Practical Stability of Nonlinear Systems

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Preface

An interesting and fruitful technique that has gained increasing significance and has given decisive impetus for modern development of stability theory of differential equations is the second method of Lyapunov. A manifest advantage of this method is that it does not require the knowledge of solutions and therefore has great power in applications. It is now well recognized that the concept of Lyapunov-like function and the theory of differential and integral inequalities can be utilized to investigate qualitative and quantitative properties of nonlinear differential equations.

In the stabilization of nonlinear systems interesting set of problems deals with bringing states close to certain sets rather than to the particular state $x = 0$. From a practical point of view, a concrete system will be considered stable if the deviations of the motions from the equilibrium remain within certain bounds determined by the physical situation, in case the initial values and/or the disturbances are bounded by suitable constraints. The desired state of a system may be mathematically unstable and yet the system may oscillate sufficiently near this state that its performance is acceptable. Many aircraft and missiles behave in this manner. Even asymptotic stability by itself is not sufficient in practice since the domain of attraction may not be large enough to allow the desired deviations to cancel out. As a result, the system may be asymptotically stable in theory but is actually unstable in practice. Thus a notion of stability which is neither weaker nor stronger than Lyapunov stability is desired from practical considerations. LaSalle and Lefschetz in their book, [1], suggested a name for such a concept and called it "practical stability". Hahn in his book, [1], wrote that "a systematic study of this concept, the development of criteria etc., have not yet been undertaken." This is precisely what is planned in this book. It's aim is to present a systematic account of the development, describe the current state of the useful theory and provide a unified general structure applicable to a variety of nonlinear problems of diverse interest.

Some of the important features of the monograph are as follows: This is the first book that

(i) deals with practical stability and its development;

(ii) presents a systematic study of the theory of practical stability in terms of two different measures and arbitrary sets; and,

(iii) demonstrates the manifestations of general Lyapunov's method by showing how this effective technique can be adapted to investigate various apparently diverse nonlinear problems including control systems and multivalued differential equations.

In view of the existence of several excellent books on Lyapunov stability by second method, we have not included Lyapunov stability criteria but for the definitions and that too for the comparison with the definitions of practical stability. Instead of incorporating in the main body of the book, some of the needed known results have been listed in an appendix for the convenience of the reader. We do hope that this mongraph will stimulate further investigation and new thinking on this important practical concept.

We wish to express our thanks to Donn Harnish for her excellent typing of this manuscript even though she is new for this type of technical typing.

V. Lakshmikantham
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What is practical stability?

1.0. INTRODUCTION.

Theory of stability in the sense of Lyapunov is now well known and is widely used in concrete problems of the real world. It is obvious that, in applications, asymptotic stability is more important than stability. In fact, the desirable feature is to know the size of the region of asymptotic stability so that we can judge whether or not a given system is sufficiently stable to function properly and may be able to see how to improve its stability. On the other hand, the desired system may be unstable and yet the system may oscillate sufficiently near this state that its performance is acceptable. Thus it is clear that we need a notion of stability that is more suitable in several situations than Lyapunov stability. Such a concept is called practical stability. In this chapter we shall introduce the concept of practical stability, obtain some simple criteria for such stability to hold and discuss extensions to other nonlinear systems.

We begin section 1.1 by defining various notions of Lyapunov stability and boundedness so that we can compare and contrast with practical stability concepts. Section 1.2 is devoted to the definition of practical stability. By means of examples, we demonstrate that although practical stability is neither weaker nor stronger than Lyapunov stability, it is more suitable and desirable in dealing with concrete problems of the real world. In section 1.3, we offer simple criteria for practical stability of differential equations. We utilize variation of parameters formulae and norms as candidates. Several interesting results employing differential and integral inequalities are presented.

Section 1.4 deals with practical stability criteria for the delay differential equations where employing a unified approach the study of delay differential equations is reduced to the study of ordinary differential equations. In section 1.5, we extend practical stability considerations to integro-differential systems of Volterra type. Utilizing the idea of finding an equivalent linear differential system for a given linear integro-differential system of Volterra type, we discuss practical stability of linear and nonlinear integro-differential systems.

Section 1.6 investigates difference equations and Volterra type difference equations relative to practical stability. The latter type equations arise when we discretize integro-differential equations of Volterra type. Here also we use the idea of finding a linear difference system that is equivalent to a given linear Volterra difference system and then employ comparison technique. Finally, in section 1.7, we present impulsive differential equations, prove necessary tools and discuss several criteria for practical stability.

1.1. DEFINITIONS OF LYAPUNOV STABILITY.

Consider the differential system

$$x' = f(t, x), \; x(t_0) = x_0, \; t_0 \geq 0 \tag{1.1.1}$$

where $f \in C[\mathbf{R}_+ \times \mathbf{R}^n, \mathbf{R}^n]$. Suppose that the function f is smooth enough to guarantee existence, uniqueness and continuous dependence of solutions $x(t) = x(t, t_0, x_0)$ of (1.1.1). Let us first define the stability concepts in the sense of Lyapunov. For that purpose, we assume that $f(t, 0) \equiv 0$ so that $x(t) \equiv 0$ is the (trivial) solution of (1.1.1) through $(t_0, 0)$. We now list various definitions of stability:

Definition 1.1.1

The trivial solution of (1.1.1) is said to be

(S_1) equi-stable if for each $\epsilon>0$, $t_0\in \mathbf{R}_+$, there exists a positive function $\delta=\delta(t_0, \epsilon)$ which is continuous in t_0 for each ϵ such that $|x_0| < \delta$ implies $|x(t)| <\epsilon$, $t\geq t_0$;

(S_2) uniformly stable if the δ in (S_1) is independent of t_0;

(S_3) quasi-equi asymptotically stable, if for each $\epsilon > 0$, $t_0\in \mathbf{R}_+$, there exist positive numbers $\delta_0=\delta_0(t_0)$ and $T=T(t_0, \epsilon)$ such that $|x_0| <\delta_0$ implies $|x(t)|<\epsilon$, $t\geq t_0+T$;

(S_4) quasi-uniformly asymptotically stable if the numbers δ_0 and T in (S_3) are independent of t_0;

(S_5) equi-asymptotically stable if (S_1) and (S_3) hold together;

(S_6) uniformly asymptotically stable if (S_2) and (S_4) hold together;

(S_7) quasi-equi asymptotically stable in the large if for each $\epsilon > 0$, $\alpha > 0$, $t_0 \in \mathbf{R}_+$, there exists a positive number $T=T(t_0, \epsilon, \alpha)$ such that $|x_0| \leq\alpha$ implies $|x(t)|<\epsilon$, $t\geq t_0 + T$;

(S_8) quasi-uniformly asymptotically stable (in the large) if the number T in (S_7) is independent of t_0;

(S_9) completely stable if (S_1) holds and (S_7) is satisfied for all α, $0\leq\alpha< \infty$;

(S_{10}) uniformly completely stable if (S_2) holds and (S_8) is verified for all α, $0\leq\alpha < \infty$;

(S_{11}) unstable if (S_1) fails to hold.

Remark 1.1.1

We note that the existence of the trivial solution of (1.1.1) is not necessary for the notions (S_3), (S_4), (S_7) and (S_8) to hold. Furthermore, even when the trivial solution does not exist, we may have stability eventually which is a generalization of Lyapunov stability. We shall define such a concept below.

Definition 1.1.2

The system (1.1.1) is said to be

(E_1) eventually stable, if for each $\epsilon>0$, there exist two positive numbers $\delta=\delta(\epsilon)$ and $\tau=\tau(\epsilon)$ such that $|x_0|<\delta$ implies $|x(t)|<\epsilon$, $t\geq t_0\geq\tau$;

(E_2) eventually asymptotically stable if (E_1) and (S_4) hold simultaneously.

It is clear that, in applications, asymptotic stability is more important than stability. It is therefore necessary to know the size of the domain of asymptotic stability so that based on estimates of the conditions under which the system will actually operate, requirements on its performance etc., we can judge whether or not the system is sufficiently stable to function properly and may be able to see how to improve its stability. Thus for practical purposes, complete stability seems desirable.

Corresponding to different types of stability, we can define concepts of boundedness. To do this, we do not need the existence of the trivial solution.

Definition 1.1.3

The differential system (1.1.1) is said to be

(B_1) equi-bounded if, for each $\alpha\geq 0$, $t_0\in\mathbf{R}_+$, there exists a positive function $\beta=\beta(t_0,\alpha)$ that is continuous in t_0 for each α such $|x_0|\leq\alpha$ implies $|x(t)|<\beta$, $t\geq t_0$;

(B_2) uniformly bounded if the β in (B_1) is independent of t_0;

(B_3) quasi-equi ultimately bounded if, for each $\alpha \geq 0$, $t_0 \in \mathbf{R}_+$, there exist positive numbers N and $T = T(t_0, \alpha)$ such that $| x_0 | \leq \alpha$ implies $| x(t) | < N$, $t \geq t_0 + T$;

(B_4) quasi-uniform ultimately bounded if the T in (B_3) is independent of t_0;

(B_5) equi-ultimately bounded if (B_1) and (B_3) hold together;

(B_6) uniform-ultimately bounded if (B_2) and (B_4) hold simultaneously;

(B_7) equi-Lagrange stable if (B_1) and (S_7) are satisfied;

(B_8) uniform-Lagrange stable if (B_2) and (S_8) are satisfied;

(B_9) eventually bounded if, for each $\alpha \geq 0$ there exist two positive numbers $\tau = \tau(\alpha)$ and $\beta = \beta(\alpha)$ such that $| x_0 | \leq \alpha$ implies $| x(t) | < \beta$, $t \geq t_0 \geq \tau$;

(B_{10}) eventually Lagrange stable if (B_9) and (S_8) hold together.

We note that if $f(t,0) \equiv 0$ and β occurring in (B_1), (B_2) has the property that $\beta \to 0$ as $\alpha \to 0$, then the definitions (B_1), (B_2) imply (S_1), (S_2) respectively.

Let us illustrate the definitions given above with some examples.

Example 1.1.1

Consider the system of equations

$$(1.1.2) \quad \begin{cases} x'(t) = n(t)y + m(t)\, x(x^2 + y^2),\ x(t_0) = x_0, \\ y'(t) = -n(t)x + m(t)y(x^2 + y^2),\ y(t_0) = y_0, \end{cases}$$

where n, $m \in C[\mathbf{R}_+, \mathbf{R}]$. The general solution of (1.1.2) is given by

$$x(t)=\frac{x_0\cos(\int_{t_0}^{t} n(t)dt)+y_0\sin(\int_{t_0}^{t} n(t)dt)}{(1-2(x_0^2+y_0^2)\int_{t_0}^{t} m(t)dt)^{\frac{1}{2}}},$$

$$y(t)=\frac{y_0\cos(\int_{t_0}^{t} n(t)dt)-x_0\sin(\int_{t_0}^{t} n(t)dt)}{(1-2(x_0^2+y_0^2)\int_{t_0}^{t} m(t)dt)^{\frac{1}{2}}}$$

which reduces to

(1.1.3) $$r^2(t)=x^2(t)+y^2(t)=r_0^2(1-2\,r_0^2\int_{t_0}^{t} m(t)dt)^{-1},$$

where $r_0^2=x_0^2+y_0^2$. It follows from (1.1.3) that the trivial solution of (1.1.2) is stable if $m(t)\leq 0$, $t\geq t_0$. If $m(t)>0$, $t\geq t_0$, then the trivial solution of (1.1.2) is stable when the integral

(1.1.4) $$\int_{t_0}^{t} m(t)dt$$

is bounded and unstable when (1.1.4) is unbounded.

Example 1.1.2

Consider the differential system

(1.1.5) $$\begin{cases} x'(t)=-x-y+k(x-y)(x^2+y^2),\ x(t_0)=x_0, \\ y'(t)=x-y+k(x+y)(x^2+y^2),\ y(t_0)=y_0, \end{cases}$$

where $k>0$ is a constant. The general solution of (1.1.5) is given by

$$x(t)=\frac{1}{\sqrt{k\mu}}(x_0\cos\theta-y_0\sin\theta),\ y(t)=\frac{1}{\sqrt{k\mu}}(x_0\sin\theta+y_0\cos\theta)$$

with $\theta=2(t-t_0)-\frac{1}{2}\ln\mu$ and $\mu=r_0^2+(\frac{1}{k}-r_0^2)\exp(2(t-t_0))$ This

reduces to

$$(1.1.6)\qquad (r(t))^2=\frac{1}{k\mu}r_0^2.$$

It is clear that if $r_0^2=x_0^2+y_0^2<\frac{1}{k}$, then the trivial solution of (1.1.5) is asymptotically stable.

Example 1.1.3

Consider the equation of perturbed motion of a mechanical system with one degree of freedom given by

$$\frac{d^2x}{dt^2}-2h\frac{dx}{dt}+gx=0,\ x(0)=x_0,\ x'(0)=y_0,$$

which can be rewritten as the system

$$(1.1.7)\qquad \begin{cases} \dfrac{dx}{dt}=y,\ x(0)=x_0 \\ \dfrac{dy}{dt}=2hy-gx,\ y(0)=y_0 \end{cases}$$

with $g-h^2>0$. Since the characteristic roots of $\lambda^2-2h\lambda+g=0$ are complex conjugate, let $\lambda_1=\alpha+i\beta$, $\lambda_2=\alpha-i\beta$ and the general solution of (1.1.7) is of the form

$$x(t)=[x_0\cos\beta t+\frac{y_0-\alpha x_0}{\beta}\sin\beta t]\exp(\alpha t).$$

Consequently, the trivial solution of (1.1.7) is unstable if $\alpha>0$, $h<0$ and $g>0$.

Example 1.1.4

Consider the differential equation

$$x' = -\lambda'(t),\ x(t_0) = x_0, \tag{1.1.8}$$

where $\lambda \in C^1[\mathbf{R}_+, \mathbf{R}_+]$ with $\lambda'(t) \geq 0$. Then $x(t) = x_0 + \lambda(t_0) - \lambda(t)$ and hence, we have

$$|\,x(t)\,| < |\,x_0\,| + \lambda(t_0),\ t \geq t_0.$$

It is easy to see that (B_1) holds with $\beta = \alpha + \lambda(t_0)$ and that β does not tend to zero as $\alpha \to 0$. If $\lambda(t)$ is decreasing to zero as $t \to \infty$, then, given $\epsilon > 0$, there exists a $\tau(\epsilon)$ such that $\lambda(t_0) < \frac{\epsilon}{2}$ if $t_0 \geq \tau(\epsilon)$. As a result, (E_1) holds with $\delta = \frac{\epsilon}{2}$ and $\tau(\epsilon)$ defined above.

Example 1.1.5

Consider the equation

$$x' = \frac{-x}{1+t},\ x(t_0) = x_0 \tag{1.1.9}$$

whose solution $x(t) = \dfrac{x_0(1+t_0)}{(1+t)}$ does not tend to zero uniformly with respect to t_0.

1.2. DEFINITIONS OF PRACTICAL STABILITY.

We have seen that complete stability is a more desirable feature in applications than asymptotic stability. Sometimes even instability may be good enough. Since the desired state of a system may be mathematically unstable but the system may oscillate sufficiently near this state so that its performance is considered acceptable. For example, an aircraft or a missile may oscillate around a mathematically unstable course yet its performance may be acceptable. Many problems fall into this category including the travel of a space vehicle between two points and the problem, in a chemical process, of keeping the temperature within certain bounds. To deal with such situations, the notion of practical stability is more useful, which we define below.

Definition 1.2.1

The system (1.1.1) is said to be

(PS_1) practically stable if, given (λ, A) with $0<\lambda<A$, we have $|x_0|<\lambda$ implies $|x(t)|<A$, $t\geq t_0$ for some $t_0\in \mathbf{R}_+$;

(PS_2) uniformly practically stable if (PS_1) holds for every $t_0\in\mathbf{R}_+$;

(PS_3) practically quasi stable if given (λ, B, T)>0 and some $t_0\in \mathbf{R}_+$, we have $|x_0|<\lambda$ implies $|x(t)|<B$, $t\geq t_0+T$;

(PS_4) uniformly practically quasi stable if (PS_3) holds for all $t_0\in \mathbf{R}_+$;

(PS_5) strongly practically stable if (PS_1) and (PS_3) hold simultaneously;

(PS_6) strongly uniformly practically stable if (PS_2) and (PS_4) hold together;

(PS_7) practically asymptotically stable if (PS_1) and (S_7) hold with $\alpha=\lambda$;

(PS_8) uniformly practically asymptotically stable if (PS_2) and (S_8) hold at the same time with $\alpha=\lambda$;

(PS_9) practically unstable if (PS_1) does not hold;

(PS_{10}) eventually practically stable if given (λ, A), $0<\lambda<A$, there exists a $\tau=\tau(\lambda, A)$ such that $|x_0|<\lambda$ implies $|x(t)|<A$, $t\geq t_0\geq\tau$;

(PS_{11}) eventually strongly practically stable if (PS_{10}) and (PS_3) hold;

(PS_{12}) eventually uniformly strongly practically stable if (PS_{10}) and (PS_4) are satisfied simultaneously.

In (PS_5) and (PS_6), if $0<B<\lambda<A$, then we say that the system (1.1.1) is contractively practically stable, while for $0<\lambda<B<A$, the system is said to be expansively practically stable.

Sometimes, in physical problems, one is interested in the behavior of systems within specified bounds during a fixed time interval. The concept of "finite time stability" is appropriate to cover such situations. For example, the notion (PS_5) would translate into the following: the differential system (1.1.1) is strongly practically stable if, given positive numbers λ, A, B and T we have $|x_0|<\lambda$ implies $|x(t)|<A$, $t_0 \le t \le t_0 + T$ and $|x(t_0+T)|<B$.

Let us consider some examples to illustrate the notions of practical stability.

Example 1.2.1

Consider the example 1.1.1 and let $A=2\lambda$. Suppose that $\int_{t_0}^{t} m(s)ds=\beta>0$. Then, we get from (1.1.3)

$$\lim_{t\to\infty} r^2(t)=r_0^2(1-2r_0^2\beta)^{-1}. \tag{1.2.1}$$

It therefore follows from (1.2.1) that the system (1.1.2) is practically stable if $\beta \le \frac{3}{8\lambda^2}$ and practically unstable if $\beta>\frac{3}{8\lambda^2}$.

Example 1.2.2

Consider the example 1.1.2. Recall that $r_0^2=x_0^2+y_0^2<\frac{1}{k}$ is the region of asymptotic stability. Let λ, A be such that $\frac{1}{\sqrt{k}}<\lambda<A$. Then for the initial values (x_0, y_0) such that $\frac{1}{k} \leq r_0^2<\lambda^2$, the system (1.1.5) is not practically stable which shows that asymptotic stability is not sufficient for practical stability to hold. In other words, the presence or absence of practical stability in a system does not depend on asymptotic stability of the system.

Example 1.2.3

Consider the system (1.1.7) of Example 1.1.3. Recall that the system (1.1.7) is unstable if $\alpha>0$, $h<0$ and $g>0$. Define the sets

$$S_0=\left\{(x, y):\ x^2 + \left(\frac{y-hx}{\beta}\right)^2<\lambda^2\right\}$$

$$S=\left\{(x, y):\ x^2 + \left(\frac{y-hx}{\beta}\right)^2<\delta\lambda^2\right\}$$

with $\delta>0$ and $\beta=\sqrt{g-h^2}$. Take $T< \frac{\ln\delta}{2\alpha}$. Then, it is easy to check that the system (1.1.7) is practically stable with respect to the sets S_0, S and the finite time interval $[t_0, t_0 + T]$.

Example 1.2.3 indicates that it is more natural to define practical stability in terms of arbitrary sets rather than neighborhoods of the origin. We shall introduce such concepts in the next chapter. We have seen that practical stability is neither weaker nor stronger than Lyapunov stability. Before we speak of practical stability we must decide on

(i) how near the desired state (that is, set S) it is necessary to have the system operate; and

(ii) how well the initial set (that is, set S_0) can be controlled.

Also, practical stability is somewhat similar to uniform boundedness. It is, however, not merely that a bound exists but that the bound be preassigned. Note that Lagrange stability is somewhat similar to practical asymptotic stability and ultimate boundedness is a necessary condition for the system to possess strong practical stability.

Sometimes, the interdependence of (λ, A, B, T) may be useful in practice. For example, (PS_3) may be weakened as follows. The system (1.1.1) is said to be

(PS_3*) practically quasi-stable if given $(\lambda, B)>0$ and $t_0 \in R_+$, there exists a $T=T(t_0, \lambda, B)$ such that $|x_0|<\lambda$ implies $|x(t)|<B$, $t \geq t_0 + T$.

If (PS_1) and (PS_3*) hold together, we can identify that as (PS_5*) and other similar concepts may be introduced. Occasionally, it is advantageous to restrict the initial times t_0 to a given set $T_0 \subset \mathbf{R}_+$, instead of allowing the initial set (set T_0) to be the whole real line.

1.3. STABILITY CRITERIA.

Corresponding to Definition 1.2.1, we can define the practical stability notions for the scalar differential equation

(1.3.1) $$u' = g(t, u),\ u(t_0) = u_0 \geq 0,$$

where $g \in C[\mathbf{R}_+^2, \mathbf{R}]$. For example, the equation (1.3.1) is said to be practically stable if, given $0<\lambda<A$, we have

(1.3.2) $$u_0 < \lambda \text{ implies } u(t) < A,\ t \geq t_0 \text{ for some } t_0 \in \mathbf{R}_+,$$

where $u(t, t_0, u_0)$ is any solution of (1.3.1).

We can now discuss some simple practical stability results.

Let us define

$$[x, y]_{\pm} = \lim_{h \to 0^{\pm}} \left[\frac{1}{h} |x + hy| - |x| \right].$$

Theorem 1.3.1

Assume that for $(t, x) \in \mathbf{R}_+ \times \mathbf{R}^n$,

$$[x, f(t, x)]_+ \leq g(t, |x|), \tag{1.3.3}$$

where $g \in C[\mathbf{R}_+^2, \mathbf{R}]$. Then the practical stability properties of (1.3.1) imply the corresponding practical stability properties of the system (1.1.1).

Proof

Let the equation (1.3.1) be practically stable. Then for some $t_0 \in \mathbf{R}_+$ and $0 < \lambda < A$, (1.3.2) holds. It is easy to show that for these λ, A the system (1.1.1) is also practically stable. If this were false, there would exist a solution $x(t) = x(t, t_0, x_0)$ of (1.1.1) with $|x_0| < \lambda$ and a $t_1 > t_0$ such that

$$|x(t_1)| = A \text{ and } |x(t)| \leq A, \ t \in [t_0, t_1].$$

Setting $m(t) = |x(t)|$ and using (1.3.3), we obtain the differential inequality

$$D^+ m(t) \leq g(t, m(t)), \ t \in [t_0, t_1],$$

which yields, by comparison Theorem A.1.1, the estimate

$$m(t) \leq r(t, t_0, m(t_0)), \ t \in [t_0, t_1],$$

where $r(t, t_0, u_0)$ is the maximal solution of (1.3.1). At $t = t_1$, we therefore arrive at the contradiction

$$A = |\, x(t_1) \,| \leq r(t_1, t_0 \mid x_0 \mid) < A,$$

proving our claim. Similarly, one can prove other concepts of practical stability and hence the proof is complete.

As an example of Theorem 1.3.1, consider the linear nonhomogeneous differential system

$$x' = A(t)\, x + F(t),\ x(t_0) = x_0, \tag{1.3.4}$$

where A(t) is a n × n continuous matrix and $F \in C\,[\mathbf{R}_+, \mathbf{R}^n]$. The assumption (1.3.3) shows that

$$g(\, t, u) = \mu[A(t)]\, u + |\, F(t) |,$$

where $\mu[A(t)]$ is the logarithmic norm of A(t) defined by

$$\mu[A(t)] = \lim_{h \to 0^+} \frac{1}{h}[\,|\, I + h\, A(t)\,| - 1],$$

I being the identity matrix. Hence the solution of (1.3.1) is

$$u(t, t_0, u_0) = u_0 \exp\Big(\int_{t_0}^{t} \mu[A(s)]\, ds\Big)$$

$$+ \int_{t_0}^{t} \exp\Big(\int_{s}^{t} \mu[A(\sigma)]\, d\sigma\Big) \,|\, F(s) \,|\, ds,$$

for $t \geq t_0$ and consequently, if $\mu[A(t)] \leq 0$ for $t \geq t_0$, we get

$$u(t, t_0, u_0) \leq u_0 + \int_{t_0}^{t} |\, F(s) \,|\, ds,\ t \geq t_0,$$

which implies practical stability of the system (1.3.4) provided $\int_{t_0}^{\infty} |F(s)| ds < A - \lambda$ for some t_0.

It is important to remember that the value of $\mu[A(t)]$ depends on the particular norm used for vectors and matrices. For example, if $|x|$ represents the Euclidean norm, then $\mu[A]$ is the largest eigenvalue of $\frac{1}{2}(A+A^*)$, A^* being the transpose of A. For more details on $\mu[A]$, see Lakshmikantham and Leela [1].

Let us next consider the differential system

$$x' = A(t)x + F(t, x),\ x(t_0) = x_0, \tag{1.3.5}$$

where A(t) is a n × n continuous matrix and $F \in C[\mathbf{R}_+ \times \mathbf{R}^n, \mathbf{R}^n]$. Let $\Phi(t, t_0)$ be the fundamental matrix solution of

$$x' = A(t)x \tag{1.3.6}$$

with $\Phi(t_0, t_0) = I$ (Identity matrix). Suppose that

$$\left| \begin{array}{l} |\Phi(t, t_0)| \leq \eta(t)\mu(t_0),\ t \geq t_0, \\ |F(t, x)| \leq b(t)\,|x|^\alpha,\ \alpha > 0,\ (t, x) \in \mathbf{R}_+ \times \mathbf{R}^n, \end{array} \right. \tag{1.3.7}$$

where η, μ, $b \in C[\mathbf{R}_+, (0, \infty)]$. We then have the following result.

Theorem 1.3.2

Consider the system (1.3.5) and suppose that (1.3.7) holds with $0 < \alpha < 1$. Assume that for any given $0 < \lambda < A$, for all $t_0 \in \mathbf{R}_+$ and $t \geq t_0$

$$[\eta(t)\mu(t_0)]^{1-\alpha}[1 + (1-\alpha)(\mu(t_0)\lambda)^{\alpha-1} \times \tag{1.3.8}$$
$$\times \int_{t_0}^{t} \mu(s)b(s)\eta^\alpha(s)ds] < \left(\frac{A}{\lambda}\right)^{1-\alpha}.$$

Then the system (1.3.5) is uniformly practically stable.

Proof

By Corollary A.1.2, we have

$$x(t)=\Phi(t, t_0)\, x_0+\int_{t_0}^{t} \Phi(t, s)\, F(s, x(s))ds,\ t\geq t_0,$$

where $x(t)=x(t, t_0, x_0)$ is any solution of (1.3.5). Because of (1.3.7) we then arrive at

$$|\, x(t)\,| \leq \eta(t)\mu(t_0)\,|\, x_0\,| + \eta(t)\int_{t_0}^{t} \mu(s)b(s)\,|\, x(s)\,|^{\alpha}ds,\ t\geq t_0,$$

which yields by Lemma A.1.1,

$$(1.3.9)\qquad |\, x(t)| \leq \eta(t)\mu(t_0)\,|\, x_0\,|\, [1+(1-\alpha)(\mu(t_0)|\, x_0|)^{\alpha-1}\times$$

$$\times \int_{t_0}^{t} b(s)\mu(s)\eta^{\alpha}(s)ds]^{\frac{1}{1-\alpha}}$$

for $t\geq t_0$. Suppose that $0<\lambda<A$ is given. Then (1.3.8) and (1.3.9) yield the relation

$$|\, x(t)| < A,\ t\geq t_0,\ \text{provided}\ |\, x_0\,| < \lambda,$$

which proves uniform practical stability of the system (1.3.5) for $\alpha\in(0, 1)$.

For the case $\alpha\in(1, \infty)$, we have the following result which gives practical stability over a finite time interval.

Theorem 1.3.3

Consider the system (1.3.5) and assume that (1.3.7) holds with $\alpha\in(1, \infty)$. Suppose that for any given $0<\lambda<A$, for all $t_0\in\mathbf{R}_+$ and $t\in t_0, t_0+T]$,

(1.3.8) is satisfied where

$$t_0 + T < \beta = \sup\Big\{t \geq t_0 : \ (\alpha-1)(\mu(t_0)|\, x_0\, |)^{\alpha-1} \times \times \int_{t_0}^{t} \mu(s)b(s)\eta^{\alpha}(s)ds < 1\}.$$

Then the system (1.3.5) is uniformly practically stable.

Proof

Since in this case Lemma A.1.1 holds only for $t_0 \leq t < \beta$, consequently (1.3.9) holds on $t_0 \leq t \leq t_0 + T$. This proves, as before, uniform practical stability of (1.3.5).

Corollary 1.3.1

In addition to the assumptions of Theorem 1.3.2, if we have

$$[\eta(t_0+T)\mu(t_0)]^{1-\alpha}[1+(1-\alpha)(\mu(t_0)\lambda)^{\alpha-1} \times \times \int_{t_0}^{t_0+T} \mu(s)b(s)\eta^{\alpha}(s)ds] < \Big(\frac{B}{\lambda}\Big)^{1-\alpha}$$

where $0 < B < A$ is any given number, then the system (1.3.5) is strongly uniformly practically stable.

Corollary 1.3.2

Suppose that the assumptions of Theorem 1.3.2 hold. Let $\sigma(t-t_0) = \eta(t)\mu(t_0)$, $\sigma(t)$ is decreasing in t and $\sigma(t-t_0) < \frac{B}{A}$ for $t \geq t_0 + T$. Then the system is strongly uniformly practically stable.

We shall now employ results of nonlinear variation of parameters to discuss practical stability. Corresponding to the system (1.1.1), let us consider its perturbed system

(1.3.10) $$y' = f(t, y) + R(t, y), \ y(t_0) = x_0$$

where f, $R \in C[\mathbf{R}_+ \times \mathbf{R}^n, \mathbf{R}^n]$. Then we can prove the following results.

Theorem 1.3.4

Assume that

(i) the system (1.1.1) admits unique solutions $x(t, t_0, x_0)$ for $t \geq t_0$;

(ii) $\Phi(t, t_0, x_0) = \frac{\partial x}{\partial x_0}(t, t_0, x_0)$ exists, is continuous and $\Phi^{-1}(t, t_0, x_0)$ exists for all $t \geq t_0$;

(iii) $v(t)$ is any solution of

(1.3.11) $$v' = \Phi^{-1}(t, t_0, x_0) R(t, x(t, t_0, v(t))), \ v(t_0) = x_0,$$

which exists for $t \geq t_0$;

(iv) $|\Phi^{-1}(t, t_0, x_0) R(t, x(t, t_0, x_0))| \leq g(t, |x_0|)$ where $g \in C[\mathbf{R}_+^2, \mathbf{R}_+]$ and $r(t, t_0, u_0)$ is the maximal solution of

(1.3.12) $$u' = g(t, u), \ u(t_0) = u_0 \geq 0,$$

existing for $t \geq t_0$;

(v) $|x(t, t_0, x_0)| \leq a(|x_0|)\sigma(t - t_0)$, $t \geq t_0$ where $a \in C[\mathbf{R}_+, \mathbf{R}_+]$ is strictly increasing with $a(0) = 0$ and $\sigma \in C[\mathbf{R}_+, \mathbf{R}_+]$ is decreasing such that, given $(\lambda, A, B, T) > 0$, $\lambda < A$, $B < A$ and $\sigma(t - t_0) < \frac{B}{a(A)}$, for $t \geq t_0 + T$;

(vi) the system (1.3.12) is uniformly practically stable with respect to (λ, A).

Then the perturbed system (1.3.10) uniformly practically quasi-stable.

Proof

By Theorem A.1.2, any solution $y(t, t_0, x_0)$ of (1.3.10) satisfies

$$y(t, t_0, x_0)=x(t, t_0, v(t))$$

where $v(t)$ is a solution of (1.3.11). Setting $m(t)=|v(t)|$ and using condition (iv), we obtain the inequality

$$D^+m(t)\leq g(t, m(t)),\ t\geq t_0,$$

which, by Theorem A.1.1, yields the relation

$$|v(t)|\leq r(t, t_0, |x_0|),\ t\geq t_0.$$

Then, in view of (v), we have

$$\begin{aligned} |y(t, t_0, x_0)| &\leq a(|v(t)|)\sigma(t-t_0) \\ &\leq a(r(t, t_0, |x_0|))\sigma(t-t_0),\ t\geq t_0, \end{aligned} \tag{1.3.13}$$

using the monotonicity of $a(u)$. Since the equation (1.3.12) is uniformly practically stable, give $0<\lambda<A$, we have

$$u_0<\lambda \text{ implies } u(t, t_0, u_0)<A,\ t\geq t_0, \text{ for all } t_0\geq 0.$$

It then follows from (1.3.13) and (v) that

$$|y(t, t_0, x_0)|\leq a(A)\sigma(t-t_0)<B,\ t\geq t_0+T$$

which proves uniform practical quasi-stability of (1.3.10).

Corollary 1.3.3

Under the assumptions of Theorem 1.3.4, if $\sigma(t)\leq 1$, then the perturbed system (1.3.10) is uniformly strongly practically stable with respect to $(\lambda, a(A), B)$.

Theorem 1.3.5

Assume that conditions (i), (ii) and (iii) of Theorem 1.3.4 hold. Suppose further

(A_1) $|\Phi(t, t_0, x_0)|$, $|\Phi^{-1}(t, t_0, x_0)| \leq a(|x_0|)$, $t \geq t_0$, where $a \in C[\mathbf{R}_+, \mathbf{R}_+]$ and $a(u)$ is nondecreasing in u;

(A_2) $|R(t, y)| \leq g(t, |y|)$ where $g \in C[\mathbf{R}_+^2, \mathbf{R}_+]$ and $g(t, u)$ is nondecreasing in u for each $t \in \mathbf{R}_+$;

(A_3) $\tilde{r}(t, t_0, u_0)$, $r_0(t, t_0, v_0)$ are the maximal solutions of

$$u' = \tilde{g}(t, u),\ u(t_0) = u_0 \geq 0,$$
$$v' = g_0(t, v),\ v(t_0) = v_0 \geq 0,$$

existing for $t \geq t_0$, where $\tilde{g}(t, u) = a(u)\, g(t, u\, a(u))$, $g_0(t, u) = a^2(\tilde{r}(t, t_0, u_0))\, g(t, u)$, $a^2(u)$ being the composition $a(a(u))$;

(A_4) given $0 < \lambda < A$, $u_0 < a(\lambda)\lambda$ implies $r_0(t, t_0, a(\lambda)\lambda) < A$, $t \geq t_0$.

Then the perturbed system (1.3.10) is practically stable.

Proof

The solution $x(t, t_0, x_0)$ of (1.1.1) is related to $\Phi(t, t_0, x_0)$ by

$$x(t, t_0, x_0) = \int_0^1 \Phi(t, t_0, x_0 s)\, ds\, x_0$$

as can be seen by integrating $\frac{dx}{ds}(t, t_0, x_0 s) = \Phi(t, t_0, x_0 s) x_0$ from $s = 0$ to $s = 1$. Hence, by (A_1), we have

$$|x(t, t_0, x_0)| \leq a(|x_0|)\, |x_0|,\ t \geq t_0. \tag{1.3.14}$$

Moreover, it follows from (1.3.11), (A_1) and (A_2) that

$$|\, v(t)\, | \leq |\, x_0\, | + \int_{t_0}^{t} \tilde{g}(s, |\, v\,(s)|)ds,\ t \geq t_0.$$

Since $\tilde{g}(t, u)$ is also nondecreasing in u, we get, by Corollary A.1.1,

$$(1.3.15) \qquad |\, v(t)\, | \leq \tilde{r}(t, t_0, |\, x_0\, |),\ t \geq t_0.$$

Now, using Theorem A.1.2, (1.3.14) and (1.3.15), we arrive at

$$|\, y(t, t_0, x_0)\, | \leq a(|\, x_0\, |)|\, x_0\, | + \int_{t_0}^{t} g_0(s, |\, y(s, t_0, x_0)\, |)\, ds,$$

which implies by Corollary A.1.1, the estimate

$$|\, y(t, t_0, x_0)| \leq r_0(t, t_0, a(|\, x_0\, |)\, |\, x_0\, |,\ t \geq t_0.$$

Given $0<\lambda<A$, we see immediately, because of (A_4), that if $|\, x_0\, |<\lambda$, we have $|\, y(t, t_0, x_0)|<A$, proving the claim.

Theorem 1.3.6

Assume that

(i) f has continuous partial derivatives $\frac{\partial f}{\partial x}$ on $\mathbf{R}_+ \times \mathbf{R}^n$ and $x(t, t_0, x_0,)$ is the unique solution of (1.1.1) existing for $t \geq t_0$;

(ii) $|\, \Phi(t, t_0, x_0)| \leq a(|\, x_0|)$, $t \geq t_0$, where $a \in C[\mathbf{R}_+, \mathbf{R}_+]$, a(u) is nondecreasing in u and $\Phi(t, t_0, x_0) = \frac{\partial x}{\partial x_0}(t, t_0, x_0)$;

(iii) $|\, R(t, y)| \leq g(t, |\, y|)$, where $g \in C[\mathbf{R}_+^2, \mathbf{R}_+]$, g(t, u) is nondecreasing in u for each $t \in \mathbf{R}_+$ and $r(t, t_0, u_0)$ is the maximal solution of

$$u' = a(u)g(t, u),\ u(t_0) = u_0 \geq 0,$$

existing for $t \geq t_0$;

(iv) given $0<\lambda<A$, it follows that $u_0<a(\lambda)\lambda$ implies $r(t, t_0, u_0)<A$ for $t\geq t_0$.

