Quadratic Logic

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1 Abstract

The well-known linear logic system represents a world where, whenever a proposition is used, it is also consumed. However, this system does not generalize well to the real world, where economies of scale dictate that producing more copies of something at one time will take less resources than producing each copy individually. To model this, we remove the strict one-to-one correspondence between propositions on the left-hand side of a sequent and the propositions they produce on the right-hand side when consumed by rule application. By replacing the linear function with a quadratic function, we are able to model the real world more effectively under this system, while still preserving linearity in some cases.

2 Introduction

The system of linear logic stipulates that when an left rule is used to create a new proposition, any propositions used in the creation process are removed. This means that throughout the course of a linear proof, things are consumed and reused in a one-for-one correspondence. Having an A and using it to come to the conclusion of B is sufficient to create the proposition $A \multimap B$, consuming the A and the B in the process. Likewise, having an $A \multimap B$ and an A is sufficient to remove both and create a B, but the A and the $A \multimap B$ cannot be used later on. The meaning of linear logic becomes clear from these examples – put in one copy of the necessary ingredients to get one copy of a proposition.

It follows that multiple copies – for example, n copies – of a proposition can be created by putting in n copies of each of the necessary premises. However, this does not match up with reality. It is often more equivalent to create things in large batches, or otherwise utilize fewer resources to create more things than could be individually made using the same amount of resources.[1] This reflects how things work in the real world.[2] Therefore, we seek to model this with a new set of rules, which better reward proof authors for efficiently using left rules by rewarding them with additional copies of the proposition based on the number of copies of the premises used.

[1] This is where the common grocery store advertisement of "the more you spend, the more you save" becomes relevant.

[2] The author recognizes that computer science theory often does not seek to represent the real world, as evidenced by the amount of time spent on [insert problem you hate here].

The next question to consider is how much of a reward one should get from a multiple-introduction of propositions. Increasing the number of propositions created by a linear factor proves problematic, as a device comes about to generate infinitely many copies of a proposition as long as the cut and identity rules hold. This tends to over-reward introducing small numbers of propositions, while not truly appreciating the genius that comes with consuming 10 copies of A and B each to introduce some number of copies of $A \otimes B$.[3] Therefore, we need to consider what function $f(n)$ should be used to determine how many copies of a proposition should be introduced when n copies of each of the premises are used. Functions of degree 3 and above, as well as exponential functions, grow to concerningly large numbers even on small n.[4] Based on some preliminary experiments, it seems that functions in the neighborhood of $\Theta(n^2)$ form the ideal candidates. For now, we proceed with the most basic function in this class – $f(n) = n^2$.

3 Rules

The following set of rules, shown in Figure 1 on the next page, are adapted from Frank Pfenning's rules for linear logic. [1] We introduce the notation A true $[n]$ to indicate having $n > 0$ copies of proposition A, but the rules otherwise remain quite similar to their original formulation. Additionally, since we are trying to focus on quadratic functions in this work, we remove the exponential entirely since $2^n \notin \Theta(n^2)$.

The notable rules in this set are the id rule, which enables us to square the quantity of an atomic proposition should it be necessary. It also moves it to the right-hand side. The split rule is also new, and allows us to split up a group of n propositions into two groups of size x and $n - x$, where $0 < x < n$. Combine inverts a split, combining two groups of the same proposition which have sizes m and n into a single group with size $m + n$ on either side of the sequent. For simplicity, we only permit positive integer numbers of propositions to be taken at a time. The remainder of the rules operate more or less as expected, with left rules using n copies of each premise and creating n^2 copies of whatever is being introduced.

4 Identity and Cut

As it turns out, the linear identity and cut theorems still hold here. To prove these, we begin by proving a pair of very important lemmas to relate this back to the original system of linear logic.

> **Lemma** (Quadratic Evaluation)
> If we let $f(n) = n^2$, then $f(1) = 1$.
> **Proof:** $f(1) = 1^2 = 1 \cdot 1 = 1$. The remainder of the proof is left as an exercise to the reader, or to any friends of the reader who happen to have an interest in foundational mathematics.

[3] We needed some $A \otimes B$s to give out to 2023 SIGBOVIK attendees as souvenirs, but did not have enough for everyone. :(

[4] Since this is a computer science paper, we opt to not acknowledge the existence of numbers above 2,147,483,647.

