
1 The Desert of the Reals: Floating-Point Arithmetic on 2 Deterministic Systems

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5 Abstract

6 Floating-point calculations are critical to a number of special domains in mod-
7 ern computing, including machine learning, graphics, and scientific computing.
8 Numerical calculations are particularly susceptible to opaque and system-local
9 optimizations, which can break certain guarantees for deterministic computers.
10 We consider the background and implementation of IEEE 754 floating-point arith-
11 metic and options for implementing mathematics compatibly with fully repro-
12 ducible and portable computing. We consider hardware-based and software-based
13 proposals.

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Manuscript submitted for review.

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33 1 Introduction

34 Floating-point operations are a technically complex subject and the ex-
 35 tent to which developers or source code alter or test this information will
 36 depend on many factors. Apart from the general exhortation to develop-
 37 ers to be careful and to make sure they know what they are doing, there
 38 is little of practical use that can be recommended. [Jones, 2008, p. 197]

39 Modern digital computers deal, at their root, in binary representation, entirely
 40 zeros and ones.¹ These are often formally considered to be whole numbers in a num-
 41 ber base of two. However, numerical calculations very frequently require the use of
 42 numbers with a fractional part to adequately represent the elements of a computation.

43 Early numeric computing tended to focus on problems of interest to military and
 44 national security applications, such as partial differential equations and numerical
 45 optimization. Such calculations typically involve arrays, and linear algebra was elab-
 46 orated hand-in-hand with digital computing techniques in software and hardware.
 47 Numerics assumed prominence for a wider audience with the rise of gaming on per-
 48 sonal computers, although these algorithms emphasized speed over exactness.² To
 49 this point in the history of computing, most software either ran on a single platform
 50 for its lifetime (as with supercomputing) or did not require portably deterministic
 51 algorithms (as with gaming).³

52 On the other hand, deterministic computing describes the ability for a given com-
 53 putation be reproducible exactly. Such reproducibility permits referential transparency
 54 and more powerful reasoning about a program’s results and dependencies. This in-
 55 cludes, for Urbit as a state machine, that the event log replay be portable across plat-
 56 forms to yield the same result. Conceptual guarantees must be backstopped by actual
 57 implementation guarantees for determinism to hold.

58 2 The Field of the Reals

59 Binary computer values are at root easily represented as non-negative integers. How-
 60 ever, it is frequently convenient when working with human applications to either
 61 use other numeric bases (notably decimal and hexadecimal) or to permit non-integer
 62 mathematics.

¹Analog computers may operate on a continuum of value. Computer logic systems may be architected on other numeric bases for their logic, such as the 1837 Analytical Engine’s decimal system and the 1958 Setun’s (Сетунь) ternary system.

²This is reflected in algorithms such as the “fast inverse square root”, which permits a degree of inaccuracy in exchange for a substantial speedup.

³To be clear, 3D gaming algorithms are deterministic (assuming no random sources are used), but exact error values are often not reproducible across platforms, nor was this a design criterion. The reasons for this are discussed below.

In the historical development of mathematics, logical problems in each set of numbers drove the discovery and elaboration of more elaborate algebras. For instance, in the field of natural numbers \mathbb{N} , the operation of addition (+) or multiplication (\times) produces a value within the set; however, permitting subtraction ($-$) of a larger number from a smaller number can result in a value inexpressible in \mathbb{N} . This motivated the introduction of the integers \mathbb{Z} , augmenting the numbers from zero to (positive) infinity with the negative numbers. Division ($/$) similarly produced a crisis when applied to values which did not have a whole-number ratio between them, a situation resolved by the Pythagorean⁴ innovation of the rational numbers or fractions as a class \mathbb{Q} . Ultimately, the common reference set for engineering mathematics (and the human understanding of the continuum such as measurement) is the set of real numbers. The set of real numbers, denoted by \mathbb{R} , is characterized by its continuity, implying that for any two distinct values within this set, there exists a difference, no matter how small.

Since the operations and conventions of \mathbb{R} have been found to be so useful, it is desirable to extend the semantics to computer programming. However, digital computers, by virtue of their binary representation, effectively use natural numbers \mathbb{N} to represent numbers (to the limit of memory rather than positive infinity $+\infty$). Several schemes permit a computer integer to be interpreted as if it were a number with a fractional part, including a scaling factor, fixed-point representation,⁵ and floating-point representations.

The basic concept of floating point arithmetic is that it permits the representation of a discrete subset of \mathbb{R} by composing a significand, a base, and an exponent. The significand is the set of significant digits, possibly including the sign; the base is the understood number base (typically 2); and the exponent is the power to which that base is put before multiplying by the significand to yield the result.⁶ The most ubiquitous floating-point format in use today is defined by the IEEE 754 standard, but certain hardware platforms such as GPUs utilize alternative floating-point arithmetic representations.⁷

To summarize, given an abstract description of a floating point system, there are several practical implementations that can be derived. We need to specify at least four quantities: sign,⁸ significand, base, and exponent.⁹ The base is presumably fixed by the protocol, leaving three free values for the implementation to economically encode.

⁴A legendary attribution, alas, predicated on the Pythagorean discovery of the irrational numbers as a separate class thereby inducing a crisis. <https://plato.stanford.edu/entries/pythagoras/>

⁵Hand-in-hand with the development of linear algebra, machines such as ENIAC and MANIAC were employed in the 1940s for solving thermonuclear reaction calculations and neutron diffusion equations. Under the direction of John von Neumann, it appears that some calculations did experimentally involve a floating-point scheme, although this was later rejected definitively in favor of fixed-point arithmetic. [Kahan, 1997b, p. 3]

⁶Compare so-called “engineering” notation, such as $1\text{e}5$ for 10,000, which compactly represents the significand 1 and the exponent 5 with an understood base of 10 indicated by e and a sign, implicit for positive.

⁷We cite `bfloat16` [Wang and Kanwar, 2019] and `TensorFloat-32` [Kharva, 2020], among others.

⁸We could omit the sign by introducing an offset or only allowing positive values.

⁹The exponent has a bias so that it in turn does not need a sign.

2.1 IEEE 754 Basics

Early computer systems with floating-point units chose bespoke but incompatible representations, ultimately leading to the IEEE 754 (primarily architected by William Kahan). IEEE 754 reconciled considerations from many floating-point implementations across hardware manufacturers into an internally consistent set of fixed-width representations.¹⁰ For instance, the 32-bit “single precision” C `float`/Fortran `REAL*4` specification denotes particular bit positions as meaningful,

```
SEEE.EEEE.EFFF.FFFF.FFFF.FFFF.FFFF.FFFF
```

where S is the sign bit, 0 for positive (+) and 1 for negative (−); E is the exponent in base-2 (8 bits); and F is the significand (23 bits). The exponent is actually calculated at an offset bias of 127 (2^7) so that a more expressive range of orders of magnitude can be covered. The significand has an implied leading 1 bit unless all are zero. To wit,

$$(-1)^S \times 2^{E-127} \times 1.F$$

IEEE 754 specifies operations between numbers, including of different magnitudes. The standard dictates behavior and provides outlines for arithmetic, but leaves algorithmic details to the implementation. Numbers are normalized by adjusting the exponent of the smaller operand and aligning the significands, then the operations are carried out. In practice, extended precision values are used in the intermediate steps of many algorithms, leading to greater accuracy than would otherwise be expected.¹¹

Since the IEEE 754 floating-point format packs values of different kind together bitwise, conventional integer operations such as left shift (`<<`) and addition (`+`) do not trivially apply.¹²

Floating-point addition (`add`) proceeds per the following algorithm:

1. Compare exponents of the two numbers. Shift the smaller number rightwards until its exponent matches the larger exponent.
2. Add the significands together.
3. Normalize the sum by either shifting right and incrementing the exponent, or shifting left and decrementing the exponent.
4. If an overflow or an underflow occurs, yield an exception.
5. Round the significand to the appropriate number of bits.

¹⁰We ignore the decimal representations introduced in IEEE 754-2008, which do not materially change our argument.

