A Genius Solution

Applications of the Sprague-Grundy Theorem to Korean Reality TV

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Abstract

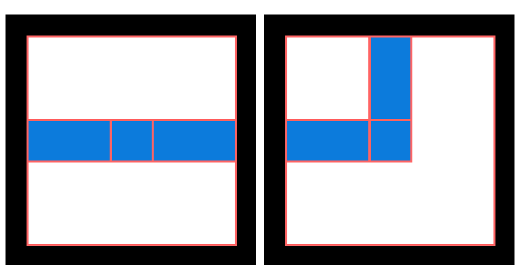
`Monorail` is a two-person tile placing game that was popularized by the South Korean reality show `The Genius` in 2014. This paper uses the Sprague-Grundy theorem to demonstrate a winning strategy for the first player.

Background

`The Genius` was a South Korean reality show that aired for 4 seasons, from 2013 to 2015, where players were eliminated through a series of social/strategy games. The prize for winning each season ranged from approximately $50,000 to $100,000. Additionally, a Dutch version of the show aired for 1 season in 2022 (which had no prize). In both versions of the show, each episode would culminate in a "Death Match", a game that was used to determine which of two contestants would be eliminated. `Monorail` is a tile-placing game that was used as a Death Match twice during the Korean series, and two additional times during the Dutch series (where it was called "On Track"). While some players claimed to know a winning strategy, their strategies were never rigorously presented on the program and they never played `Monorail` during the series, so it is impossible to determine whether those strategies were correct and complete.

Rules

`Monorail` is a tile-placing game where the objective is to finish a looping train track. The game is played with a set of 16 identical double-sided square tiles with train tracks on them. One side of the tile has the train tracks continuing in a straight line, while the other side has the train tracks turning 90 degrees (Figure 1).



(a) Straight Side (b) Corner Side

Figure 1: Sides of tiles

Before the start of the game, a special double-wide tile is placed that represents the train station (Figure 2).

On each player's turn, they may place 1, 2 or 3 tiles, subject to the following conditions:

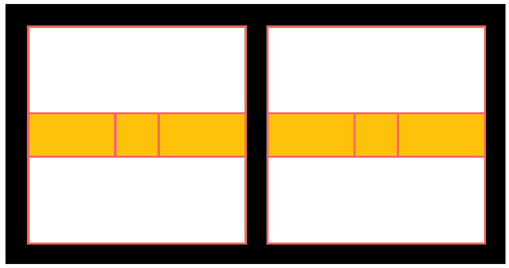


Figure 2: Station

- At least 1 of the placed tiles must be orthogonally adjacent to a previously placed tile
- All placed tiles must be adjacent and in a straight line (either horizontally, or vertically)

The player chooses which side of the tile is face-up and its rotation before placing it. This is equivalent to choosing which two of the tile's four sides to connect with train tracks.

Additionally, the tiles placed by the player do not need to connect to the existing track, nor does the track need to be connected between the placed tiles (e.g. Figure 3). The move restrictions are on where the tiles are placed, not the orientation of the tiles.

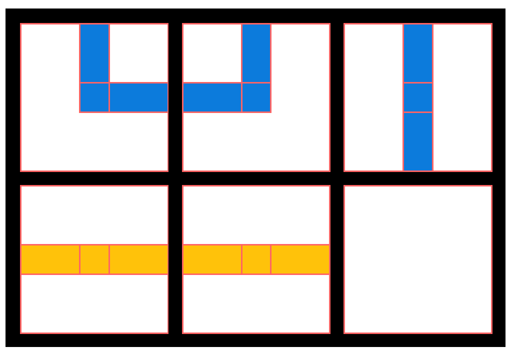


Figure 3: Example starting move

The goal of the game is to be the player that finishes a loop that connects the start and end of the station (Figure 4).

A player wins if they place tiles that finish a loop, provided that the loop uses all the tiles that have been placed. It is ok if some tiles remain unplaced.

