UNIVERSITY OF CALGARY

MATH 433 TERM ESSAY

Efficient Space Travel

Siegfried Peschke

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Contents

1 Introduction

Space travel is inherently extremely expensive due to the cost of designing and manufacturing a rocket, fuel, and the payload. For example, the Space Shuttle program run by NASA which ran over the course of 39 years, adjusted for inflation, cost an estimated $196 billion. According to NASA, launching a Space Shuttle costs an average of $450 million per mission. If space travel were cheaper it would be more accessible and could be done at a higher frequency. This motivates innovators and scientists to explore ways of bringing the cost of space travel down.

One area to explore in lowering the cost of space travel is reducing fuel consumption. Although propellant is typically a small fraction of the total cost of a space mission, saving pennies anywhere possible without sacrificing the mission is still an effective way to bring the cost of a mission down. A few ways to lower fuel consumption would be to make more efficient engines, which would result in a lower requirement for propellant. Finding the best propellant, whether it means higher energy density or a different method of propellant, would also reduce the cost. There are also many trajectories in which to send an object between celestial bodies within the solar system. For this paper, let us focus on the most energy efficient trajectory, the associated cost and furthermore, explore when this trajectory is viable.

2 The Hohmann Transfer

In 1925, Walter Hohmann found the most energy efficient method of transferring between two circular orbits around the same central body which lie in the same plane, called the Hohmann transfer. In order to calculate the Hohmann transfer between two orbits, we must make some simplifying assumptions. The first and fundamental assumption is that there is only one body which exerts a gravitational force on the object of interest. The second assumption will be that accelerations or changes in velocity are instantaneous. This is an acceptable assumption since the time in which the rocket burns fuel to accelerate will be much less than the period of the orbit. A third and final assumption will be that the initial and final orbits lie in the same plane, i.e. are co-planar.

The Hohmann Transfer is a two-impulse transfer between two circular orbits which orbit the same central body. Let us consider each impulse, $\Delta\vec{v}$. Let $\overrightarrow{v_i}$ be the initial velocity and $\overrightarrow{v_f}$ be the final velocity of the impulse. Then

$$\Delta\vec{v} = \overrightarrow{v_f} - \overrightarrow{v_i} \tag{1}$$

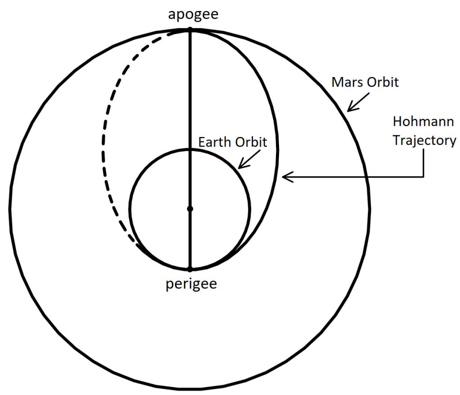
where the magnitudes are given by

$$\Delta v^2 = v_f^2 + v_i^2 - 2v_f v_i cos\theta \tag{2}$$

where θ is the angle between the initial velocity, $\overrightarrow{v_i}$, and the impulse, $\Delta\vec{v}$.

The impulse is an acceleration due to the burning of propellant. We want to minimize the impulse and thus the amount of energy required in order to transfer between orbits. To do this, we can see that we want to keep the $cos\theta$ term negative with the largest magnitude possible. Therefore the impulses are performed in the direction of $\theta = 0$. This in fact, is exactly why the Hohmann transfer method is the most energy efficient trajectory to transfer between two circular orbits. Each impulse is performed when the transfer orbit is tangent to the departure or target body's orbit. This means that the relative velocity is at a minimum and thus the impulse requires the least amount of energy.

