☐ Prove that the set of rational numbers $\mathbb{Q}$, equiped with the two binary operations of addition and multiplication forms a field.

Solⁿ: A set F with two binary operations + and $\cdot$ is a field if the following hold:

1. $(F, +)$ is an abelian (commutative) group:

(a) Closure under +,

(b) associativity of +,

(c) identity element 0,

(d) additive inverse

(e) commutativity of +

2. $(F \setminus \{0\}, \cdot)$ is an abelian group:

(a) closure under $\cdot$

(b) associativity of $\cdot$

(c) identity element 0

(d) inverse multiplicative for every non zero element.

(e) commutivity of $\cdot$

3. Distributivity: $x \cdot (y+z) = x \cdot y + x \cdot z$ for all $x, y, z \in F$

Finally, $0 \neq 1$ must hold (so the two identifiers are distinct)

Verification for $\mathbb{Q}$:

Every rational number can be written as $\frac{a}{b}$ with $a \in \mathbb{Z}$, $b \in \mathbb{Z} \setminus \{0\}$.

1. $(\mathbb{Q}, +)$ is an abelian group.

• Closure under addition:

if $x = a/b$ and $y = c/d$ then

$$x + y = \frac{a}{b} + \frac{c}{d} = \frac{ad+bc}{bd}$$

and $ad+bc$ and bd are integers with $bd \neq 0$

Thus $x+y \in \mathbb{Q}$

→ Associativity: addition of rationals is associative because it follows from associativity of integer addition

for rational x, y, z $(x+y)+z = x+(y+z)$.

→ Additive Identity: 0 satisfies $x+0=x$ for every rational x.

→ Additive inverse: for $x=a/b$, the additive inverse is $-x=\frac{-a}{b}$, which is rational and

- satisfies $x+(-x)=0$

→ Commutativity:

$$\frac{a}{b}+\frac{c}{d}=\frac{c}{d}+\frac{a}{b}$$

2. $(Q \setminus \{0\})$ is an abelian group.

- Closure under multiplication:

With $x=\frac{a}{b}$, $y=c/d$

$$x\cdot y=\frac{a}{b}\cdot\frac{c}{d}=\frac{ac}{bd}$$

and ac, bd are integer with $bd\neq 0$, so the product is in Q, If neither x nor y is zero then $ac\neq 0$, so the product is non zero.

- **Associativity**: Multiplication of rationals is associative.
- **Multiplicative identity**: 1 satisfy $1 \cdot x = x$ for all $x \in Q$
- Multiplicative inverse: For a non zero rational $x = \frac{a}{b}$ with $a \neq 0$, the inverse is b/a (an element of Q) and $\frac{a}{b} \cdot \frac{b}{a} = 1$

Commutitivity: $\frac{a}{b} \cdot \frac{c}{d} = \frac{c}{d} \cdot \frac{a}{b}$ because integer multiplication is commutative. Thus $(Q \setminus \{0\})$ is an abelian group.

3. **Distributivity**: For rationals $x = \frac{a}{b}, y = \frac{c}{d}, z = \frac{e}{f}$

$$x \cdot (y+z) = \frac{a}{b}\left(\frac{c}{d} + \frac{e}{f}\right) = \frac{a}{b} \cdot \frac{cf+de}{df} = \frac{acf}{bdf} + \frac{ade}{bdf}$$

$$= \frac{a}{b} \cdot \frac{c}{d} + \frac{a}{b} \cdot \frac{e}{f}$$

$$= x \cdot y + x \cdot z$$

4. $0 \neq 1$

In $\mathbb{Q}$, 0 is $0/1$ and 1 is $1/1$. There are different rationals so $0 \neq 1$, This prevents the degenerate one - element ring.

All field axioms hold for $\mathbb{Q}$: $(\mathbb{Q}, +)$ is a abelian group, $(\mathbb{Q} \setminus \{0\}, \cdot)$ is an abelian group multiplication distributes over addition, and $0 \neq 1$. Therefore $\mathbb{Q}$ with usual addition and multiplication is a field.