In some cases, it may become impossible to complete a loop with the remaining tiles (e.g. Figure 5). Therefore, a player also has the option of declaring that the track cannot be finished instead of placing tiles. If they are correct, then they win. However, if their opponent can demonstrate that the track could have been finished, the opponent wins instead.

Rule Modifications for Analysis

It is clear that if the track cannot be completed, it is in the active player's best interest to declare this fact and immediately win. Thus, in a game between perfect strategists, it would never be to a player's benefit to make a move that resulted in a track that cannot be completed[1]. Additionally, if a player starts their turn in a position where the track is completable, they may always make a move that results in a track that

[1]Of course, real world players can make mistakes. One player won an otherwise losing game by confidently playing a move that made the track impossible to finish. This confused their opponent who did not claim the win on their turn.

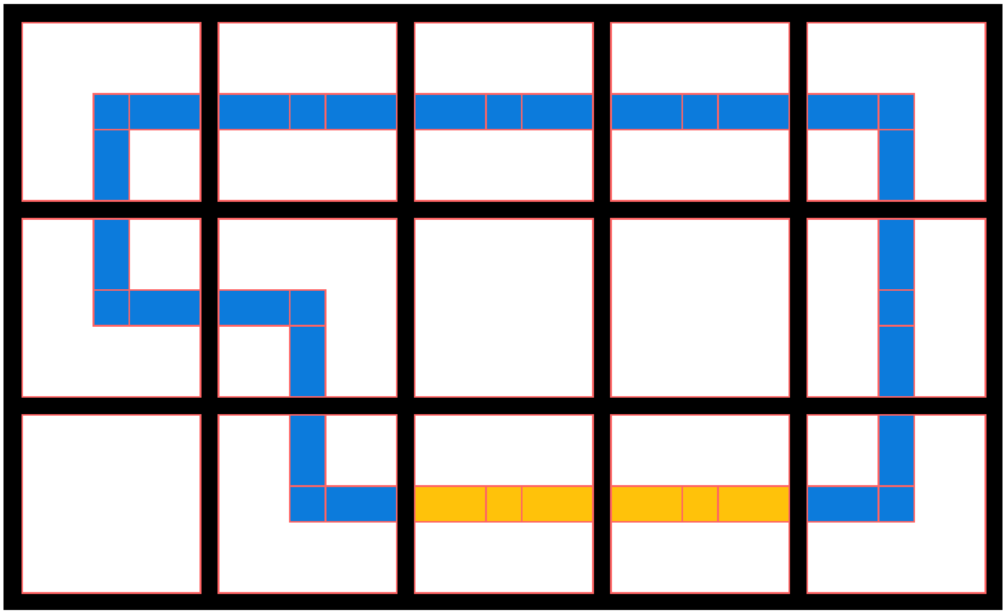
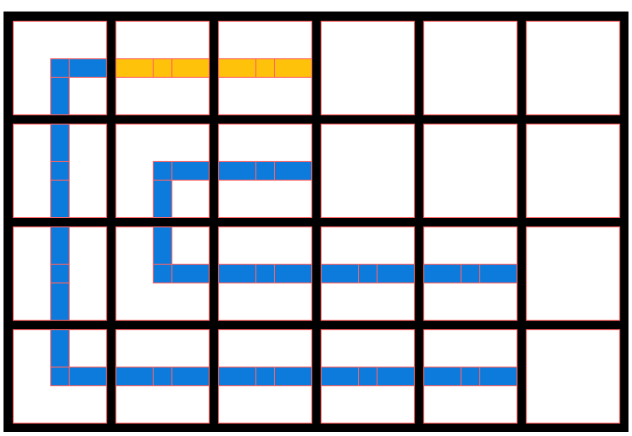
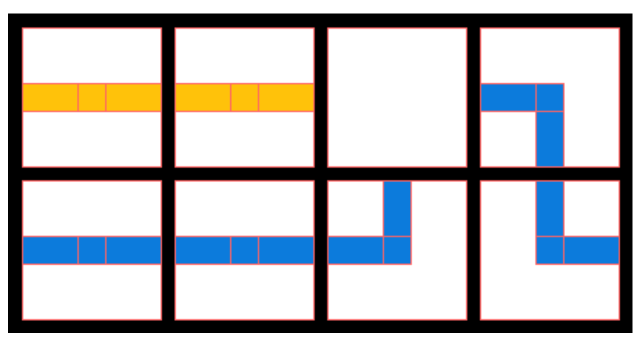


