DIGITAL SIGNAL PROCESSING

LECTURE (3)

Discrete-Time Signals and Systems

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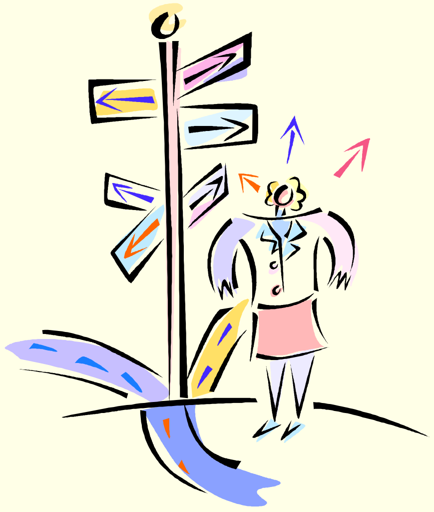
Agenda



- Discrete-time signals: sequences
- Discrete-time system
- Linear Time-Invariant Causal Systems (LTIC System)
- Linear Constant-Coefficient Difference Equations

Discrete-Time Signals---Sequences

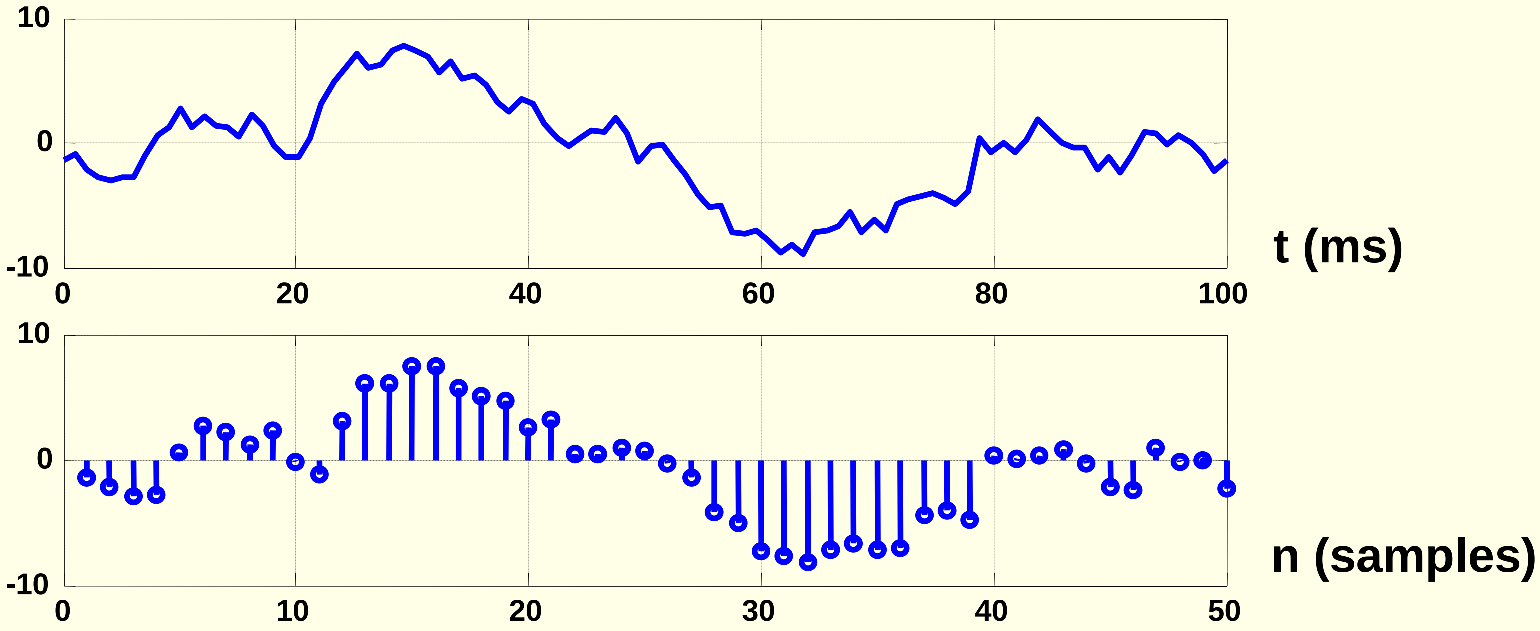
The Taxonomy of Signals

- Signal: A function that conveys information

		Amplitude: Continuous	Amplitude: Discrete
Time	Continuous	analog signals	continuous-time signals
	Discrete	**discrete-time signals**	**digital signals**

Discrete-Time Signals: Sequences

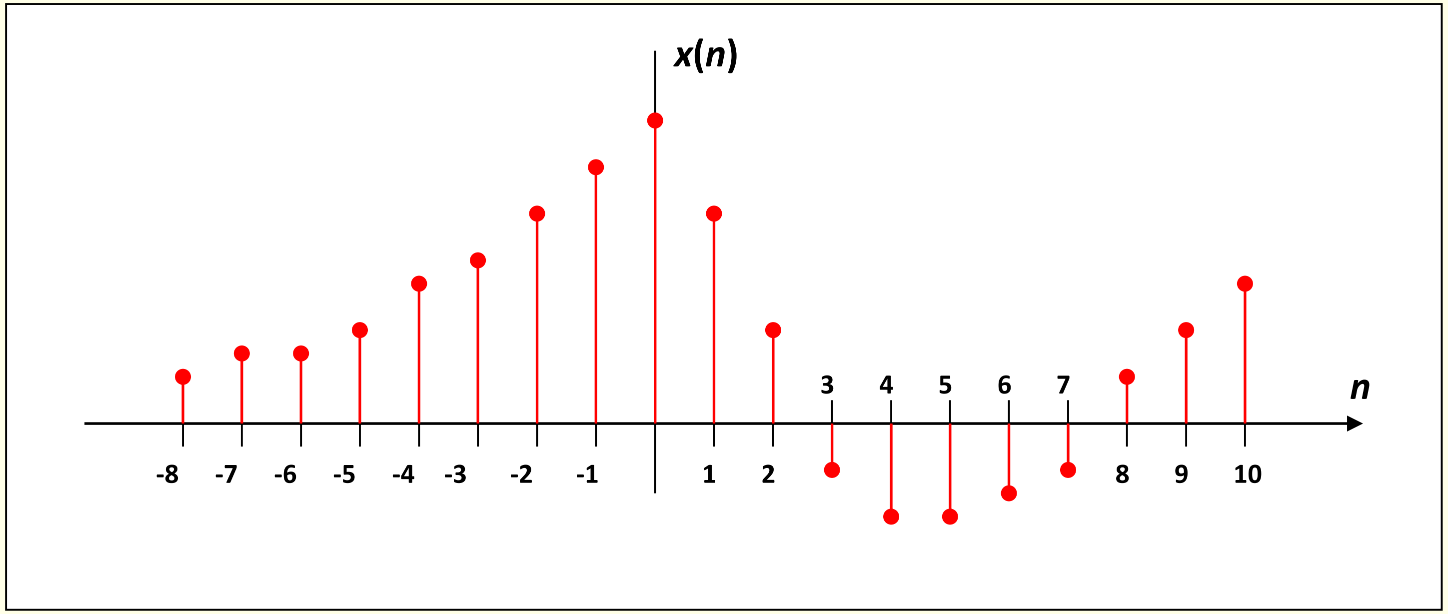
- Discrete-time signals are represented by sequence of numbers
  - The n^{th} number in the sequence is represented with $x[n]$
- Often times sequences are obtained by sampling of continuous-time signals
  - In this case $x[n]$ is value of the analog signal at $x_c(nT)$
  - Where T is the sampling period



Representation by a Sequence

- Discrete-time system theory
  - Concerned with processing signals that are represented by *sequences*.

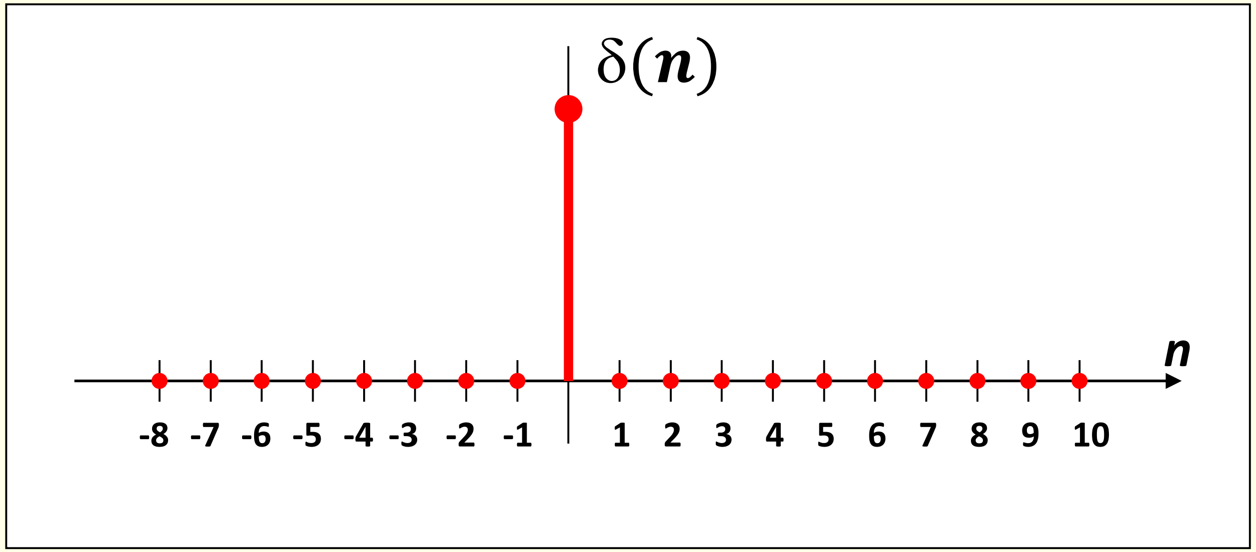
$$x = \{x(n)\}, \qquad -\infty < n < \infty$$



Important Sequences

- Unit-sample sequence $\delta(n)$
- Sometime call
  - a discrete-time impulse; or
  - an impulse

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$



$$Note: \;\; x(n)\,\delta(n-m) = x(m)\,\delta(n-m)$$

Important Sequences

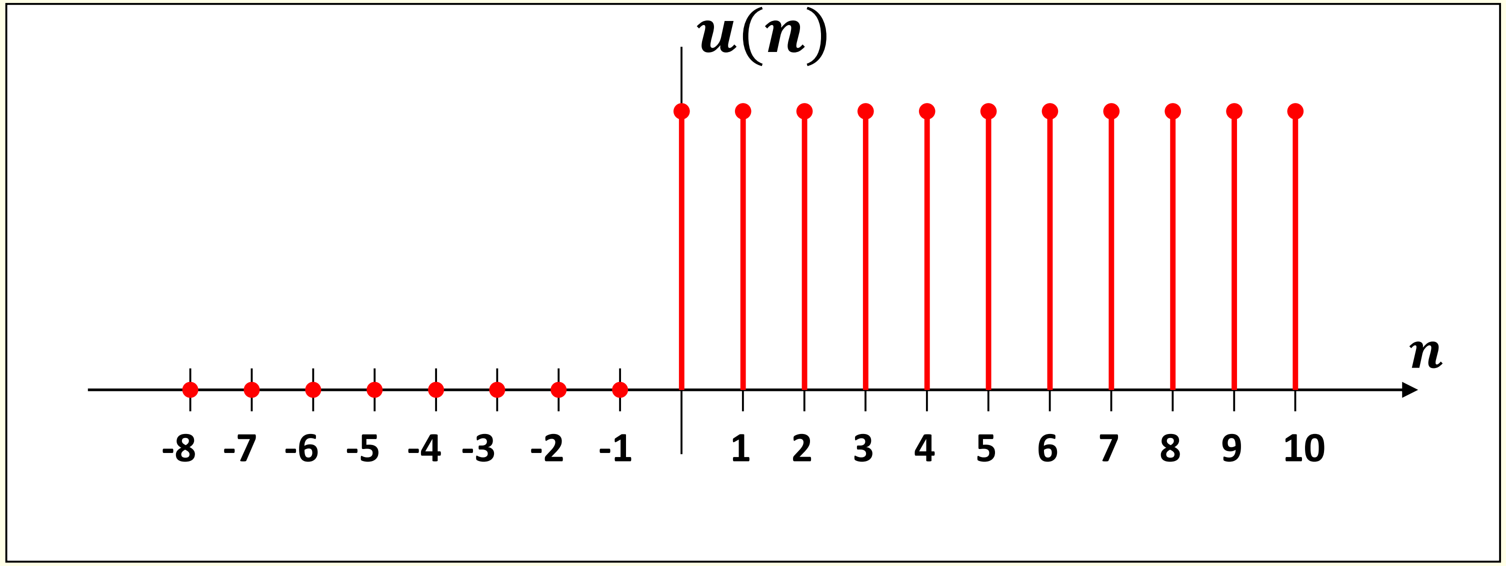
- Unit-step sequence $u(n)$

$$u(n)=\begin{cases}1 & n\geq 0\\ 0 & n<0\end{cases}$$

- Fact:

$$u(n)=\sum_{m=0}^{\infty}\delta(n-m)$$

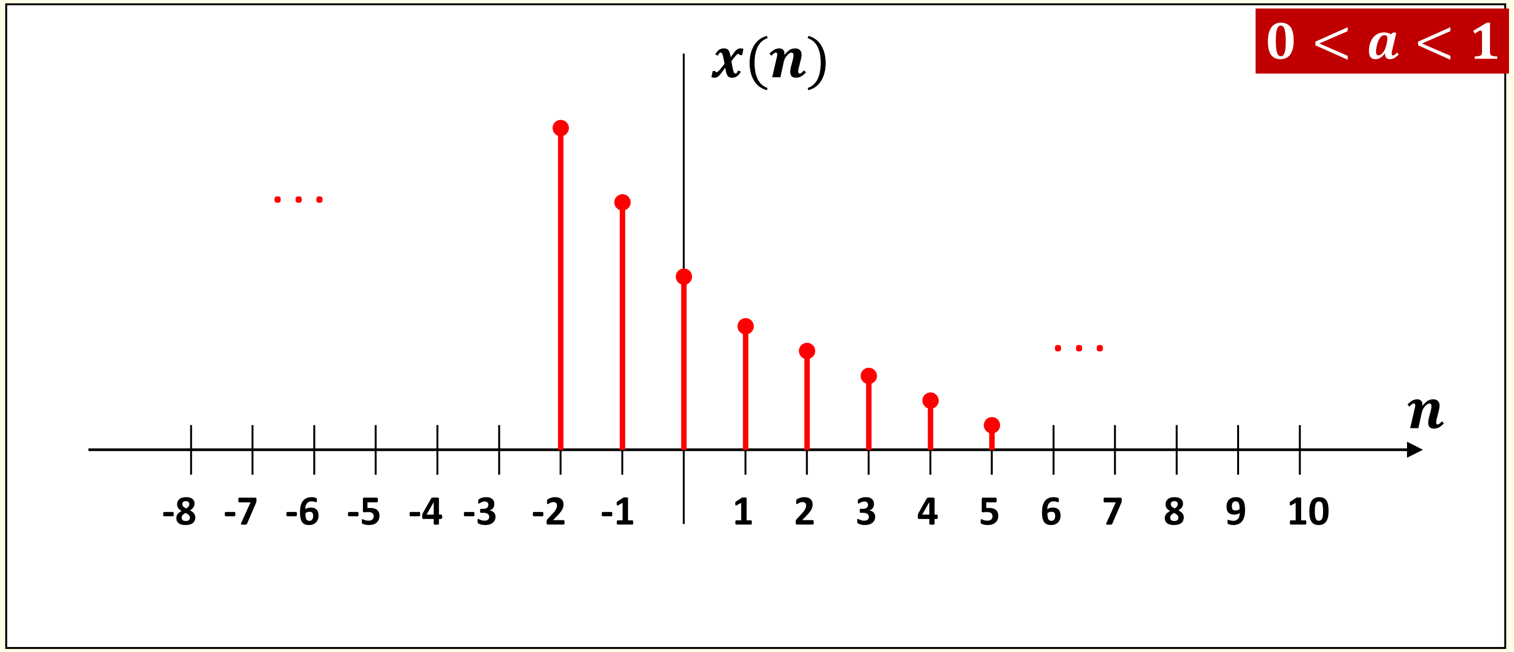
$$\delta(n)=u(n)-u(n-1)$$



Important Sequences

- Real exponential sequence

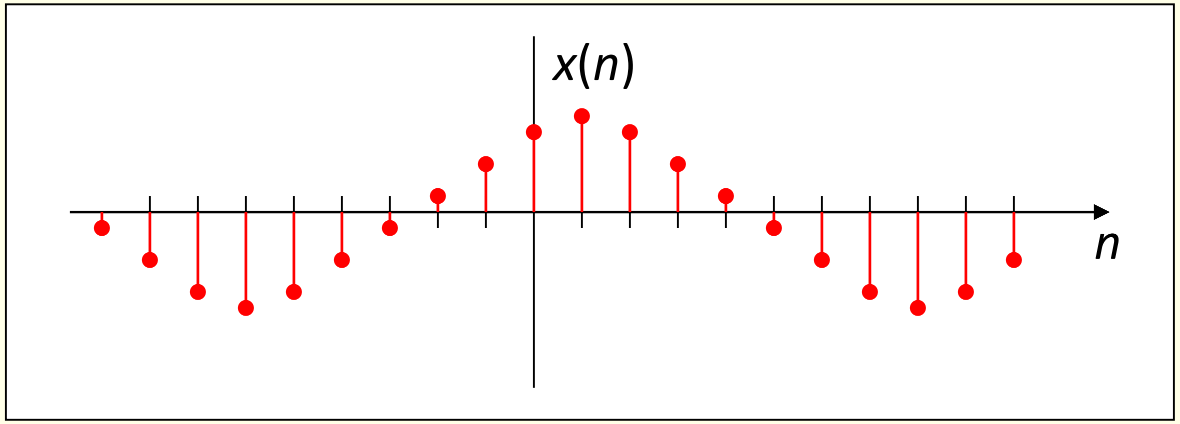
$$x(n) = a^n$$



Important Sequences

- Sinusoidal sequence

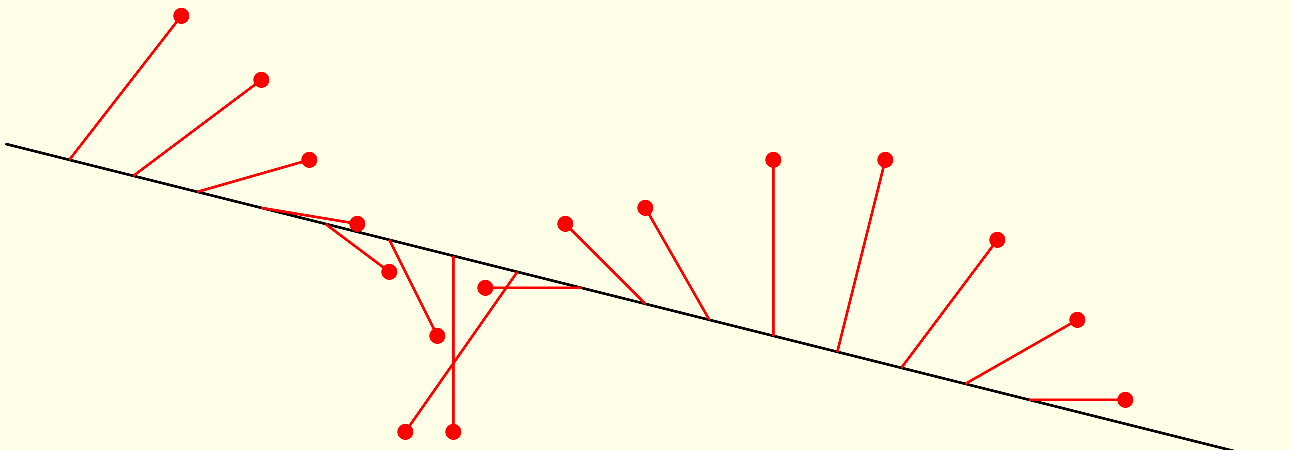
$$x(n) = A\cos(n\omega_0 + \phi)$$



Important Sequences

- Complex exponential sequence

$$x(n) = e^{(\sigma + j\omega_0)n} = e^{\sigma n} e^{j\omega_0 n} = a^n e^{j\omega_0 n}$$



Periodic Sequences

- A sequence $x(n)$ is defined to be periodic with period N if

$$\boxed{x(n) = x(n+N) \quad \text{for all } N}$$

- Example: consider $x(n) = e^{j\omega_0 n}$

$$x(n) = e^{j\omega_0 n} = e^{j\omega_0 (n+N)} = e^{j\omega_0 N} e^{j\omega_0 n} = x(n+N)$$

$\Rightarrow$ $\omega_0 N = 2k\pi$ $\Rightarrow$ $N = \dfrac{2k\pi}{\omega_0}$ $\Rightarrow$ $\dfrac{2\pi}{\omega_0}$ must be a rational number

Examples of Periodic Sequences

$$x_1[n] = \cos(\pi n / 4)$$

- Suppose it is periodic sequence with period N

$$x_1[n] = x_1[n+N]$$

$$\cos(\pi n / 4) = \cos[\pi(n+N)/4]$$

$$\pi n / 4 + 2\pi k = \pi n / 4 + N\pi / 4, \quad k : \text{integer}$$

$$N = 2\pi k / (\pi / 4) = 8k$$

$$k = 1, \rightarrow N = 8 = 2\pi / w_0$$

Examples of Periodic Sequences

$$\frac{2\pi}{8} \rightarrow \frac{3\pi}{8} \quad \Longrightarrow \quad x_1[n] = \cos(3\pi n/8)$$

- Suppose it is periodic sequence with period N

$$x_1[n] = x_1[n+N]$$

$$\cos(3\pi n/8) = \cos[3\pi(n+N)/8]$$

$$3\pi n/8 + 2\pi k = 3\pi n/8 + 3N\pi/8, \quad k : \text{integer}$$

$$N = 2\pi k / w_0 = 2\pi k / (3\pi/8)$$

$$k = 3, \rightarrow N = 16$$

Example of Non-Periodic Sequences

$$x_2[n] = \cos\left(n\right)$$

- Suppose it is periodic sequence with period N

$$x_2[n] = x_2[n+N]$$

$$\cos\left(n\right) = \cos(n+N)$$

$$for \quad n + 2\pi k = n + N, \quad k : \text{integer},$$

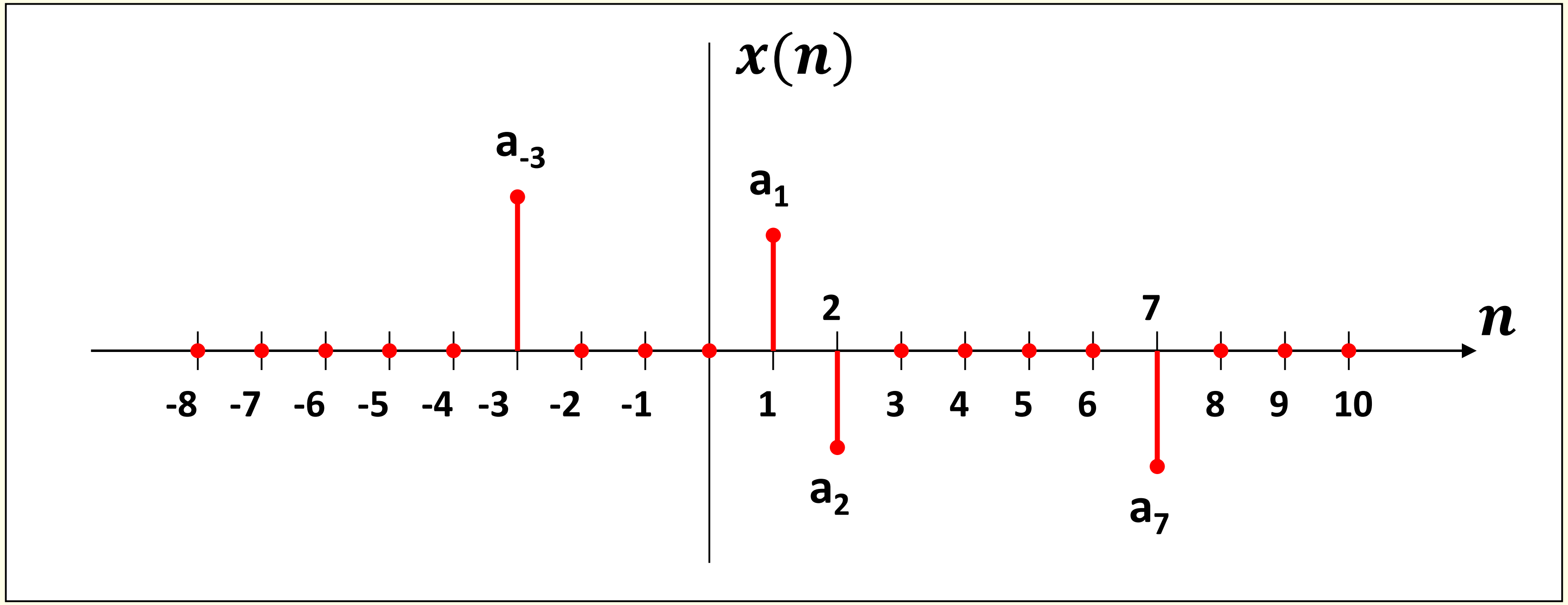
$$there\ is\ no\ \text{integer}\ N$$

Operations on Sequences

- Sum $x+y=\{x(n)+y(n)\}$
- Product $x\cdot y=\{x(n)y(n)\}$
- Multiplication $\alpha x=\{\alpha x(n)\}$
- Shift $y(n)=x(n-n_0)$
- Reflection $y(n)=x(-n)$

