Mandatory homework 7 - 8 Stat 220 - H20

October 29, November 5 : 1015 - 1200

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Due at the end of November 08

Problem 8.1

Simulate a Poisson process from the renewal representation. Let $\{T_j,\, j = 1, \ldots, N\}$ be random sample from the exponential distribution with intensity $\lambda = 1/2$.

$$X(t) = \max\left\{k\colon S_k \le t\right\}, \qquad S_k = T_1 + \cdots + T_k,$$

a) Use $N = 10$.

b) Use $N = 100$.

c) Use $N = 1000$.

```
lb<-0.5
N<-10
T<-rexp(N,lb)
S<-rep(0,N)
S[1]<-T[1]
for(k in 2:N) S[k]<- S[k-1] + T[k]
delta<-0.1
tmax<- S[N]+ 0.5/lb
tvalues<-seq(delta, tmax,delta)
ntvalues<- length(tvalues)
X<-rep(0, ntvalues)
for(t in 1: ntvalues) X[t]<- length( which( S<=tvalues[t] ))
plot(tvalues, X, type="s")
```

d) Can you extend the source code and plot $n = 1000$ Poisson processes with $N = 100$ in one frame? It is an advantage for the visual presentation that you use more than one color here.

Problem 8.2

[PK, Exercise 5.1.6, page 229]

Messages arrive at a telegraph office as a Poisson process with mean rate of 3 messages pr hour.

a) What is the probability that no messages arrive during the morning hours 8.00 AM to noon?

b) What is the distribution of the time at which the first afternoon message arrives?

Problem 8.3
[PK, Exercise 5.1.7, page 229]

Suppose that customes arrive at a facility according to a Poisson process having rate $\lambda = 2$. Let $X(t)$ be the number of customers that have arrived up to time t. Determine the following probabilities and conditional probabilites.

a) $\mathbb{P}\big(X(1) = 2\big)$.
b) $\mathbb{P}\big(X(1) = 2 \text{ and } X(3) = 6\big)$.
c) $\mathbb{P}\big(X(3) = 6 \,|\, X(1) = 2\big)$.

Problem 8.4
[PK, Exercise 5.1.9, page 229]

Let $\{X(t), t \geq 0\}$ be a Poisson process having rate paramter $\lambda = 2$. Determine the following expectations:

a) $\mathbb{E}\, X(2)$.
b) $\mathbb{E}\, X^2(1)$.
c) $\mathbb{E}\, X(1)\, X(2)$.

Problem 8.5
[PK, Problem 5.1.9, page 231]

Arrivals of passengers at a buss stop form a Poisson process $\{X(t), t \geq 0\}$ with rate $\lambda = 2$ pr unit time. Assume that a bus departed at time $t = 0$ leaving no customers behind. Let T denote the arrival time of the next bus. Then the number of passengers present when it arrives is $X(T)$. Suppose that the bus arrival times T is independent of the Poisson process and thus T has a uniform probability density function

$$f_T(t) = \begin{cases} 1, & 0 \leq t \leq 1; \\ 0, & \text{otherwise}. \end{cases}$$

a) Determine the conditional moments $\mathbb{E}\,[X^2(T) \,|\, T = t]$ and $\mathrm{Var}[X(T) \,|\, T = t]$.
b) Determine the mean $\mathbb{E}\, X(T)$ and the variance $\mathrm{Var}(X(T))$.

Problem 8.6
[SR, Exercise 37, p. 363]
Let $\{N(t), t \geq 0\}$ be a Poisson process with rate. For $i \leq n$ and $s < t$.

a) Find $\mathbb{P}\big(N(t) = n \,|\, N(s) = i\big)$.
b) Find $\mathbb{P}\big(N(t) = i \,|\, N(t) = n\big)$.

Problem 8.7
[PK, Problem 5.1.11, page 231]

Assume that a device fails when a cumulative effect of k shocks occurs. If the shocks happen according to a Poisson process of parameter λ, what is the density function for the life T of the device?

Problem 8.8
[PK, Exercise 5.3.2, page 245]

A radioactive source emitts particles according to a Poisson process of rate $\lambda = 2$ particles per minute.

a) What is the probability that the first particle appears some time after 3 minutes but before 5 minutes?
b) What is the probability that exactly one particle is emitted in the interval from 3 minutes to 5 minutes?

Problem 8.9
[PK, Exercise 5.3.3, page 245]

Customers enter a store accoring to a Poisson process of rate $\lambda = 6$ per hour. Suppose that it is known that only a single customer entered during the first hour. What is the conditional probability that this person entered during the first 15 minutes?

Problem 8.10
[PK, Exercise 5.3.4, page 245]

Let $\{X(t), t \geq 0\}$ be a Poisson process with rate $\xi = 3$ per hour. Find the conditional probability that there were two events in the first hour given that there were five events in the first 3h.

Problem 8.11
[PK, Problem 5.3.8, page 247]

Consider a Poisson process with parameter λ. Given that $X(t) = n$ find the density function for S_r , the rth arrival time. Assume that $r \leq n$.

Problem 8.12
[PK, Exercise 5.4.3, page 257]

Customers arrive a certain facility according to a Poisson process of rate λ. Suppose that it is known that five customers arrived in the first hour. Determine the total waiting time $\mathbb{E}(S_1 + S_2 + S_3 + S_4 + S_5)$.