¹¹“Each of the computational operations that return a numeric result specified by this standard shall be performed as if it first produced an intermediate result correct to infinite precision and with unbounded range, and then rounded that intermediate result, if necessary, to fit in the destination’s format.” [IEEE, 2008] Note that, per Risse, “there is no indication whether or not a computation with IEEE 754 is exact even if all arguments are.” The ISO/IEC9899 C standard confesses its own fallibility: “The floating-point model is intended to clarify the description of each floating-point characteristic and does not require the floating-point arithmetic of the implementation to be identical” [ISO/IEC, 2018, fn. 21].

¹²For instance, left-shifting `1.0` does not result in `2.0` but `1.7014118e38`; we leave the mathematics of why as an exercise to the reader.

127 6. Renormalize as necessary (back to step 3).

128 IEEE 754 floating-point arithmetic and its predecessors have some significant math-
 129 ematical compromises even in its formal specification.¹³ For instance, as a result of
 130 the discrete nature of the bitwise representation in E and F, floating-point mathemat-
 131 ics are actually a subset of discrete mathematics masquerading as real mathematics.
 132 This has non-trivial consequences for certain aspects of calculations, including error
 133 accrual. In particular, three facts dominate the resolution:

- 134 1. The distance between two adjacent values changes based on the magnitude of
 135 the exponent and the distance from zero. (The significand resolution stays the
 136 same but the exponent changes.)
- 137 2. There is a relative approximation error for a given bitwidth in IEEE 754, called
 138 the *machine epsilon*.¹⁴
- 139 3. Operations between numbers of different magnitudes are particularly affected
 140 by their relative numerical horizon.

Variable precision and truncation error. For most values of the exponent E, the difference between two discrete values is determined by the absolute magnitude of the significand S. The difference between serial values is

$$\begin{aligned}\Delta S &= 000.0000.0000.0000.0000.0001_2 \\ &= 1.00000011920928955078125 - 1.0 \\ &= 0.00000011920928955078125 \\ &= 2^{-23}.\end{aligned}\tag{1}$$

141 This is multiplied by the the result of the exponent E and the bias, meaning that for
 142 each exponent value the difference between subsequent values changes. (Figure 1
 143 represents this schematically.)

144 However, for normalized numbers, or numbers that are left-shifted or right-shifted
 145 in order to carry out a calculation, values are determined by the *relative shift* in ex-
 146 ponent ΔE . For $E = 2$, for instance, the difference between serial values is 2^{-21} .
 147 This variable precision means that the precision of floating-point values varies across
 148 the range of representable numbers when operations take place. Operations between
 149 two numbers of fairly different precisions are particularly vulnerable to accuracy loss,
 150 although some numerical techniques can be employed to mitigate.

151 Truncation error results from terminating repeating “binaries” (by analogy with
 152 “decimals”). Just as $\frac{1}{3} = 0.\bar{3} = 0.3333\dots$ has a finite precision when written in base-
 153 10, numbers that are not precise powers of two result in repeating fractions. These
 154 necessarily terminate at the resolution of the significand. The number and nature of
 155 truncation and rounding can significantly affect the accuracy of floating-point arith-
 156 metic and algorithms [Izquierdo and Polhill, 2006].

¹³None of this intends to demean the impressive technical accomplishment of IEEE 754 and its architects.

¹⁴Note that this is different from the smallest representable value for a given bit width; e.g., for 32-bit single-precision `float` the smallest representable value is $0000.0000.0000.0000.0000.0000.0000.0001 = 1 \times 10^{-45}$.

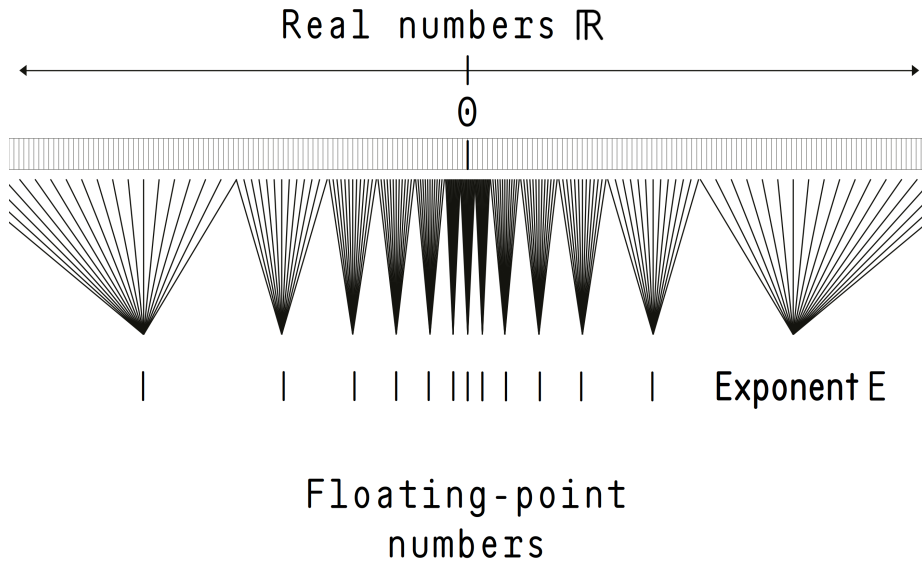


Figure 1: Schematic representation of granularity and variable precision of floating-point values and their relationship to the (continuous) set of real numbers.

Machine epsilon. The machine epsilon, or smallest value discernable from 1.0, is determined by the precision of the floating-point representation. The machine epsilon for a particular bit width is determined by setting two to the negative power of the number of bits used for the magnitude of the mantissa and accounting for the leading implicit bit 1; for 32-bit single-precision `float` this is $2^{-23-1} = 2^{-22}$. Differences from 1.0 smaller than this cannot be represented in this bit width.

Sequence ordering. In situations in which floating-point operations may occur in different orders, even the basic guarantee of commutativity breaks. For instance, in 64-bit FP arithmetic, the following holds true (example in Python):

```
In [1]: (1.1-0.3)-0.8
Out[1]: 0.0

In [2]: (1.1-0.8)-0.3
Out[2]: 5.551115123125783e-17
```

This occurs since operations of different magnitude can affect the resulting significant, a sort of horizon of resolution leading to differences in the outcome. Sequence order can be changed (and thus commutativity broken) as a result of many common programmer design patterns, including compiler optimizations, race conditions, and parallelization.

Another problem in numerical analysis, error accrual is likewise due to the horizon of resolution. The accrual of error due to summing sequences of numbers (whether

180 in parallel or serially) occurs in the summation of sequences of numbers since the
181 error term can grow as n . Kahan–Babuška compensated summation can be used to
182 track a separate error term (*de facto* extending the precision during the operations)
183 and adding it back in to the sum before yielding the final result [Kahan, 1965, Babuška,
184 1969].

185 Formally neither associative nor commutative for the above reasons, floating-
186 point arithmetic can break our mathematical intuitions in interesting ways. However,
187 this is a consistent and well-understood phenomenon. For our purposes as designers
188 of deterministic computers, the most damning indictment has to do not with IEEE 754
189 itself but with manufacturer deviation in hardware implementation. In 1997, William
190 Kahan himself complained (justly) about the compromises inherent in the standard
191 for compiler implementers:

192 Most computer linguists find floating-point arithmetic too disruptive [due
193 to] [t]heir predilection for “referential transparency” Computer lin-
194 guists also dislike functions with side-effects and functions affected by
195 implicit variables not explicit in argument lists. But floating-point oper-
196 ations can raise IEEE 754 exception flags as side-effects, and operations
197 are affected implicitly by exception-handling and rounding modes eligi-
198 ble at run-time according to IEEE 754. Alas, that standard omitted to bind
199 flags and modes to locutions in standard programming languages, and
200 this omission grants computer linguists a licence for inaction. [Kahan,
201 1997a]

202 There are several sources of trouble for even single-threaded deterministic com-
203 putation using hardware IEEE 754 floating-point units (FPUs):¹⁵

- 204 1. Optional, discretionary, or advisory aspects.
- 205 2. Gaps or omissions in the specification.
- 206 3. Failure to implement the specification exactly.
- 207 4. Out-of-sequence computations.

