Calculation formula of the viability kernel in convex and linear dynamical systems

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Abstract. The most critical problem of viability theory is determining the initial states of a dynamical system for which at least one solution remains in the same set. This problem is analogous to many real situations, particularly management problems modelled as high-dimensional dynamic systems. However, the high dimensionality hampers the practical utility of the theory. The existing literature reports difficulties with models of more than four dimensions. Therefore, we investigate the current algorithms and study a calculation formula for high-dimensional linear and convex problems. We found a formula for viable sets in case of backward invariance of the environment set. Finally, since many dynamical systems are neither convex nor linear, extending our work to such systems is the immediate challenge.

Keywords: Viability Theory, Viability Kernel, computing viability kernel

1 Introduction

The most important practical problem of viability theory is determining the *viability kernel*, the set of initial states of a dynamical system for which at least one solution remains in the same set. This problem occurs in many real high-dimensional situations. However, existing algorithms report difficulties solving linear models in more than four dimensions and non-convex ones, even for low dimensions.

The *Saint Pierre Algorithm* [14] is by far the most influential. The tangential condition (equivalence (b) of the Viability Theorem 1) is the fundament of this approach. In our bibliographical research, we found a free implementation of the algorithm with several applications in economics, although in low dimensions (see [9,10] and [8]). Another, more recent, influential algorithm is [13], which reports experiments on linear and convex systems with at least 40 state variables. We found some extensions: [7,15,12] and recently [16]. Our work is a theoretical deepening of this line of research. We must also cite theoretical developments that seek to support the numerical problem, although we did not find practical algorithms that accompany them. Between these developments are those based on the value function [1,4], on simulation [2], and on classification using the Support Vector Machine [3]. We also found works for discrete stochastic viability [5] and, recently, a paper proposed a set-level approach [6].

As mentioned, this work represents a theoretical extension of [13]. We try to find a calculation formula instead of the implementation of an algorithm. In the next section, we present the Viability Theorem in an abstract context of Hilbert spaces, in Section 3 the basic general formula and in Section 4 a couple of examples applied to the case linear that verifies our theoretical results.

2 The viability theorem

We will formulate the problem in the Hilbert space context. Let $\mathcal{H}$ be the *space of evolutions* a Hilbert space with inner product $\langle \cdot \mid \cdot \rangle$, induced norm $\|\cdot\|$. Let Id be the identity operator defined on $\mathcal{H}$. Let C be subset of $\mathcal{H}$ called *environment*. When C is nonempty, we define a set-valued *operator* $\varphi : C \to 2^{\mathcal{H}}$. We say that the pair (C, φ) is a *viable system* if, for every evolution $x \in C$, there exists an evolution $y \in C$ such that $y - x \in \varphi y$. Equivalently, we say that C is a *viable set* under φ if (C, φ) is a viable system.

Furthermore, recall some basic definitions that we use in this article. Let $\mathbb{N}$ be the set of natural numbers including the 0, C a nonempty subset of $\mathcal{H}$, and fix $x \in C$. Let $\mathcal{N}(x)$ be the set of neighbourhoods of x and B the unitary open ball of $\mathcal{H}$. We say that the operator $\varphi : C \to 2^{\mathcal{H}}$ is *upper semicontinuous* at x if $(\forall \epsilon \in \mathbb{R}_{++})(\exists V \in \mathcal{N}(x))(\forall y \in V)\ \varphi y \subset \varphi x + \epsilon B$. φ is upper semicontinuous if it is upper semicontinuous at all evolutions $x \in C$. The *inverse* of φ, denoted by φ^{-1}, is defined by the set $\{(u, x) \in \mathcal{H}^2 : u \in \varphi x\}$. Additionally, if C is nonempty and convex, the *normal cone* to C at x is

$$N_C x \doteq \begin{cases} \{u \in \mathcal{H} : \sup\ \langle C - x \mid u \rangle \leq 0\} & \text{if } x \in C \\ \emptyset & \text{otherwise} \end{cases}.$$

