$

$$v[n] - \frac{m}{m+\rho\Delta} v[n-1] = \frac{\Delta}{m+\rho\Delta} f[n]$$

Order of System

- Order of the Continuous System is the highest power of the derivative associated with the output in the differential equation
- For example the order of the system shown is 1.

$$m\frac{dv(t)}{dt}+\rho v(t)=f(t)$$

Order of System Contd.

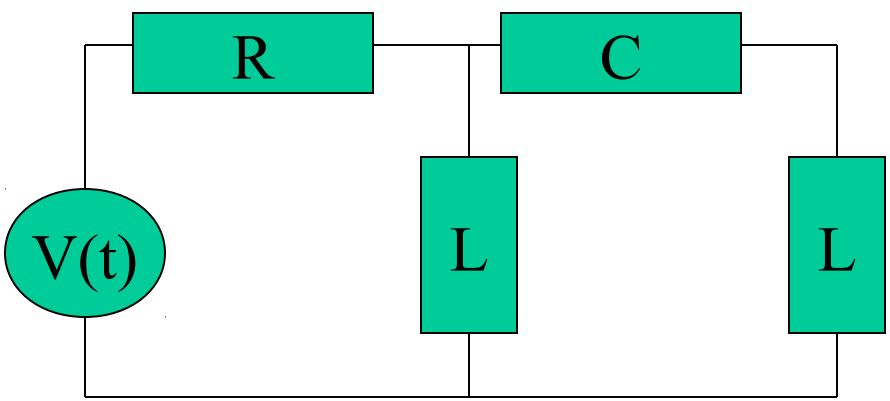
- Order of the Discrete Time system is the highest number in the difference equation by which the output is delayed
- For example the order of the system shown is 1.