Rules for Quadratic Logic

$$\frac{(P\ \text{atomic})}{P\ \mathsf{true}\ [n] \Vdash P\ \mathsf{true}\ \left[n^2\right]}\ id \qquad \frac{\Gamma, A\ \mathsf{true}\ [x]\,, A\ \mathsf{true}\ [n-x] \Vdash B\ \mathsf{true}\ [m]}{\Gamma, A\ \mathsf{true}\ [n] \Vdash B\ \mathsf{true}\ [m]}\ split$$

$$\frac{\Gamma_1 \Vdash A\ \mathsf{true}\ [m] \qquad \Gamma_2 \Vdash A\ \mathsf{true}\ [n]}{\Gamma_1, \Gamma_2 \Vdash A\ \mathsf{true}\ [m+n]}\ combine\ R \qquad \frac{\Gamma, A\ \mathsf{true}\ [m+n] \Vdash B\ \mathsf{true}\ [p]}{\Gamma, A\ \mathsf{true}\ [m]\,, A\ \mathsf{true}\ [n] \Vdash B\ \mathsf{true}\ [p]}\ combine\ L$$

$$\frac{\Gamma_1 \Vdash A\ \mathsf{true}\ [n] \qquad \Gamma_2 \Vdash B\ \mathsf{true}\ [n]}{\Gamma_1, \Gamma_2 \Vdash A \otimes B\ \mathsf{true}\ \left[n^2\right]}\ \otimes R \qquad \frac{\Gamma, A\ \mathsf{true}\ [n]\,, B\ \mathsf{true}\ [n] \Vdash C\ \mathsf{true}\ [m]}{\Gamma, A \otimes B\ \mathsf{true}\ [n] \Vdash C\ \mathsf{true}\ [m]}\ \otimes L$$

$$\frac{\Gamma \Vdash A\ \mathsf{true}\ [n] \qquad \Gamma \Vdash B\ \mathsf{true}\ [n]}{\Gamma \Vdash A \& B\ \mathsf{true}\ \left[n^2\right]}\ \& R \qquad \frac{\Gamma, A\ \mathsf{true}\ [n] \Vdash C\ \mathsf{true}\ [m]}{\Gamma, A \& B\ \mathsf{true}\ [n] \Vdash C\ \mathsf{true}\ [m]}\ \& L1 \qquad \frac{\Gamma, B\ \mathsf{true}\ [n] \Vdash C\ \mathsf{true}\ [m]}{\Gamma, A \& B\ \mathsf{true}\ [n] \Vdash C\ \mathsf{true}\ [m]}\ \& L2$$

$$\frac{\Gamma \Vdash A\ \mathsf{true}\ [n]}{\Gamma \Vdash A \oplus B\ \mathsf{true}\ \left[n^2\right]}\ \oplus R1 \qquad \frac{\Gamma \Vdash B\ \mathsf{true}\ [n]}{\Gamma \Vdash A \oplus B\ \mathsf{true}\ \left[n^2\right]}\ \oplus R2$$

$$\frac{\Gamma, A\ \mathsf{true}\ [n] \Vdash C\ \mathsf{true}\ [m] \qquad \Gamma, B\ \mathsf{true}\ [n] \Vdash C\ \mathsf{true}\ [m]}{\Gamma, A \oplus B\ \mathsf{true}\ [n] \Vdash C\ \mathsf{true}\ [m]}\ \oplus L$$

$$\frac{\Gamma, A\ \mathsf{true}\ [n] \Vdash B\ \mathsf{true}\ [n]}{\Gamma \Vdash A \multimap B\ \mathsf{true}\ \left[n^2\right]}\ \multimap R \qquad \frac{\Gamma_1 \Vdash A\ \mathsf{true}\ [n] \qquad \Gamma_2, B\ \mathsf{true}\ [n] \Vdash C\ \mathsf{true}\ [m]}{\Gamma_1, \Gamma_2, A \multimap B\ \mathsf{true}\ [n] \Vdash C\ \mathsf{true}\ [m]}\ \multimap L$$

$$\frac{}{\cdot \Vdash 1\ \mathsf{true}\ [n]}\ 1R \qquad \frac{\Gamma \Vdash C\ \mathsf{true}\ [n]}{\Gamma, 1 \Vdash C\ \mathsf{true}\ [n]}\ 1L$$

$$\frac{}{\Gamma \Vdash \top\ \mathsf{true}\ [n]}\ \top R \qquad \frac{}{\Gamma, 0\ \mathsf{true}\ [n] \Vdash C\ \mathsf{true}\ [m]}\ 0L$$

Figure 1: The inference rules for quadratic logic.

Lemma (Linear-Quadratic Correspondence[a])

Any valid proof under linear logic without the exponential is also a valid proof under quadratic logic when $f(n) = n^2$ if we treat a proposition judged as true in linear logic to have the judgement $\text{ true }[1]$ in quadratic logic.