Then the perturbed system (1.3.10) is practically stable.

Proof

By Theorem A.1.2 and the estimate (1.3.15), we have

$$|y(t, t_0, x_0)|\leq a(|x_0|)|x_0|+\int_{t_0}^{t} a(|y(s, t_0, x_0)|)\, g(s,|y(s, t_0, x_0)|)ds$$

which yields by Corollary A.1.1, the relation

$$|y(t, t_0, x_0)| \leq r(t, t_0, a(|x_0|)|x_0|),\ t\geq t_0.$$

The assumption (iv) now implies practical stability of the perturbed system (1.3.10) proving the theorem.

1.4. DELAY DIFFERENTIAL EQUATIONS.

Let $\mathfrak{C}=C[[-\tau, 0], \mathbf{R}^n]$ and for any $\phi\in\mathfrak{C}$, let us use the norm $|\phi|_0=\max_{-\tau\leq s\leq 0}|\phi(s)|$. If $x\in C[[t_0-\tau, \infty), \mathbf{R}^n]$, $t_0\geq 0$, we define $x_t\in\mathfrak{C}$ by $x_t(s)=x(t+s)$, $-\tau\leq s\leq 0$. Consider the initial value problem for the delay differential system

$$x'(t)=f(t, x_t),\ x_{t_0}=\psi_0\in\mathfrak{C}, \tag{1.4.1}$$

where $f\in C[\mathbf{R}_+\times\mathfrak{C}, \mathbf{R}^n]$. If f maps bounded sets into bounded sets, then for each $t_0\in\mathbf{R}_+$, $\phi_0\in\mathfrak{C}$, there exists a solution $x(t)=x(t_0, \phi_0)(t)$ defined on an interval $[t_0, t_0+\alpha)$, $\alpha>0$.

Theorem 1.4.1

Assume that

(i) g_0, $g \in C[\mathbf{R}_+^2, \mathbf{R}]$, $g_0(t, u) \le g(t, u)$, $r(t, t_0, u_0)$ is the right maximal solution of

$$u' = g(t, u),\ u(t_0) = u_0 \ge 0. \tag{1.4.2}$$

existing on $[t_0, \infty)$ and $\eta(t, t^0, v_0)$ is the left maximal solution of

$$v' = g_0(t, v),\ v(t^0) = v_0 \ge 0, \tag{1.4.3}$$

existing on $t_0 \le t \le t^0$;

(ii) $[\phi(0), f(t, \phi)]_- \le g(t, |\phi(0)|)$ for $\phi \in \Omega$ and $t \in \mathbf{R}_+$, where

$$\Omega = \{\phi \in \mathcal{C}:\ |\phi(s)| \le \eta(s, t, |\phi(0)|),\ -\tau \le s \le 0\}. \tag{1.4.4}$$

Then the practical stability properties of (1.4.2) imply the corresponding practical stability properties of the delay differential system (1.4.1).

Proof

Let $x(t) = x(t_0, \phi_0)(t)$ be any solution of (1.4.1) on $[t_0, \infty)$. Set $m(t) = |x(t)|$ and $m_{t_0} \le u_0$. We shall show that

$$m(t) \le r(t, t_0, u_0),\ t \ge t_0. \tag{1.4.5}$$

To prove (1.4.5), it is enough to prove that $m(t) < u(t, \epsilon)$, $t \ge t_0$, for sufficiently small $\epsilon > 0$, since $\lim_{\epsilon \to 0} u(t, \epsilon) = r(t, t_0, u_0)$, $u(t, \epsilon)$ being any solution of $u' = g(t, u) + \epsilon$, $u(t_0) = u_0 + \epsilon$.

If $m(t) < u(t, \epsilon)$, $t \ge t_0$, is not true, then there exists a $t_1 > t_0$ such that

$$m(t_1) = u(t_1, \epsilon) \text{ and } m(s) < u(s, \epsilon),\ t_0 \le s < t_1.$$

This implies that

$$D_- m(t_1) \ge u'(t_1, \epsilon) = g(t_1, u(t_1, \epsilon)) + \epsilon. \tag{1.4.6}$$

Now consider the the left maximal solution $\eta(s, t_1, m(t_1))$ on $t_0 \leq s \leq t_1$ of the problem

$$v' = g_0(t, v), \ v(t_1) = m(t_1).$$

By Lemma A.1.2, we obtain

$$r(s, t_0, u_0) \leq \eta(s, t_1, m(t_1,)), \ t_0 \leq s \leq t_1.$$

Since $r(t_1, t_0, u_0) = \lim_{\epsilon \to 0} u(t_1, \epsilon) = m(t_1) = \eta(t_1, t_1, m(t_1))$ and $m(t) \leq u(t, \epsilon)$, $t \in [t_0, t_1]$, it follows that

$$m(s) \leq r(s, t_0, u_0) \leq \eta(s, t_1, m(t_1)), \ t_0 \leq s \leq t_1.$$

Since $m_{t_0} \leq u_0$, we have $m(t_1+s) \leq \eta(t_1+s, t_1, m(t_1))$, which implies that $x_{t_1} \in \Omega$. Consequently, condition (ii) yields that

$$D_-m(t_1) \leq g(t_1, m(t_1))$$

which contradicts (1.4.6). Hence we have proved that

$$| x(t_0, \phi_0)(t) | \leq r(t, t_0, | \phi_0 |_0), \ t \geq t_0.$$

From this estimate it is easy to see that any practical stability property of (1.4.2) yields the corresponding practical stability property of (1.4.1). The proof is therefore complete.

Let us now discuss some simple special cases of Theorem 1.4.1:

(1) Suppose that $g_0(t, u) \equiv 0$ and $g(t, u) \geq 0$. Then $\eta(s, t^0, v_0) \equiv v_0$ and $\Omega = \{\phi \in \mathbb{C}: \ |\phi(s)| \leq | \phi(0))|, \ -\tau \leq s \leq 0\}$. Since $g(t, u) \geq 0$, we can only expect practical stability for (1.4.2) which shows that the system (1.4.1) is also practically stable.

(2) Suppose that $g_0(t, u) = -\frac{A'(t)}{A(t)} u$ where $A(t) > 0$ is continuous and

differentiable on $[t_0, \infty)$ and $A(t) \to \infty$ as $t \to \infty$. Let

$$g(t, u) = g_0(t, u) + \frac{1}{A(t)} g_1(t, A(t)\, u)$$

where $g_1 \in C[\mathbf{R}_+^2, \mathbf{R}_+]$. Now $\eta(s, t, v_0) = v_0 \frac{A(t)}{A(s)}$, $t_0 \le s \le t$ and

$$\Omega = \{\phi \in \mathfrak{C}:\ A(s)\, |\phi(s)| \le |\phi(0)|\, A(t),\ -\tau \le s \le 0\}.$$

It is easy to verify that $r(t, t_0, u_0) = r_1(t, t_0, u_0)\,(A(t))^{-1}$ where $r_1(t, t_0, u_0)$ is the maximal solution of $u' = g_1(t, u)$, $u(t_0) = u_0$. In this case (1.4.2) can be strongly practically stable and so the system (1.4.1) can also be strongly practically stable.

(3) If $g_0(t, u) = g(t, u) = -\gamma(u)$ where $\gamma \in C[\mathbf{R}_+, \mathbf{R}_+]$, $\gamma(u)$ is increasing with $\gamma(0) = 0$, the left maximal solution is given by

$$\eta(s, t^0, v_0) = G^{-1}(G(v_0) - (s - t^0)),\ t_0 \le t \le t^0,$$

where $G'(u) = \frac{1}{\gamma(u)}$ and G^{-1} is the inverse function of G.

Since $\eta(s, t^0, v_0)$ is increasing in s to the left of t^0, choosing a fixed $s_0 < t^0$ and defining $L(u) = \eta(s_0, t^0, u)$, it is clear that $L(u)$ is increasing and $L(u) > u$ for $u > 0$. Hence, in this case,

$$\Omega = \{\phi \in \mathfrak{C}:\ |\phi(s)| \le L(|\phi(0)|),\ -\tau \le s \le 0\}$$

and clearly the equation (1.4.2) is practically asymptotically stable. Therefore the system (1.4.1) is also practically asymptotically stable.

(4) Suppose that $g_0(t, u) = -\gamma(u)$ and $g(t, u) = -\gamma(u) + \sigma(t)$ where γ is as in (3) and $\sigma \in C[\mathbf{R}_+, \mathbf{R}_+]$ with $\int_t^{t+1} \sigma(s)ds \to 0$ as $t \to \infty$. Ω remains

the same as in (3). We shall show that (1.4.2) is eventually practically stable so that the system (1.4.1) is also eventually practically stable. Let $0<\lambda<A$ be given and suppose that $u_0<\lambda$. If the claim is not true, there would exist $t_0<t_1<t_2$ such that

$$u(t_1)=\lambda,\ u(t_2)=A \text{ and } \lambda \leq u(t) \leq A,\ t \in [t_1, t_2].$$

Choose a $\tau=\tau(\lambda, A)>0$ such that $Q(\tau)<\min(\gamma(\lambda), A-\lambda)$, where

$$Q(t)=\sup\{\hat{G}(s):\ t-1 \leq s<\infty\},\ \hat{G}(t)=\int_t^{t+1} \sigma(s)ds.$$

Then, it follows that, for $t_0 \geq \tau$,

$$A=u(t_2)=u(t_1)-\int_{t_1}^{t_2} \gamma(u(s))ds+\int_{t_1}^{t_2} \sigma(s)ds$$

$$\leq \lambda-\gamma(\lambda)(t_2-t_1)+\int_{t_1-1}^{t_2} \hat{G}(s)ds$$

$$\leq \lambda+(t_2-t_1)(-\gamma(\lambda)+Q(\tau))+Q(\tau)$$

$$\leq \lambda+Q(\tau)<A$$

which is a contradiction. Hence the claim is established.

1.5. INTEGRO-DIFFERENTIAL EQUATIONS.

In this section, we shall extend practical stability considerations for integro-differential equations. Utilizing the idea of finding an equivalent linear differential system for a given linear integro-differential system, we investigate the stability behavior.

Consider the linear initial value problem for integro-differential equations given by

$$(1.5.1) \qquad x'(t)=A(t)\,x(t)+\int_{t_0}^{t} B(t,\,s)\,x(s)ds+F(t),\; x(t_0)=x_0,$$

where $A(t)$, $B(t,\,s)$ are continuous $n \times n$ matrices on $\mathbf{R}_+$ and $\mathbf{R}_+^2$ respectively and $F\in C[\mathbf{R}_+,\,\mathbf{R}^n]$. By Theorem A.2.1, system (1.5.1) is equivalent to the linear differential system

$$(1.5.2) \qquad v'(t)=B(t)v(t)+L(t,\,t_0)x_0+H(t),\; v(t_0)=x_0,$$

where $B(t)=A(t)-L(t,\,t)$, $H(t)=F(t)+\int_{t_0}^{t} L(t,s)F(s)ds$ and $L(t,\,s)$ satisfies

$$(1.5.3) \qquad B(t,\,s)+L_s(t,\,s)+L(t,\,s)A(s) + \int_{s}^{t} L(t,\,\sigma)B(\sigma,s)ds=0.$$

Concerning the system (1.5.1), we can prove the following result on practical stability.

Theorem 1.5.1

Suppose that there exists an $n \times n$ continuous matrix function $L(t,\,s)$ on $\mathbf{R}_+^2$ such that $L_s(t,\,s)$ exists and satisfies (1.5.3). Let $\psi(t,\,t_0)$ be the fundamental matrix solution of

$$(1.5.4) \qquad v'=B(t)v,\; \psi(t_0,\,t_0)=I \text{ (Identity matrix).}$$

Then the system (1.5.1) is practically quasi-stable if, given $(\lambda,\,B,\,T)>0$,

$$(1.5.5) \qquad |\psi(t,\,t_0)|+\int_{t_0}^{t} |\psi(t,\,s)|\;|\,L(s,\,t_0)|ds<\frac{B}{2\lambda},$$

$$\int_{t_0}^{t} |\psi(t, s)|\,|H(s)|ds < \frac{B}{2}, \tag{1.5.6}$$

for $t \geq t_0 + T$.

Proof

In view of Theorem A.2.1, it is enough to consider the system (1.5.2). By Corollary A.1.2, we have

$$v(t) = \psi(t, t_0)x_0 + \int_{t_0}^{t} \psi(t, s)[L(s, t_0)x_0 + H(s)]ds, \; t \geq t_0,$$

and therefore it follows that

$$|v(t)| \leq |x_0|[|\psi(t, t_0)| + \int_{t_0}^{t} |\psi(t, s)|\,|L(s, t_0)|ds] +$$

$$\int_{t_0}^{t} |\psi(t, s)|\,|H(s)|\,ds,$$

for $t \geq t_0$. The conclusion of theorem is immediate because of (1.5.5) and (1.5.6).

Next we shall consider the nonlinear integro-differential system

$$x'(t) = f(t, x(t)) + \int_{t_0}^{t} F(t, s, x(s))\,ds, \; x(t_0) = x_0, \tag{1.5.7}$$

where $f \in C[\mathbf{R}_+ \times \mathbf{R}^n, \mathbf{R}^n]$ and $F \in C[\mathbf{R}_+^2 \times \mathbf{R}^n, \mathbf{R}^n]$. Relative to (1.5.7) we can prove the following result.

Theorem 1.5.2

Assume that

$$(1.5.8) \qquad [x, f(t, x)]_+ \le -\beta |x|, \; (t, x) \in \mathbf{R}_+ \times \mathbf{R}^n,$$

where $\beta > 0$ and

$$(1.5.9) \qquad |F(t, s, x)| \le g(t, s, |x|), \; (t, s, x) \in \mathbf{R}_+^2 \times \mathbf{R}^n,$$

where $g \in C[\mathbf{R}_+^2 \times \mathbf{R}_+, \mathbf{R}_+]$ and $g(t, s, u)$ is nondecreasing in u. Suppose further that $r(t, t_0, u_0)$ is the maximal solution of

$$(1.5.10) \qquad u' = G(t, u), \; u(t_0) = u_0 \ge 0,$$

where

$$G(t, u) = \int_{t_0}^{t} e^{\beta(t-t_0)} g(t, s, u\, e^{-\beta(s-t_0)}) ds, \; t \ge t_0.$$

Then the system (1.5.7) is strongly practically stable if $r(t, t_0, \lambda) < A$, $t \ge t_0$ and $e^{-\beta T} < \frac{B}{A}$, $T > 0$ being given.

Proof

Let $(\lambda, A, B, T) > 0$ be given such that $B < A$ and $x(t)$ be any solution of (1.5.7) with $|x_0| < \lambda$. Then, setting $m(t) = |x(t)|$, we get using (1.5.8) and (1.5.9),

$$D^+m(t) \le -\beta m(t) + \int_{t_0}^{t} g(t, s, m(s)) ds, \; t \ge t_0.$$

Letting $v(t) = m(t)\, e^{\beta(t-t_0)}$, we see that

$$D^+v(t) \le \int_{t_0}^{t} e^{\beta(t-t_0)} g(t, s, v(s) e^{-\beta(s-t_0)})\, ds$$

which yields, on integration

$$v(t)\leq m(t_0)+\int_{t_0}^{t}[\int_{t_0}^{s} e^{\beta(s-t_0)}g(s,\ \xi,\ v(\xi)^{-\beta(\xi-t_0)})d\xi]ds$$
$$\equiv\eta(t).$$

Then $\eta'(t)=\int_{t_0}^{t} e^{\beta(t-t_0)}g(t,\ s,\ v(s)\ e^{-\beta(s-t_0)})ds$

$$\leq\int_{t_0}^{t} e^{\beta(t-t_0)}g(t,\ s,\ \eta(s)e^{-\beta(s-t_0)})\ ds,$$

in view of the fact that g is nondecreasing. Note that $\eta'(t)\geq 0$ and hence $\eta(t)$ is nondecreasing. Consequently we arrive at

$$\eta'(t)\leq\int_{t_0}^{t} e^{\beta(t-t_0)}g(t,\ s,\ \eta(t)e^{-\beta(s-t_0)})ds\equiv G(t,\ \eta(t)),$$

which yields the estimate, by Theorem A.1.1,

$$\eta(t)\leq r(t,\ t_0,\ m(t_0)),\ t\geq t_0.$$

It therefore follows that

$$|\ x(t)\ |\leq r(t,\ t_0,\ |\ x_0\ |)e^{-\beta(t-t_0)},\ t\geq t_0,$$

which implies that

$$|\ x(t)\ |\leq r(t,\ t_0,\ \lambda)<A,\ t\geq t_0$$

and

$$|\ x(t)\ |\leq r(t,\ t_0,\ \lambda)e^{-\beta T}<B,\ t\geq t_0+T$$

proving the claim of the theorem.

1.6. DIFFERENCE EQUATIONS.

In this section, we shall investigate practical stability relative to difference equations. We shall also consider difference equations of Volterra type. Let $N^+_{n_0}=\{n_0, n_0+1, \cdots, n_0+k, \cdots\}$, $n_0\geq 0$.

Consider the difference equation given by

$$x_{n+1}=f(n, x(n)),\ x(n_0)=x_0, \tag{1.6.1}$$

where f: $N^+_{n_0}\times \mathbf{R}^d\rightarrow\mathbf{R}^d$ and $x(n)=x_n$. Let us begin with the following result.

Theorem 1.6.1

Let $g(n, u)$ be a nonnegative function, nondecreasing in u. Suppose that f: $N^+_{n_0}\times \mathbf{R}^d\rightarrow\mathbf{R}^d$ and $|f(n, x)|\leq g(n, |x|)$. Then the practical stability properties of

$$u_{n+1}=g(n, u_n),\ u(n_0)=u_0\geq 0 \tag{1.6.2}$$

imply the corresponding practical stability properties of (1.6.1).

Proof

Let $x_n=x(n)$ be the solution of (1.6.1). Then, by assumption, we have

$$|x_{n+1}|\leq|f(n, x_n)|\leq g(n, |x_n|),\ n\geq n_0.$$

Now we claim that

$$|x_n|\leq u_n,\ n\geq n_0 \tag{1.6.3}$$

where u_n is the solution of (1.6.2). If (1.6.3) is not true, there exists a $k\in N^+_{n_0}$ such that $|x_k|\leq u_k$ and $|x_{k+1}|>u_{k+1}$. It then follows, in view of the monotone nature of g, that

$$g(k, u_k)=u_{k+1}<|x_{k+1}|\leq g(k, |x_k|)\leq g(k, u_k)$$

which is a contradiction. Hence (1.6.3) holds. The conclusion of the theorem is therefore immediate from (1.6.3) and the proof is complete.

As an example, consider the case $g(n, u)=a_n u$, $a_n \geq 0$. Then the solution of (1.6.2) is given by

$$u_n = u_0 \prod_{i=n_0}^{n-1} a_i.$$

Consequently, given $(\lambda, A, B, T)>0$, if

$$\prod_{i=n_0}^{n-1} a_i < \frac{A}{\lambda} \text{ and } \prod_{i=n_0}^{n-1} a_i < \frac{B}{\lambda}, \; n \geq n_0 + T,$$

then the difference equation (1.6.1) is strongly practically stable.

Theorem 1.6.2

Let $\Phi(n, n_0)$ be the fundamental matrix solution of the linear difference system

(1.6.4) $$x_{n+1} = A(n)x_n,$$

where $A(n)$ is a $d \times d$ nonsingular matrix. Let F: $N_{n_0}^+ \times \mathbf{R}^d \to \mathbf{R}^d$ and

(1.6.5) $$|\Phi^{-1}(n+1, n_0)\, F(n, \Phi(n, n_0)\, y_n)| \leq g(n, |\, y_n \,|)$$

where $g(n, u) \geq 0$ is nondecreasing in u. Assume that the solutions u_n of

(1.6.6) $$u_{n+1} = u_n + g(n, u_n),\; u(n_0) = u_0 \geq 0$$

are bounded for $n \geq n_0$. Then the practical stability properties of the linear difference system (1.6.4) imply the corresponding practical stability properties of

(1.6.7) $$x_{n+1} = A(n)x_n + F(n, x_n),\; x(n_0) = x_0.$$

Proof

The linear transformation $x_n = \Phi(n, n_0)\, y_n$ reduces (1.6.7) to

$$y_{n+1} = y_n + \Phi(n+1, n_0)\, F(n, \Phi(n, n_0)\, y_n).$$

Hence, we have, because of (1.6.5)

$$|\, y_{n+1} \,| \le |\, y_n \,| + g(n, |\, y_n \,|),$$

which yields, arguing as before,

$$|\, y_n \,| \le u_n$$

where u_n is the solution of (1.6.6) provied $|\, y_0 \,| \le u_0$. It then follows that, for $n \ge n_0$,

$$|\, x_n \,| \le |\, \Phi(n, n_0) \,|\, u_n.$$

The conclusion of the theorem is now immediate completing the proof.

We can also use the variation of parameters formula to discuss the system (1.6.7). The following result is to that effect.

Theorem 1.6.3

Let $\Phi(n, n_0)$ be the fundamental matrix solution of (1.6.4) such that $|\, \Phi(n, n_0) \,| \le M$ for $n \ge n_0$. Suppose that

$$|\, F(n, x) | \le g_n |\, x \,|,$$

where $g_n \ge 0$ and $\sum_{n=n_0}^{\infty} g_n < \infty$. Then, if

$$\exp(M \sum_{i=n_0}^{n-1} g_i) < \frac{A}{\lambda M},$$

the system (1.6.7) is uniformly practically stable.

Proof

Let $y(n, n_0, x_0)$ be the solution of (1.6.7). Set

$$y_n=y(n, n_0, x_0)=\Phi(n, n_0)\, x_n,\ x(n_0)=x_0. \tag{1.6.8}$$

Then, substituting in (1.6.7), we get

$$\Phi(n+1, n_0)\, x_{n+1}= A(n)\Phi(n, n_0)\, x_n+F(n, y_n)$$

from which we see that

$$x_{n+1}-x_n=\Phi^{-1}(n+1, n_0)\, F(n, y_n)$$

and

$$x_n=\sum_{i=n_0}^{n-1}\Phi(n_0, i+1)\, F(i, y_i)+x_0.$$

By (1.6.8), it then follows that

$$y(n, n_0, x_0)=\Phi(n, n_0)\, x_0+\sum_{i=n_0}^{n-1}\Phi(n, i+1)\, F(i, y_i), \tag{1.6.9}$$

which is the variation of parameters formula. We therefore obtain, using the assumption,

$$|y_n|\le M|x_0|+M\sum_{i=n_0}^{n-1} g_i\, |y_i|.$$

So, the comparison equation is

$$u_n=M\,|x_0|+M\sum_{i=n_0}^{n-1} g_i\, u_i$$

which is equivalent to $\Delta u_n=Mg_nu_n$, the solution of which is given by

$$u_n=M\,|x_0|\prod_{i=n_0}^{n-1}(1+Mg_i).$$

Since $1+g_i\le\exp(g_i)$, we get

$$|\, y_n \,| \le M \,|\, x_0 \,| \exp\left(M \sum_{i=n_0}^{n-1} g_i\right),\ n \ge n_0.$$

This, in turn, shows that $|\, y_n \,| < A$ provided $|\, x_0 \,| < \lambda$, proving practical stability of (1.6.7).

Next, consider the linear difference system of Volterra type

$$\text{(1.6.10)} \quad \left|\ \begin{aligned} &\Delta x(n) = x(n+1) - x(n) = A(n)x(n) \\ &\qquad + \sum_{s=n_0}^{n-1} K(n, s)x(s) + F(n), \\ &\qquad x(n_0) = x_0, \end{aligned}\right.$$

where $A(n)$, $K(n, s)$ are $d \times d$ matrices for each $n, s \in N$ and $F\colon\ N^+_{n_0} \to \mathbf{R}^d$.

Let us first state a result corresponding to Fubini's theorem which can be proved by induction.

Lemma 1.6.1

Let $L(n, s)$, $K(n, s)$ be $d \times d$ matrices defined for $s, n \ge n_0$ such that L, K are zero matrices for $n, s < n_0$. Then the following relation

$$\sum_{s=n_0}^{n-1} L(n, s+1) \sum_{\sigma=n_0}^{s-1} K(s, \sigma)\, x(\sigma) = \sum_{s=n_0}^{n-1} \sum_{\sigma=s}^{n-1} L(n, \sigma+1) K(\sigma, s) x(s)$$

holds where $x\colon\ N^+_{n_0} \to \mathbf{R}^d$.

Next, we prove a comparison result.

Lemma 1.6.2

Let $n \in N^+_{n_0}$, $r > 0$ and $g(n, r)$, $G(n, s, r)$ be nonnegative functions, nondecreasing with respect to r for fixed n, s. Suppose that

$$\Delta u(n) \le g(n, u(n)) + \sum_{s=n_0}^{n-1} G(n, s, u(s)),\ u(n_0) = u_0 \ge 0$$

for any u: $N^+_{n_0} \to \mathbf{R}_+$. Then

$$u(n) \leq r(n),\ n \geq n_0$$

where $r(n) = r(n, n_0, u_0)$ is the solution of the difference equation

$$\Delta r(n) = g(n, r(n)) + \sum_{s=n_0}^{n-1} G(n, s, r(n)),\ r(n_0) = u_0.$$

Proof

By setting

$$v(n) = g(n, u(n)) + \sum_{s=n_0}^{n-1} G(n, s, u(s))$$

we see that

$$u(n) \leq \Delta^{-1} v(n) + w(n) \equiv z(n)$$

where Δ^{-1} is the anti-difference operator, w(n) is an arbitrary function of period 1 and z(n) is such that $\Delta z(n) = v(n) \geq 0$ and therefore nondecreasing. Consequently,

$$\Delta z(n) \leq g(n, z(n)) + \sum_{s=n_0}^{n-1} G(n, s, z(n)),\ z(n_0) = u_0$$

which implies, using the arguments as in the proof of Theorem 1.6.1,

$$z(n) \leq r(n),\ n \geq n_0.$$

Since $u(n) \leq z(n)$, the proof of Lemma is complete.

Now, we are in a position to prove a result which gives a linear difference equation equivalent to (1.6.10).

Theorem 1.6.4

Assume that there exists a d×d matrix L(n, s) defined on $N^+_{n_0} \times N^+_{n_0}$ satisfying

(1.6.11) $$K(n, s)+L(n, s+1)-L(n, s)+L(n, s+1)\ A(s) + \sum_{\sigma=s}^{n-1} L(n, \sigma+1)\ K(\sigma, s)=0.$$

Then (1.6.10) is equivalent to the linear difference equation

(1.6.12) $$\Delta y(n)=B(n)\ y(n)+L(n, n_0)\ x_0+H(n),\ y(n_0)=x_0,$$

where

(1.6.13) $$B(n)=A(n)-L(n, n),\ H(n)=F(n) + \sum_{s=n_0}^{n-1} L(n, s+1)F(s).$$

Proof

We first prove that every solution of (1.6.10) satisfies (1.6.12). Let $x(n)=x(n, n_0, x_0)$ be the solution of (1.6.10). Setting $p(s)=L(n, s)\ x(s)$, we have

$$p(s+1)-p(s)=[\ L(n, s+1)-L(n, s)]\ x(s)+L(n, s+1)\ [x(s+1)-x(s)].$$

Substituting from (1.6.10), we get

(1.6.14) $$p(s+1)-p(s)=[\ L(n, s+1)-L(n, s) + L(n, s+1)A(s)]\ x(s)+\ L(n, s+1)\ [\sum_{\sigma=n_0}^{n-1} K(s, \sigma)\ x(\sigma)+F(s)].$$

Summing both sides of (1.6.14) from n_0 to $n-1$, we obtain

$$L(n, n)\ x(n)-\ L(n, n_0)\ x_0= \sum_{s=n_0}^{n-1} [\ L(n, s+1)-L(n, s)+L(n, s+1)A(s) + \sum_{\sigma=s}^{n-1} L(n, \sigma+1)\ K((\sigma, s)]\ x(s)+\sum_{s=n_0}^{n-1} L(n, s+1)\ F(s).$$

Using Lemma 1.6.1 and (1.6.12), we get

$$L(n, n)\ x(n)-L(n, n_0)\ x_0=-\sum_{s=n_0}^{n-1} K(n, s)\ x(s)$$

which, in view of (1.6.10) and (1.6.13), yields

$$\Delta x(n)=B(n)\,x(n)+H(n)+L(n,\,n_0)\,x_0,$$

proving that x(n) is a solution of (1.6.12).

To prove that every solution of (1.6.12) is also a solution of (1.6.10), let $y(n)=y(n,\,n_0,\,x_0)$ be any solution of (1.6.12) for $n\geq n_0$. Define

$$z(n)=\Delta y(n)-A(n)y(n)-F(n)-\sum_{s=n_0}^{n-1}K(n,\,s)\,y(s).$$

In view of (1.6.11), we then obtain

$$z(n)=\Delta y(n)-A(n)y(n)-F(n)+\sum_{s=n_0}^{n-1}[L(n,\,s+1)-L(n,\,s)+L(n,\,s+1)A(s)$$
$$+\sum_{\sigma=s}^{n-1}L(n,\,\sigma+1)K(\sigma,\,s)]\,y(s).$$

By Lemma 1.6.1, it follows that

$$z(n)=\Delta y(n)-A(n)y(n)-F(n)+\sum_{s=n_0}^{n-1}[L(n,\,s+1)$$
$$-L(n,\,s)+L(n,\,s+1)A(s)]\,y(s)$$
$$+\sum_{s=n_0}^{n-1}L(n,\,s+1)\sum_{\sigma=n_0}^{n-1}K(s,\,\sigma)\,y(\sigma). \tag{1.6.15}$$

Setting $p(s)=L(n,\,s)\,y(s)$, we can obtain as before

$$L(n,\,n)\,y(n)-L(n,\,n_0)\,x_0=\sum_{s=n_0}^{n-1}[L(n,\,s+1)$$
$$-L(n,\,s)]\,y(s)+\sum_{s=n_0}^{n-1}L(n,\,s+1)\Delta y(s). \tag{1.6.16}$$

Using (1.6.12), (1.6.15) and (1.6.16), we see that

$$z(n)=-\sum_{s=n_0}^{n-1}L(n,\,s+1)[\Delta y(s)-A(s)\,y(s)-F(s)-\sum_{\sigma=n_0}^{n-1}K(s,\,\sigma)\,y(\sigma)]$$

which, in view of the definition of z(n), yields

$$z(n)=-\sum_{s=n_0}^{n-1}L(n,\,s+1)\,z(s).$$

It is easy to see that $z(n_0)=0$ and therefore it follows that $z(n)=0$ for all $n \geq n_0$. Hence the proof is complete.

Now we can prove the following result on practical asymptotic stability relative to the linear difference equation of Volterra type (1.6.10).

Theorem 1.6.5

Assume that the assumptions of Theorem 1.6.4 are satisfied and for $n \geq n_0$, $0<\alpha<1$, the following estimates hold:

(i) $|L(n, s)| \leq k_0 \alpha^{n-s}$,

(ii) $|F(n)| \leq k_0 \alpha^{n}$,

(iii) $|\Phi(n, s)| \leq k_0 \alpha^{n-s}$.

Then the system (1.6.10) is practically asymptotically stable.

Proof

Let $y(n, n_0, x_0)$ be any solution of (1.6.10). Then, by Theorem 1.6.4, it is also a solution of (1.6.12). Hence, in view of (1.6.9), we get

$$(1.6.17) \qquad |y(n, n_0, x_0)| \leq |\Phi(n, n_0)|\,|x_0| + \sum_{s=n_0}^{n-1} |\Phi(n, s+1)|\,|p(s)|$$

where $p(s)=H(s)+L(s, n_0)\,x_0$. Now, using the estimates (i), (ii), (iii), we obtain successively

$$| p(s) | \leq | F(s) | + | L(s, n_0) | \, | x_0 | + \sum_{\sigma=n_0}^{s-1} | L(s, \sigma+1) | \, | F(\sigma) |$$

$$\leq k_0[\alpha^s + \alpha^{s-n_0} | x_0 | + \sum_{\sigma=n_0}^{s-1} k_0 \alpha^{s-\sigma-1} \alpha^{\sigma}]$$

$$= k_0 \alpha^s [\, 1 + \alpha^{-n_0} | x_0 | + \alpha^{-1}(k_0 \sum_{\sigma=n_0}^{s-1} (1))]$$

$$= k_0 \alpha^s [1 + \alpha^{-n_0} | x_0 | + \alpha^{-1} k_0 (s - n_0)].$$

Hence (1.6.17) yields

$$| y(n, n_0, x_0) | \leq k_0 \alpha^{n-n_0} | x_0 | + \Big(\sum_{s=n_0}^{n-1} k_0 \alpha^{n-s-1} [k_0 \alpha^s (1 + \alpha^{-n_0} | x_0 |$$

$$+ \alpha^{-1} k_0 (s - n_0) \Big)],$$

which after simplification gives the estimate

$$| y(n, n_0, x_0) | \leq | x_0 | \, (k_0 + k_1 n) \alpha^{n-n_0} + k_2 (n + n^2) \alpha^{n-n_0}, \; n \geq n_0,$$

where k_1, k_2 are positive constants. It is clear that as $n \to \infty$, the right hand side tends to zero and therefore it is easy to see that (1.6.10) is practically asymptotically stable. The proof is complete.

1.7. IMPULSIVE DIFFERENTIAL EQUATIONS.

Many evolution processes, such as biological phenomena involving thresholds, bursting rhythm models in medicine and biology, optimal control models in economics, pharmacokinetics, frequency modulated systems and motion of missiles or airplanes are characterized by the fact that at certain moments of time, they experience a change of state abruptly. These perturbations act instantaneously, that is, in the form of impulses. Therefore, differential equations involving impulses appear as a natural description of observed evolution phenomena of several real world problems. We shall consider, in this section, differential systems with fixed moments of impulse effects and investigate practical stability of such systems.

Consider the linear system with fixed moments of impulse effect

$$\text{(1.7.1)} \qquad \begin{cases} x' = A(t)x, \ t \neq t_k, \ x(t_0^+) = x_0, \\ \Delta x \equiv x(t_k^+) - x(t_k^-) = B_k(x), \ t = t_k, \end{cases}$$

where $A(t)$, B_k are $n \times n$ matrices and $\{t_k\}$ is such that

$$\text{(1.7.2)} \qquad 0 < t_1 < t_2 < \cdots < t_k < \cdots \text{ and } \lim_{k \to \infty} t_k = \infty.$$

Suppose that $A(t)$ is piecewise continuous from $\mathbf{R}_+$ to $\mathbf{R}$ with discontinuities of the first kind at $t = t_k$ and $A(t)$ is left continuous at $t = t_k$, $k = 1, 2, \cdots$. Then, it is easy to see that for any (t_0, x_0) there exists a unique solution $x(t, t_0, x_0)$ of (1.7.1) which is left continuous at $t = t_k$, for $t \geq t_0$. If we denote by $U_k(t, s)$ the fundamental matrix solution of the linear differential system

$$\text{(1.7.3)} \qquad x' = A(t)x, \ t_{k-1} < t < t_k,$$

then the solution of (1.7.1) can be written in the form

$$\text{(1.7.4)} \qquad x(t, t_0, x_0) = W(t, t_0)\, x_0,$$

where $W(t, s)$ is given by

$$
(1.7.5) \qquad W(t,s)=\begin{cases} U_k(t, s) \text{ for } t, s\in(t_{k-1}, t_k], \\[2ex] U_{k+1}(t, t_k)(I+B_k)U_k(t, s) \text{ for } t_{k-1}<s\le t_k<t\le t_{k+1}, \\[2ex] U_{k+1}(t, t_k) \prod_{j=k}^{i+1} (I+B_j)U_j(t_j, t_{j+1})(I+B_i)U_i(t_i, s) \end{cases}
$$

$$\text{for } t_{i-1}<s\le t_i<t_k<t\le t_{k+1},$$

I being the unit matrix.

As a direct corollary of the formula (1.7.5) we can obtain an analogue of Liouville formula for linear impulsive system (1.7.1) in the form

$$
(1.7.6) \qquad \det W(t, t_0)=\prod_{t_0<t_k<t} \det (I+B_k)\exp\times \left(\int_{t_0}^{t} \operatorname{tr} A(s)ds\right),\ t\ge t_0.
$$

Let us next consider the nonlinear impulsive differential system

$$
(1.7.7) \qquad \begin{cases} x'=f(t, x),\ t\ne t_k,\ x(t_0^+)=x_0, \\ \Delta x=I_k(x),\ t=t_k. \end{cases}
$$

Concerning the differentiability of solutions of (1.7.7) with respect to initial values, we have the following result whose proof is very much similar to the proof of corresponding result for differential equations without impulse effects. We merely state such a result.

Theorem 1.7.1

Assume that

(i) f: $\mathbf{R}_+ \times \mathbf{R}^n \to \mathbf{R}^n$ is continuous in $(t_{k-1}, t_k] \times \mathbf{R}^n$ and possesses continuous partial derivatives $\frac{\partial f}{\partial x}$ in $(t_{k-1}, t_k] \times \mathbf{R}^n$ for each $k=1, 2, 3, \cdots$;

(ii) for every $x \in \mathbf{R}^n$ and $k=1, 2, \cdots$, there exist finite limits of the functions f and $\frac{\partial f}{\partial x}$ as $(t, y) \to (t_k, x)$, $t > t_k$;

(iii) I_k: $\mathbf{R}^n \to \mathbf{R}^n$ is continuously differentiable.