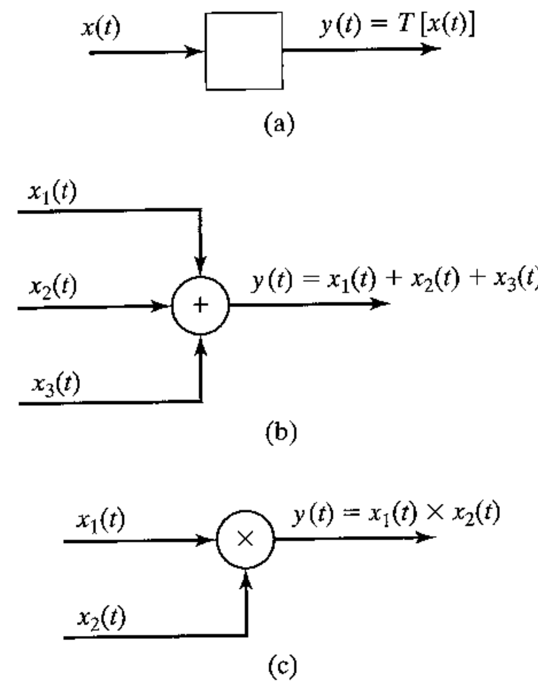
$$y[n] = 1.01y[n-1] + x[n]$$

Interconnected Systems

- Parallel
- Serial (cascaded)
- Feedback

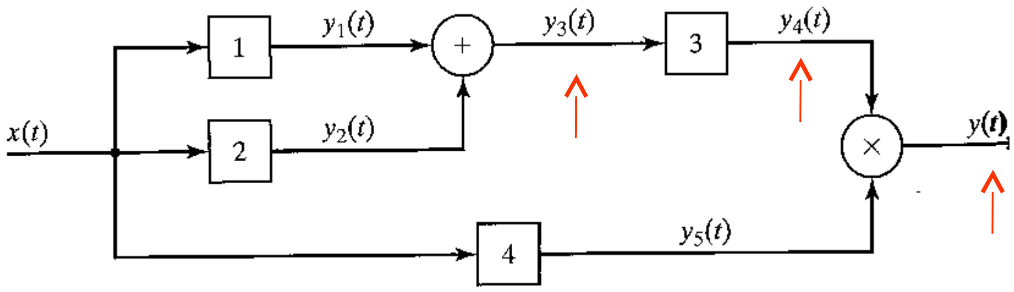
notes





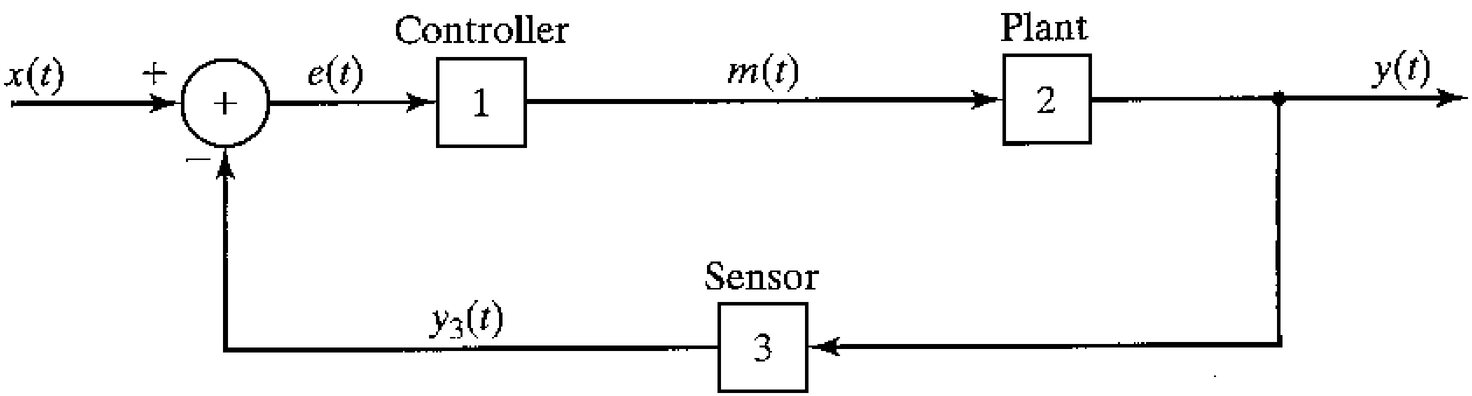
Interconnected System Example

- Consider the following systems with 4 subsystem
- Each subsystem transforms it input signal
- The result will be:
  - y3(t)=y1(t)+y2(t)=T1[x(t)]+T2[x(t)]
  - y4(t)=T3[y3(t)]= T3(T1[x(t)]+T2[x(t)])
  - y(t)= y4(t)* y5(t)= T3(T1[x(t)]+T2[x(t)])* T4[x(t)]



Feedback System

- Used in automatic control
  - e(t)=x(t)-y3(t)= x(t)-T3[y(t)]=
  - y(t)= T2[m(t)]=T2(T1[e(t)])
  - → y(t)=T2(T1[x(t)-y3(t)])= T2(T1([x(t)] - T3[y(t)])) =
  - =T2(T1([x(t)] –T3[y(t)]))



Types of Systems

- Causal & Anticausal
- Linear & Non Linear
- Time Variant &Time-invariant
- Stable & Unstable
- Static & Dynamic
- Invertible & Inverse Systems

Causal & Anticausal Systems

- Causal system : A system is said to be *causal* if the present value of the output signal depends only on the present and/or past values of the input signal.
- Example: $y[n]=x[n]+1/2x[n-1]$

Causal & Anticausal Systems Contd.

- Anticausal system : A system is said to be *anticausal* if the present value of the output signal depends only on the future values of the input signal.
- Example: $y[n]=x[n+1]+1/2x[n-1]$

Linear & Non Linear Systems

- A system is said to be linear if it satisfies the principle of superposition
- For checking the linearity of the given system, firstly we check the response due to linear combination of inputs
- Then we combine the two outputs linearly in the same manner as the inputs are combined and again total response is checked
- If response in step 2 and 3 are the same,the system is linear othewise it is non linear.

Time Invariant and Time Variant Systems

- A system is said to be *time invariant* if a time delay or time advance of the input signal leads to a identical time shift in the output signal.

$$y_i(t) = H\{x(t-t_0)\}$$

$$= H\{S^{t0}\{x(t)\}\} = HS^{t0}\{x(t)\}$$

$$y_0(t) = S^{t0}\{y(t)\}$$

$$= S^{t0}\{H\{x(t)\}\} = S^{t0}H\{x(t)\}$$

Stable & Unstable Systems

- A system is said to be *bounded-input bounded-output stable* (BIBO stable) iff every bounded input results in a bounded output.

i.e.

$$\forall t \quad |x(t)| \leq M_x < \infty \rightarrow \forall t \quad |y(t)| \leq M_y < \infty$$

Stable & Unstable Systems Contd.

