We Found the Best Shuffled Deck[*]

Philip Mallory[†], Gabriel Carvalho-Silva, Juba Ziani[‡], Yani Ziani, Gennie Mansi, Michael A. Specter, James Zhu, Jake Abernethy[§], Justin Burkett, John Blum, Deven Desai, Henry Segerman, Sabetta Matsumoto[¶], Danny Smyl[‖], Lukas Graber, Meg Millhouse, Clio Andris, Mathieu Dahan, John "Bean" Mains, Ashwin Kanhere, Graham Gussack, Mark Mote[**], Joseph Andress, Lily Turaski, DeeDee Turaski, I. P. Freely, Anthony Voutas, Rachel Schwind[††], Daniel Bruder, Patrick Kastner, Patrick Holmes, Matt Abate, and Shreyas Kousik[‡‡]

Abstract

Any person with a basic understanding of combinatorics should understand that there are many shuffled decks, so the task implied by the title of this paper was incredibly difficult to achieve. However, it was managed to be done in this paper, and conveyed here in word form in passive voice without personal pronouns, because this is an abstract. Indeed, this paper proposes the most shuffled deck, and associated metrics by which to determine its shuffledness. In particular, the following contributions are made: the word shuffledness, a deterministic shuffling algorithm, several means by which to measure shuffledness, some pretty shuffling visualization stuff, and other miscellaneous thoughts.

[*] and goaded many people into being coauthors

[†] All authors are first author [Demaine and Demaine, 2023], but Phil is the firstest. The author list has been shuffled deterministically.

[‡] Juba takes credit for the bottom-k most terrible puns in the paper, for any k.

[§] We are, in fact, not quite sure whether Jake has consented to being an author of this paper.

[¶] Has just acquired her first deck of french-suited playing cards and hopes this research will give her an understanding of what they are good for.

[‖] A.K.A. Daddy Cool

[**] Oh! Hi Mark!

[††] because every paper needs an explosives expert

[‡‡] Unbeknownst to the other authors, this paper is not even a legitimate illegitimate paper! This is actually all part of a social experiment for the impending SIGBOVIK 2025 paper, "How many Ph.D.s does it take to write a sh*tpost?"

5 Introduction

It is erstwhile observed that there are many, many possible shufflings of a deck of cards [Brunson, 1969]. The problem is bad for a normal deck of cards [Brunson, 1969], but also for a more fancy deck of cards [Churchill et al., 2019]. This challenge naturally leads to the question, *what is the most shuffled deck?* Other, lesser researchers have attempted to tackle this age-old problem [Diaconis et al., 1983], and have fallen far short.

Forsaking sanity, we restrict ourselves to the standard 52 card deck of French playing cards[1]. In this case, there are 52! potential sequences of cards [Brunson, 1969]. That's so many! We visualize a shuffle of such a deck with the preeminent deck visualization software, MATLAB[2], as shown in Figs. 1–3.

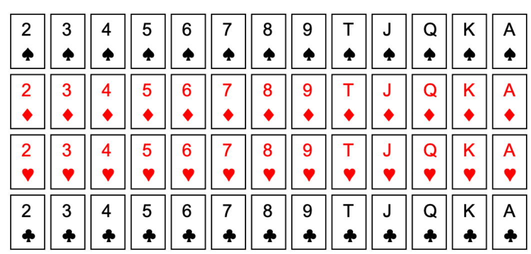


Figure 1: Good old reliable unshuffled deck. Not the *best* deck, but certainly one of the decks. The "T" stands for 10 because we were too lazy to change the spacing for just one of the ranks.

[1] Also often called "freedom cards".

[2] `https://github.com/pmallory/shufflemetrics`

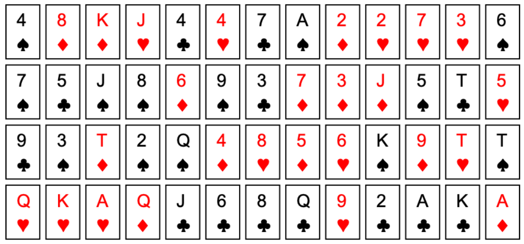


Figure 2: This is terrible. This loathsome pile of filth is an affront to the sensibilities of any life-hardened shuffle enthusiast [Rogers, 1976].

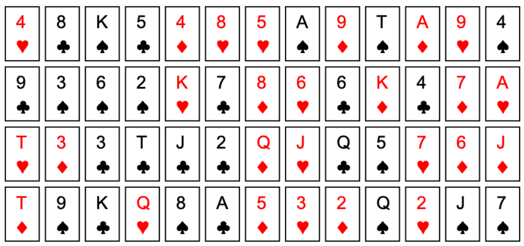


Figure 3: Now this is what we're talkin' about! Just *look* at this son of a gun. This is a mightily fine shuffled deck, if we do say so ourselves (and we do).

2 Related Work

Related work or no related work? That is the question [Shakespeare, 1596] (also see Fig. 4)! But there is this one cool paper about shuffling that Jake told us about [Bayer and Diaconis, 1992], and Sabetta found a cool one too [Diaconis et al., 1983] (nevermind that we totally dissed this paper earlier). So, because there actually were some papers in the aether, we decided to write a related work section.

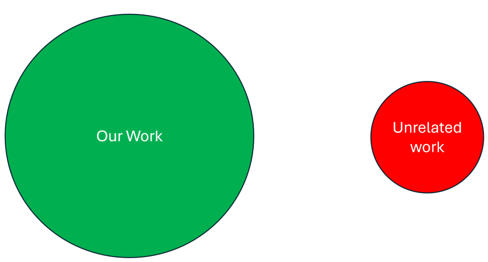


Figure 4: An Euler diagram, *not a garbage Venn diagram* [Euler, 1782], about our related work.