We then design the orbit to be elliptical with it's perigee at the inner orbit and the apogee at the outer orbit as shown below



Now we want to find the velocity of an object in an elliptical orbit. This is given by the vis-viva equation (orbital-energy-invariance law) as

$$v^2 = GM\left(\frac{2}{r} - \frac{1}{a}\right) \tag{3}$$

where v is the relative speed between the two bodies, G is the gravitational constant, M is the mass of the central body, r is the distance between the two bodies, and a is the length of the semi-major axis. In this case, the central body is the sun, which we will consider to be stationary. The semi-major axis is related to the eccentricity of the ellipse through the following

$$\frac{P}{1-\epsilon} = a = \frac{A}{1+\epsilon} \tag{4}$$

where P is the radius of the inner circular orbit and A is the radius of the outer circular orbit. This yields the speeds at the perigee and apogee of the Hohmann transfer ellipse to be

$$v_{per}^2 = GM\left(\frac{2}{P} - \frac{1-\epsilon}{P}\right) = GM\left(\frac{1+\epsilon}{P}\right) \tag{5}$$

and

$$v_{apo}^2 = GM\left(\frac{2}{A} - \frac{1+\epsilon}{A}\right) = GM\left(\frac{1-\epsilon}{A}\right) \tag{6}$$

We know that velocities of circular orbits is the same as that of an ellipse with eccentricity $\epsilon = 0$ which yields the velocities of the two circular orbits to be

$$v_{inner}^2 = \frac{GM}{P} \tag{7}$$

and

$$v_{outer}^2 = \frac{GM}{A} \tag{8}$$

We will consider the case where the initial orbital radius is smaller than the final orbital radius. In order to fulfill our requirements for the impulses, we will perform the first impulse at perigee and the second at apogee. These two impulses will carry out the Hohmann transfer and are given by equation (2) as follows

$$\Delta v_1 = \sqrt{v_{per}^2 + v_{inner}^2 - 2v_{per}v_{inner}} \tag{9}$$

and

$$\Delta v_2 = \sqrt{v_{outer}^2 + v_{apo}^2 - 2v_{outer}v_{apo}} \tag{10}$$

Since the Hohmann transfer is a two-impulse maneuver, the total impulse is

$$\Delta v_T = \Delta v_1 + \Delta v_2 \tag{11}$$

3 Broken Assumptions

Unfortunately, the underlying assumptions of the Hohmann transfer are not met by our system of the Sun, Earth, Mars and the payload rocket. The first assumption that is not met by our system is that Earth and Mars' orbits are not co-planar but rather Mars has an ecliptic inclination of $i = 1.85181869°$. Therefore we must take this into account and introduce a plane change maneuver as well.

The second assumption that is not met by our system is that the Earth does have a significant gravitational pull on the rocket. So we must first get significantly far away from the Earth's gravitational potential well so it may be ignored, and then we can use the Hohmann transfer to calculate a trajectory to get to Mars. Upon arrival to Mars, we also wish to land the payload. In order to perform all of these stages to get to Mars, we will utilize what is called the patched conic approach. The trajectory is broken into phases that are connected by patches where each patch is the solution of a two body problem. In order to escape Earth's gravitational field, we will use a hyperbolic escape orbit which will give us an initial velocity away from the Earth. Once we are sufficiently far away as to ignore the Earth's gravitational field, we can then use the Hohmann transfer to find the elliptical orbit to get to Mars. Once at Mars, the reverse of the hyperbolic escape is performed, called hyperbolic capture. It turns out that a Hohmann transfer orbit with hyperbolic escape and capture trajectories can be shown to be the minimum energy trajectory for an interplanetary mission, just as the Hohmann transfer itself is the minimum energy trajectory for a change of circular orbit around a single body.

4 Finding the Plane Change

Plane changes are very expensive in terms of the required change in velocity and resulting propellant consumption. To minimize this, we should change the plane at one of two nodes in the elliptical orbit. The first node in our elliptical trajectory is the first impulse of the Hohmann transfer, at perigee of the ellipse. The second node is then at the apogee of the elliptical orbit, which is arrival at Mars. Therefore of the two nodes, we can only perform the plane change at the first which leads to combining the plane change with the first impulse in the Hohmann transfer. The magnitude of this impulse is given by

$$\Delta v^2 = v_f^2 + v_i^2 - 2v_f v_i \cos\theta \tag{12}$$

where v_i is the initial speed, v_f is the final speed, and θ is the angle change required. Note how the meaning of θ in equation (2) is the same as here. Since we know that if the orbits are co-planar, the desired impulse is in the $\theta = 0$ direction. Now we modify this to have θ be the angle change between the two planes of orbit. Thus we simply modify equation (9) to obtain an impulse of

$$\Delta v_1 = \sqrt{v_{per}^2 + v_{inner}^2 - 2v_{per}v_{inner}\cos\theta} \tag{13}$$