Let $x(t) = x(t, t_0, x_0)$ be the solution of (1.7.7) existing on $[t_0, \infty)$ and let $H(t, t_0, x_0) = \frac{\partial f}{\partial x}(t, x(t, t_0, x_0))$. Then, the following statements hold:

(a) $\Phi(t, t_0, x_0) = \frac{\partial x}{\partial x_0}(t, t_0, x_0)$ exists and is the solution of

$$\text{(1.7.8)} \qquad \begin{cases} y' = H(t, t_0, x_0)y, \ t \neq t_k, \ y(t_0^+) = x_0 \\ \Delta y = \frac{\partial I_k}{\partial x}(x(t_k))y, \ t = t_k \end{cases}$$

such that $\Phi(t_0, t_0, x_0)$ is the identity matrix;

(b) $\frac{\partial x}{\partial t_0}(t, t_0, x_0)$ exists, and is the solution of (1.7.8), satisfying

$$\text{(1.7.9)} \qquad \frac{\partial x}{\partial t_0}(t, t_0, x_0) = -\Phi(t, t_0, x_0)\, f(t_0, x_0), \ t \geq t_0.$$

Now let us consider the impulsive differential system

$$\text{(1.7.10)} \qquad \begin{cases} x' = A(t)x + f(t, x), \ t \neq t_k, \ x(t_0^+) = x_0, \\ \Delta x = B_k x + I_k(x), \ t = t_k \end{cases}$$

under the following assumptions:

(i) $0 < t_1 < t_2 < \cdots < t_k < \cdots$ and $\lim_{k \to \infty} t_k = \infty$;

(ii) $A(t)$ is a $n \times n$ matrix, piecewise continuous from $\mathbf{R}_+$ to $\mathbf{R}$ with discontinuities of the first kind at $t=t_k$ and $A(t)$ is left continuous at $t=t_k$, $k=1, 2, 3, \cdots$;

(iii) f: $\mathbf{R}_+ \times \mathbf{R}^n \to \mathbf{R}^n$ is continuous in $(t_{k-1}, t_k] \times \mathbf{R}^n$ and for every $x \in \mathbf{R}^n$, $k=1, 2, \cdots$ $\lim\limits_{(t, y) \to (t_k, x)} f(t, y)$ exists for $t > t_k$;

(iv) for $k=1, 2, \cdots$, B_k is a $n \times n$ matrix and I_k: $\mathbf{R}^n \to \mathbf{R}^n$ is continuous.

By direct verification, we can prove the following variation of parameters formula.

Theorem 1.7.2

Assume that assumptions (i)−(iv) listed above hold. Let $x(t)$ be any solution of (1.7.10) existing on $[t_0, \infty)$ and let $W(t, s)$ be the fundamental matrix solution of (1.7.1) given by (1.7.5). Then $x(t)$ satisfies the integral equation, for $t \geq t_0$,

(1.7.11)
$$x(t) = W(t, t_0)x_0 + \int_{t_0}^{t} W(t, s)f(s, x(s))ds$$
$$+ \sum_{t_0 < t_k < t} W(t, t_k^+)\, I_k(x(t_k)).$$

We shall next discuss an analogue of Alekseev's nonlinear variation of parameters formula. For this purpose, we consider the following system:

(1.7.12)
$$\begin{cases} y' = f(t, y) + R(t, y), \ t \neq t_k, \ y(t_0^+) = x_0, \\ \Delta y = h_k(y), \ t = t_k. \end{cases}$$

Theorem 1.7.3

Suppose that assumptions (i) to (iii) of Theorem 1.7.1 hold. Assume further that R: $\mathbf{R}_+\times\mathbf{R}^n\to\mathbf{R}^n$ is continuous in $(t_{k-1}, t_k]\times\mathbf{R}^n$ for $k=1,2,\cdots$, $\lim\limits_{(t, y)\to(t_k, x)} R(t, y)$ exists for $t>t_k$, $k= 1, 2, \cdots$ and h_k: $\mathbf{R}^n\to\mathbf{R}^n$ is continuous. If $x(t)=x(t, t_0, x_0)$ is the solution of (1.7.7) existing on $[t_0,\infty)$, then any solution $y(t)=y(t, t_0, x_0)$ of (1.7.12) satisfies the integral equation

(1.7.13)
$$y(t)=[\, x(t)+\int_{t_0}^{t}\Phi(t, s, y(s))\, R(s, y(s))ds$$
$$+\sum_{t_0<t_k<t}\int_0^1\Phi(t, t_k, y(t_k)+sh_k(y(t_k)))ds\, h_k(y(t_k))]$$

for $t\geq t_0$, where $\Phi(t, t_0, x_0)=\frac{\partial x}{\partial x_0}(t, t_0, x_0)$.

Proof

Set $v(s)=x(t, s, y(s))$ for $t_0\leq s\leq t$ so that by Theorem 1.7.1

$$\frac{dv}{ds}=\frac{\partial x}{\partial t_0}(t, s, y(s))+\frac{\partial x}{\partial x_0}(t, s, y(s))\, y'(s)$$
$$=\Phi(t, s, y(s))[y'(s)-f(s, y(s))],\ s\neq t_k,$$

and

$$\Delta v|_{s=t_k}=x(t, t_k^+, y(t_k^+))-x(t, t_k^-, y(t_k^-))$$
$$=x(t, t_k, y(t_k)+h_k(y(t_k)))-x(t, t_k, y(t_k))$$
$$=\int_0^1\Phi(t, t_k, y(t_k)+sh_k(y(t_k)))ds\, h_k(y(t_k)).$$

Then, for $t \neq t_k$, we have

$$v(t)=v(t_0)+\int_{t_0}^{t}\frac{dv}{ds}+\sum_{t_0<t_k<t}\Delta v(t_k),\ t \geq t_0,$$

which yields the desired relation (1.7.13) since $v(t)=x(t, t, y(t))$ and $(v_0)=x(t, t_0, x_0)$. The proof is complete.

Having the necessary tools at our disposal, let us begin with a simple result yielding practical stability properties.

Theorem 1.7.4

Assume that condition (i) of Theorem 1.7.1 holds. Suppose further that

(a) $[x, f(t, x)]_+ \leq g(t, |x|)$, $t \neq t_k$, $(t, x) \in \mathbf{R}_+ \times \mathbf{R}^n$, where $g \in C[\mathbf{R}_+^2, \mathbf{R}]$;

(b) $|x+I_k(x)| \leq G_k(|x|)$, where G_k: $\mathbf{R}_+ \to \mathbf{R}_+$ and $G_k(u)$ is nondecreasing in u.

Then the practical stability properties of

$$(1.7.14)\qquad \begin{cases} u'=g(t, u),\ t \neq t_k,\ u(t_0^+)=u_0 \geq 0, \\ u(t_k^+)=G_k(u(t_k)),\ t=t_k \end{cases}$$

Imply the corresponding practical stability properties of (1.7.7).

Proof

Setting $m(t)=|x(t, t_0, x_0)|$ and using the assumptions (a), (b), we get

$$D^+m(t) \leq g(t, m(t)),\ t \neq t_k,\ m(t_0)=|x_0|,$$

$$m(t_k^+) \leq G_k(m(t_k)),$$

where $x(t, t_0, x_0)$ is any solution of (1.7.7). Then, by Theorem A.3.1, we obtain

$$m(t) \le r(t, t_0, |x_0|), \quad t \ge t_0, \tag{1.7.15}$$

where $r(t, t_0, u_0)$ is the maximal solution of (1.7.14). The conclusion of theorem is now immediate and the proof is complete.

Let us demonstrate the significance of the assumptions of Theorem 1.7.4 by considering some simple examples of g and G_k.

Example 1.7.1

Let $f(t, x) = Ax$, where A is a $n \times n$ matrix. It is easy to compute $[x, Ax]_+ \le \mu[A] \, |x|$ where $\mu[A]$ is the logarithmic norm of A, so that $g(t, u) = \mu[A]u$. If $I_k(x) = B_k x$ where B_k is a $n \times n$ matrix for each k and $|x + I_k(x)| \le d_k |x|$ so that $G_k(u) = d_k u$, then the solution of (1.7.14) is given by

$$u(t, t_0, u_0) = u_0 \prod_{t_0 < t_k < t} d_k \exp(\mu[A](t - t_0)), \quad t \ge t_0.$$

Consequently, if we assume that $\prod_{k=1}^{\infty} d_k$ converges and less than $\frac{A}{\lambda}$, then $\mu[A] \le 0$ implies that the system (1.7.14) is practically stable.

Example 1.7.2

Consider (1.7.14) with $g(t, u) = p(t)\phi(u)$ where $p \in C[\mathbf{R}_+, \mathbf{R}_+]$, ϕ, $G_k \in C[\mathbf{R}_+, \mathbf{R}_+]$ and nondecreasing. Suppose further, for every $\sigma \ge \lambda > 0$,

$$\int_{t_k}^{t_{k+1}} p(s)ds + \int_{\sigma}^{G_k(\sigma)} \frac{ds}{\phi(s)} \le 0, \quad k = 1, 2, \cdots. \tag{1.7.16}$$

Let $t_0 \in (t_j, t_{j+1}]$ and $0<\lambda<A$ be given such that $\lambda<\min(a, G_k(A))$. Suppose that $0 \leq u_0 < \lambda$. We claim that $u(t)<A$, $t \geq t_0$, where $u(t)$ is any solution of (1.7.14). If it is not true, there exists a $t^* \in (t_0, t_{j+1}]$ and we have $u(t^*) \geq A$. Then we get

$$\int_{G_j(A)}^{A} \frac{ds}{\phi(s)} < \int_{\lambda}^{A} \frac{ds}{\phi(s)} \leq \int_{u_0}^{A} \frac{ds}{\phi(s)} \leq \int_{u_0}^{u(t^*)} \frac{ds}{\phi(s)} \leq \int_{t_0}^{t^*} p(s)ds \leq \int_{t_j}^{t_{j+1}} p(s)ds,$$

which implies $\int_{t_j}^{t_{j+1}} p(s)ds + \int_{A}^{G_j(A)} \frac{ds}{\phi(s)} > 0$ contradicting (1.7.16). Hence $u(t)<A$ for $t \in (t_j, t_{j+1}]$ whenever $u_0 < \lambda$.

Let $i \geq j+2$ and assume that $u(t)<A$ for $t \in (t_{j+1}, t_i]$. Then for $t \in (t_i, t_{i+1}]$, we have

$$\int_{u(t_i^+)}^{u(t)} \frac{ds}{\phi(s)} \leq \int_{t_i}^{t} p(s)ds \leq \int_{t_i}^{t_{i+1}} p(s)ds. \tag{1.7.17}$$

Since $u(t_i^+) \leq G_i(u(t_i))$, it follows that

$$\int_{u(t_i)}^{u(t_i^+)} \frac{ds}{\phi(s)} \leq \int_{u(t_i)}^{G_i(u(t_i))} \frac{ds}{\phi(s)},$$

and consequently, we get from (1.7.17),

$$\int_{u(t_i)}^{u(t)} \frac{ds}{\phi(s)} \leq \int_{t_i}^{t_{i+1}} p(s)ds + \int_{u(t_i)}^{G_i(u(t_i))} \frac{ds}{\phi(s)} \leq 0.$$

Hence $u(t) \le u(t_i) < A$ for $t \in (t_j, t_{i+1}]$ and therefore by induction, $u(t) < A$ for $t \ge t_0$. We have thus practical stability of (1.7.14). If, in particular, $p(t) = \frac{1}{t}$, $t \ge 1$, $\phi(u) = 2u$, $G_k(u) = (1+\alpha_k)^2 u$ with $|1+\alpha_k| \le \frac{k}{k+1}$ for $k = 2, 3, 4, \cdots$, then it is easy to check that condition (1.7.16) is satisified and (1.7.14) is practically stable. We note that the corresponding differential equation $x' = \frac{2x}{t}$ is not practically stable.

Let us consider the impulsive differential systems (1.7.1) and (1.7.10) under the assumptions (i) to (iv) of Theorem 1.7.2. We shall use the variation of parameters formula to discuss practical stability properties of (1.7.10). For this purpose, let us suppose that $U_k(t, s)$ is the fundamental matrix solution of (1.7.3) and the following assumptions hold:

(A_1) $|U_k(t, s)| \le \phi(t)\psi(s)$, $t_{k-1} < s \le t \le t_k$, $k = 1, 2, \cdots$, with $\psi(t_k^+)$ and $\phi(t_k^+) > 0$;

(A_2) $|f(t, x)| \le a(t)|x|^m$ for $(t, x) \in \mathbf{R}_+ \times \mathbf{R}^n$, $m > 1$;

(A_3) $|I + B_k x| \le \gamma_k$, $|I_k(x)| \le p_k |x|$ for $x \in \mathbf{R}^n$, $\gamma_k, p_k \ge 0$;

where the functions a, ϕ, ψ are piecewise continuous from $\mathbf{R}_+$ to $(0, \infty)$ with discontinuities of the first kind only at $t = t_k$ such that

(A_4) $\phi(t)\psi(t_0^+) \prod_{t_0 < t_k < t} r_k < N$, $t \ge t_0$;

(A_5) $D(t_0, \infty) < \infty$ where $D(t_0, t) = \int_{t_0}^{t} \Big(\prod_{t_0 < t_k < s} r_k \Big)^{m-1} \times \phi^m(s)\psi(s)a(s)ds$;

(A_6) $\phi(t)\psi(t_0^+) \prod_{t_0 < t_k < t} (r_k) < B$, $t \ge t_0 + T$,

where $(B, T) > 0$ are given numbers and $r_k = (\gamma_k + p_k)\phi(t_k)\psi(t_k^+)$.

We can now prove the following result on practical stability properties (1.7.10).

Theorem 1.7.5

Suppose that the assumptions (i) to (iv) of Theorem 1.7.2 and (A_1) to (A_3) are satisifed. Then the system (1.7.10) is (a) practically stable if (A_4) and (A_5) hold; (b) strongly practically stable if (A_5) and (A_6) hold.

Proof

Let W(t, s) be the fundamental matrix solution of (1.7.1) so that any solution $x(t)=x(t, t_0, x_0)$ of (1.7.10) satisfies the variation of parameters formula (1.7.11). Hence, in view of assumptions (A_1) to (A_3), setting $v(t)=\frac{|x(t)|}{\phi(t)}$ we arrive at the inequality

$$\begin{aligned} v(t) &\leq |x_0|\psi(t_0^+)\prod_{t_0<t_k<t}\gamma_k\phi(t_k)\psi(t_k^+) \\ &+\int_{t_0}^{t}[\prod_{s\leq t_k<t}\left(\gamma_k\phi(t_k)\psi(t_k^+)\right)\phi^m(s)\psi(s)a(s)v(s)]ds \\ &+\sum_{t_0<t_k<t}p_k\phi(t_k)\psi(t_k^+)\prod_{t_k<t_i<t}\gamma_i\phi(t_i)\psi(t_i^+)v(t_k),\ t\geq t_0. \end{aligned}$$

Then, in view of Theorem A.3.2 with $g(u)=u^m$, Lemma A.1.1 and the relations for $r_k$and $D(t_0, t)$, after some computation we get

$$(1.7.18)\quad \begin{cases} v(t) \le |x_0|\psi(t_0^+) \prod_{t_0<t_k<t} r_k[1-(m-1)\times \\ \times(\psi(t_0^+)|x_0|)^{m-1}D(t_0,t)]^{\frac{-1}{m-1}}, \end{cases}$$

for all $t \ge t_0$ for which $[(m-1)(\psi(t_0^+)|x_0|)^{m-1}D(t_0,t)] < 1$.

Suppose that (A_4) and (A_5) hold. Let $|x_0| < \lambda$ where

$$\lambda = \min[\{2(m-1)(\psi(t_0^+))^{m-1}D(t_0,\infty)\}^{\frac{-1}{m-1}},\ \frac{A}{N}2^{-\frac{1}{m-1}}].$$

Then from (1.7.18) we see that $|x(t)| < A$, $t \ge t_0$ proving that the system (1.7.10) is practically stable. Since (A_6) implies practical quasi-stability, (A_4), (A_5) and (A_6) show that the system (1.7.10) is strongly practically stable. This proves the theorem.

In order to avoid repetition, we shall not discuss the results corresponding to the nonlinear variation of parameters formula (1.7.13).

1.8. NOTES.

The classical notion of stability known as Lyapunov stability is, of course, due to Lyapunov [1]. The notion of practical stability was suggested in the book of LaSalle and Lefschetz [1].

The definitions of Lyapunov stability and boundedness described in Section 1.1 are taken from Lakshmikantham and Leela [1]. These definitions are based on several works on Lyapunov stability. See, for example, Barbashin [1], Barbashin and Krasovski [1], Persidski [1], Antosiewicz [1], Massera [1], Krasovski [1], Hahn [1], Yoshizawa [1], LaSalle and Lefschetz [1], Rouche, Habets and Laloy [1], Zubov [1,2], Grujic, Martynyuk and Ribbens-Pavella [1]. For the examples 1.1.1 to 1.1.3 see Doboshin [1] and also Martynyuk [3, 14 - 16].

The notions of practical stability defined in Section 1.2 are adapted from Martynyuk [3, 14-16] which are based on post- Lyapunov notions of stability such as stability in the large, stability in the whole, complete stability, finite time stability, technical stability, etc. See Chetayev [1], LaSalle and Lefschetz [1], Hahn [1], and Weiss and Infante [1, 2].

Section 1.3 offers several results on practical stability of ordinary differential equations employing comparison principle and a norm as candidate. Here we utilize various forms of variation of parameters formulae. Theorems 1.3.2 and 1.3.3 are similar in spirit to those given in Martynyuk [4], while the rest of the results are new. See also Bernfeld, Lakshmikantham [1], Grujic [2], Martynyuk [1, 2, 4 -8], and Zubov [3].

Most of the results in Sections 1.4 to 1.7 which are extensions of practical stability to other nonlinear systems are presented for the first time and are new. See for the corresponding results on Lyapunov stability, Lakshmikantham, Leela and Martynyuk [1], Lakshmikantham and Trigiante [1], Lakshmikantham, Bainov and Simeonov [1], Leela and Zouyousefain [1], Martynyuk [2, 11, 13], and Samoilenko and Perestyuk [1].

2
Method of Lyapunov Functions

2.0. INTRODUCTION.

This chapter is devoted essentially to the investigation of practical stability properties using Lyapunov-like functions and the theory of differential inequalities. It also provides more general concepts of practical stability in terms of two measures and arbitrary sets, and stresses the importance of employing several Lyapunov-like functions.

In section 2.1, we formulate basic comparison results in terms of Lyapunov-like functions and the comparison systems that are necessary for later discussion. As a consequence, we also consider a global existence result. Section 2.2 offers various practical stability criteria in terms of Lyapunov-like functions using comparison principle. Several special cases of importance and general examples are given to clarify the results obtained. Section 2.3 deals with nonuniform practical stability considerations under weaker assumptions, utilizing the idea of perturbing Lyapunov functions. The method of vector Lyapunov functions is discussed in Section 2.4, where it is demonstrated that employing several Lyapunov functions, than a single one, is more advantageous in certain situations. In Section 2.5, large scale dynamic systems are investigated by the method of decomposition-aggregation in which several Lyapunov functions occur in a natural way.

We introduce, in Section 2.6, general definitions of practical stability in terms of arbitrary sets. As an important special case, we consider practical stability concepts in terms of two measures which, in turn, unify a variety of notions of practical stability. Section 2.7 provides practical stability criteria in terms of two measures, while Section 2.8 considers global results relative to arbitrary sets which can be utilized to derive several types of stability results. Finally, in Section 2.9, we prove practical stability properties relative to arbitrary sets.

2.1. BASIC COMPARISON THEOREMS.

Consider the differential system

$$x' = f(t, x),\ x(t_0) = x_0,\ t_0 \in \mathbf{R}_+, \tag{2.1.1}$$

where $f \in C[\mathbf{R}_+ \times \mathbf{R}^n, \mathbf{R}^n]$. For any Lyapunov-like function $V \in C[\mathbf{R}_+ \times \mathbf{R}^n, \mathbf{R}_+]$, we define the function

$$D^+V(t, x) = \lim_{h \to 0^+} \sup \frac{1}{h}[V(t+h, x+hf(t, x)) - V(t, x)]. \tag{2.1.2}$$

for $(t, x) \in \mathbf{R}_+ \times \mathbf{R}^n$. Occasionally we shall denote (2.1.2) by $D^+V(t, x)_{(2.1.1)}$ to emphasize the definition of D^+V with respect to the system (2.1.1). One could also utilize other generalized derivatives of V, for example,

$$D^-V(t, x) = \lim_{h \to 0^-} \inf \frac{1}{h}[V(t+h, x+hf(t, x)) - V(t, x)]. \tag{2.1.3}$$

We note that if $V \in C^1[\mathbf{R}_+ \times \mathbf{R}^n, \mathbf{R}_+]$, then $D^+V(t, x) = V'(t, x)$, where

$$V'(t, x) = V_t(t, x) + V_x(t, x)f(t, x).$$

We can now formulate basic comparison results that we need in terms of Lyapunov function V.

Theorem 2.1.1

Let $V \in C[\mathbf{R}_+ \times \mathbf{R}^n, \mathbf{R}_+]$ and be locally Lipschitzian in x. Assume that $D^+V(t, x)$ satisfies

(2.1.4) $$D^+V(t, x) \leq g(t, V(t, x)),\ (t, x) \in \mathbf{R}_+ \times \mathbf{R}^n,$$

where $g \in C[\mathbf{R}_+^2, \mathbf{R}]$. Let $r(t, t_0, u_0)$ be the maximal solution of the scalar differential equation

$$u' = g(t, u),\ u(t_0) = u_0 \geq 0,$$

existing for $t \geq t_0$. Then $V(t_0, x_0) \leq u_0$ implies

(2.1.5) $$V(t, x(t, t_0, x_0)) \leq r(t, t_0, u_0),\ t \geq t_0$$

where $x(t, t_0, x_0)$ is any solution of (2.1.1) existing for $t \geq t_0$.

Proof

Let $x(t) = x(t, t_0, x_0)$ be any solution of (2.1.1) existing for $t \geq t_0$ such that $V(t_0, x_0) \leq u_0$. Set $m(t) = V(t, x(t))$, so that for sufficiently small $h > 0$, we have

$$m(t+h) - m(t) = V(t+h, x(t+h)) - V(t+h, x(t) + hf(t, x(t)))$$

$$+ V(t+h, x(t) + hf(t, x(t))) - V(t, x(t)).$$

Since $V(t, x)$ is locally Lipschitzian in x, we get using (2.1.4) the inequality

$$D^+m(t) \leq g(t, m(t)),\ t \geq t_0$$

with $m(t_0) \leq u_0$. Theorem A.1.1 now yields the desired estimate (2.1.5).

Corollary 2.1.1

Suppose that $g(t, u) \equiv 0$ in Theorem 2.1.1. Then $V(t, x(t))$ is nonincreasing in $t \geq t_0$ and $V(t, x(t)) \leq V(t_0, x_0)$ for $t \geq t_0$.

In some situations, estimating D^+V in terms of a function of t, x and V is more suitable. The next comparison result is in that direction, whose proof is similar to that of Theorem 2.1.1.

Theorem 2.1.2

Let $V \in C[\mathbf{R}_+ \times \mathbf{R}^n, \mathbf{R}_+]$ and $V(t, x)$ be locally Lipschitzian in x. Assume that $g \in C[\mathbf{R}_+ \times \mathbf{R}^n \times \mathbf{R}_+, \mathbf{R}]$ and for $(t, x) \in \mathbf{R}_+ \times \mathbf{R}^n$,

$$D^+V(t, x) \leq g(t, x, V(t, x)).$$

Let $x(t) = x(t, t_0, x_0)$ be any solution of (2.1.1) existing on $[t_0, \infty)$ and $r(t, t_0, x_0, u_0)$ be the maximal solution of

$$u' = g(t, x(t), u), \quad u(t_0) = u_0 \geq 0,$$

existing on $[t_0, \infty)$. Then $V(t_0, x_0) \leq u_0$ implies

$$V(t, x(t)) \leq r(t, t_0, x_0, u_0), \quad t \geq t_0.$$

Sometimes, in applications, the following variants of Theorem 2.1.1 are more useful.

Theorem 2.1.3

Assume that the hypotheses of Theorem 2.1.1 hold except that the inequality (2.1.4) is replaced by

$$A(t)D^+V(t, x) + D^+A(t)\, V(t, x) \leq g(t, V(t, x)A(t)) \tag{2.1.6}$$

for $(t, x) \in \mathbf{R}_+ \times \mathbf{R}^n$, where $A \in C[\mathbf{R}_+, (0, \infty)]$ and

$D^+A(t) = \lim_{h \to 0^+} \sup \frac{1}{h}[A(t+h) - A(t)]$. Then $V(t_0, x_0)A(t_0) \leq u_0$ implies the estimate

$$(2.1.7) \qquad V(t, x(t))A(t) \leq r(t, t_0, u_0),\ t \geq t_0.$$

Proof

Defining $L(t, x) = A(t)V(t, x)$, we find that for small $h > 0$,

$$L(t+h, x+hf(t, x)) - L(t, x) = V(t+h, x+hf(t, x))[A(t+h) - A(t)]$$
$$+A(t)[V(t+h, x+hf(t, x)) - V(t, x)],$$

and therefore, using (2.1.6), we get

$$D^+L(t, x) \leq g(t, L(t, x)).$$

The desired estimate (2.1.7) follows immediately from Theorem 2.1.1.

Theorem 2.1.4

Let the hypotheses of Theorem 2.1.1 hold except that in place of (2.1.4), we assume that

$$(2.1.8) \qquad D^+V(t, x) + C(t, |x|) \leq g(t, V(t, x)),\ (t, x) \in \mathbf{R}_+ \times \mathbf{R}^n,$$

where $C(t, u) \geq 0$ is continuous in $(t, u) \in \mathbf{R}_+^2$ and $g(t, u)$ is nondecreasing in u for each $t \in \mathbf{R}_+$. Then $V(t_0, x_0) \leq u_0$ implies that

$$(2.1.9) \qquad V(t, x(t)) + \int_{t_0}^{t} C(s, |x(s)|)ds \leq r(t, t_0, u_0),\ t \geq t_0.$$

Proof

Define $m(t)=V(t, x(t))+\int_{t_0}^{t} C(s, |x(s)|)ds$ so that $V(t, x(t))\leq m(t)$. The monotone character of g together with (2.1.8) now gives

$$D^+m(t)\leq g(t, m(t)),\ t\geq t_0$$

and the claim follows from Theorem A.1.1.

The next comparison result plays an important role whenever we utilize a vector Lyapunov function and its proof is similar to that of Theorem 2.1.1 and uses Theorem A.1.3.

Theorem 2.1.5

Let $V\in C[\mathbf{R}_+\times\mathbf{R}^n,\ \mathbf{R}^N]$ and V be locally Lipschitzian in x. Assume that

$$D^+V(t, x)\leq g(t, V(t, x)),\ (t, x)\in\mathbf{R}_+\times\mathbf{R}^n,$$

where $g\in C[\mathbf{R}_+\times\mathbf{R}_+^N,\ \mathbf{R}^N]$ and $g(t, u)$ is quasimonotone nondecreasing in u. Let $r(t)=r(t, t_0, u_0)$ be the maximal solution of

$$u'=g(t, u),\ u(t_0)=u_0\geq 0, \tag{2.1.10}$$

existing for $t\geq t_0$ and $x(t)=x(t, t_0, x_0)$ be any solution of (2.1.1) for $t\geq t_0$. Then $V(t_0, x_0)\leq u_0$ implies

$$V(t, x(t))\leq r(t),\ t\geq t_0.$$

We recall that inequalities between vectors are componentwise and quasimonotonicity of $g(t, u)$ means that $u\leq v$, $u_i=v_i$ for $1\leq i\leq N$ implies $g_i(t, u)\leq g_i(t, v)$.

The function $V(t, x)$ is said to be mildly unbounded, if for every $T>0$, $V(t, x) \to \infty$ as $|x| \to \infty$ uniformly for $t \in [0, T]$.

The mild unboundedness of $V(t, x)$ guarantees that, whenever $V(t,x(t))$ is finite, $|x(t)|$ is also finite. Consequently, the assumption that the solutions $x(t)$ of (2.1.1) exist for $t \geq t_0$ becomes superfluous if $V(t, x)$ is assumed further to be mildly unbounded in foregoing theorems. From this observation results the following global existence theorem.

Theorem 2.1.6

Let $V(t, x)$ be mildly unbounded in addition to the hypotheses of Theorem 2.1.1. Then, every solution $x(t)$ of (2.1.1) exists for $t \geq t_0$ and satisfies (2.1.5).

Proof

Suppose that the assertion that every solution $x(t)$ of (2.1.1) exists for $t \geq t_0$ is false. Then, there exists a $t_1 > t_0$ such that $x(t)$ cannot be continued to the closed interval $[t_0, t_1]$. This implies that there exists a sequence $\{t_k\}$ with $t_k \to t_1^-$. By Theorem 2.1.1, we have

$$V(t, x(t)) \leq r(t, t_0, V(t_0, x_0)), \; t_0 \leq t < t_1.$$

Since $V(t, x)$ is mildly unbounded and $r(t, t_0, u_0)$ exists for $t \geq t_0$, there arises a contradiction as $t_k \to t_1^-$. Hence $x(t)$ exists for $t \geq t_0$, which in turn yields the estimate (2.1.5). The proof is complete.

2.2. STABILITY CRITERIA.

We begin by defining the following classes of functions:

$K = \{a \in C[\mathbf{R}_+, \mathbf{R}_+]$: $a(u)$ is strictly increasing in u and
$a(u) \to \infty$ as $u \to \infty\}$,
$CK = \{a \in C[\mathbf{R}_+^2, \mathbf{R}_+]$: $a(t, u) \in K$ for each $t \in \mathbf{R}_+\}$,
$L = \{\sigma \in C[\mathbf{R}_+, \mathbf{R}_+]$: $\sigma(u)$ is strictly decreasing in u and
$\sigma(u) \to 0$ as $u \to \infty\}$,
$LK = \{a \in C[\mathbf{R}_+^2, \mathbf{R}_+]$: $a(t, u) \in K$ for each $t \in \mathbf{R}_+$ and
$a(t, u) \in L$ for each $u \in \mathbf{R}_+\}$.

Let $S(\rho) = \{x \in \mathbf{R}^n\colon \ | x | < \rho\}$. We shall now establish some sufficient conditions for practical stability of system (2.1.1).

Theorem 2.2.1

Assume that

(A_0) λ, A are given such that $0 < \lambda < A$;

(A_1) $V \in C[\mathbf{R}_+ \times \mathbf{R}^n, \mathbf{R}_+]$ and $V(t, x)$ is locally Lipschitzian in x;

(A_2) for $(t, x) \in \mathbf{R}_+ \times S(A)$, $b(| x |) \leq V(t, x) \leq a(| x |)$ and

$$D^+V(t, x) \leq g(t, V(t, x)), \tag{2.2.1}$$

where $a, b \in K$ and $g \in C[\mathbf{R}_+^2, \mathbf{R}]$;

(A_3) $a(\lambda) < b(A)$ holds.

Then, the practical stability properties of

$$u' = g(t, u),\ u(t_0) = u_0 \geq 0 \tag{2.2.2}$$

imply the corresponding practical stability properties of the system (2.1.1).

Proof

Let us first suppose that (2.2.2) is practically stable. Then, given

$(a(\lambda),b(A))$, it follows, because of (A_3), that

(2.2.3) $$u_0<a(\lambda) \text{ implies } u(t,\, t_0,\, u_0)<b(A),\ t\geq t_0.$$

Let $|\, x_0\, |<\lambda$. We claim that $|\, x(t)|<A$, $t\geq t_0$, where $x(t)=x(t,\, t_0,\, x_0)$ is any solution with $|\, x_0\, |<\lambda$. If this is not true, there would exist a $t_1>t_0$ and a solution $x(t)=x(t,\, t_0,\, x_0)$ of (2.1.1) with $|\, x_0\, |<\lambda$ such that

$$|\, x(t_1)\, |=A \text{ and } |\, x(t)|\leq A \text{ for } t_0\leq t\leq t_1.$$

By (A_2) and the continuity of the functions involved, we now have

(2.2.4) $$V(t_1,\, x(t_1))\geq b(A).$$

Choosing $V(t_0,\, x_0)=u_0$, we then have by Theorem 2.1.1, the estimate

(2.2.5) $$V(t,\, x(t))\leq r(t,\, t_0,\, u_0),\ t_0\leq t<t_1,$$

where $r(t,\, t_0,\, u_0)$ is the maximal solution of (2.2.2). The relations (2.2.3), (2.2.4) and (2.2.5) together with (A_2) now lead to the contradiction

$$b(A)=b(|\, x\, (t_1)|)\leq V(t_1,\, x(t_1))\leq r(t_1,\, t_0,\, u_0)<b(A),$$

since $u_0\leq a(|\, x_0\, |)<a(\lambda)$. This proves that the system (2.1.1) is practically stable.

We shall next prove that (2.1.1) is strongly practically stable for $(\lambda,A,B,T)>0$. To prove this, let us suppose that (2.2.2) is strongly practically stable for $(a(\lambda),\, b(A),\, b(B),\, T)>0$. This means we need to prove only practical quasi-stability of the system (2.1.1). Since (2.2.2) is practically quasi-stable, we have

(2.2.6) $$u_0<a(\lambda) \text{ implies } u(t,\, t_0,\, u_0)<b(B) \text{ for } t\geq t_0+T,$$

where $u(t, t_0, u_0)$ is any solution of (2.2.2). Suppose that $|x_0| < \lambda$ so that we have $|x(t)| < A$ for $t \geq t_0$, because of the practical stability of (2.2.2). As a result, it follows that the estimate (2.2.5) is true for $t \geq t_0$, that is,

$$V(t, x(t)) \leq r(t, t_0, u_0),\ t \geq t_0, \tag{2.2.7}$$

which implies that

$$b(|x(t)|) \leq V(t, x(t)) \leq r(t, t_0, u_0) < b(B),\ t \geq t_0 + T.$$

Thus we see that $|x(t)| < B$ for $t \geq t_0 + T$ whenever $|x_0| < \lambda$ and therefore strong practical stability of the system (2.1.1) is proved.

One can similarly prove other practical stability properties of the system (2.1.1) and hence the proof is complete.

Corollary 2.2.1

In Theorem 2.2.1,

(i) $g(t, u) \equiv 0$ is admissible to yield uniform practical stability;

(ii) $g(t, u) = -\alpha u + k$, α, $k > 0$ is admissible to imply uniform strong practical stability;

(iii) $g(t, u) = -\sigma'(t)$, $\sigma \in L$ and σ is differentiable, is admissible to guarantee eventual practical stability

Proof

The proof of (i) is immediate. Concerning (ii), we observe that the solutions $u(t, t_0, u_0)$ of (2.2.2) are given by

$$u(t, t_0, u_0) = u_0 e^{-\alpha(t-t_0)} + \frac{k}{\alpha}[1 - e^{-\alpha(t-t_0)}],\ t \geq t_0.$$

Suppose that $(\lambda, A, B, T)>0$ are given such that $\lambda<A$, $B<A$. Then, if $a(\lambda)+\frac{k}{\alpha}<b(A)$ and $a(\lambda)e^{-\alpha T}+\frac{k}{\alpha}<b(B)$, we have uniform strong practical stability of (2.1.1).

To prove (iii), note that $u(t, t_0, u_0)\leq u_0+\sigma(t_0)$, $t\geq t_0$ and hence it is enough to have $\sigma(t_0)\leq b(A)-a(\lambda)$. Since $\sigma\in L$, there exists a $\tau=\tau(\lambda,A)$ such that $\sigma(t_0)\leq b(A)-a(\lambda)$ if $t_0\geq\tau$. Thus, the equation (2.2.2) is eventually practically stable. Hence, Theorem 2.2.1 implies the desired properties.

We have assumed in Theorem 2.2.1 stronger requirements on V only to unify all the practical stability results in one theorem. This clearly puts burden on the comparison equation (2.2.2). However, to obtain only nonuniform practical stability criteria, we could weaken certain assumptions of Theorem 2.2.1 as in the next result, which we merely state.

Theorem 2.2.2

Assume that conditions (A_0) - (A_3) of Theorem 2.2.1 hold with the following changes in (A_2) and (A_3):

$(A_2{}^*)$ for $(t, x)\in \mathbf{R}_+\times S(A)$, $b(|x|)\leq V(t, x)\leq a(t, |x|)$, where $b\in K$ and $a\in CK$;

$(A_3{}^*)$ $a(t_0, \lambda)<b(A)$ for some $t_0\in\mathbf{R}_+$.

Then uniform or nonuniform practical stability properties of (2.2.2) imply the corresponding nonuniform practical stability properties of the system (2.1.1).

We shall next consider a result which gives practical asymptotic stability of (2.1.1).

Theorem 2.2.3

Suppose that (A_0), (A_1), $(A_2{}^*)$ and $(A_3{}^*)$ hold. Assume further that

(A_4) $W \in C[\mathbf{R}_+ \times \mathbf{R}^n, \mathbf{R}_+]$, $W(t, x)$ is locally Lipschitzian in x, $D^+W(t, x)$ is bounded from above or from below for $(t, x) \in \mathbf{R}_+ \times S(A)$, $W(t, x)$ is positive definite and for $(t, x) \in \mathbf{R}_+ \times S(A)$,

$$D^+V(t, x) \leq -C(W(t, x)),\ C \in K.$$

Then, the system (2.1.1) is practically asymptotically stable.

Proof

By Theorem 2.2.2 with $g(t, u) \equiv 0$, it follows, because of (A_4), that the system (2.1.1) is practically stable. Hence it is enough to prove that every solution $x(t)$ of (2.1.1) with $|x_0| < \lambda$ satisfies $\lim_{t\to\infty} |x(t)| = 0$. Since $W(t, x)$ is assumed to be positive definite, it is enough to show that $\lim_{t\to\infty} W(t, x(t)) = 0$ for any solution $x(t)$ of (2.1.1). We first note that $\liminf_{t\to\infty} W(t, x(t)) = 0$. For otherwise, in view of (A_4) we obtain a contradiction that $V(t, x(t)) \to -\infty$ as $t \to \infty$.