Theorem 1 (Viability Theorem). *Let C be a nonempty compact convex subset of $\mathcal{H}$ and φ an upper semicontinuous operator with nonempty closed convex values. The following assertions are equivalent: (a) The system (C, φ) is viable; (b) For all $x \in C$ and for any $u \in N_C x$, $\sup\ \langle \varphi x | -u \rangle \geq 0$ holds.*

3 Computing the viability kernel

Given a system (C, φ), the viability theory's critical problem is determining if the pair satisfies the viability conditions. Moreover, if this is not the case, the crucial issue consists of finding the largest subset of C, say $\mathcal{V}_\varphi(C)$, such that the system $(\mathcal{V}_\varphi(C), \varphi)$ is viable. The subset $\mathcal{V}_\varphi(C)$ is called *viability kernel* of the set C for the operator φ.

Lemma 1. *Let C be a nonempty subset of $\mathcal{H}$. The following assertions are equivalent: (a) The system (C, φ) is viable; (b) $C \cap (\mathrm{Id} - \varphi)^{-1}(x) \neq \emptyset$ for any $x \in C$; and (c) $C \subset (\mathrm{Id} - \varphi)(C)$.*

Proof. Let C be a nonempty subset of $\mathcal{H}$, such that the system (C, φ) is viable. Then

$$\begin{aligned}(\forall x \in C)(\exists y \in C)\, x \in (\mathrm{Id} - \varphi)(y) &\leftrightarrow C \subset \cup_{y \in C}(\mathrm{Id} - \varphi)(y) = (\mathrm{Id} - \varphi)(C) \\ &\leftrightarrow (\forall x \in C)(\exists y \in C)\, y \in (\mathrm{Id} - \varphi)^{-1}(x) \\ &\leftrightarrow (\forall x \in C)\, (\mathrm{Id} - \varphi)^{-1}(x) \cap C \neq \emptyset.\end{aligned}$$

What proves the equivalence between (a), (b), and (c).

Now, we pose a formula for computing the viability kernel of a subset C of $\mathcal{H}$ for φ by using the equivalent definition (c) of viability expressed in Lemma 1. Fix C a nonempty compact convex subset of $\mathcal{H}$, an let $\{C_n\}_{n \in \mathbb{N}}$ be the sequence given by

$$(\forall n \in \mathbb{N})\ C_{n+1} \doteq C_n \cap (\mathrm{Id} - \varphi)(C_n) \text{ and } C_0 \doteq C.$$

The following Proposition ensures that the intersection of all these sets is the viability kernel of C for the operator φ.

Proposition 1. *Let C be a nonempty compact convex subset of $\mathcal{H}$, $\varphi : C \to 2^{\mathcal{H}}$ be an upper semicontinuous operator with nonempty closed convex values. Given the subsets sequence defined above, then the system $(\cap_{n \in \mathbb{N}} C_n, \varphi)$ is viable.*

Proof. Assume that $\mathcal{V}_\varphi(C) \neq \emptyset$. What we need to demonstrate is (1) $\mathcal{V}_\varphi(C) \subset \cap_{n \in \mathbb{N}} C_n$ and (2) $\cap_{n \in \mathbb{N}} C_n \cap (\mathrm{Id} - \varphi)^{-1}(x) \neq \emptyset$ for any $x \in \cap_{n \in \mathbb{N}} C_n$. To demonstrate (1), assume $\mathcal{V}_\varphi(C) \subset C_k$ for an arbitrary integer $k \in \mathbb{N}$. Moreover, let $x \in \mathcal{V}_\varphi(C)$, then $x \in C_k$. Since the system $(\mathcal{V}_\varphi(C), \varphi)$ is viable, then there exists $y \in \mathcal{V}_\varphi(C)$ such that $x \in (\mathrm{Id} - \varphi)(y)$, thus $y \in C_k$ and $x \in (\mathrm{Id} - \varphi)(C_k)$, which implies that $x \in C_{k+1}$ according to the definition of the sequence $\{C_n\}_{n \in \mathbb{N}}$. Therefore, $x \in \cap_{n \in \mathbb{N}} C_n$. Because $x \in \mathcal{V}_\varphi(C)$, we obtain $\mathcal{V}_\varphi(C) \subset \cap_{n \in \mathbb{N}} C_n$. To demonstrate (2), assume $x \in \cap_{n \in \mathbb{N}} C_n$ and some integer $k \in \mathbb{N}$, then $x \in C_k$ and $x \in C_{k+1}$. According to the definition of the sequence $\{C_n\}_{n \in \mathbb{N}}$, we obtain $x \in C_k$ and there exists $y \in C_k$ such that $x \in (\mathrm{Id} - \varphi)(y)$, which is equivalent to $y \in (\mathrm{Id} - \varphi)^{-1}(x)$. Then $C_k \cap (\mathrm{Id} - \varphi)^{-1}(x) \neq \emptyset$ for any $k \in \mathbb{N}$. Therefore, $(\cap_{n \in \mathbb{N}} C_n) \cap (\mathrm{Id} - \varphi)^{-1}(x) \neq \emptyset$ for any $x \in \cap_{n \in \mathbb{N}} C_n$. Thus, by Lemma 1 (b), we obtain that the system $(\cap_{n \in \mathbb{N}} C_n, \varphi)$ is viable. $\square$

