Oblig 5

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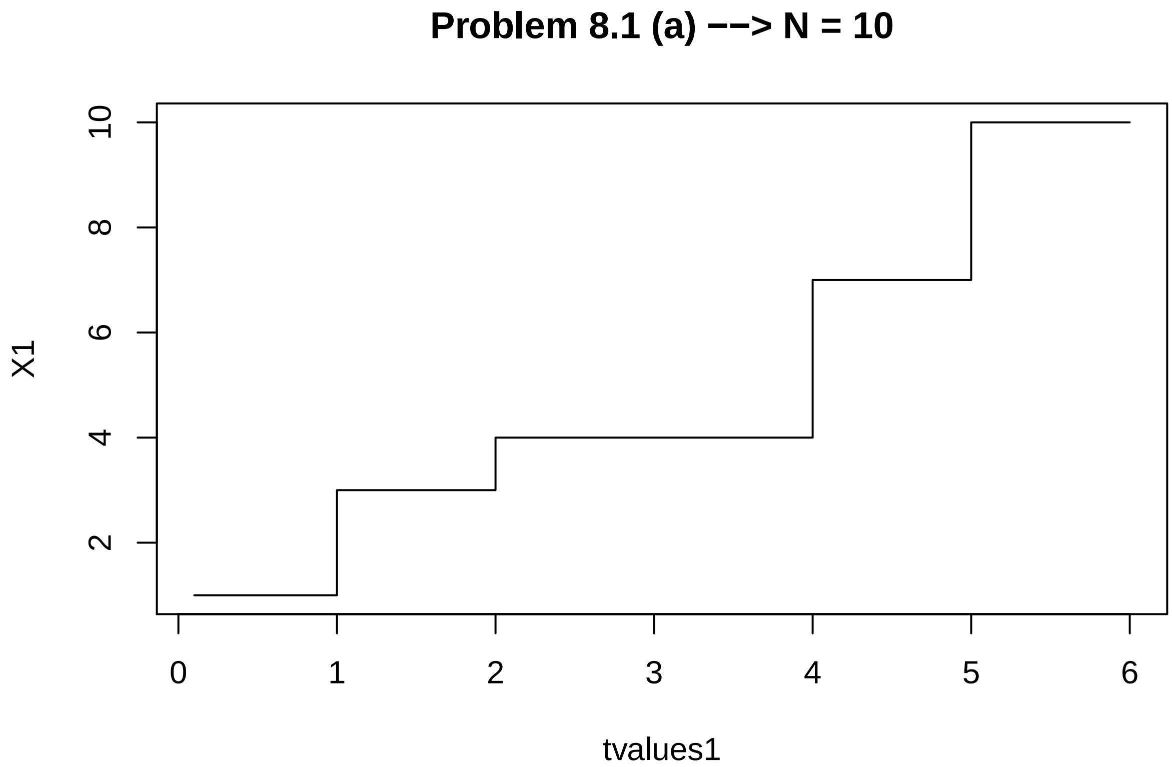
Problem 8.1

```
set.seed(123)
lb<-0.5
delta<-0.1
N<-c(10, 100, 1000)

T1<-rpois(N[1],lb)
S1<-rep(0,N[1])
S1[1]<-T1[1]

for(k in 2:N[1]) S1[k]<- S1[k-1] + T1[k]

tmax1<- S1[N[1]]+ 0.5/lb
tvalues1<-seq(delta, tmax1,delta)
ntvalues1<- length(tvalues1)
X1<-rep(0, ntvalues1)
for(t in 1: ntvalues1) X1[t]<- length( which( S1<=tvalues1[t] ))
plot(tvalues1, X1, type="s", main = 'Problem 8.1 (a) --> N = 10')
```