Figure 4: Example finished game



(a) Not enough tiles remaining



(b) Impossible junction

Figure 5: Impossible to complete a loop

is completable by extending the track one tile along one of the possible routes. Thus, a player will never be forced to make a move that turns a completable track into one that cannot be completed. Therefore, the rules can be modified so that a player may not make a move that results in a track that's impossible to complete. This simplifies the assessment of positions without affecting whether each position is winning or losing for the player to move.

Mathematical background

Impartial games

The Sprague-Grundy Theorem simplifies evaluating impartial games, which are games that meet the following conditions:

- There are two players
- The players alternate turns
- There are a finite number of possible gamestates
- There is a maximum number of turns
- The last player to move wins and the other player loses
- There is no randomness
- A player's possible moves depend only on the state of the board, not which player's turn it is.[2]

It should be clear that `Monorail` satisfies each of these properties.

Impartial subgames There are times when the state of an impartial game may be described as the combination of multiple non-interacting sections of the game state. For example, consider the game of `Nim`. In `Nim`, there are some number of distinct piles, each containing stones. On a player's turn, they may remove any number of stones from any single pile. The player that removes the final stone from the last remaining pile wins.

Because removing stones from one pile doesn't affect how many stones are in any other pile, a game of `Nim` with multiple piles could be considered a combination of multiple games of `Nim` that each have fewer piles.

In general, we can say that an impartial game G is the sum of impartial subgames $G = G_1 \bigoplus G_2 \bigoplus ... \bigoplus G_n$ if it satisfies the following conditions:

- On a player's turn, they must choose a particular subgame (pile) G_i where a legal move is possible and make a move in that subgame
- Making a move in one subgame never affects which moves are available in any other subgame
- The player who makes the last move wins (i.e. after their move, no subgame has a legal move remaining)

In fact, this process also generalizes in reverse. Any finite subset of impartial games $\{G_1, ..., G_n\}$ can be used to create a larger impartial game G if players are able to choose which subgame to play in for each turn.

The Sprague-Grundy Theorem The Sprague-Grundy theorem establishes the existence of a function N that maps game-states of impartial games to the non-negative integers with the following properties:

- $N(G) = 0 \iff$ The player to move is losing
- $N(G_1 \bigoplus G_2) = \mathrm{XOR}(N(G_1), N(G_2))$ (where XOR is the bitwise exclusive-or function).
- If $N(G) = k$, and $k \geq m > 0$ there exists a move to a position $G^{'}$ such that $N(G^{'}) = m$[3]

Table 1: Bitwise XOR sums ((0-7) x (0-7))

XOR	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**
0	0	1	2	3	4	5	6	7
1	1	0	3	2	5	4	7	6
2	2	3	0	1	6	7	4	5
3	3	2	1	0	7	6	5	4
4	4	5	6	7	0	1	2	3
5	5	4	7	6	1	0	3	2
6	6	7	4	5	2	3	0	1
7	7	6	5	4	3	2	1	0

Because there is a bijection between the equivalence classes of N and the sizes of a single pile game of `Nim`, $N(G)$ is often called the `Nimber` of G. In particular, a game of `Nim` with 1 pile and K stones has a nimber of K.

These properties lead to a few immediate important consequences:

1. If $N(G) \neq 0$, the player is winning and they can make a move that leaves their opponent with a losing state, i.e. the new state G' is such that $N(G') = 0$.
2. Because $\text{XOR}(A, B) = 0 \iff A = B$,
   $N(G_1 \bigoplus G_2) = 0 \iff N(G_1) = N(G_2)$
3. In any position in an impartial game, if $N(G) \neq 0$, one may create a new game combining G with a single pile game of `Nim` with $N(G)$ stones. In this new game, the first player to move will be losing.