We start from the beginning. The creation of 52-card shuffling is not explicitly mentioned in Genesis, however, the intention was clear: there can only be one master shuffle [God, 2009]. This theme underscores the premise of the majority of related works. Nonetheless, one can metaphorically interpret the act of creation itself as a precursor to such shuffling phenomena. According to the biblical account, God created the world and all its intricacies, establishing order from chaos. In this context, one might envision the creation of shuffling as a reflection of God's divine ingenuity and the inherent complexity of the universe. Just as God fashioned the universe with diverse elements and systems, the intricate patterns of shuffling exemplify the creative potential inherent in the world. Each shuffle represents a unique arrangement, mirroring the endless possibilities woven into the fabric of existence. While the specifics of card shuffling are not detailed in Genesis, its conceptual parallel can be found in the broader theme of creation, illustrating the marvels of divine craftsmanship and the rich tapestry of life.

Having considered the Judeo-Christian perspective on the theologically ideal shuffle, we turn to consider an alternative perspective from the Akkadian Pantheon. One can interpret the flood of Enlil as a Great Shuffling and the warning of Ea to Atra-Hasis as a rebuke of the leader of the gods' decision to shuffle the world without the counsel of the other gods. As such, the transformation of Atra-Hasis to the immortal Uta-Napishtim serves as a perpetual reminder to the great Enlil to avoid brash unconscious shuffling. It would be easy to conclude that this is a general forbearance against all shuffling. On closer inspection though, Enlil's true crime was of not considering the counsel of his advisors. Thus a more precise lesson can be learned from the story of the great flood. That is: one must always seek advice on the best way to shuffle. It is precisely this advice that we, the authors, have achieved with this paper.

Contemporary works attempted to capture the divine craftsmanship of well-advised shuffling. For example, and perhaps most prominently, acclaimed artists Laugh My F*cking Ass Off (LMFAO) successfully integrated shuffling into an electronic stereophonic framework [Stroud, 2022]. In their efforts, the thematic presence of "shuffling everyday" or, colloquially, "every day I'm shuffling" formed an emblematic silhouette for the modern representation or manifestation of millennial shuffling[3]. Possibly most appropriately, LMFAO's work demonstrates the potentiality for a master shuffle, as depicted both robot-

[3] Because we millennials don't possess real skill sets.

ically and theistically in their inimicable opus[4]. Obviously, this will be solved herein.

1 Problem Statement

Just Say "Yes" to No-tation

We are not sure how this happened, but we do indeed have a notation section. The integers from 1 to n are $\mathbb{N}_n$ (note that we take the earliest historical mathematical precedent available and exclude 0 from the integers). The integers from a to b are $\mathbb{N}_{a:b}$. Consider a 52 card deck D_k represented as a list. The subscript $n \in \mathbb{N}_{52!}$ indicates that D_n is the n^{th} deck in the ordered set $\mathcal{D} = \{D_k\}_{n=1}^{52!}$ of all possible shuffled decks. The i^{th} card in deck k is $D_k(i)$. We define D_1 as the deck shown in Fig. 1 and $D_{52!}$ as D_1 in reverse order:

$$\begin{aligned} D_1 &= (1, 2, \cdots, 52) \text{ and} \\ D_{52!} &= (52, 51, \cdots, 1). \end{aligned} \tag{1}$$

Problem Statement Statement

Pre-Problem Statement

Before stating our problem, we establish context for the reader. There are many problems [Brunson, 1969], and this is but one of them. We found this one especially perplexing, enough to spend hours of our life solving it, and then even more hours writing about it.

Problem Statement

Now we state our problem:

Problem 1 (Existence)**.** *Find the best shuffled deck[5], which we denote $D^{\heartsuit}$. That is, under any reasonable metric μ, for any other deck $D \in \mathcal{D}$, we have that $\mu(D^{\heartsuit})$ is a better number than $\mu(D)$.*

The heart clearly marks how this is the best, most lovely shuffled deck (also impersonally called optimal). But of course this leads immediately to the problem of how to measure optimality:

Problem 2 (Shuffledness)**.** *How can we quantify the goodness of that shuffle in Fig. 3? That is, construct some μ as in Prob. 1.*

If you feel that this isn't enough problems, dear reader, fret not. We will discover *even more problems* along the way!

6 Finding the Deck

We do not want to reveal how we found the deck, but in the name of openness and science, we begrudgingly provide some insight.

Method: To solve Prob. 1, we mailed a team of graduate students[6] and unsuspecting bypassers[7] to a secret archaeological dig site in As Sabakh al Kabirah, just southeast of Ras Lanuf, Libya, near the coordinates 30°11′49.7″ N, 18°51′52.1″ E. There, we carefully dug many big holes.

Results: In most of the holes, we found useless random old things, but in one of them, we found a bunch of ancient playing card related content, as shown in Fig 5. Amongst this trove was $D^{\heartsuit}$.

Discussion: One of the bypassers realized inadvertently that $D^{\heartsuit}$ was indeed perfect and promptly vanished in a puff of logic [Adams, 1979]. The remainder of the team recognized the risk and began allowing themselves to be aware of only small portions of the deck's perfection at a time, enabling us to prepare this manuscript. We have since discovered that, while the original deck does indeed cause spontaneous butterflication, simply viewing a picture of the deck (see Fig. 3) is safe [qntm, 2021].

Next, we propose a novel approach to shuffling to enable the systematic creation of inferior decks across the entire spectrum of decks, thereby enabling analysis of $D^{\heartsuit}$.

8 Deterministic Shuffling

We seek a way to systematically generate inferior decks to $D^{\heartsuit}$ and thereby prove that we indeed found the most shuffled deck. To do this, we propose a novel[8] *deterministic* approach to shuffling, which bypasses shuffling entirely and just produces a shuffled deck. We define determinism using the DEQN approach [Müller and Placek, 2018]. Because enumerating all shuffled decks would take way too much memory [Ulhaq, 2022, Brunson, 1969], we represent them implicitly as a function:

[4] `https://youtu.be/KQ6zr6kCPj8?si=TDvivn1KPe5G23qo`

[5] We wanted to call her the "very best deck, like no-one ever was; to characterize her is our real test; to find her is our cause". However, this was a bit inconvenient.