The following property of the pair (C, φ) is a sufficient condition of our formula. We say the environment C is *backward invariant* under φ if $(\mathrm{Id} - \varphi)(C) \subset C$.

Corollary 1. *Let C be a nonempty compact convex subset of $\mathcal{H}$, $\varphi : C \to 2^{\mathcal{H}}$ be an upper semicontinuous operator with nonempty closed convex values. If C is backward invariant under φ, then the viability kernel of C under φ is given by*

$$\mathcal{V}_\varphi(C) = \lim_{n \to \infty} (\mathrm{Id} - \varphi)^n(C).$$

4 Examples

Example 1. Let H be an Euclidean space, let $T \in \mathbb{R}_{++}$, and suppose $\mathcal{H} \doteq L^2([0,T];\mathsf{H})$. Furthermore, let C be a nonempty compact subset of H, thus the environment is given by

$$C \doteq \{x \in W^{1,2}([0,T];\mathsf{H}) \cap C^0([0,T];\mathsf{H}) : x(t) \in \mathsf{C} \text{ a.e. } t \in [0,T]\}. \tag{1}$$

Now, let $A \in \mathcal{B}(\mathcal{H})$ be a linear bounded operator and P be a closed convex subset of $\mathcal{H}$. Then the operator φ is given by $\varphi : C \to 2^{\mathcal{H}} : \varphi \doteq A - P$, or $\dot{x} \in Ax - P$. The problem consists in to find the viability kernel of the system $(C,(A,P))$ denoted by $\mathcal{V}_{A,P}(C)$, i.e., to select those functions of C, such that satisfy $\dot{x} \in Ax - P$ and $x(t) \in \mathsf{C}$ for the space-time $[0,T]$ a.e. Thus, we have the following Corollary

Corollary 2. *Let the system $(C,(A,P))$, where C is defined by (1), the operator defined by $\dot{x} \in Ax - P$, where $A \in \mathcal{B}(\mathcal{H})$ a linear bounded operator and P a closed convex subset of $\mathcal{H}$. If the condition $(\mathrm{Id} - A)(C) + P \subset C$ is satisfied, then the viability kernel of the system $(C,(A,P))$ is given by*

$$\mathcal{V}_{A,P}(C) = \lim_{n\to\infty}\left((\mathrm{Id} - A)^{n+1}(C) + \sum_{k=0}^{n}(\mathrm{Id} - A)^k(P)\right).$$

Proof. This result is an application of Corollary 1, after verifying that C is a closed convex subset of $\mathcal{H}$ and $\varphi = A - P$ is a nonempty upper semicontinuous operator with nonempty closed convex values. Furthermore, we can verify the sufficient condition of Corollary 1, since $(\mathrm{Id} - \varphi)(C) = (\mathrm{Id} - A)(C) + P \subset C$. Thus, the viability kernel is given by

$$\mathcal{V}_{A,P}(C) = \lim_{n\to\infty}\left((\mathrm{Id} - A)^{n+1}(C) + \sum_{k=0}^{n}(\mathrm{Id} - A)^k(P)\right).$$

□

Remark 1. Let us note that if $\|\mathrm{Id} - A\| < 1$, then A is invertible. Furthermore, we conclude that, in this case, the viability kernel of the system $(C,(A,P))$ is $\mathcal{V}_{A,P}(C) = A^{-1}(P)$.