Monorail Solution.

A computer search of gamestates in Monorail found a winning move for the first player. An examination of the move showed that it uniquely determined the loop that could be finished with the remaining tiles (Figure 6).

The remaining locations where tiles can be placed are separated into 2 non-adjacent areas, which can be thought of as separate subgames because moves in one area do not affect possible legal moves in the other area. After this point, the location where a tile is placed also uniquely determines how it must be rotated. Therefore, for the purposes of analyzing subgames it is sufficient merely to look at the positions where tiles need to be placed and not their rotation.

It is clear that rotating, translating or mirroring a subgame does not meaningfully affect the moves available. If two positions are made up of equivalent subgames, their evaluation must be the same. For example, the 2 positions in Figure 7 have 8 tiles remaining unplaced, and a move must be either in the group of 5 tiles forming a long L, or the set of 3 tiles in a line. These two positions must therefore have the same nimber.

However, care must be taken when determining which subgames are identical. A move may only be made if at least one of the tiles placed is adjacent to an already placed tile. For example, the two positions in Figure 8 look similar because the three remaining spots to play a tile are in an L shape. However, in position a, it is legal to place only the corner piece, while in position b it is not. In fact, playing in the corner is the only winning move in position a, while b is a losing position.

[2]The term “impartial game” is often used to refer only to this last condition. However, game theory discussion often focuses on the class of games that meet the other criteria, and discusses Sprague-Grundy’s application to impartial games in that narrower context, so for brevity this paper uses the term “impartial game” to refer to games that meet all the listed criteria

[3]Technically, it should be specified whether the domain of N is games or positions. In practice, any position could also be the starting position of a game with the same rules, so the ambiguity doesn’t cause issues.