5 Calculating Hyperbolic Escape

Performing a hyperbolic escape is simply setting the object into what is called a hyperbolic trajectory. A hyperbolic trajectory is the trajectory of an object around a central body with more than enough speed to escape the central object's gravitational influence. This is really just an elliptical orbit where $\epsilon > 1$. Therefore the equations for a Hohmann transfer, which is an elliptical orbit, apply here as well.

Since we launch from either the surface of the Earth or an orbit around Earth, we have a rotational speed, v_{rot}, associated with each. If launching from the surface of a rotating body, the rotational speed is specific to where off the surface the rocket is launched. The rotational speed at a given point on the body is dependent on the latitude, α. This speed is given by

$$v_{rot_s} = \frac{2\pi R\cos\alpha}{S} \tag{14}$$

where R is the radius of the body, S is the number of seconds per rotation, and α is the latitude angle where 0 is at the equator and 90 degrees is at either pole.

If launching from an orbit around a central body, the rotational speed is specific to the distance of the orbit from the body. The rotational speed is given by

$$v_{rot_o}{}^2 = \frac{GM}{R_o} \tag{15}$$

where R_o is the distance of the orbit from the center of mass of the body and M is the mass of the central body.

The escape velocity is the minimum speed required for an object to escape from the gravitational influence of a massive central body without the aid of thrust and ignoring frictional forces such as air friction. The escape velocity is given by solving for the speed from the kinetic energy required to escape the potential well of the central body. The escape velocity at the surface is

$$v_{esc_s} = \sqrt{\frac{2GM}{R} - v_{rot_s}^2} \tag{16}$$

where M is the mass of the central body and R is the the distance from the object to the center of mass of the central body.

The escape velocity for an object in orbit around the central body of mass M at a distance of radius R is given by

$$v_{esc_o} = v_{rot_o}\sqrt{2} \tag{17}$$

Therefore the launch impulse is simply equal to the escape velocity.

6 Calculating Hyperbolic Capture

Hyperbolic Capture is simply the reverse of hyperbolic escape. The rocket approaches with an encounter velocity, v_{enc} and wishes to either land on the surface or go into orbit around a central body. We will set the encounter velocity to be the speed after the second impulse of the Hohmann transfer. This way, we perform the second impulse of the Hohmann transfer, which sets the rocket to match Mars' orbit around the Sun. Once that impulse is complete, then proceed to either land on the surface or fall into an orbit around Mars using equations (14) through (17) to do so.

7 Rockets and Propellant Consumption

In order to calculate propellant usage of the trip to Mars we will assume the rocket does not have any drag forces such as air friction acting on it. We will then utilize the Tsiolkovsky rocket equation which describes the motion of a device that can apply acceleration to itself using thrust by expelling part of its mass and thereby move due to the conservation of momentum. The equation relates Δv, the maximum change of velocity of the rocket, with the effective exhaust velocity, v_e, and the initial and final mass of the rocket, m_i and m_f as follows

$$\Delta v = v_e \ln\left(\frac{m_i}{m_f}\right) \tag{18}$$

The effective exhaust velocity is related to the specific impulse through the equation

$$v_e = g_0 I_{sp} \tag{19}$$

where g_0 is the gravitational acceleration constant and I_{sp} is the specific impulse.

The specific impulse of a rocket is a measure of how effectively the rocket uses propellant or in more technical terms, the thrust produced per unit of propellant expended. The specific impulse is used as the standard comparison parameter between rocket engines firstly, because it gives us a quick way to determine the thrust of a rocket given we know the mass flow rate through the nozzle. Secondly, it is an indication of engine efficiency. Two different rocket engines have different values of specific impulse. The engine with the higher value of specific impulse is more efficient because it produces more thrust for the same amount of propellant.