Suppose that $\limsup_{t\to\infty} W(t, x(t)) \neq 0$. Then, for any $\epsilon > 0$, there exist divergent sequences $\{t_n\}$, $\{t_n{}^*\}$ such that $t_i < t_i{}^* < t_{i+1}$, $i = 1, 2, \ldots$, and

$$\text{(2.2.8)} \qquad \left\{ \begin{aligned} & W(t_i, x(t_i)) = \tfrac{\epsilon}{2},\ W(t_i{}^*, x(t_i{}^*)) = \epsilon \text{ and} \\ & \tfrac{\epsilon}{2} < W(t, x(t)) < \epsilon,\ t \in (t_i, t_i{}^*),\ i = 1, 2, \ldots. \end{aligned} \right.$$

Of course, we could have, instead of (2.2.8),

$$(2.2.9)\qquad \begin{cases} W(t_i, x(t_i))=\epsilon,\ W(t_i^*, x(t_i^*))=\frac{\epsilon}{2} \text{ and} \\ \frac{\epsilon}{2}<W(t, x(t))<\epsilon \text{ for } t\in(t_i, t_i^*),\ i=1, 2, \ldots. \end{cases}$$

Suppose that $D^+W(t, x)\leq M$. Then it is easy to obtain, using (2.2.8), the relation $t_i^*-t_i>\frac{\epsilon}{2M}$. In view of (A_4), we have for large n,

$$0\leq V(t_n^*, x(t_n^*))\leq V(t_0, x_0)+\sum_{1\leq i\leq n}\int_{t_i}^{t_i^*} D^+V(s, x(s))ds$$

$$\leq V(t_0,x_0)-nC(\tfrac{\epsilon}{2})\,\frac{\epsilon}{2M}<0$$

which is a contradiction. Thus $W(t, x(t))\rightarrow 0$ as $t\rightarrow\infty$ and hence $x(t)\rightarrow 0$ as $t\rightarrow\infty$. The case when D^+W is bounded from below can be proved using (2.2.9) with similar arguments. The proof is therefore complete.

Corollary 2.2.2

In Theorem 2.2.3, the following choices of $W(t, x)$ are admissible to yield the same conclusion:

(i) $W(t, x)=|x|$ provided that f is bounded on $\mathbf{R}_+\times S(A)$;

(ii) $W(t, x)=V(t, x)$.

If we desire only uniform properties of practical stability of the system (2.1.1), we can relax the assumptions of Theorem 2.2.3 by employing Lyapunov-like functions which satisfy less restrictive conditions.

Theorem 2.2.4

Assume that

(i) $0<\lambda<A$;

(ii) $V\in C[\mathbf{R}_+\times S(A)\cap S^C(\lambda), \mathbf{R}_+]$, $V(t, x)$ is locally Lipschitzian in x and for $(t, x)\in\mathbf{R}_+\times S(A)\cap S^C(\lambda)$,

$$b(|x|)\le V(t, x)\le a(|x|),\ a, b\in K,$$

$$D^+V(t, x)\le g(t, V(t, x)),$$

where $g\in C[\mathbf{R}_+^2, \mathbf{R}]$ and $S^C(\lambda)$ is the complement of the set $S(\lambda)$;

(iii) $a(\lambda)<b(A)$ holds.

Then uniform practical stability of (2.2.2) implies the uniform practical stability of the system (2.1.1).

Proof

Suppose that the equation (2.2.2) is uniformly practically stable. Then, given $(a(\lambda), b(A))$, we have

(2.2.10) $$u_0<a(\lambda) \text{ implies } u(t, t_0, u_0)<b(A),\ t\ge t_0,$$

for all $t_0\in\mathbf{R}_+$. Let $|x_0|<\lambda$. We claim that $|x(t)|<A$, $t\ge t_0$. If this is not true, there would exist a solution $x(t)=x(t, t_0, x_0)$ of (2.1.1) and $t_2>t_1>t_0$ such that

(2.2.11) $$|x(t_2)|=A,\ |x(t_1)|=\lambda \text{ and } \lambda\le|x(t)|\le A,\ t_1\le t<t_2.$$

By Theorem 2.1.1, we now have

(2.2.12) $$V(t, x(t))\le r(t, t_1, V(t_1, x(t_1))),\ t_1\le t\le t_2,$$

where $r(t, t_1, u_0)$ is the maximal solution of (2.2.2) with $r(t_1, t_1, u_0)=u_0$.

In view of assumption (ii), the relations (2.2.10), (2.2.11) and (2.2.12), we are lead to the contradiction

$$b(A) \le V(t_2, x(t_2)) \le r(t_2, t_1, a(\lambda)) < b(A)$$

which proves the uniform practical stability of the system (2.1.1).

Corollary 2.2.3

The function $g(t, u) \equiv 0$ is admissible in Theorem 2.2.4 to yield uniform practical stability of the system (2.1.1).

Theorem 2.2.5

Assume that the hypotheses (i) and (iii) of Theorem 2.2.4 hold. Suppose further that for each $0 < \eta < A$, there exist $\mu_\eta > 0$ and $V_\eta \in C[\mathbf{R}_+ \times S(A) \cap S^C(\eta), \mathbf{R}_+]$, $V_\eta(t, x)$ is locally Lipschitzian in x and for $(t, x) \in \mathbf{R}_+ \times S(A) \cap S^C(\eta)$,

$$\text{(2.2.13)} \quad \begin{cases} b(|x|) \le V_\eta(t, x) \le a(|x|), \ a, b \in K \text{ with } a(0) = b(0) = 0, \\ D^+ V_\eta(t, x) \le -\mu_\eta. \end{cases}$$

Then the system (2.1.1) is uniformly practically asymptotically stable.

Proof

Since (i) and (iii) of Theorem 2.2.4 hold, taking $\eta = \lambda$ and $g(t, u) \equiv 0$, we get uniform practical stability of the system (2.1.1), by Corollary 2.2.3. Let $0 < \epsilon < A$ and $t_0 \in \mathbf{R}_+$ be given. Choose a $\delta = \delta(\epsilon) > 0$ such that $a(\delta) < b(\epsilon)$. Then, under the assumptions, it easily follows that $|x_0| < \delta(\epsilon)$ implies $|x(t)| < \epsilon$, $t \ge t_0$. Now, let us show that there exists a

$t^* \in [t_0, t_0+T]$, where $T = \frac{a(\lambda)}{\mu_\eta}$, $\eta = \delta(\epsilon)$, such that $|\, x(t^*)| < \delta$, where $x(t) = x(t, t_0, x_0)$ is any solution of (2.1.1) with $|\, x_0 \,| < \lambda$.

Suppose that this is not true. The $\delta \leq |\, x(t) \,|$ for $t_0 \leq t \leq t_0 + T$. Since we have $|\, x(t)| < A$, $t \geq t_0$ whenever $|\, x_0 \,| < \lambda$, we get from (2.2.13),

$$V_\eta(t_0+T,\, x(t_0+T)) \leq V_\eta(t_0,\, x_0) - \mu_\eta T,\ \eta = \delta(\epsilon),$$

and consequently we arrive at the contradiction

$$0 < b(\delta) \leq a(\lambda) - \mu_\eta T = 0.$$

Thus, $|\, x_0 \,| < \lambda$ implies $|\, x(t) \,| < \epsilon$, $t \geq t_0 + T$ and the proof of the theorem is complete.

Sometimes, a judicious selection of $V(t, x)$ reflecting more closely certain properties of the given system leads to more precise result rather than choosing $V(t, x)$ as simple as possible such as $V(t, x) = |\, x \,|$. As an example, let us consider the differential system

(2.2.14) $$x' = A(t)x + \tilde{R}(t, x),\ x(t_0) = x_0$$

concerning which we shall assume the following:

(i) there exists a continuously differentiable matrix $G(t)$, which is self-adjoint and positive, that is, the Hermitian form (Gx, x) is positive definite and $\lambda_1, \lambda_2 > 0$ are the smallest and the largest eigenvalues of $G(t)$;

(ii) the function $v \in C[\mathbf{R}_+, \mathbf{R}]$ is the largest eigenvalue of the matrix $G^{-1}(t)Q(t)$, where

$$Q(t) = \frac{dG(t)}{dt} + G(t)A(t) + A^*(t)G(t),$$

$A(t)$ being a continuous matrix on $\mathbf{R}_+$ and $A^*(t)$ is its transpose;

(iii) $\tilde{R} \in C[\mathbf{R}_+ \times S(A), \mathbf{R}^n]$ and $|\tilde{R}(t, x)| \leq \beta(t) |x|^\alpha$, $0 < \alpha \leq 1$, where $\beta \in C[\mathbf{R}_+, \mathbf{R}_+]$.

Choosing the Lyapunov function defined by $V(t, x) = (G(t)x, x)$, we obtain

$$\begin{aligned} V'(t, x) &= \frac{\partial V(t, x)}{\partial t} + \frac{\partial V(t, x)}{\partial x}[A(t)x + \tilde{R}(t, x)] \\ &= (G'x, x) + (GAx, x) + (Gx, Ax) + (G\tilde{R}, x) + (Gx. \tilde{R}) \\ &= (G'x, x) + (GAx, x) + (A^*Gx, x) + (G\tilde{R}, x) + (Gx, \tilde{R}) \\ &= (Q(t)x, x) + (G\tilde{R}, x) + (Gx, \tilde{R}). \end{aligned}$$

In view of the definition of $Q(t)$, we also obtain

$$G^{-1}(t)Q(t) = G^{-1}\frac{dG}{dt} + A + G^{-1}A^*G$$

and hence,

$$(Qx, x) \leq v(t)(Gx, x) = v(t)V(t, x).$$

Moreover, it is easy to infer that

$$(Gx, \tilde{R}) + (G\tilde{R}, x) \leq 2[(Gx, x)(G\tilde{R}, \tilde{R})]^{\frac{1}{2}}.$$

Since

$$\lambda_1(x, x) \leq (Gx, x) \leq \lambda_2(x, x),$$

it follows, on account of (iii), that

$$\begin{aligned} (G\tilde{R}, \tilde{R}) &\leq \lambda_2(\tilde{R}, \tilde{R}) \leq \lambda_2(\beta(t))^2(x, x)^\alpha \\ &= \lambda_2\lambda_1^{-\alpha}(\beta(t))^2\, \lambda_1^\alpha(x, x)^\alpha \end{aligned}$$

$$\leq \lambda_2\ \lambda_1^{-\alpha}(\beta(t))^2(V(t, x))^{\alpha}.$$

Using all these inequalities, we arrive at

$$V'(t, x) \leq v(t)V(t, x) + h(t)[V(t, x)]^{p},$$

where $h(t) = 2\beta(t)\ (\lambda_2\lambda_1^{-\alpha})^{\frac{1}{2}}$, $0 < \alpha \leq 1$, $p = \frac{1+\alpha}{2}$. The corresponding comparison equation is therefore

$$u' = v(t)u + h(t)u^{p},\ u(t_0) = u_0 \geq 0,$$

whose solution is given by

$$u(t) = [u_0^{q} + q\int_{t_0}^{t} h(s)\ \exp(q\int_{t_0}^{s} v(\xi)d\xi)ds]^{\frac{1}{q}}\ \exp(\int_{t_0}^{t} v(s)ds),$$

for $t \geq t_0$, where $q = 1 - p$.

Taking into account the definition of $u(t)$, we can now use Theorems 2.2.1, 2.2.2, 2.2.3 to prove various practical stability properties. For example, given $0 < \lambda < A$, if the functions v and h satisfy the inequality

$$[\lambda_2^{q}\lambda^{2q} + q\int_{t_0}^{t} h(s)\ \exp(q\int_{t_0}^{s} v(\xi)d\xi)]^{\frac{1}{q}}\ \exp\ (\int_{t_0}^{t} v(s)ds) < \lambda_1 A^2,$$

then the system (2.2.14) is practically stable.

2.3. PERTURBING LYAPUNOV FUNCTIONS.

We shall discuss, in this section, nonuniform practical stability of (2.1.1) under weaker assumptions, which indicates that in those situations when the Lyapunov function chosen does not satisfy all the desired conditions, it is fruitful to perturb that Lyapunov function than to discard it.

Theorem 2.3.1.

Assume that

(i) $0<\lambda<A$;

(ii) $V_1 \in C[\mathbf{R}_+ \times S(A), \mathbf{R}_+]$, $V_1(t, x)$ is locally Lipschitzian in x and for $(t, x) \in \mathbf{R}_+ \times S(A)$, $V_1(t, x) \leq a_1(t, |x|)$, $a_1 \in CK$ and

$$D^+V_1(t, x) \leq g_1(t, V_1(t, x)),$$

where $g_1 \in C[\mathbf{R}_+^2, \mathbf{R}]$;

(iii) $V_2 \in C[\mathbf{R}_+ \times S(A) \cap S^C(\lambda), \mathbf{R}_+]$, $V_2(t, x)$ is locally Lipschitzian in x and for $(t, x) \in \mathbf{R}_+ \times S(A) \cap S^C(\lambda)$,

$$b(|x|) \leq V_2(t, x) \leq a_2(|x|),\ b, a_2 \in K,$$

$$D^+V_1(t, x) + D^+V_2(t, x) \leq g_2(t, V_1(t, x) + V_2(t, x)),$$

where $g_2 \in C[\mathbf{R}_+^2, \mathbf{R}]$;

(iv) $a_1(t_0, \lambda) + a_2(\lambda) < b(A)$ for some $t_0 \in \mathbf{R}_+$;

(v) $u_0 < a_1(t_0, \lambda)$ implies $u(t, t_0, u_0) < a_1(t_0, \lambda)$ for $t \geq t_0$, where $u(t, t_0, u_0)$ is any solution of

$$u' = g_1(t, u),\ u(t_0) = u_0, \tag{2.3.1}$$

and $v_0 < a_1(t_0, \lambda) + a_2(\lambda)$ implies $v(t, t_0, v_0) < b(A)$, $t \geq t_0$, for every $t_0 \in \mathbf{R}_+$, where $v(t, t_0, v_0)$ is any solution of

$$v' = g_2(t, v),\ v(t_0) = v_0 \geq 0. \tag{2.3.2}$$

Then, the system (2.1.1) is practically stable.

Proof

We claim that if $|x_0|<\lambda$ we have $|x(t)|<A$, $t\geq t_0$ where $x(t)=x(t,t_0,x_0)$ is any solution of (2.1.1). If this is not true, there would exist a $t_2>t_1>t_0$ and a solution $x(t)=x(t, t_0, x_0)$ of (2.1.1) such that

(2.3.3) $\quad |x(t_1)|=\lambda,\ |x(t_2)|=A$ and $\lambda\leq|x(t)|\leq A,\ t_1\leq t\leq t_2.$

Hence we get by Theorem 2.1.1, using (iii), the estimate

(2.3.4) $\quad V_1(t, x(t))+V_2(t, x(t))\leq r_2(t, t_1, V_1(t_1,x(t_1))+V_2(t_1,x(t_1)))$

for $t_1\leq t\leq t_2$, where $r_2(t, t_1, v_0)$ is the maximal solution of (2.3.2) through (t_1, v_0). Similarly, condition (ii) gives the estimate

$$V_1(t, x(t))\leq r_1(t, t_0, V_1(t_0, x_0)),\ t_0\leq t\leq t_1,$$

where $r_1(t, t_0, u_0)$ is the maximal solution of (2.3.1). Since $|x_0|<\lambda$, we have, because of (ii),

$$V_1(t_0, x_0)\leq a_1(t_0, |x_0|)<a_1(t_0, \lambda)$$

and hence, (v) shows that

$$V_1(t_1, x(t_1))<a_1(t_0, \lambda).$$

Also, $V_2(t_1, x(t_1))\leq a_2(|x(t_1)|)=a_2(\lambda)$, because of (ii) and consequently

$$V_1(t_1, x(t_1))+V_2(t_1, x(t_1))\leq a_1(t_0, \lambda)+a_2(\lambda).$$

Now, using (2.3.4) and (v), we obtain

(2.3.5) $$\begin{cases} V_1(t_2, x(t_2))+V_2(t_2, x(t_2))\leq r_2(t_2, t_1, a_1(t_0, \lambda)+a_2(\lambda)) \\ \qquad\qquad <b(A). \end{cases}$$

But, in view of (2.3.3), (ii) and (iii), we have

$$V_1(t_2, x(t_2))+V_2(t_2, x(t_2))\geq V_2(t_2, x(t_2))\geq b(|\, x(t_2)\, |)=b(A)$$

which contradicts (2.3.5). This proves the claim.

If $V_1(t, x)\equiv 0$, $g_1(t, u)\equiv 0$ so that $a_1(t, u)\equiv 0$, Theorem 2.3.1 reduces to Theorem 2.2.4. If, on the other hand, $V_2(t, x)\equiv 0$, $g_2(t, u)\equiv 0$ so that $a_2(u)\equiv 0$, $b(u)\equiv 0$ and $V_1(t, x)\geq b_1(|\, x\, |)$, $b_1\in K$, then Theorem 2.3.1 reduces to Theorem 2.2.1 provided $a_1(t_0, \lambda)<b_1(A)$. As it is, Theorem 2.3.1 shows the advantage of perturbing Lyapunov functions in obtaining nonuniform practical stability.

We shall next consider, in the same spirit, a result which yields practical asymptotic stability.

Theorem 2.3.2

Suppose that the hypotheses of Theorem 2.3.1 hold except that the estimate on $D^+V_1(t, x)$ is strengthened to

(2.3.6) $\qquad D^+V_1(t, x)\leq -C(W(t, x)),\ C\in K,\ (t, x)\in \mathbf{R}_+\times S(A),$

where $W\in C[\mathbf{R}_+\times\mathbf{R}^n, \mathbf{R}_+]$, $W(t, x)$ is locally Lipschitzian in x and positive definite, and $D^+W(t, x)$ is bounded from above or from below. Then, the system (2.1.1) is practically asymptotically stable.

Proof

It is clear from (2.3.6) that $D^+V_1(t, x)\leq 0$ which means that $g_1(t, u)\equiv 0$ so that condition (v) corresponding to (2.3.1) is satisfied. Consequently, we have because of Theorem 2.3.1, practical stability of the system (2.1.1).

By repeating the proof of Theorem 2.2.3 relative to showing $\lim_{t\to\infty} W(t,x(t))=0$ for all the solutions $x(t)$ such that $|x_0|<\lambda$ and noting that $W(t, x)$ is positive definite, we see that $\lim_{t\to\infty} x(t)=0$. Hence the proof is complete.

2.4. SEVERAL LYAPUNOV FUNCTIONS.

As we have seen, utilizing a single Lyapunov function, one can investigate several results of practical stability in a unified way. Also, it is beneficial to use more than one Lyapunov-like function in obtaining results under weaker conditions. Hence, it is natural to ask whether employing several Lyapunov functions is more advantageous, in some situations. The answer is affirmative and the corresponding approach offers a more flexible mechanism with each Lyapunov function satisfying less rigid requirements. Naturally, when we employ vector Lyapunov functions, Theorem 2.1.5 plays an important role. As a typical result, we shall merely state a theorem that gives sufficient conditions, in terms of vector Lyapunov functions, for practical stability properties of the system (2.1.1).

Theorem 2.4.1

Assume that

(i) $0<\lambda<A$;

(ii) $V\in C[\mathbf{R}_+\times\mathbf{R}^n, \mathbf{R}_+^N]$, $V(t, x)$ is locally Lipschitzian in x, the function

$$V_0(t, x)=\sum_{i=1}^{N} V_i(t, x) \tag{2.4.1}$$

satisfies the estimates

$$b(|x|) \le V_0(t, x) \le a(|x|),\ a,\ b \in K$$

and

$$D^+V(t, x) \le g(t, V(t, x)),$$

for $(t, x) \in \mathbf{R}_+ \times S(A)$, where $g \in C[\mathbf{R}_+ \times \mathbf{R}_+^N, \mathbf{R}^N]$ and $g(t, u)$ is quasimonotone nondecreasing in u;

(iii) $a(\lambda) < b(A)$ holds.

Then the practical stability properties of the system

$$u' = g(t, u),\ u(t_0) = u_0 \ge 0, \tag{2.4.2}$$

imply the corresponding practical stability properties of the system (2.1.1).

We note that in Theorem 2.4.1, we have used the function $V_0(t, x)$ defined by (2.4.1) as a measure and consequently, we need to modify the definition of practical stability of (2.4.2) as follows: for example, (2.4.2) is practically stable with respect to $(a(\lambda), b(A))$ if $\sum_{i=1}^{N} u_{0i} < a(\lambda)$ implies $\sum_{i=1}^{N} u_i(t, t_0, u_0) < b(A)$, $t \ge t_0$, where $u(t, t_0, u_0)$ is any solution of the comparison system (2.4.2). We could use other convenient measures such as

$$V_0(t, x) = \max_{1 \le i \le N} V_i(t, x),$$

$$V_0(t, x) = \sum_{i=1}^{N} d_i V_i(t, x), \text{ where } d > 0 \text{ is a vector,}$$

or

$$V_0(t, x) = Q(V(t, x)), \text{ where } Q: \mathbf{R}_+^N \to \mathbf{R}_+ \text{ and } Q(u) \text{ is nondecreasing in } u,$$

and appropriate modifications of practical stability definitions are employed for the system (2.4.2).

To demonstrate the advantage of employing several Lyapunov functions, let us consider the following example.

Example 2.4.1

Consider the system

$$(2.4.3)\qquad \begin{cases} x' = e^{-t}x + y\,\sin t - (x^3 + xy^2)\sin^2 t, \\ y' = x\,\sin t + e^{-t}y - (x^2y + y^3)\sin^2 t. \end{cases}$$

Suppose that we choose a single Lyapunov function $V(t, x, y) = x^2 + y^2$. Then, using the inequality $2|ab| \le a^2 + b^2$ and observing that $(x^2+y^2)^2 \sin^2 t \ge 0$, we get

$$D^+V(t, x, y) \le 2[e^{-t} + |\sin t|]V(t, x, y).$$

It is clear that

$$u' = 2[e^{-t} + |\sin t|]u$$

is not practically stable and consequently, we cannot deduce any information about the practical stability of the system (2.4.3) from Theorem 2.2.1, even though the system (2.4.3) is practically stable.

Now, let us take two functions V_1, V_2 defined by $V_1(t,x,y) = \frac{1}{2}(x+y)^2$, $V_2(t, x, y) = \frac{1}{2}(x-y)^2$ so that $V_0(t, x, y) = x^2 + y^2$. This means that we can take $a(u) = b(u) = u^2$. Furthermore, the vectorial inequality

$$D^+V(t, x, y) \le g(t, V(t, x, y))$$

is satisfied with $g=(g_1, g_2)$, where

$$g_1(t, u_1, u_2)=2(e^{-t}+ \sin t)u_1,$$

$$g_2(t, u_1, u_2)=2(e^{-t}- \sin t)u_2.$$

It is obvious that the function $g(t, u)$ is quasimontone nondecreasing in u and the comparison system (2.4.2) is practically stable for any $0<\lambda<A$ which satisfy, for example, $\exp(e^{-t_0}+2)<(\frac{A}{\lambda})^2$. Hence Theorem 2.4.1

implies that the system (2.4.3) is also practically stable.

2.5. LARGE SCALE DYNAMIC SYSTEMS.

We develop, in this section, a decomposition-aggregation method for practical stability analysis of large scale dynamic systems in the context of the method of vector Lyapunov functions. Let us consider dynamic systems that are composed of a number of interconnected subsystems. Mathematical descriptions of such systems are obtained by adding more structure to the system (2.1.1) and writing it in the form

$$\text{(2.5.1)} \qquad \begin{cases} x_i' = F_i(t, x_i)+R_i(t, x),\ x_i(t_0)=x_{i0}, \\ i= 1, 2, \ldots, k. \end{cases}$$

In this description, the functions $F_i(t, x_i)$ represent isolated (uncoupled) subsystems and $R_i(t, x)$ the interactions. Here $x=(x_1, x_2, \ldots, x_k)$ with each $x_i \in \mathbf{R}^{n_i}$ such that $\sum_{i=1}^{k} n_i = n$.

In order to construct an aggregate model for the overall system (2.1.1), we shall impose constraints on the interaction functions $R_i(t, x)$. We define

$$R_i(t, x) \equiv R_i(t, e_{i1}x_1, e_{i2}x_2, \ldots, e_{ik}x_k), \tag{2.5.2}$$

where $e_{ij}: \mathbf{R}_+ \to [0, 1]$ are continuous functions representing the elements of a $k \times k$ interconnection matrix E. Then we can prove the following result.

Theorem 2.5.1

(i) $0 < \lambda < A$;

(ii) $V_i \in C[\mathbf{R}_+ \times \mathbf{R}^{ni}, \mathbf{R}_+]$, $V_i(t, x_i)$ is locally Lipschitzian in x_i for a constant $L_i > 0$ and for $(t, x) \in \mathbf{R}_+ \times S(A)$, $i = 1, 2, \ldots, k$,

$$b_i(|x_i|) \le V_i(t, x_i), \ b_i \in K,$$

$$D^+V_i(t, x_i)_{(2.5.3)} \le G_i(t, V_i(t, x_i)),$$

$$\sum_{i=1}^{k} b_i(|x_i|) \ge B(|x|), \ \sum_{i=1}^{k} V_i(t, x_i) \le C(|x|),$$

where $C, B \in K$ and $G_i \in C[\mathbf{R}_+^2, \mathbf{R}]$, with $D^+V_i(t, x_i)_{(2.5.3)}$ being the generalized derivative of $V_i(t, x_i)$ with respect to the subsystems

$$x_i' = F_i(t, x_i), \ x_i(t_0) = x_{i0}; \tag{2.5.3}$$

(iii) $|R_i(t, x)| \le H_i(t, e_{i1}|x_1|, e_{i2}|x_2|, \ldots, e_{ik}|x_k|)$ where $H_i \in C[\mathbf{R}_+ \times \mathbf{R}^k, \mathbf{R}_+]$ and $H_i(t, u)$ is nondecreasing in u, $i = 1, 2, \ldots, k$;

(iv) $C(\lambda) < B(A)$ holds.

Then the practical stability properties of

$$(2.5.4)\qquad u' = g(t, u),\ u(t_0) = u_0 \geq 0,$$

where $g_i(t, u_1, \ldots, u_k) = G_i(t, u_i) + L_i H_i(t, e_{i1} b_1^{-1}(u_1), \ldots, e_{ik} b_k^{-1}(u_k))$, b_i^{-1} being the inverse function of b_i and L_i are the Lipschitz constants for V_i, imply the corresponding practical stability properties of the large scale dynamic system (2.5.1).

Proof

For $(t, x) \in \mathbf{R}_+ \times S(A)$, we compute $D^+V_i(t, x_i)_{(2.5.1)}$ so that in view of the assumptions, we get

$$(2.5.5)\qquad \left|\begin{aligned} & D^+V_i(t, x_i)_{(2.5.1)} \leq D^+V_i(t, x_i)_{(2.5.3)} + L_i |R_i(t, x)| \\ & \leq G_i(t, V_i(t, x_i)) + L_i H_i(t, e_{i1}|x_1|, \ldots, e_{ik}|x_k|) \\ & \leq G_i(t, V_i(t, x_i)) + L_i H_i(t, e_{i1} b_1^{-1}(V_1(t, x_1)), \ldots) \\ & \equiv g_i(t, V_1(t, x_1), \ldots, V_k(t, x_k)). \end{aligned}\right.$$

Note that $g(t, u)$ is quasimonotone nondecreasing in u because of the assumptions on $H(t, u)$ and $G_i(t, u_i)$. Now, given $(C(\lambda), B(A))$, if we suppose that (2.5.4) is practically stable, then we have

$$\sum_{i=1}^{k} u_{0i} < C(\lambda) \text{ implies } \sum_{i=1}^{k} u_i(t, t_0, u_0) < B(A),\ t \geq t_0.$$

Let $|x_0| < \lambda$. We claim that $|x(t)| < A$, $t \geq t_0$. If this is not true, then there exist a $t_1>t_0$ and a solution $x(t, t_0, x_0)$ of (2.5.1) such that

$$|x(t_1, t_0, x_0)|=A \text{ and } |x(t, t_0, x_0)|\leq A, \; t_0\leq t\leq t_1.$$

This means that

$$\sum_{i=1}^{k} V_i(t_1, x_i(t_1, t_0, x_0))\geq B(A).$$

choosing $u_{0i}=V_i(t_0, x_{i0})$, we have by Theorem 2.1.5 and the inequality (2.5.5), the estimate

$$V_i(t, x_i(t, t_0, x_0))\leq r_i(t, t_0, V(t_0, x_0)), \; t_0\leq t\leq t_1,$$

where $r(t, t_0, u_0)$ is the maximal solution of (2.5.4). By (ii), we then get the contradiction

$$B(A)\leq\sum_{i=1}^{k} V_i(t_1, x_i(t_1, t_0, x_0))\leq\sum_{i=1}^{k} r_i(t_1, t_0, u_0)<B(A),$$

proving the practical stability of the system (2.5.1). Hence the proof is complete.

In some situations, it may be more convenient to define $V(t,x)=\sum_{i=1}^{k} d_i V_i(t, x_i)$, for some vector $d>0$, as a single Lyapunov function and compute $D^+V(t, x)$ relative to the system (2.5.1). In this case, the comparison system (2.5.4) reduces to a single comparison equation. In some cases, one may be able to test directly the sign definiteness of $D^+V(t, x)$ to prove practical stability.

2.6. GENERAL DEFINITIONS OF PRACTICAL STABILITY.

In Definition 1.2.1, the concepts of practical stability have been defined in terms of neighborhoods of the origin. This, in general, is not necessary, since there is no need to assume the existence of the trivial solution of the differential system involved to discuss practical stability. Consequently, it is useful to define practical stability notions relative to arbitrary sets or more generally, arbitrary tubes.

For any set $A \subset \mathbf{R}^n$, we denote by A^0 and ∂A the interior and the boundary of A respectively. Let Ω denote the set of all non-empty closed subsets of $\mathbf{R}^n$. Together with the Hausdorff distance ρ given by

$$\rho(A, B) = \max\left[\sup_{x \in A} \rho(x, B), \quad \sup_{y \in B} \rho(y, A)\right]$$

whose values are in the extended real line, the set Ω becomes a metric space. Let $C[\mathbf{R}_+, \Omega]$ denote the space of continuous set-valued functions defined form $\mathbf{R}_+$ into Ω. Let $x(t, t_0, x_0)$ be any solution of (2.1.1).

Definition 2.6.1.

Let $S_0, S \in C[\mathbf{R}_+, \Omega]$ such that $S_0(t) \subset S(t)$ and $\partial S_0(t) \cap \partial S(t) = \phi$ (empty set) for $t \in \mathbf{R}_+$. Then the system (2.1.1) is said to be

(PS_1) practically stable, if for some $t_0 \in \mathbf{R}_+$,

$$x_0 \in S_0(t_0) \text{ implies } x(t, t_0, x_0) \in S^0(t), \ t \geq t_0;$$

(PS_2) uniformly practically stable if (PS_1) holds for all $t_0 \in \mathbf{R}_+$;

(PS_3) practically quasi-stable, if for some $t_0 \in \mathbf{R}_+$ and a given $T > 0$

$$x_0 \in S_0(t_0) \text{ implies } x(t, t_0, x_0) \in S^0(t), \ t \geq t_0 + T;$$

(PS_4) uniformly practically quasi-stable if (PS_3) holds for all $t_0 \in \mathbf{R}_+$.

Based on these definitions and the ones given in Definition 1.2.1, it is easy to formulate other concepts of practical stability. For example, strong practical stability of the system (2.1.1) would imply that, given sets S_0, S, $D \in C[\mathbf{R}_+, \Omega]$ and $T > 0$ satisfying $S_0(t) \subset S(t)$, $\partial S_0(t) \cap \partial S(t) = \phi$ for $t \in \mathbf{R}_+$, it follows that

$$x_0 \in S_0(t_0) \text{ implies } x(t) \in S^0(t),\ t \geq t_0 \text{ and } x(t) \in D(t),\ t \geq t_0 + T,$$

where $x(t) = x(t, t_0, x_0)$ is any solution of (2.1.1). As mentioned earlier, if $D(t) \subset S_0(t)$, $t \in \mathbf{R}_+$, we have contractive practical stability and if $S_0(t) \subset D(t) \subset S(t)$, $t \in \mathbf{R}_+$, we have expansive practical stability. Other remarks follow as before.

To unify various Lyapunov stability concepts and to offer a general framework for study, the stability concepts defined in terms of two measures have been employed fruitfully. This idea is also useful in the study of practical stability since it can be used to describe various sets and to provide sufficient conditions for practical stability. We need the following class of functions:

$$\Gamma = \{h \in C[\mathbf{R}_+ \times \mathbf{R}^n, \mathbf{R}_+]: \quad \inf_{(t,\, x)} h(t, x) = 0\}.$$

Definition 2.6.2

Let h_0, $h \in \Gamma$. Then the system (2.1.1) is said to be

(PS_1) (h_0, h) - practically stable if given $0 < \lambda < A$, $h_0(t, x) < \lambda$ implies $h(t, x(t, t_0, x_0)) < A$, $t \geq t_0$, for some $t_0 \in \mathbf{R}_+$;

(PS_2) (h_0, h) - uniformly practically stable if (PS_1) holds for all $t_0 \in \mathbf{R}_+$.

Based on this definition, it is easy to formulate other notions of practical stability in terms of two measures. Furthermore, the sets $S_0(t)$, $S(t)$ given in Definition 2.6.1 can be described by means of h_0, h, as follows:

$$S_0(t)=\{x\in\mathbf{R}^n:\ h_0(t,x)\le\lambda,\ t\in\mathbf{R}_+\},$$

$$S(t)=\{x\in\mathbf{R}^n:\ h(t,x)\le A,\ t\in\mathbf{R}_+\}.$$

A few choices of two measures (h_0, h) will help to emphasize the generality of the Definition 2.6.2.

(i) $h(t, x)=h_0(t, x)=|x|$ yields practical stability notions defined in Definition 1.2.1;

(ii) partial practical stability concepts are obtained if $h_0(t,x)=|x|$ and $h(t, x)=|x|_s$, $0\le s<n$;

(iii) eventual practical stability concepts are realized if $h(t,x)=h_0(t,x)=|x|+\sigma(t)$, $\sigma\in L$;

(iv) conditional practical stability notions are valid if $h(t,x)=d(x,A)$, $h_0(t, x)=d(x, B)$ where $A\subset B\subset \mathbf{R}^n$ and d is the distance function.

When we utilize the two measures h_0, h we need to assume suitable relation between these functions so that the definitions make sense.

Definition 2.6.3

Let $h_0, h\in\Gamma$. Then we say that

(i) h_0 is finer than h, if given $\lambda>0$ there exists a function $\Phi\in CK$ such that $h(t, x)\le\Phi(t, h_0(t, x))$ whenever $h_0(t, x)<\lambda$;

(ii) h_0 is uniformly finer than h, if in (*i*), $\Phi\in K$;

(iii) h_0 is asymptotically finer than h, if given $\lambda>0$ there exists a function $\Phi\in LK$ such that $h(t, x)\le\Phi(t, h_0(t, x))$ provided $h_0(t,x)<\lambda$.

We also require to relate Lyapunov functions to the two measures.

Definition 2.6.4

Let $V \in C[\mathbf{R}_+ \times \mathbf{R}^n, \mathbf{R}_+]$ and $h \in \Gamma$. Then V is said to be

(i) h-positive definite if given $\lambda > 0$, there exists a function $b \in K$ such that $b(h(t, x)) \leq V(t, x)$ provided $h(t, x) < \lambda$;

(ii) weakly h-decrescent if given $\lambda > 0$, there exists a function $a \in CK$ such that $V(t, x) \leq a(t, h(t, x))$ provided $h(t, x) < \lambda$;

(iii) h-decrescent if in (ii), $a \in K$;

(iv) h-asymptotically decrescent if given $\lambda > 0$ there exists a function $a \in LK$ such that $V(t, x) \leq a(t, h(t, x))$ whenever $h(t, x) < \lambda$.

2.7. STABILITY CRITERIA IN TERMS OF TWO MEASURES.

Let us now establish some sufficient conditions for the (h_0, h)- practical stability of the system (2.1.1).

Theorem 2.7.1

Assume that

(i) $0 < \lambda < A$;

(ii) $h_0, h \in \Gamma$ and h_0 is finer than h i.e. $h(t, x) \leq \Phi(h_0(t, x))$, $\Phi \in K$ whenever $h_0(t, x) < \lambda$;

(iii) $V \in C[\mathbf{R}_+ \times \mathbf{R}^n, \mathbf{R}_+]$, $V(t, x)$ is locally Lipschitzian in x and satisfies

(2.7.1) $$b(h(t, x)) \leq V(t, x) \text{ if } h(t, x) < A, \; b \in K,$$

(2.7.2) $$V(t, x) \leq a(h_0(t, x)) \text{ if } h_0(t, x) < \lambda, \; a \in K,$$

and the differential inequality

(2.7.3) $$D^+V(t, x) \leq g(t, V(t, x)), \; (t, x) \in S(h, A),$$

where $g \in C[\mathbf{R}_+^2, \mathbf{R}]$ and $S(h, A) = \{(t, x) \in \mathbf{R}_+ \times \mathbf{R}^n : h(t, x) < A\}$;

(iv) $\Phi(\lambda) < A$ and $a(\lambda) < b(A)$ hold.

Then the practical stability properties of (2.2.2) imply the corresponding (h_0, h) - practical stability properties of the system (2.1.1).