Sequence Representation Using delay unit

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\ \delta(n-k)$$



$$x(n) = a_{-3}\delta(n+3) + a_1\delta(n-1) + a_2\delta(n-3) + a_7\delta(n-7)$$

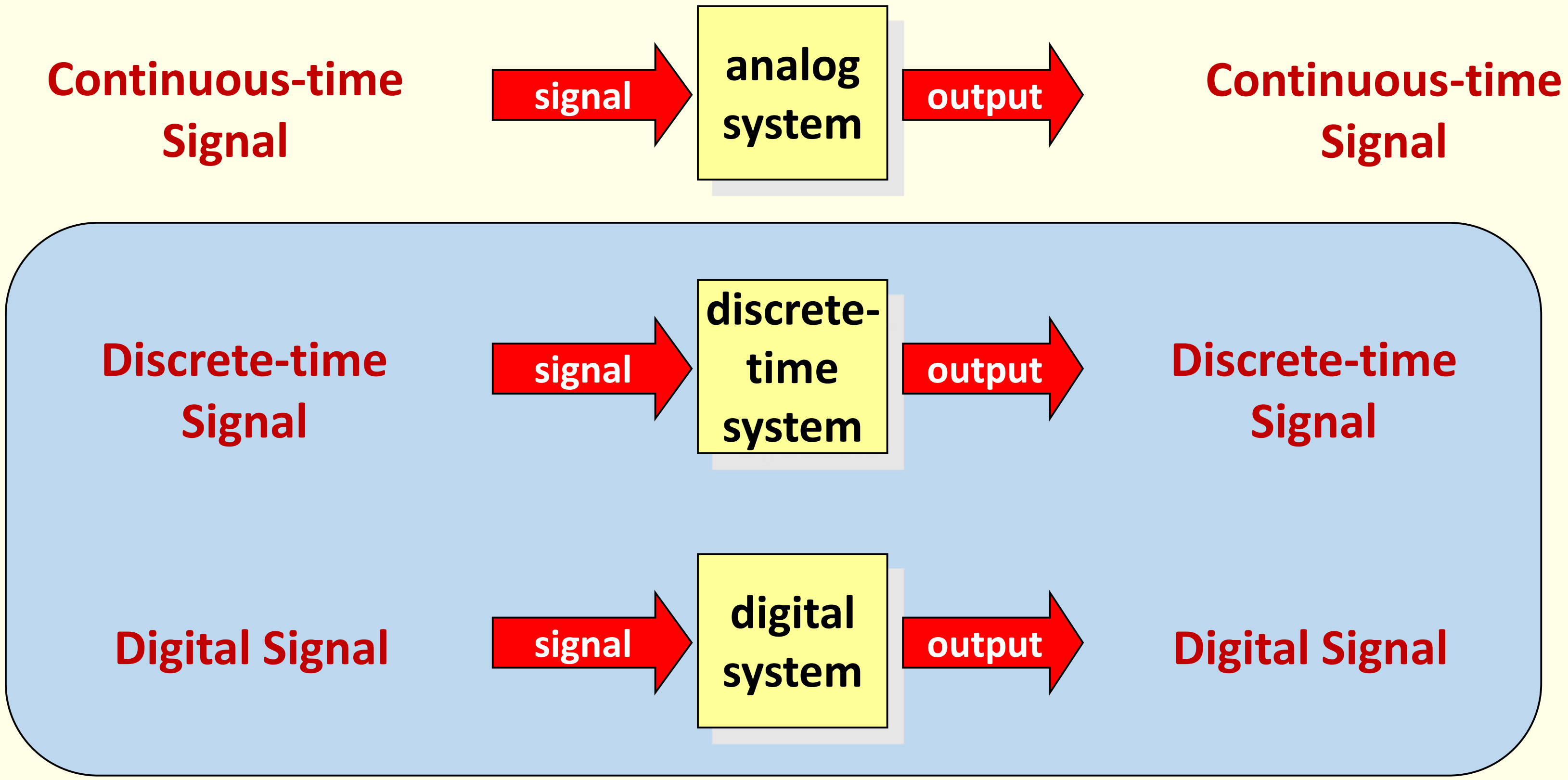
Energy of a Sequence

- Energy of a sequence is defined by

$$E = \sum_{n=-\infty}^{n=\infty} |x(n)|^2$$

DISCRETE-TIME SYSTEMS

Signal Process Systems



Discrete-Time Systems

- **Definition**: A discrete-time system is a device or algorithm that operates on a discrete-time signal called the input or excitation (e.g. $x(n)$), according to some rule (e.g. $T[.]$) to produce another discrete-time signal called the output or response (e.g. $y(n)$).

$$x(n) \longrightarrow \boxed{T\,[\,]} \longrightarrow y(n) = T[x(n)]$$

input signal or excitation

output signal or response

- This expression denotes also the transformation $T[.]$, (also called operator or mapping) or processing performed by the system on x(n) to produce y(n).

Discrete–Time Systems

- Discrete-Time System Analysis
  - It is the process of determining the response, y(n) of that system described by an operator, transformation T[.] or its impulse response, h(n) to a given excitation, x(n).
  - This process could done also in z- or frequency domains.
- Discrete-Time System Design
  - It is the process of synthesizing the system parameters that satisfy the input output specification.
  - This process could done also in time, z- or frequency domains
- Digital Filter
  - It is a digital system that can be used to filter discrete –time signal.
  - Filtering is a process by which the frequency spectrum of the signal could be modified or manipulated according to some desired specifications.

Classification of Discrete-Time Systems

- Static (Memoryless) Vs Dynamic system (Memory)
- Time varying Vs Time Invariant system
- Linear Vs Nonlinear System
- Causal Vs Noncausal System
- Stable vs. Unstable System

Static Vs Dynamic System

- **Definition**: A discrete-time system is called *static* or *memoryless* if its output at any time instant *n* depends on the input sample at the same time, but not on the past or future samples of the input.

- In the other case, the system is said to be *dynamic* or to have *memory*. If the output of a system at time *n* is completely determined by the input samples in the interval from *n-N* to *n* ($N \geq 0$), the system is said to have memory of ***duration N***.

- If $N \geq 0$, the system is ***static*** or ***memoryless***.

- If $0 < N < \infty$, the system is said to have ***finite memory***.

- If $N \to \infty$, the system is said to have ***infinite memory***.

Examples:

- The static (memoryless) systems:

$$y(n) = nx(n) + bx^3(n)$$

- The dynamic systems with finite memory:

$$y(n) = \sum_{k=0}^{N} h(k)x(n-k)$$

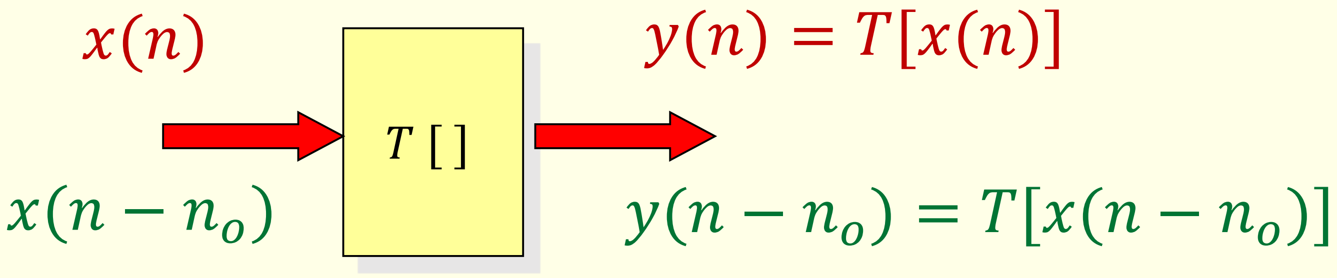
- The dynamic system with infinite memory:

$$y(n) = \sum_{k=0}^{\infty} h(k)x(n-k)$$

Time Varying Vs Time Invariant System

- **Definition**: A discrete-time system is called ***time-invariant*** if its input-output characteristics do not change with time. In the other case, the system is called ***time-variable***.

- Time-Invariant (shift-invariant) Systems
  - A time shift at the input causes corresponding time-shift at output



Examples:

- The time-invariant systems:

$$y(n) = x(n) + bx^3(n)$$

$$y(n) = \sum_{k=0}^{N} h(k)x(n-k)$$

- The time-variable systems:

$$y(n) = nx(n) + bx^3(n-1)$$

$$y(n) = \sum_{k=0}^{N} h^{N-n}(k)x(n-k)$$

Example of Time-Invariant System

- Accumulator system $y(n) = \sum_{k=-\infty}^{n} x(k)$
- Shift the system by (m)
  - then the output: $\mathrm{y}(n - \mathrm{m}) = \sum_{k=-\infty}^{n-\mathrm{m}} x_1(k)$
- Let the input: $x(n - m)$,
  - then the output: $y_1(n) = \sum_{k=-\infty}^{n} x(k - m)$
  - Let: $K = k - m$, then $y_1(n) = \sum_{k=-\infty}^{n-m} x(k) = \mathrm{y}(n - \mathrm{m})$

- **Hence , the system is Time-Invariant System**

Example of Time-Varying System

- Accumulator system $y(n) = nx(n)$
- Shift the system by (m)
  - then the output: $\mathrm{y}(n-\mathrm{m}) = (n-m)x(n-m)$
- Let the input: $x(n-m)$,
  - then the output: $y_1(n) = nx(n-m) \neq \mathrm{y}(n-\mathrm{m})$

- **Hence , the system is Time-Varying System**

Linear vs. Non-linear Systems

- **Definition**: A discrete-time system is called ***linear*** if only if it satisfies the ***linear superposition principle***. In the other case, the system is called ***non-linear***.

$$x(n) \longrightarrow \boxed{T\,[\;]} \longrightarrow y(n) = T[x(n)]$$

$$\boxed{T[ax_1(n)+bx_2(n)] = aT[x_1(n)]+bT[x_2(n)]}$$

Examples:

- The linear systems:

$$y(n) = \sum_{k=0}^{N} h(k)x(n-k)$$

$$y(n) = x(n^2) + bx(n-k)$$

- The non-linear systems:

$$y(n) = nx(n) + bx^3(n-1)$$

$$y(n) = \sum_{k=0}^{N} h(k)x(n-k)x(n-k+1)$$

Example of Linear System

- Accumulator system $y(n) = \sum_{k=-\infty}^{n} x(k)$
- For arbitrary input: $x_1(n)$,
  - then the output: $y_1(n) = \sum_{k=-\infty}^{n} x_1(k)$

- For arbitrary input: $x_2(n)$,
  - then the output: $y_2(n) = \sum_{k=-\infty}^{n} x_2(k)$

- Let the input: $x(n) = ax_1(n) + bx_2(n)$,
  - then the output:
    - $y(n) = \sum_{k=-\infty}^{n} ax_1(k) + bx_2(k)$
    - $y(n) = a\sum_{k=-\infty}^{n} x_1(k) + b\sum_{k=-\infty}^{n} x_2(k)$
    - Then, $y(n) = ay_1(n) + by_2(n)$

- **Hence , the system is Linear System**

Example of Nonlinear Systems

- Accumulator system $y(n) = x^2(\mathrm{n})$
- For arbitrary input: $x_1(n)$,
  - then the output: $y_1(n) = x_1^2\,(\mathrm{n})$

- For arbitrary input: $x_2(n)$,
  - then the output: $y_2(n) = x_2^2\,(\mathrm{n})$

- Let the input: $x(n) = ax_1(n) + bx_2(n)$,
  - then the output:
    - $y(n) = [ax_1(n) + bx_2(n)]^2$
    - $y(n) = a^2x_1^2\,(\mathrm{n}) + \mathrm{ab}x_1(n)x_2(n) + b^2x_2^2\,(\mathrm{n})$
    - Then, $y(n) \neq ay_1(n) + by_2(n)$

- **Hence , the system is Nonlinear System**

Causal vs. Non-causal Systems

- **Definition**: A system is said to be ***causal*** if the output of the system at any time *n* (i.e., *y(n)*) depends only on present and past inputs (i.e., *x(n)*, *x(n-1)*, *x(n-2)*, …). In mathematical terms, the output of a ***causal*** system satisfies an equation of the form

$$y(n) = F\left[x(n), x(n-1), x(n-2), \ldots\right]$$

- where $F[.]$ is some arbitrary function. If a system does not satisfy this definition, it is called ***non-causal***.