The amount of propellant burned for an impulse Δv will be given by solving equation (18) for m_i and and finding the difference between initial and final masses to give

$$m_{fuel} = m_i - m_f = m_f(e^{\frac{\Delta v}{v_e}} - 1) \tag{20}$$

It is typical of a space-bearing rocket to depart and burn it's propellant in stages. To account for this, we will simply calculate the impulse Δv from equation (18) for each stage in order to find the amount of propellant consumed. If between stages, a piece of the rocket is detached, the m_i for the following stage is the difference between m_f and the mass of the piece that detached.

8 Viability of the Hohmann Trajectory

Let us explore situations where the Hohmann transfer method is the desired trajectory. A maximally energy efficient trajectory is best suited in the case of limited fuel reserves or if the propellant cost is a significant cost for the mission. It should also be noted, the Hohmann transfer sacrifices time efficiency for energy efficiency. Therefore the Hohmann transfer is also suitable for situations in which the time it takes to arrive at the destination is not of importance.

Limited fuel reserves have historically been a problem for objects already in space who want to change their orbit. This has happened often with satellites, where there is a need for a change in the orbital distance but very limited fuel resources to do so. In this case, the Hohmann transfer is the best trajectory because it requires the least amount of fuel. On top of that, satellites changing orbital distance does not have to be done in a time efficient manner, therefore the Hohmann transfer is perfectly suited. In regards to interplanetary travel, this could be useful for a satellite studying a planet and then wanting to transfer to a different planet. Again, time effectiveness is not an issue, so one would want to be as energy efficient as possible. It should be noted that the Hohmann transfer is not always the most energy efficient trajectory for interplanetary travel. This is because the Hohmann transfer is the most energy efficient direct trajectory. However utilizing phenomena such as gravity assists from other planets along the way may be more energy efficient, although not a direct trajectory.

Propellant cost is typically a small fraction of the total cost for a mission. Logistics costs, such as wages, building the rocket, research, and so on usually dominate the cost of space travel. However, if time efficiency is not an issue, why not save a few pennies on propellant.

Space missions are generally very time consuming due to distances being extremely large. When a crew is involved on a mission, time efficiency can become a huge issue for many reasons. If the travel time is longer than necessary, the cost of the mission may skyrocket due to the increase need of supplies for the crew. More supplies for the crew can be very expensive and also increases the amount of payload mass dedicated to crews resources. Another situation would be having a crew on the surface of either the Moon or Mars. If the crew runs into a problem and requires supplies, time efficiency is usually an issue. Therefore the Hohmann transfer is typically not suitable when a crew is involved on the mission. Missions where time efficiency is not important usually concern machinery, such as telescopes, rovers, or satellites. On these missions, the machinery does not consume resources like a crew does, but rather remains idle. Unless there is a special case for moving a satellite quickly, the Hohmann transfer is then most viable for missions concerning only machinery, as it saves on fuel cost.

9 The Space Shuttle

In order to get some numbers associated with performing the Hohmann transfer between Earth and Mars, let us take one of the most popular and efficient rockets that has been used. The Aerojet Rocketdyne RS-25 otherwise known as the Space Shuttle main engine (SSME) has one of the largest specific impulses of any space rocket of $I_{sp} = 452$ seconds in a vacuum and $I_{sp} = 366$ seconds at sea level. The Space Shuttle Orbiter utilizes three of these rockets in conjunction to carry the Space Shuttle orbiter to it's destination.

However, the Aerojet Rocket is only used to propel the Orbiter once in space. The Space Shuttle is launched into space using a pair of the Space Shuttle Solid Rocket Boosters (SRBs) along with an expendable external tank containing the propellant. Each SRBs has a specific impulse of $I_{sp} = 242$ seconds at sea level. The SRBs work in conjunction with the SSME to perform lift off. The design is such that the net force horizontally is 0, so that the stack of Shuttle and SRBs would not tip. It should be noted that the Space Shuttle was retired in 2016 and will be replaced by the Space Launch System (SLS) in 2019.