Proof: Observe that any linear left rule is the same as a quadratic left rule where the parameters for all the $\text{ true }[]$ judgements are set to 1. Any linear right rule is the same as a quadratic right rule where the parameters of both the inputs and the outputs are 1. By the Quadratic Evaluation Lemma, it holds that any quadratic logic rule is the same as a linear logic rule in this case, because $n = 1$ means $f(n) = n^2 = 1$. This is good, because it allows us to use quadratic rules as linear rules in this case by replacing any instance of the true judgement with the $\text{ true }[1]$ judgement. Since all left rules now consume one copy of each premise and produce one copy of each result, simply substituting any linear rules with the corresponding quadratic rules transforms the linear proof into a quadratic proof.

[a] The authors do not think this is much of a correspondence, but every PL paper seems to be required to feature at least one correspondence.

From this second lemma, some useful results from linear logic can be translated to quadratic logic.

Corollary (Linear Identity)

It is true for any proposition A that $A \text{ true } [1] \Vdash A \text{ true } [1]$.

Proof: Once again, we take the inductive proof of identity in linear logic and translate it to quadratic logic via the Linear-Quadratic Correspondence. Since the corollary that we are proving is just a linear logic sequent rephrased as a quadratic logic sequent, we are done.

Corollary (Linear Cut)

If $\Gamma_1 \Vdash A \text{ true } [1]$ and $\Gamma_2, A \text{ true } [1] \Vdash B \text{ true } [1]$, then $\Gamma_1, \Gamma_2 \Vdash B \text{ true } [1]$.

Proof: This is the same as the linear cut theorem, only translated to use the quadratic syntax for "one copy of a proposition being true". The proof is the same as in linear logic, after conversion via the Linear-Quadratic Correspondence lemma.

4.1 Linear Identity Revisited

There are two versions of the identity theorem that can hold. One of them is the standard linear identity, applied to propositions in any quantity.

Theorem (Linear Identity 2: Electric Boogaloo)

$A \text{ true } [n] \Vdash A \text{ true } [n]$ for any proposition A and any positive integer n.

To prove this, we proceed by a nested induction. Firstly, as with normal linear identity, we take as inductive hypothesis that for any smaller propositions, this theorem holds.[5] We then proceed by induction on n.

The base case comes when $n = 1$. In this case, we have no choice but to actually break down the proposition and prove the identity rule, using the proof from the earlier Linear Identity proof.

The inductive step is slightly more interesting. Let $\mathcal{D}$ represent the derivation of the $n = 1$ case from linear identity. Then the following represents a proof that A true $[n] \Vdash A$ true $[n]$:

$$A[1] \Vdash A[1] \qquad \text{(by derivation } \mathcal{D}\text{)}$$

$$A[n-1] \Vdash A[n-1] \qquad \text{(inductive hypothesis)}$$

$$A[n-1], A[1] \Vdash A[n] \qquad \text{(combine } R \text{ rule)}$$

$$A[n] \Vdash A[n] \qquad \text{(split rule)}$$

4.2 Quadratic Identity

Linear identity is a nice theorem to have around, however, for the purposes of proving things like cut later, it proves to be too weak. In the spirit of asking "can we do better?", the proof of identity once again can be reconsidered to strengthen the theorem more.

Theorem (Quadratic Identity)
A true $[n] \Vdash A$ true $\left[n^2\right]$ for any proposition A and any positive integer n.

Even only using the old linear identity theorem, we have sufficient machinery to prove this theorem! The atomic case comes about from the *id* rule, which directly proves the theorem for atomic propositions. For propositions built up with one or more connectives, we must perform more proof using the previous linear identity theorem. One example of such an proof is provided. The remainder are similar.

Example ($\oplus$ Case)
We aim to show $A \multimap B$ true $[n] \Vdash A \multimap B$ true $\left[n^2\right]$.
Proof:

$$A \text{ true } [n] \Vdash A \text{ true } [n] \qquad \text{(by linear identity)}$$

$$B \text{ true } [n] \Vdash B \text{ true } [n] \qquad \text{(by linear identity)}$$

$$A \multimap B \text{ true } [n], A \text{ true } [n] \Vdash B \text{ true } [n] \qquad (\multimap L \text{ rule})$$

$$A \vdash B \text{ true } [n] \Vdash A \multimap B \text{ true } \left[n^2\right] \text{ true } \left[n^2\right] \qquad (\multimap R \text{ rule})$$

Observe that these proofs of quadratic identity never invoke the quadratic identity rule as an inductive hypothesis. Doing so would result in having many extra copies of propositions from multiple layers of squaring, which could lead to undesirable outcomes since there is nowhere for the extra copies of propositions to go.[6]

[5] This is not directly used, but is necessary to allow the $n = 1$ case to invoke linear identity.
[6] These extra propositions could eventually reach critical mass, creating undesirable side effects such as forming a black hole. Black holes are not specified in the rules and would be likely considered undefined behaviour.