208 **Optional aspects.** Several aspects of IEEE 754 are optional or advisory, includ-
209 ing:

- 210 1. Exception handling means that the hardware may specify rounding via an over-
211 flow flag.
- 212 2. Extended precisions formats are not a huge deal to leave out, but extended pre-
213 cision arithmetic (used for intermediate results) can materially change results.

¹⁵We do not lay blame at the feet of any particular party; the facts are the facts. Indeed, a more recent revision of IEEE 754 leads with a call for portability: “This standard provides a discipline for performing floating-point computation that yields results independent of whether the processing is done in hardware, software, or a combination of the two” [IEEE, 2008].

214 3. Subnormals are optional;¹⁶ some platforms may flush them to zero or (worse)
 215 allow subnormal support to be disabled in certain cases.¹⁷

216 **Omissions.** Whether something is a gap or optional is something of a philosophical
 217 question for us, but some parts are underspecified in a way that makes portability
 218 impossible. E.g., mixed-precision operations can yield unpredictable results depend-
 219 ing on the compiler and hardware. This is a function of rounding modes, precision
 220 loss, precision of intermediate results, and the presence or absence of dedicated hard-
 221 ware support for certain precision combinations.

222 **Inexact implementation.** Failure to implement IEEE 754 correctly may happen
 223 inadvertently, as with the Pentium FDIV bug in the 1990s [Edelman, 1997]. Alter-
 224 natively, chipset designers may deviate from the specification for reasons of perfor-
 225 mance or limitations in the architecture.

226 For instance, IEEE 754 defines a range of numbers as “not-a-number” values, or
 227 NaNs. Per the specification, a NaN can be a signalling NaN, meaning that it intends
 228 to flag and possibly disrupt a problematic computation;¹⁸ or a quiet NaN, which does
 229 not raise such an exception and merely yields a result with the NaN propagated to
 230 the final result.¹⁹ Not all processors implement this part of IEEE 754 correctly: “The
 231 Motorola DSP563CCC does not support NaNs or infinities. Floating-point arithmetic
 232 operations do not overflow to infinity; they saturate at the maximum representable
 233 value” [Jones, 2008, p. 338].

234 As a further example, fused multiply-add (FMA) ($a \times b + c$) is implemented on
 235 certain hardware to favor `double` operations and not quadruple-precision operations
 236 [Kahan, 1997a, p. 5].

237 **Out-of-sequence computations.** A modern compiler using optimization flags
 238 or even modest parallelism can easily cause a floating-point calculation to rely on
 239 operands that were produced in an order different than that specified in the code.
 240 This is largely opaque to the programmer, aside from some simple heuristics, and
 241 makes it difficult to reproduce or reason about the fine details of computations.

242 As demonstrated above, out-of-sequence or resequenced computations can affect
 243 results due to rounding behavior and the “numerical horizon” which results between
 244 values. These can happen due to multithreaded computation or an optimizing com-
 245 piler.

¹⁶Subnormals are a convention that allows values smaller than the “normal” IEEE 754 smallest non-zero value. They permit a graceful underflow behavior, and can prevent an unintentional division by zero.

¹⁷“Some processors do not support subnormal numbers in hardware” [Jones, 2008, p. 338]. (The risk is that this permits inadvertent division by zero.) Various chipsets solve this exceptional behavior differently.

¹⁸“C support for signaling NaNs, or for auxiliary information that could be encoded in NaNs, is problematic. Trap handling varies widely among implementations. Implementation mechanisms may trigger signaling NaNs, or fail to, in mysterious ways. The IEC 60559 floating-point standard recommends that NaNs propagate; but it does not require this and not all implementations do.” [Jones, 2008, p. 339]

¹⁹In Urbit, the Vere runtime unifies NaNs, meaning that any bitwise information which may be encoded in the significand field—the “NaN payload”—is thrown away.

246 **Rounding mode.** IEEE 754 floating-point operations take place using one of four
247 rounding modes.

- 248 1. Round to nearest, ties to even. Set ties to the last bit as zero (even). The default.
- 249 2. Round to zero. Truncate, effectively rounding positive numbers down and neg-
250 ative numbers up.
- 251 3. Round toward positive infinity. Up regardless of sign.
- 252 4. Round toward negative infinity. Down regardless of sign.

253 The rounding mode can affect the result of computations, and if other processes are
254 changing the mode (which can even be set per-thread), results may not be reliably
255 reproducible.

256 “Obtaining the correctly rounded result of an addition or subtraction operation
257 requires an additional bit in the significand (as provided by the IEC 60559 guard bit)
258 to hold the intermediate result” [Jones, 2008, p. 65].

259 3 Urbit's Implementation of IEEE 754

260 Urbit implements a subset of IEEE 754 functionality in `/sys/hoon`, the Hoon lan-
261 guage specification. The Nock operations formally take place on integers. In practice,
262 we could imagine several ways of implementing such operations: bitmasking the in-
263 tegers or breaking them apart into three components, for instance. We take Urbit's
264 implementation of `@rs` (single-precision `float`) as representative.²⁰

265 `++rs` is a wrapper core to instrument arithmetic arms like `++add` using the `++ff`
266 floating-point functionality core. Ultimately this resolves to breaking out the compo-
267 nents (sign, exponent, and significand) into separate numbers for the actual opera-
268 tion.²¹

269 The `++fn` core offers a generalized interface for a superset of IEEE 754-style floating-
270 point implementations, permitting bit width, precision, bias, and rounding mode to
271 be freely specified.²² The actual implementation on `+$fn`-typed values is rather dense
272 and features numerous rounding and overflow checks:

```

273 ++  add
274   | =  [a=[e=@s a=@u] b=[e=@s a=@u] e=?] ^-  fn
275   +=  q=(dif:si e.a e.b)
276   |-  ?. (syn:si q) $(b a, a b, q +(q))
277   ? :  e
278       [%f & e.b (^add (lsh [0 (abs:si q)] a.a) a.b)]
279
```

²⁰The only significant variation in the other real types in Hoon arises for quadruple-precision floating-point values `@rq` which are represented in the runtime by a pair of `doubles`.

²¹The decimal output is produced using the traditional Steele–White Dragon4 algorithm [Jr. and White, 1991]. It is worth considering upgrading Hoon from Dragon4 to Errol [Andryso et al., 2016] for speed and accuracy.

²²In practice, of course, Urbit hews to recognized types, but the temptation to design new floating-point layouts is intriguing.

```

280   += [ma=(met 0 a.a) mb=(met 0 a.b)]
281   += ^= w %+ dif:si e.a %- sun:si
282   ?: (gth prc ma) (^sub prc ma) 0
283   += ^= x %+ sum:si e.b (sun:si mb)
284   ?: =((cmp:si w x) --1)
285   ?- r
286   %z (lug %fl a &) %d (lug %fl a &)
287   %a (lug %lg a &) %u (lug %lg a &)
288   %n (lug %na a &)
289   ==
290   (rou [e.b (^add (lsh [0 (abs:si q)] a.a) a.b)])
291

```

There is, of course, a feint in the foregoing discussion. Nock is a virtual machine specification, and in practice operations that would benefit from more direct expression in C are *jetted*.²³ Thus the actual call in this case will correspond to some C code using the SoftFloat library:²⁴

```

296 u3_noun u3qet_add(u3_atom a, u3_atom b, u3_atom r) {
297     union sing c, d, e;
298     // set IEEE 754 rounding mode
299     _set_rounding(r);
300     // unwrap nouns into C-typed values
301     c.c = u3r_word(0, a);
302     d.c = u3r_word(0, b);
303     // perform addition and unify NaN
304     e.s = _nan_unify(f32_add(c.s, d.s));
305
306     // wrap C value back into noun
307     return u3i_words(1, &e.c);
308 }
309
310

```

Why SoftFloat? Enter, stage left, the problem of platform-portable determinism.

4 Deterministic Computation with a Fractional Part

Non-real arithmetic is less significant for many of the core operations of Urbit as a personal server platform. However, gaming, machine learning, graphics, and other applications rely on floating-point calculations—preferably as fast as possible. In fact, not only applications-oriented processes rely on determinism: guarantees in cryptography and contractual correctness for web3; verification and validation; and code correctness analysis all require reproducible determinism.²⁵

²³A shorthand for *jet-accelerated code*.