Examples:

- The causal system:

$$y(n) = \sum_{k=0}^{N} h(k)x(n-k)$$

$$y(n) = x^2(n) + bx(n-k)$$

- The non-causal system:

$$y(n) = nx(n+1) + bx^3(n-1)$$

$$y(n) = \sum_{k=-10}^{10} h(k)x(n-k)$$

Example :

- Check the discrete-time system for causality if its response is of the form:

  a) $y(n) = R[x(n)] = 3\,x(n - 2) + 3\,x(n + 2)$

  b) $y(n) = R[x(n)] = 3\,x(n - 1) - 3\,x(n - 2)$

- Solution:

  a) Let $x1(n)$ and $x2(n)$ be distinct excitations that satisfy Eq. (2.4b) and assume that $x1(n) \neq x2(n)$ for $n > k$

  For $n = k$

  $R[x1(n)]|_{n=k} = 3x1(k - 2) + 3x1(k + 2)$

  $R[x2(n)]|_{n=k} = 3x2(k - 2) + 3x2(k + 2)$

  and since we have assumed that $x1(n) \neq x2(n)$ for $n > k$, it follows that:

  $x1(k + 2) \neq x2(k + 2)$ and thus $3\,x1(k + 2) \neq 3\,x2(k + 2)$

  Therefore, $R[x1(n)] = Rx2(n)$ for $n = k$

  that is, the system is noncausal.

  b) For this case

  $R[x1(n)]|_{n=k} = 3x1(k - 1) + 3x1(k - 2)$

  $R[x2(n)]|_{n=k} = 3x2(k - 1) + 3x2(k - 2)$

  If $n \leq k$,

  $x1(k - 1) = x2(k - 1)$ and $x1(k - 2) = x2(k - 2)$

  for $n \leq k$ or

  $R[x1(n)] = Rx2(n)$ for $n \leq k$

  that is, the system is causal.

Stable vs. Unstable of Systems

- **Definition**: An arbitrary relaxed system is said to be ***bounded input - bounded output (BIBO) stable*** if and only if every bounded input produces the bounded output. It means, that there exist some finite numbers say M_x and M_y , such that

$$|x(n)| \leq M_x < \infty \Rightarrow |y(n)| \leq M_y < \infty$$

- for all *n*. If for some bounded input sequence *x(n)* , the output *y(n)* is unbounded (infinite), the system is classified as ***unstable***.

Testing for Stability or Instability

$$y[n] = (x[n])^2$$

is stable

if $\quad |x[n]| \le B_x < \infty, \quad \text{for all } n$

then $\quad |y[n]| \le B_y = B_x^2 < \infty, \quad \text{for all } n$

Testing for Stability or Instability

- Accumulator system $$y[n]=\sum_{k=-\infty}^{n} x[k]$$

$$x[n]=u[n]=\begin{cases}0 & n<0\\ 1 & n\geq 0\end{cases} \quad : bounded$$

$$y[n]=\sum_{k=-\infty}^{n} x[k]=\sum_{k=-\infty}^{n} x[k]=\begin{cases}0 & n<0\\ n+1 & n\geq 0\end{cases} \quad : not\ bounded$$

- Accumulator system is not stable

Characterization of Discrete Time System

- Any Discrete-time can be characterized by one of the following representations:
  1) Difference Equation
  2) Impulse Response
  3) Transfer Function
  4) Frequency Response

Difference Equation

- Continuous-time systems are characterized in terms of differential equations. Discrete-time systems, on the other hand, are characterized in terms of difference equations.
- A LTI system can be described by a linear constant coefficient difference equation of the form:

$$\sum_{i=0}^{M} b_i y(n-i) = \sum_{i=0}^{N} a_i x(n-i)$$

or

$$y(n) = \sum_{i=0}^{N} a_i x(n-i) - \sum_{i=1}^{M} b_i y(n-i)$$

- This equation describes a recursive approach for computing the current output, y(n) given the input values, x(n) and previously computed output values y(n-i). b_0 =1, M and N are called the order of the system, a_i and b_i are constant coefficients . Two types of discrete-time systems can be identified: nonrecursive and recursive.

Recursive vs. Non-recursive Systems

- **Definition**: A system whose output *y(n)* at time *n* depends on any number of the past outputs values (e.g. *y(n-1)*, *y(n-2)*, …), is called a ***recursive system***. Then, the output of a causal recursive system can be expressed in general as

$$y(n) = F[y(n-1), y(n-2), \dots, y(n-N), x(n), x(n-1), \dots, x(n-m)]$$

- where *F[.]* is some arbitrary function.
- In contrast, if *y(n)* at time *n* depends only on the present and past inputs then such a system is called ***Nonrecursive***.

$$y(n) = F[x(n), x(n-1), \dots, x(n-m)]$$

Examples:

- The Non-recursive system:

$$y(n) = \sum_{k=0}^{N} h(k)x(n-k)$$

- The recursive system:

$$y(n) = \sum_{k=0}^{N} b(k)x(n-k) - \sum_{k=1}^{N} a(k)y(n-k)$$

Solving the difference equation

- Example: compute impulse response of LTICS that described by the following Difference Equation.

$$y(n) - ay(n-1) = x(n)$$

- Solution:
  - Since the system is Causal-System, then $y(n) = 0 \quad for\ n < 0.$
  - The system input is $x(n) = \delta(n)$
  - Then the output can be calculated as follows:

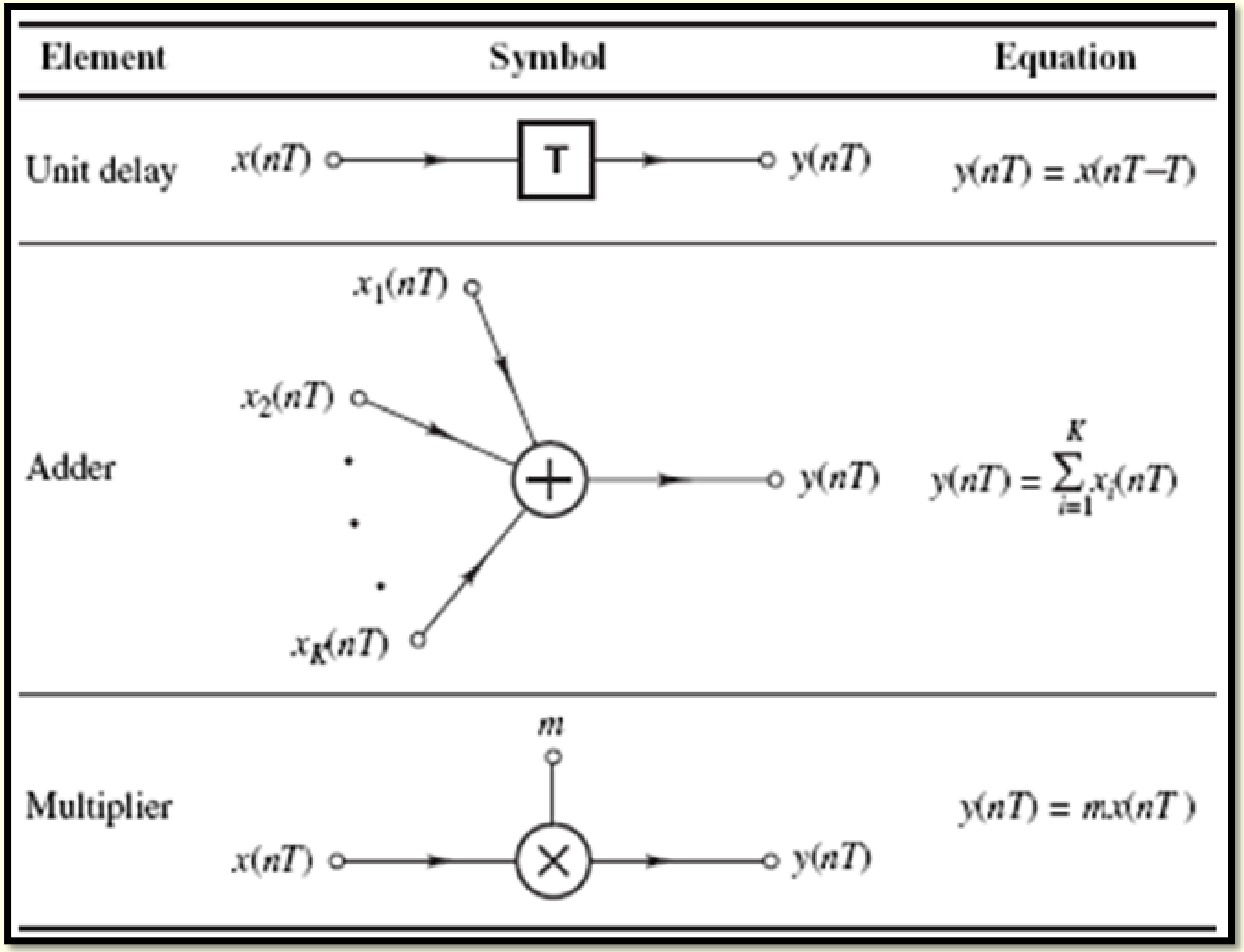
$$y(n) = ay(n-1) + \delta(n)$$

  - At n=0: $y(0) = ay(-1) + \delta(0) = 1 = a^0$
  - At n=1: $y(1) = ay(0) + \delta(1) = a = a^1$
  - At n=2: $y(2) = ay(1) + \delta(2) = a^2$
  - At n=3: $y(3) = ay(2) + \delta(3) = a^3$

$$Then\ the\ general\ form\ of\ the\ output\ y(n) = a^n \quad for\ n > 0$$

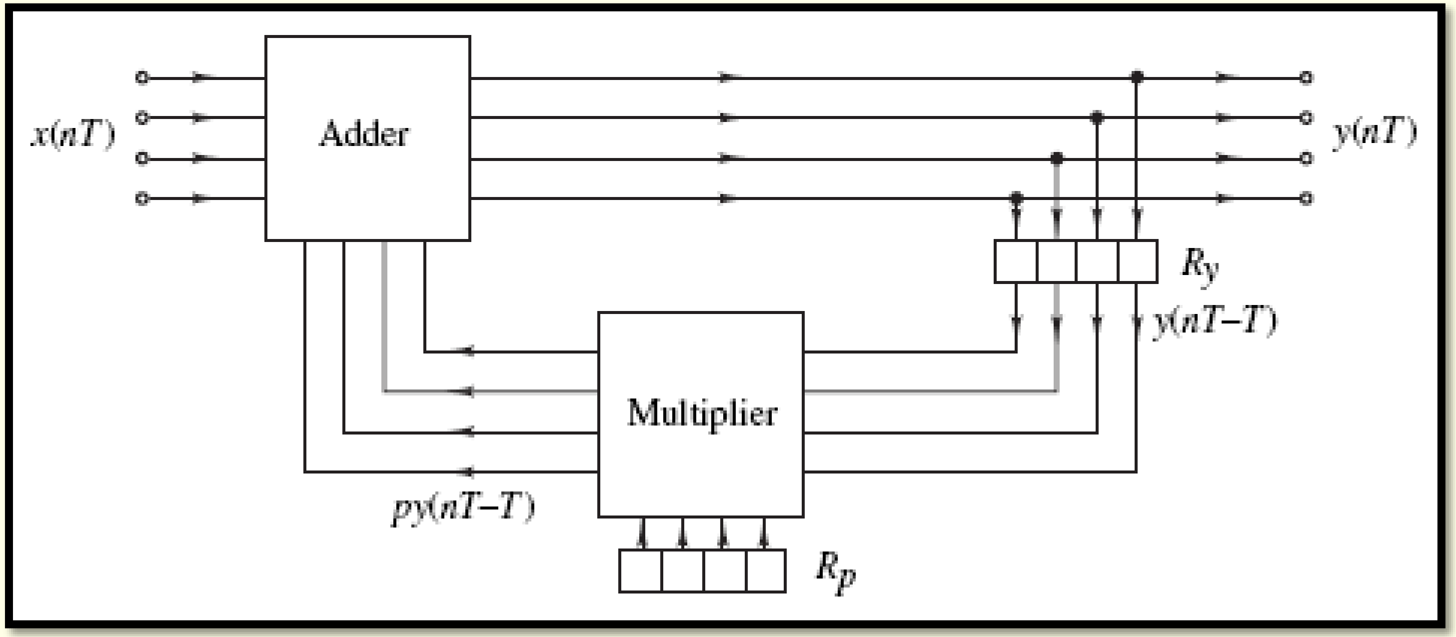
Discrete-Time system Networks

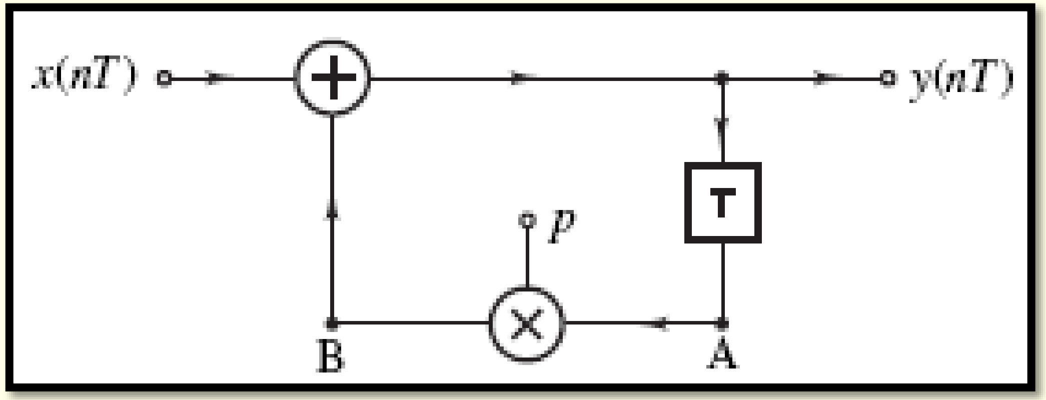
- The basic elements of discrete-time systems are the adder, the multiplier, and the unit delay.
- Ideally, the adder produces the sum of its inputs and the multiplier multiplies its input by a constant instantaneously.
- The unit delay, on the other hand, is a memory element that can store just one number. At instant (nT), in response to a synchronizing clock pulse, it delivers its content to the output and then updates its content with the present input.
- The device freezes in this state until the next clock pulse. In effect, on the clock pulse, the unit delay delivers its previous input to the output.



