# Mandatory homework 5 - 6 Stat 220 - H20

October 08, 15: 1015 - 1200

Version-I-200823 2020-10-06-21-13-40

Due at the end of October 18

## Problem 6.1

$$\mathbb{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \end{array} \begin{array}{c} \begin{array}{ccc} 0 & 1 & 2 \end{array} \\ \begin{pmatrix} 0.7 & 0.2 & 0.1 \\ 0.0 & 0.6 & 0.4 \\ 0.5 & 0.0 & 0.5 \end{pmatrix} \end{array}.$$

a) Calculate $\mathbf{P}(X_3 = 1|X_0 = 0)$ and $\mathbf{P}(X_4 = 1|X_0 = 0)$. Hint: Save time and use R for these calculations.

b) Also explain why the chain is irreducible and aperiodic. Is it recurrent or transient?

## Problem 6.2

An urn initially contains a single red ball and a single green ball. A ball is drawn at random, removed, and replaced by a ball of the opposite color, and this process repeats so that there are always exactly two balls in the urn. Let $X_n$ be the number of red balls in the urn after n draws, with $X_0 = 1$.

a) Specify the transition probabilities for the Markov chain $\{X_n,\, n \geq 0\}$.

b) What is the period for the different states?

## Problem 6.3

a) Make a simple drawing of (1)

b) Which states are transient and which are recurrent in the Markov chain whose transition probability matrix is

$$\mathbb{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccccc} 0 & 1 & 2 & 3 & 4 & 5 \end{array} \\ \begin{pmatrix} \frac{1}{3} & 0 & \frac{1}{3} & 0 & 0 & \frac{1}{3} \\ \frac{1}{2} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ \frac{1}{4} & \frac{1}{4} & \frac{1}{4} & 0 & 0 & \frac{1}{4} \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix} \end{array}. \tag{1}$$

c) Find all communicating classes; which classes are transient and which are recurrent?
d) Express $\mathbb{P}$ when the state space is written as $\mathcal{S} = \{1,3,0,2,4,5\}$.
e) Let $T = \min\{n \geq 0\colon X_n \in \text{set of recurrent states}\}$. Find $\mathbb{P}_0(X_T = 5)$ and $\mathbb{E}_0 T$. Find $\mathbb{P}^\infty$ with R.

Problem 6.4
Let $\mathbb{P}$ be a Markov transition matrix

$$\mathbb{P} = \begin{pmatrix} P_{00} & P_{01} & \cdots & P_{0,N-1} & P_{0N} \\ P_{10} & P_{11} & \cdots & P_{0,N-1} & P_{1N} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ P_{i_0-1,0} & P_{i_0-1,1} & \cdots & P_{i_0-1,N-1} & P_{i_0-1,N} \\ P_{i_0,0} & P_{i_0,1} & \cdots & P_{i_0,N-1} & P_{i_0,N} \\ P_{i_0+1,0} & P_{i_0+1,1} & \cdots & P_{i_0+1,,N-1} & P_{i_0+1,N} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ P_{N0} & P_{N1} & \cdots & P_{N,N-1} & P_{NN} \end{pmatrix}$$

with state space $\mathcal{S} = \{0, \ldots, N\}$ and let $i_0$ be a fixed state. Define the taboo matrix, $\mathbb{R}$, which equals $\mathbb{P}$ except that the $i$th row vector is replaced by zeros. This means that $\mathbb{R} = \{R_{ij}\}$ with

$$R_{ij} = \begin{cases} 0, & \text{if } i = i_0; \\ P_{ij}, & \text{otherwise,} \end{cases} \tag{2}$$

which is

$$\mathbb{R} = \begin{pmatrix} P_{00} & P_{01} & \cdots & P_{0,N-1} & P_{0N} \\ P_{10} & P_{11} & \cdots & P_{0,N-1} & P_{1N} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ P_{i_0-1,0} & P_{i_0-1,1} & \cdots & P_{i_0-1,N-1} & P_{i_0-1,N} \\ 0 & 0 & \cdots & 0 & 0 \\ P_{i_0+1,0} & P_{i_0+1,1} & \cdots & P_{i_0+1,,N-1} & P_{i_0+1,N} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ P_{N0} & P_{N1} & \cdots & P_{N,N-1} & P_{NN} \end{pmatrix}.$$

We see that $\mathbb{R}$ is equal to $\mathbb{P}$ minus a matrix that consists of zeros all places except the $i_0$-th row that equals the $i_0$-th row of $\mathbb{P}$. This $i_0$-th row defines a probability measure on the state space, and we denote it by

$$\nu \stackrel{\text{def}}{=} \left(P_{i_0,0}, P_{i_0,1}, \ldots, P_{i_0,N}\right) = \{P_{i_0 j}, \quad j \in \mathcal{S}\} \quad \text{row vector.}$$

For building the simple matrix described we also need a column vector. Let $s = (0, \ldots, 0, 1, 0, \ldots, 0)' = e_{i_0}$, be the $i_0$-th unit vector. Then we get an expression for the wanted matrix,

$$s \otimes \nu \overset{\text{def}}{=} s\,\nu = \begin{pmatrix} 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \\ P_{i_0,0} & P_{i_0,1} & \cdots & P_{i_0,N-1} & P_{i_0,N} \\ 0 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{pmatrix}.$$