²⁴u3 functions are Urbit noun library functions. The `sing` union is a union of `uint32_t` and `SoftFloat float32_t` types.

²⁵Support for IEEE 754 is similar for support for the Markdown markup language. Many platforms support a subset of Markdown coupled with platform-specific extensions. (See also SQL.) Internal references, HTML, inline \LaTeX math mode, code block language specification, and other features see varying levels of support with GitHub, Pandoc, Obsidian, and other editors and converters.

319 Why can't we just allow different results in the last binary places of the signifi-
 320 cand? Philosophically, Urbit holds the following statements as bedrock truth [Monk,
 321 2020a]:

- 322 1. A10. Correctness is more important than performance.
- 323 2. A12. Correctness is more important than optimality.
- 324 3. A14. Deterministic beats heuristic.
- 325 4. F1. If it's not deterministic, it isn't real.

326 Urbit makes much of avoiding the “ball of mud” “standard software architecture”
 327 [Foote and Yoder, 1999]. In this design anti-pattern, a lack of guarantees and pre-
 328 dictable behavior leads *inevitably* to haphazard and illegible software bloat [Foote
 329 and Yoder, 1999]. We can thus understand why Urbit as a platform considers even de-
 330 viations in the last bit of a significand to be threads fraying the edge of sanity Monk
 331 [2020b]:

332 If you do the same thing twice, your computer should react the same way.
 333 This is comforting. This is also what makes it easy to reason about and
 334 use effectively. If you're not sure what your computer will do, you'll be
 335 afraid of it and act defensively toward it. This inevitably leads to a big
 336 ball of mud.

337 For most purposes in the broader software world, tightly reproducible precision
 338 has not been a high priority. Precision having already been sacrificed, the gist of
 339 the calculation is more important than the fourth decimal place (e.g. in realtime 3D
 340 graphics). This leads the phrase “implements the IEEE 754 standard” to be interpreted
 341 erroneously to imply full reproducibility [Figuerola del Cid, 2000].

342 For example, consider the expression $(a \times b) + c$. If a compiler permits the two op-
 343 erations to be evaluated sequentially (a multiplication followed by an addition), then
 344 rounding occurs twice. If a compiler optimizes the operation into an FMA, or fused
 345 multiply-add, then a single rounding occurs. Peters presents a pathological case for
 346 32-bit single-precision floating-point values: $a = 1.00000011920929$, $b = 53400708$,
 347 and $c = -b$. In this case, the two-stage operation wipes out the 0.00000011920929
 348 component of a , yielding a as an integer. Then c is added and the result is 8. With
 349 FMA as a single-step operation, the (correct) answer 6.365860462 is obtained. The
 350 optimization is more correct than the naïve route in this case.

351 However, in another example due to Dawson, FMA yields incorrect results: for
 352 $a \times b + c \times d$ with $a = c$ and $b = -d$, the answer should be zero, and calculated in two
 353 steps will typically be zero. With a fused multiply-add, however, the code becomes
 354 `fmad(a, b, c*d)`, rounding the multiplication of c and d but not that of a and b ; the
 355 answer will likely not be zero.

356 The situation grows more ambiguous across architectures. Jones [2008, p. 346]
 357 presents the pathological case of a compliant platform that may use extended preci-
 358 sion bits in the calculation of $a + b$:

```

359 #include <stdio.h>
360
361 extern double a, b;
362
363 void f(void) {
364     double x;
365     x = a + b;
366     if (x != a + b)
367         printf("x != a + b\n");
368 }

```

369 In this hypothetical case, “any extended precision bits will be lost in the first calculation of $a+b$ when it is assigned to x . The result of the second calculation of $a+b$ may be held in a working register at the extended precision and potentially contain additional value bits not held in x , the result of the equality test then being false.” Higham [2002] provides further examples of pathological cases.

374 K&R C permitted the compiler to re-order floating-point expressions by associativity, which could run afoul of our limitations. ANSI C (C89), recognizing the issue introduced by this innocuous change, forbade such re-ordering [MacDonald, 1991]. Compiler optimizations (e.g. GCC’s -O3) can bypass this restriction, once again breaking determinism;²⁶ for instance, floating-point operations can be pipelined, leading to out-of-order execution.

380 The fly in the ointment for Urbit’s deterministic computing is that jet-accelerated Nock equivalents must reliably produce the same results (both to each other and to Nock) regardless of the runtime on which it is being evaluated. Thus even small irregularities in floating-point implementations have macroscopic ramifications for deterministic computing. Any guarantee broken breaks them all, just as it would for a formal correctness proof.²⁷

386 The challenge of the lack of determinacy for certain critical applications has been acknowledged before, such as by James Demmel and the ReproBLAS team [Demmel et al., Ahrens et al., 2018] and by Dawson. Dawson makes much of the effect of rounding modes and the option to disable subnormals, both of which would have major effects on computational reproducibility. The situation is worse for transcendental functions, because there is necessarily truncation and/or rounding error [Dawson, 2013].

393 The field of debate for possible solutions for implementing floating-point arithmetic which is portable across platforms includes:

- 395 1. Hardware-supported floating-point arithmetic.
- 396 2. Software-defined floating-point library.
- 397 3. Opaque calculations.

²⁶This can be mitigated in turn by the use of the `volatile` designation, but this is sufficient to illustrate the problem.

²⁷I was once asked by a retired computer science professor if such guarantees would make things easier. Well, at the end developer level!

4. Stored results.

5. Proscribing IEEE 754.

We consider each in turn, with its ramifications for a deterministic computing platform and in particular its prospects for adoption in Nock-based systems.

4.1 Hardware-supported floating-point arithmetic

As outlined above, execution of software-equivalent floating-point computations produced from source by different compilers on different hardware architectures may lead to small differences in outcome, non-negligible for a deterministic computer. Thus, for this and a constellation of related reasons, hardware-supported floating-point arithmetic seems to be *prima facie* unviable for deterministic computing.²⁸

We do not know the field of possible future hardware architectures which Nock as a deterministic computing platform may be called upon to execute. Jet-accelerated code should be intelligently robust about its the hardware, but Hoon and Nock code should be completely agnostic to the hardware.

That’s the problem. What are some possible hardware-targeted solutions?

1. Control the compiler and runtime stack top to bottom.

2. Store a hardware and compiler tag and simulate when not on that platform.

3. Support only a single hardware for the lifetime of a ship.

4. Dock floating-point results.

5. Check consistency of results.

4.1.1 Control the stack

If you controlled the compiler and runtime execution stack to a sufficient degree, could you yield deterministic floating-point arithmetic from the hardware? “A translator that generates very high performance code is of no use if the final behavior is incorrect” [Jones, 2008, p. 189]; that is, optimizations often come at the cost of correctness.

To start off, what must be considered part of the stack in this sense? At a minimum, the compiler and linker toolchain (including flags and options) and the actual runtime

²⁸Although not of grave consequence, the C language (as of C23) does not implement at least two types specified by IEEE 754-2019 and recent predecessors: `binary128` quadruple precision and `binary256` octuple precision. While neither are significant losses, we also note that Urbit does not currently support a C-style `long double` type. C’s `long double` is 80 bits wide on some common consumer hardware, such as the x86-64 architecture, but is 128 bits wide on the 64-bit ARM architecture. (The situation is worse for Python, whose `numpy.float128` type eponymously advertises itself as quadruple precision but is in fact a regular 80-bit `long double`.) Some compilers and libraries do support quadruple-precision floating-point mathematics, such as GCC’s `__float128` type. We note that IEEE 754 80-bit extended-precision could be implemented using the `++fn` core should demand arise.

must be included. (This explicitly introduces a dependence between Martian software and Earthling software, repugnant to the Urbit ethos.)

We also must decide what the target is. Do we aim for the most portable configuration (as determined by number of consumer or enterprise users)? Do we aim for the “closest” to IEEE 754 adherence? Do we aim for simplicity, or compilation speed, or any of a half-dozen other optimizable variables?