Proof

Suppose that the equation (2.2.2) is practically stable with respect to $(a(\lambda), b(A))$ so that we have

(2.7.4) $$u_0 < a(\lambda) \text{ implies } u(t, t_0, u_0) < b(A),\ t \geq t_0.$$

Then we claim that the system (2.1.1) is (h_0, h) - practically stable with respect to (λ, A). If this is not true, then there would exist a $t_1 > t_0$ and a solution $x(t) = x(t, t_0, x_0)$ of (2.1.1) such that

(2.7.5) $$h_0(t_0, x_0) < \lambda,\ h(t_1, x(t_1)) = A \text{ and } h(t, x(t)) < A,\ t_0 \leq t < t_1.$$

Since in view of (ii) and (iv), we have $h(t_0, x_0 \leq \Phi(h_0(t_0, x_0)) < \Phi(\lambda) < A$. Hence, by Theorem 2.7.1, we get because of (2.7.3),

(2.7.6) $$V(t, x(t)) \leq r(t, t_0, u_0),\ t_0 \leq t < t_1,$$

where $r(t, t_0, u_0)$ is the maximal solution of (2.2.2) such that $u_0 = V(t_0, x_0)$. Now, the relations (2.7.2) - (2.7.6) yield, using the continuity of the functions involved, the following contradiction.

$$b(A) = b(h(t_1, x(t_1))) \leq V(t_1, x(t_1)) \leq r(t_1, t_0, V(t_0, x_0))$$
$$\leq r(t_1, t_0, a(h_0(t_0, x_0))) \leq r(t_1, t_0, a(\lambda)) < b(A).$$

This proves (h_0, h) - practical stability of the system (2.1.1). Other

(h_0,h)- practical stability properties may be proved similarly based on the foregoing proof and the corresponding proofs of theorems in Section 2.2. We omit the details. Hence the proof is complete.

Theorem 2.7.1 is clearly a generalization of Theorem 2.2.1. As mentioned earlier, for various choices of h_0, h, Theorem 2.7.1 provides sufficient conditions for several different practical stability concepts, most important of which is partial practical stability. Other theorems of Section 2.2 can easily be stated and proved in the framework of Theorem 2.7.1 which we shall omit to avoid repetition. However, we shall prove a result giving (h_0, h) - practical asymptotic stability, which is in the spirit of results in Section 2.3. Furthermore, its assumptions are less restrictive and yet yields more information than Theorem 2.2.3.

Theorem 2.7.2

Assume that (i), (ii) of Theorem 2.7.1 hold. Suppose further

(iii) $V_1 \in C[S(h, A), \mathbf{R}_+]$, $V_1(t, x)$ is locally Lipschitzian in x and $V_1(t, x) \leq \psi(t, h_0(t, x))$ if $h_0(t, x) < \lambda$, where $\psi \in CK$;

(iv) $V_2 \in C[S(h, A) \cap S^C(h_0, \lambda), \mathbf{R}_+]$, $V_2(t, x)$ is locally Lipschitzian in x and for $(t, x) \in S(h, A) \cap S^C(h_0, \lambda)$,

$$b(h(t, x) \leq V_2(t, x) \text{ if } h(t, x) < A, \; b \in K,$$

$$V_2(t, x) \leq a(h_0(t, x)) \text{ if } h_0(t, x) < \lambda, \; a \in K$$

and

$$D^+V_1(t, x) + D^+V_2(t, x) \leq 0;$$

(v) $\Phi(\lambda) < A$ and $\psi(t_0, \lambda) + a(\lambda) < b(A)$ hold;

(vi) V_3, $V_4 \in C[S(h, A), \mathbf{R}_+]$ such that $V_1 = V_3 + V_4$, $V_3(t, x)$ is h - positive definite and on $S(h, A)$,

$$D^+V_1(t, x) \leq -\lambda(t)C(V_3(t, x))$$

where $C \in K$ and $\lambda \in C[\mathbf{R}_+, \mathbf{R}_+]$ is integrally positive, that is,

$\int_I \lambda(s)ds = \infty$ whenever $I = \bigcup_{i=1}^{\infty}[\alpha_i, \beta_i]$, $\alpha_i < \beta_i < \alpha_{i+1}$ and

$\beta_i - \alpha_i \geq \delta > 0$;

(vii) for every $y \in C[\mathbf{R}_+, \mathbf{R}^n]$ with $h(t, y(t)) < A$, the function

$\int_0^t [D^+V_4(s, y(s))]_{\pm} ds$ is uniformly continuous on $\mathbf{R}_+$, where

$[\cdot]_{\pm}$ means that either positive or negative part is considered for all $s \in \mathbf{R}_+$.

Then the system (2.1.1) is (h_0, h) - practically asymptotically stable and $\lim_{t\to\infty} V_4(t, x(t))$ exists and is finite for any solution $x(t)$ of (2.1.1).

Proof

We shall first show that the system (2.1.1) is (h_0, h) - practically stable, which means,

(2.7.7) $$h_0(t_0, x_0) < \lambda \text{ implies } h(t, x(t)) < A,\ t \geq t_0.$$

We note that whenever $h_0(t_0, x_0) < \lambda$, we have because of (ii) and (iv),

$$h(t_0, x_0) \leq \Phi(h_0(t_0, x_0)) \leq \Phi(\lambda) < A.$$

Hence, if the (h_0, h) - practical stability of (2.1.1) does not hold, then there would exist a solution $x(t)=x(t, t_0, x_0)$ of (2.1.1) and $t_2>t_1>t_0$ such that

$$(2.7.8) \qquad \begin{cases} h(t_2, x(t_2))=A,\ h_0(t_1, x(t_1))=\lambda \text{ and} \\ x(t)\in S(h, A)\cap S^C(h_0, \lambda) \text{ for } t_1\le t\le t_2. \end{cases}$$

Setting $m(t)=V_1(t, x(t))+V_2(t, x(t))$, we obtain, in view of (iv),

$$D^+m(t)\le 0,\ t_1\le t\le t_2,$$

from which results

$$(2.7.9) \qquad m(t)\le m(t_1),\ t_1\le t\le t_2.$$

Similarly, (vi) gives

$$V_1(t, x(t))\le V_1(t_0, x_0),\ t_0\le t\le t_1,$$

which implies by (iii)

$$(2.7.10) \qquad V_1(t_1, x(t_1))\le \psi(h_0(t_0, x_0))<\psi(t_0, \lambda).$$

Also by (iv), we have

$$V_2(t_1, x(t_1))\le a(h_0(t_1, x(t_1)))=a(\lambda).$$

Hence it follows that

$$m(t_1)\le \psi(t_0, \lambda)+a(\lambda)<b(A)$$

because of (2.7.10) and (v). But $m(t_2)\ge b(h(t_2, x(t_2)))=b(A)$ by (2.7.8) and (iv). Consequently, we arrive at a contradiction

$$b(A) \le m(t_2) \le m(t_1) \le \psi(t_0, \lambda) + a(\lambda) < b(A)$$

because of (2.7.9). Thus, (h_0, h) - practical stability of the system (2.1.1) is established.

To prove the theorem, it remains to be proved that $\lim\limits_{t\to\infty} h(t,x(t))=0$ and $\lim\limits_{t\to\infty} V_4(t, x(t))=\sigma<\infty$. For this purpose, let x(t) be any solution of (2.1.1) satisfying (2.7.7). Define the functions $m_1(t)=V_1(t, x(t))$, $m_3(t)=V_3(t, x(t))$ and $m_4(t)=V_4(t, x(t))$ so that $m_1(t)=m_3(t)+m_4(t)$. Assumptions (iii) and (vi) yield that $m_1(t)$ is nonincreasing and bounded from below and therefore $\lim\limits_{t\to\infty} m_1(t)=\sigma<\infty$. We claim that $\lim\limits_{t\to\infty} \inf m_3(t)=0$. If this is not true, there would exist a $\beta>0$ and a $T>t_0$ such that

$$m_3(t) \ge \beta,\ t \ge T. \tag{2.7.11}$$

By (vi) and (2.7.11), it then follows that

$$D^+m_1(t) \le -\lambda(t)C(m_3(t)) \le -\lambda(t)C(\beta),\ t \ge T.$$

Thus, for $I=\bigcup\limits_{i=1}^{\infty}[\alpha_i, \beta_i]$ such that $T<\alpha_i<\beta_i<\alpha_{i+1}$, $\beta_i-\alpha_i \ge \beta$, we get a contradiction

$$\lim_{t\to\infty} m_1(t) \le m_1(T) - C(\beta \int_T^{\infty} \lambda(s)ds \le m_1(T) - C(\beta) \int_I \lambda(s)ds = -\infty.$$

Suppose now that $\lim\limits_{t\to\infty} \sup m_3(t)>0$. Then there exists a $\gamma>0$ such that $\lim\limits_{t\to\infty} \sup m_3(t)>3\gamma$. Since $\lim\limits_{t\to\infty} m_1(t)=\sigma$ and $m_1(t)$ is nonincreasing, there exists an $M>0$ such that

$$\sigma \le m_1(t) \le \sigma + \gamma,\ t \ge t_1 + M. \tag{2.7.12}$$

For definiteness, suppose that assumption (vii) holds with $[\ \cdot\ ]_+$. Since $m_3(t)$ is continuous, we can choose a sequence

$$t_0 + M < t_1^{(1)} < t_1^{(2)} < \ldots < t_i^{(1)} < t_i^{(2)} < \ldots$$

such that for $i = 1, 2, \ldots,$

$$\left\{ \begin{array}{l} m_3(t_i^{(1)}) = 3\gamma,\ m_3(t_i^{(2)}) = \gamma \text{ and} \\ \gamma \le m_3(t) \le 3\gamma,\ t \in [t_i^{(1)}, t_i^{(2)}]. \end{array} \right. \tag{2.7.13}$$

From (2.7.12) and (2.7.13), it is easy to see that

$$\left\{ \begin{array}{l} m_1(t_i^{(1)}) - m_3(t_i^{(1)}) \le \sigma - 2\gamma, \\ m_1(t_i^{(2)}) - m_3(t_i^{(2)}) \ge \sigma - \gamma. \end{array} \right. \tag{2.7.14}$$

Since $m_1(t) = m_3(t) + m_4(t)$, it follows from (2.7.14) that

$$0 < \gamma \le m_4(t_i^{(2)}) - m_4(t_i^{(1)}) \le \int_{t_i^{(1)}}^{t_i^{(2)}} [D^+ m_4(s)]_+ ds,$$

which shows by (vii) that there exists a $d > 0$ such that

$$t_i^{(2)} - t_i^{(1)} \ge d,\ i = 1, 2, \ldots. \tag{2.7.15}$$

By (2.7.13), (2.7.15) and (vi), we then get

$$\lim_{t \to \infty} m_1(t) \le m_1(t_0 + M) - \int_{t_0 + M}^{\infty} C(\gamma)\lambda(s) ds$$

$$\le m_1(t_0 + M) - C(\gamma) \int_I \lambda(s) ds = -\infty,$$

where $I=\bigcup_{i=1}^{\infty}[t_i^{(1)}, t_i^{(2)}]$. This contradiction implies $\lim_{t\to\infty} m_3(t)=0$ and since V_3 is h - positive definite, we get in turn $\lim_{t\to\infty} h(t,x(t))=0$. Thus, we conclude the system (2.1.1) is (h_0, h) - practically asymptotically stable. To prove the last assertion of the theorem, note that

$$\lim_{t\to\infty} m_1(t)=\sigma \text{ and } \lim_{t\to\infty} m_3(t)=0$$

and consequently $\lim_{t\to\infty} m_4(t)=\sigma$. The proof is therefore complete.

2.8. GLOBAL RESULTS IN TERMS OF SETS.

As we have seen, the proofs of many results in the theory of practical stability depend on showing that solutions cannot leave some prescribed sets or to estimate the escape time. This observation makes it possible to give some global results in terms of arbitrary sets which can be employed as tools in dealing with various problems of stability. In applications, these tools enlarge the class of useful Lyapunov functions and also offer more flexibility.

We need the following notion concerning the comparison equation (2.2.2) before we proceed further.

Let for some $a\in C[\mathbf{R}_+, \mathbf{R}]$, $S(t)$ denote the set

$$S(t)=\{u\in\mathbf{R}:\ u<a(t),\ t\in\mathbf{R}_+\}.$$

We say that the set $S(t)$ is practically stable for the equation (2.2.2) if for some $t_0\in\mathbf{R}_+$,

$$u_0\in S(t_0) \text{ implies } u(t, t_0, u_0)\in S^0(t),\ t\geq t_0,$$

where for each $t\in\mathbf{R}_+$, $S^0(t)$ is the interior of $S(t)$.

Other notions of practical stability can be defined similarly with respect to the set S(t). Recall that Ω denotes the set of all nonempty closed subsets of $\mathbf{R}^n$ and ρ is the Hausdorff distance. We are now in a position to prove some global results.

Theorem 2.8.1

Assume that

(i) the sets A, B, $E \in C[\mathbf{R}_+, \Omega]$, for each $t \in \mathbf{R}_+$, $A(t) \subset B^0(t) \subset B(t) \subset E^0(t)$ and $E^0(t)$ is a region;

(ii) $V \in C[\mathbf{R}_+ \times E^0(t), \mathbf{R}]$ and $V(t, x)$ is locally Lipschitzian in x;

(iii) $a \in C[\mathbf{R}_+, \mathbf{R}]$, $F(t) \subset \partial B(t)$, $t \in \mathbf{R}_+$ and for $(t, x) \in \mathbf{R}_+ \times F(t)$, $V(t,x) \geq a(t)$;

(iv) for some $t_0 \in \mathbf{R}_+$ and $x_0 \in A(t_0)$, $V(t_0, x_0) < a(t_0)$;

(v) $g \in C[\mathbf{R}_+ \times \mathbf{R}, \mathbf{R}]$ and $(t, x) \in \mathbf{R}_+ \times B^0(t)$, $D^+V(t, x) \leq g(t, V(t,x))$;

(vi) the set S(t) is practicaly stable with respect to (2.2.2).

Then

(a) there exists no $t^* > t_0$ such that

$$x(t, t_0, x_0) \in B^0(t),\ t \in [t_0, t^*) \text{ and } x(t^*, t_0, x_0) \in F(t^*);$$

(b) if in addition $F(t) \equiv \partial B(t)$, $t \in \mathbf{R}_+$, then the system (2.1.1) is (A(t),B(t)) - practically stable i.e. for any solution $x(t,t_0,x_0)$ of (2.1.1). $x_0 \in A(t_0)$ implies $x(t,t_0,x_0) \in B(t)$, $t \geq t_0$, for some $t_0 \in \mathbf{R}_+$.

Proof

Let $t_0 \in \mathbf{R}_+$ be some given initial time and let $x_0 \in A(t_0)$. This implies by (iv), $V(t_0, x_0)<a(t_0)$. Setting $u_0=V(t_0, x_0)$, we see that $u_0\in S(t_0)$. Consequently, we have by (vi),

$$u(t, t_0, u_0)<a(t),\ t\geq t_0, \tag{2.8.1}$$

where $u(t, t_0, u_0)$ is any solution of (2.2.2). Suppose that there exists a $t^*>t_0$ satisfying $x(t, t_0, x_0)\in B^0(t)$, $t\in[t_0, t^*)$ and $x(t^*, t_0, x_0)\in F(t^*)$. This, in view of (iii), gives

$$V(t^*, x(t^*, t_0, x_0))\geq a(t^*). \tag{2.8.2}$$

The choice of u_0 together with (ii) and (v) yields, on the basis of Theorem A.1.1, the inequality

$$V(t, x(t, t_0, x_0))\leq r(t, t_0, u_0),\ t\in[t_0, t^*), \tag{2.8.3}$$

where $r(t, t_0, u_0)$ is the maximal solution of (2.2.2). The continuity of functions involved in (2.8.3) show that

$$V(t^*, x(t^*, t_0, x_0))\leq r(t^*, t_0, u_0)$$

which, because of (2.8.1) and (2.8.2) leads to the contradiction

$$a(t^*)\leq V(t^*, x(t^*, t_0, x_0))\leq r(t^*, t_0, u_0)<a(t^*),$$

proving (a). If $F(t)\equiv\partial B(t)$, $t\in\mathbf{R}_+$, then the conclusion (b) is immediate and the proof is complete.

Theorem 2.8.2

Let the assumptions (i), (ii) and (iii) of Theorem 2.8.1 hold. Suppose that

(iv*) for every $(t, x)\in\mathbf{R}_+\times\partial A(t)$, $V(t, x)<a(t)$;

(v*) $g\in C[\mathbf{R}_+\times\mathbf{R}, \mathbf{R}]$ and for $(t, x)\in\mathbf{R}_+\times B^0(t)/A(t)$,

$D^+V(t, x) \leq g(t, V(t, x))$;

(vi*) the set $S(t)$ is uniformly practically stable with respect to (2.2.2).

Then, (a) there exists no $t^* > t_0$, for every $t_0 \in \mathbf{R}_+$ and $x_0 \in A(t_0)$ such that

(2.8.4) $\quad x(t, t_0, x_0) \in B^0(t),\ t \in [t_0, t^*)$ and $x(t^*, t_0, x_0) \in F(t^*)$;

if, in addition $F(t) \equiv \partial B(t)$, $t \in \mathbf{R}_+$, the system (2.1.1) is $(A(t), B(t))$ - uniformly practically stable.

Proof

Although the proof of this theorem is similar to that of Theorem 2.8.1, it requires a few technical details that are not apparent and hence, we sketch the proof.

Suppose, if possible, that there exists a $t^* > t_0$ such that (2.8.4) holds, where $(t_0, x_0) \in \mathbf{R}_+ \times A(t_0)$. Then, by continuity of the functions $A(t)$, $B(t)$ and $x(t, t_0, x_0)$, it follows that there is a $t_1 \in (t_0, t^*)$ satisfying

$$x_1 = x(t_1, t_0, x_0) \in \partial A(t_1),\ x(t, t_0, x_0) \in B^0(t)/A^0(t),\ t \in (t_1, t^*)$$

and

(2.8.5) $$x(t^*, t_0, x_0) \in F(t^*).$$

Setting $u_1 = V(t_1, x_1)$, we have because of (iv*), $V(t_1, x_1) < a(t_1)$. As before, the assumptions (ii) and (v*) give

(2.8.6) $$V(t, x(t, t_1, x_1)) \leq r(t, t_1, u_1),\ t \in [t_1, t^*).$$

where $x(t, t_1, x_1)$ is any solution of (2.1.1) through (t_1, x_1) and $r(t, t_1, u_1)$ is the maximal solution of (2.2.2) starting at (t_1, u_1). It follows that (2.8.6) is true for $x(t, t_0, x_0)$ on $[t_1, t^*)$. By continuity of the functions involved, (2.8.6) leads to

$$a(t^*) \leq V(t^*, x(t^*, t_0, x_0)) \leq r(t^*, t_1, u_1) < a(t^*)$$

in view of the assumptions (iii), (vi*) and (2.8.4). Hence (a) is proved and the part (b) is immediate.

The next result deals with strong practical stability.

Theorem 2.8.3

Assume that

(i) the sets A, B, $E \in C[\mathbf{R}_+, \Omega]$, for each $t \in \mathbf{R}_+$, $B(t) \subset E^0(t)$, $A(t) \subset E^0(t)$ and $E^0(t)$ is a region;

(ii) $V \in C[\mathbf{R}_+ \times E^0(t), \mathbf{R}]$ and $V(t, x)$ is locally Lipschitzian in x;

(iii) the differential system (2.1.1) is $(A(t), E(t))$ - practically stable;

(iv) $a \in C[\mathbf{R}_+, \mathbf{R}]$ and for $(t, x) \in \mathbf{R}_+ \times E^0(t)/B(t)$, $V(t, x) \geq a(t)$;

(v) $g \in C[\mathbf{R}_+ \times \mathbf{R}, \mathbf{R}]$ and for $(t, x) \in \mathbf{R}_+ \times E^0(t)$, $D^+V(t, x) \leq g(t, V(t, x)$;

(vi) the set $S(t)$ is practically quasi-stable relative to equation (2.2.2).

Then, the differential system (2.1.1) is $(A(t), E(t), B(t), T)$ - strongly practically stable.

Proof

Let $x_0 \in A(t_0)$ for some $t_0 \in \mathbf{R}_+$ and let $x(t, t_0, x_0)$ be any solution of (2.1.1). Then, by (iii), $x(t, t_0, x_0) \in E^0(t)$, $t \geq t_0$. Moreover, by (vi), given $S(t)$, $T>0$, we have

(2.8.7) $\quad u_0 < a(t_0)$ implies $u(t, t_0, u_0) < a(t)$, $t \geq t_0 + T$,

where $u(t, t_0, u_0)$ is any solution of (2.2.2). We now choose $u_0 = V(t_0, x_0)$.

Let $\{t_k\}$ be a sequence such that $t_k \geq t_0+T$ and $t_k \to \infty$ as $k \to \infty$. Assume, if possible, that there exists a solution $x(t, t_0, x_0)$ of (2.1.1) satisfying $x(t_k, t_0, x_0) \in E^0(t_k)/B(t_k)$. Then, in view of (iv), we get

$$V(t_k, x(t_k, t_0, x_0)) \geq a(t_k). \tag{2.8.8}$$

As a consequence of (ii) and (v), we have

$$V(t, x(t, t_0, x_0)) \leq r(t, t_0, u_0),\ t \geq t_0 \tag{2.8.9}$$

where $r(t, t_0, u_0)$ is the maximal solution of (2.2.2). Using (2.8.7), (2.8.8) and (2.8.9), we obtain

$$a(t_k) \leq V(t_k, x(t_k, t_0, x_0)) \leq r(t_k, t_0, u_0) < a(t_k)$$

which is a contradiction. This proves that $x(t, t_0, x_0) \in B(t)$, $t \geq t_0+T$, whenever $(t_0, x_0) \in \mathbf{R}_+ \times A(t_0)$ and in view of (iii), this shows that (2.1.1) is $(A(t), E(t), B(t), T)$ - strongly practically stable. The proof is complete.

Theorem 2.8.4

Let the assumptions (i), (ii), (iv) and (v) of Theorem 2.8.3 hold. Suppose further that

(iii*) the differential system (2.1.1) is $(A(t), E(t))$ - uniformly practically stable;

(vi*) the set $S(t)$ is uniformly practically quasi-stable relative to (2.2.2).

Then the differential system (2.1.1) is $(A(t), E(t), B(t), T)$ - strongly uniformly practically stable.

The proof is very similar to the proof of Theorem 2.8.3 and hence we omit the details.

2.9. STABILITY CRITERIA IN TERMS OF SETS.

Let the sets S_0, S of Definition 2.6.1 be the neighborhoods of the origin in $\mathbf{R}^n$. We need the following definition concerning Lyapunov functions.

Definition 2.9.1

A function $V \in C[\mathbf{R}_+ \times \mathbf{R}^n, \mathbf{R}_+]$ is said to be locally large if V(t, x) is positive definite and for any $\sigma \in (0, \delta)$, there exists a neighborhood $S_\delta \subseteq S$ of the origin such that $V(t, x) = \sigma$ for $(t, x) \in \mathbf{R}_+ \times S_\delta$ is a closed surface with respect to origin.

It is clear that $V = \frac{x^2}{(1+x^2)^2} + \frac{y^2}{(1+y^2)^2}$ is locally large on the set

$S_{1/5} = \{(x, y): \ | x | < 1/5, \ | y | < 1/5\}$, where as the function $V = x^2 + y^2 + \frac{1}{2}xy \sin t$ is not locally large but positive definite.

For a given set S, let us define

$$V_M^S(t) = \sup_{x \in S(t)} V(t, x), \text{ and } V_m^S(t) = \inf_{x \in S(t)} V(t, x).$$

we can prove the following results.

Theorem 2.9.1

Assume that

(i) $V \in C[\mathbf{R}_+ \times S(t), \mathbf{R}_+]$, V(t, x) is locally Lipschitzian in x and locally large;

(ii) $D^+V(t, x) < D^+\eta(t)$ for $(t, x) \in \mathbf{R}_+ \times S(t)$, where $\eta \in C[\mathbf{R}_+, (0, \infty)]$ and $\eta(t)$ is nondecreasing in $t \in \mathbf{R}_+$;

(iii) $\eta(t_0) \leq V_M^{S_0}(t_0)$ for some $t_0 \in \mathbf{R}_+$ and $\eta(t) \leq V_m^{\partial S}(t)$ for $t \geq t_0$.

Then the system (2.1.1) is practically stable for the sets (S_0, S).

Proof

Let $x(t, t_0, x_0)$ be any solution of (2.1.1) such that $x_0 \in S_0(t_0)$. We claim that the system (2.1.1) is (S_0, S) - practically stable. If this is not true, there would exist a $t_1 > t_0$ and a solution $x(t) = x(t, t_0, x_0)$ of (2.1.1) such that

$$x_0 \in S_0(t_0) \text{ and } x(t_1) \in \partial S(t_1).$$

The assumption (ii) then yields that

$$V(t_1, x(t_1)) < V(t_0, x_0) + \eta(t_1) - \eta(t_0).$$

Since $V(t, x)$ is locally large, the level surfaces of $V(t, x)$ are bounded and hence we get

$$V(t_1, x(t_1)) < \eta(t_1)$$

which implies that

$$V(t_1, x(t_1)) < V_m^S(t_1).$$

This is a contradiction and therefore, the theorem is proved.

Theorem 2.9.2

Assume that (i) of Theorem 2.9.1 holds. Suppose further

(ii*) $D^+V(t, x) < D^+\eta(t)$ for $t \in \mathbf{R}_+$ and $x \in S(t)/\overline{S}_0(t)$, η being the same function as defined in Theorem 2.9.1;

(iii*) $\eta(t_2) - \eta(t_1) \leq V_m^S(t_2) - V_M^{S_0}(t_1)$ for every $t_2 > t_1$, $t_1, t_2 \in \mathbf{R}_+$.

Then the system (2.1.1) is uniformly practically stable for (S_0, S).

Proof

As before, if the conclusion is not true, there would exists a $t_2>t_1>t_0$ and a solution $x(t)=x(t, t_0, x_0)$ of (2.1.1) with $x_0 \in S_0(t_0)$ such that

$$x(t_1)\in\partial S_0(t_1),\ x(t_2)\in\partial S(t_2) \tag{2.9.1}$$

and

$$x(t)\in S(t)/\overline{S}_0(t),\ t\in(t_1, t_2).$$

Then, we have by (ii*), the estimate

$$V(t_2, x(t_2))<V(t_1, x(t_1))+\eta(t_2)-\eta(t_1).$$

By (iii*), we get

$$V(t_2, x(t_2))<V(t_1, x(t_1))+V_m^S(t_2)-V_M^{S_0}(t_1).$$

Using the fact that $V(t, x)$ is locally large, we obtain

$$V(t_2, x(t_2))<V_m^S(t_2),$$

which implies $x(t_2)\notin\partial S(t_2)$ which contradicts (2.9.1). Hence the proof is complete.

To prove a result concerning practical instability, we need the following notation.

Let $S_1(t)\subset S_0(t)$ such that $\partial S_1\cap\partial S_0=\phi$ for $t\in\mathbf{R}_+$. Set

$$G(t)=\overline{S}(t)/S_1(t),$$

$$H(t)=\{x\in\mathbf{R}^n:\ V(t, x)>V_M^{S_1}(t),\ t\in\mathbf{R}_+\},$$

and

$$Q(t)=G(t)\cap H(t) \text{ which is nonempty for } t\in \mathbf{R}_+,$$

such that

$$Q(t+\tau_1)\cap \partial S(t_0+\tau_1) \text{ is not empty for } \tau_1\in[t_0, \infty).$$

Theorem 2.9.3

Let the assumption (i) of Theorem 2.9.1 hold. Suppose further that

(a) $D^+V(t, x)>D^+\eta(t)$, $t\in\mathbf{R}_+$ and $x\in Q(t)$;

(b) $\eta(t_0)\le V_m^{S_1}(t_0)$, $\eta(t_1)\ge V_M^{\partial S_1}(t_1)$ for all $t_1\in[t_0, t_0+\tau_1]$,

$\eta(t_0+\tau_1)\ge V_M^{\partial S}(t_0+\tau_1)$ and $V(t_0+\tau_1, x)\le V_M^{\partial S}(t_0+\tau_1)$ for

all $x\in Q(t_0+\tau_1)$ where $\tau_1\in[t_0, \infty)$.

Then the system (2.1.1) is practically unstable with respect to sets (S_0, S).

Proof

Consider any solution $x(t)=x(t, t_0, x_0^*)$ of (2.1.1) such that

$$x_0^*\in Q(t_0) \text{ and } x_0^*\in S_0(t_0)/\bar{S}_1(t_0).$$

Let $t_*\in[t_0, t_0+\tau_1]$ be the first time for which $V(t_*, x(t_*))=V_M^{\partial S_1}(t_*)$. From (a), we then get

$$V(t_*, x(t_*))>V_M^{S_1}(t_0)+\eta(t_*)-\eta(t_0),$$

and using (b), we have

$$V(t_*, x(t_*))>V_m^{S_1}(t_*).$$

This is a contradiction and hence t_* does not exist on $[t_0, t_0+\tau_1]$.
Since τ_1 is arbitrary,

$$V(t, x(t)) > V_M^{\partial S_1}(t)$$

for all $t \in [t_0, t_0+\tau_1]$. This shows that if $x(t, t_0, x_0^*) \in S(t)$ for all $t \in [t_0, t_0+\tau_1]$, then $x(t, t_0, x_0^*) \in Q(t)$ for all $t \in [t_0, t_0+\tau_1]$.

Suppose now that for all $t \in [t_0, t_0+\tau_1]$, $x(t, t_0, x_0^*) \in S(t)$. Then,

$$V(t_0+\tau_1, x(t_0+\tau_1)) > V_m^{\partial S}(t_0) + \eta(t_0+\tau_1) - \eta(t_0).$$

Hence by(b), we find

$$V(t_0+\tau_1, x(t_0+\tau_1)) > V_M^{\partial S}(t_0+\tau_1)$$

which is again a contradiction to (b) which implies that $x(t, t_0, x_0^*) \notin S(t)$ for all $t \in [t_0, t_0+\tau_1]$. Consequently, we can find a $t^* \in [t_0, t_0+\tau_1]$ such that $x(t^*, t_0, x_0^*) \in \partial S(t^*)$ and this proves practical instability of the system (2.1.1) proving the theorem.

2.10. NOTES.

In Section 2.1, we present necessary comparison results in terms of scalar and vector Lyapunov functions which are taken from Lakshmikantham and Leela [1], and Lakshmikantham, Leela and Martynyuk [1]. All the results on practical stability considered in Section 2.2 are new. The idea of perturbed Lyapunov functions introduced in Lakshmikantham and Leela [3] is utilized in Section 2.3 to investigate practical stability and thus all the results discussed in Section 2.3 are new. Theorem 2.4.1 establishes general connection between practical stability of the comparison system and the given system in terms of vector Lyapunov function. Example 2.4.1 and the idea of vector Lyapunov functions is due to Matrosov, see Voronova and Matrosov [1]. See also, Martynyuk [4] for the use of several

Lyapunov functions in practical stability. The analysis of practical stability of large scale systems is discussed by many authors. See Martynyuk [2, 3, 8, 15], Michel and Miller [1] and Siljak [1]. Theorem 2.5.1 is new in this direction.

Section 2.6 presents rather general definitions of practical stability in terms of arbitrary sets and two measures which include many well known definitions of practical stability. It is interesting to note that Definition 2.6.1 is very much similar to the definition in Ladde and Leela [1] where the analysis of invariant sets is discussed. Section 2.7 considers practical stability in terms of two different measures and all the results of this section are new. Section 2.8 is based on the results of Ladde and Leela [1]. Finally, Section 2.9 discusses results on practical stability in terms of two arbitary sets which are adapted from Martynyuk [3, 4, 12, 13, 16]. For several allied results, see Abdullin and Anapolsky [1], Bernfeld and Lakshmikantham [1], Gunderson [1], Grujic [1-5], Hallam and Komkov [1], Kayande [1], Kayande and Wong [1], Kamenkov and Lebedev [1], Kamenkov [1], Lebedev [1], Martynyuk [1, 4-6], Moiseyev [1], Tsokos and Ramamohana Rao [1], Weiss [1, 2], Weiss and Infante [1, 2], Windeknecht and Mesarovic [1], and Zubov [2].

3
Perturbed Systems

3.0. INTRODUCTION.

In this chapter, we extend practical stability considerations to a variety of nonlinear systems, including perturbed systems. Of course, the concept of Lyapunov-like functions and the corresponding theory of inequalities play a decisive role.

In Section 3.1, we discuss perturbed differential systems utilizing coupled comparison functions which result when unperturbed systems have nice stability properties. Sometimes, it may be convenient to involve perturbations in the definition of practical stability itself since one can then deal with perturbations directly as constraints. This is the content of Section 3.2. Section 3.3 is devoted to the development of a comparison theorem which connects the solutions of perturbed and unperturbed differential systems in a useful manner. This comparison result blends, in a certain sense, the two approaches, namely, the method of variation of parameters and the Lyapunov's method and consequently is more suitable to deal with the practical stability considerations so as to gain from perturbations. In Section 3.4, we extend the Lyapunov method for practical stability to difference equations. Section 3.5 is concerned with delay differential equations where we employ Lyapunov functions on product spaces to obtain practical stability criteria which is more helpful than using either Lyapunov functions or Lyapunov functionals and incorporates the advantages of both the methods. Integro-differential equations of Volterra type are investigated in Section 3.6, where a unified approach is presented via Lyapunov functions to extend results on practical stability.

Section 3.7 considers impulsive differential equations while Section 3.8 discusses impulsive integro-differential equations of Volterra type. In both cases, we utilize piecewise continuous Lyapunov functions and the theory of impulsive differential inequalities to prove practical stability properties. Finally, Section 3.9 investigates weakly coupled reaction-diffusion equations relative to practical stability criteria where vector Lyapunov functions occur in a natural way.

3.1. STABILITY OF PERTURBED SYSTEMS.

In order to unify the investigation of practical stability properties of perturbed differential systems, it is useful to utilize coupled comparison functions as in Theorem 2.1.2. Of course, the use of coupled functions is also beneficial in the study of stability properties of unperturbed systems, since estimating $D^+V(t, x)$ by means of a function of t, x and $V(t, x)$ is more advantageous than by a function of t and $V(t, x)$ only.

Let us consider the perturbed differential system

$$x' = f(t, x) + R(t, x), \; x(t_0) = x_0, \tag{3.1.1}$$

where $f, R \in C[\mathbf{R}_+ \times \mathbf{R}^n, \mathbf{R}^n]$ together with the unperturbed system (2.1.1). Of course, the perturbation term $R(t, x)$ can enter the system in other ways, for example,

$$x' = f(t, x, R(t, x)),$$

$$x' = f(t, x)R_0(t, x) + R_1(t, x),$$

where $R_0(t, x)$ is an $n\times n$ matrix and $R_1(t, x)$ is a n - vector. We shall only be concerned with the system (3.1.1).

As mentioned earlier, in this approach, we need to consider the coupled comparison equation

$$u' = g(t, x(t), u), \quad u(t_0) = u_0 \geq 0, \tag{3.1.2}$$

where $g \in C[\mathbf{R}_+ \times \mathbf{R}^n \times \mathbf{R}_+, \mathbf{R}]$ and $x(t) = x(t, t_0, x_0)$ is any solution of (3.1.1) existing on $[t_0, \infty)$. We require suitable practical stability concepts relative to the coupled system (3.1.2).

Definition 3.1.1

The coupled system (3.1.2) is said to be conditionally practically stable if, given $0<\lambda<A$, a, $b\in K$ such that $a(\lambda)<b(A)$, $|\ x_0\ |<\lambda$ and $u_0<a(\lambda)$ implies either

(i) $u(t, t_0, x_0, u_0)<b(A)$, $t\geq t_0$

or

(ii) $u(t, t_0, x_0, u_0)<b(A)$ on any interval $t_0\leq t\leq t_1$, for which $|x(t)|\leq A$,

where $u(t, t_0, x_0, u_0)$ is any solution of (3.1.2).

The other practical stability notions can be defined in a similar way. Note that in the above definition, part (i) may be considered as partial practical stability of the combined system (3.1.1) and (3.1.2) with respect to the component u.

Let us prove the following typical result.

Theorem 3.1.1.

Assume that (A_0) - (A_3) of Theorem 2.2.1 hold except that condition

(2.2.1) is replaced by

$$D^+V(t, x)_{(3.1.1)} \leq g(t, x, V(t, x)), \text{ for } (t, x) \in \mathbf{R}_+ \times S(A),$$

where $g \in C[\mathbf{R}_+ \times \mathbf{R}^n \times \mathbf{R}_+, \mathbf{R}]$. Then, conditional practical stability properties of the coupled equation (3.1.2) imply the corresponding practical stability properties of the perturbed system (3.1.1).

Proof

The proof is very much similar to the proof of Theorem 2.2.1 except that we now employ Theorem 2.1.2 instead of Theorem 2.1.1. As a result, in place of the relations (2.2.3), we now have

$$|x_0| < \lambda \text{ and } u_0 < a(\lambda) \text{ implies } u(t, t_0, x_0, u_0) < b(A),\ t \geq t_0,$$

if we use (i) of definition 3.1.1. Then, Theorem 2.1.2 yields

$$V(t, x(t)) \leq r(t, t_0, x_0, u_0),\ t_0 \leq t \leq t_1,$$

and therefore we get a contradiction as before, proving the practical stability of the system (3.1.1). Based on these changes, it is easy to construct the proofs of other practical stability properties. Hence the theorem is proved.