Example

- For the discrete-time system shown in the Figure, if the system is initially relaxed, that is, y(n) = 0 for n < 0, and p is a real constant, do the following:
  a) Derive the difference equation.
  b) Derive the impulse response, h(n).
  c) Check the system stability.
  d) Find the unit-step response of this system





Solution

a) From the Figure, the signals at node A is y(nT – T) and at node B is [p y(nT – T)], respectively. Thus, the difference equation has the form:

$$y(nT) = x(nT) + py(nT - T) \text{ or simply } y(n) = x(n) + py(n-1)$$

b) The impulse response, h(n) is the response of the system, y(n) when it is excited by input $x(n) = \delta(n)$. With $x(nT) = \delta(nT)$, we can write:

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & n \neq 0 \end{cases}$$

$y(n) = x(n) + p\,y(n-1)$

$y(n) = h(n) = \delta(n) + p\,y(n-1)$

$y(0) = h(0) = 1 + p\,y(-1) = 1 + 0 = 1$

$y(1) = h(1) = 0 + p\,y(0) = p$

$y(2) = h(2) = 0 + p\,y(T\,) = p^2$

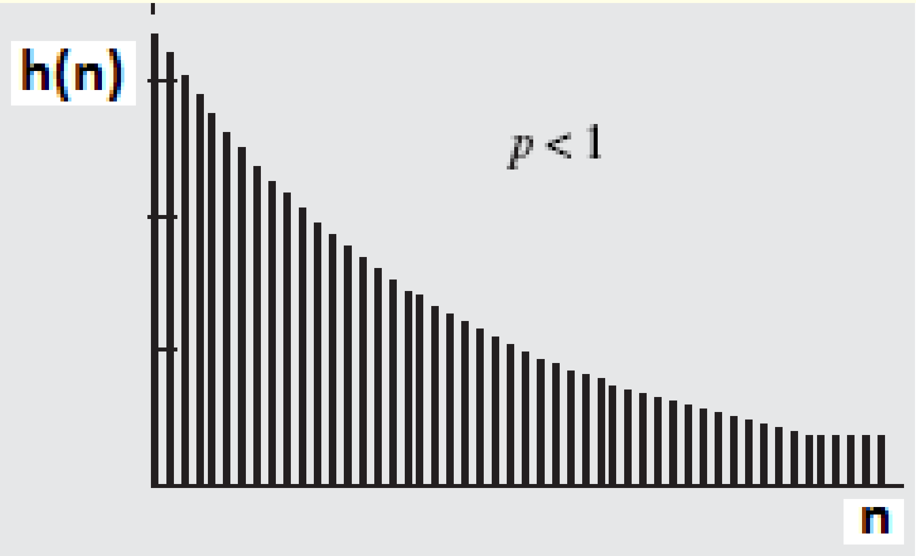
. . . .

$h(n) = p^n$

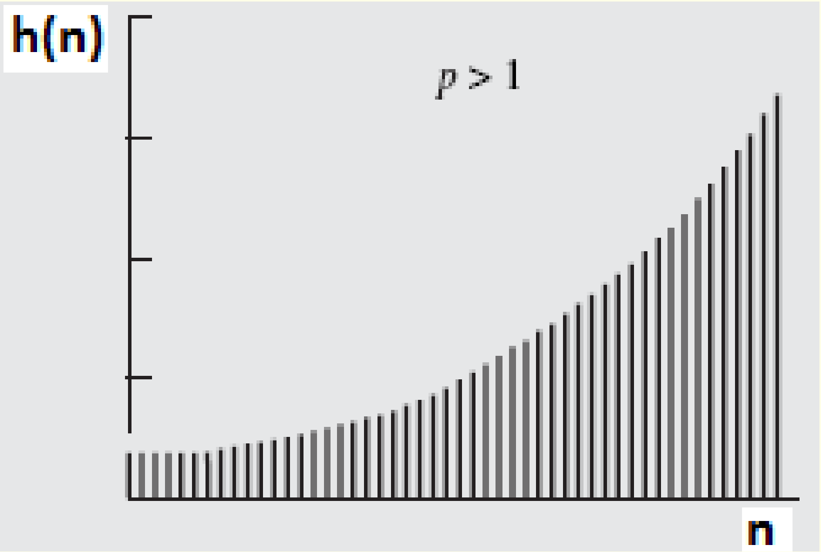
and since $y(n) = 0$ for $n \leq 0$, we have $\rightarrow h(n) = p^n\,u(n)$

Solution (Cont.)

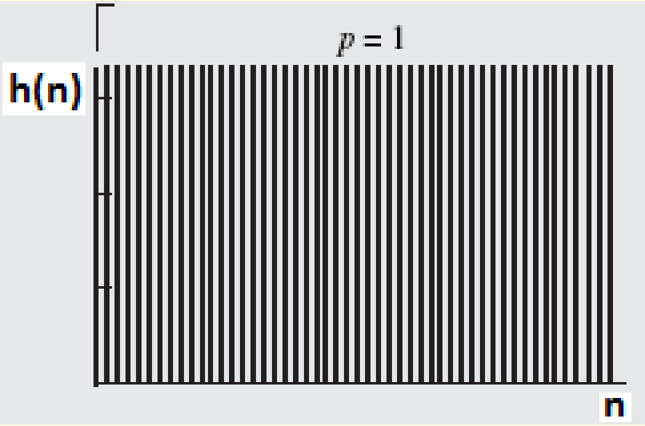
- The impulse response, $h(n)$ of this system is illustrated in the Figure.



(a) Stable system



(b) Unstable system



(c) Unstable system

Solution (Cont.)

c) check the stability: $\sum_{k=0}^{n} |h(k)| = \sum_{k=0}^{n} |p^k| = \frac{1-p^{(n+1)}}{1-p}$

For p< 1 and n $\to \infty$ the $p^{(n+1)}$) $\to$ 0 and $\sum_{k=0}^{\infty} |h(k)| = \sum_{k=0}^{\infty} |p^k| = \frac{1}{1-p} < \infty$

In this case this system is stable one.

d) With x(n) = u(n), we get: y(n) = x(n) + py(n – 1)

$y(0) = 1 + p\,y(-1) = 1$

$y(1) = 1 + p\,y(0) = 1 + p$

$y(2) = 1 + p\,y(1) = 1 + p + p^2$

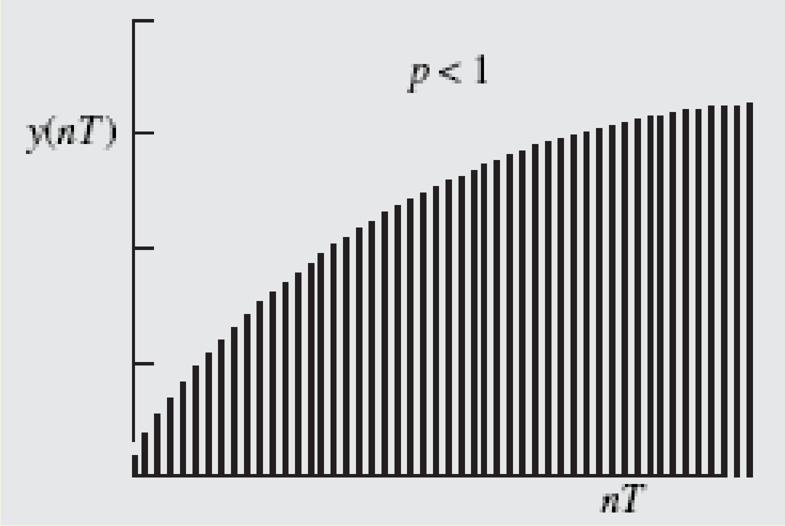
. . . .

$$y(n) = u(n) \sum_{k=0}^{n} p^k = u(n) \frac{1-p^{(n+1)}}{1-p}$$

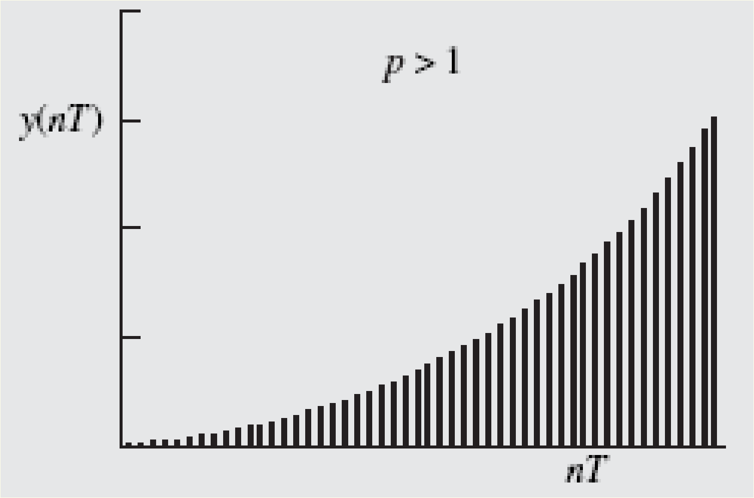
Solution (Cont.)

- The unit-step response for the three values of p is illustrated in the Figure. Evidently, the response converges if $p < 1$ and diverges if $p \geq 1$.

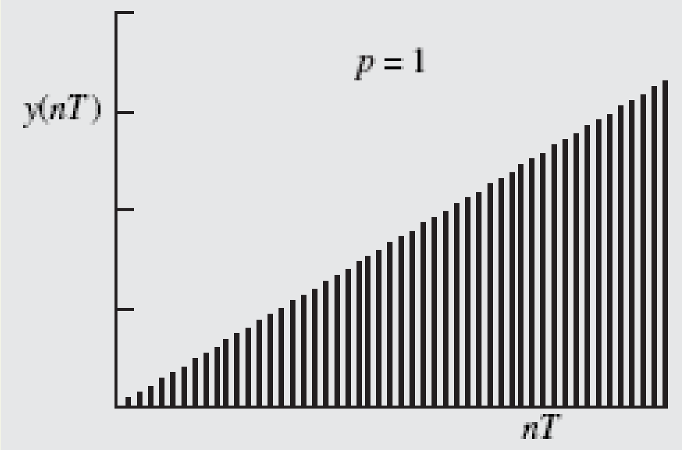
(a) P < 1



(b) P > 1



(c) P = 1



LINEAR TIME-INVARIANT CAUSAL SYSTEMS (LTIC SYSTEM)

Signal Process Systems

- A important class of systems

  Linear Shift-Invariant Systems.

- In particular, we'll discuss

  Linear Shift-Invariant Discrete-Time Systems.