For instance, suppose that one intended to use the C keyword `volatile` to block certain common optimizations on a floating-point value.²⁹ The runtime at the level of Nock does not know if a value is considered floating-point or not. At the level of a jet, the use of `volatile` can correctly bar certain hardware optimizations, but these need to be carefully enumerated and understood in the light of the other toolchain concerns enumerated in this section. Strictly speaking, `volatile` only seeks to guarantee that stale calculations are not inadvertently reused due to optimization. Without hardware optimization, the utility of an FLU for fast floating-point computations is questionable. The risk of a jet mismatch remains high, as does a nonportable jet.³⁰

Can the C-defined floating-point environment (as supplied by `fenv.h`) answer to this need? This affords the ability to specify not only rounding modes and access floating-point exception status flags, but it is not clear whether this environmental control portably spans the entire output of floating-point computations.³¹

Finally, “[a]n implementation is not required to provide a facility for altering the modes for translation-time arithmetic, or for making exception flags from the translation available to the executing program” [Jones, 2008, p. 200]. The information we purport to gain by controlling the stack in the manner above outlined is possibly not even available to the compiler and the runtime executable.

We suggest that deterministically correct stack control in the sense we have described here is impossible for an arbitrary configuration of the modern hardware stack.³²

²⁹In any case, this assumes a legible and enumerable set of behaviors for `volatile` which is, alas, not the case. “`volatile` is a hint to the implementation to avoid aggressive optimization involving the object because the value of the object might be changed by means undetectable by an implementation” [Jones, 2008, p. 472]. “Actions on objects so declared shall not be ‘optimized out’ by an implementation or reordered except as permitted by the rules for evaluating expressions” [Jones, 2008, p. 1500]. “The `volatile` qualifier only indicates that the value of an object may change in ways unknown to the translator (therefore the quality of generated machine code may be degraded because a translator cannot make use of previous accesses to optimize the current access)” [Jones, 2008, p. 963]. The same author provides examples of C code that is ambiguous in `volatile`’s semantics, pp. 1290–1291; and undefined in `volatile`’s semantics, pp. 1482–1483.

³⁰“What constitutes an access to an object that has `volatile`-qualified type is implementation-defined” [Jones, 2008, p. 1488]. “Volatile-qualified objects can also be affected by translator optimizations” [Jones, 2008, p. 1490]. The C novice may at this point wonder what the intended utility of `volatile` in fact is: “[a] `volatile` declaration may be used to describe an object corresponding to a memory-mapped input/output port or an object accessed by an asynchronously interrupting function” [Jones, 2008, p. 1499].

³¹“The floating-point environment access and modification is only meaningful when `#pragma STDC FENV_ACCESS` is set to `ON`. ... In practice, few current compilers, such as HP aCC, Oracle Studio, and IBM XL, support the `#pragma` explicitly, but most compilers allow meaningful access to the floating-point environment anyway.” [CPP, 2023]

³²The possibility of circumscribing the set of permissible IEEE 754 operations, which may afford a different approach to this problem but seems similarly susceptible of shipwreck, is explored in a subsequent section, *quod vide*.

453 4.1.2 Simulate the hardware

454 If you knew what the compiler and execution stack behavior looked like when a cal-
455 culation was performed, could you reproduce it in software at need on a different
456 platform?

457 Hardware simulation faces some difficulties in the same vein as controlling the
458 stack. The proposal yields a combinatorial explosion when considering the combina-
459 tions of hardware chips, compilers, and compiler flags. Nor is it clear that hardware
460 documentation can be accrued in sufficient quantity and detail to guarantee the suc-
461 cess of such a project.

462 The Urbit runtime provides an epoch system, meaning that the event log is sepa-
463 rated into snapshots and subsequent events [~mastyr-bottec, 2020]. This is currently
464 used to monitor the use of old binaries which could potentially have a jet mismatch. It
465 would be moderately straightforward to extend this functionality to record the com-
466 pilation flags and architecture of that Vere binary, which could be useful in event
467 playback. However, this remains an unsatisfactory solution because it would lead
468 to Urbit runtime instances intentionally producing different code (rather than a jet
469 mismatch which would require correction).

470 4.1.3 Support a single hardware platform

471 A permanent commitment to a single hardware platform—either for the Urbit plat-
472 form as a whole or for a particular running instance—could solve the determinism
473 problem. This configuration would be tenable for single-purpose ships with lifetime
474 control (likely moons or comets), but inconvenient for the “hundred-year computer”
475 model touted for planets and superior ranks in Urbit.

476 Marriage is a fine institution, but I’m not ready for an institution. (Mae
477 West)

478 To make a lifelong commitment to a particular hardware platform when the life-
479 time of a deterministic computer is unknown is therefore deemed foolhardy.

480 4.1.4 Dock floating-point results

481 What about trimming floating-point values of their least significant bits? When would
482 this take place—at each step of a multi-step computation? At the level of single-bit
483 rounding errors, this would potentially work, and amounts to selecting a rounding
484 mode towards even (last digit 0). Accrual across multiple calculations could poten-
485 tially render this unreliable, particularly if different computational paths are supposed
486 to lead to the same result and do not as a result of docking.

487 One could also envision docking more than the last bit. This introduces a step to
488 check and adjust the floating-point value, and in addition breaks IEEE 754 compliance—
489 at which point the trouble of trying to reconcile IEEE 754 with determinism fails.

490 In general, we cannot assign a high degree of significance to figures beyond the
491 first few, but accruals across large data sets (such as large language models) can be-
492 come significant (as attested to by the need for compensated summation).

493 A related technique could pack bits of larger floating-point values into smaller
 494 ones, but this is functionally a software-defined solution (see, e.g., Brun [2018]).

495 4.1.5 Consistency checks

496 Another option is to compare Nock and jet code for every computation and only ac-
 497 cept the C code if it is “correct”. This immediately runs into a very undesirable charac-
 498 teristic: every floating-point calculation is run twice, obviating at least one calculation
 499 and destroying any efficiency gains from jetting the code.

500 One could cache floating-point computations somewhere in the system.³³ This is
 501 liable to become prohibitively large for systems as every individual FP calculation of
 502 all time becomes archived against future need.

503 We conclude that, at the current time, naïve hardware-defined floating point is
 504 not viable for deterministic systems.

505 4.2 Software-defined floating-point library

506 In the absence of a dedicated floating-point unit (FPU) and floating-point assembly
 507 instructions, floating-point computations are carried out in software. The type can be
 508 decomposed from bits, operated on, then packed back into the single type of appropri-
 509 ate value. For instance, prior to the widespread advent of 64-bit consumer hardware,
 510 applications running on PC architecture that needed `double` values utilized software
 511 emulation using two 32-bit numbers together.

512 Urbit’s current solution for floating-point computation is to utilize a software-
 513 defined floating-point library, the SoftFloat library by Hauser. SoftFloat is an imple-
 514 mentation in software of a subset of IEEE 754 for five floating-point types.³⁴ Urbit
 515 statically links the library into its runtime binary so it is always available for Nock to
 516 utilize as a jet.

517 While formally correct, software FP is slower than hardware FP, and likely pro-
 518 hibitively slow for many large matrix applications such as LLMs. (“Correctness is
 519 more important than performance.”) Performance is the dolorous stroke against software-
 520 defined floating point. (On the other hand, some early versions of the Apple-IBM-
 521 Motorola PowerPC RISC architecture did not have dedicated hardware floating-point
 522 units (FPUs) or floating-point assembler instructions at all, requiring full software
 523 implementation.³⁵)

524 An optimized portable deterministic software library for floating-point calcula-
 525 tions may be a sufficiently fast solution to meet Urbit’s needs even for vector com-
 526 putations. A different avenue worthy of investigation is to take IEEE 754 compliant

³³Indeed, something like this cache system was employed on Sun SPARC architecture, as discussed in Section 4.4.

³⁴“The current release supports five binary formats: 16-bit half-precision, 32-bit single-precision, 64-bit double-precision, 80-bit double-extended-precision, and 128-bit quadruple-precision” [Hauser, 2018].