[6] We vastly underpaid them, as is the norm in our field.

[7] We did not pay them, as is the norm in our field.

[8] We did not check.

Figure 5: One of our intrepid graduate students excavating a variety of playing card related content.

Problem 3. *Find a monotonically increasing map,* shuffle : $[0, 1] \to \mathcal{D}$*, for which*

$$\begin{aligned} D_1 &= \text{shuffle}(0) \text{ and} \\ D_{52!} &= \text{shuffle}(1). \end{aligned}$$

We denote an arbitrary shuffle as $D_k = \text{shuffle}(x)$. In other words, the shuffle operator should map a real number to one of all possible shuffled 52 card decks.

Remark 4. *We pronounce D_k as "deck", not "D.K.", who is a Nintendo character and has little to do with shuffling.*

Remark 5. *We stopped using "D_k" as notation because finding k is actually pretty hard. But we left the above remark because we thought it was still funny.*

8.1 A Combinatorial Problem

It turns out implementing the shuffle map is tricky. At least, it took more than a couple of hours to think about, and required three separate conversations with computer science professors, each of whom asked, "Why are you wasting your time on this?!" Two of them are now co-authors on this paper.

Anyways, the rough idea is to get something like $k = \text{round}(x \cdot 52!)$, so that the deck index k is monotonically increasing in $x \in [0, 1]$. The issue with this of course is that we would then need to map $k \mapsto D_k$, which is hard. Instead, recall that we labeled all the cards from 1 to 52 in the notation section. So, we can smoosh the card numbers together into a big integer:

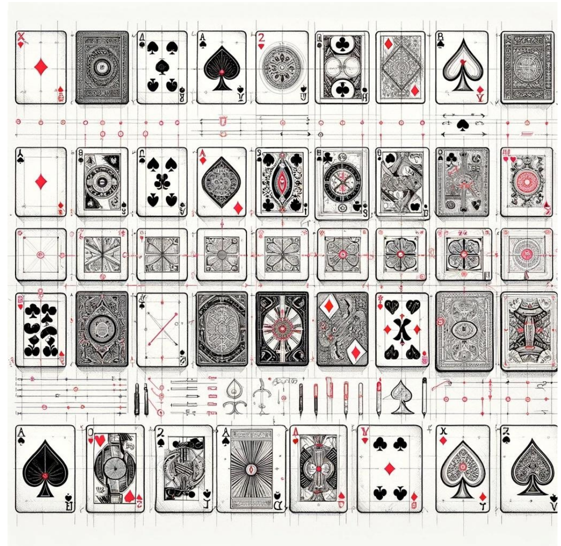


Figure 6: Sketches of some of the cards found at the dig site, displaying their ancient and rich and very historical character, clearly imbued with importance and dignity.

Definition 6 (Valid Deck Integer)**.** *Consider a map* int : $\mathcal{D} \to \mathbb{N}$ *for which*

$$\text{int}(D) = \sum_{i=1}^{52} D(i) \cdot 10^{(104-2i)}, \tag{2}$$

which is an integer with either 103 or 104 digits. An integer given by $\text{int}(D)$ *for some* $D \in \mathcal{D}$ *is called a* valid deck integer.

Note that the deck index $k \in \mathbb{N}_{52!}$ is *not* the same as a valid deck integer. In fact, we give up looking for k, and instead just directly look for a map from $x \in [0, 1]$ to the corresponding valid deck integer.

To do this, first notice that we have smallest and largest valid deck integers. The lower bound is

$$\begin{aligned} L &= \text{int}(D_1) \\ &= 1 \times 10^{102} + 2 \times 10^{100} + \cdots + 52 \times 10^0 \\ &= 1,020,304,050,607,080,910,111,213,141, \cdots \\ &\quad \cdots 516,171,819,202,122,232,425,262,728, \cdots \\ &\quad \cdots 293,031,323,334,353,637,383,940,414 \cdots \\ &\quad \cdots 243,444,546,474,849,505,152, \end{aligned}$$

and the upper bound is

$$\begin{aligned} H &= \mathrm{int}(D_{52!}) \\ &= 52 \times 10^{102} + 51 \times 10^{100} + \cdots + 1 \times 10^{0} \\ &= 52,515,049,484,746,454,443,424,140,393,\cdots \\ &\quad 837,363,534,333,231,302,928,272,625,242,\cdots \\ &\quad 322,212,019,181,716,151,413,121,110,090,\cdots \\ &\quad 807,060,504,030,201 \end{aligned}$$

These bounds turn out to be quite useful:

Lemma 7 (Enumerate ALL the decks!)**.** *Every possible shuffled deck $D \in \mathcal{D}$ can be represented as a valid deck integer* $\mathrm{int}(D) \in [L, H]$.

Proof. Oops, this follows from (2). ♠♥♣♦

The problem is that *most* of the integers from L to H are not *valid deck integers*. For example, consider the integer $222\cdots2$ (i.e., 2 repeated 104 times). To resolve this issue, we need to map $x \in [0,1]$ to some $n \in [L, H]$ and then find the closest valid deck integer to n. We call this *deterministic shuffling*:

$$\arg\min_{D \in \mathcal{D}} q_{52}\Big(\mathrm{int}(D), \mathrm{round}\,\big((H-L)\cdot x\big) + L\Big), \quad (3)$$

where q_{52} is a valid quasimetric on the integers, meaning it obeys the triangle inequality and identity but not necessarily symmetry. That is, for any $a, b, c \in \mathbb{N}$, we have

$$q_{52}(a,a) = 0 \quad (4)$$

$$q_{52}(a,b) = q_{52}(b,a) \iff a = b, \text{ and} \quad (5)$$

$$q_{52}(a,c) \le q_{52}(a,b) + q_{52}(b,c). \quad (6)$$

Since (3) searches over valid deck integers, it is a combinatorial optimization problem, which is hard to solve [Brunson, 1969]. But somehow, we kind of did it!