If the unperturbed system (2.1.1) satisfies the estimate

$$D^+V(t, x)_{(2.1.1)} \leq -C(V(t, x)),\ (t, x) \in \mathbf{R}_+ \times S(A),$$

where $C \in K$ and if we assume that $V(t, x)$ is Lipschitzian in x for a function $L(t) \geq 0$ for $x \in S(A)$, then it is easy to show that

$$D^+V(t, x)_{(3.1.1)} \leq -C(V(t, x)) + L(t)\,|R(t, x)|,\ (t, x) \in \mathbf{R}_+ \times S(A).$$

Consequently, the coupled equation reduces to

$$u' = -C(u) + w(t, x(t)),\ u(t_0) = u_0 \geq 0, \tag{3.1.3}$$

where $w(t, x) = L(t) \mid R(t, x) \mid$. Assume now that

$$\begin{cases} \mid w(t, x) \mid \leq \sigma(t), \text{ whenever } \mid x \mid \leq A \text{ and} \\ \displaystyle\int_t^{t+1} \sigma(s)ds \to 0 \text{ as } t \to \infty. \end{cases}$$

Then we claim that (3.1.3) is uniformly practically stable. Let $0 < \lambda < A$ b be given. Using the assumption on σ, we note that

$$\int_{t_0}^{t} \sigma(s)ds = \int_{t_0}^{t} [\int_{s-1}^{s} \sigma(\xi)d\xi]ds \leq \int_{t_0-1}^{t} [\int_{s}^{s+1} \sigma(\xi)d\xi]ds$$

$$= \int_{t_0-1}^{t} G(s)ds,$$

where $G(t) = \int_t^{t+1} \sigma(s)ds$. Let

$$Q(t) = \sup\{G(s) : t-1 \leq s < \infty\}$$

so that $Q \in L$. Choose a $\tau = \tau(\lambda, A) > 0$ such that

$$Q(\tau) < \min[C(a(\lambda)),\ b(A) - a(\lambda)]$$

and let $\mid x_0 \mid < \lambda$ with $t_0 \geq \tau$. Suppose that $\mid x(t) \mid \leq A$ for $t_0 \leq t \leq t_1$ and let $u_0 < a(\lambda)$. If possible, let $u(t_1) = u(t_1, t_0, x_0, u_0) \geq b(A)$. Then there would exist a $t_2 > t_0$ such that

$$u(t_2)=a(\lambda) \text{ and } a(\lambda)\le u(t)\le b(A),\ t_2\le t\le t_1.$$

we then have from (3.1.3),

$$b(A)\le u(t_1)\le u(t_2)-\int_{t_2}^{t_1} C(u(s))ds+\int_{t_2}^{t_1} |\, w(s, x(s))\, |\, ds$$

$$\le a(\lambda)-C(a(\lambda))(t_1-t_2)+\int_{t_2-1}^{t_1} G(s)ds$$

$$\le a(\lambda)-(t_1-t_2)[-C(a(\lambda))+Q(\tau)]+Q(\tau)$$

$$\le a(\lambda)+Q(\tau)<b(A)$$

which is a contradiction. Hence (3.1.2) is uniformly practically stable and consequently, the perturbed system (3.1.1) is uniformly practically stable.

3.2. STABILITY OF PERTURBED SYSTEMS (CONTINUED).

We can introduce constraints on constantly acting perturbations and modify the notions of practical stability so as to deal with perturbations directly. This leads to the following definition.

Definition 3.2.1

The perturbed system (3.1.1) is said to be practically stable under constantly acting perturbations, if given the sets S_0, S, $P\in C[\mathbf{R}_+, \Omega]$ such that $S_0(t)\subset S(t)$, $S_0(t)\cap S(t)=\phi$, $t\in\mathbf{R}_+$, we have

$$\begin{cases} x_0 \in S_0(t_0) \text{ and } R(t, x) \in P(t) \text{ for } x \in S(A) \text{ imply} \\ \qquad x(t) \in S^0(t),\ t \geq t_0, \end{cases}$$

where $x(t)$ is any solution of (3.1.1).

Other definitions can be modified similarly.

We can now prove some typical results concerning practical stability of the perturbed system (3.1.1).

Theorem 3.2.1

Assume that

(i) $V \in C[\mathbf{R}_+ \times S(t), \mathbf{R}_+]$, $V(t, x)$ is locally Lipschitzian in x and $D^+V(t, x)_{(3.1.1)} \leq g(t, V(t, x))$ for $R(t, x) \in P(t)$, $x \in S(t)$ and $t \in \mathbf{R}_+$, where $g \in C[\mathbf{R}_+^2, \mathbf{R}]$;

(ii) the maximal solution $r(t, t_0, u_0)$ of

(3.2.1) $$u' = g(t, u),\ u(t_0) = \geq 0,$$

satisfies $u_0 < V_M^{S_0}(t_0)$ and $r(t, t_0, u_0) < V_M^{\partial S}(t)$, $t \geq t_0$.

Then, the perturbed system (3.1.1) is practically stable.

Proof

Let $R(t, x) \in P(t)$ whenever $x \in S(t)$, $t \in \mathbf{R}_+$ and let $x_0 \in S_0(t_0)$. We claim that $x(t) \in S^0(t)$, $t \geq t_0$. If this is not true, then there would exist a $t_1 > t_0$ and a solution $x(t) = x(t, t_0, x_0)$ of (3.1.1) such that

(3.2.2) $$x(t_1) \in \partial S(t_1) \text{ and } x(t) \in S(t) \text{ for } t_0 \leq t \leq t_1.$$

Hence Theorem 2.1.1 yields, because of (3.2.2), the estimate

$$V(t, x(t)) \le r(t, t_0, u_0),\ t_0 \le t \le t_1,$$

where $r(t_1, t_0, u_0)$ is the maximal solution of (3.2.1). In view of the conditions on $r(t, t_0, u_0)$, we then have

$$V_M^{\partial S}(t_1) \le V(t_1, x(t_1)) \le r(t_1, t_0, u_0)$$
$$\le r(t_1, t_0, V_M^{S_0}(t_0)) < V_M^{\partial S}(t_1)$$

which is a contradiction. This proves the theorem.

A variant of Theorem 3.2.1 in which assumptions on the perturbation are separated is the following.

Theorem 3.2.2

Assume that

(i) $V \in C[\mathbf{R}_+ \times S(t), \mathbf{R}_+]$, $V(t, x)$ is Lipschitzian in x for a function $L(t) > 0$ and $D^+V(t, x)_{(2.1.1)} \le g(t, V(t, x))$, $(t, x) \in \mathbf{R}_+ \times S(t)$, where $g \in C[\mathbf{R}_+^2, \mathbf{R}]$;

(ii) $L(t) \mid R(t, x) \mid \le D^+\eta(t)$, whenever $R(t, x) \in P(t)$ and $x \in S(t)$, $t \in \mathbf{R}_+$, where $\eta \in C[\mathbf{R}_+, (0, \infty)]$ with $\eta(t)$ nondecreasing;

(iii) the maximal solution $r(t, t_0, u_0)$ of

$$u' = g(t, u + \eta(t)),\ u(t_0) = u_0 \ge 0$$

satisfies $r(t, t_0, u_0) < V_M^{\partial S}(t) - \eta(t)$, $t \ge t_0$, whenever $u_0 \le V_M^{S_0}(t_0) - \eta(t_0)$.

Then, the perturbed system (3.1.1) is practically stable.

Proof

Let $R(t, x) \in P(t)$, $x \in S(t)$, $t \in \mathbf{R}_+$ and let $x_0 \in S_0(t_0)$. As in the proof of Theorem 3.2.1, if the conclusion is not true, we arrive at (3.2.2). Now, for $t_0 \leq t \leq t_1$,

$$D^+V(t, x)_{(3.1.1)} \leq D^+V(t, x)_{(2.1.1)} + L(t) \mid R(t, x) \mid$$
$$\leq g(t, V(t, x)) + D^+\eta(t),$$

and consequently, setting $q(t) = V(t, x(t)) - \eta(t)$, we get

$$D^+q(t) \leq g(t, q(t) + \eta(t)),\ q(t_0) = u_0.$$

Hence Theorem 2.1.1 yields

$$V(t, x(t)) \leq r(t, t_0, u_0) + \eta(t),\ t_0 \leq t \leq t_1,$$

from which we obtain a contradiction in view of the assumptions on $r(t,t_0,u_0)$. Thus, the proof is complete.

Finally, we shall merely state a result whose proof is similar to Theorem 2.8.1.

Theorem 3.2.3

Assume that conditions (i) and (iii) of Theorem 2.8.1 hold. Suppose that

(ii*) $D^+V(t, x)_{(3.1.1)} < D^+\eta(t)$, whenever $R(t, x) \in P(t)$, $x \in S(t)$ and $t \in \mathbf{R}_+$, with η being the same function.

Then the system (3.1.1) is practically stable.

3.3. A TECHNIQUE IN PERTURBATION THEORY.

In this section, we develop a new comparison theorem that connects the solutions of perturbed and unperturbed differential systems in a manner useful in the theory of perturbations. This comparison result blends, in a sense, the two approaches mentioned earlier and consequently provides a flexible mechanism to preserve the nature of perturbations. The results that are given in this section show that the usual comparison theorem (Theorem 2.1.1) in terms of a Lyapunov function is included as a special case and that perturbation theory could be studied in a more fruitful way.

Consider the two differential systems

$$y' = f(t, y), \; y(t_0) = x_0, \tag{3.3.1}$$

and

$$x' = F(t, x), \; x(t_0) = x_0, \tag{3.3.2}$$

where $f, F \in C[\mathbf{R}_+ \times \mathbf{R}^n, \mathbf{R}^n]$. Relative to the system (3.3.1), let us assume that the following assumption (H) holds:

(H) the solutions $y(t, t_0, x_0)$ of (3.3.1) exist for all $t \geq t_0$, unique and continuous with respect to the initial data and $|y(t,t_0,x_0)|$ is locally Lipschitzian in x_0.

For any $V \in C[\mathbf{R}_+ \times \mathbf{R}^n, \mathbf{R}_+]$ and any fixed $t \in [0, \infty)$, we define

$$D_-V(s, y(t, s, x)) \equiv \liminf_{h \to 0^-} \frac{1}{h}[V(s+h, y(t, s+h, x+hF(s,x)) - V(s, y(t, s, x))] \tag{3.3.3}$$

for $t_0 < s \leq t$ and $x \in \mathbf{R}^n$.

The following comparison result which relates the solutions of (3.3.2) to the solutions of (3.3.1) is an important tool in the subsequent discussion.

Theorem 3.3.1

Assume that the assumption (H) holds. Suppose that

(i) $V \in C[\mathbf{R}_+ \times \mathbf{R}^n, \mathbf{R}_+]$, $V(s, x)$ is locally Lipschitzian in x and for $t_0 < s \leq t$, $x \in \mathbf{R}^n$,

$$D_-V(s, y(t, s, x)) \leq g(t, V(s, y(t, s, x))); \tag{3.3.4}$$

(ii) $g \in C[\mathbf{R}_+^2, \mathbf{R}]$ and the maximal solution $r(t, t_0, u_0)$ of

$$u' = g(t, u), \; u(t_0) = u_0 \geq 0 \tag{3.3.5}$$

exists for $t \geq t_0$.

Then, if $x(t) = x(t, t_0, x_0)$ is any solution of (3.3.2), we have

$$V(t, x(t, t_0, x_0)) \leq r(t, t_0, u_0), \; t \geq t_0, \tag{3.3.6}$$

provided $V(t_0, y(t, t_0, x_0)) \leq u_0$.

Proof

Let $x(t) = x(t, t_0, x_0)$ be any solution of (3.3.2). Set

$$m(s) = V(s, y(t, s, x(s))), \; t_0 \leq s \leq t$$

so that $m(t_0) = V(t_0, y(t, t_0, x_0))$. Then using the assumptions (H) and (i), it is easy to obtain

$$D^+m(s) \leq g(s, m(s)), \; t_0 \leq s \leq t,$$

which yields by Theorem A.1.1 the estimate

$$m(s) \leq r(s, t_0, u_0), \; t_0 \leq s \leq t \tag{3.3.7}$$

provided $m(t_0) \leq u_0$. Since $m(t) = V(t, y(t, t, x(t))) = V(t, x(t, t_0, x_0))$, the desired result (3.3.6) follows from (3.3.7) by setting $s=t$.

Taking $u_0=V(t_0, y(t, t_0, x_0))$, the inequality (3.3.6) becomes

(3.3.8) $\quad V(t, x(t, t_0, x_0))\leq r(t, t_0, V(t_0, y(t, t_0, x_0))), \; t\geq t_0,$

which shows the connection between the solutions of systems (3.3.1) and (3.3.2) in terms of the maximal solution of (3.3.5).

A number of remarks can now be made:

(1) The trivial function $f(t, y)\equiv 0$ is admissible in Theorem 3.3.1 to yield the estimate (3.3.6) provided $V(t_0, x_0)\leq u_0$. In this case $y(t, t_0, x_0)=x_0$ and the hypothesis (H) is trivially verified. Since $y(t, s, x)=x$, the definition (3.3.3) reduces to

(3.3.9) $\quad D\_V(s, x)\equiv\lim_{h\to 0^-}\inf\frac{1}{h}[V(s+h, x+hF(s, x))-V(s, x)]$

which is the usual definition of generalized derivative of the Lyapunov function relative to the system (3.3.2). Consequently, Theorem 3.3.1 reduces, in this special case, to Theorem 2.1.1.

(2) Suppose that $f(t, y)=A(t)y$ where $A(t)$ is a $n\times n$ continuous matrix. The solution $y(t, t_0, x_0)$ of (3.3.1) then satisfy $y(t,t_0,x_0)=\Phi(t,t_0)x_0$, where $\Phi(t, t_0)$ is the fundamental matrix solution of $y'=A(t)y$, with $\Phi(t_0, t_0)=I$ (identity matrix). The assumption (H) is clearly verified. Suppose also that $g(t, u)\equiv 0$ Then (3.3.6) yields

(3.3.10) $\quad V(t, x(t, t_0, x_0))\leq V(t_0, \Phi(t, t_0)x_0), \; t\geq t_0.$

If $V(t, x)=|x|$, (3.3.10) leads to

(3.3.11) $\quad |x(t, t_0, x_0)|\leq|\Phi(t, t_0)x_0|, \; t\geq t_0.$

If, on the other hand, $g(t, u) = -\alpha u$, $\alpha > 0$, we get a sharper estimate

$$V(t, x(t, t_0, x_0)) \le V(t_0, \Phi(t, t_0)\, x_0) \exp(-\alpha(t - t_0)), \ t \ge t_0, \tag{3.3.12}$$

which, in the special case $V(t, x) = |\, x\,|$ reduces to

$$|\, x(t, t_0, x_0)\,| \le |\, \Phi(t, t_0)\, x_0|\, \exp(-\alpha(t - t_0)), \ t \ge t_0. \tag{3.3.13}$$

Clearly the relation (3.3.13) helps in improving the behavior of solutions of (3.3.2) relative to the behavior of solutions of (3.3.1). This is a great asset in perturbation theory and it can be seen by setting $F(t, x) = f(t, x) + R(t, x)$ where $R(t, x)$ is the perturbation term.

(3) Suppose that $f(t, y)$ is nonlinear, $f_y(t, y)$ exists and is continuous for $(t, y) \in \mathbf{R}_+ \times \mathbf{R}^n$. Then, it is well known that the solutions $y(t, t_0, x_0)$ are differentiable with respect to (t_0, x_0) and we have

$$\begin{cases} \dfrac{\partial y}{\partial t_0}(t, t_0, x_0) = -\Phi(t, t_0, x_0) f(t_0, x_0), \ t \ge t_0, \\[2ex] \dfrac{\partial y}{\partial x_0}(t, t_0, x_0) = \Phi(t, t_0, x_0) \end{cases} \tag{3.3.14}$$

where $\Phi(t, t_0, x_0)$ is the matrix solution of the variational equation

$$z' = f_y(t, y(t, t_0, x_0))z.$$

If $V(s, x)$ is also assumed to be differentiable, then by (3.3.14), we have, for a fixed t,

$$D_-V(s, y(t, s, x)) \equiv V_s(s, y(t, s, x)) \tag{3.3.15}$$

$$+V_x(s, y(t, s, x)) \cdot \Phi(t, s, x) \cdot [F(s, x)-f(s,x)]$$

The relation (3.3.15) gives an intuitive feeling of the definition (3.3.3). If, in addition, $V(t, x)=|x|^2$ and $F(t, x)=f(t, x)+R(t, x)$, (3.3.15) yields

$$D_-V(s, y(t, s, x)) \equiv 2y(t, s, x) \cdot \Phi(t, s, x) \cdot R(s, x)$$

which exhibits how the perturbation term is involved in the computation.

(4) When the solutions of (3.3.1) are known, a possible Lyapunov function for (3.3.2) is

$$W(s, x)=V(s, y(t, s, x)) \tag{3.3.16}$$

where $V(s, x)$ and $y(t, s, x)$ are as before. One could use $V(s, x)=|x|$ so that $W(s, x)=|y(t, s, x)|$ is a candidate for Lyapunov function for (3.3.2). If $y(t, s, x) \equiv x$, condition (3.3.14) reduces to

$$\lim_{h \to 0^-} \inf \frac{1}{h}[|x+hF(t, x)|-|x|] \le g(t, |x|)$$

which is an often used assumption in comparison results.

As an application of Theorem 3.3.1, we shall consider some results on practical stability of the system (3.3.2).

Theorem 3.3.2

Assume that (H) holds and (i) of Theorem 3.3.1 is verified. Suppose that $g \in C[\mathbf{R}_+^2, \mathbf{R}]$ and for $(t, x) \in \mathbf{R}_+ \times \mathbf{R}^n$,

$$b(|x|) \le V(t, x) \le a(|x|), \ a, b \in K. \tag{3.3.17}$$

Furthermore, suppose that $0<\lambda<A$ are given and $a(\lambda)<b(A)$. If the unperturbed system (3.3.1) is (λ, λ) practically stable, then the practical stability properties of (3.3.1) imply the corresponding practical stabilty properties of the perturbed system (3.3.2).

Proof

Assume that (3.3.5) is strongly practically stable. Then, we have, given $(\lambda, A, B, T)>0$ such that $\lambda<A$, $B<A$,

(3.3.18) $$u(t, t_0, u_0)<b(A),\ t\geq t_0 \text{ if } u_0<a(\lambda)$$

and

(3.3.19) $$u_0<a(\lambda) \text{ implies } u(t, t_0, u_0)<b(B),\ t\geq t_0+T.$$

Since (3.3.1) is (λ, λ) practically stable, we have

(3.3.20) $$|\, y(t, t_0, x_0)\, |<\lambda,\ t\geq t_0, \text{ if } |\, x_0\, |<\lambda.$$

We claim that $|\, x_0\, |<\lambda$ also implies that $|\, x(t, t_0, x_0)\, |<A$, $t\geq t_0$, where $x(t, t_0, x_0)$ is any solution of (3.3.2). If this is not true, there would exist a solution $x(t, t_0, x_0)$ of (3.3.2) with $|\, x_0\, |<\lambda$ and a $t_1>t_0$ such that $|\, x(t_1, t_0, x_0)\, |\leq A$, $t_0\leq t\leq t_1$. Then by Theorem 3.3.1, we have

$$V(t, x(t, t_0, x_0))\leq r(t, t_0, V(t_0, y(t, t_0, x_0))),\ t_0\leq t\leq t_1.$$

Consequently, we get

$$b(A)\leq V(t_1, x(t_1, t_0, x_0))\leq r(t_1, t_0, a(|\, y(t_1, t_0, x_0)\, |))$$
$$\leq r(t_1, t_0, a(\lambda))<b(A).$$

This contradiction proves that

$$|x_0|<\lambda \text{ implies } |x(T)|<A,\ t\geq t_0.$$

To show strong practical stability, we see from the foregoing argument that we have

$$b(|x(t, t_0, x_0)|)\leq V(t, x(t, t_0, x_0))\leq r(t, t_0, V(t_0, y(t, t_0, x_0)))$$

for all $t\geq t_0$, if $|x_0|<\lambda$. From this it follows that

$$b(|x(t, t_0, x_0)|)\leq r(t, t_0, a(\lambda)),\ t\geq t_0.$$

Now (3.3.19) yields the strong practical stability of the system (3.3.2) and the proof is complete.

Setting $F(t, x)=f(t, x)+R(t, x)$ in Theorem 3.3.2, we see that although the unperturbed system (3.3.2) is strongly practically stable, the perturbed system (3.3.2) is strongly practicaly stable, a improvement caused by the perturbing term.

Let us present a simple but illustrative example. Consider

$$y'=e^{-t}y^2,\ y(t_0)=x_0, \tag{3.3.21}$$

whose solutions are given by $y(t, t_0, x_0)=\dfrac{x_0}{1+x_0(e^{-t}-e^{-t_0})}$.

The fundamental matrix solution of the corresponding variational equation is

$$\Phi(t, t_0, x_0)=\frac{1}{[1+x_0(e^{-t}-e^{-t_0})]^2}.$$

Consequently, choosing $V(t, x)=x^2$, we see that

$$D\_V=2y(t, s, x)\Phi(t, s, x)R(s, x)$$

where R(t, x) is the perturbation. Let $R(t, x)=\frac{-x^2}{2}$ so that the perturbed equation is

$$x'=e^{-t}x^2-\frac{x^2}{2},\ x(t_0)=x_0. \tag{3.3.22}$$

Accordingly, it is easily seen that $g(t, u)=-u^{3/2}$ and hence the solutions of

$$u'=-u^{3/2},\ u(t_0)=u_0\geq 0$$

are $u(t, t_0, u_0)=\frac{4u_0}{[2+u_0^{1/2}(t-t_0)]^2}$. Thus, by Theorem 3.3.1, we get

the relation

$$|x(t, t_0, x_0)|^2\leq\frac{|x_0|^2}{[1+x_0(e^{-t}-e^{-t_0}+\frac{t-t_0}{2})]^2},\ t\geq t_0$$

which shows that all solutions $x(t, t_0, x_0)\to 0$ as $t\to\infty$, although only some solutions $y(t, t_0, x_0)$ are bounded. For example, setting $t_0=0$ and $x_0=1$ shows that e^t is the corresponding solution of (3.3.21) where as for the same initial conditions, the solutions of (3.3.22) is $\frac{2}{2+t+2e^{-t}}$.

3.4. DIFFERENCE EQUATIONS.

The most common way to solve ordinary differential equations numerically is by discretization, that is, by approximating it by a difference equation. Moreover, much of the theory of difference equations is more challenging and deeper than the corresponding theory of differential equations. We shall therefore develop, in this section, Lyapunov methods for practical stability.

Let us consider the difference equation

$$y_{n+1}=f(n, y_n) \tag{3.4.1}$$

where $f:N^+_{n_0}\times\mathbf{R}^d\to\mathbf{R}^d$ and $f(n, x)$ is continuous in x. Let $y(n, n_0, y_0)$ be the solution of (3.4.1) having (n_0, y_0) as the initial condition and defined for $n\in N^+_{n_0}$.

Let $V:N^+_{n_0}\times\mathbf{R}^d\to\mathbf{R}_+$ and define the variation of the function V relative to (3.4.1) by

$$\Delta V(n, y_n)=V(n+1, y_{n+1})-V(n, y_n). \tag{3.4.2}$$

Then we can prove the following results concerning the practical stability of the system (3.4.1).

Theorem 3.4.1

Suppose that

(i) $0<\lambda<A$ are given;

(ii) $V:N^+_{n_0}\times\mathbf{R}^d\to\mathbf{R}_+$, $V(n, x)$ is continuous in x,

$$b(|x|)\le V(n, x)\le a(|x|),\ a, b\in K,$$

and

$$\Delta V(n, y_n)\le w(n, V(n, y_n)),\ \text{for } n\in N^+_{n_0},\ y_n\in\mathbf{R}^d,$$

where $w:N^+_{n_0}\times\mathbf{R}_+\to\mathbf{R}$ and $w(n, u)$ is continuous in u;

(iii) $g(n, u)=u+w(n, u)$ and $g(n, u)$ is nondecreasing in u;

(iv) $a(\lambda)<b(A)$ holds

Then the practical stability properties of

$$u_{n+1}=g(n, u_n),\ u_{n_0}=u_0\geq 0 \tag{3.4.3}$$

imply the corresponding practical stability properties of the system (3.4.1).

Proof

Let us suppose that (3.4.3) is practically stable for $(a(\lambda), b(A))$. Then we have

$$u_0<a(\lambda) \text{ implies } u(n, n_0, u_0)<b(A),\ n\geq n_0, \tag{3.4.4}$$

where $u(n)=u(n, n_0, u_0)$ is the solution of (3.4.3). Let $|\ y_0\ |<\lambda$. We claim that $|\ y(n, n_0, y_0)\ |<A$ for $n\geq n_0$. If this is not true, then there would exist a $n_1>n_0$ and a solution $y(n)=y(n, n_0, y_0)$ of (3.4.1) such that

$$y(n_1)>A \text{ and } y(n)<A \text{ for } n_0\leq n<n_1. \tag{3.4.5}$$

Choose $u_0=V(n_0, y_0)$ so that we get from condition (ii) the relation

$$V(n, y(n))\leq u(n),\ n\geq n_0, \tag{3.4.6}$$

using the same arguments as we did in the proof of Theorem 1.6.1. Hence (3.4.4), (3.4.5) and (3.4.6) yield

$$b(A)\leq V(n_1, y(n_1))\leq u(n_1, n_0, u_0)\leq u(n_1, n_0, a(\lambda))<b(A)$$

which is a contradiction. Hence the system (3.4.1) is practically stable.

If we suppose that (3.4.3) is practically asymptotically stable, it is clear from (3.4.6) that

$$0=\lim_{n\to\infty} V(n, y(n)) \leq \lim_{n\to\infty} u(n)=0$$

and consequently, condition (ii) gives $\lim_{n\to\infty} | y(n) |=0$. Thus, we get practical asymptotic stability of the system (3.4.1). Other concepts may be proved similarly. Hence the proof is complete.

The following corollary of Theorem 3.4.1 is interesting.

Corollary 3.4.1

(i) The function $w(n, u)\equiv 0$ is admissible in Theorem 3.4.1 to yield uniform practical stability of (3.4.1);

(ii) the function $w(n, u)=-C(u)$, $C\in K$ is admissible in Theorem 3.4.1 to give uniform practical asymptotic stability of (3.4.1).

3.5. DELAY DIFFERENTIAL EQUATIONS.

Let $\mathcal{C}=C[[-\tau, 0], \mathbb{R}^n]$ and for any $\phi\in\mathcal{C}$, let $| \phi |_0=\max_{-\tau\leq s\leq 0} | \phi(s)|$. If $x\in C[[t_0-\tau, \infty), \mathbb{R}^n]$, $t_0\in\mathbb{R}_+$, we define $x_t\in\mathcal{C}$ by $x_t(s)=x(t+s)$, $-\tau\leq s\leq 0$. Consider the intial value problem

$$x'(t)=f(t, x_t),\ x_{t_0}=\phi_0\in\mathcal{C}, \tag{3.5.1}$$

where $f\in C[\mathbb{R}_+\times\mathcal{C}, \mathbb{R}^n]$. It is known that if f maps bounded sets into bounded sets, then for each (t_0, ϕ_0), there exists a solution $x(t)=x(t_0,\phi_0)(t)$ defined on an interval $[t_0, t_0+\alpha)$, $\alpha>0$. We shall employ Lyapunov functions on the product space $\mathbb{R}^n\times\mathcal{C}$ and develop corresponding theory for practical stability criteria for the system (3.5.1).

If $V\in C[\mathbb{R}_+\times\mathbb{R}^n\times\mathcal{C}, \mathbb{R}_+]$, define

$$
(3.5.2)\quad \begin{cases} D^{+}V(t,\,\phi(0),\,\phi)=\lim\limits_{h\to 0^{+}}\sup\frac{1}{h}[V(t+h,\,\phi(0)+hf(t,\,\phi), \\ \qquad x_{t+h}(t,\,\phi))-V(t,\,\phi(0),\,\phi)], \end{cases}
$$

where $x(t, \phi)$ denotes any solution of (3.5.1) with the initial function ϕ at time t.

Theorem 3.5.1

Assume that

(i) $0<\lambda<A$ are given;

(ii) $V\in C[\mathbf{R}_+\times\mathbf{R}^n\times\mathcal{C},\,\mathbf{R}_+]$, $V(t, x, \phi)$ is locally Lipschitzian in x and for $(t, \phi(0), \phi)\in\mathbf{R}_+\times S(A)\times\mathcal{C}(A)$,

$$b(|\,\phi(0)\,|)\le V(t,\,\phi(0),\,\phi)\le a(|\,\phi\,|_0),\ a,\ b\in K,$$

and

$$D^{+}V(t,\,\phi(0),\,\phi)\le g(t,\,V(t,\,\phi(0),\,\phi)),$$

$g\in C[\mathbf{R}_+^2,\,\mathbf{R}]$ and $\mathcal{C}(A)=\{\phi\in\mathcal{C}:\ |\,\phi\,|_0<A\}$;

(iii) $a(\lambda)<b(A)$ holds.

Then the practical stability properties of the comparison equation

$$(3.5.3)\qquad u'=g(t,\,u),\ u(t_0)=u_0\ge 0$$

imply the corresponding practical stability properties of the system (3.5.1).

Proof

The proof is very much similar to the proof of Theorem 2.2.1 with suitable modifications. The practical stability of (3.5.1) would be as follows:

$$|\,\phi_0\,|_0<\lambda \textit{ implies } |\,x(t_0,\,\phi_0)(t)|<A \text{ for } t\ge t_0,$$

where $x(t_0, \phi_0)(t)$ is any solution of (3.5.1). Furthermore, if we let $m(t)=V(t, x(t_0, \phi_0)(t), x_t(t_0, \phi_0))$, then condition (ii) and Lipschitzian nature of V yields the differential inequality

$$D^+m(t)\leq g(t, m(t))$$

and hence Theorem A.1.1 shows that whenever $V(t_0, \phi_0(0), \phi_0)\leq u_0$,

$$V(t, x(t_0, \phi_0)(t), x_t(t_0, \phi_0))\leq r(t, t_0, u_0),\ t\geq t_0.$$

The rest of the proof follows standard arguments.

We shall next discuss a result which is more interesting and is similar to Theorem 2.2.3.

Theorem 3.5.2

Assume that conditions (i) and (iii) of Theorem 3.5.1 hold. Suppose also that

(ii*) $V, W\in C[\mathbf{R}_+\times\mathbf{R}^n\times\mathcal{C}, \mathbf{R}_+]$, $V(t, x, \phi)$, $W(t, x, \phi)$ are locally Lipschitzian in x and for $(t, \phi(0), \phi)\in\mathbf{R}_+\times S(A)\times\mathcal{C}(A)$,

$$b(|\phi(0)|)\leq V(t, \phi(0), \phi)\leq a_0(|\phi(0)|)+a_1(W(t, \phi(0), \phi))$$

and

$$D^+V(t, \phi(0), \phi)\leq -C(W(t, \phi(0), \phi)),$$

where $b, a_0, a_1, C\in K$;

(iv) $D^+W(t, \phi(0), \phi)$ is bounded from above or from below and $\lim_{t\to\infty} W(t, x(t_0, \phi_0)(t), x_t(t_0, \phi_0))=0$ implies

$\lim_{t\to\infty} \inf|x(t_0,\phi_0)(t)|=0$ for any solution $x(t_0, \phi_0)(t)$ of (3.5.1) such that

$$|\ \phi_0\ |_0<\lambda \text{ implies } |\ x(t_0, \phi_0)(t)\ |<A \text{ for } t\geq t_0.$$

Then the system (3.5.1) is practically asymptotically stable if

$$W(t, \phi(0), \phi)\leq\psi(|\ \phi_0\ |_0),\ \psi\in K.$$

Proof

Since $V(t, \phi(0), \phi)\leq a(|\ \phi\ |_0)$ for some $a\in K$ in view of assumptions on W and $D^+V(t, \phi(0), \phi)\leq 0$, we see immediately from Theorem 3.5.1, the system (3.5.1) is practically stable for (λ, A). If we show

$$\lim_{t\to\infty} W(t, x(t_0, \phi_0)(t), x_t(t_0, \phi_0))=0 \tag{3.5.4}$$

then we have

$$\lim_{t\to\infty} V(t, x(t_0, \phi_0)(t), x_t(t_0, \phi_0))\leq \lim_{t\to\infty} \inf a_0(|\ x(t_0, \phi_0)(t)\ |)$$
$$+\lim_{t\to\infty} W(t, x(t_0, \phi_0)(t), x_t(t_0, \phi_0))=0$$

since V is nondecreasing and has a finite limit as $t\to\infty$ and hence it follows that $\lim_{t\to\infty} |\ x(t_0, \phi_0)(t)\ |=0$. Showing that (3.5.4) holds is exactly the same as in the proof of Theorem 2.2.3 and thus, the proof of Theorem 3.5.2 is complete.

The following corollary is interesting in itself.

Corollary 3.5.1

The function $W(t, \phi(0), \phi)=\int_{-\tau}^{0} |\ \phi(s)\ |^2 ds$ is admissible in Theorem 3.5.2

to give the same conclusion.

Proof

Since $W(t, x(t_0, \phi_0)(t), x_t(t_0, \phi_0)) = \int_{t-\tau}^{t} | x(t_0, \phi_0)(s) | ds$, we see that

$$D^+W = x^2(t_0, \phi_0)(t) - x^2(t_0, \phi_0)(t-\tau)$$

and practical stability implies that D^+W is bounded. Furthermore, it is clear from the form of W that $\liminf_{t\to\infty} | x(t_0, \phi_0)(t) | = 0$.

Hence all the assumptions of Theorem 3.5.2 hold, proving the corollary.

Consider the scalar equation

$$x'(t) = b(t)x(t-\tau), \ \tau > 0, \tag{3.5.5}$$

where $b:[-\tau, \infty) \to [-1, 0]$ is continuous and

$$\left\{ \begin{array}{l} \int_{t-\tau}^{t} | b(u) | du + \tau \leq 2, \ b(t+\tau) = b(t), \\ \int_{t-\tau}^{t} |b(u)| du > 0 \text{ and } \int_{t-\tau}^{t} [1 - |b(s)|] \, ds > 0. \end{array} \right. \tag{3.5.6}$$

Taking the Lyapunov function

$$V(t, x, \phi) = [x + \int_{-\tau}^{0} b(t+s)\phi(s)ds]^2 + \int_{-\tau}^{0} \int_{s}^{0} | b(t+u) | \phi^2(u) du \, ds,$$

we obtain, using the assumptions (3.5.6),

$$D^+V(t, x(t_0, \phi_0)(t), x_t(t_0, \phi_0))$$
$$\leq[|b(t)|-1]\int_{t-\tau}^{t} |b(s)|x^2(t_0, \phi_0)(s)ds.$$

Clearly V satisfies all the assumptions of Theorem 3.5.2 and hence (3.5.5) is practically asymptotically stable.

Instead of Lyapunov functions on product spaces, we could have used only Lyapunov functions on $\mathbf{R}_+\times\mathbf{R}^n$, in which case, we need to specify minimal classes of functions over which the derivative of the Lyapunov function can be estimated so that the study of delay differential equations can be reduced to the study of differential equations without delay. This would extend the discussion of Section 1.4 to the case of Lyapunov functions. We shall consider such a technique in the next section where we deal with integro-differential equations of Volterra type.

3.6. INTEGRO-DIFFERENTIAL EQUATIONS OF VOLTERRA TYPE.

As indicated at the end of the last section, we shall develop, in this section, the method of Lyapunov functions for the integro-differential system of Volterra type, namely,

(3.6.1) $$x'=f(t, x, Tx),\ x(t_0)=x_0,\ t_0\geq 0,$$

where $f\in C[\mathbf{R}_+\times\mathbf{R}^n\times\mathbf{R}^n, \mathbf{R}^n]$, $Tx=\int_{t_0}^{t} K(t, s, x(s))ds$ and the kernel

$K\in C[\mathbf{R}_+^2\times\mathbf{R}^n, \mathbf{R}^n]$. Before we proceed to state the results, let us list the following hypotheses:

(H_0) $V \in C[\mathbf{R}_+ \times \mathbf{R}^n, \mathbf{R}_+]$, $V(t, x)$ is locally Lipschitzian in x and

$$b(|x|) \le V(t, x) \le a(|x|),\ (t, x) \in \mathbf{R}_+ \times \mathbf{R}^n,\ a, b \in K;$$

(H_1) $g_0, g \in C[\mathbf{R}_+^2, \mathbf{R}]$, $g_0(t, u) \le g(t, u)$, $r(t, t_0, u_0)$ is the right maximal solution of

$$u' = g(t, u),\ u(t_0) = u_0 \ge 0, \tag{3.6.2}$$

existing on $[t_0, \infty)$ and $\eta(t, t^0, v_0)$ is the left maximal solution of

$$v' = g_0(t, v),\ v(t^0) = v_0 \ge 0,\ t^0 > t_0, \tag{3.6.3}$$

existing on $t_0 \le t \le t^0$;

(H_2) $$D_-V(t, x, Tx) \equiv \lim_{h \to 0^-} \inf \frac{1}{h}[V(t+h, x+hf(t, x, Tx)) - V(t, x)]$$
$$\le g(t, V(t, x)),\ (t, x) \in \Omega,$$

where $\Omega = \{x \in C[\mathbf{R}_+, \mathbf{R}^n]:\ V(s, x(s)) \le \eta(s, t, V(t, x(t))),\ t_0 \le s \le t\}$.

We are now in a position to prove the following general comparison theorem which permits a unified theory of stability for the integro-differential system (3.6.1).

