$P(X(t)=k) = \frac{\lambda^k}{k!} e^{-\lambda}$

Problem 8.3

Customers arrive acc. Poisson process

rate $\lambda = 2$

$X(t)$ number of customer arrived up to t.

a) $P(X(1)=2) = \frac{2^2}{2!} e^{-2} = 2e^{-2} = \underline{0{,}2706}$

$P(X(1)=2 \cap X(3)=6)$

$= P(X(1)=2) \cdot \frac{(3\cdot 2)^6}{6!} e^{-(3\cdot 2)} = \frac{6^5}{5!} \cdot e^{-6}$

$= 0{,}2706 \cdot 0{,}1606 = \underline{\underline{0{,}0434}}$

c)

$$P(X(3)=6 \mid X(1)=2) = \frac{P(X(3)=6 \cap X(1)=2)}{P(X(1)=2)}$$

$$= \frac{\cancel{0{,}2706} \cdot 0{,}1606}{\cancel{0{,}2706}}$$

$$= \underline{\underline{0{,}1606}}$$

Problem 8.4

$$\{X(t), t > 0\} \qquad \lambda = 2$$

a) $E(X_i(t)) = \lambda \cdot t$

$\Rightarrow E(X(2)) = 2 \cdot 2 = \underline{\underline{4}}$

b) $E(X^2) = (E(X))^2 + V(X)$

$$E(X(1)^2) = (E(X(1)))^2 + V(X(1))$$

$$\downarrow \qquad \downarrow$$

$$(\lambda \cdot 1)^2 + \lambda \cdot 1$$

$$= (2 \cdot 1)^2 + \lambda \cdot 1$$

$$= 4 + 2 = \underline{\underline{6}}$$

c) Due to indipendence:

$$\mathrm{Cov}[X(1), X(2)] = 0$$

$$\Rightarrow E[X(1) \cdot X(2)] = E[X(1)] \cdot E[X(2)]$$

$$= 4 \cdot 2 = \underline{\underline{8}}$$

Problem 8.5

Arrival of passengers Pois $\{X(t), t > 0\}$

Rate : $\lambda = 2$ pr. unit time

Depart $t = 0$

Next Arrive at $T \to$ Passengers $X(T)$

$T \sim$ Uniform dist.

$$f_T(t) = \begin{cases} 1, & 0 < t < 1 \\ 0, & \text{otherwise} \end{cases}$$

a) $E[X^2(T) \mid T=t] = E[X(t)] = \lambda \cdot t = 2t$

$V[X(T) \mid T=t] = V[X(t)] = \lambda \cdot t = 2t$

$E[X(T) \mid T=t] = E[X^2(t)] = (E[X(t)])^2 + V[X(t)]$

$= (2t)^2 + 2t = 4t^2 + 2t = 2t(2t+1)$

b) Siden $T \sim U(0,1)$

$$\Rightarrow E(T) = \frac{1+0}{2} = \underline{\frac{1}{2}}$$

og

$$V(T) = \frac{1}{12} \cdot (1-0)^2 = \frac{1}{12}$$

$$\Rightarrow \quad E(X(t)) = \lambda \cdot E(T) = 2 \cdot \frac{1}{2} = \underline{\underline{1}}$$

$$V(X(t)) = \lambda \cdot V(T) = 2 \cdot \frac{1}{12} = \underline{\underline{\frac{1}{6}}}$$

Problem 8.6

$\{N(t),\ t \geq 0\}$ $\qquad$ $i \leq n$

$s \leq t$

a)

$$P(N(t) = n \mid N(s) = i)$$

$$= P(N(t) - N(s) = n - i).$$

$$= P(N(t-s) = n - i)$$

$$= \frac{\lambda(t-s)^{(n-i)}}{(n-i)!} \cdot e^{-(\lambda(t-s))}$$

b) $i < n$ og $s < t$

$$P(N(s) = i \mid N(t) = n)$$

$$= \frac{P(N(s) = i) \cap P(N(t) - N(s) = n - i)}{P(N(t) = n)}$$

$$= \frac{(\lambda s)^i \cdot \cancel{e^{-(\lambda s)}} \cdot (\lambda(t-s))^{(n-i)} \cdot \cancel{e^{-(\lambda t)}} \cdot n!}{i! \cdot (n-i)! \cdot (\lambda t)^n \cdot \cancel{e^{-(\lambda(t-s))}}}$$