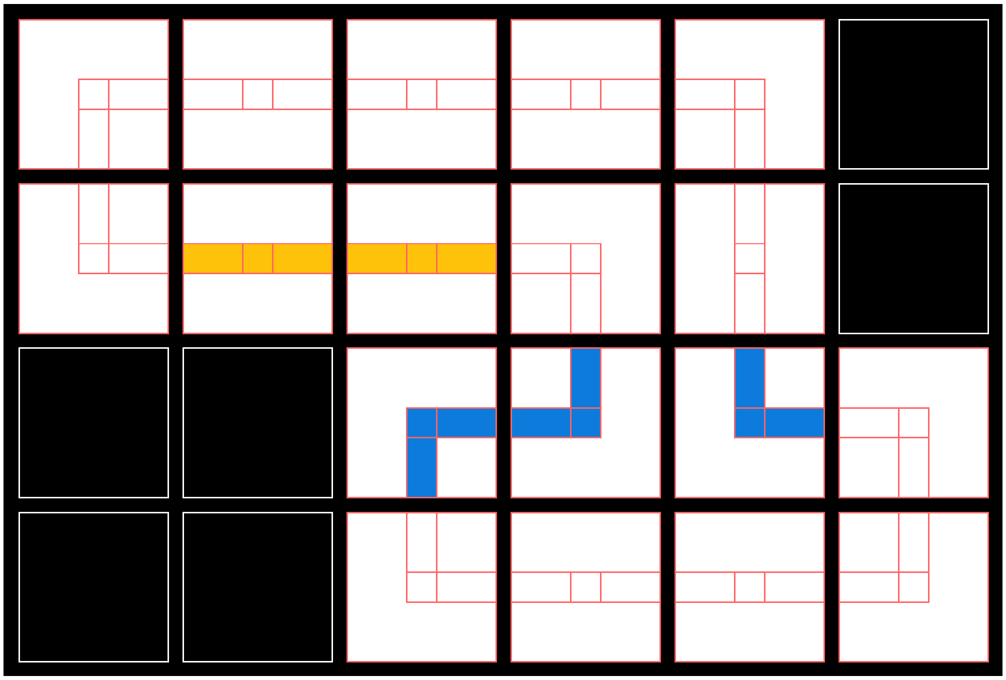


Figure 6: Winning move with only possible path

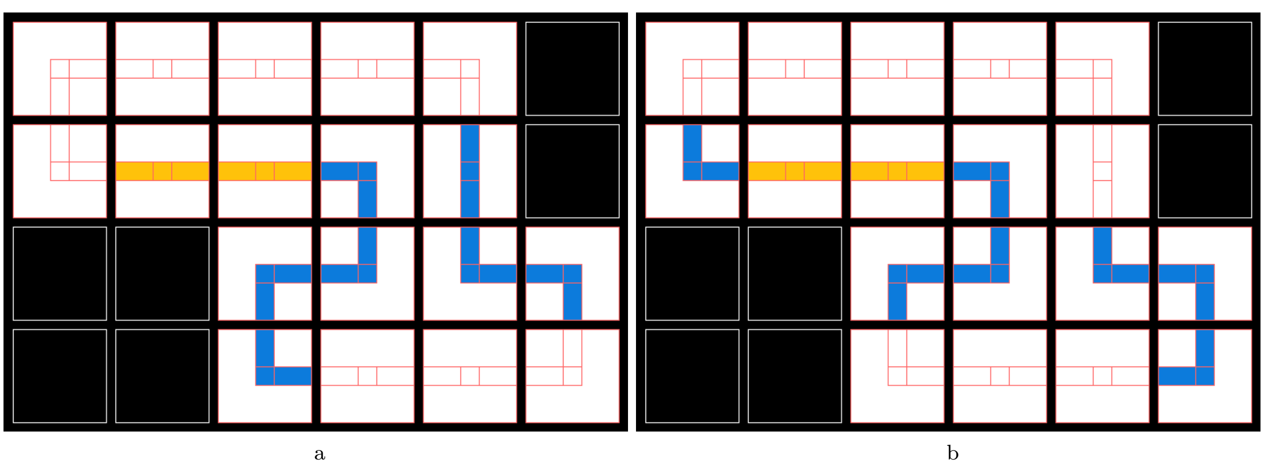
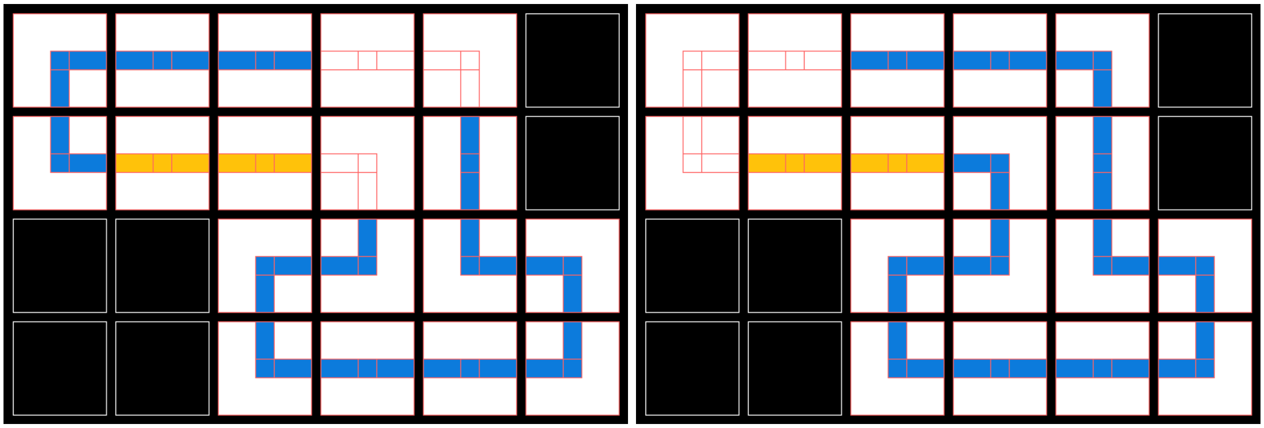