Theorem 3.6.1

Assume that (H_0), (H_1) and (H_2) hold. Let $x(t, t_0, x_0)$ be any solution of (3.6.1) such that $V(t_0, x_0) \le u_0$. Then

$$V(t, x(t, t_0, x_0)) \le r(t, t_0, u_0),\ t \ge t_0.$$

Proof

Set $m(t)=V(t, x(t, t_0, x_0))$, $t\geq t_0$, so that $m(t_0)\leq u_0$. Since $r(t,t_0,u_0)=\lim_{\xi\to 0^+} u(t, \xi)$ where $u(t, \xi)$ is any solution of

$$u'=g(t, u)+\xi, \; u(t_0)=u_0+\xi,$$

for $\xi>0$, sufficiently small, it is enough to prove that $m(t)<u(t, \xi)$, for $t\geq t_0$. If this is not true, there exists a $t_1>t_0$ such that

$$m(t_1)=\xi \text{ and } m(t)<u(t, \xi), \; t_0\leq t<t_1.$$

This implies that

$$D_-m(t_1)\geq u'(t_1, \xi)=g(t_1, m(t_1))+\xi. \tag{3.6.4}$$

Consider now the left maximal solution $\eta(s, t_1, m(t_1))$ of (3.6.3) with $v(t_1)=m(t_1)$ on the interval $t_0\leq s\leq t_1$. Since

$$r(t_1, t_0, u_0)=\lim_{\xi\to 0^+} u(t_1, \xi)=m(t_1)=\eta(t_1, t_1, m(t_1))$$

and

$$m(s)\leq u(s, \xi) \text{ for } t_0\leq s\leq t_1,$$

it follows by Lemma A.1.2 that

$$m(s)\leq r(s, t_0, u_0)\leq \eta(s, t_1, m(t_1)), \; t_0\leq s\leq t_1.$$

This inequality implies that (H_2) holds for $x(s, t_0, x_0)$ on $t_0\leq s\leq t_1$ and as a result, standard computation yields

$$D_-m(t_1)\leq g(t_1, m(t_1))$$

which contradicts (3.6.4). Thus $m(t) \leq r(t, t_0, u_0)$, $t \geq t_0$ and the proof is complete.

Having this comparison theorem at our disposal, we merely state the following theorem that offers various practical stability criteria in a single setup, since we shall prove later a more general result relative to impulsive integro-differential equations.

Theorem 3.6.2

Let (H_0), (H_1) and (H_2) hold. Then the practical stability properties of (3.6.2) imply the corresponding practical stability properties of (3.6.1).

The following special cases of Theorem 3.6.2 are important.

Case I Suppose that $g_0(t, u) \equiv 0$ so that $g(t, u) \geq 0$. Then $\eta(s, t^0, v_0) = v_0$ and the set Ω in (H_2) reduces to

$$\Omega = \{x \in C[\mathbf{R}_+, \mathbf{R}^n]: \; V(s, x(s)) \leq V(t, x(t)), \; t_0 \leq s \leq t\}.$$

Case II Suppose that $g_0(t, u) = -[\frac{A'(t)}{A(t)}]\, u$, wher $A(t) > 0$ is continuously differentiable on $[t_0, \infty)$ and $A(t) \to \infty$ as $t \to \infty$. Let $g(t, u) = g_0(t, u) + [\frac{1}{A(t)}]\, g_1(t, A(t)\, u)$ with $g_1 \in C[\mathbf{R}_+^2, \mathbf{R}_+]$. Evidently $\eta(s, t^0, v_0) = \frac{v_0 A(t)}{A(s)}$ for $t_0 \leq s \leq t^0$. Hence

$$\Omega = \{x \in C[\mathbf{R}_+, \mathbf{R}^n]: \; V(s, x(s))A(s) \leq V(t, x(t))\, A(t), \; t_0 \leq s \leq t\}.$$

Case III Suppose that $g_0(t, u) = -\gamma(u)$ where $\gamma \in K$ and $g(t, u) \equiv g_0(t, u)$. Computing $\eta(s, t^0, v_0)$ we see that

$$\eta(s, t^0, v_0) = J^{-1}[J(v_0) - (s - t^0)],\ 0 \le s \le t^0,$$

where $J(u) - J(u_0) = \int_{u_0}^{u} \frac{1}{\gamma(s)} ds$ and J^{-1} is the inverse function of J.

Since $\eta(s, t^0, v_0)$ is increasing in s to the left of t^0, choosing a fixed $s_0 < t^0$ and defining $L(u) = \eta(s_0, t^0, u)$, it is clear that $L(u) > u$ for $u > 0$, $L(u)$ is continuous and increasing in u. Hence the set reduces to

$$\Omega = \{x \in C[\mathbf{R}_+, \mathbf{R}^n]:\ V(s, x(s)) \le L(V(t, x(t))),\ t_0 \le s \le t\}.$$

3.7. IMPULSIVE DIFFERENTIAL SYSTEMS.

We shall devote this section to extend the method of Lyapunov functions to impulsive differential equations considered in Section 1.7. In the study of stability of impulsive differential systems, the use of classical Lyapunov functions restricts the flexibility of the method. The fact that the solutions of such systems are piecewise continuous functions makes it necessary to introduce certain analogues of Lyapunov functions which has the first kind of discontinuities. By means of such functions, the extension of Lyapunov's method to impulsive differential systems is much more effective.

Consider the impulsive differential system

$$(3.7.1)\qquad \begin{cases} x' = f(t, x),\ t \ne t_k,\ x(t_0^+) = x_0,\ t_0 \ge 0, \\ \Delta x = I_k(x),\ t = t_k,\ k = 1, 2, \ldots, \end{cases}$$

under the following assumptions:

(A_0) (i) $0<t_1<t_2<\ldots<t_k<\ldots$ and $t_k\to\infty$ as $k\to\infty$;

(ii) $f\in \mathbf{R}_+\times\mathbf{R}^n\to\mathbf{R}^n$ is continuous in $(t_{k-1}, t_k]\times\mathbf{R}^n$ and for each $x\in\mathbf{R}^n$, $k=1,2,\ldots,$ $\lim_{(t,y)\to(t_k^+,x)} f(t,y)=f(t_k^+,x)$ exists

(iii) I_k: $\mathbf{R}^n\to\mathbf{R}^n$.

Let V: $\mathbf{R}_+\times\mathbf{R}^n\to\mathbf{R}_+$. Then V is said to belong to class V_0 if

(i) V is continuous in $(t_{k-1}, t_k]\times\mathbf{R}^n$ and for each $x\in\mathbf{R}^n$, $k=1, 2, \ldots,$ $\lim_{(t,\,y)\to(t_k^+,\,x)} V(t, y)=V(t_k^+, x)$ exists;

(ii) V is locally Lipschitzian in x for $(t_{k-1}, t_k]$.

For $(t, x)\in(t_{k-1}, t_k]\times\mathbf{R}^n$, define

$$D^+V(t, x)=\lim_{h\to0^+}\sup\frac{1}{h}[V(t+h, x+hf(t, x))-V(t, x)]. \tag{3.7.2}$$

We can now formulate the following comparison result.

Theorem 3.7.1

Let V: $\mathbf{R}_+\times\mathbf{R}^n\to\mathbf{R}_+$ and $V\in V_0$. Assume that

$$\left\{\begin{array}{l} D^+V(t, x)\le g(t, V(t, x)),\ t\ne t_k, \\ V(t, x+I_k(x))\le\psi_k(V(t, x)),\ t=t_k,\ k=1, 2, \ldots, \end{array}\right. \tag{3.7.3}$$

where $g:\mathbf{R}_+^2\to\mathbf{R}$ satisfies $(A_0(ii))$ and $\psi_k:\mathbf{R}_+\to\mathbf{R}_+$ is nondecreasing. Let $r(t)=r(t, t_0, u_0)$ be the maximal solution of the scalar impulsive differential equation

$$(3.7.4) \qquad \begin{cases} u' = g(t, u), \ t \neq t_k, \ u(t_0^+) = u_0 \geq 0, \\ u(t_k^+) = \psi_k(u(t_k)), \ k = 1, 2, \ldots, \end{cases}$$

existing on $[t_0, \infty)$. Then $V(t_0^+, x_0) \leq u_0$ implies that

$$(3.7.5) \qquad V(t, x(t)) \leq r(t), \ t \geq t_0,$$

where $x(t) = x(t, t_0, x_0)$ is any solution of (3.7.1) existing on $[t_0, \infty)$.

Proof

Let $x(t) = x(t, t_0, x_0)$ be any solution of (3.7.1) existing for $t \geq t_0$ such that $V(t_0^+, x_0) \leq u_0$. Define $m(t) = V(t, x(t))$ for $t \neq t_k$ so that for small $h > 0$, we have

$$m(t+h) - m(t) = V(t+h, x(t+h)) - V(t+h, x(t) + hf(t, x(t)))$$
$$+ V(t+h, x(t) + hf(t, x(t))) - V(t, x(t)).$$

Since $V(t, x)$ is locally Lipschitzian in x for $t \in (t_k, t_{k+1}]$, using (3.7.3), we arrive at

$$D^+ m(t) \leq g(t, m(t)), \ t \neq t_k, \ m(t_0^+) \leq u_0,$$
$$m(t_k^+) = V(t_k^+, x(t_k) + I_k(x(t_k))) \leq \psi_k(m(t_k)), \ k = 0, 1, 2, \ldots.$$

Hence by Theorem A.3.1, we obtain the desired estimate (3.7.5).

The following corollary of Theorem 3.7.1 is useful.

Corollary 3.7.1

If in Theorem 3.7.1, we suppose that the following holds:

(a) $g(t, u) \equiv 0$, $\psi_k(u) = u$ for all k;

(b) $g(t, u) \equiv 0$, $\psi_k(u) = d_k u$, $d_k \geq 0$ for all k;

(c) $g(t, u) = -\alpha u$, $\alpha > 0$, $\psi_k(u) = d_k u$, $d_k \geq 0$ for all k;

(d) $g(t, u) = \lambda'(t)u$, $\psi_k(u) = d_k u$, $d_k \geq 0$ for all k and $\lambda \in C^1[\mathbf{R}_+, \mathbf{R}_+]$,

then the following conclusions are valid respectively:

(a) $V(t, x(t))$ is nonincreasing in t and $V(t, x(t)) \leq V(t_0^+, x_0)$, $t \geq t_0$;

(b) $V(t, x(t)) \leq V(t_0^+, x_0) \prod_{t_0 < t_k < t} d_k$, $t \geq t_0$;

(c) $V(t, x(t)) \leq [V(t_0^+, x_0) \prod_{t_0 < t_k < t} d_k] \exp(-\alpha(t - t_0))$, $t \geq t_0$;

(d) $V(t, x(t)) \leq [V(t_0^+, x_0) \prod_{t_0 < t_k < t} d_k] \exp(\lambda(t) - \lambda(t_0))$, $t \geq t_0$.

Let us consider the impulsive differential system (3.7.1) subject to (A_0). We shall prove the following result which offers sufficient conditions in a unified way for various practical stability criteria.

Theorem 3.7.2

Assume that

(i) $0 < \lambda < A$ are given;

(ii) $V \in \mathbf{R}_+ \times \mathbf{R}^n \to \mathbf{R}_+$, $V \in V_0$ and satisfies (3.7.3) for $x \in S(A)$;

(iii) there exists a $\rho = \rho(A) > 0$ such that $x \in S(A)$ implies $x + I_k(x) \in S(\rho)$ for all k;

(iv) for $(t, x) \in \mathbf{R}_+ \times S(\rho)$, $b(|x|) \leq V(t, x) \leq a(|x|)$, $a, b \in K$;

(v) $a(\lambda) < b(A)$ holds.

Then the practical stability properties of (3.7.4) imply the corresponding practical stability properties of (3.7.1).

Proof

Suppose that (3.7.4) is practically stable. Then, we have

$$u_0 < a(\lambda) \text{ implies } u(t, t_0, u_0) < b(A),\ t \geq t_0,$$

where $u(t, t_0, u_0)$ is any solution of (3.7.4). Let $u_0 = a(|\ x_0\ |)$. We claim that if $|\ x_0\ | < \lambda$, then $|\ x(t)\ | < A$ for $t \geq t_0$, where $x(t) = x(t, t_0, x_0)$ is any solution of (3.7.1). If this is not true, there would exist a solution $x(t)$ of (3.7.1) with $|\ x_0\ | < \lambda$ and a $t^* > t_0$ such that $t_k < t^* \leq t_{k+1}$ for some k, satisfying

$$|\ x(t^*)\ | \geq A \text{ and } |\ x(t)\ | < A \text{ for } t_0 \leq t \leq t_k.$$

Condition (iii) shows that $|\ x_k^+\ | = |\ x_k + I_k(x_k)\ | < \rho$ whenever $|\ x_k\ | < A$ where $x_k = x(t_k)$. Hence we can find a t^0 such that $t_k < t^0 \leq t^*$ and $A \leq |x(t^0)\ | < \rho$. Now, setting $m(t) = V(t, x(t))$ for $t_0 \leq t \leq t^0$ and using (ii) and (iii), we get by Theorem 3.7.1, the estimate

$$V(t, x(t)) \leq r(t, t_0, a(|\ x_0\ |)),\ t_0 \leq t \leq t^0, \tag{3.7.6}$$

where $r(t, t_0, u_0)$ is the maximal solution of (3.7.4). We are then lead to the contradiction

$$b(A) \leq b(|\ x(t^0)\ |) \leq V(t^0, x(t^0)) \leq r(t^0, t_0, a(|\ x_0\ |)) < b(A)$$

because of (iv) and (v). This proves that (3.7.1) is practically stable.

Let us suppose next that (3.7.4) is strongly practically stable which implies that (3.7.1) is practically stable. Consequently, we have

(3.7.7) $$| x_0 | < \lambda \text{ implies } | x(t) | < A,\ t \geq t_0.$$

Let $0<B<A$ and $T>0$ be given. Since (3.7.4) is practically quasi-stable, given $b(B)>0$ and $T>0$, it follows that

$$0 \leq u_0 < a(\lambda) \text{ implies } u(t, t_0, u_0) < b(B),\ t \geq t_0+T.$$

Let $| x_0 | < \lambda$. In view of (3.7.7), arguments leading to (3.7.6) yield

$$V(t, x(t)) \leq r(t, t_0, a(| x_0 |)),\ t \geq t_0,$$

from which it follows that

$$b(| x(t) |) \leq V(t, x(t)) \leq r(t, t_0, a(| x_0 |)) < b(B),\ t \geq t_0+T,$$

which proves that $| x(t) | < B$, $t \geq t_0+T$. Hence the proof of Theorem 3.7.2 is complete.

Corollary 3.7.2

In Theorem 3.7.2,

(a) the functions $g(t, u) \equiv 0$, $\psi_k(u)=d_k u$, $d_k \geq 0$ for all k are admissible to yield uniform practical stability of (3.7.1) provided the infinite product $\prod_{i=1}^{\infty} di$ converges. In particular, $d_k=1$ for all k is admissible;

(b) $g(t, u)=\lambda'(t)u$, $\lambda \in C^1[\mathbf{R}_+, \mathbf{R}_+]$, $\psi_k(u)=d_k u$, $d_k \geq 0$ for all k are admissible to imply practical stability of (3.7.1) provided

(3.7.8) $$\lambda(t_k)+\ln d_k \leq \lambda(t_{k-1}) \text{ for all k;}$$

(c) the functions in (b) are admissible to assure practical asymptotic stability of (3.7.1) provided (3.7.8) is strengthened to

(3.7.9) $\lambda(t_k)+\ln(\alpha\, d_k)\leq\lambda(t_{k-1})$ for all k, where $\alpha>1$.

Proof

The claim in (a) is immediate. To prove (b) and (c), we see that any solution $u(t, t_0, u_0)$ of

(3.7.10)
$$\begin{cases} u'=\lambda'(t)u,\ t\neq t_k,\ u(t_0^+)=u_0\geq 0 \\ u(t_k^+)=d_k u(t_k),\ k=1, 2, \ldots, \end{cases}$$

is given by

$$u(t, t_0, u_0)=u_0\Big(\prod_{t_0<t_k<t} d_k\Big)\exp[\lambda(t)-\lambda(t_0)],\ t\geq t_0$$

Since $\lambda(t)$ is nondecreasing, it follows from (3.7.8) that

$$u(t, t_0, u_0)\leq u_0\exp[\lambda(t_1)-\lambda(t_0)],\ t\geq t_0,$$

provided $0<t_0<t_1$. Hence if λ, A satisfy the relation

$$\exp[\lambda(t_1)-\lambda(t_0)]<\frac{b(A)}{a(\lambda)}$$

then practical stability of (3.7.10) follows.

If on the other hand (3.7.9) holds, then we get

$$u(t, t_0, u_0)\leq u_0\exp[\lambda(t_1)-\lambda(t_0)]\frac{1}{\alpha^k},\ t_{k-1}<t\leq t_k,$$

from which $\lim_{t\to\infty} u(t, t_0, u_0)=0$ follows. Thus Theorem 3.7.2 implies the claim of Corollary 3.7.2

Let us next consider an example.

Example

Consider

$$(3.7.11)\quad \begin{cases} x' = [\sin(\ln(t+1)) + \cos(\ln(t+1)) - 2]x, \ t \neq t_k, \ x(t_0^+) = x_0, \\ \Delta x = \beta_k x, \ \ t = t_k, \ k = 1, 2, \ldots, \end{cases}$$

with $-1 \leq \beta_k \leq 0$. Taking $V = |x|^2$, we get

$$\begin{cases} D^+V(x) \leq \lambda'(t)V(x), \ t \neq t_k \\ V(x + I_k(x)) \leq (1+\beta_k)^2 V(x) \leq V(x), \ t = t_k \end{cases}$$

where $\lambda(t) = \exp[-2(t+1)(2 - \sin \ln(t+1))]$, so that

$$g(t, u) = \lambda'(t)u \text{ and } \psi_k(u) = u.$$

Consequently the general solution is given by

$$u(t, t_0, u_0) = u_0 \exp[\lambda(t) - \lambda(t_0)], \ t \geq t_0.$$

Hence (3.7.4) is practically asymptotically stable and therefore (3.7.11) is also practically asymptotically stable.

3.8. IMPULSIVE INTEGRO-DIFFERENTIAL EQUATIONS.

In this section, we wish to extend Lyapunov's method to impulsive integro-differential equations using piecewise continuous Lyapunov functions and the theory of impulsive differential inequalities. This important technique which reduces the study of impulsive integro-differential equations to the study of impulsive differential equations crucially depends on choosing suitable minimal sets of functions along which the derivative of the Lyapunov function allows a convenient estimate.

Consider the impulsive integro-differential system

$$\text{(3.8.1)} \qquad \begin{cases} x' = f(t, x, Tx), \ t \neq t_i, \ x(t_0^+) = x_0, \ t_0 \geq 0, \\ \Delta x = I_i(x(t_i)), \ i = 1, 2, \ldots, \end{cases}$$

where the following conditions hold for f, T and I:

(i) $0 < t_1 < t_2 < \ldots < t_i < \ldots$, $t_i \to \infty$ as $i \to \infty$ and I_i: $\mathbf{R}^n \to \mathbf{R}^n$;

(ii) f: $\mathbf{R}_+ \times \mathbf{R}^n \times \mathbf{R}^n \to \mathbf{R}^n$ is continuous on $(t_{i-1}, t_i] \times \mathbf{R}^n \times \mathbf{R}^n$, $i = 1, 2, \ldots$;

(iii) $Tx = \int_{t_0}^{t} K(t, s, x(s))ds$ where K: $\mathbf{R}_+^2 \times \mathbf{R}^n \to \mathbf{R}^n$ is continuous on $(t_{i-1}, t_i] \times (t_{i-1}, t_i] \times \mathbf{R}^n$, $i = 1, 2, \ldots$.

We shall assume existence and uniqueness of solutions of (3.8.1) and note that the solutions $x(t) = x(t, t_0, x_0)$ of (3.8.1) are piecewise continuous with discontinuities of the first type at $t = t_i$, at which they are left continuous.

We need the following comparison result before we proceed to extend Lyapunov's method to (3.8.1). We shall employ Lyapunov functions introduced in Section 3.7 and the comparison Theorem 3.6.1. For convenience let us state the following hypothesis:

(H_1) g_0, g: $\mathbf{R}_+^2 \to \mathbf{R}$ are continuous on $(t_{i-1}, t_i] \times \mathbf{R}_+$,

$\lim_{(t,v)\to(t_i,u)} g(t, v)=g(t_i^+, u)$, $\lim_{(t,v)\to(t_i,u)} g_0(t, v)=g_0(t_i^+, u)$,

$t>t_i$, $g_0(t, u)\le g(t, u)$, $r(t, t_0, u_0)$ is the right maximal solution of the impulsive differential equation

$$\text{(3.8.2)} \quad \begin{cases} u'=g(t, u),\ t\neq t_i,\ u(t_0^+)=u_0\geq 0,\ t_0\geq 0, \\ u(t_i^+)=\psi_i(u(t_i)),\ i=1, 2, \ldots, \end{cases}$$

existing on $[t_0, \infty)$ and $\eta(t, t^0, v_0)$ is the left maximal solution of

$$v'=g_0(t, v),\ v(t^0)=v_0\geq 0$$

$$v(t_i^+)=\psi_i(u(t_i)),\ i=1, 2, \ldots,$$

existing on $t_0\le t\le t^0$, ψ_i: $\mathbf{R}_+\to\mathbf{R}_+$ is nondecreasing.

Theorem 3.8.1

Assume that (H_1) is satisfied and

(i) $V\in V_0$, $V(t, x)$ is locally Lipschitzian in x and for $t>t_0$, $x\in\Omega$,

$$D_-V(t, x)\le g(t, V(t, x)),\ t\neq t_i,$$

where $\Omega=\{x\in PC[\mathbf{R}_+, \mathbf{R}^n]:\ V(s, x(s))\le\eta(s, t, V(t, x(t))),\ t_0\le s\le t\}$, PC denoting the class of piecewise continuous functions;

(ii) $$V(t^+, x+I_i(x))\le\psi_i(V(t, x)),\ t=t_i.$$

Then, if $x(t)=x(t, t_0, x_0)$ is any solution of (3.8.1) existing on $[t_0,\infty)$, we have

$$V(t, x(t))\le r(t, t_0, u_0),\ t\geq t_0$$

provided $V(t_0^+, x_0) \le u_0$.

Proof

Let $t_0 \ge 0$ and $t_0 \in (t_{j-1}, t_j]$ for some $j \ge 1$. For convenience, we designate

$$t_i = t_{j+i-1} \text{ if } t_0 \ne t_j,$$

$$t_i = t_{j+i} \text{ if } t_0 = t_j,\ i = 1, 2, \ldots.$$

Let $x(t) = x(t, t_0, x_0)$ be any solution of (3.8.1) existing on (t_0, ∞) and set $m(t) = V(t, x(t))$. Then for $t \in (t_0, t_1]$, Theorem 3.6.1 implies that $m(t) \le r_1(t, t_0, u_0)$ where $r_1(t, t_0, u_0)$ is the right maximal solution of (3.8.2) on $(t_0, t_1]$ such that $r_1(t_0^+, t_0, u_0) = u_0$. Since $\psi_1(u)$ is nondecreasing in u and $m(t_1) \le r_1(t_1, t_0, u_0)$, we get from (ii), $m(t_1^+) \le u_1^+$ where $u_1^+ = \psi_1(r_1(t_1, t_0, u_0))$.

Using again Theorem 3.6.1, we obtain

$$m(t) \le r_2(t, t_1, u_1^+),\ t \in (t_1, t_2],$$

where $r_2(t, t_1, u_1^+)$ is the right maximal solution of (3.8.2) on $(t_1, t_2]$, such that $r_2(t_1^+, t_1, u_1^+) = u_1^+$. We therefore have successively

$$m(t) \le r_{i+1}(t, t_i, u_i^+),\ t \in (t_i, t_{i+1}],$$

where $r_{i+1}(t, t_i, u_i^+)$ is the maximal solution of (3.8.2) existing on $(t_i, t_{i+1}]$ such that $r_{i+1}(t_i^+, t_i, u_i^+) = u_i^+$. Thus if we define

$$u(t)=\begin{cases} u_0, \ t=t_0 \\ r_1(t, t_0, u_0), \ t\in(t_0, t_1], \\ r_2(t, t_1, u_i^+), \ t\in(t_1, t_2], \\ \cdots\cdots \\ \cdots\cdots \\ r_{i+1}(t, t_i, u_i^+), \ t\in(t_i, t_{i+1}], \\ \cdots\cdots \\ \cdots\cdots \end{cases}$$

then it is easy to see that u(t) is a solution of (3.8.2) and

$$m(t)\leq u(t), \ t\geq t_0.$$

Since $r(t, t_0, u_0)$ is the maximal solution of (3.8.2), we get immediately

$$m(t)\leq r(t, t_0, u_0), \ t\geq t_0$$

and the proof is complete.

Let us collect several interesting and useful special cases of Theorem 3.8.1 in the following corollary.

Corollary 3.8.1

If in Theorem 3.8.1,

(i) $g_0(t, u)=g(t, u)\equiv 0$ and $\psi_i(u)=u$ for all i, then $V(t, x(t))$ is nondecreasing in t and

$$V(t, x(t)) \leq V(t_0^+, x_0),\ t \geq t_0.$$

(ii) $g_0(t, u) = g(t,u) \equiv 0$ and $\psi_i(u) = d_i u$, $d_i \geq 0$ for all i, then

$$V(t, x(t)) \leq V(t_0^+, x_0) \prod_{t_0 < t_i < t} d_i,\ t \geq t_0.$$

(iii) $g_0(t, u) \equiv 0$, $g(t, u) = \lambda'(t)u$ where $\lambda \in C^1[\mathbf{R}_+, \mathbf{R}_+]$, $\lambda'(t) \geq 0$ and $\psi_i(u) = d_i u$, $d_i \geq 0$ for all i, then

$$V(t, x(t)) \leq [V(t_0^+, x_0) \prod_{t_0 < t_i < t} d_i] \exp[\lambda(t) - \lambda(t_0)],\ t \geq t_0.$$

(iv) $g_0(t, u) = g(t, u) = -\dfrac{A'(t)}{A(t)}u$, where $A(t) > 0$ is continuously differentiable on $\mathbf{R}_+$, and $A(t) \to \infty$, and $\psi_i(u) = d_i u$, $d_i \geq 0$ for i, then

$$V(t, x(t)) \leq [V(_0^+, x_0) \prod_{t_0 < t_i < t} d_i] \frac{A(t_0)}{A(t)},\ t \geq t_0.$$

In particular, $A(t) = e^{\alpha t}$, $\alpha > 0$, is admissible.

(v) $g_0(t, u) = g(t, u) = -\gamma(u)$ where $\gamma \in K$, $\psi_i(u) = u$ for all i, then

$$V(t, x(t)) \leq J^{-1}[J(V(t_0^+, x_0) - (t - t_0)],\ t \geq t_0,$$

where J^{-1} is the inverse function of J and $J'(u) = \dfrac{1}{\gamma(u)}$.

It is useful to know how the minimal class of functions Ω of assumption (i) change depending on the choice of g_0 since the derivative of Lyapunov function has to be estimated along these sets. Due to the fact these special cases are precisely similar to the special cases considered for

Theorem 3.6.2, we shall not repeat them here.

Having the comparison Theorem 3.8.1 at our disposal, it is now easy to prove in a unified way various sufficient conditions for practical stability of the system 3.8.1. We shall merely state such a result, since its proof can be constructed based on Theorems 2.2.1 and 2.7.1.

Theorem 3.8.2

Suppose that the assumptions of Theorem 3.8.1 hold. Assume further that $0<\lambda<A$ are given, and there exists a $\rho=\rho(A)>0$ such that if $x\in S(A)$, then $x+I_k(x)\in S(\rho)$ for each k. Suppose also that

$$b(|x|)\leq V(t,x)\leq a(|x|) \text{ for } (t,x)\in \mathbf{R}_+\times S(\rho),$$

where a, $b\in K$ and that $a(\lambda)<b(A)$ holds. Then the practical stability properties of the scalar differential equation (3.8.2) imply the corresponding practical stability properties of the system (3.8.1).

Finally, we consider a simple situation of (3.8.1) where

$$f(t,x,Tx)=Ax+\int_{t_0}^{t} K(t,s,x(s))ds$$

and suppose that

$$|K(t,s,x)|\leq H(t,s)|x| \text{ on } \mathbf{R}_+\times \mathbf{R}^n,$$

other conditions being the same. Then taking $V(t,x)=e^{\beta t}|x|$, $\beta>0$ and using the set

$$\Omega=[x\in PC[\mathbf{R}_+,\mathbf{R}^n]:\ e^{\beta s}|x(s)|\leq e^{\beta t}|x(t)|,\ t_0\leq s\leq t],$$

we easily compute

$$D_-V \leq [\beta + \mu(A) + \int_{t_0}^{t} e^{\beta(t-s)} H(t, s)ds]V,$$

where $\mu(A)$ is the logarithmic norm of A defined by

$$\mu(A) = \lim_{h \to 0} \frac{1}{h}[\| I + hA| - 1],$$

I being the identity matrix. Thus, we see that $g(t, u) = \sigma(t)u$, where

$\sigma(t) = \beta + \mu(A) + \int_{t_0}^{t} e^{\beta(t-s)} H(t, s)ds$. If $\psi_i(u) = d_i u$, $d_i \geq 0$ for all i,

then one can conclude practical stability of (3.8.1) based on Corollary 3.7.1 and Theorem 3.8.2.

3.9. REACTION-DIFFUSION EQUAITONS.

In ths section, we investigate practical stability properties of weakly coupled reaction-diffusion systems by means of vector Lyapunov functions and show that this effective approach is also a natural setting for the discussion of such systems.

Let Ω be a bounded domain in $\mathbf{R}^n$ and let $H = (t_0, \infty) \times \Omega$, $t_0 \geq 0$. Suppose that the boundary ∂H of H is split into two parts ∂H_0, ∂H_1, such that $\partial H = \partial H_0 \cup \partial H_1$, $\{t_0\} \times \partial\Omega \subset \partial H_0$ and $\partial H_0 \cap \partial H_1$ is empty.

A vector ν is said to be an outer normal at $(t, x) \in \partial H_1$ if $(t, x - h\nu) \in H$ for small $h > 0$. The outer normal derivative is then given by

$$\frac{\partial u(t, x)}{\partial \nu} = \lim_{h \to 0} \frac{u(t, x) - u(t, x - h\nu)}{h}$$

for any $u \in C[\overline{H}, \mathbf{R}^N]$. We shall always assume that an outer normal exists on ∂H_1 and the functions in question have outer normal derivatives on ∂H_1. If $u \in C[\overline{H}, R^N]$ is such that the partial derivatives u_t, u_x, u_{xx} exist and are continuous in H, then we shall say that $u \in C^*(J)$, $J=[t_0, \infty)$.

Let $f \in C[H \times \mathbf{R}^N \times \mathbf{R}^n \times \mathbf{R}^{n^2}, \mathbf{R}^N]$. (Here f represents the vector $f_k(t, x, u, u_x^k, u_{xx}^k)$, $k= 1, 2, \ldots, N$; it is important to note that each component f_k contains partial derivatives of kth component of u only. For convenience, we shall use the notation $f(t, x, u, u_x, u_{xx})$ to represent such vector functions). Then f is said to be elliptic at $(t_1, x_1) \in H$ if for any u, P, Q, R, the quadratic form

$$\sum_{i,j=1}^{N} (Q_{ij}-R_{ij})\lambda_i\lambda_j \leq 0, \ \lambda \in \mathbf{R}^n$$

implies

$$f(t, {}_1, x_1, u, P, Q) \leq f(t_1, x_1, u, P, R).$$

If this property holds for every $(t, x) \in H$, then f is said to be elliptic in H. Here and in what follows, the inequalities between vectors are understood component wise. Also f is said to be quasimonotone nondecreasing in u if

$$u \leq v, \ u_i = v_i \text{ for } ,i \leq N \text{ implies } f_i(t, x, u, P, Q) \leq f_i(t, x, v, P, Q).$$

The following comparison result is crucial to our discussion; for the proof, see Lakshmikantham [2].

Theorem 3.9.1

Assume that

(i) $v, w \in C^*(J)$, $f \in C[H \times \mathbf{R}^n \times \mathbf{R}^n \times \mathbf{R}^{n^2}, \mathbf{R}^N]$, f is elliptic in H and quasimontone nondecreasing in u and

$$v_t \le f(t, x, v, v_x, v_{xx}),\ w_t \ge f(t, x, w, w_x, w_{xx}) \text{ in } H;$$

(ii) $Z \in C^*(J)$, $Z > 0$ on $\overline{H}$, $\partial Z/\partial\nu \ge \beta > 0$ on ∂H_1 and for sufficiently small $\epsilon >$, either

a) $\epsilon Z_t > f(t, x, v, v_x, v_{xx}) - f(t, x, v-\epsilon Z, v_x - \epsilon Z_{xx})$ or

b) $\epsilon Z_t > f(t, x, w+\epsilon Z, w_x + \epsilon Z_x, w_{xx}) - f(t, x, w, w_x, w_{xx})$ on h;

(iii) $v \le w$ on ∂H_0 and $\dfrac{\partial v}{\partial \nu} \le \dfrac{\partial w}{\partial \nu}$ on ∂H_1.

Then, $v \le w$ on $\overline{H}$.

As an example, consider the interesting special case

$$f_k(t, x, u, u_x^k, u_{xx}^k) = a^k u_{xx}^k + b^k u_x^k + F_k(t, x, u) \tag{3.9.1}$$

where

$$a^k u_{xx}^k = \sum_{i,j=1}^{n} a_{ij}^k u_{x_i x_j},\ b^k u_x^k = \sum_{j=1}^{n} b_j^k u_{x_j}^k,\ k = 1, 2, \ldots, N$$

and F is Lipschitzian and quasimontone nondecreasing in u. That is, F satisfies

$$|\, F_k(t, x, u) - F_k(t, x, v)\, | \le L \sum_{\mu=1}^{n} |\, u_\mu - v_\mu\, |\ ,\ (t, x) \in H.$$

Assume also that the boundary H is smooth enough, that is, there exists a $h \in C^2[\Omega, \mathbf{R}_+]$ such that $\partial h/\partial\nu \ge 1$ on ∂H_1 and h_x, h_{xx} are bounded.

Let $M > 1$ and define $H(x) = \exp((mLh(x))$, $Z(x) = (\exp(N_0 t))H(x)$ and $\widetilde{Z}(x) = \widetilde{e} Z(t)$, where $\widetilde{e} = (1, 1, \ldots, 1)$, $N_0 = MLN + A$, L is the Lipschitz constant for F and $|a^k H_{xx} + b^k H_x| \le A_k \le A$, $k = 1, 2, \ldots, N$. Then

$$\frac{\partial Z}{\partial \nu}=ML\left(\frac{h(x)}{\partial \nu}\right)\tilde{e}\geq ML\tilde{e}>0 \text{ on } \partial H_1$$

and

$$\epsilon(\widetilde{Z}^k_t-a^k\,\widetilde{Z}^k_{xx}-b^k\,\widetilde{Z}^k_x)\geq\epsilon(N_0-A)\,\widetilde{Z}^k=\epsilon MLN\,\widetilde{Z}^k>\epsilon LNZ.$$

Using the Lipschitz condition of F, we have

$$|\,F^k(t, x, w+\epsilon\widetilde{Z})-F^k(t, x, w)\,|\leq L\sum_{\mu=1}^{n}\epsilon\widetilde{Z}_\mu=\epsilon LNZ$$

and consequently, we get for $\epsilon>0$,

$$\epsilon\widetilde{Z}_t>\epsilon[a\widetilde{Z}_{xx}+b\widetilde{Z}_{xx}]+F(t, x, w+\epsilon\widetilde{Z})-F(t, x, w)$$

which is exactly condition (b) of (ii) in Theorem 3.9.1. Similarly (a) of (ii) is also verified.

We note that if H_1 is empty so that $\partial H=\partial H_0$, then assumption (ii) can be replaced by a one-sided Lipschitz condition of the form

$$\text{(3.9.2)}\qquad f^k(t, x, u, P, Q)-f^k(t, x, \overline{u}, P, Q)\leq\sum_{\mu=1}^{n}(u_\mu-\overline{u}_\mu)\quad u\geq\overline{u}$$

where L is a positive constant. Even when ∂H_1 is non-empty assumption (3.9.2) is enough if (iii) is strengthened to

$$\frac{\partial v}{\partial \nu}+\psi(t, x, v)\leq\frac{\partial w}{\partial \nu}+\psi(t, x, w) \text{ on } \partial H_1$$

where $\psi\in C[\partial H_1\times\mathbf{R}^N, \mathbf{R}^N]$ and $\psi(t, x, u)$ is strictly increasing in u. Of course, if ψ is not strictly increasing in u or $\psi\equiv0$, then we need condition (ii) which implies that we require smooth boundary information when considering such reaction-diffusion systems.

We consider the system of reaction-diffusion equations

$$(3.9.3)\qquad \begin{cases} u_t = Lu + f(t, x, u) \text{ in } J\times\Omega,\ J=(t_0, \infty),\ t_0 \geq 0 \\ u(t_0, x) = \phi_0(x) \text{ in } \overline{\Omega},\ \dfrac{\partial u(t, x)}{\partial \nu} = 0 \text{ on } J\times\partial\Omega, \end{cases}$$

where the elliptic operator L is such that

$$L_k u^k = \sum_{i,j=1}^{n} a_{ij}^k u_{x_i x_j}^k + \sum_{j=1}^{n} b_j^k u_{x_j}^k,\ k = 1, 2, \dots, N,$$

$$\sum_{i,j=1}^{n} a_{ij}^k \lambda_i \lambda_j \geq \beta |\lambda|^2,\ \lambda \in \mathbf{R}^n;$$

and $f \in C[\mathbf{R}_+ \times \overline{\Omega} \times \mathbf{R}^N, \mathbf{R}^N]$. Here $\Omega \subset \mathbf{R}^n$ is assumed to be bounded, open connected region equiped with a smooth boundary. We assume existence and uniqueness of solutions of (2.1) in $C^*(J)$. For existence results, see Amann [1].

On the basis of Theorem 3.9.1, we can now extend the method of vector Lyapunov functions to study practical stability properties of solutions of (3.9.3).

