FUNMANAbstraction

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1 Stratification Abstraction

The SIERHD model from the July monthly demo uses the model summarized by the Petrinet diagram in Figure 1.

The following transitions connect the variables S_u, S_v, E_u, E_v, I_u, and I_v:

$$
\begin{aligned}
(I_u, S_u) &\xrightarrow{r_1} (I_u, E_u)\\
(I_u, S_v) &\xrightarrow{r_2} (I_u, E_v)\\
(I_v, S_u) &\xrightarrow{r_3} (I_v, E_u)\\
(I_v, S_v) &\xrightarrow{r_4} (I_v, E_v)\\
(S_u) &\xrightarrow{r_5} (S_v)\\
(S_v) &\xrightarrow{r_6} (S_u)\\
(E_u) &\xrightarrow{r_7} (I_u)\\
(E_v) &\xrightarrow{r_8} (I_v)
\end{aligned}
$$

Recovering the original, unstratified model corresponds to an abstraction where $S = (S_u, S_v)$, $I = (I_u, I_v)$, and $E = (E_u, E_v)$:

$$
\begin{aligned}
(I, S) &\xrightarrow{r_1} (I, E)\\
(I, S) &\xrightarrow{r_2} (I, E)\\
(I, S) &\xrightarrow{r_3} (I, E)\\
(I, S) &\xrightarrow{r_4} (I, E)\\
(S) &\xrightarrow{r_5} (S)\\
(S) &\xrightarrow{r_6} (S)\\
(E) &\xrightarrow{r_7} (I)\\
(E) &\xrightarrow{r_8} (I)
\end{aligned}
$$

In order for the abstraction to preserve the semantics of the stratified model, it must define $S^t = S^t_u + S^t_v$, $I^t = I^t_u + I^t_v$, and $E^t = E^t_u + E^t_v$ for all time points t. If we look at the definitions for these terms, we have:

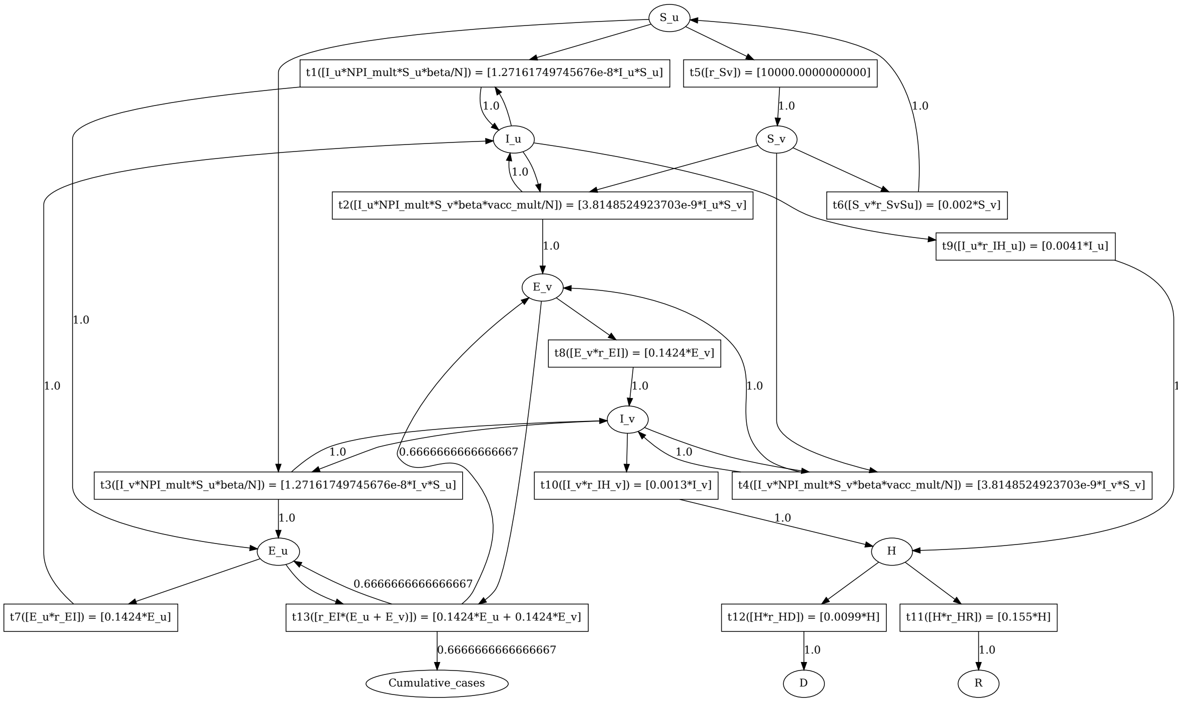

Figure 1: SEIRHD Model Petrinet

$$
\begin{aligned}
\frac{\partial S_u}{\partial t} &= -I_u S_u r_1 - I_v S_u r_3 - S_u r_5 + S_v r_6 \\
\frac{\partial S_v}{\partial t} &= -I_u S_v r_2 - I_v S_v r_4 + S_u r_5 - S_v r_6 \\
\frac{\partial S}{\partial t} &= \frac{\partial S_u}{\partial t} + \frac{\partial S_v}{\partial t} \\
&= -I_u S_u r_1 - I_v S_u r_3 - S_u r_5 + S_v r_6 - I_u S_v r_2 - I_v S_v r_4 + S_u r_5 - S_v r_6 \\
&= -I_u S_u r_1 - I_v S_u r_3 - I_u S_v r_2 - I_v S_v r_4
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial I_u}{\partial t} &= I_u S_u r_1 - I_u S_u r_1 + I_u S_v r_2 - I_u S_v r_2 + E_u r_7 \\
&= E_u r_7 \\
\frac{\partial I_v}{\partial t} &= I_v S_u r_3 - I_v S_u r_3 + I_v S_v r_4 - I_v S_v r_4 + E_v r_8 \\
&= E_v r_8 \\
\frac{\partial I}{\partial t} &= \frac{\partial I_u}{\partial t} + \frac{\partial I_v}{\partial t} \\
&= E_u r_7 + E_v r_8
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial E_u}{\partial t} &= I_u S_u r_1 + I_v S_u r_3 - E_u r_7 \\
\frac{\partial E_v}{\partial t} &= I_u S_v r_2 + I_v S_v r_4 - E_v r_8 \\
\frac{\partial E}{\partial t} &= \frac{\partial E_u}{\partial t} + \frac{\partial E_v}{\partial t} \\
&= I_u S_u r_1 + I_v S_u r_3 - E_u r_7 + I_u S_v r_2 + I_v S_v r_4 - E_v r_8
\end{aligned}
$$

Abstraction implies that we allow additional behaviors in the more abstract model (i.e., overapproximate). In Petrinet models, overapproximation corresponds to cases where the abstract compartment may take on additional values beyond those possible when aggregating the corresponding refined compartments.

$$
\begin{aligned}
\underline{\frac{\partial S_u}{\partial t}} &\geq -\overline{I_u S_u} r_1 - \overline{I_v S_u} r_3 - \overline{S_u} r_5 + \underline{S_v} r_6 \\
&= -N^2 r_1 - N^2 r_3 - N r_5 + 0 r_6 \\
&= -S_u^0 (N r_1 + N r_3 + r_5)
\end{aligned}
$$

$$
\begin{aligned}
\overline{\frac{\partial S_u}{\partial t}} &\leq -\underline{I_u S_u} r_1 - \underline{I_v S_u} r_3 - \underline{S_u} r_5 + \overline{S_v} r_6 \\
&= -00 r_1 - 00 r_3 - 0 r_5 + S_v^0 r_6 \\
&= S_v^0 r_6
\end{aligned}
$$

$$
\begin{aligned}
\underline{\frac{\partial S_v}{\partial t}} &\geq -\overline{I_u S_v} r_2 - \overline{I_v S_v} r_4 + \underline{S_u} r_5 + -\overline{S_v} r_6 \\
&= -N S_v^0 r_1 - N S_v^0 r_4 - 0 r_5 + S_v^0 r_6 \\
&= -S_v^0 (N r_1 + N r_4 + r_6)
\end{aligned}
$$

$$
\begin{aligned}
\overline{\frac{\partial S_v}{\partial t}} &\leq -\underline{I_u S_v} r_2 - \underline{I_v S_v} r_4 + \overline{S_u} r_5 - \underline{S_v} r_6 \\
&= -00 r_1 - 00 r_4 + S_u^0 r_5 - 0 r_6 \\
&= S_u^0 r_5
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial S}{\partial t} &\leq \overline{\frac{\partial S_u}{\partial t}} + \overline{\frac{\partial S_v}{\partial t}} \\
&\leq S_v^0 r_6 + S_u^0 r_5 \\
\frac{\partial S}{\partial t} &\geq \underline{\frac{\partial S_u}{\partial t}} + \underline{\frac{\partial S_v}{\partial t}} \\
&\geq -S_u^0 (N r_1 + N r_3 + r_5) - S_v^0 (N r_1 + N r_4 + r_6)
\end{aligned}
$$

$$
\begin{aligned}
\underline{\frac{\partial I_u}{\partial t}} &\geq \underline{I_u S_u} r_1 - \overline{I_u S_u} r_1 + \underline{I_u S_v} r_2 - \overline{I_u S_v} r_2 + \underline{E_u} r_7 \\
&= 00 r_1 - N S_u^0 r_1 + 00 r_2 - N S_v^0 r_2 + 0 r_7 \\
&= -N(S_u^0 r_1 + S_v^0 r_2)
\end{aligned}
$$