Example 2. The second example is a special case of the first one. Here $\mathsf{H} \doteq \mathbb{R}$ and let $a, \mu \in \mathbb{R}_{++}$, thus

$$C \doteq \{x \in W^{1,2}([0,T];\mathbb{R}) : x(t) \in [-a,a] \text{ a.e. } t \in [0,T]\} \tag{2}$$

and

$$P \doteq \{p \in W^{1,2}([0,T];\mathbb{R}) : p(t) \in [-\mu,\mu] \text{ a.e. } t \in [0,T]\}.$$

The operator is given by $\varphi x \doteq \alpha x + p$, where $p \in P$ and $\alpha \in \mathbb{R}_{++}$. The problem is to find the viability kernel $\mathcal{V}_{\alpha,\mu}(C)$, where C is given by (2). Although the problem is simple, it is valuable to apply the results of this paper and compare them with the following theoretical solution.

In this case $A \doteq \alpha$, $C \doteq [-a, a]$, and $P \doteq [-\mu, \mu]$. We distinguish two cases: $\mu \leq \alpha a$ and $\mu > \alpha a$. **(1)** In case that $\mu \leq \alpha a$, we have $(1-a)\alpha + \mu \leq a$, i.e. $(1-\alpha)[-a, a] + [-\mu, \mu] \subset [-a, a]$, or $(\mathrm{Id} - A)(C) + P \subset C$, therefore, according to Corollary 2

$$\mathcal{V}_{\alpha,\mu}([-a, a]) = \sum_{n \in \mathbb{N}} (1-\alpha)([-\mu, \mu]),$$

Furthermore, the assumption $\alpha \in \mathbb{R}_{++}$ implies that $\|\mathrm{Id} - A\| = \sup_{|x| \leq 1} |x - \alpha x$ or $\alpha \in]0, 2[$, then $A = \alpha$ is invertible (which is obvious in this case) and thus the viability kernel is given by

$$\mathcal{V}_{\alpha,\mu}([-a, a]) = [-\frac{\mu}{\alpha}, \frac{\mu}{\alpha}].$$

(2) When $\mu > \alpha a$, Proposition (1) is applicable, but no formula is available in this work.

Finally, it is also possible to compute the viability kernel by using the assertion (b) of the Viability Theorem 1. Let $d \in \mathbb{R}_{++}$ a number $d \leq a$. Since $x \in]d, d[\leftrightarrow N_{]-d,d[} = \{0\}$, Condition (b) of Theorem B.2 of [11] is equivalent to

$$(\forall x \in]-d, d[)(\forall u \in N_{]-d,d[}x) \ \sup_p \ \{(x-p)(-u) : p \in [\mu, \mu]\} \geq 0.$$

Then, by Lemma B.1 of [11], the above inequality is equivalent to

$$(\forall u \in \mathbb{R} \setminus \{0\})(\forall x \in \{-d, d\}) \ \sup_p \ \{(x-p)(-u) : p \in [-\mu, \mu]\} \geq 0.$$

Therefore, if $u \in \mathbb{R}_{++}$ and $x \doteq a$, then

$$0 \leq \sup_p \ \{(d-p)(-u) : p \in [-\mu, \mu]\} = -du + \sup_p \ pu = -du + \mu u,$$

what implies that the viability kernel is $\mathcal{V}_{\alpha,\mu}([-a, a]) = [-d, d] = [-\frac{\mu}{\alpha}, \frac{\mu}{\alpha}]$ for any $d \in]0, a]$. This result coincides with our formula of Remark 1 of Corollary B 2.

5 Conclusions

Algorithms for determining the viability kernel report difficulties in problems with more than four dimensions. That is why we propose a calculation formula for linear and convex systems. This formula is valid when the environment set is backward invariant for the operator. An immediate extension of this result consists in relaxing the invariance condition, while a further extension is to study neither convex nor linear systems.

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