PROBLEM 8.13
[PK, Problem 5.4.1, page 257]

Let $\{S_j,\, j \geq 1\}$ denote the event times for a Poisson process $\{X(t),\, t \geq 0\}$ of rate λ. Suppose that it is known that $X(1) = n$. For $k < n$, what is the conditional density function for $S_1, \ldots, S_{k-1}, S_{k+1}, \ldots, S_n$ given that $S_k = s$.

PROBLEM 8.14
Let X_i be exponential (λ_i) for $i = 1, 2$ and independent. We use $U = \min(X_1, X_2) = X_1 \wedge X_2$, $V = \max(X_1, X_2) = X_1 \vee X_2$ and $Z = i$ iff $X_i = U$. We know that $U \sim$ exponential $(\sum_i \lambda_i)$ and

$$\mathbb{P}(Z = i) = \frac{\lambda_i}{\sum_j \lambda_j}, \quad i = 1, 2.$$

Here we have the focus on the maximum. For the exponential densities we use $f_i = f_{X_i}$.

a) Suppose that the two rates are different. Explain that the tail of the distribution has to be dominated by the smallest rate and conclude that V is not exponential distributed.

b) Show that

$$f_V(t) = F_1(t)f_2(t) + F_2(t)f_1(t).$$

HINT: $\{\max_i X_i \leq t\} = \bigcap_i \{X_i \leq t\}$.

c) Make a plot of the density when $\lambda_1 = 2$ and $\lambda_2 = 3$. Do you have a comment that says something more than a)?

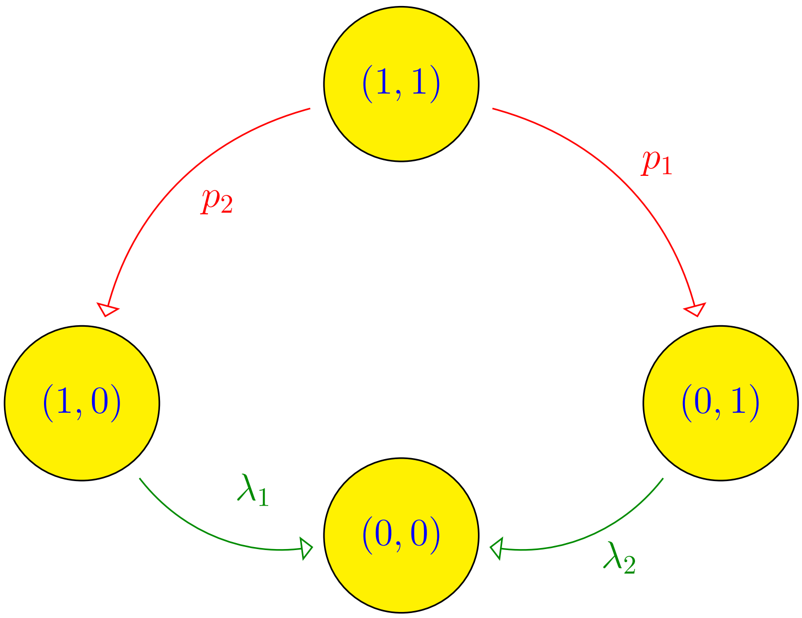

Figure 1: A simple queueing model for the maximum

The graph illustrate a situation with two servers with exponetial service times where each of them has exactly one customer at time zero. The state (r, s) means the two serves (1,2) has (r,s) customers in service.

i) First we wait for U to happen.

ii) If $U = X_1$ then we go from $(1,1)$ to $(0,1)$ with probability $p_1 = \mathbb{P}(Z = 1)$. Likewise, $U = X_2$ then we go from $(1,1)$ to $(1,0)$ with probability $p_2 = \mathbb{P}(Z = 1)$.

iii) We then have to wait for remaining variable. Due to lack of memory the distribution of the remaining variable at time $U = u$ is unchanged.

d) Explain that

$$V \overset{d}{=} U + X_1'(1 - Z) + X_2' Z = U + W, \quad \text{say,} \tag{1}$$

where all variables on the right hand side are independent and $X_i' \overset{d}{=} X_i$. This means that we first draw the independent variables U and Z and if $Z = 1$ we draw $X_2' \sim \text{exponential}\,(\lambda_2)$ and vice versa for $Z = 2$. This means that

$$f_W = p_2 f_1 + p_1 f_2, \qquad f_V(t) = \int_0^t f_U(t - s) f_W(s) ds.$$

e) Find the expectation and the variance for V from (1). What if the rates are equal?

f) Verify the identity; $X_1 \vee X_2 = X_1 \wedge X_2 + |X_1 - X_2|$ and relate it to (1).

Problem 8.15

Prove that the superposition of two independent Poisson processes with respective rates $\{\lambda_i,\, i = 1.2\}$ is a Poisson process and find its rate.

Problem 8.16

Let $\{I_j,\, j \geq 1\}$ be a Bernoulli sequence with success probability p_1 and let $Y \sim \text{Poisson}(\lambda)$ be independent of that sequence. Let $X_1 = \sum_{j=1}^{Y} I_j$ show $X_2 = \sum_{j=1}^{Y} (1 - I_j)$. Show that $X_i \sim \text{Poisson}(\lambda p_i)$ for $i = 1, 2$ and independent where $\sum_i p_i = 1$.

References

Howard M Pinsky and Samuel Karlin. *An introduction to stochastic modeling*. Academic Press, Inc., San Diego, CA, fourth edition, 2011. ISBN 978-0-12-381416-6. http://www.sciencedirect.com/science/book/9780123814166.

Sheldon Ross. *Introduction to Probability Models, Eleventh Edition*. Academic Press, 2014. ISBN 9780124079489.