8.2 The Shuffle Algorithm

We implemented deterministic shuffling as shown in Alg. 1. The algorithm takes in a value $x \in [0,1]$ and outputs a deck $D \in \mathcal{D}$. It first scales x up and rounds it to be an integer n in $[L, H]$. Then it finds the nearest valid deck integer by iterating through each pair of digits of n, starting from highest to lowest. For each pair of digits, we convert it to an integer between 1 and 52, then find the nearest card available from an unshuffled source deck D_0, and put that card into the output deck. The notion of "nearest card" depends on how one implements the distance function q_{52}, which we discuss below.

Remark 8. *You may ask why we did not just do this in base 52. We are wondering the same thing, and in fact, we are just sad that you did not ask us before we implemented everything in base 10. It would have been so much easier.*

Algorithm 1 Deterministic Shuffle: $D = \mathrm{shuffle}(x)$

input: $x \in [0,1]$, H, L
$n \leftarrow \mathrm{round}((H-L)\cdot x) + L$
$D_0 \leftarrow (1, 2, \cdots, 52)$ ▷ initialize source deck
$D \leftarrow (\emptyset)$ ▷ initialize empty output deck
for $i = 52, 51, \cdots, 1$ **do** ▷ iterate digits of n
  // isolate card digits and rescale to $[1, 52]$
  if $i = 1$ **then**
    $c_n \leftarrow \lfloor n \times 10^{-2i+1} \rfloor$ ▷ 1st digit is OK
  else
    $c_n \leftarrow \lfloor n \times 10^{-2i+1} \times \frac{51}{100} \rfloor + 1$
  end if
  $c_i \leftarrow \arg\min_{c_0 \in D_0} q_{52}(c_n, c_0)$ ▷ nearest card
  $D.\mathrm{append}(c_i)$ ▷ add card to D
  $D_0.\mathrm{delete}(c_i)$ ▷ remove used card
  $n \leftarrow n - (c_n \times 10^{2i-2})$ ▷ clear used digits
end for
return D

8.3 The Hunchback of Notre-Distance

It turns out that, depending on how one implements q_{52}, one can get all kinds of different (bad) shuffles. And, in the worst case, we end up having to reintroduce randomness, which defeats the whole point of deterministic shuffling! For example, consider $q_{52}(a,b) = |a-b|$. Suppose that $a = 2$ (i.e., the second card in the unshuffled deck). Then both $b = 1$ and $b = 3$ are equidistant from a, which means we need to implement a random tiebreaker.

To avoid this, we implement a quasimetric that measures the distance from card a to card b as an increasing number in an unshuffled deck that loops around at 52:

$$q_{52}(a,b) = \begin{cases} b - a, & b \ge a \\ (52 - a) + b, & a > b. \end{cases} \quad (7)$$

To understand this, consider the following examples: $q_{52}(1, 52) = 51$ from first to last card, $q_{52}(52, 1) = 1$ from last to first card, and $q_{52}(3, 1) = 50$ from third to first card.

8.4 The Inverse Shuffle

Of course, to enable anything truly useful, we also need a handle on shuffle^{-1}. We don't actually need to hold on to the whole preimage, just an element of the preimage for any deck D. It turns out this is pretty easy, as Alg. 2 shows[9].

Algorithm 2 Inverse Shuffle: $x = \text{shuffle}^{-1}(D)$

```
input: D ∈ 𝒟, H, L
n ← int(D)                          ▷ see (2)
return x ← (n − L)/(H − L)
```

8.5 Our Algorithm is Unfair!

Really we should just end this section, but there was one last interesting question that we wanted to squeeze in: is the deterministic shuffle *fair*? That is, if we draw x uniformly from $[0, 1]$, is every deck equally likely via $D = \text{shuffle}(x)$? Unfortunately, no:

Proposition 9. *Suppose* $x \in [0, 1]$ *is drawn randomly from a uniform distribution, and suppose* shuffle *is implemented as in Alg. 1. Then* $P(\text{shuffle}(x)) \neq \frac{1}{52!}$.

Proof. The only way this would have a chance of being true is if $\frac{H-L}{52!} \in \mathbb{N}$, but unfortunately, it is not. ♠♥♣♦

The remainder is on the order of 10^{66}, whereas $H - L$ is on the order of 10^{103} and $52! \approx 8.7 \times 10^{67}$, so we are not *too far* from a fair deterministic shuffling algorithm. We could probably get the algorithm to be fair by setting $L = 0$ and $H = 52! \times 10^{36}$, for example. But we leave that to future work.

Now that we can generate *and order* decks with Alg. 1, we are ready to evaluate $D^{\heartsuit}$.

4 Investigating Perfection

The most shuffled deck is presented in Fig. 3. Besides its immediate perfection, which is readily apparent even to naïve viewers, we now confirm its perfection via mathematical proof.

4.1 Proof of God's Love

We begin with a simple mathematical test, where we invert the best shuffle. Surprisingly to us, but not to God [Lennon and McCartney, 1963], we get the following result.

Theorem 10. *Consider a function*

$$f(n) = \frac{\pi}{e + \frac{\varphi}{n}}, \tag{8}$$

where $\varphi = \frac{1+\sqrt{5}}{2}$ *(i.e., the golden ratio). Then*

$$\text{shuffle}^{-1}(D^{\heartsuit}) = \underbrace{f \circ f \circ \cdots \circ f(-e^{i\pi})}_{52 \text{ times}}, \tag{9}$$

where i am the imaginary unit. That is, the most shuffled deck is the inverse shuffle of f composed with itself 52 times.

Proof. This surprising result follows directly from the Big Bang. ♠♥♣♦

And boy, if that isn't proof of God's love well then you just need to have a conversation with Her.

4.2 Another Core Result

It is clear that the shuffle operation is idempotent. This has two important implications. First, it eliminates the need for any unfashionable *reshuffling*, thereby reducing its carbon footprint. Second, it brings us to the section's titular "Core Result":

Theorem 11. *Theorem left as an exercise to reader.*

Proof. First, suppose it isn't. But observe that if it isn't then it can't. Therefore it must. ♠♥♣♦

Next, we proceed away from pure theoretical results into empirical territory.