Linear Shift-Invariant Systems

$$x(n) = \delta(n) \xrightarrow{\quad} \boxed{T[\,]} \xrightarrow{\quad} y(n) = T[x(n)] = h(n)$$

unit impulse

impulse response

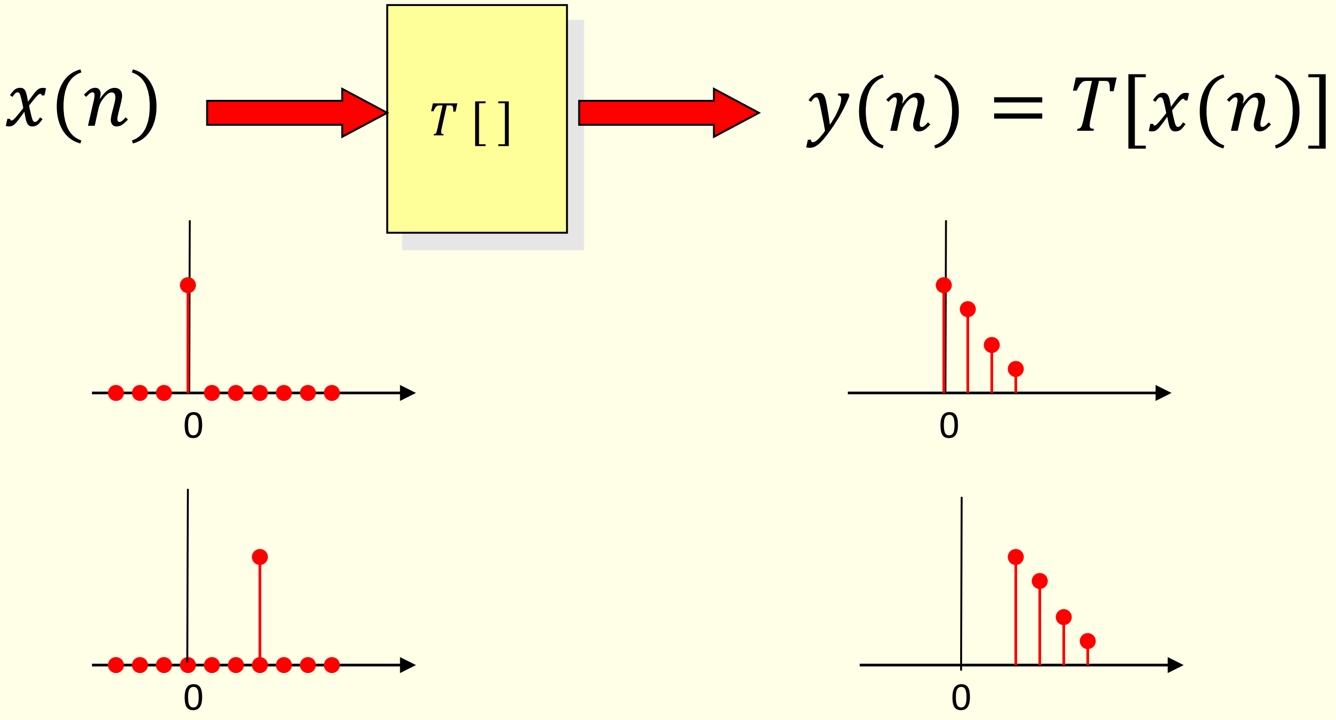
$$\boxed{x(n) = \sum_{k=-\infty}^{\infty} x(k)\ \delta(n-k)}$$

$$\boxed{y(n) = T\left[\sum_{k=-\infty}^{\infty} x(k)\ \delta(n-k)\right]}$$

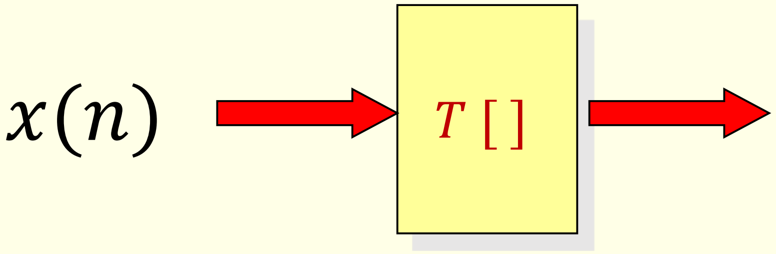
- LTI system description by **convolution** (convolution sum):

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\ \underbrace{T[\delta(n-k)]}_{\text{Time k impulse; The output value at time n}} = \sum_{k=-\infty}^{\infty} x(k)\ \underbrace{h(n-k)}_{\text{Only with the time difference}} = x(n) * h(n)$$

Impulse Response



Convolution Sum



$$y(n) = T[x(n)]$$

$$y(n) = x(n) * h(n)$$

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\ h(n-k) = \underbrace{x(n) * h(n)}_{\text{Convolution}}$$

A linear shift-invariant system is completely characterized by its impulse response.

Properties of Convolution (Cumulative)

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\ h(n-k) = x(n) * h(n)$$

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)\ x(n-k) = h(n) * x(n)$$

$$\boxed{x(n) * h(n) = h(n) * x(n)}$$

Properties of Convolution (Associative):

Cascaded Connection

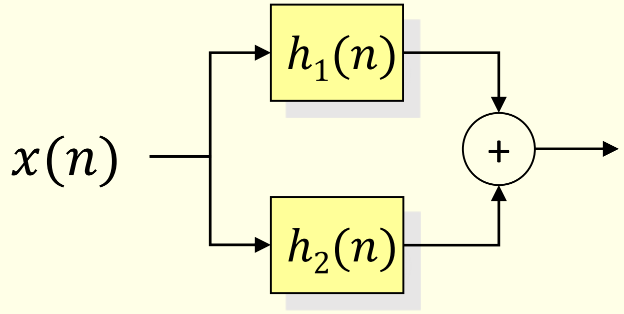
$$x(n) \rightarrow \boxed{h_1(n)} \rightarrow \boxed{h_2(n)} \rightarrow \qquad y(n) = x(n) * [h_1(n) * h_2(n)]$$

$$x(n) \rightarrow \boxed{h_2(n)} \rightarrow \boxed{h_1(n)} \rightarrow \qquad y(n) = x(n) * [h_2(n) * h_1(n)]$$

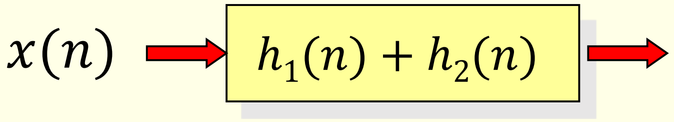
$$x(n) \rightarrow \boxed{h_1(n) * h_2(n)} \rightarrow \qquad y(n) = x(n) * h_1(n) * h_2(n)$$

These systems are identical.

Properties of Convolution (Distributive):
Parallel Connection



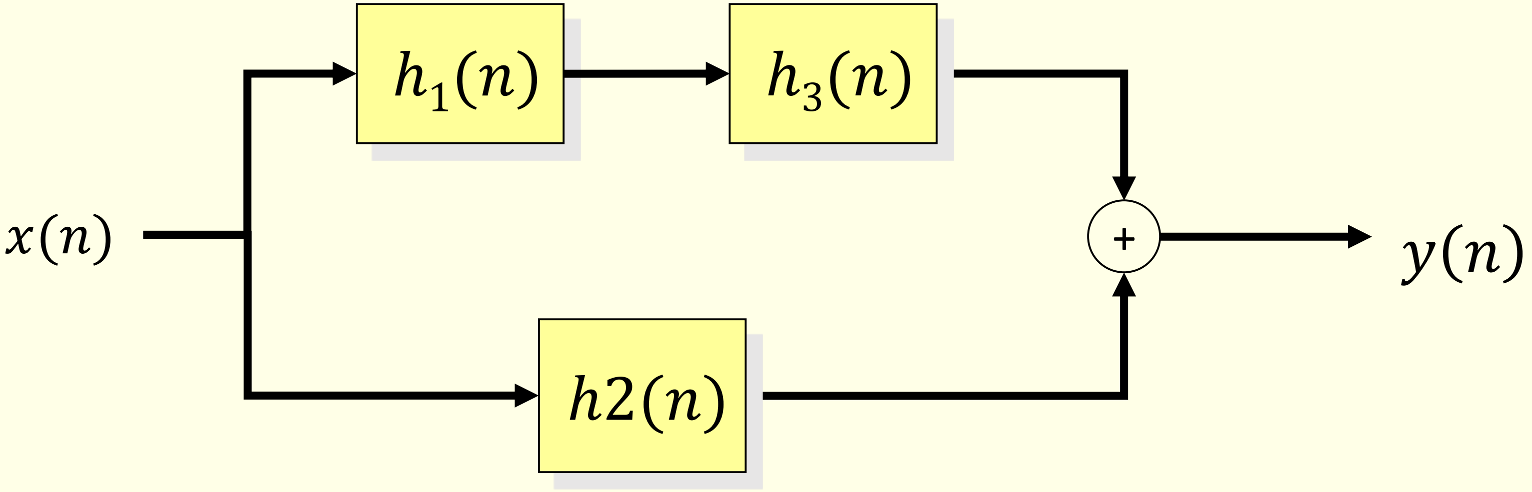
$$y(n) = x(n) * h_1(n) + x(n) * h_2(n)$$



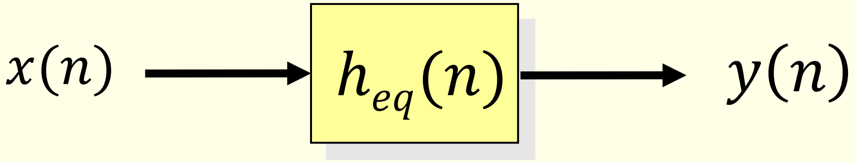
$$y(n) = x(n) * [h_1(n) + h_2(n)]$$

These two systems are identical.

Example: Parallel and Cascade



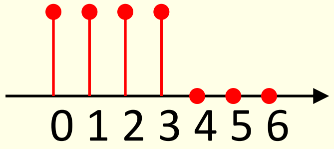
$$y(n) = x(n) * [h_1(n) * h_3(n) + h_2(n)]$$



$$h_{eq}(n) = [h_1(n) * h_3(n) + h_2(n)]$$

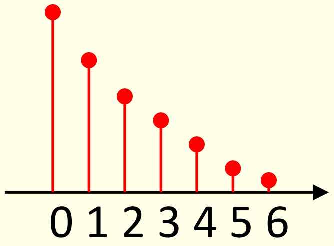
Example

$$x(n) = u(n) - u(n - N)$$



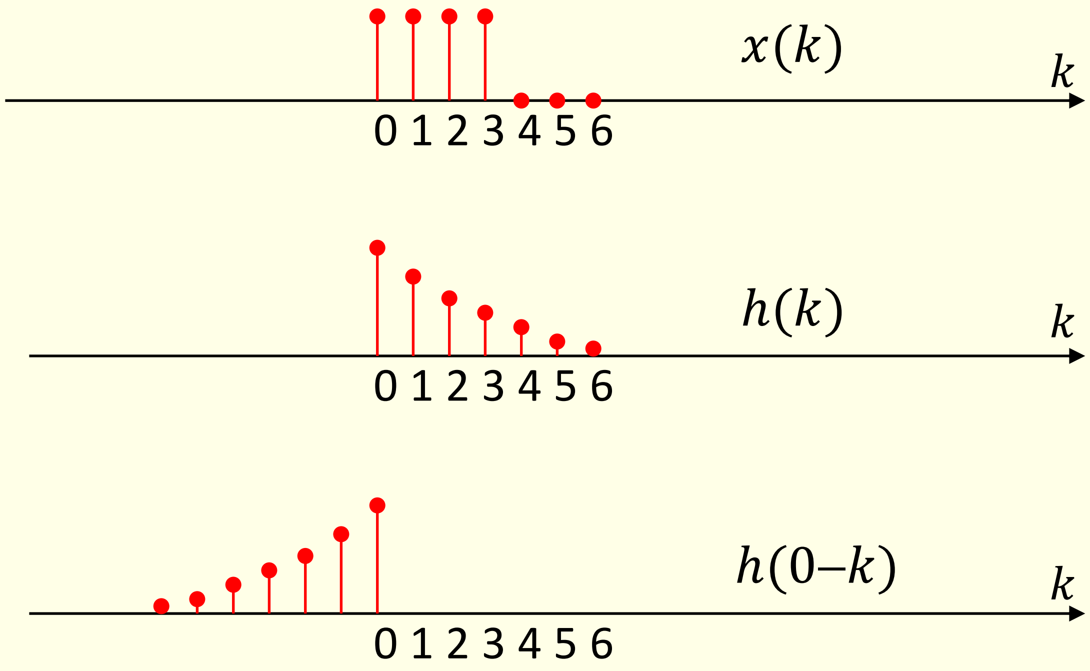
$$h(n) = \begin{cases} a^n & n \geq 0 \\ 0 & n < 0 \end{cases}$$