NASA has two possible locations on Earth two launch the Space Shuttle from. The first is the Vanderberg AFB, which has a latitude of $\alpha = 34.7420°$. The second is the Kennedy Space Center with latitude $\alpha = 28.5729°$. However, SpaceX has a new private launch site in South Texas with a latitude of $\alpha = 25.9920°$. Since we want to be as close the equator as possible, the south Texas launch site is preferred and one can imagine NASA would be quite convincing in having SpaceX allow them to use this launch site.

When launching from an orbit around Earth, the escape velocity is significantly less than at the surface. For simplicity of the paper, we will select the medium earth orbit (MEO) to be the orbital launch distance. The most common altitude for MEO is approximately $20,000km$ although it ranges from $2,000km$ to nearly $36,000km$.

Upon arrival to Mars, we shall descend to the surface. Just like launching from Earth, we have to consider the latitude of our arrival location on the surface of Mars. The closer to the equator of Mars, the less fuel necessary to land. However, this produces many problems with entry direction. So we take the simple landing, and land on a pole which will have a latitude of $\alpha = \pm 90°$.

10 Results

10.1 Total Impulse and Propellant Requirements

Stage	Symbol	Impulse	Fuel Mass
Launch from Surface of Earth	Δv_{surf}	$11,172\ m/s$	$1,466,681\ kg$
Launch from MEO Orbit	Δv_{orb}	$6,313\ m/s$	$127,994\ kg$
Hohmann Transfer Impulse #1	Δv_1	$2,945\ m/s$	$41,314\ kg$
Transfer Impulse Corrected	Δv_{corr}	$3,113\ m/s$	$43,960\ kg$
Hohmann Transfer Impulse #2	Δv_2	$2,649\ m/s$	$30,103\ kg$
Total Hohmann Transfer	Δv_{Hohmm}	$5,762\ m/s$	$74,063\ kg$
Landing on Surface of Mars	Δv_{land}	$5,027\ m/s$	$42,993\ kg$
Total from MEO to Mars		$17,102\ m/s$	$245,050\ kg$
Total from Surface to Mars		$21,960\ m/s$	$1,583,737\ kg$

Table 1: A table of the impulses and associated propellant required to perform each impulse for the trip to Mars. Calculations may be found in the Appendix.

10.2 Trip Duration

The duration of the trip is hugely dominated by the Hohmann trajectory. Both hyperbolic escape and capture take a relatively short amount of time to where they can be ignored. Therefore the duration of the trip is simply half the period of the ellipse of the Hohmann trajectory which is given by Kepler's 3rd Law as follows

$$\begin{aligned} T &= \pi\sqrt{\frac{a^3}{GM_S}} \\ &= \pi\sqrt{\frac{(1.8877 \times 10^{11})^3}{(6.67408 \times 10^{-11})(1.98847 \times 10^{30})}} \\ &= 2.2366... \times 10^7 sec \end{aligned}$$

Therefore the duration of the trip is then roughly 259 days or 8.6 months.

11 Conclusion

The Hohmann transfer could also be used as a benchmark tool. It serves for calculating the minimum mass of propellant a rocket would need in order to perform an orbital transfer. This is useful for cases when a rocket has limited fuel and one is exploring possible trajectories for the rocket. If a rocket does not have the necessary fuel to perform the Hohmann transfer to a different orbit, there is no other way of arriving and maintaining said orbit.

In the example in this paper, we found that the simplified model of the Hohmann transfer requires a total impulse of $5,594 m/s$. When the plane change is accounted for, the total impulse becomes $5,762 m/s$. The difference is small, but not quite negligible. However, what the simplified model does not account for is launching and landing the rocket. Clearly, from table 2, we can see that launching and landing a rocket is even more propellant demanding than the Hohmann transfer itself. Therefore these stages of a trajectory cannot be ignored.

Therefore the Hohmann transfer, on its own, is really only useful for satellites in orbit around a central body, wanting to change orbital distance. What a coincidence that, the Hohmann method is only viable for the special case it was derived for. However, with the method of patched conics, we can see that, depending on the degree of accuracy one wishes to achieve, accounting for plane change and performing hyperbolic escape and capture is relatively easy to do.