3 Measuring Shuffledness

There are many potential metrics by which one could quantify a shuffle. The most obvious is the sum of the face values of the cards in the shuffle. This is not an effective means of comparing different sequences of cards however as this metric bears no relation to the sequence of cards in the shuffle, due to the commutativity of addition.

We therefore seek a noncommutative operation by which we can reduce a sequence of cards into an easily digestible number like 8 [Silverstein, 2008]. We consulted Wikipedia, ChatGPT, and some computer scientists for sophisticated sounding math that might be applicable to the problem. Finding little that wasn't

[9] This is a theory paper. Numerically unstable code is available. Managing really really big numbers is hard.

muddied immediately with "probability theory" and "randomness" we instead determined to forge our own path [Rogers, 1976]. Note, that by "metric" we do not mean an actual *metric* in the formal sense, but just a function that measures a sense of quality[10] and shuffly goodness.

3.1 L1 Shuffledness

Imagine you want to compute the ℓ_1-distance from any permutation of cards to our "son of a gun" (or equivalently "mightily fine shuffled") deck. Indeed, we can write the ℓ_1-distance from any deck D to our perfectest deck as

$$\mu_{\mathrm{L1}}(D) = \sum_{i=1}^{52} \left| D(i) - D^{\heartsuit}(i) \right| . \qquad (10)$$

Algorithmically, to implement this, we just need to compute 52 differences and add them up. However, as we have been told over and over, "if you want to publish in a top CS theory conference, you need to over-complicate the proof so that people think you're smart even when you're doing something trivial". Sadly, this initial approach does not fit the requirements of the top, elitist publications we want this paper to appear in, and we therefore omit proof as our proof.

This metric is shown for 1,001 different decks, generated using Alg. 1 with evenly-spaced $x \in [0, 1]$, plus the best deck (see Thm. 10) in Fig. 10.

3.2 Differential Shuffledness

We propose a differential metric function that measures how shuffled a deck is by looking at relationships between consecutive cards in our shuffle. As our bartender and co-author Graham Gussack[11] mentioned: "well if I see a 4 of clubs next to a 4 of spades, I'm gonna raise hell." So, we are interested in finding decks where there are not too many very similar consecutive cards.

First, we want to be able to talk about the distance between two consecutive cards $D(i)$ and $D(i+1)$ in a deck D. We write

$$d_{\mathrm{P}}(D; i, i+1) = |D(i) - D(i+1)| \qquad (11)$$

[10]"Are you teaching your students quality?" is a useful question to prompt someone to have a mental breakdown and ride a motorcycle across the USA [Pirsig, 1974].

[11]He makes the best drinks and we love him.

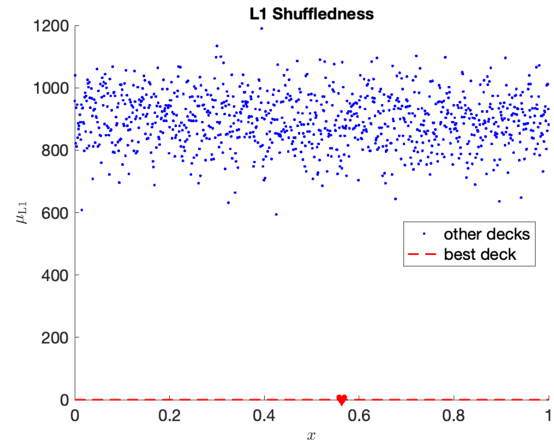


Figure 7: L1 shuffledness, with the best deck shown as a heart. Isn't she lovely?

We named our differential function d_{P}, after differential privacy for the strong privacy-preserving protections provided by d_{P}.

We now define our differential metric over the entire deck:

$$\mu_{\mathrm{DP}} = \frac{1}{52} \sum_{i=1}^{51} d_{\mathrm{P}}(D; i, i+1) \qquad (12)$$

Indeed, it does not encode any information about the very best deck. In fact, the metric function has actually absolutely no relationship whatsoever to the best deck. You don't need to be *close* to the best deck to be a *well-shuffled* deck[12], just be yourself man!

This metric is illustrated for 1,001 decks in Fig. 8.

3.3 Card Distance Shuffledness

Inspired by the above blasphemy about differential shuffledness, we propose a similar metric that measures the cumulative card distance via q_{52} as in (7):

$$\mu_{52}(D) = \sum_{i=1}^{51} q_{52}(D(i), D(i+1)). \qquad (13)$$

An evaluation of this metric on 1,001 decks is shown in Fig. 9.

3.4 Inverse Shuffle Shuffledness

We have an inverse shuffle, so we'll use it!

$$\mu^{-1}(D) = \left| \mathrm{shuffle}^{-1}(D^{\heartsuit}) - \mathrm{shuffle}^{-1}(D) \right| . \qquad (14)$$

[12]This author has been punished for heresy.

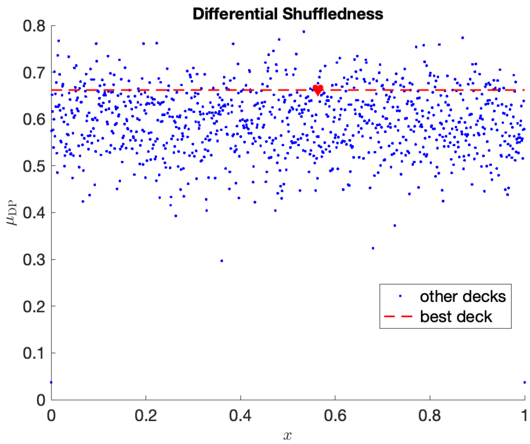


Figure 8: DP shuffledness, with the best deck shown as a heart. Look at how good it is!

Evaluation of this metric on 1,001 decks is shown in Fig. 10. It creates a very pretty pattern. We will investigate why in future work.