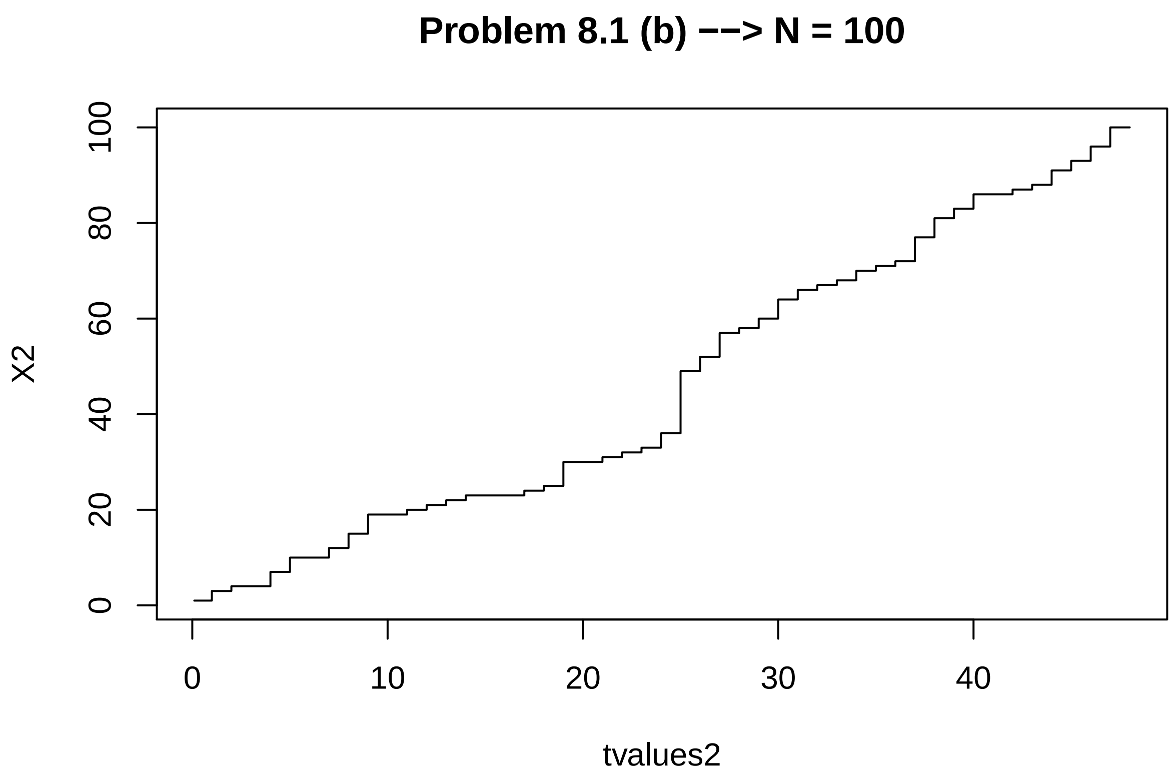

Problem 8.1 (a) --> N = 10
X1
10
8
6
4
2
0
1
2
3
4
5
6
tvalues1

```r
set.seed(123)
T2<-rpois(N[2],lb)
S2<-rep(0,N[2])
S2[1]<-T2[1]

for(k in 2:N[2]) S2[k]<- S2[k-1] + T2[k]

tmax2 <- S2[N[2]]+ 0.5/lb
tvalues2<-seq(delta, tmax2, delta)
ntvalues2<- length(tvalues2)
X2<-rep(0, ntvalues2)
for(t in 1: ntvalues2) X2[t]<- length( which( S2<=tvalues2[t] ))

plot(tvalues2, X2, type="s", main = 'Problem 8.1 (b) --> N = 100')
```



$$P(X(t)=k)=\frac{\lambda^k}{k!}e^{-\lambda}$$

Problem 8.3

Customers arrive acc. Poisson Process

rate $\lambda=2$

$X(t)$ number of customer arrived up to t.