Figure 7: Positions from the same class



(a) Corner piece placeable (b) Corner piece not placeable

Figure 8: Distinct L Groups

Analyzing positions in terms of their subgames greatly reduces the number of cases that must be evaluated. From the position following the first move, a total of 4900 positions can be reached (including the one shown in Figure 6). However, these 4900 positions are made up of just 23 distinct subgames

Endgame technique

Given a winning position made up of subgames $G_1, ..., G_k$ the following strategy will find a winning move

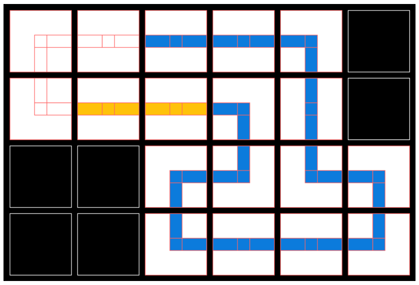
1. For each subgame determine its corresponding nimber, $N_i = N(G_i)$
2. Find the nimber of the overall position, $N(G) = \text{XOR}(N_1, ..., N_k)$
3. Find an i such that $N_i > \text{XOR}(N_i, N(G))$[4]
4. Make a move in G_i resulting in G_i^* such that $N(G_i^*) = \text{XOR}(N_i, N(G))$[5]

A representative position of each subgame is listed in the appendix with its corresponding nimber. Note that the two subgames in Figure 6 have a nimber of 5, which makes the nimber of the overall position 0. Thus, the player to move (the second player) is in a losing position. Any move they make will result in a winning position (with a non-0 nimber) for the first player. Therefore, `Monorail` is a win for the first player with perfect play.

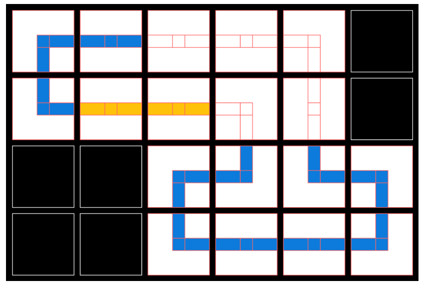
[4]The existence of i is implied by the Sprague-Grundy theorem