$$= \frac{(\lambda s)^i \cdot (\lambda t - \lambda s)^{(n-i)} \cdot n!}{i! \quad (\lambda t)^n \quad (n-i)!}$$

$$= \frac{\cancel{\lambda^i} \cdot s^i \cdot \cancel{\lambda^n} \cdot \cancel{\lambda^i} \cdot (t-s)^{n-i} \cdot n!}{\cancel{\lambda^n} \cdot t^n \cdot (n-i)!}$$

$$= \frac{s^i \cdot (t-s)^{n-i} \cdot n!}{i!\, t^n (n-i)!}$$

Problem 8.7

λ: freq of shocks

n: Events $n = \{1, 2, \ldots, k\}$

k: Shocks (k'th shock terminate device)

T: Lifetime of device

$$P(N(T) - N(t) = n) = \frac{(\lambda T)^n}{n!} e^{-(\lambda T)}$$

cdf of pmf will be

the sum:

$$F(t) = \sum_{i=0}^{k} \frac{(\lambda t)^n}{n!} e^{-(\lambda t)}$$

Problem 8.8

Radio active source emit:

$\lambda = 2$

Probability for occuring between

N_1: Ingen partikel fra 0 → 3 min

N_2: Partikkel lost ut etter
mellom $T_1 = 3$ og $T_2 = 5$

Ventetiden er $\exp(\lambda, t)$

$$P(T_1 < t < T_2) = \int_{T_1}^{T_2} \lambda e^{-\lambda t}$$

$$= 2 \int_3^5 e^{-2t}\, dt$$

$$= 2\int_3^5 e^u \, dt \qquad\qquad u = -2t$$

$$\frac{du}{dt} = -2$$

$$= -\frac{1}{\cancel{2}}\cancel{2}\int_3^5 e^u \, du \qquad\qquad dt = -\frac{1}{2}$$

$$= -\left[e^{-2t}\right]_3^5$$

$$= -\left[e^{-2\cdot5} - e^{-2\cdot3}\right] = e^{-6} - e^{-10}$$

$$= \underline{\underline{2{,}43\cdot10^{-3}}}$$

b) Sanns. for en partikkel mellom 3 og 5 min er sanns. for 0 fra 0 til 3 og 1 fra 3 til 5. ved $\lambda = 2$

$$\Rightarrow P(N(3)-N(0)=0 \cap N(5)-N(3))$$

$$= \frac{(2\cdot 2)^1}{1!} \cdot e^{-2\cdot 2} = 4e^{-4}$$

$$\underline{\underline{\approx 0{,}073}}$$

Problem 8.4

customers enter acc. Pois ~ $\lambda = 6$

$$P(N(t+s) - N(s) = n) = \frac{(\lambda t)^n}{n!} e^{-(\lambda t)}$$

We shall find:

$$P(N(\tfrac{1}{4}) = 1 \mid N(1) = 1)$$

$$= \frac{P(\frac{1}{4} = 1) \cdot P(\frac{3}{4} = 0)}{P(N(1) = 1)}$$

$$= \frac{(6 \cdot \frac{1}{4})^1}{1!} e^{-6 \cdot \frac{1}{4}} \cdot \frac{(6 \cdot \frac{3}{4})^0}{0!} e^{-6 \cdot \frac{3}{4}} \cdot \frac{1!}{(6 \cdot 1)^1} \cdot e^{6 \cdot 1}$$

$$= \frac{\cancel{6}}{4} \cdot 1 \cdot \frac{1}{\cancel{6}} \cdot e^{-6(\frac{1}{4} + \frac{3}{4}) + 6}$$