So far as we've solved it, we simply need to know a few parameters and then we can find the minimum mass or propellant and the impulses required. However, one can begin to take down these assumptions in order to achieve higher degrees of accuracy. For example, we assumed an instantaneous impulse. In reality, there is burn times for the propellant, and the impulse is not instantaneous but rather achieved over an acceleration period. Accounting for this alone can already make solving the problem quite tedious as one must consider the timing of the acceleration, the duration and the constantly decreasing mass of propellant as it is consumed. Another thing one could account for is the eccentricity of Earth's and Mars' orbits around the Sun, since they are not actually circular. This again makes finding the Hohmann transfer more difficult. If one wishes to achieve only a rough benchmark of the minimum cost of fuel and mass of propellant, the method as shown in this paper is sufficient. If higher degrees of accuracy are desired, accounting for the phenomena stated above could achieve this.

12 Appendix

Variable	Symbol	Value
Gravitational Constant	G	$6.67408 \times 10^{-11}\ \frac{m^3}{kgs}$
Earth's Gravitational Acceleration	g_0	$9.80655\ \frac{m}{s^2}$
Mass of Sun	M_S	$1.98847 \times 10^{30}\ kg$
Mass of Earth	M_E	$5.9722 \times 10^{24}\ kg$
Mass of Mars	M_M	$0.64171 \times 10^{24}\ kg$
Radius of Earth	R_E	$6.3781 \times 10^{6}\ m$
Radius of Mars	R_M	$3.3895 \times 10^{6}\ m$
Distance from Sun to Earth	D_E	$1.4960 \times 10^{11}\ m$
Distance from Sun to Mars	D_M	$2.2794 \times 10^{11}\ m$
Seconds per Day on Earth	S_E	$86,164\ s$
Inclination between Earth and Mars	θ	1.85181869°
Mass of Space Shuttle Orbiter	M_{Orb}	$78,000\ kg$
Mass of Main Engine	M_{SSME}	$3,527\ kg$
Mass of Solid Rocket Boosters	M_{SRBs}	$590,000\ kg$
Mass of Space Shuttle Payload	M_{load}	$5,000\ kg$
Specific Impulse of SRBs	I_{SRBs}	242 *seconds*
I_{sp} of Main Engine @ vacuum	I_{SSME}	452 *seconds*
I_{sp} of Main Engine @ sea level	I_{SSME}	366 *seconds*
Distance of Orbital Launch	R_o	$20 \times 10^{6}\ m$
Latitude of Earth Surface Launch	α_E	25.9920°
Latitude of Mars Surface Landing	α_M	± 90°

Table 2: A table of standard values used to make the calculations

12.1 Determining the Hohmann Trajectory

First we find the semi-major axis of our system

$$
\begin{aligned}
a &= \frac{1}{2}(D_E + D_M) \\
&= \frac{1}{2}(1.4960 \times 10^{11} + 2.2794 \times 10^{11}) \\
&= 1.8877 \times 10^{11} m
\end{aligned}
$$

which yields an eccentricity of the Hohmann trajectory of

$$
\begin{aligned}
\epsilon &= 1 - \frac{D_E}{a} \\
&= 1 - \frac{1.4960 \times 10^{11}}{1.8877 \times 10^{11}} \\
&= 0.2075012...
\end{aligned}
$$

12.2 Determining the Hohmann Impulses

We begin by finding the speeds of Earth and Mars from equations (7) and (8)

$$
\begin{aligned}
v_{inner} = v_{Earth} &= \sqrt{\frac{GM_S}{D_E}} \\
&= \sqrt{\frac{(6.67408 \times 10^{-11})(1.98847 \times 10^{30})}{1.4960 \times 10^{11}}} \\
&= 29,784 m/s
\end{aligned}
$$

and

$$
\begin{aligned}
v_{outer} = v_{Mars} &= \sqrt{\frac{GM_S}{D_M}} \\
&= \sqrt{\frac{(6.67408 \times 10^{-11})(1.98847 \times 10^{30})}{2.2794 \times 10^{11}}} \\
&= 24,129 m/s
\end{aligned}
$$