³⁵“There are several 680x0-based Macintosh computers that do not contain floating-point coprocessors” [Beesley and Elzinga, 1994]; on the other hand, “floating-point calculations are performed even faster under the ... emulator than on a real 680x0-based Macintosh computer,” indicating that optimized software acceleration is possible, modulo chipset versions and tuned libraries. The PowerPC 601, introduced in 1991, had a native FPU.

floating-point values as inputs and outputs, then transform into a local representation for an optimized portable deterministic calculation. For instance, Thall [2007] presents the concept of “unevaluated sums”, a generalized technique for accruing error in situations where additional precision is necessary for accuracy. However, even with an agreed-upon standard library like SoftFloat, it is important to keep in mind that exact floating-point results for transcendental functions are still not correctly known in many cases.³⁶ This particular poses a problem for functions like `sin` which may be calculated by different routes in Hoon/Nock and in C/Rust. For the time being, we conclude that Urbit’s discipline requires only using Hoon/Nock implementations of transcendental functions.

4.3 Opaque calculations

When a request for data is made over the network, one is not certain what the resulting data will be. Their value is epistemically opaque. In Urbit’s event log, the results of network calls are persisted as effects in the modified state (for successful events).

What if Urbit treated a call that had a floating-point computation as if it were a network call, that is, as if it were a referentially opaque injection into Urbit’s state? One difference is that network calls result as side effects from hints to the runtime which then handles the plumbing, as it were, and injects the resulting `gift` task back into Arvo as if a *deus ex machina*, from Arvo’s perspective. (It should of course know how to handle such a contingency.) There are two main objections that can be made here:

1. From the programmer’s standpoint, every floating-point computation would need to be bundled as if it were a network call, and the result treated as if it were a new move passed back into the kernel. This destroys synchronicity and changes floating-point computations from lightweight programmer choices into heavy and occasional calls.
2. The storage of every result of every floating-point computation could become prohibitively large. Work on large matrices in numerical analysis or machine learning could rapidly balloon the event log since every intermediate state would also become part of the ship’s immutable history.

To the first objection, we can point to the current design pattern utilized in scrying (or the request for values from the bound `scry` namespace). Local `scry` values (such as values exposed by a system service or vane) are accessed synchronously using the `.^` dotket operator. This is straightforward and easy to integrate into a program. Remote `scry` values must be requested asynchronously from another ship, and return at an indeterminate future time as `gifts` to be processed in another part of the vane or application.

³⁶To correctly calculate a trigonometric function for `double` may take over a hundred bits of precision before correct rounding can be determined. Furthermore, the C `math.h` implementation of `sin` may or may not use `f_sin`.

To the second, we observe that although Urbit is a state machine whose history is part of its state, in practice we can mitigate event log growth by either *chopping* the event log by storing its state and permitting replay forward from that point or *tombstoning* data which should never be available again.³⁷

In this proposal, however, one could imagine a situation like that which obtains in scrying: fast software implementation treated synchronously, slow hardware acceleration treated asynchronously.

4.4 Stored results

Instead of repeating computations that have been made in the past, what if we cached the result of all of them, so that any new computations with the same values are guaranteed to result in the same value via a cache lookup instead of a calculation? Urbit uses memoization frequently in Arvo and in the runtime, so this is an aesthetically compatible option; we consider its feasibility.

A recently proposed hardware acceleration technique is to store the results of previous multiplication and division operations in a cache, reusing rather than recalculating the result whenever possible. (Dynamic profiling has found that a high percentage of these operations share the same operands as previous operations.) [Jones, 2008, p. 1148]³⁸

On Urbit, this introduces an $O(1)$ average-case/ $O(n)$ worst-case cache lookup from a MurmurHash3 hash key calculation (what Urbit calls a `++mug`). This must be weighed against the floating-point algorithm in consideration, as well as what is actually hashed (likely the Nock of the calculation contained in the dynamic hint).

This bears some similarities to aspects of the network call suggestion above, in that the second objection to that one holds here. Event log and state bloat (via the cache) are liabilities. Such a cache would be a feature of the Arvo instance, not the runtime VM. Unlike a truncated event log, the cache must be a permanent feature of the ship’s state rather than a convenience.

“Storing results” could also be met by the use of SPARC-style logging. In that hardware platform, suspicious computations are flagged and hashed into a lookup table by site in the originating program. Such events are logged not by timestamp or by computation hash but by callsite in the originating program [Kahan, 1997a, p. 6].³⁹ Sun implemented this in SPARC for “retrospective diagnostics” but the technique could allow a more lapidary operation for Urbit. (Follow-on considerations include whether such computations should now be considered “bound” in a sense like that of the scry namespace.)

³⁷The memory implications of these are not necessary here, but take place in different arenas: the runtime versus the Arvo noun arena.

³⁸Cf. Jones’ citation of Citron et al.

³⁹What constitutes “suspicion” is only sparsely elaborated by Kahan in that article.

4.5 Proscribing IEEE 754

What if the Scylla of IEEE 754 is avoided for some other Charybdis? We can approach this solution space at two levels: either by sector or entirely.

Proscribe by sector. One solution to the speed-vs.-reproducibility dilemma is to permit hardware-accelerated IEEE 754 operations, but only in a verified subset permissible for jets. This would require careful vetting of the hardware stack and compiler options to define a permissible subset of IEEE 754 operations as “known good”. Coupled with the epoch system, it may be a feasible solution.

What degree of vetting will reliably answer the gap between IEEE 754 and hardware implementation for any particular operation? Jones [2008, pp. 330ff.] and Goldberg [1991] provide a careful analysis of accuracy errors inherent to IEEE 754 as a standard, but due to the variety of possible scenarios do not treat of real compilers and chipsets much.⁴⁰ Trivially, as demonstrated above in the Python example, $(a + b) + c \neq a + (b + c)$, and even modest reordering of operations by a zealous compiler optimization is susceptible of introducing nonportable and thus nondeterministic (in our sense) behavior.

Having identified an appropriate subset of operations, we may imagine that the use of `#ifdef`, Autotools’ `configure`, and a jetting library may answer to our need. Any jet library would have to be carefully constructed to avoid imposing tight discipline directly on the end user (modal Hoon author). We cannot recommend this path today but do not consider the way to be shut, especially given liberal use of `volatile`.

In particular, fused multiply-add operations are subject to reordering by an optimizing compiler. Avoiding these would require some discipline on the part of the jet developer, since code that does not explicitly `fma` may yet reduce to it in a compiler pass. A jetting library would be advantageous in this case.

As an example of a refactoring of IEEE 754 operations for determinism, consider the ReproBLAS project (last update ~2016.2.21). ReproBLAS seeks to produce a set of reproducible deterministic algorithms reflecting the standard operations of BLAS [Ahrens et al., 2018]. It accomplishes this by introducing a binned data type and a set of basic operations carefully built on IEEE 754 for the objective of completely portable reproducibility.⁴¹ This is similar to our proposal for a vetted jetting library and may be worth attention, particularly in association with requirements around -00.

Proscribe by replacement. Finally, we face the possibility of jettisoning decades of floating-point libraries entirely and forging a new trail. We explicitly omit attempting to implement a new standard as hubristic, but would like to explore some alternatives.

Posits. In 2015, John Gustafson proposed a new standard for representing values drawn from \mathbb{R} called universal numbers or unums [Gustafson, 2015, 2017a,b, Group,

⁴⁰See particularly the note on “Common Implementations” on p. 346 of Jones [2008].

⁴¹“Using our default settings, reproducibly summing n floating point types requires approximately $9n$ floating point operations (arithmetic, comparison, and absolute value). In theory, this count can be reduced to $5n$ using the new “augmented addition” and “maximum magnitude” instructions in the proposed IEEE Floating Point Standard 754-2018.” [Ahrens et al., 2018]

2022]. The current version of unums, called posits, supports interval arithmetic and greater resolution near 1.0, at the cost of decreased resolution for extremely large and extremely small values. Unums also guarantee associativity and distributivity of operations.

Gustafson’s criticisms of IEEE 754 focused on determinism and exactness; underflow and overflow; fixed bit widths for mantissa and exponent; rounding; and the large wasted block of NaNs [Risse, 2016]. Unums likewise must provide sign, exponent with bias, and significand; they additionally signal whether the value is an interval. Unlike IEEE 754’s use of multiple bit widths, 32-bit posits are argued to be sufficient for almost all applications.