$$y(n) = ?$$



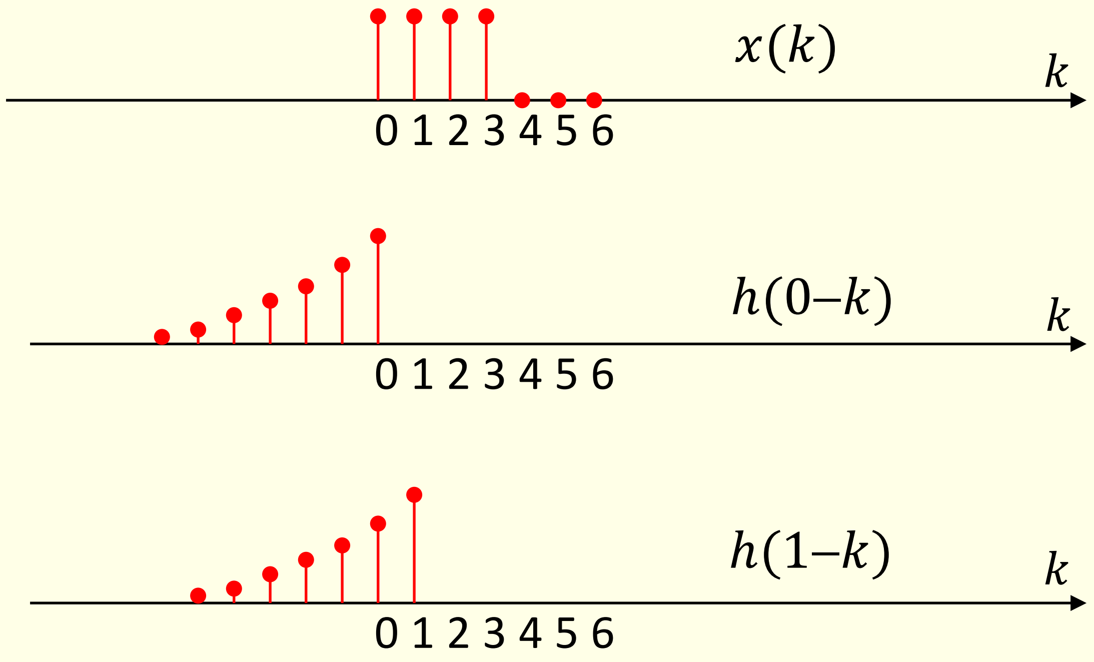
Example

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)\ h(n-k)$$



Example

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)\ h(n-k)$$



compute $y(0)$

compute $y(1)$

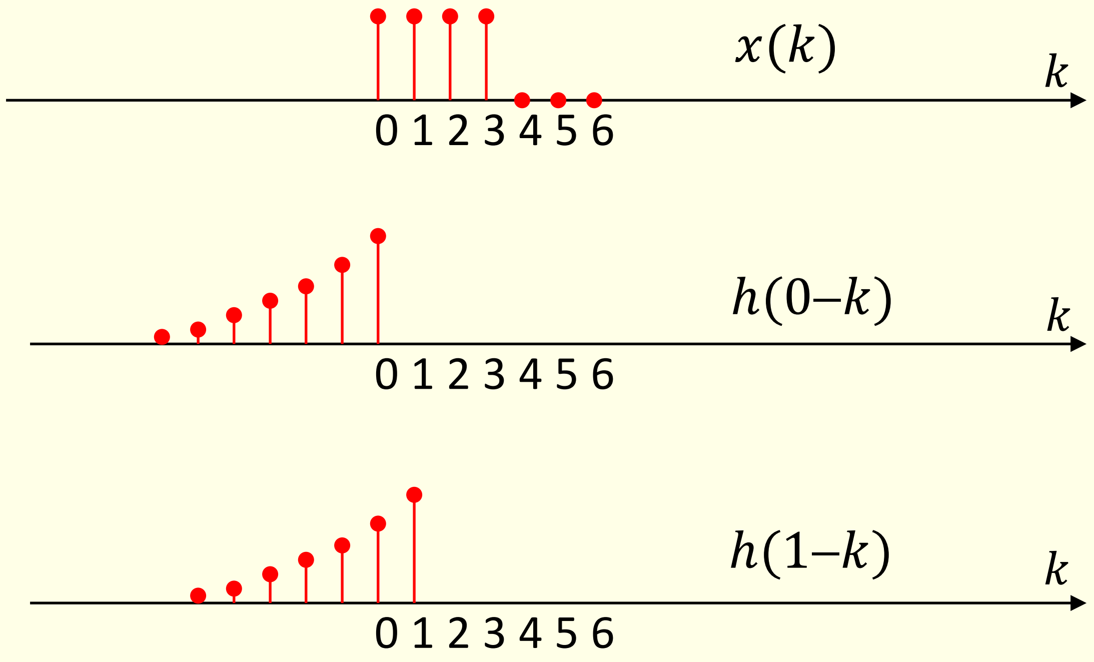
How to computer $y(n)$?

Example

Two conditions have to be considered.

n<*N* and *n*≥*N*.

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)\ h(n-k)$$



compute $y(0)$

compute $y(1)$

How to computer $y(n)$?

Ways to find D.T. Convolution

- Three ways to perform digital convolution
  - Graphical method
  - Table method
  - Analytical method

1- Graphical Method

- Example: Find the convolution of the two sequences x[n] and h[n] given by $\boldsymbol{x[n]} = [\underset{\uparrow}{\mathbf{3}}, \mathbf{1}, \mathbf{2}]$ and $\boldsymbol{h[n]} = [\underset{\uparrow}{\mathbf{3}}, \mathbf{2}, \mathbf{1}]$
- **Solution: On the Board**

2- Table Method

- Example: Find the convolution of the two sequences x[n] and h[n] given by $x[n] = [\underset{\uparrow}{3}, 1, 2]$ and $h[n] = [\underset{\uparrow}{3}, 2, 1]$

K	-2	-1	0	1	2	3	4	5		
x[k]:			3	1	2				y[n<0]:	0
h[0-k]:	1	2	3						y[0]:	9
h[1-k]:		1	2	3					y[1]:	9
h[2-k]:			1	2	3				y[2]:	11
h[3-k]:				1	2	3			y[3]:	5
h[4-k]:					1	2	3		y[4]:	2
h[5-k]:						1	2	3	y[n≥5]:	0

Example:

- Find the convolution of the two sequences x[n] and h[n] represented by,

$$x[n] = [2, \underset{\uparrow}{1}, -2, 3, -4] \quad \text{and} \quad h[n] = [\underset{\uparrow}{3}, 1, 2, 1]$$

- **Solution: On the Board**

3- Analytical Method (Example)

$$x(n)=u(n)-u(n-N) \rightarrow \boxed{h(n)=\begin{cases} a^n & n\geq 0 \\ 0 & n<0 \end{cases}} \rightarrow y(n)=?$$

$$y(n)=x(n)*h(n)=\sum_{k=-\infty}^{\infty} x(k)\; h(n-k)$$

For n < N: $$y(n)=\sum_{k=0}^{n} a^{n-k} = a^n\sum_{k=0}^{n} a^{-k} = a^n\,\frac{1-a^{-(n+1)}}{1-a^{-1}} = \frac{a^n-a^{-1}}{1-a^{-1}}$$

For n ≥ N: $$y(n)=\sum_{k=0}^{N-1} a^{n-k} = a^n\sum_{k=0}^{N-1} a^{-k} = a^n\,\frac{1-a^{-N}}{1-a^{-1}} = \frac{a^n-a^{n-N}}{1-a^{-1}}$$

3- Analytical Method (Example) Cont…

$$x(n) = u(n) - u(n-N) \rightarrow \boxed{h(n) = \begin{cases} a^n & n \ge 0 \\ 0 & n < 0 \end{cases}} \rightarrow y(n) = ?$$

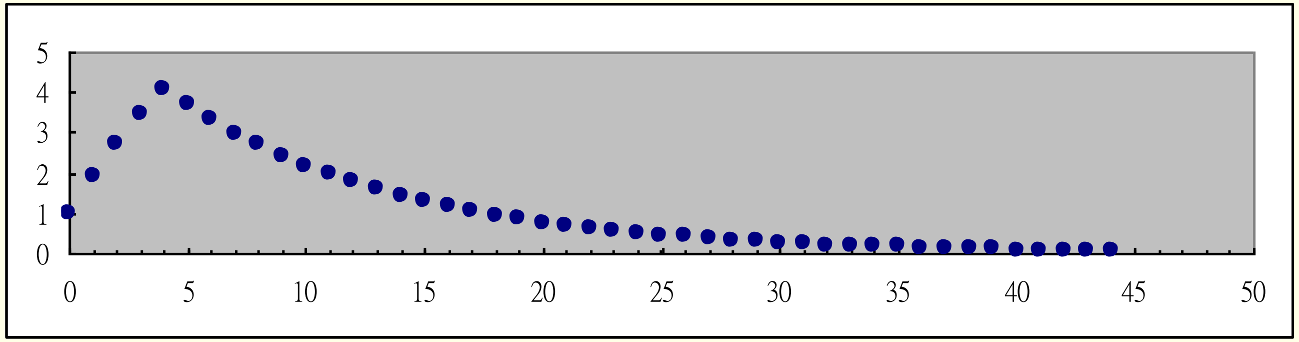
$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)\, h(n-k)$$

For n < N:

$$y(n) = \sum_{k=0}^{n} a^{n-k} = a^n \sum_{k=0}^{n} a^{-k} = a^n \frac{1-a^{-(n+1)}}{1-a^{-1}} = \frac{a^n - a^{-1}}{1-a^{-1}}$$

For n ≥ N:

$$y(n) = \sum_{k=0}^{N-1} a^{n-k} = a^n \sum_{k=0}^{N-1} a^{-k} = a^n \frac{1-a^{-N}}{1-a^{-1}} = \frac{a^n - a^{n-N}}{1-a^{-1}}$$



Example:

- Find the output $y[n]$ of a Linear, Time-Invariant system having an impulse response $h[n]$, when an input signal $x[n]$ is applied to it $h[n] = a^n u(n)$ *and* $x[n] = u(n)$, where $|a| < 1$.
- **Solution**:
- By definition of Convolution sum, the output y[n] is given as

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)\ h(n-k)$$

$$y(n) = \sum_{k=-\infty}^{\infty} a^k u(k).u(n-k) = \sum_{k=0}^{\infty} a^k .u(n-k)$$

$$y(n) = \sum_{k=0}^{n} a^k = \frac{1-a^{(n+1)}}{1-a}$$

Causal LTI Systems

- A relaxed LTI system is ***causal*** if and only if its impulse response is zero for negative values of *n* , i.e.

$$\boxed{h(n)=0 \quad \text{for } n<0}$$

- Then, the two equivalent forms of the convolution formula can be obtained for the causal LTI system:

$$y(n)=\sum_{k=0}^{\infty} h(k)x(n-k)=\sum_{k=-\infty}^{n} x(k)h(n-k)$$

Stable LTI Systems

- A LTI system is ***stable*** if its impulse response is absolutely summable, [i.e. every *bounded input* produce a *bounded output* (BIBO)]

$$S \stackrel{\Delta}{=} \sum_{k=-\infty}^{\infty} |h(k)| < \infty$$

Example:

- Show that the linear shift-invariant system with impulse response $h(n) = a^n\, u(n)$ where $|a| < 1$ is stable.

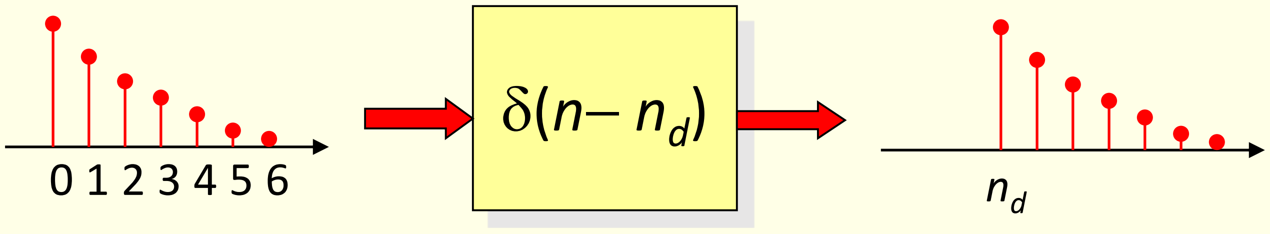
$$S = \sum_{k=0}^{\infty} |\, h(k)\,| = \sum_{k=0}^{\infty} a^k = \frac{1}{1-a} < \infty$$

Impulse Response of the Ideal Delay System

<u>**Ideal Delay System:**</u> $$y(n) = x(n - n_d)$$

By letting $x(n) = \delta(n)$ and $y(n) = h(n)$,

$$\Rightarrow \boxed{h(n) = \delta(n - n_d)}$$

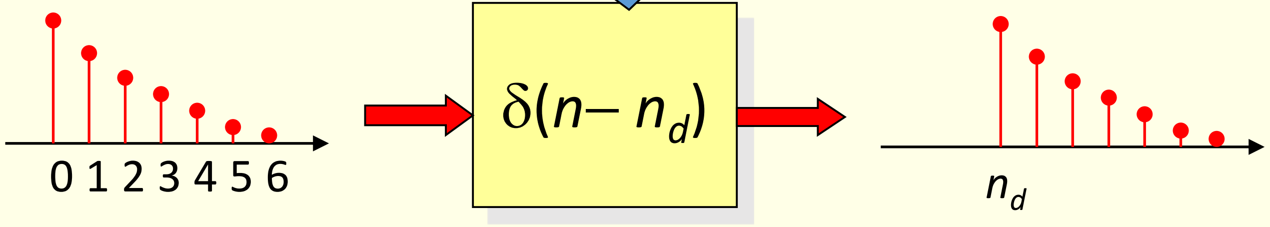


Impulse Response of the Ideal Delay System

You must know: $x(n) * \delta(n - n_d) = x(n - n_d)$

$\delta(\boldsymbol{n} - \boldsymbol{n_d})$ plays the following functions:

- Shift; or
- Copy

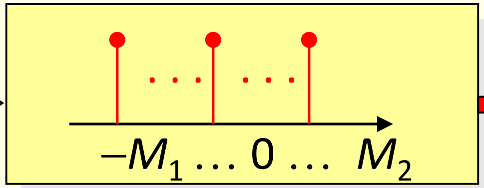


Impulse Response of the Moving Average

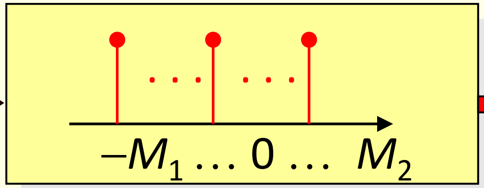
Moving Average: $$y(n) = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{k=M} x(n-k)$$

$$h(n) = \frac{1}{M_1 + M_2 + 1} \sum_{k=-M_1}^{M} \delta(n-k)$$

$$h(n) = \begin{cases} \dfrac{1}{M_1 + M_2 + 1} & M_1 \le n \le M_2 \\ 0 & otherwise \end{cases}$$



Can you explain with $\delta(n-k)$?