[5]Such a move must exist because $N_i > \text{XOR}(N_i, N(G))$

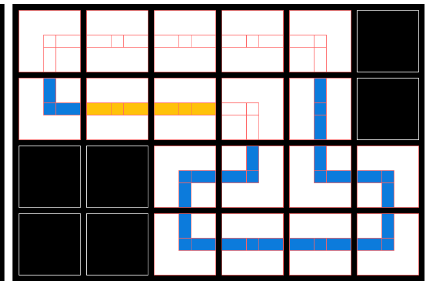
Appendix



Lx

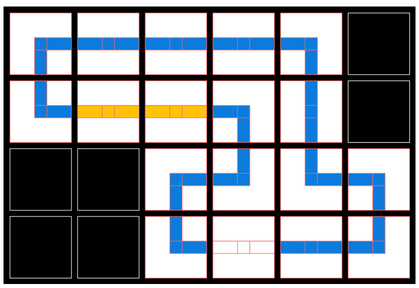


S+1

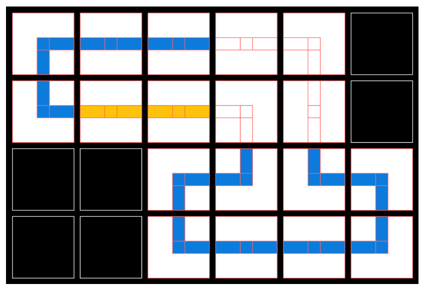


T5

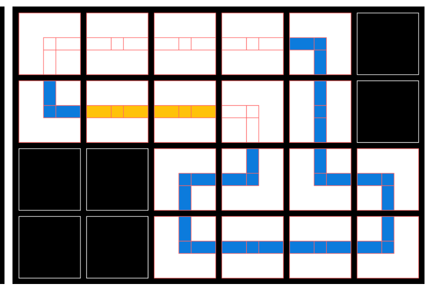
Figure 9: subgames with nimber 0



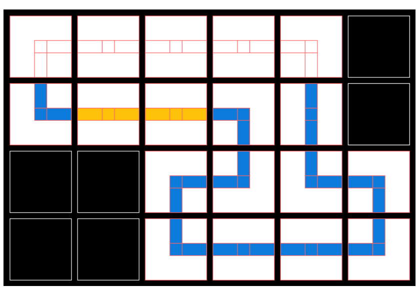
1



S

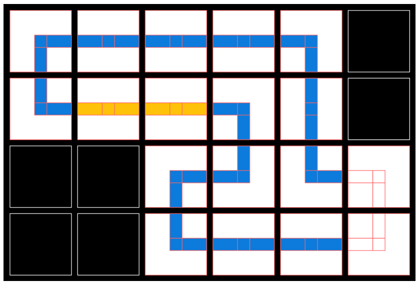


L4

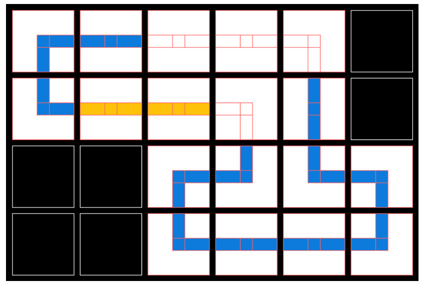


5

Figure 10: subgames with nimber 1

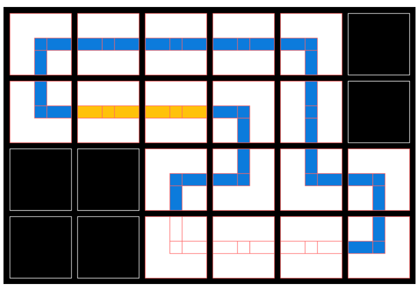


2



T

Figure 11: subgames with nimber 2



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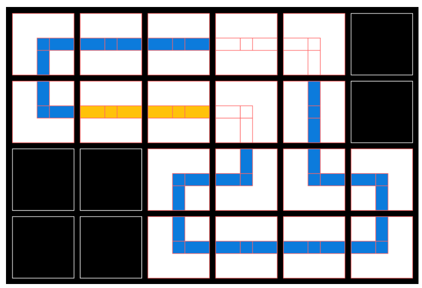
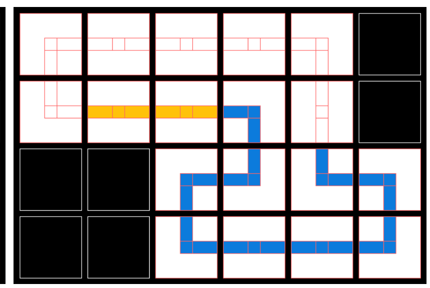
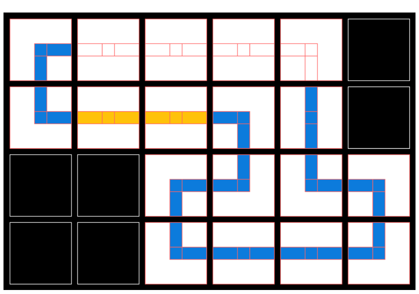