The speeds of the Hohmann trajectory at perigee and apogee are given by equations (5) and (6)

$$\begin{aligned}
v_{per} &= \sqrt{GM_S\frac{1+\epsilon}{D_E}} \\
&= \sqrt{(6.67408\times10^{-11})(1.98847\times10^{30})\frac{1+(0.2075012...)}{1.4960\times10^{11}}} \\
&= 32,729 m/s
\end{aligned}$$

and

$$\begin{aligned}
v_{apo} &= \sqrt{GM_S\frac{1-\epsilon}{D_M}} \\
&= \sqrt{(6.67408\times10^{-11})(1.98847\times10^{30})\frac{1-(0.2075012...)}{2.2794\times10^{11}}} \\
&= 21,480 m/s
\end{aligned}$$

which yields the impulses from equations (9) and (10)

$$\begin{aligned}
\Delta v_1 &= \sqrt{v_{per}^2+v_{Earth}^2-2v_{per}v_{Earth}} \\
&= \sqrt{32729^2+29784^2-2(32729)(29784)} \\
&= 2,945 m/s
\end{aligned}$$

and

$$\begin{aligned}
\Delta v_2 &= \sqrt{v_{Mars}^2+v_{apo}^2-2v_{Mars}v_{apo}} \\
&= \sqrt{24129^2+21480^2-2(24129)(21480)} \\
&= 2,649 m/s
\end{aligned}$$

However, we must apply the plane change from equation (13). This yields the corrected Δv_1 impulse of

$$\begin{aligned}
\Delta v_{corr} &= \sqrt{v_{per}^2+v_{Earth}^2-2v_{per}v_{Earth}\cos\theta} \\
&= \sqrt{32729^2+29784^2-2(32729)(29784)\cos 1.85181869} \\
&= 3,113 m/s
\end{aligned}$$

12.3 Finding Launch Impulse

12.3.1 Launching from the Surface of Earth

Finding the rotational speed provided by the Earth from equation (14) at the launch location in south Texas yields

$$\begin{aligned}
v_{rot_s} &= \frac{2\pi R_E \cos(\alpha_E)}{S} \\
&= \frac{2\pi(6.3781 \times 10^6)\cos(25.9920)}{86164} \\
&= 418 m/s
\end{aligned}$$

The escape velocity and thus the launch impulse required is then calculated from equation (16) to be

$$\begin{aligned}
\Delta v_{surf} = v_{esc} &= \sqrt{\frac{2GM_E}{R_E} - v_{rot_s}^2} \\
&= \sqrt{\frac{2(6.67408 \times 10^{-11})(5.9722 \times 10^{24})}{(6.3781 \times 10^6)} - 418^2} \\
&= 11,172 m/s
\end{aligned}$$

12.3.2 Launching from MEO Orbit around Earth

Finding the orbital speed from equation (15) in MEO yields

$$\begin{aligned}
v_{rot_o} &= \sqrt{\frac{GM_E}{R_o}} \\
&= \sqrt{\frac{(6.67408 \times 10^{-11})(5.9722 \times 10^{24})}{(20 \times 10^6)}} \\
&= 4,464 m/s
\end{aligned}$$

The escape velocity and thus the launch impulse required is then calculated from equation (17) to be

$$\begin{aligned}
\Delta v_{orb} = v_{esc} &= v_{rot_o}\sqrt{2} \\
&= 4464\sqrt{2} \\
&= 6,313 m/s
\end{aligned}$$

12.4 Finding Landing Impulse

Since we land at a pole of Mars, there is no rotational speed contribution, $v_{rot_s} = 0$. Therefore the landing impulse is simply the escape velocity at the surface which is given by equation (16)

$$\begin{aligned}\Delta v_{land} = v_{esc} &= \sqrt{\frac{2GM_M}{R_M} - v_{rot_s}^2}\\ &= \sqrt{\frac{2(6.67408 \times 10^{-11})(0.64171 \times 10^{24})}{(3.3895 \times 10^6)} - 0^2}\\ &= 5,027 m/s\end{aligned}$$