a) $P(X(1)=2)=\frac{2^2}{2!}e^{-2}=2e^{-2}=\underline{0,2706}$

$$P(X(1)=2 \cap X(3)=6)$$

$$=P(X(1)=2)\cdot\frac{(3\cdot 2)^6}{6!}e^{-(3\cdot 2)}=\frac{6^5}{5!}\cdot e^{-6}$$

$$=0,2706\cdot 0,1606=\underline{\underline{0,0434}}$$

c)

$$P(X(3)=6 \mid X(1)=2) = \frac{P(X(3)=6 \cap X(1)=2)}{P(X(1)=2)}$$

$$= \frac{\cancel{0{,}2706} \cdot 0{,}1606}{\cancel{0{,}2706}}$$

$$= \underline{\underline{0{,}1606}}$$

Problem 8.4

$$\{X(t), t>0\} \qquad \lambda = 2$$

a) $E(X_i(t)) = \lambda \cdot t$

$\Rightarrow E(X(2)) = 2 \cdot 2 = \underline{\underline{4}}$

b) $E(X^2) = (E(X))^2 + V(X)$

$$E(X(1)^2) = (E(X(1)))^2 + V(X(1))$$

$$\downarrow \qquad\qquad \downarrow$$

$$(\lambda \cdot 1)^2 \quad + \quad \lambda \cdot 1$$

$$= (2 \cdot 1)^2 + \lambda \cdot 1$$

$$= 4 + 2 = \underline{\underline{6}}$$

c) Due to indipendence:

$$\mathrm{Cov}[X(1), X(2)] = 0$$

$$\Rightarrow E[X(1) \cdot X(2)] = E[X(1)] \cdot E[X(2)]$$

$$= 4 \cdot 2 = \underline{\underline{8}}$$

Problem 8.5

Arrival of passengers Pois $\{X(t), t > 0\}$

Rate : $\lambda = 2$ pr. unit time

Depart $t = 0$

Next Arrive at $T \to$ Passenges $X(T)$

$T \sim$ Uniform dist.

$$f_T(t) = \begin{cases} 1, & 0 < t < 1 \\ 0, & \text{otherwise} \end{cases}$$

a) $E[X^2(T) \mid T = t] = E[X(t)] = \lambda \cdot t = 2t$

$V[X(T) \mid T = t] = V[X(t)] = \lambda \cdot t = 2t$

$E[X(T) \mid T = t] = E[X^2(t)] = (E[X(t)])^2 + V[X(t)]$

$= (2t)^2 + 2t = 4t^2 + 2t = 2t(2t+1)$

b) Siden $T \sim U(0,1)$

$$\Rightarrow E(T) = \frac{1+0}{2} = \underline{\frac{1}{2}}$$

og

$$V(T) = \frac{1}{12} \cdot (1-0)^2 = \frac{1}{12}$$

$$\Rightarrow \quad E(X(t)) = \lambda \cdot E(T) = 2 \cdot \frac{1}{2} = \underline{\underline{1}}$$

$$V(X(t)) = \lambda \cdot V(T) = 2 \cdot \frac{1}{12} = \underline{\underline{\frac{1}{6}}}$$

Problem 8.6

$\{N(t),\ t \geq 0\}$ $\qquad i \leq n$

$s \leq t$

a)

$$P(N(t) = n \mid N(s) = i)$$

$$= P(N(t) - N(s) = n - i).$$

$$= P(N(t-s) = n-i)$$

$$= \frac{\lambda(t-s)^{(n-i)}}{(n-i)!} \cdot e^{-(\lambda(t-s))}$$

b) $i < n$ og $s < t$

$$P(N(s) = i \mid N(t) = n)$$

$$= \frac{P(N(s) = i) \cap P(N(t) - N(s) = n - i)}{P(N(t) = n)}$$

$$= \frac{(\lambda s)^i \cdot \cancel{e^{-(\lambda s)}} \cdot (\lambda(t-s))^{(n-i)} \cdot \cancel{e^{-(\lambda t)}} \cdot n!}{i! \cdot (n-i)! \cdot (\lambda t)^n \cdot \cancel{e^{-(\lambda(t-s))}}}$$

$$= \frac{(\lambda s)^i \cdot (\lambda t - \lambda s)^{(n-i)} \cdot n!}{i! \quad (\lambda t)^n \quad (n-i)!}$$

$$= \frac{\cancel{\lambda^i} \cdot s^i \cdot \cancel{\lambda^n} \cdot \cancel{\lambda^i} \cdot (t-s)^{n-i} \cdot n!}{\cancel{\lambda^n} \cdot t^n \cdot (n-i)!}$$

$$= \frac{s^i \cdot (t-s)^{n-i} \cdot n!}{i!\, t^n (n-i)!}$$

Problem 8.7

λ: freq of shocks

n: Events $n = \{1, 2, \ldots, k\}$

k: Shocks (k'th shock terminate device)