L

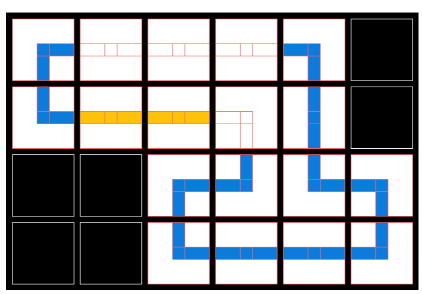


C5

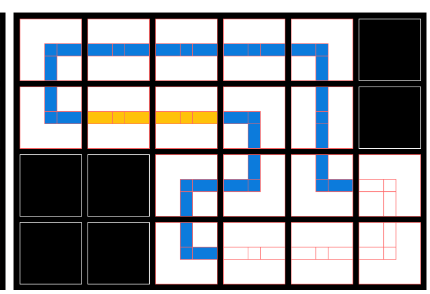
Figure 12: subgames with nimber 3



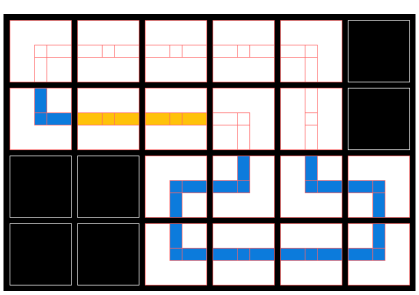
4



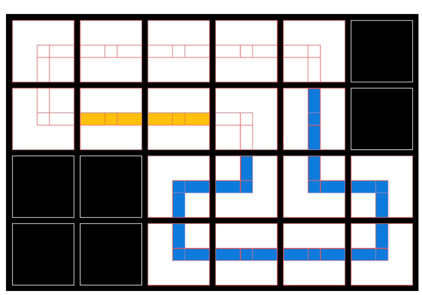
L3



L3x

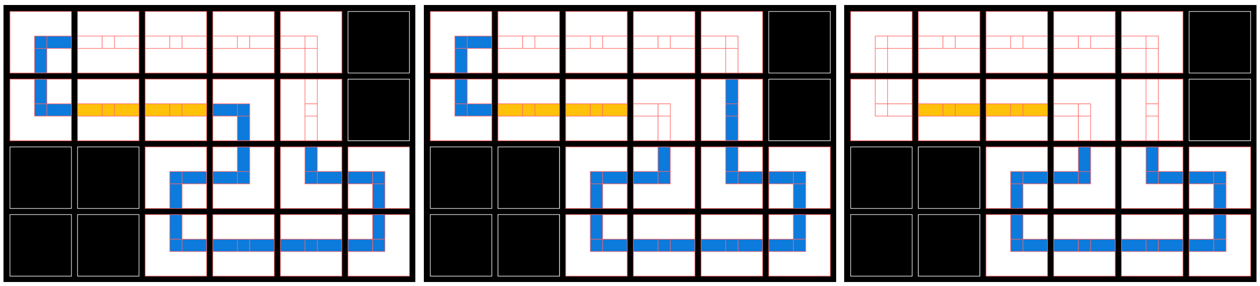


S+3



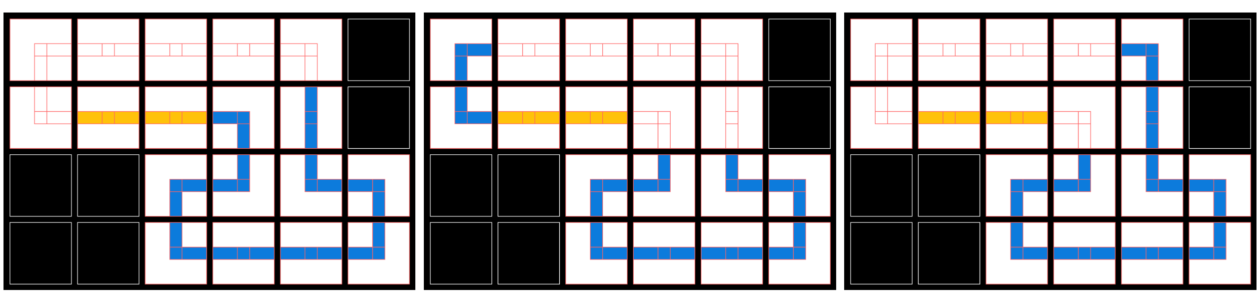
C4+1

Figure 13: subgames with nimber 4



L4x T4 S+L3x

Figure 14: subgames with nimber 5



L5x S+2 C4

Figure 15: subgames with nimber 6