$$
\begin{aligned}
\overline{\frac{\partial I_u}{\partial t}} &\leq \overline{I_u S_u} r_1 - \underline{I_u S_u} r_1 + \overline{I_u S_v} r_2 - \underline{I_u S_v} r_2 + \overline{E_u} r_7 \\
&= N S_u^0 r_1 - 00 r_1 + N S_v^0 r_2 - 00 r_2 + N r_7 \\
&= N(S_u^0 r_1 + S_v^0 r_2 + r_7)
\end{aligned}
$$

$$
\begin{aligned}
\underline{\frac{\partial I_v}{\partial t}} &\geq \underline{I_v S_u} r_3 - \overline{I_v S_u} r_3 + \underline{I_v S_v} r_4 - \overline{I_v S_v} r_4 + \underline{E_v} r_8 \\
&= 00 r_3 - N S_u^0 r_3 + 00 r_4 - N S_v^0 r_4 + 0 r_8 \\
&= -N(S_u^0 r_3 + S_v^0 r_4)
\end{aligned}
$$

$$
\begin{aligned}
\overline{\frac{\partial I_v}{\partial t}} &\leq \overline{I_v S_u} r_3 - \underline{I_v S_u} r_3 + \overline{I_v S_v} r_4 - \underline{I_v S_v} r_4 + \overline{E_v} r_8 \\
&= N S_u^0 r_3 - 00 r_3 + N S_v^0 r_4 - 00 r_4 + N r_8 \\
&= N(S_u^0 r_3 + S_v^0 r_4 + r_8)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial I}{\partial t} &\leq \overline{\frac{\partial I_u}{\partial t}} + \overline{\frac{\partial I_v}{\partial t}} \\
&\leq N(S_u^0 r_1 + S_v^0 r_2 + r_7) + N(S_u^0 r_3 + S_v^0 r_4 + r_8) \\
&= N(S_u^0 r_1 + S_v^0 r_2 + r_7 + S_u^0 r_3 + S_v^0 r_4 + r_8) \\
\frac{\partial I}{\partial t} &\geq \underline{\frac{\partial I_u}{\partial t}} + \underline{\frac{\partial I_v}{\partial t}} \\
&\geq -N(S_u^0 r_1 + S_v^0 r_2 + S_u^0 r_3 + S_v^0 r_4)
\end{aligned}
$$

$$
\begin{aligned}
\underline{\frac{\partial E_u}{\partial t}} &\geq \underline{I_u S_u} r_1 + \underline{I_v S_u} r_3 - \overline{E_u} r_7 \\
&= 00 r_1 + 00 r_3 - N r_7 \\
&= -N r_7
\end{aligned}
$$

$$
\begin{aligned}
\overline{\frac{\partial E_u}{\partial t}} &\leq \overline{I_u S_u} r_1 + \overline{I_v S_u} r_3 - \underline{E_u} r_7 \\
&= N S_u^0 r_1 + N S_u^0 r_3 - 0 r_7 \\
&= N S_u^0 (r_1 + r_3)
\end{aligned}
$$

$$
\begin{aligned}
\underline{\frac{\partial E_v}{\partial t}} &\geq \underline{I_u S_v} r_2 + \underline{I_v S_v} r_4 - \overline{E_v} r_8 \\
&= 00 r_2 + 00 r_4 - N r_8 \\
&= -N r_8
\end{aligned}
$$

$$
\begin{aligned}
\overline{\frac{\partial E_v}{\partial t}} &\leq \overline{I_u S_v} r_2 + \overline{I_v S_v} r_4 - \underline{E_v} r_8 \\
&= N S_v^0 r_2 + N S_v^0 r_4 - 0 r_8 \\
&= N S_v^0 (r_2 + r_4)
\end{aligned}
$$

$$
\begin{aligned}
\frac{\partial E}{\partial t} &\leq \overline{\frac{\partial E_u}{\partial t}} + \overline{\frac{\partial E_v}{\partial t}} \\
&\leq N S_u^0 (r_1 + r_3) + N S_v^0 (r_2 + r_4) \\
\frac{\partial E}{\partial t} &\geq \underline{\frac{\partial E_u}{\partial t}} + \underline{\frac{\partial E_v}{\partial t}} \\
&\geq -N (r_7 + r_8)
\end{aligned}
$$

$$
\begin{aligned}
S^{t+dt} &= S^t + \frac{\partial S}{\partial t} dt \\
S^{t+dt} &\leq S^t + \overline{\frac{\partial S}{\partial t}} dt \\
&= S^t + (-\underline{I_u S_u} r_1 - \underline{I_v S_u} r_3 - \underline{S_u} r_5 + \overline{S_v} r_6) dt
\end{aligned}
$$

Assume that all compartments are population constrained. Use information about monotonicity.

$$
\begin{aligned}
\frac{\partial S_u}{\partial t} \leq 0, 0 \leq S_u \leq N \\
\frac{\partial S_v}{\partial t} \leq 0, 0 \leq S_v \leq N \\
0 \leq I_u \leq N \\
0 \leq I_v \leq N
\end{aligned}
$$