3.5 Rounding Error Shuffledness

Our rushed numerical implementation has resulted in the fun fact that, for most $x \in [0, 1]$,

$$x \neq \text{shuffle}^{-1}(\text{shuffle}(x)), \tag{15}$$

mostly due to rounding errors. So, we propose to measure how shuffled a deck is by how bad our numerical implementation is:

$$\mu_{\text{err}}(D) = \sum_{i=1}^{52} \left| D(i) - \hat{D}(i) \right| \tag{16}$$

where $\hat{D} = \text{shuffle}(\text{shuffle}^{-1}(D))$. Values of this metric for 1,001 decks are shown in Fig. 11.

3.6 Shuffledness via Scarcity

According to the most basic laws of economics [Moneymaker, 1970], something that is more scarce is more valuable. Thus, we propose a *scarcity metric*[13]:

$$\mu_{\text{btc}}(D) = \begin{cases} 1 \text{ ฿} & D = D^{\heartsuit} \\ 0 & \text{otherwise.} \end{cases} \tag{17}$$

It should be immediately clear to a reader why this metric is valuable: it is not only simply and crisply defined, but also enjoys strong privacy-preserving properties, as it is implemented using the blockchain [Bankman-Fried, 2021]. Computing and releasing the metric does not reveal any information about which non best deck (carefully defined as a deck that is not $D^{\heartsuit}$), an agent is computing the distance from. We leave the study of the actual usefulness of this metric to future work.

As you probably have observed, this metric is shown for 1,001 decks in Fig. 12.

[13]Please don't mine more bitcoin, as this may decrease the value of our metric.

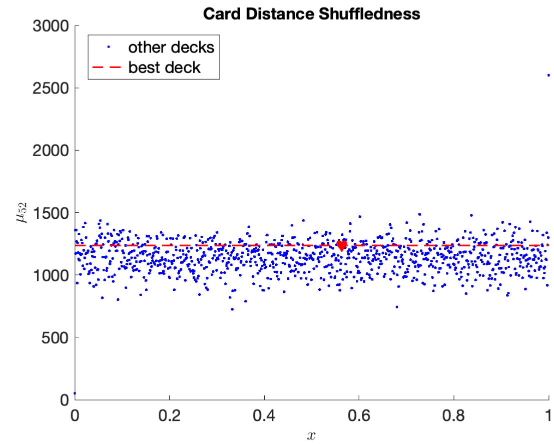


Figure 9: Card distance shuffledness, with the best deck shown as a heart. They're all pretty well shuffled, but the best deck is the best. Notice how poorly shuffled D_1 is in terms of card distance – and notice how well shuffled $D_{52!}$ is! Fascinating!

3.7 Combat Shuffledness

As is commonly known across human cultures, the only true constant is war, specifically the card game, "war" [Tzu, 499 BC]. We played a game of war with $D^{\heartsuit}$ and it took 76 turns, which was fun. So, we propose the following combat-based metric:

$$\mu_{\text{war}}(D) = (\text{\# of turns of war with } D). \tag{18}$$

This metric is illustrated for 1,001 decks in Fig. 13.

3.8 E-Shuffledness

My coauthors have taken a fairly strange definition of shuffling above—and I intend to protest that decision here. It's true, the shuffled deck of Figure 3 will provide to you a fairly stimulating game of Go Fish (if you're into that sort of thing). But we also must

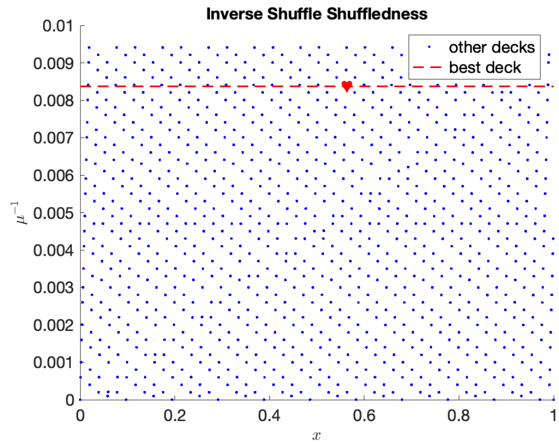


Figure 10: Inverse shuffle shuffledness, with the best deck shown as a heart. This one turned out super weird and we are not sure why, but it is pretty.

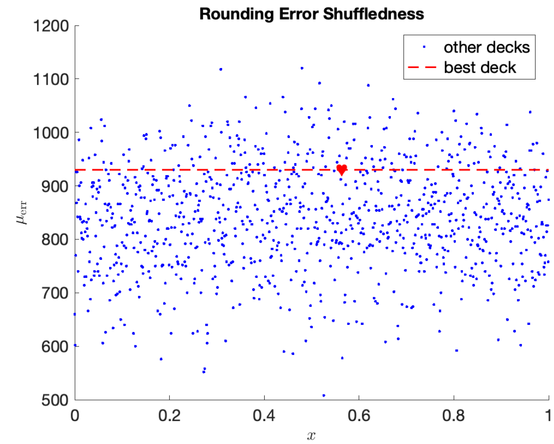


Figure 11: Rounding error shuffledness, with the best deck shown as a heart. The best deck is, as always, clearly the best.

be forward looking in our study to the games and rule sets yet to be invented. Recently, many games have emerged aiming to mix in word-play style rules with french deck playing card games: *Parlay* [Oviedo, 2009] for example, pitches itself as a cross between poker and word making, and *Tryce* [Oviedo, 2009] pitches itself as contract a rummy variant where the necessary contracts consist of the normal sets and runs but also adds words[14]. In this section, we move at least five steps further by providing the best *e-shuffling* of a french deck, when the shuffled deck is written out as a string and where the shuffledness of that deck is assessed based on the distribution of the letter E. In doing so we enable a suite of new french deck games, whose rules will be the subject of future work to be developed over the coming century.