[Gustafson, 2015] [Gustafson, 2017a] [Group, 2022] [Risse, 2016]

A unum/posit implementation for Urbit would be as straightforward as the implementation of IEEE 754. For jetting, there is a software library for posits called SoftPosit based on the SoftFloat library [Cerlane, 2018]. A number of other software implementations exist, but at the time of writing no hardware support has been noted. (Since there are no hardware implementations, the effect of optimizations on determinism cannot yet be assessed; it is presumed that the situation will be better than IEEE 754 given the advantages of a clean slate.)

Hand-rolled floats. If IEEE 754 presents too many difficulties to be viable at high speed, then hand-rolling a custom hybrid hardware–software scheme via bit-masking could be attractive. This returns to the more “Wild West” days before IEEE 754’s introduction, but is presaged by the recent introduction of `bfloat16`, `TensorFlow-32`, and other types designed for machine learning applications. Without access to hardware manufacturers, however, this amounts in the end to software-defined floating point and seems unlikely to be competitive speedwise. (We cite the idea put forth previously in this article to convert to an intermediate representation for computation, yielding IEEE 754 as necessary.)

It may also be worth considering the use of a 3-tuple of sign, exponent, and significand (with only software jetting), and leave details of jet implementation to library authors.

Fixed-point and \mathbb{Q} . A fixed-point representation differs from a floating-point scheme in that the exponent is fixed by the protocol or metadata and thus only the sign and significand need be included in the bit representation. (With an offset, even the sign can be elided.) The advantage of such a scheme is that it affords the benefits of floating-point mathematics at near-integer operation speeds (e.g. left-shift to multiply by two). One disadvantage is that there is a smallest representable value; this lack of subnormals requires either an underflow handler or the possibility of inadvertent division by zero. Fixed-point operations could also be used as intermediates in calculations. (This echoes once again the idea of conversion to an intermediate representation then conversion back out to IEEE 754.)⁴²

⁴²“One solution to implementing floating-point types on processors that support fixed-point types is to convert the source containing floating-point data operations to make calls to a fixed-point library” [Jones, 2008, p. 346]. Note that the sense of our current interest is reversed.

676 If a rational number scheme is implemented, then a variety of possible implemen-
677 tations are possible, ranging from bitpacked fixed-width integers to pairs of arbitrary-
678 width integers. Reduction to “simplest” values introduces some overhead; fractions
679 are formally an ordered pair (a, b) with $b \neq 0$, but there is an equivalence class of
680 multiples. (That is, if we write $\frac{1}{2}$ as $(1, 2)$, we have also to consider $(2, 4)$, $(3, 6)$, in-
681 deed an infinite sequence of such ordered pairs.) Rational numbers are a superset of
682 floating-point numbers and fixed-point numbers, but accrue processing overhead due
683 to dereferencing arbitrary integers and other aspects of computation on operations.

684 However, deviation from the proscription scheme, even inadvertently, would mean
685 that a ship is considered invalid in a sense equivalent to double-booting or breaking
686 the scry namespace. This option is deemed worth investigation, likely viable, but
687 bearing unknown risks.

688 4.6 Irregularities

689 Any approach to modeling real numbers runs the risk that different calculation path-
690 ways will yield a different kind of inexactness in the result. These can be mitigated by
691 some of the approaches suggested above, and also by checking the correspondence of
692 the Hoon code and the underlying jet, particular for known edge cases in behavior.
693 While Hoon-jet compliance is an open research problem,⁴³ we can apply principles
694 of unit testing together with a period of testing Nock and jet compliance.⁴⁴

695 Jet mismatches have been rare in the current era.⁴⁵ Some jet “mismatches” occur
696 because the runtime raises a different error than the corresponding Hoon—these are
697 relatively innocuous. Others may occur because actually different results are pro-
698 duced for different input. These are grave, and ultimately motivated the introduction
699 of the epoch system so that event log replays can take into account the previous less-
700 perfect jet version in the runtime [*mastyx-bottec*, 2020].

701 5 Linear Algebra in Hoon

702 Lagoon⁴⁶ is an Urbit library to facilitate Hoon-native mathematical operations. It
703 envisions six native types,

- 704 1. `%real`, an IEEE 754 floating-point value
- 705 2. `%uint`, an unsigned integer
- 706 3. `%int2`, a twos-complement signed integer
- 707 4. `%cplx`, a BLAS-compatible ordered pair
- 708 5. `%unum`, a unum/posit value

⁴³Particularly as regards co-generation of Hoon and C/Rust, or formal proofs.

⁴⁴The Vere runtime supports a debugging flag which runs both the Nock and the jet and checks for identical results.

⁴⁵By “current era”, we mean after the last global network breach in ~2020.12 [*Urbit*, 2020].

⁴⁶Linear AlGebra in hOON

709 6. `%fixp`, a fixed-precision value

710 for which `%real` allows the rounding mode to be specified; `%cplx` consists of a pair
 711 of two values, real and imaginary parts; and `%fixp` requires the expected precision.

712 Lagoon implements algorithmically correct reference implementation in Hoon
 713 with the expectation that `/lib/lagoon` will be jetted. Operations include basic arith-
 714 metic, vector and matrix row/column operations, matrix multiplication, and matrix
 715 inversion. The jetting scheme may take advantage of software libraries or appropri-
 716 ate hardware, but must hew to the dictum that “if it’s not deterministic, it isn’t real.”⁴⁷

717 Lagoon has passed through several implementations and remains in active de-
 718 velopment. The current implementation is the `lagoon` branch of the `urbit/urbit`
 719 repository [Urbit, 2023].

720 6 Conclusion

721 To summarize, the most promising solutions for floating-point mathematics on Urbit
 722 per the above analysis include:

- 723 1. Hardware FP on single machine for entire lifetime.
- 724 2. Optimized software FP with vetted jetting library.
- 725 3. Opaque calculation as callback.
- 726 4. Cached results by callsite.
- 727 5. Utilizing a subset of IEEE 754 in hardware.
- 728 6. Replacing IEEE 754 with another approach of sufficient speed, fixed-point and
 729 unum/posits chief among these.

730 Several recent efforts on Urbit have encountered the difficulties of producing fast
 731 and reliable floating-point calculations on a Nock-based system.⁴⁸ We anticipate that,
 732 water finding its own level, each will adopt a suitable deterministic solution for eval-
 733 uation in Nock. We do not anticipate these to be the last foundational numerical
 734 libraries built on Urbit, but instead among the first. Thus we have documented the
 735 paths we have explored as an annotated map for future travelers in search of a one
 736 true representation for continuous mathematics.

⁴⁷It is a worth a final digression to address reproducibility on parallel systems. We do not consider this a design goal for Lagoon at the current time, since Urbit is inherently single-threaded. Operations like reduction must take place on a single computer; while jets may in principle utilize parallelism their points of entry and exit are unique. However, we note that the `ReproBLAS` project has addressed this issue in the context of reproducible parallelism [Ahrens et al., 2018], as has Chohra et al.

⁴⁸`UWasm` or `UrWasm`, which as a `WebAssembly` implementation directly relies on floating-point computations [~dozreg-toplud, 2023] (see e.g. KloudKoder [2022] for a discussion of the issues involved); `lazytrig`, which is slow and implemented purely in Hoon (Nock); Lagoon, which includes IEEE 754 operations with its linear algebra library (`q.v`); and MNIST.