Can you explain with $u(n)$?

Forward Difference Vs Backward Difference

- Impulse response of Forward Difference

$$h[n] = \delta[n+1] - \delta[n]$$

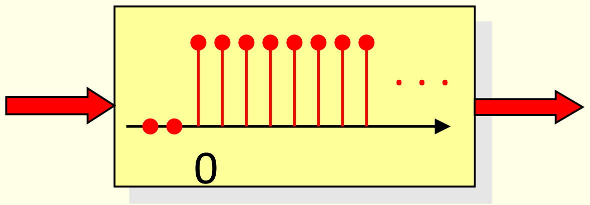
- Impulse response of Backward Difference

$$h[n] = \delta[n] - \delta[n-1]$$

Impulse Response of the Accumulator

Accumulator:
$$y(n)=\sum_{k=-\infty}^{n}x(k)$$

$$h(n)=\sum_{k=-\infty}^{n}\delta(k)==\begin{cases}1, & n\geq 0\\ 0, & n<0\end{cases}=u(n)$$



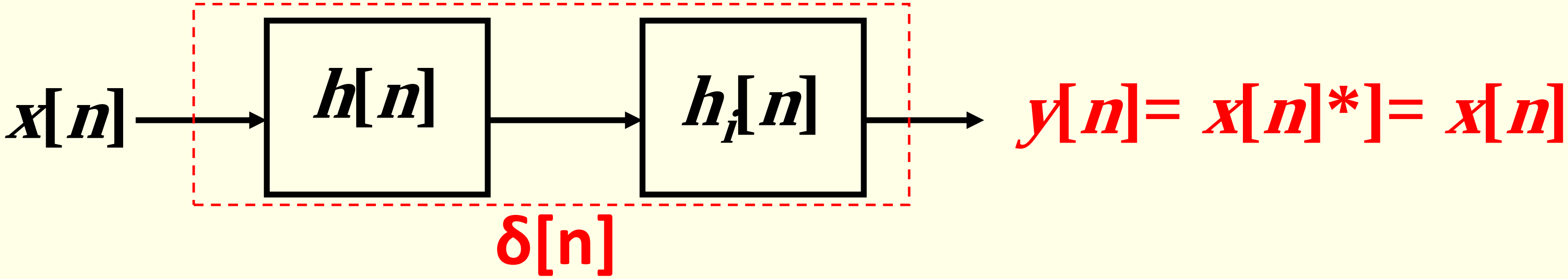
Can you explain?

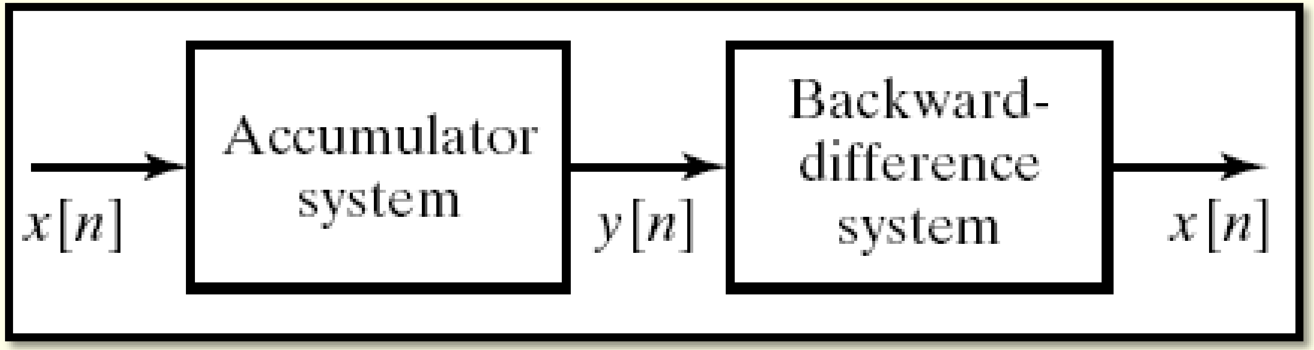
$$S=\sum_{n=-\infty}^{\infty}\left|u[n]\right|=\infty$$

Unstable system

Inverse system

$$h[n]*h_i[n]=h_i[n]*h[n]=\delta[n]$$





$$h[n]=u[n]*(\delta[n]-\delta[n-1])$$
$$=u[n]-u[n-1]=\delta[n]$$

Discrete-Time Frequency Response (DTFT)

FREQUENCY-DOMAIN REPRESENTATION OF DISCRETE-TIME SIGNALS AND SYSTEMS

Frequency Response

- DTFT, Discrete-Time Fourier Transform of $h(n)$ is

$$H\left(e^{jw}\right) = \sum_{n=-\infty}^{\infty} h[n]e^{-jwn},$$

- IDTFT, Inverse Discrete-Time Fourier Transform is

$$h(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(e^{j\omega})\, e^{j\omega n} d\omega$$

$x(n) \rightarrow \boxed{h(n)} \rightarrow y(n) = x(n) * h(n)$

$X(e^{jw}) \rightarrow \boxed{\mathrm{H}(e^{jw})} \rightarrow \mathrm{Y}(e^{jw}) = X(e^{jw}).H(e^{jw})$

Example: DTFT of The Ideal Delay System

$$\boxed{h(n) = \delta(n - n_d)}$$

$$Since,\ \ H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h(k) e^{-j\omega n} = \sum_{k=-\infty}^{\infty} \delta(k - n_d) e^{-j\omega n}$$

$$then,\ \ H(e^{j\omega}) = e^{-j\omega n_d}$$

The Magnitude: $|H(e^{j\omega})| = 1$

The phase: $\angle H(e^{j\omega}) = -\omega n_d$

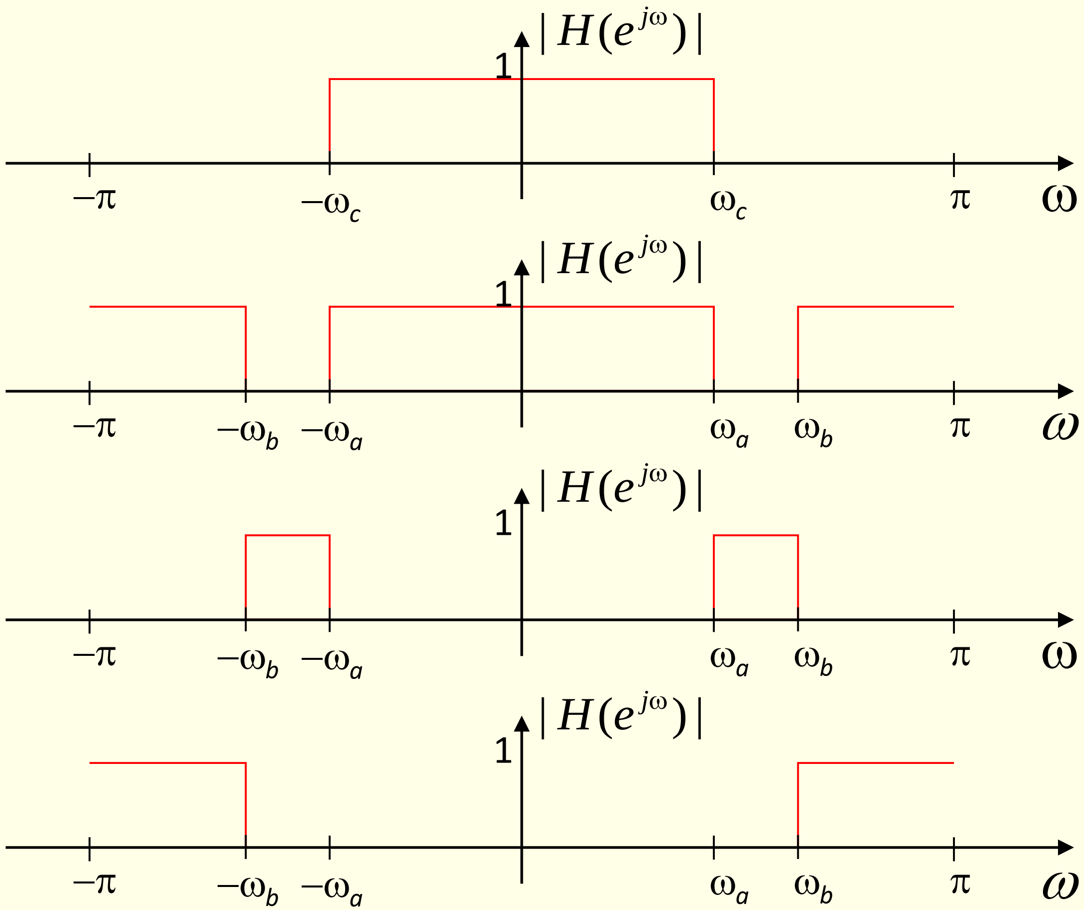
Ideal Frequency-Selective Filters

Lowpass Filter

Bandstop Filter

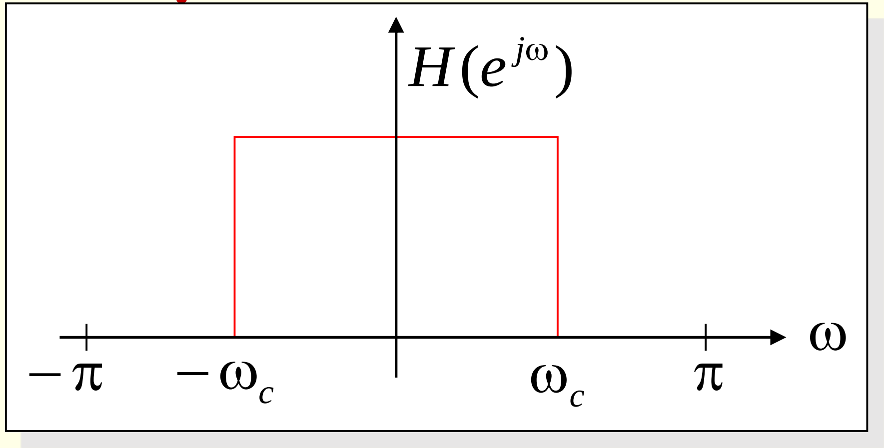
Bandpass Filter

Highpass Filter



Example: IDTFT of Ideal Lowpass Filter

$$H(e^{j\omega}) = \begin{cases} 1 & |\omega| < \omega_c \\ 0 & \omega_c < \omega \le \pi \end{cases}$$



$$h(n) = \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} H(e^{j\omega})e^{j\omega n}d\omega$$

$$= \frac{1}{2\pi}\int_{-\omega_c}^{\omega_c} e^{j\omega n}d\omega == \frac{1}{2j\pi n}\int_{-\omega_c}^{\omega_c} e^{j\omega n}d(j\omega n)$$

$$= \frac{1}{2j\pi n}e^{j\omega n}\Big|_{-\omega_c}^{\omega_c} = \frac{\sin\omega_c n}{\pi n}$$

Example

- Find the response of the ideal delay system $h(n) = \delta(n - n_d)$, if the input is $x(n) = \delta(n) + 2\delta(n-1)$.
- **Solution**:
  - By DTFT of $h(n)$:

$$H(e^{j\omega}) = \sum_{n=-\infty}^{\infty} h(n)\, e^{-j\omega n} = \sum_{n=-\infty}^{\infty} \delta(n - n_d)\, e^{-j\omega n} = e^{-j\omega n_d}$$

  - By DTFT of $x(n)$:

$$X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j\omega n} = \sum_{n=-\infty}^{\infty} [\delta(n) + 2\delta(n-1)]\, e^{-j\omega n}$$

$$X(e^{j\omega}) = 1 + 2e^{-j\omega}$$

  - The system output in frequency domain:

$$Y(e^{j\omega}) = X(e^{j\omega}).H(e^{j\omega}) = e^{-j\omega n_d} + 2e^{-j\omega(n_d+1)}$$

Example (Cont…..)

- By the IDTFT, The system output in Time domain:

$$y(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} Y\left(e^{j\omega}\right).\, e^{j\omega n}\,.\, d\omega$$

$$y(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left[e^{-j\omega n_d} + 2e^{-j\omega(n_d+1)}\right].\, e^{j\omega n}\,.\, d\omega$$

$$y(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} \left[e^{j\omega(n-n_d)} + 2e^{j\omega(n-n_d-1)}\right].\, d\omega$$

- Then, y(n) is

$$\boldsymbol{y(n) = \delta(n - n_d) + 2\delta(n - n_d - 1)}$$

Thank you for your attention