$$= 1/4$$

Problem 8.10

$$\{X(t),\ t \geq 0\} \qquad E = \lambda = 3 \quad (\text{Pr. nr.})$$

$$P(X(1) = 2 \mid X(3) = 5)$$

$$= \frac{P(X(1) = 2 \cap X(3-1) = 3)}{P(X(3) = 5)}$$

$$= \frac{(3\cdot 1)^2}{2!} e^{-(3\cdot 1)} \cdot \frac{(3\cdot 2)^3}{3!} e^{-(3\cdot 2)} \cdot \frac{5!}{(3\cdot 3)^5} \cdot e^{(3\cdot 3)}$$

$$= \frac{9 \cdot 6^3 \cdot 5!}{9^5 \ 2! \cdot 3!} \cdot e^{-3+(-6)+9}$$

$$= \frac{6^3 \cdot 5 \cdot 4}{9^4 \cdot 2!} = \frac{80}{243} = \underline{\underline{0{,}329}}$$

Problem 8.12

$Pois(\lambda) \sim X(t) = n$

time pdf. for S_r for r'th. Arrival

$r \leq n$.

cdf: $P(S_r \leq t) = P(X(t) \geq n)$

$$= \sum_{r=n}^{\infty} \frac{(\lambda t)^r}{r!} \cdot e^{-\lambda t}$$

$$= \sum_{r=0}^{n-1} 1 - \frac{(\lambda t)^r}{r!} \cdot e^{-\lambda t} \quad \forall \quad r = \{1, 2, \ldots, n\}$$

pdf: $\frac{d}{dt}\left[1 - e^{-\lambda t}\left[1 + \frac{\lambda t}{1!} + \frac{(\lambda t)^2}{2!} + \dots + \frac{(\lambda t)^{n-1}}{(n-1)!}\right]\right]$

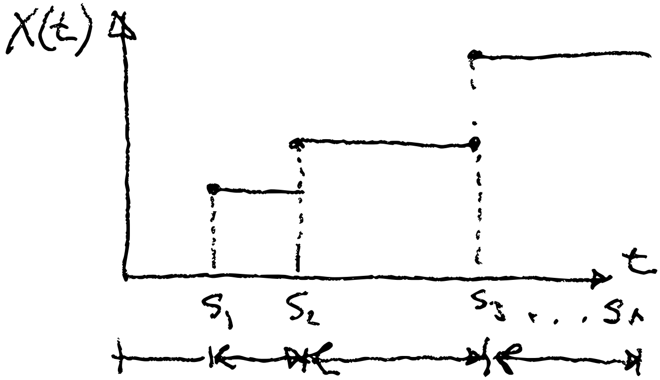
$$= \lambda e^{-\lambda t}\left[1 + \frac{\lambda t}{1!} + \frac{(\lambda t)^2}{2!} + \dots + \frac{(\lambda t)^{n-1}}{(n-1)!}\right]$$

$$- e^{-\lambda t}\left[\lambda + \lambda\frac{(\lambda t)}{1!} + \lambda\cdot\frac{(\lambda t)^2}{2!} + \dots + \lambda\frac{(\lambda t)^{n-2}}{(n-2)!}\right]$$

$$= \frac{\lambda^n \cdot t^{n-1}}{(n-1)!} e^{-\lambda t}$$

$\blacksquare$

Problem 8.12



- Customers occur at time S_n also known as waiting times

  $X(t)$ count the number of events occurring at time t.

- We introduce independent RV $U_n \sim \mathrm{Uni}(0,t)$

- and $R(.)$ symmetric function for n variables.

$$\Rightarrow E(\eta(S_1, S_2, \ldots, S_n \mid X(t) = n)$$
$$= E(\eta(U_1, U_2, \ldots, U_n).$$

- $U_1, U_2, \ldots, U_n$ is the order statistics such that U_n and S_n have the same distribution.

- $E(U_1) = E(U_2) = \ldots = E(U_n) = \frac{t}{\lambda}$

$$\Rightarrow E(S_1, S_2, \ldots, S_n \mid X(t) = S)$$
$$= E(U_1 + U_2 + \ldots + U_n)$$
$$= \underline{n \cdot \frac{t}{2}} \longrightarrow \text{Mean waiting time}$$

$\blacksquare$

Problem 8.13

Og så langt kom en

denne gang...