References

- Floating-point environment, 2023. URL <https://en.cppreference.com/w/c/numeric/fenv>.
- Peter Ahrens, Hong Diep Nguyen, and James Demmel. ReproBLAS: Reproducible basic linear algebra subprograms, 2018. URL <https://bebop.cs.berkeley.edu/reproblas>.
- Marc Andryso, Ranjit Jhala, and Sorin Lerner. Printing floating-point numbers: A faster, always correct method. In *Proceedings of the 43rd Annual ACM SIGPLAN-SIGACT Symposium on Principles of Programming Languages (POPL '16)*, pages 555–567, 2016. doi: 10.1145/2837614.2837654. URL <https://dl.acm.org/doi/10.1145/2837614.2837654>.
- Ivo Babuška. Numerical stability in mathematical analysis. *Information Processing*, 68:11–23, 1969.
- Kenneth R. Beesley and Dirk Elzinga. *Inside Macintosh: PowerPC System Software*. Addison-Wesley, Boston, 1994. URL https://developer.apple.com/library/archive/documentation/mac/pdf/PPC\System\System_Software/Intro_to_PowerPC.pdf.
- Laurent Le Brun. Making floating point numbers smaller, 2018. URL <https://www.ctrl-alt-test.fr/2018/making-floating-point-numbers-smaller/>.
- Leong Cerlane. Softposit, 2018. URL <https://gitlab.com/cerlane/SoftPosit>. Git repository.
- Chemseddine Chohra, Philippe Langlois, and David Parello. Efficiency of reproducible level 1 blas. In Marco Nehmeier, Jürgen Wolff von Gudenberg, and Warwick Tucker, editors, *Scientific Computing, Computer Arithmetic, and Validated Numerics*, pages 99–108, Cham, 2016. Springer International Publishing. ISBN 978-3-319-31769-4.
- D. Citron, D. Feitelson, and L. Rudolph. Accelerating multi-media processing by implementing memoing in multiplication and division units. In *Proceedings of 8th International Conference on Architectural Support for Programming Languages and Operating Systems (ASPLOS-VIII)*, pages 252–261, 1998.
- Bruce Dawson. Floating-point determinism, 2013. URL <https://randomascii.wordpress.com/2013/07/16/floating-point-determinism/>.
- James Demmel, M. Gates, G. Henry, X. S. Li, J. Riedy, and P. T. P. Tang.
- Alan Edelman. The mathematics of the pentium division bug. *SIAM Review*, 39(1): 54–67, 1997. doi: 10.1137/S0036144595293959.
- S. A. Figueroa del Cid. *A Rigorous Framework for Fully Supporting the IEEE Standard for Floating-Point Arithmetic in High-Level Programming Languages*. PhD thesis, New York University, January 2000.

- 774 Brian Foote and Joseph Yoder. Big ball of mud, 1999. URL [http://laputan.org/
775 mud/mud.html](http://laputan.org/mud/mud.html).
- 776 David Goldberg. What every computer scientist should know about floating-point
777 arithmetic. *ACM Computing Surveys*, 23(1):5–48, 1991. URL [https://dl.acm.
778 org/doi/pdf/10.1145/103162.103163](https://dl.acm.org/doi/pdf/10.1145/103162.103163).
- 779 Posit Working Group. Standard for posit™ arithmetic. Technical report, 2022. URL
780 https://posithub.org/docs/posit_standard-2.pdf.
- 781 John L. Gustafson. *The End of Error: Unum Computing*. A K Peters/CRC Press, 2015.
782 ISBN 978-1-4822-3986-7.
- 783 John L. Gustafson. Posit arithmetic, 2017a. URL [https://posithub.org/docs/
784 Posits4.pdf](https://posithub.org/docs/Posits4.pdf).
- 785 John L. Gustafson. Beyond floating point: Next generation computer arithmetic (stan-
786 ford seminar), 2017b. URL <https://www.youtube.com/watch?v=aPOY1uAA-2Y>.
- 787 John R. Hauser. Berkeley softfloat release 3e, 2018. URL [http://www.jhauser.us/
788 arithmetic/SoftFloat-3/doc/SoftFloat.html](http://www.jhauser.us/arithmetic/SoftFloat-3/doc/SoftFloat.html).
- 789 Nicholas J. Higham. *Accuracy and Stability of Numerical Algorithms*. SIAM, Philadel-
790 phia, 2nd ed edition, 2002.
- 791 IEEE. 754-2008 IEEE standard for floating-point arithmetic, 2008. URL [https://
792 ieeexplore.ieee.org/document/4610935](https://ieeexplore.ieee.org/document/4610935).
- 793 ISO/IEC. Iso/iec 9899:2018 information technology – programming languages – C,
794 2018. URL <https://www.iso.org/standard/74528.html>.
- 795 Luis R. Izquierdo and J. Gary Polhill. Is your model susceptible to floating point errors?
796 *Journal of Artificial Societies and Social Simulation*, 9(4), 2006. URL [https://www.
797 jasss.org/9/4/4.html](https://www.jasss.org/9/4/4.html).
- 798 Derek M. Jones. *The New C Standard: An Economic and Cultural Commentary*. 2008.
- 799 Guy L. Steele Jr. and Jon L. White. How to print floating-point numbers accurately. In
800 *Proceedings of the 1990 ACM SIGPLAN Conference on Programming Language Design
801 and Implementation (PLDI '90)*, pages 372–389, 1991. URL [http://kurtstephens.
802 com/files/p372-steele.pdf](http://kurtstephens.com/files/p372-steele.pdf).
- 803 William Kahan. Further remarks on reducing truncation errors. *Communications of
804 the ACM*, 8(1):40, 1965. doi: 10.1145/363707.363723.
- 805 William Kahan. Lecture notes on the status of IEEE Standard 754 for binary floating-
806 point arithmetic, 1997a. URL [https://people.eecs.berkeley.edu/~wkahan/
807 ieee754status/IEEE754.PDF](https://people.eecs.berkeley.edu/~wkahan/ieee754status/IEEE754.PDF).
- 808 William Kahan. The baleful effect of computer languages and benchmarks upon ap-
809 plied mathematics, physics and chemistry (John von Neumann lecture), 1997b. URL
810 <https://people.eecs.berkeley.edu/~wkahan/SIAMjvn1.pdf>.

- 811 Paresh Kharva. TensorFlow-32 in the A100 GPU accelerates AI training, HPC
812 up to 20x, 2020. URL [https://blogs.nvidia.com/blog/2020/05/14/
813 tensorfloat-32-precision-format/](https://blogs.nvidia.com/blog/2020/05/14/tensorfloat-32-precision-format/).
- 814 KloudKoder. Floating-point rounding mode control prototyping (WebAssembly issue
815 #1456), 2022. URL <https://github.com/WebAssembly/design/issues/1456>.
816 Git repository.
- 817 Tom MacDonald. C for numerical computing. *Journal of Supercomputing*, 5(1):31–48,
818 1991. doi: 10.1007/BF00155856.
- 819 Philip C. Monk. Urbit precepts, 2020a. URL <https://urbit.org/blog/precepts>.
- 820 Philip C. Monk. Urbit precepts (discussion), 2020b. URL [https://urbit.org/blog/
821 precepts-discussion](https://urbit.org/blog/precepts-discussion).
- 822 Christoph Peters. fma: A faster, more accurate instruction, 2021. URL [https://
823 momentsingraphics.de/FMA.html](https://momentsingraphics.de/FMA.html).
- 824 Thomas Risse. It’s time for unums—an alternative to IEEE 754 floats and doubles. In
825 *Proceedings of the Fifth International Conference on Signal & Image Processing (SIP-
826 2016)*, pages 50–51, 2016.
- 827 ~dozreg-toplud. Uwasm, 2023. URL <https://github.com/Quodss/urwasm>. Git
828 repository.
- 829 ~mastyrbottec. Epoch system, 2020. URL [https://roadmap.urbit.org/
830 project/epoch-system](https://roadmap.urbit.org/project/epoch-system).
- 831 Andrew Thall. Extended-precision floating-point numbers for GPU computation.
832 Technical Report CIM-007-01, The University of North Carolina at Chapel Hill,
833 2007. URL http://andrewthall.org/papers/df64_qf128.pdf.
- 834 Urbit. The last network breach, 2020. URL [https://roadmap.urbit.org/
835 project/last-network-breach](https://roadmap.urbit.org/project/last-network-breach).
- 836 Urbit. Urbit, 2023. URL <https://github.com/urbit/urbit>. Git repository.
- 837 Shibo Wang and Pankaj Kanwar. BFloat16: The secret to high performance on cloud
838 TPUs, 2019. URL [https://cloud.google.com/blog/products/ai-machine-
839 learning/bfloat16-the-secret-to-high-performance-on-cloud-tpus](https://cloud.google.com/blog/products/ai-machine-learning/bfloat16-the-secret-to-high-performance-on-cloud-tpus).