Example

- $y[n]=1/3(x[n]+x[n\text{-}1]+x[n\text{-}2])$

$$
\begin{aligned}
y[n] &= \frac{1}{3}\left|x[n]+x[n-1]+x[n-2]\right| \\
&\leq \frac{1}{3}(|\,x[n]\,|+|\,x[n-1]\,|+|\,x[n-2]\,|) \\
&\leq \frac{1}{3}(M_x+M_x+M_x)=M_x
\end{aligned}
$$

Stable & Unstable Systems Contd.

Example: The system represented by

$$y(t) = A\ x(t) \text{ is unstable ; } A > 1$$

Reason: let us assume x(t) = u(t), then at every instant u(t) will keep on multiplying with A and hence it will not be bonded.

Static & Dynamic Systems

- A static system is memoryless system
- It has no storage devices
- its output signal depends on present values of the input signal
- For example

$$i(t) = {}^{1}\!/_{R}\, v(t)$$

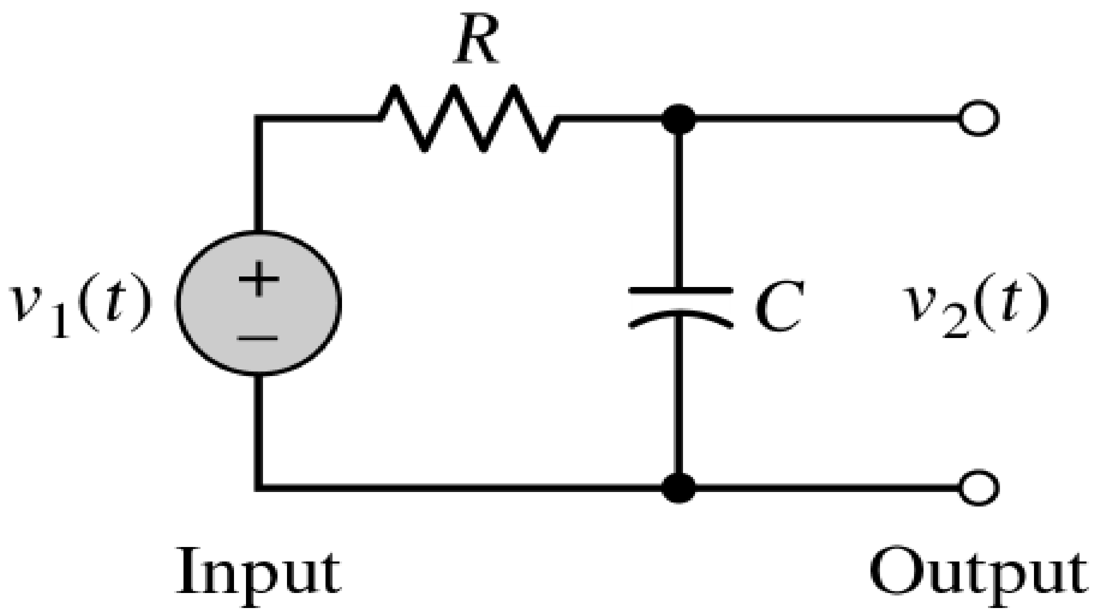
Static & Dynamic Systems Contd.

- A dynamic system possesses memory
- It has the storage devices
- A system is said to possess *memory* if its output signal depends on past values and future values of the input signal

$$i(t) = \frac{1}{L}\int_{-\infty}^{t} v(\tau)d\tau$$

$$y[n] = x[n] + x[n-1]$$

Example: Static or Dynamic?



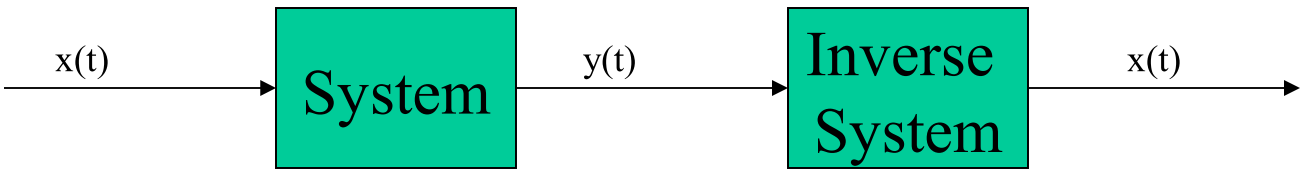
Example: Static or Dynamic?