T: Lifetime of device

$$P(N(T) - N(t) = n) = \frac{(\lambda T)^n}{n!} e^{-(\lambda T)}$$

cdf of pmf will be

the sum:

$$F(t) = \sum_{i=0}^{k} \frac{(\lambda t)^n}{n!} e^{-(\lambda t)}$$

Problem 8.8

Radio active source emit:

$\lambda = 2$

Probability for occuring between

N_1: Ingen partikkel fra 0 → 3 min

N_2: Partikkel loast ut etter
mellom $T_1 = 3$ og $T_2 = 5$

Ventetiden er $\exp(\lambda, t)$

$$P(T_1 < t < T_2) = \int_{T_1}^{T_2} \lambda e^{-\lambda t}$$

$$= 2 \int_3^5 e^{-2t}\, dt$$

$$= 2 \int_3^5 e^u \, dt$$

$$u = -2t$$

$$\frac{du}{dt} = -2$$

$$= -\frac{1}{\cancel{2}} \cancel{2} \int_3^5 e^u \, du$$

$$dt = -\frac{1}{2}$$

$$= -\left[e^{-2t} \right]_3^5$$

$$= -\left[e^{-2 \cdot 5} - e^{-2 \cdot 3} \right] = e^{-6} - e^{-10}$$

$$\underline{\underline{= 2{,}43 \cdot 10^{-3}}}$$

b) Sanns. for en partikkel mellom 3 og 5 min er sanns. for 0 fra 0 til 3 og 1 fra 3 til 5. ved $\lambda = 2$

$\Rightarrow P(N(3)-N(0)=0 \cap N(5)-N(3))$

$$= \frac{(2\cdot 2)^1}{1!} \cdot e^{-2\cdot 2} = 4e^{-4}$$

$$\underline{\underline{\approx 0{,}073}}$$

Problem 8.4

customers enter acc. Pois ~ $\lambda = 6$

$$P(N(t+s) - N(s) = n) = \frac{(\lambda t)^n}{n!} e^{-(\lambda t)}$$

We shall find:

$$P(N(\tfrac{1}{4}) = 1 \mid N(1) = 1)$$

$$= \frac{P(\frac{1}{4} = 1) \cdot P(\frac{3}{4} = 0)}{P(N(1) = 1)}$$

$$= \frac{(6 \cdot \frac{1}{4})^1}{1!} e^{-6 \cdot \frac{1}{4}} \cdot \frac{(6 \cdot \frac{3}{4})^0}{0!} e^{-6 \cdot \frac{3}{4}} \cdot \frac{1!}{(6 \cdot 1)^1} \cdot e^{6 \cdot 1}$$

$$= \frac{\cancel{6}}{4} \cdot 1 \cdot \frac{1}{\cancel{6}} \cdot e^{-6(\frac{1}{4} + \frac{3}{4}) + 6}$$

$$\underline{\underline{= 1/4}}$$

Problem 8. 10

$\{X(t), t \geq 0\}$ $\quad \varepsilon = \lambda = 3$ (Pr. nr.)

$$P(X(1) = 2 \mid X(3) = 5)$$

$$= \frac{P(X(1) = 2 \cap X(3-1) = 3)}{P(X(3) = 5)}$$

$$= \frac{(3\cdot 1)^2}{2!} e^{-(3\cdot 1)} \cdot \frac{(3\cdot 2)^3}{3!} e^{-(3\cdot 2)} \cdot \frac{5!}{(3\cdot 3)^5} \cdot e^{(3\cdot 3)}$$

$$= \frac{9 \cdot 6^3 \cdot 5!}{9^5 \; 2! \cdot 3!} \cdot e^{-3 + (-6) + 9}$$

$$= \frac{6^3 \cdot 5 \cdot 4}{9^4 \cdot 2!} = \frac{80}{243} = \underline{\underline{0{,}329}}$$