Our convention is to encode a deck using a string of card names, each separated using a comma-space: a deck might begin, e.g., "ACE OF SPADES, TWO OF HEARTS," and may end with, "FOUR OF CLUBS, QUEEN OF DIAMONDS". Using this convention, any e-shuffled deck will produce a string of exactly 843 characters and all possible deck-strings will have exactly the same character frequencies regardless of the amount of e-shuffling[15].

Non-surprisingly, the most common letters in any deck-string will be O (appearing 73 times), S (73 times), E (70 times) and F (60 times). The most likely letters to appear within a card name are O (appearing in the names of all 52 cards), F (all 52 cards), S (all 52 cards), and E (42 cards). Reiterating however, the inflated frequencies of the letters O, F, and S are non-surprising since OF appears in every card name and since a suit, such as CLUBS, will always be pluralized.

What is indeed surprising is the prevalence of the letter E in card names and deck-strings. There are card names such as KING OF CLUBS or FOUR OF DIAMONDS which do not contain an E at all, and nonetheless it is true that there are more E's in a deck than F's[16].

To that end, in this section, we present the most e-shuffled french deck of cards, determined using the distribution of the letter E in the deck. Now let's walk this statement back:

Lemma 12. *It is possible to find two cards that are of the same string length and have the same placement of Es:*

Proof. Check it out:

- FIVE OF HEARTS, and NINE OF HEARTS
- FOUR OF SPADES, JACK OF SPADES, and KING OF SPADES, and
- TWO OF CLUBS, and SIX OF CLUBS.

[14] The author has played neither game, and thus is unable to comment fully on how forward-looking either game is.

[15] This project is funded by NSF CAREER Award 0.46334538254 (apply the inverse shuffle function in Algorithm 1 to find the actual proposal number, sorry)

[16] There are even more E's in the string "FRENCH DECK" than O's F's and S's combined.

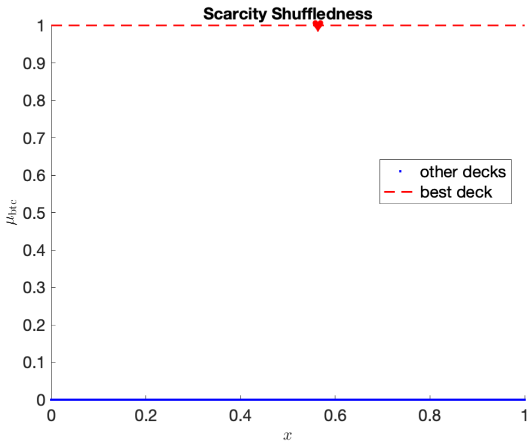


Figure 12: Scarcity shuffledness, with the best deck shown as a heart. It's actually the best this time! Buy more bitcoin!

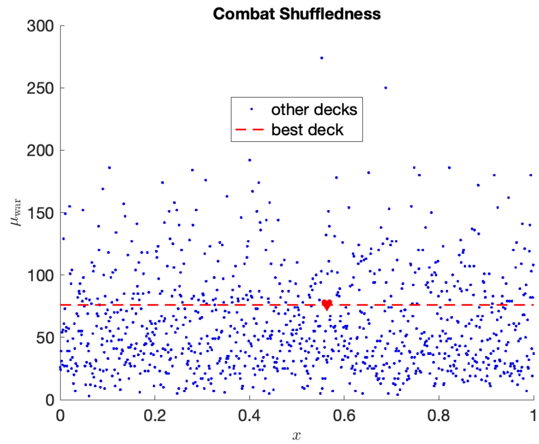


Figure 13: Combat shuffledness, with the best deck shown as a heart. We also have a histogram of the number of turns all the games took floating around somehewere.

So, for every deck that is most e-shuffled there are also $2^4 3^4 2^4 = 20736$ other decks which are also most e-shuffled. ♠♥♣♦

Anyway, consider $F : \{D \,|\, D \text{ is a deck}\} \to \mathbb{N}^{70}$ that operates on a deck D and produces an ordered tuple of 70 integers that are the locations of the letter E in the string version of D:

$$F_i(D) = (\text{location of the } i^{\text{th}} \text{ instance of “E” in the string } D). \tag{19}$$

We refer to $F(D)$ as the distribution corresponding to D, and for any $F(D)$ there will be many D' with $F(D) = F(D')$: we show above at least $2^4 3^4 2^4 = 20736$ decks satisfy each feasible distribution.

Regardless, we study two notions of distribution e-shuffledness: entropy and distribution variance. To compute the distribution variance σ^2 we use:

$$\mu(D) = \frac{1}{70} \sum_{i \in F(D)} i \tag{20}$$

$$\sigma^2(D) = \frac{1}{70} \sum_{i \in F(D)} (i - \mu(D))^2 \tag{21}$$

To compute the entropy, we compute a distribution of the inter-E distances, compute the variance of that distribution, and then multiply by -1. Using this approach, a high-entropy deck will better approximate a uniform distribution of Es. The most e-shuffled deck for each criterion is presented in Figure 14, and I've included three such most e-shuffled decks as to allow the reader to choose the best most e-shuffled deck for their setting.

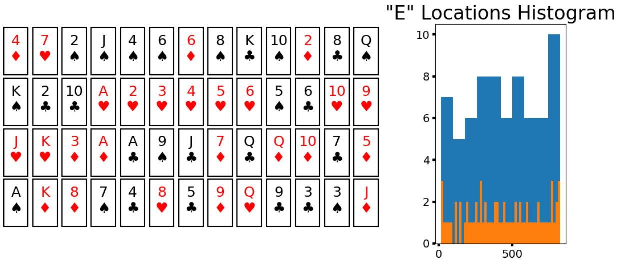
Each deck of Figure 14 was identified computationally, using a brute force approach that was stopped early to accommodate this paper's submission deadline. In this way, the reader should treat each provided deck as though the true most shuffled deck is at least as e-shuffled as that one, and we're already aware of 20735 other decks which are[17].