12.5 Fuel Requirements

We must calculate the fuel requirements in backwards order of the trajectory due to knowing the final mass of each stage. Finding the mass of the propellant from equation (20) for each stage we find

12.5.1 Landing on Mars

$$\begin{aligned}m_{land} &= (M_{orb} + 3M_{SSME} + M_{load}) \left(exp\left(\frac{\Delta v_{Hohmm}}{3g_0 I_{SSME}}\right) - 1\right)\\ &= (93581) \left(exp\left(\frac{5027}{3(9.80655)(452)}\right) - 1\right)\\ &= 42,993 kg\end{aligned}$$

12.5.2 The Hohmann Transfer

The required mass of propellant for the second impulse of the Hohmann transfer is

$$\begin{aligned}m_2 &= (M_{orb} + 3M_{SSME} + M_{load}) \left(exp\left(\frac{\Delta v_{Hohmm}}{3g_0 I_{SSME}}\right) - 1\right)\\ &= (93581 + 42993) \left(exp\left(\frac{2649}{3(9.80655)(452)}\right) - 1\right)\\ &= 30,103 kg\end{aligned}$$

Therefore the fuel requirement for the first impulse of the Hohmann transfer is

$$\begin{aligned} m_1 &= (M_{orb} + 3M_{SSME} + M_{load}) \left(exp\left(\frac{\Delta v_{Hohmm}}{3g_0 I_{SSME}} \right) - 1 \right) \\ &= (93581 + 42993 + 30103) \left(exp\left(\frac{2945}{3(9.80655)(452)} \right) - 1 \right) \\ &= 41,314kg \end{aligned}$$

The fuel requirement for the corrected first impulse, which includes a plane change is

$$\begin{aligned} m_{corr} &= (M_{orb} + 3M_{SSME} + M_{load}) \left(exp\left(\frac{\Delta v_{Hohmm}}{3g_0 I_{SSME}} \right) - 1 \right) \\ &= (93581 + 42993 + 30103) \left(exp\left(\frac{3113}{3(9.80655)(452)} \right) - 1 \right) \\ &= 43,960kg \end{aligned}$$

Therefore the fuel requirement for the total Hohmann transfer, is simply

$$m_{Hohmm} = m_{corr} + m_2 \qquad = 43960 + 30103 = 74,063kg$$

12.5.3 Launching from MEO orbit around Earth

Since we are already in MEO orbit, we only use the SSME rocket and therefore the mass of propellant is

$$\begin{aligned} m_{orb} &= (M_{orb} + 3M_{SSME} + M_{load}) \left(exp\left(\frac{\Delta v_{orb}}{3g_0 I_{SSME}} \right) - 1 \right) \\ &= (93581 + 42993 + 74,063) \left(exp\left(\frac{6313}{3(9.80655)(452)} \right) - 1 \right) \\ &= 127,994kg \end{aligned}$$

12.5.4 Launching from the surface of Earth

Since from launch at the Surface, the SRBs perform the Δv_{surf}, we use equation (20) twice. Once for the launch where SRBs are used, and then another for the rest of the trajectory where the SSME is used. Note that between these two stages, the fuel from launch is completely used and the SRBs detach from the Shuttle. The final mass is then given by

$$\begin{aligned} m_f &= M_{Orb} + M_{load} + 3M_{SSME} + 2MSRBs + m_{land} + m_{Hohmm} \\ &= 78000 + 5000 + 3(3527) + 2(590000) + 42993 + 74,063 \\ &= 1,390,637kg \end{aligned}$$

Therefore the mass of propellant required for launch is

$$
\begin{aligned}
m_{surf} &= m_f \left(exp\left(\frac{\Delta v_{surf}}{g_0(2I_{SRBs} + 3I_{SSME})} \right) - 1 \right) \\
&= (1390637) \left(exp\left(\frac{11172}{(9.80655)(2(242) + 3(366))} \right) - 1 \right) \\
&= 1,466,681kg
\end{aligned}
$$