Problem 8.12

$Pois(\lambda) \sim X(t) = n$

time pdf. for S_r for r'th. Arrival

$r \leq n$.

cdf: $P(S_r \leq t) = P(X(t) \geq n)$

$$= \sum_{r=n}^{\infty} \frac{(\lambda t)^r}{r!} \cdot e^{-\lambda t}$$

$$= \sum_{r=0}^{n-1} 1 - \frac{(\lambda t)^r}{r!} \cdot e^{-\lambda t} \quad \forall \quad r = \{1, 2, .., n\}$$

pdf: $\frac{d}{dt}\left[1-e^{-\lambda t}\left[1+\frac{\lambda t}{1!}+\frac{(\lambda t)^2}{2!}+\dots+\frac{(\lambda t)^{n-1}}{(n-1)!}\right]\right]$

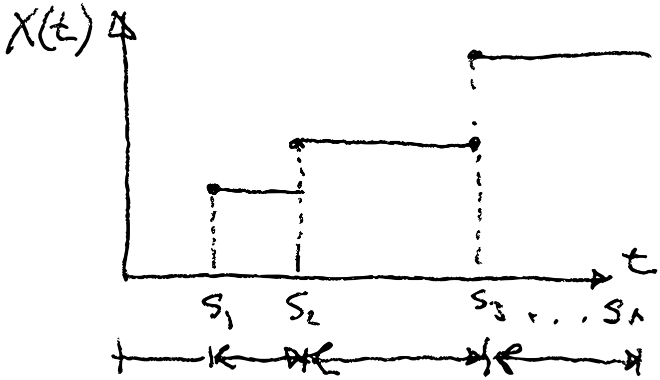
$$= \lambda e^{-\lambda t}\left[1+\frac{\lambda t}{1!}+\frac{(\lambda t)^2}{2!}+\dots+\frac{(\lambda t)^{n-1}}{(n-1)!}\right]$$

$$- e^{-\lambda t}\left[\lambda+\lambda\frac{(\lambda t)}{1!}+\lambda\cdot\frac{(\lambda t)^2}{2!}+\dots+\lambda\frac{(\lambda t)^{n-2}}{(n-2)!}\right]$$

$$= \frac{\lambda^n \cdot t^{n-1}}{(n-1)!}e^{-\lambda t}$$

$\boxtimes$

Problem 8.12



- Customers occur at time S_n also known as waiting times

  $X(t)$ count the number of events occurring at time t.

- We introduce independent RV $U_n \sim \text{Uni}(0, t)$

- and $R(.)$ symmetric function for n variables.

$$\Rightarrow E(\,\ell(S_1, S_2, \ldots, S_n \mid X(t) = n\,)$$
$$= E(\,\ell(U_1, U_2, \ldots, U_n)\,.$$

- $U_1, U_2, \ldots, U_n$ is the order statistics such that U_n and S_n have the same distribution.

- $E(U_1) = E(U_2) = \ldots = E(U_n) = \frac{t}{\lambda}$

$$\Rightarrow E(S_1, S_2, \ldots, S_n \mid X(t) = S)$$
$$= E(U_1 + U_2 + \ldots + U_n)$$
$$= \underline{n \cdot \frac{t}{2}} \longrightarrow \text{Mean waiting time}$$