Answer:

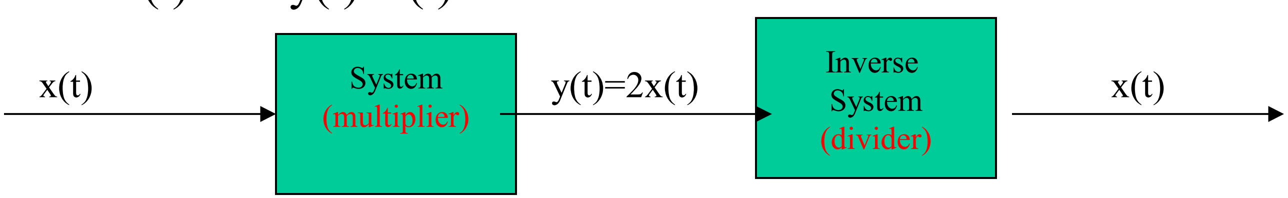
- The system shown above is RC circuit
- R is memoryless
- C is memory device as it stores charge because of which voltage across it can’t change immediately
- Hence given system is dynamic or memory system

Invertible & Inverse Systems

- If a system is invertible it has an Inverse System



- Example: y(t)=2x(t)
  - System is invertible→ must have inverse, that is:
  - For any x(t) we get a distinct output y(t)
  - Thus, the system must have an Inverse
    - x(t)=1/2 y(t)=z(t)



LTI Systems

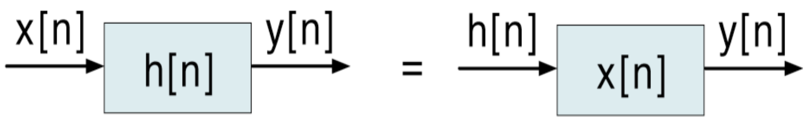
- LTI Systems are *completely characterized* by its unit sample response
- The output of *any* LTI System is a convolution of the input signal with the unit-impulse response, *i.e.*

$$y[n] = x[n] * h[n]$$
$$= \sum_{k=-\infty}^{+\infty} x[k]h[n-k]$$

Properties of Convolution

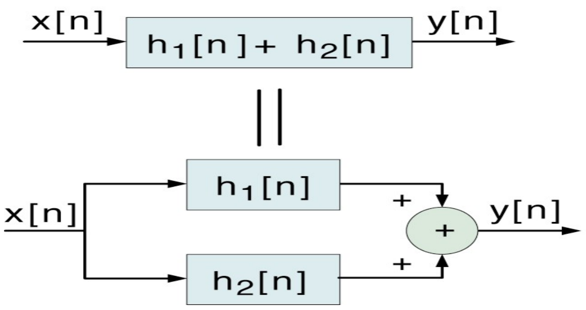
Commutative Property

$$x[n]*h[n]=h[n]*x[n]$$



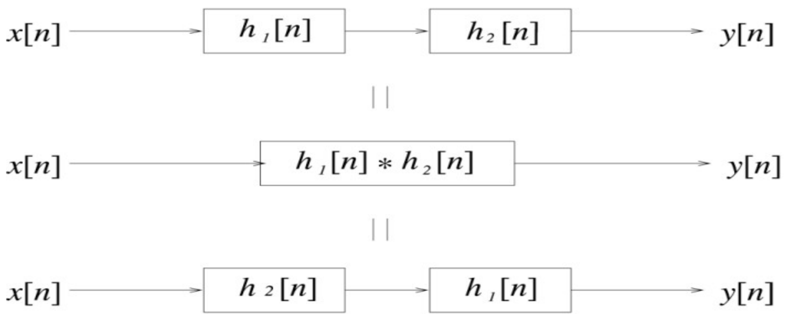
Distributive Property

$$x[n]*(h_1[n]+h_2[n])=$$
$$(x[n]*h_1[n])+(x[n]*h_2[n])$$



Associative Property

$$x[n]*h_1[n]*h_2[n]=$$
$$(x[n]*h_1[n])*h_2[n]=$$
$$(x[n]*h_2[n])*h_1[n]$$



Useful Properties of (DT) LTI Systems

- **Causality:** $h[n] = 0 \qquad n < 0$

- **Stability:** $\sum_{k=-\infty}^{\infty} |h[k]| < \infty$

Bounded Input $\leftrightarrow$ Bounded Output

for $|x[n]| \le x_{\max} < \infty$

$$|y[n]| = \left| \sum_{k=-\infty}^{\infty} x[k]h[n-k] \right| \le x_{\max} \left| \sum_{k=-\infty}^{\infty} h[n-k] \right| < \infty$$

THANKS