Thus $\mathbb{R} = \mathbb{P} - s \otimes \nu$.
As an example we use the rain transition matrix,

$$\mathbb{P} = \begin{array}{c} \\ 0 \\ 1 \\ 2 \\ 3 \end{array} \begin{array}{c} \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \begin{pmatrix} 0.62 & 0.00 & 0.38 & 0.00 \\ 0.55 & 0.00 & 0.45 & 0.00 \\ 0.00 & 0.24 & 0.00 & 0.76 \\ 0.00 & 0.19 & 0.00 & 0.81 \end{pmatrix} \end{array} = \begin{pmatrix} a & 0 & 1-a & 0 \\ b & 0 & 1-b & 0 \\ 0 & c & 0 & 1-c \\ 0 & d & 0 & 1-d \end{pmatrix} \tag{3}$$

and here we choose $i_0 = 0$.

a) Put the the matrix $\mathbb{P}$ in R and compute $s$, $\nu$ and $\mathbb{R} = \mathbb{P} - s \otimes \nu$ in R with $i_0 = 0$. (In R the state space is represented as $\{1, 2, 3, 4\}$ and $i_0 = 1$). Write

```
nu<-t(mP[1,])                   # The first row vector of  mP
mI<-diag(rep(1,4))              # unitmatrix of order 4
s<-mI[,1]                       # first unit vector, e_1, in R**4
msnu<-s%*%nu
mR<- mP-msnu
```

where `t()` is the function in R that transpose a matrix or a vector and `%*%` is matrix mulitplication. Print $\mathbb{P}$, $\mathbb{R}$, $s$, $\nu$ and $s \otimes \nu$.

b) Choose $M > 1$ and compute the approximations;

$$\begin{aligned} \widetilde{\mathbb{G}}_{s,\nu} &= \sum_{n=0}^{M} \mathbb{R}^n = \mathbb{I} + \mathbb{R} + \mathbb{R}^2 + \cdots + \mathbb{R}^M, \\ \widetilde{\pi}^{(s)} &= \nu\, \widetilde{\mathbb{G}}_{s,\nu}, \\ \widetilde{\pi} &= \widetilde{\pi}^{(s)} / \widetilde{\pi}^{(s)} 1, \end{aligned}$$

where $\widetilde{\pi}^{(s)} 1$ is a normalisation so that we get a probability. In R the normalisation looks like

```
wtpi<- pis/sum(pis)
```

Compare $\widetilde{\pi}$ with the exact $\pi$ for $M$ large enough. You find $\pi$ from (3).

c) Use your computer and compute the distribution of the return time for $i_0$.

PROBLEM 6.5

Let $\mathbb{P}$ be a finite transition matrix. Suppose that all $P_{ij}$'s are strictly positive, i.e. there is a $\delta > 0$ so that $P_{ij} \geq \delta$ for all $i, j$. We can write this shortly as $\mathbb{P} > 0$.

a) Show that this implies that $n$ step transition matrix is bounded below in the same way; $P_{ij}^{(n)} \geq \delta$ for all $n \geq 1$ and all $(i, j)$.
b) Suppose that $\mathbb{P}$ is finite, irreducible and aperiodic. Explian that for some finite $n_0$; $\mathbb{P}^{n_0} > 0$. Hint: Explain first, with reference to known theory, that $\mathbb{P}^n \to \mathbf{1} \otimes \pi$.

PROBLEM 6.6

Let $\mathbb{P}$ be irreducible with state space $\mathcal{S} = \{0, \ldots, N\}$ and periodic with period equal to $d$. Let $d = 2$. Define

$$E_0 = \big\{j\colon \sum_{h \geq 1} P_{0j}^{(hd)} > 0\big\}, \qquad E_1 = \big\{j\colon \sum_{h \geq 1} P_{0j}^{(hd+1)} > 0\big\}.$$

a) Explain that $E_0 \cup E_1 = \mathcal{S}$.
b) Prove that $E_0 \cap E_1 = \emptyset$.

With eventually a possible relabelling of the state space we can write the transition matrix as

$$\mathbb{P} = \begin{array}{c c} & \begin{array}{cc} E_0 & E_1 \end{array} \\ \begin{array}{c} E_0 \\ E_1 \end{array} & \begin{pmatrix} \mathbb{P}_{00} & \mathbb{P}_{01} \\ \mathbb{P}_{10} & \mathbb{P}_{11} \end{pmatrix} \end{array}.$$

c) What do you know about $\mathbb{P}_{ii}$ for $i = 0, 1$.
d) If $N$ is finite, is true that $|E_0| = |E_1|$? You may construct an example. Is $\mathbb{P}^2$ irreducible?

PROBLEM 6.7

Let $\mathbb{P}$ have finite state space of size $N + 1$. Suppose that $\mathbb{P}$ is irreducible. Let «$i$» and «$j$» be a fixed and different states.

a) Prove that $P_{ij}^{(n)} > 0$ for some $n \leq N$.
b) Prove that $P_{ii}^{(n)} > 0$ for some $n \leq N + 1$.
c) If $\mathbb{P}$ is periodic, explain that $d \leq N + 1$.

PROBLEM 6.8

Let $\mathbb{P}$ be a transition matrix that contains the classes $A$ and $B$.

a) Suppose $B$ is accessible for $A$. Prove that $A$ is transient.
b) Let $\{X_t, t \geq 0\}$ be a simple random walk on the non-negative integers with $p = P_{i,i+1} = 1 - P_{i,i-1} = 1 - q$ for $i > 0$ and with «0» as an absorbing barrier. Assume that $p > q > 0$. Classify the equivalence classes for this chain.