$\blacksquare$

Problem 8.13

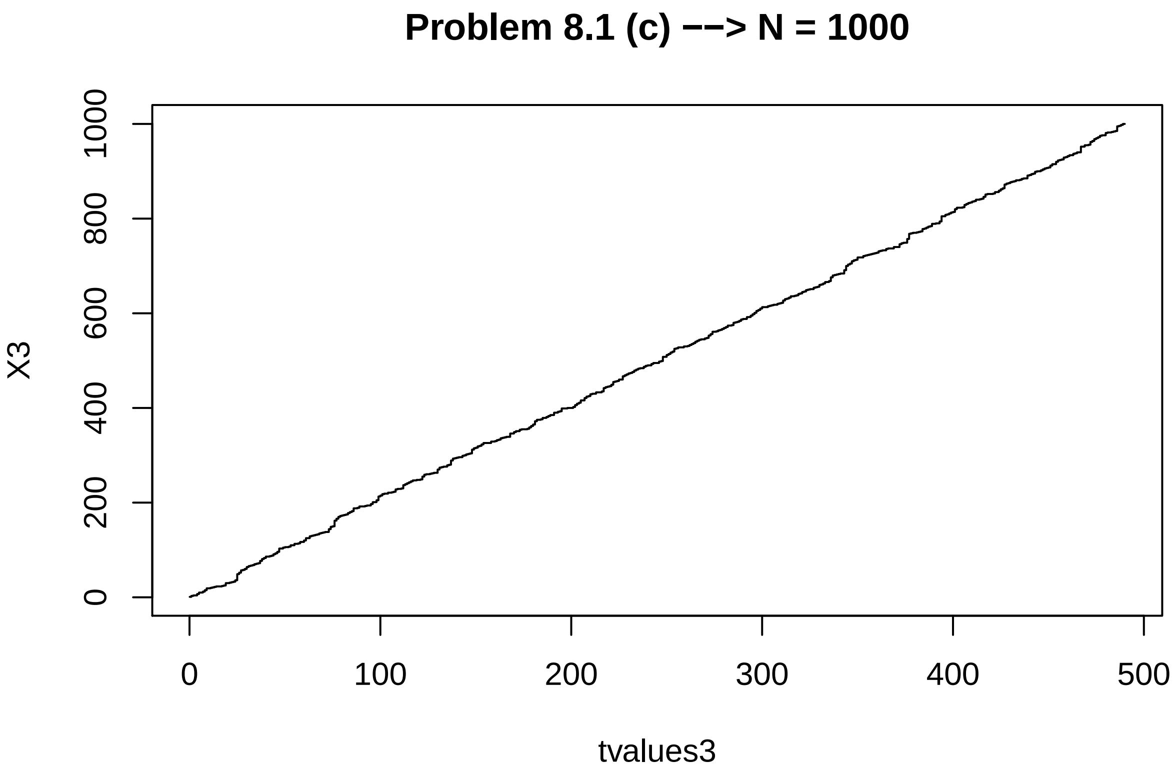
Og så langt kom en

denne gang...

```r
set.seed(123)
T3<-rpois(N[3],lb)
S3<-rep(0,N[3])
S3[1]<-T3[1]

for(k in 2:N[3]) S3[k]<- S3[k-1] + T3[k]

tmax3 <- S3[N[3]]+ 0.5/lb
tvalues3<-seq(delta, tmax3,delta)
ntvalues3<- length(tvalues3)
X3 <-rep(0, ntvalues3)
for(t in 1: ntvalues3) X3[t]<- length( which( S3<=tvalues3[t] ))
plot(tvalues3, X3, type="s", main = 'Problem 8.1 (c) --> N = 1000')
```