4.3 Quadratic Cut Theorem

Now that quadratic identity has been proven, we are now ready to approach the problem of cut. As it turns out, the cut theorem not only holds when $n = 1$, it also holds in the general case!

Theorem (Cut)
If $\mathcal{D} = \Gamma_1 \Vdash A \text{ true } [n]$ and $\mathcal{E} = \Gamma_2, A \text{ true } [n] \Vdash C \text{ true } [m]$ then $\Gamma_1, \Gamma_2 \Vdash C \text{ true } [m]$.

The proof of this is via induction. For a particular case, the inductive hypothesis is that the cut theorem holds on any proposition A' and two derivations $\mathcal{D}'$ and $\mathcal{E}'$, where one of the following hold: A' is smaller than A, or $A' = A$ and $\mathcal{D}'$ is smaller than $\mathcal{D}$, or $A' = A$, $\mathcal{D}' = \mathcal{D}$, and $\mathcal{E}'$ is smaller than $\mathcal{E}$.

This is a fairly standard setup for a cut proof. [2] As such, we present two cases for the $\otimes$ connective as examples of how this proof proceeds. The remainder of the proof is similar to these cases.[7]

Example (Principal Case)

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}_1}{\Gamma_1 \Vdash A \text{ true } [n]} \quad \overset{\mathcal{D}_2}{\Gamma_2 \Vdash B \text{ true } [n]}}{\Gamma_1, \Gamma_2 \Vdash A \otimes B \text{ true } \left[n^2\right]} \otimes R \qquad \mathcal{E} = \dfrac{\overset{\mathcal{E}_1}{\Gamma_3, A \text{ true } \left[n^2\right], B \text{ true } \left[n^2\right] \Vdash C \text{ true } [m]}}{\Gamma_3, A \otimes B \text{ true } \left[n^2\right] \Vdash C \text{ true } [m]} \otimes L$$

We then show that $\Gamma_1, \Gamma_2, \Gamma_3 \Vdash C \text{ true } [m]$.

$$\begin{aligned}
\Gamma_2 &\Vdash B \text{ true } [n] && \text{(1 - by } \mathcal{D}_2\text{)} \\
B \text{ true } [n] &\Vdash B \text{ true } \left[n^2\right] && \text{(2 - by identity)} \\
\Gamma_2 &\Vdash B \text{ true } \left[n^2\right] && \text{(3 - by IH on } (B, (1), (2)) \text{ since } B < A \otimes B\text{)} \\
\Gamma_2, \Gamma_3, A \text{ true } \left[n^2\right] &\Vdash C \text{ true } [m] && \text{(4 - by IH on } (B, (3), \mathcal{E}_1) \text{ since } B < A \otimes B\text{)} \\
\Gamma_1 &\Vdash A \text{ true } [n] && \text{(5 - by } \mathcal{D}_1\text{)} \\
A \text{ true } [n] &\Vdash A \text{ true } \left[n^2\right] && \text{(6 - by identity)} \\
\Gamma_1 &\Vdash A \text{ true } \left[n^2\right] && \text{(7 - by IH on } (A, (5), (6)) \text{ since } A < A \otimes B\text{)} \\
\Gamma_1, \Gamma_2, \Gamma_3 &\Vdash C \text{ true } [m] && \text{(by IH on } (A, (7), (4)) \text{ since } A < A \otimes B\text{)}
\end{aligned}$$

Example (Side Case)

$$\mathcal{D} = \dfrac{\overset{\mathcal{D}_1}{\Gamma_1, A \text{ true } [n], B \text{ true } [n] \Vdash D \text{ true } [k]}}{\Gamma_1, A \otimes B \text{ true } \left[n^2\right] \Vdash D \text{ true } [k]} \otimes L \qquad \overset{\mathcal{E}}{\Gamma_2, D \text{ true } [k] \Vdash C \text{ true } [m]}$$

We then show that $\Gamma_1, \Gamma_2, A \otimes B \text{ true } \left[n^2\right] \Vdash C \text{ true } [m]$.

$$\begin{aligned}
\Gamma_1, \Gamma_2, A \text{ true } [n], B \text{ true } [n] &\Vdash C \text{ true } [m] && \text{(by IH on } (A \otimes B, \mathcal{D}_1, \mathcal{E}) \text{ since } \mathcal{D}_1 < \mathcal{D}\text{)} \\
\Gamma_1, \Gamma_2, A \otimes B \text{ true } \left[n^2\right] &\Vdash C \text{ true } [m] && \text{(by } \otimes L \text{ rule)}
\end{aligned}$$

[7] This would normally be left as an exercise to the reader, but having to prove this many cases would likely make any readers stop reading.