10 Applied Shuffledness

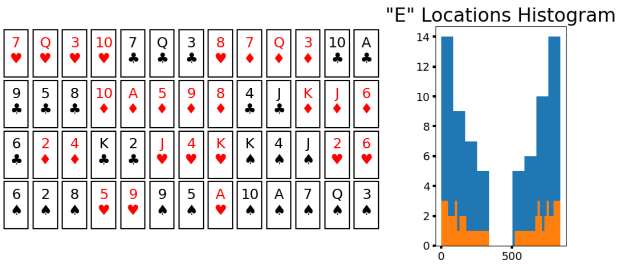
these its Firstly, algorithm order on elements, establish results. to process mathematical deterministic reliability a algorithm Creating patterns. guarantee consistent perhaps to systematic the ensures Lastly, operations fixed test and repeatability rigorously a or across shuffling based elements devise Next, design various predefined meticulous scenarios. for rearranging a randomness. for eliminate involves

unraveling fulfillment, akin profound realms, beauty intricate to a exhilarating joy in mathematics fueling in pleasure Engaging every of problem patterns, sense The a and equation elegant uncharted there's an insatiable conquered, discovery. solutions. curiosity for finding proof of journey unique for With a more. each from traversing It's stems feeling.

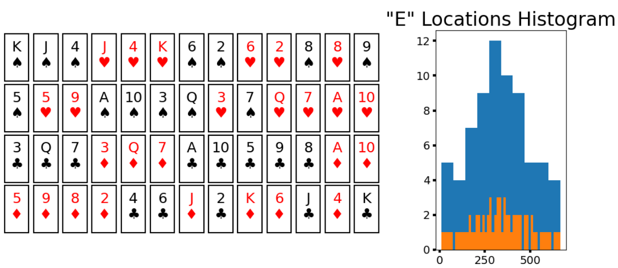
[17] At `https://github.com/mattabate/wordplay/tree/main/sigbovik2024`, one can find the code used for this search, and a python implementation of our deck plotting function.



(a) Maximum entropy of the letter E.



(b) Maximum variance of the letter E.



(c) Minimum variance of the letter E.

Figure 14: Measuring e-shuffledness using the distribution of the letter E when a deck of cards is written out as a string. Subfigures: (a) the deck that maximizes the entropy of the distribution. (b) the deck that maximizes the variance of the distribution. (c) the deck that minimizes the variance of the distribution. For each deck, the distribution of the letter E is depicted using a histograms: the x-axis is position in deck string, the y-axis is number of occurrences of the letter E where the blue histogram uses 10 buckets and the orange histogram uses 52 buckets.

evokes

depictions transforms Each imagination. fresh joy of image scientific creativity montages lies for research DALL-E shuffling Exploring concepts prowess with algorithms, DALL-E's card of artistry with in unveils touch into a a animated visualizing vibrant, deck the blending From inquiry. boundless dynamic intricate sparks experience. perspectives, whimsical crafted excitement. to and research

9 Playing with Perfection

The fact that $P = NP$ is well known in the card playing community[18], but we now know we can do better. We seek to understand just *how delightful* every gaming experience could possibly be with the most shuffled deck. As a generalization of all possible card games, we have chosen gin rummy, both for its aesthetic phonemes, and for its actual apparent relevance to other people's research [Shankar, 2022, Goldman et al., 2021, Eicholtz et al., 2021].

Experiment Design: Two participants played five games of gin rummy [Heinz, 1890] back-to-back: first with an unshuffled deck (as a control), then three with decks of inferior shuffledness, and finally one with the most shuffled deck. To ensure a controlled and fair evaluation, the participants were supplied with a shot of gin or rum before each game, consumed 30–90 seconds before play (while shuffling the deck). The games were played, and then the participants surveyed with the following question: "which game was the most fun?" Approval was obtained for this human trial from the IRB[19].

Results: Overall, the participants reported the game with the most shuffled deck as the most fun. Playing gin rummy with the control unshuffled deck was severely boring, predictable, and disappointing. The three games with decks of inferior shuffledness were split on which one was most fun; the winner reported having more fun, and the loser reported having less fun. The final game, with the most shuffled deck, ended up being the most fun, not only because it was a tiebreaker, but also because the researchers needed supporting evidence for the claims made in this paper.

Discussion: As expected, the most shuffled deck was the most fun to play with. We note that, due to budgerigary[20] restrictions, there was significant overlap ($r^2 = 0.999...$) between the researcher and participant populations. Furthermore, a record of the exact number and quantity of shots of gin and rum consumed was, for reasons that we still do not understand, lost. There was also pizza at some point.

[18] P of course meaning "Pretty shuffled" and NP "Nice to Play with."

[19] Institute of Raunchy Beverages.

[20] Budgerigars, or parakeets, are renowned for their ability to play gin rummy, but none were available for our experiments, hence we needed to use human participants.

7 Conclusion

Gee, that sure was a great paper. We're planning to write an even better follow-up for next year: "Check Your Deck Privilege".

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A Appendices

A.1 The Bestest Deck

Here she is as an integer:

$$
\begin{aligned}
29{,}461, 243, 163, 330, 132, 109, 263, 403, 470, \cdots \\
205, 013, 845, 203, 144, 254, 219, 393, 515, \cdots \\
414, 849, 402, 436, 500, 432, 182, 322, 085, \cdots \\
137, 075, 217, 281, 411, 271, 006.
\end{aligned}
$$

A.2 Classic Shuffler

This code might even compile.

```
PROGRAM CARD_SHUFFLE

      INTEGER DECK(52), I, J, K, TEMP, SEED

      DATA DECK/52*0/
      CALL SRAND(1234)

C Shuffle the deck
 10   I = 1
 20   J = INT(RAND(0) * 52) + 1
 30   K = INT(RAND(0) * 52) + 1
      TEMP = DECK(J)
      DECK(J) = DECK(K)
      DECK(K) = TEMP
      I = I + 1
      IF (I .LT. 52) GOTO 20

C Print the shuffled deck
      WRITE(*, '(A)') 'shuffled deck:'
      DO I = 1, 52
         WRITE(*, '(I3)') DECK(I)
      ENDDO

      END
```