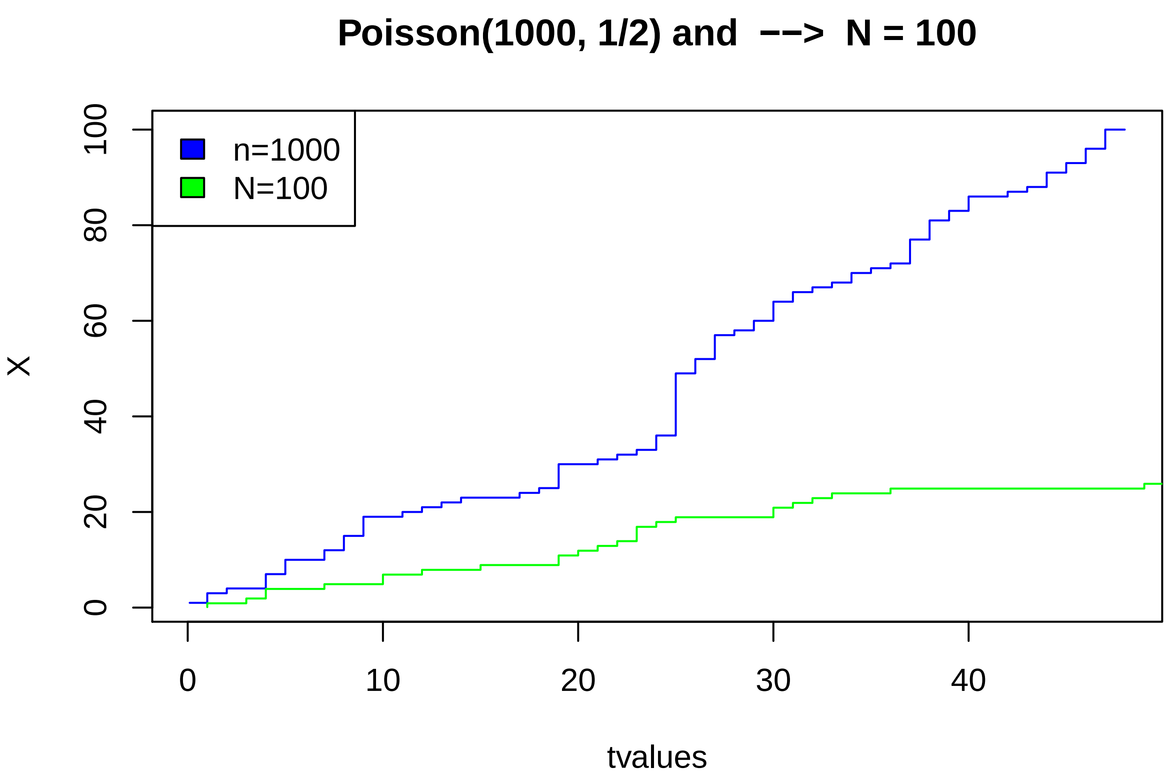


d)

```r
set.seed(123)
POI<-rpois(n=1000 ,lambda=lb)
S <-rep(0,N[2])
S[1]<-POI[1]
for(k in 2:N[2]) S[k]<- S[k-1] + POI[k]

tmax<- S[N[2]]+ 0.5/lb
tvalues<-seq(delta, tmax,delta)
ntvalues<- length(tvalues)
X<-rep(0, ntvalues)
for(t in 1: ntvalues) X[t]<- length( which( S<=tvalues[t] ))
plot(tvalues, X, type="s",col= 'blue', main = 'Poisson(1000, 1/2) and  -->  N = 100')
lines(y=tvalues2, X2, type="s", col= 'green')

legend("topleft",
c("n=1000","N=100"),
fill=c("blue","green")
)
```



Problem 8.2

a) The probability for a poisson distributed random variables is given by:

$P(X = k) = \frac{\lambda^k}{k!} \cdot exp(-\lambda)$

It is known that the mean rate is 3 messages pr hour such that
$E(X) = \lambda = 3$ and the probability that X is 0 afer 1 hour is:

$P(X = 0) = \frac{3^0}{0!} \cdot exp(-3) = 0.0497$

From $08 : 00 \rightarrow 12 : 00$ it is 4 hours and since poisson is indipendent there is 4 random variables $\{X_1, X_2, X_3, X_4\}$ wich gives us probability that X is 0 from $08 : 00 \rightarrow 12 : 00$:

$P((X_1 = 0) \cap (X_2 = 0) \cap (X_3 = 0) \cap (X_4 = 0))$
$= (\frac{3^0}{0!} \cdot e^{-3})^4 = (e^{-3})^4$
$= e^{-12} = \underline{\underline{6.14 \cdot 10^{-6}}}$

b) Since events is poisson distributed with time intervals $\{X_1, X_2, ..., X_n\}$ hrs, and rate $E(X_n) = \lambda$ of 3 with non overlaping events, the distribution is exponential with expnential parameter λ:

$f(x) = \lambda \cdot e^{-\lambda X} = 3 \cdot e^{-3 \cdot X}$

```r
x <- seq(0, 4, 0.1)
plot(x, dexp(x, 3), type = "l",
     ylab = "", lwd = 3, col = "red")
legend("topright", c(expression(paste(, lambda)), "3"),
       lty = c(0, 1), col = c("red"), box.lty = 0, lwd = 2)
```