5 Applications

The system created here might, at first glance, appear to be quite limited in its functionality. Starting with two copies of a proposition can be a massive problem when trying to create three copies – the options are to leave them as two or move up to four. To combat this problem, a new judgement can be introduced:

Definition
The judgement A so true $[n]$ is defined as A true $[n']$, where $n' \geq n > 0$.

Intuitively, we can think of A so true $[n]$ as being similar to A true $[n] \otimes \top$ true $[1]$, in that we keep the n copies of A that we need while using the remainder to prove the $\top$ part of the product. By proving conclusions as so true instead of true, creative logicians will be able to manipulate the rules of quadratic logic to give themselves a more flexible version of linear logic.

Having come from linear logic, it is no surprise that there are many similarities between quadratic and linear logic. Having one copy of a proposition makes that proposition behave more or less in a linear way – there is no way to use the identity rule to get more copies of such a proposition. What is slightly more surprising is that propositions with a count greater than 1 can have as many copies of necessary being generated through repeated use of the identity rule, as long as there is a so true judgement somewhere to take care of extra copies that may be created.

This means that propositions that are defined multiple times actually behave in a style more consistent with a standard sequent calculus, or the exponential in linear logic. This insight indicates that quadratic logic can be used to blend linear and standard styles. In fact, the way quadratic logic allows users to limit how many times something is used based on the number of calls to the identity rule means that there is a tighter degree of control afforded to the proof writer.[8]

6 Extensions to Further Quadratics

This whole paper has been, so far, devoted to using the function n^2 to determine the number of proposition copies introduced on the right side from right rules. However, what happens when we have a different quadratic as the function? There are several interesting things that could occur.

Firstly, the function could evaulate to a negative number. In this case, we go into *propositional debt* at this point, where to recover the remainder of the proof, the negative copies of a proposition must be offset by positive copies from elsewhere in the proof tree.[9] This debt cannot get swept up by a so true judgement of some form due to the definition of the judgement, meaning that the prover is actually responsible for any debt that is incurred. This debt will actually prevent the proof from being complete as well, because it will interfere

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[8] Some might go as far as to say this makes the user "resource aware".
[9] This brings a new kind of meaning to "negative type".

with the correct conclusion being reached much like an extraneous proposition in linear logic would interfere with it.

A couple further points of pain come up when using general quadratic functions of the form $f(n) = an^2+bn+c$ come about, even if we force a, b, c, n to all be integers and $n > 0$. For one, having $f(0) > 0$ allows us to pull propositions out of thin air, because we can use the id rule to generate them using this quadratic – since if a proposition does not appear on the left-hand side of the sequent, we assume there are 0 copies of it there. We therefore want $f(0) = 0$, to avoid going into debt from unused propositions, of which there are typically many in a standard proof. Aside from that, $f(1) = 1$ tends to be a desirable property, so that we preserve linearity for linear propositions as described earlier. Fortunately, parabolas need three points to be fully defined, leaving the programmer some degree of flexibility.

7 Further Research

The most pressing issue for quadratic logic, at the moment, is that the $\Vdash$ symbol being used is entirely made up of straight lines. Since quadratic functions are generally not linear, further research will likely first go toward developing a replacement for $\Vdash$ that involves more parabolas and fewer lines. This will allow the distinction between quadratic and linear logic to become more clear, as the notation for both systems currently looks very similar.[10]

Aside from the obvious notational issue, another extension the authors would like to consider comes from adding different quadratic functions into each introduction rule, causing different numbers of propositions to generated depending on the order in which some rules are used. Quadratics with imaginary roots or only one real root could also pose an interesting problem. Extending the number of copies of a proposition to the complex numbers so that there are two axes for which the count of a proposition is defined on – the real axis and the imaginary axis – could also be an interesting direction to pursue.

8 References

[1] Pfenning, F. 2017. Lecture Notes on Cut Elimination, 15-317: Constructive Logic. Carnegie Mellon University, USA. http://www.cs.cmu.edu/ crary/317-f22/lectures/10-cutelim.pdf

[2] Pfenning, F. 2023. Seeing if people will believe logical claims if Frank Pfenning's name is attached. Carnegie Mellon University, USA.

[10] The rules also look very similar in some cases.