Theorem 3.9.2

Assume that

(i) $0 < \lambda < A$ are given;

(ii) $V \in C^2[\mathbf{R}_+ \times \mathbf{R}^N, \mathbf{R}_+^N]$, $V_u(t, u)Lu \leq LV(t, u)$, and
$V_t(t, u) + V_u(t, u)f(t, x, u) \leq g(t, V(t, u)$ on $\mathbf{R}_+ \times \Omega \times S(A)$,
where $g \in C[\mathbf{R}_+ \times \mathbf{R}_+^N, \mathbf{R}_+^N]$, $g(t, u)$ is quasimontone nondecreasing and locally Lipschitzian in u;

(iii) On $\mathbf{R}_+ \times S(A)$, $b(|u|) \leq \sum_{i=1}^{n} V_i(t, u) \leq a(|u|)$, where $a, b \in K$;

(iv) $a(\lambda) < b(A)$ holds.

Then, the practical stability properties of either (a)

$$y' = g(t, u),\ y(t_0) = y_0 \geq 0, \tag{3.9.4}$$

or (b)

$$\begin{cases} u_t = Lv + g(t, v) \text{ in } J\times\Omega \\ v(t_0, x) = \psi_0(x) \geq 0 \text{ in } \overline{\Omega},\ \dfrac{\partial v(t, x)}{\partial \nu} = 0 \text{ on } J\times\partial\Omega, \end{cases} \tag{3.9.5}$$

imply the corresponding practical stability properties of (3.9.3)

Proof

Let $u(t, x)$ be any solution of (3.9.3) and $u \in C^*(J)$. Setting $m(t,x) = V(t,u(t,x))$ and using assumption (i), we get

$$\begin{cases} m_t \leq Lm + g(t, m) \text{ in } J\times\Omega \\ m(t_0, x) = V(t_0, \phi_0(x)) \text{ in } \overline{\Omega},\ \dfrac{\partial m(t, x)}{\partial \nu} = 0 \text{ in } J\times\partial\Omega. \end{cases} \tag{3.9.6}$$

Let $y(t)$ and $r(t, x)$ be the solutions of (3.9.4) and (3.9.5) respectively existing for $t \geq t_0$ and $x \in \overline{\Omega}$. Then we have

$$y' = g(t, u),\ y(t_0) = y_0 \geq m(t_0, x) \text{ in } \overline{\Omega} \tag{3.9.7}$$

and

$$\begin{cases} r_t = Lr + g(t, r) \text{ in } J\times\Omega, \\ r(t_0, x) \geq m(t_0, x) \text{ in } \overline{\Omega},\ \dfrac{\partial m(t, x)}{\partial \nu} = 0 \text{ in } J\times\partial\Omega. \end{cases} \tag{3.9.8}$$

Consequently, applying Theorem 3.9.1 yields with $v = m$ and $w = y$ or $w = r$ the estimates $V(t, u(t, x)) \leq y(t)$, or $V(t, u(t, x)) \leq r(t, x)$ in $J\times\overline{\Omega}$.

From these estimates and assumptions of Theorem 3.9.2 it is now easy to prove, the conclusion of the theorem.

If we have the same operator L, that is, the same diffusion law for all components of u in (3.9.3), then one can use a single Lyapunov function. On the other hand, if the system (3.9.3) does not enjoy this luxury, then it is not possible to employ a single Lyapunov function. We note also that if each component of V is convex then clearly $V_u(t,u)Lu \leq LV$ holds. Thus, it is clear that for general reaction-diffusion systems, utilizing vector Lyapunov functions appears to be natural and advantageous.

Let $\psi(x)$ be the solution of the steady state problem

$$(3.9.9) \qquad \begin{cases} L\psi + f(t, x\psi) = 0 \text{ in } \Omega. \\ \dfrac{\partial \psi(x)}{\partial \nu} = 0 \text{ on } \partial\Omega. \end{cases}$$

Then, setting $w = u - \psi$, we see that w satisfies

$$(3.9.10) \qquad \begin{cases} w_t = Lw + F(t, x, w) \text{ in } J \times \Omega, \\ w(t_0, x) = \phi_0(x) - \psi(x) = \overline{\psi}_0(x) \text{ in } \overline{\Omega}, \\ \dfrac{\partial w(t, x)}{\partial \nu} = 0 \text{ in } J \times \partial\Omega, \end{cases}$$

where $F(t, x, w) = f(t, x, w + \psi) - f(t, x, \psi)$. Noting that $F(t, x, 0) = 0$, we observe that practical stability properties of (3.9.10) imply the corresponding practical stability properties of (3.9.3) relative to the steady state solution $\psi(x)$.

For the purpose of illustration, let us consider a typical comparison system

$$\text{(3.9.11)}\qquad \left|\begin{array}{l} v_t=Av_{xx}-bv_x+g(t, v),\ 0<x<1,\ t\geq 0 \\ v(0, x)=\psi_0(x)\geq 0,\ 0\leq x\leq 1, \\ v_x(t, 0)=v_x(t, 1)=0,\ t\geq 0, \end{array}\right.$$

where $A>0$ is a diagonal matrix, $b>0$ and $g(t, v)$ satisfies

$$\text{(3.9.12)}\qquad g(t,u_1)-g(t, u_2)\leq L(u_1-u_2),$$

whenever $0\leq u_2\leq u_1\leq Q$, for some $Q>0$, where L is a N by N matrix. We shall consider two cases.

(i) L is a positive matrix with

$$\max_i \sum_{k=1}^{n} L_{ik}=\widetilde{L},\ 1\leq i\leq N;$$

(ii) L is a Metzeler matrix satisfying dominant diagonal condition, that is,

$$L_{ij}\geq 0 \text{ for } i\neq j \text{ and } -L_{ii}>\sum_{\substack{j=1\\ i\neq j}}^{n} L_{ij}.$$

Define $R(t, x)=(Ke^{-\alpha t+\beta(1-x)})\,\tilde{e}$, α, $K>0$ and $\beta\in R$ to be chosen and $\tilde{e}=(1, 1, \ldots, 1)$. Substituting R in (3.9.11) we get

$$\text{(3.9.13)}\quad R_t-AR_{xx}+bR_x-g(t, R)\geq r(-\alpha-a\beta^2-b\beta)-g(t, R),$$

where $a=\max A_{ii}$. Let $Q=\max_i(\max_{0\leq x\leq 1}\psi_{0i}(x))$ and $\alpha=L_0-\widetilde{L}>0$. In case (8), we now have

$$R_t-AR_{xx}+bR_x-g(t, R)\geq -R[L_0-\widetilde{L}+a\beta^2+b\beta+\widetilde{L}]=0,$$

if β is a root of $a\beta^2+b\beta+L_0=0$. But since $\beta=(-b\pm\sqrt{b^2-4aL_0})/2a$, if we suppose that $0<a<(b^2)/(4L_0)$, then β is negative. It is easy to check that R satisfies initial and boundary conditions by choosing $K=Qe^{-\beta}$. As a result, it follows from Theorem 3.9.1 with v=r and w=r that

$$0 \le r(t, x) \le R(t, x),\ t \ge 0,\ 0 \le x \le 1,$$

which implies practical asymptotic stability of (3.9.11).

In case (ii), set

$$-\gamma_i = L_{ii} + \sum_{\substack{j=1 \\ i \ne j}}^{n} L_{ij} \text{ and } \tilde{L} = \min_i \gamma_i.$$

Then, we get from (3.9.13)

$$R_t - AR_{xx} + bR_x - g(t, R) \ge -R[a\beta^2 + b\beta + L_0] = 0,$$

with $L_0 = \alpha - \tilde{L}$. We again have two cases: $\alpha > L$ and $0 < \alpha \le \tilde{L}$. If $\alpha > L$, then as in the previous situation, β is negative if we assume $0<a<b^2/4L_0$ and we obtain the same conclusion as before. If, on the other hand $0<\alpha\le\tilde{L}$ so that $L_0 \le 0$, then b^2-4aL_0 is always nonnegative and hence we have one negative root β and consequently, the conclusion remains the same as in the previous case.

It is clear that in case (i), diffusion and convection terms are contributing to practical stability and in case (ii) reaction terms are also playing a role. From these two cases, one can obtain several possibilities for the coefficients.

Instead of weakly coupled systems (3.9.3), one can also investigate certain strongly coupled systems with the help of vector Lyapunov functions. As an illustration, consider the following simple example.

Example

Consider

(3.9.14)
$$\begin{cases} u_{1t}=a_1u_{1xx}+b_1u_{1x}+b_2u_{2x}+e^{-t}u_1 \\ \quad +u_2\sin t-(u_1^3+u_1u_2^2)\sin^2 t, \\ u_{2t}=a_2u_{1xx}+a_1u_{2xx}+b_2u_{1x}+b_1u_{2x} \\ \quad +u_1\sin t+e^{-t}u_2-(u_1^2u_2+u_2^3)\sin^2 t, \text{ in } J\times\Omega \\ u_1(0, x)=\phi_{01}(x),\ u_2(0, x)=\phi_{02}(x) \text{ in } \Omega, \\ \frac{\partial u_1}{\partial \nu}=\frac{\partial u_2}{\partial \nu}=0 \text{in } J\times\partial\Omega, \end{cases}$$

with $a_1>a_2$. Then, choosing the vector Lyapunov function $V=(V_1, V_2)$ with

$$V_1(u)=\tfrac{1}{2}(u_1+u_2)^2,\ V_2(u)=\tfrac{1}{2}(u_1-u_2)^2,$$

it is easy to obtain, after some computations, the weakly coupled comparison system

(3.9.15)
$$\begin{cases} V_{1t}=A_1V_{1xx}+B_1V_{1x}+g_1(t, v) \\ V_{2t}=A_2V_{2xx}+B_2V_{2x}+g_2(t, v), \text{ in } J\times\Omega, \\ V(0, x)=\psi_{10}(x)\geq 0,\ V_2(0, x)=\psi_{20}x\geq 0, \text{ in } \overline{\Omega}, \\ \frac{\partial V_1}{\partial \nu}=\frac{\partial V_2}{\partial \nu}=0 \text{ on } J\times\partial\Omega, \end{cases}$$

where $\psi_{10}(x)=V_1(u(0, x))$, $\psi_{20}(x)=V_2(u(0, x))$, $A_1=a_1+a_2$, $B_1=b_1+b_2$, $B_2=b_1-b_2$, $g_1(t, v)=2(e^{-t}-\sin t]\ v_2$. It is now clear that the practical stability properties of the comparison system (3.9.15) imply the corresponding practical stability properties of the strongly coupled system (3.9.14).

3.10. NOTES.

The results of Section 3.1 are modelled on the corresponding results in Lakshmikantham, Leela and Martynyuk [1]. For notions of practical stability that can deal with perturbations directly, see LaSalle and Lefschetz [1]. The contents of Section 3.2 are taken from Martynyuk [3, 14]. Section 3.3 consists of results that are adapted from Lakshmikantham, Leela and Martynyuk [1] and are new. The material in Section 3.4 is based on the work of Lakshmikantham and Trigiante [1]. For the results in Section 3.5, see Lakshmikantham, Leela and Sivasundaram [1]. See also Martynyuk [11, 12]. The results of Section 3.7 and 3.8 are based on Lakshmikantham, Bainov and Simeonov [1]. For the contents of Section 3.9, see Lakshmikantham, Leela and Martynyuk [1]. See also Martynyuk [9, 10].

For several results on finite time stability, see Gunderson [1], Grujic [1, 4, 5], Hallam and Komkov [1], Kayande [1], Kayande and Rama Mohana Rao [1], Tsokos and Leela [1], Weiss [1, 2], Weiss and Infante [1], Windeknecht and Mesaravic [1] and Zubov [2].

For related results on practical stability, see Abdullin and Anapolsky [1], Bernfeld and Lakshmikantham [1], Grujic [2, 3, 4], Martynyuk [7, 9], Martynyuk and Obolensky [1], Martynyuk and Gutowski [1], Michel and Porter [1] and Moiseyev [1].

4

Control Systems

4.0. INTRODUCTION.

This chapter is devoted to the investigation of practical stability of differential systems with control and set-valued differential equations.

In Section 4.1, we offer a unified approach to specify admissible control sets corresponding to any desired property of practical stability. For this purpose, we utilize a convenient comparison result. Employing the method of vector Lyapunov functions, we discuss in Section 4.2, controllability questions in addition to practical stability criteria. Section 4.3 investigates decentralized control systems with feedback control, using the decomposition method. To find the properties of the comparison system which results when we employ vector Lyapunov method, we offer two different techniques that have been used in applications. Optimal controllability forms the content of Section 4.4, where a general example is given to illustrate the method.

Necessary theory of set value differential inequalities is developed in Section 4.5, which is then utilized to discuss weak practical stability criteria in addition to the usual criteria. The method of vector Lyapunov functions appears as a natural tool in these considerations.

4.1. CONTROL SYSTEMS.

We consider the control system

$$x'=f(t, x, u),\ x(t_0)=x_0, \tag{4.1.1}$$

where $f\in C[\mathbf{R}_+\times\mathbf{R}^n\times\mathbf{R}^m, \mathbf{R}^n]$ and u control function. Our aim is to describe a technique which provides a convenient and unified method of specifying admissible control sets corresponding to any desired practical stability behavior of the controlled motion. For this purpose, consider the scalar comparison equation

$$w'=g(t, w, w),\ w(t_0)=w_0\geq 0, \tag{4.1.2}$$

where $g\in C[\mathbf{R}_+^3, \mathbf{R}]$ and $r(t)=r(t, t_0, w_0)$ is the maximal solution of (4.1.2) existing on $[t_0, \infty)$. Let E denote the admissible control set given by

$$E=\{u\in\mathbf{R}^m:\ U(t, u)\leq r(t),\ t\geq t_0\} \tag{4.1.3}$$

where $U\in C[\mathbf{R}_+\times\mathbf{R}^m, \mathbf{R}_+]$. Corresponding to any given $u^*=u^*(t)\in E$ we denote by $x(t, t_0, x_0, u^*)$ any solution of (4.1.1). We are now in a position to prove the following result.

Theorem 4.1.1

Assume that

(i) $0<\lambda<A$ are given;

(ii) $V\in C[\mathbf{R}_+\times S(A), \mathbf{R}_+]$, $V(t, x)$ is locally Lipschitzian in x and

$$b(|x|)\leq V(t, x)\leq a(|x|),\ (t, x)\in\mathbf{R}_+\times S(A),$$

where $a, b\in K$;

(iii) for $(t, x) \in \mathbf{R}_+ \times S(A)$ and $u(t) \in E$,

$$D^+V(t, x) \equiv \lim_{h \to 0^+} \sup \frac{1}{h}[V(t+h, x+hf(t, x, u(t))) - V(t, x)]$$

$$\leq g(t, V(t, x), U(t, u(t))),$$

where $g \in C[\mathbf{R}_+^3, \mathbf{R}]$ and $g(t, w, v)$ is nondecreasing in v for each (t, w) and $U \in C[\mathbf{R}_+ \times \mathbf{R}^m, \mathbf{R}_+]$;

(iv) $a(\lambda) < b(A)$ holds.

Then the practical stability properties of (4.1.2) imply the corresponding practical stability properties of the control system (4.1.1).

Proof

Let us first prove practical stability of (4.1.1). Suppose that the comparison equation (4.1.2) is practically stable for $(a(\lambda), b(A))$. Then, because of (iv), it follows that

(4.1.4) $$w_0 < a(\lambda) \text{ implies } w(t, t_0, w_0) < b(A),\ t \geq t_0,$$

where $w(t, t_0, w_0)$ is any solution of (4.1.2) existing on $[t_0, \infty)$. Let $|x_0| < \lambda$. We claim that $|x(t)| < A$, $t \geq t_0$, where $x(t) = x(t, t_0, x_0, u^*)$ is any solution of (4.1.1) corresponding to $u^* = u^*(t) \in E$. If this is not true, then there would exist a $u^0 = u^0(t)$ and a corresponding solution $x(t) = x(t, t_0, x_0, u^0)$ of (4.1.1) satisfying

$$|x(t_1)| = A,\ |x(t)| \leq A,\ t_0 \leq t \leq t_1,$$

so that by (ii) and the continuity of the functions involved, we have

(4.1.5) $$b(A) \leq V(t_1, x(t_1)).$$

Choosing $w_0 = V(t_0, x_0)$, the assumption (iii) yields, by standard computation, the differential inequality

$$D^+m(t) \leq g(t, m(t), U(t, u(t))), \ t_0 \leq t \leq t_1,$$

where $m(t) = V(t, x(t))$. Since $u^0(t) \in E$ and $g(t, w, v)$ is nondecreasing in v, we get

(4.1.6) $$D^+m(t) \leq g(t, m(t), r(t)), \ t_0 \leq t \leq t_1,$$

where $r(t) = r(t, t_0, w_0)$ is the maximal solution of (4.1.2), which implies by Theorem A.1.4 the estimate

(4.1.7) $$m(t) \leq r(t), \ t_0 \leq t \leq t_1,$$

where $r(t)$ is also the maximal solution of

$$w' = g(t, w, r(t)), \ w(t_0) = w_0 \geq 0.$$

Now the relations (4.1.4), (4.1.5) and (4.1.7) lead to the contradiction

$$b(A) \leq V(t_1, x(t_1)) \leq r(t_1) < b(A),$$

proving that the control system (4.1.1) is practically stable.

It is now easy to construct the proof of other practical stability properties based on the foregoing proof and the corresponding proof of Theorem 2.2.1. We therefore omit the details.

As an example, consider the linear control system

(4.1.8) $$x' = Ax + Bu + \sigma(t), \ x(t_0) = x_0,$$

where A, B are n×n, n×m matrices and $\sigma \in C[\mathbf{R}_+, \mathbf{R}^n]$. Assume that

$$\mu(A) = \lim_{h\to 0^+} \frac{1}{h}[\| I+hA \| - 1] \leq -\alpha, \; \alpha > 0,$$
$$\| B \| = \beta \text{ and } \| \sigma(t) \| \leq K.$$

Take $V(t, x) = \| x \|$. Then it is easy to compute that

$$g(t, w, w) = (-\sigma + \beta)w + K. \tag{4.1.9}$$

Consequently, the solutions of (4.1.2) are of the form

$$w(t, t_0, w_0) = w_0 \exp[(\beta-\alpha)(t-t_0)] + \frac{K}{\beta-\alpha}, \; t \geq t_0,$$

and hence the admissible set E is given be

$$E = \{u \in \mathbf{R}^m: \; \| u \| \leq w(t, t_0, \| x_0 \|), \; t \geq t_0\}.$$

If $\beta - \alpha = -\gamma < 0$ and $\lambda + \frac{K}{\gamma} < A$, then the comparison equation (4.1.9) is practically stable for (λ, A). If further $\gamma\beta > K$ then (4.1.9) is practically strongly stable. Hence we get from Theorem (4.1.1) the corresponding practical stability properties of (4.1.8).

We shall next consider the admissible control set as a fixed set, namely,

$$\Omega = \{u \in \mathbf{R}^m: \; U(t, u) \leq \lambda_0(t), \; t \geq t_0\}, \tag{4.1.10}$$

where $\lambda_0 \in C[\mathbf{R}_+, \mathbf{R}_+]$ is a given function. We then have the following result.

Theorem 4.1.2

Suppose that the assumptions (i) - (iv) of Theorem 4.1.1 hold for the control set Ω instead of E. Then the practical stability properties of

$$w' = g_0(t, w), \; w(t_0) = w_0 \geq 0, \tag{4.1.11}$$

where $g_0(t, w) = g(t, w, \lambda_0(t))$ imply the corresponding practical stability properties of the control system (4.1.1).

Proof

The proof is very much similar to the proof of Theorem 4.1.1 except that we now get, instead of (4.1.6), the differential inequality

$$D^+ m(t) \leq g_0(t, m(t)), \; t_0 \leq t \leq t_1.$$

Consequently, to obtain the estimate (4.1.7), we utilize the usual comparison Theorem A.1.1 where $r(t)$ is now the maximal solution of (4.1.11) instead of (4.1.2). The rest of the proof follows with suitable modifications and the details are omitted.

As a general example, let us consider the control system

$$x' = f(t, x) + \phi(t, u), \; x(t_0) = x_0, \tag{4.1.12}$$

where $f \in C[\mathbf{R}_+ \times \mathbf{R}^n, \mathbf{R}^n]$ and $\phi \in C[\mathbf{R}_+ \times \mathbf{R}^m, \mathbf{R}^n]$. Let us pose the problem: Does there exist functions $u = u(t)$ under the presence of which the solutions of (4.1.12) be close to a given target function $\tau(t)$ for $t \geq t_0$ provided the function f is sufficiently well behaved. We shall assume that $\tau(t)$ is continuously differentiable for $t \geq t_0$ and takes values in $\mathbf{R}^n$. Consider the transformation $z = x - \tau(t)$ so that the system (4.1.12) is reduced to

$$(4.1.13) \qquad z' = F(t, z) + \xi(t),\ z(t_0) = x_0 - \tau(t_0),$$

where

$$F(t, z) = f(t, z + \tau(t)) - f(t, \tau(t)),$$

$$\psi(t, \tau(t)) = \tau'(t) - f(t, \tau(t)),$$

$$\xi(t) = \phi(t, u(t)) - \psi(t, \tau(t)).$$

Observing that $F(t, 0) \equiv 0$, we see that it is enough to study the practical stability properties of the reduced control system (4.1.13) in order to answer the posed question.

Let $V \in C[\mathbf{R}_+ \times \mathbf{R}^n, \mathbf{R}_+]$, $V(t, z)$ is Lipschitzian in z for a function $L(t) \geq 0$ and

$$D^+V(t, z)_{(4.1.14)} \leq g(t, V(t, z)),$$

where $D^+V(t, z)$ is defined relative to the system

$$(4.1.14) \qquad z' = F(t, z),$$

and $g \in C[\mathbf{R}_+^2, \mathbf{R}]$. Then it is not difficult to obtain

$$D^+V(t, z)_{(4.1.13)} \leq g(t, V(t, z)) + L(t) \mid \xi(t) \mid.$$

Setting $U(t, u) = L(t) \mid \phi(t, u) - \psi(t, \tau(t)) \mid = L(t) \mid \xi(t) \mid$, we see that $g_0(t,w)$ of (4.1.11) reduces to $g(t, w) + \lambda_0(t)$. Thus, if $g(t,w) \leq 0$, for example, and $\int_{t_0}^{\infty} \lambda_0(s)ds < b(A) - a(\lambda)$, where $\lambda_0(t)$ is the function involved in defining the control set Ω, then if follows that (4.1.11) is practically stable for $(a(\lambda), b(A))$. Then, by Theorem 4.1.2 we get the practical stability of the system (4.1.13) provided that

$$b(|z|)\le V(t, z)\le a(|z|),\ a,\ b\in K$$

and

$$a(\lambda)<b(A).$$

This then implies that

$$|x(t)-\tau(t)|<A,\ t\ge t_0 \text{ whenever } |x_0-\tau(t_0)|<\lambda,$$

relative to the system (4.1.12) and answers the problem posed.

4.2. CONTROLLABLE SYSTEMS.

In this section, we shall continue to consider the control system (4.1.1) and employ the method of vector Lyapunov functions to discuss practical stability as well as controllability of the system (4.1.1). We shall consider the control set

$$\Omega=\{u\in \mathbf{R}^m:\ U(t, u)\le v(t),\ t\ge t_0\}$$

where $U(t, u)\in C[\mathbf{R}_+\times\mathbf{R}^m, \mathbf{R}_+^m]$ and $v\in C[\mathbf{R}_+, \mathbf{R}_+^m]$ is a given function.

Theorem 4.2.1

Assume that

(i) $0<\lambda<A$ and $0<\beta<A$ are given;

(ii) $V\in C[\mathbf{R}_+\times\mathbf{R}^n, \mathbf{R}_+^N]$, $V(t, x)$ is locally Lipschitzian in x, the function $Q(V(t, x))$ where Q: $\mathbf{R}_+^N\to\mathbf{R}_+$ is continuous, strictly increasing and $Q(0)=0$ and for $a, b\in K$, $(t, x)\in\mathbf{R}_+\times S(A)$,

$$b(|x|) \leq Q(V(t, x)) \leq a(|x|);$$

(iii) for $(t, x) \in \mathbf{R}_+ \times S(A)$ and $u \in \Omega$,

$$D^+V(t, x) \leq g(t, V(t, x), U(t, u)),$$

where $g \in C[\mathbf{R}_+ \times \mathbf{R}_+^N \times \mathbf{R}_+^M, \mathbf{R}^N]$ and $g(t, w, v)$ is quasimontone nondecreasing in w and nondecreasing in v;

(iv) $a(\lambda) < b(A)$ holds;

(v) there exists a control function $v \in C[\mathbf{R}_+, \mathbf{R}_+^M]$ such that any solution $w(t, t_0, w_0, v)$ of

(4.2.1) $$w' = g(t, w, v(t)), \; w(t_0) = w_0 \geq 0$$

satisfies

(4.2.2) $Q(w_0) < a(\lambda)$ implies $Q(w(t, t_0, w_0, v)) < b(A), \; t \geq t_0$

and

(4.2.3) $Q(w(t_0 + T, t_0, w_0, v)) \leq b(\beta)$ for some $T = T(t_0, w_0) > 0$.

Then there exist admissible controls $u = u(t) \in \Omega$ such that the system (4.1.1) is practically stable and all solutions $x(t) = x(t, t_0, x_0, u)$ starting in $\Omega_1 = \{x \in \mathbf{R}^n: \; |x| < \lambda\}$ are transferred to the region $\Omega_2 = \{x \in \mathbf{R}^n: \; |x| \leq \beta\}$ in a finite time $T^* = T^*(t_0, x_0) = T(t_0, V(t_0, x_0))$, that is, the system (4.1.1) is controllable.

Proof

Let $|x_0| < \lambda$ and $U(t, u(t)) \leq v(t), \; t \geq t_0$. Then we claim that for any solution $x(t)=x(t, t_0, x_0, u)$ of (4.1.1) we have

$$| x_0 |<\lambda \text{ implies } | x(t) |<A \text{ for } t\geq t_0.$$

If this is false, there would exist a $u^*=u^*(t)\in\Omega$ and a corresponding solution $x(t)=x(t, t_0, x_0, u^*)$ of (4.1.1) such that

$$(4.2.4) \qquad | x_0 |<\lambda,\ | x(t_1) |=A \text{ and } | x(t) |\leq A,\ t_0\leq t\leq t_1.$$

Setting $m(t)=V(t, x(t))$ for $t_0\leq t\leq t_1$ and using (iii), we get

$$D^+m(t)\leq g(t, m(t), v(t)),\ t_0\leq t\leq t_1,$$

which implies, by comparison Theorem A.1.4, the estimate

$$m(t)\leq r(t, t_0, w_0, v),\ t_0\leq t\leq t_1,$$

where $r(t, t_0, w_0, v)$ is the maximal solution of (4.2.1). Choosing $w_0=V(t_0, x_0)$, we then get, because of (ii), the relation

$$(4.2.5) \quad b(| x(t) |)\leq Q(V(t, x(t)))\leq Q(r(t, t_0, w_0, v)),\ t_0\leq t\leq t_1.$$

Now we are lead to the following contradiction, in view of the relations (4.2.2), (4.2.4) and (4.2.5),

$$b(A)=b(| x(t_1) |)\leq Q(V(t_1, x(t_1)))\leq Q(r(t_1, t_0, w_0,v))$$

$$\leq Q(r(t_1, t_0, w_0, v))<b(A),$$

which proves the practical stability of the system (4.1.1). As a result, (4.2.5) holds for all $t\geq t_0$ and therefore the assumption (4.2.3) yields $|x(t_0+T^*)|\leq\beta$ where $T^*=T(t_0, V(t_0, x_0))$. The proof is therefore complete.

Consider, as an example, the linear control system

$$w' = A(t)w + B(t)v,\ w(t_0) = w_0, \tag{4.2.6}$$

where $A(t), B(t)$ are $n\times n$, $n\times m$ continuous matrices on $\mathbf{R}_+$ and $v = v(t)$ is a control function. Suppose that $Y(t)$ is the fundamental matrix solution of

$$w' = A(t)w \tag{4.2.7}$$

such that $Y(t_0)$ is the identity matrix. We shall show that we can find suitable admissible controls v to assure practical stability and controllability of the system (4.2.6). The transformation $w = Y(t)z$ reduces (4.2.6) to

$$z' = Q(t)v,\ z(t_0) = w_0 \tag{4.2.8}$$

where $Q(t) = Y^{-1}(t)B(t)$ and therefore the solutions $z(t)$ of (4.2.8) are given by

$$z(t) = w_0 + \int_{t_0}^{t} Q(s)v(s)ds,\ t \geq t_0.$$

Choosing $v(t) = Q^*(t)C + p(t)$ with $\int_{t_0}^{\infty} Q(s)p(s)ds = 0$ for some vector C where Q^* is the transpose of Q, we find that

$$z(t) = w_0 + \int_{t_0}^{t} Q(s)[Q^*(s)C + p(s)]ds. \tag{4.2.9}$$

Let $0<\lambda<A$ and $0<\beta<A$ be given. We choose C such that

$$\int_{t_0}^{\infty} | Q(s)Q^*(s) | ds \leq \frac{A-\lambda}{| C |}. \tag{4.2.10}$$

Then the estimate (4.2.9) shows that

$$| x(t) | < A, \; t \geq t_0, \text{ provided } | w_0 | < \lambda.$$

But $| w(t) | \leq | Y(t) | \, | z(t) |$, $t \geq t_0$ and therefore, if $| Y(t) | \leq 1$, $t \geq t_0$, we get the practical stability of (4.2.6). If, in addition, there exists a $T>0$ such that $| Y(t_0+T) | \leq \frac{\beta}{A}$, then we have $| w(t_0+T) | < \beta$ which shows that the system (4.2.6) is controllable.

The foregoing considerations concerning (4.2.6) prove the following result.

Theorem 4.2.2

Assume that

(i) $0<\lambda<A$ and $0<\beta<A$ are given;

(ii) $Y(t)$ is the fundamental matrix solution of (4.2.7) such that $|Y(t)| \leq 1$, $t \geq t_0$ and there exists a $T>0$ satisfying $|Y(t_0+T)| \leq \beta/A$;

(iii) there exists a C such that (4.2.10) holds.

Then there exist admissible controls v such that the system (4.2.6) is practically stable and controllable.

The linear system (4.2.6) may be taken as the comparison system (4.2.1) in which case we need to suppose that the nondiagonal elements of $A(t)$ are nonnegative, $B(t) \geq 0$, $v(t) \geq 0$ and $Q(w) = \sum_{i=1}^{N} w_i$, to guarantee that the solutions of (4.2.1) are non-negative.

4.3. DECENTRALIZED CONTROL SYSTEMS.

In this section, we shall develop an approach for designing decentralized state feed-back dead-beat control system with constraints in the state and input vectors. For this purpose, let us suppose that the control system (4.1.1) can be decomposed into N interconnected subsystems with decentralized control

$$(4.3.1) \qquad x_i' = f_i(t, x, u_i), \; i = 1, 2, \ldots, N,$$

so that $n = \sum_{i=1}^{N} n_i$ and $m = \sum_{i=1}^{N} m_i$. Any set $\Omega = \Omega_1 \times \ldots \times \Omega_N$ of functions $u_i(t, x_i) \in \Omega_i$ which do not violate existence of solutions of (4.3.1) for all $t \geq t_0$ is considered as an admissible control set. The following result determines the control laws $u_i(t, x_i)$ which minimize the settling time of any initial state x_0 belonging to a region around the origin.

Theorem 4.3.1

For a control law $u^* \in \Omega$, assume that

(A_0) conditions (i) and (ii) of Theorem 4.2.1 hold;

(A_1) $D^+V(t, x) \leq g(t, V(t, x))$, $(, x) \in \mathbf{R}_+ \times S(A)$ and $u^* \in \Omega$, where $g \in C[\mathbf{R}_+ \times \mathbf{R}_+^N, \mathbf{R}^N]$ and $g(t, w)$ is quasimonotone nondecreasing in w;

(A_2) $Q(w_0) < a(\lambda)$ implies $Q(w(t, t_0, w_0)) < b(A)$, $t \geq t_0$, where $w(t, t_0, w_0)$ is any solution of

(4.3.2) $$w' = g(t, w),\ w(t_0) = w_0 \geq 0,$$

and there exists a $\tau^* = \tau^*(t_0, w_0) > 0$ such that

(4.3.3) $$Q(w(\tau^*, t_0, w_0)) = 0.$$

Then the control system

(4.3.4) $$x' = f(t, x, u^*),\ x(t_0) = x_0,$$

is practically stable and every solution $x(t, t_0, x_0, u^*)$ of (4.3.4) such that $|x_0| < \lambda$ is transferred to the origin at time $t^* = t^*(t_0, x_0) = \tau^*(t_0, V(t_0, x_0))$, that is, $x(t^*, t_0, x_0, u^*) = 0$.

Proof

We claim that $|x_0| < \lambda$ implies that $|x(t, t_0, x_0, u^*)| < A$, $t \geq t_0$. If not, there would exist a $t_1 > t_0$ and a solution $x(t) = x(t, t_0, x_0, u^*)$ of (4.3.4) such that

$$|x(t_1)| = A,\ |x(t)| \leq A,\ t_0 \leq t \leq t_1 \text{ whenever } |x_0| < \lambda.$$

We then have, because of (A_1) and comparison Theorem A.1.3, the estimate

(4.3.5) $$V(t, x(t)) \leq r(t, t_0, w_0),\ t_0 \leq t \leq t_1,$$

and consequently, by (A_2), we get, choosing $w_0 = V(t_0, x_0)$,

$$b(A) \leq b(|x(t_1)|) \leq Q(V(t_1, x(t_1))) \leq Q(r(t_1, t_0, w_0)) < b(A).$$

This contradiction proves practical stability of (4.3.4). Since (4.3.3) holds and we now have relation (4.3.5) for all $t \geq t_0$ in view of practical stability, it follows that $x(t^*)=0$ where $t^*=\tau^*(t_0,V(t_0,x_0))$. The proof is complete.

We recall that when the method of vector Lyapunov functions was used in Section 2.4, we did employ, instead of $Q(v)$, simpler measures, namely

$$Q(V)=\sum_{i=1}^{N}V_i,\ Q(V)=\max_{1\le i\le N}\ V_i(t,x),\ Q(V)=\sum_{i=1}^{N}d_iV_i(t,x)$$

where $d_i>0$.

In order for the method of vector Lyapunov functions to be useful, we need to know the properties of the comparison system (4.3.2), which is difficult in general. There are different ways to address this problem and we shall next describe two methods.

Method 1: If the functions Q described above are differentiable, it may be easier sometimes to estimate the expression

$$\frac{\partial Q(V(t,x))}{\partial w}\,D^+V(t,x) \tag{4.3.6}$$

by means of a scalar function $G\in C[\mathbf{R}_+^2,\mathbf{R}]$, that is,

$$\frac{\partial Q(V(t,x))}{\partial w}D^+V(t,x)\le G(t,Q(V(t,x))).$$

In this case, one needs to determine the properties of the scalar differential equation

$$\rho'=G(t,\rho),\ \rho(t_0)=\rho_0\ge 0$$

and this is comparatively easier than studying (4.3.2). In some other situations, it may be possible to estimate the expression (4.3.6) directly to determine whether it is negative defnite. This technique has been fruitfully employed in large scale systems.

Method 2: We estimate $D^+V(t, x)$ by $g(t, V(t, x))$ as usual and then reduce the study of the properties of solutions of the comparison system (4.3.2) to the study of some scalar differential equation. Specifically, we have the following result.

Lemma 4.3.1

Assume that $L \in C^1[\mathbf{R}_+, \mathbf{R}_+^N]$, $g \in C[\mathbf{R}_+ \times \mathbf{R}_+^N, \mathbf{R}^N]$, $G \in C[\mathbf{R}_+^2, \mathbf{R}]$ and g, G are smooth enough to assure existence and uniqueness of solutions for $t \geq t_0$ of (4.3.2) and

(4.3.7) $$r' = G(t, r), \; r(t_0) = r_0 \geq 0.$$

Suppose further that for $(t, r) \in \mathbf{R}_+^2$,

$$g(t, L(r)) \leq \frac{dL(r)}{dr} G(t, r).$$

Then $w_0 \leq L(r_0)$ implies

$$w(t, t_0, w_0) \leq L(r(t, t_0, r_0)), \; t \geq t_0,$$

where $w(t, t_0, w_0)$, $r(t, t_0, r_0)$ are the solutions of (4.3.2), (4.3.7) respectively.

Proof

Set $m(t) = L(r(t, t_0, r_0))$ so that $m(t_0) = L(r_0) \geq w_0$ and

$$m'(t)=\frac{dL(r(t, t_0, r_0))}{dr}\, G(t, r(t, t_0, r_0))$$

$$\geq g(t, L(r(t, t_0, r_0)))\equiv g(t, m(t)).$$

Hence, by comparison Theorem A.1.3 we get the stated result in view of uniqueness of solutions.

4.4. OPTIMAL CONTROLLABILITY.

In this seciton, we shall consider the control system (4.1.1) to discuss optimal stabilization. Specifically, we shall prove the following result.

Theorem 4.4.1

(i) $0<\lambda<A$ are given;

(ii) $V\in C^1[\mathbf{R}_+\times\mathbf{R}^n, \mathbf{R}_+^N]$, $Q\in C[\mathbf{R}_+^N, \mathbf{R}_+]$, $Q(w)$ is nondecreasing in w, $Q(0)=0$, $g\in C[\mathbf{R}_+\times\mathbf{R}^N\times\mathbf{R}^n\times\mathbf{R}^m, \mathbf{R}^N]$ and $g(t, w, x, u)$ is quasimonotone nondecreasing in w;

(iii) $\Omega\subset\mathbf{R}^m$ is a convex, compact set and for $u^0(t, x)\in\Omega$, the system

$$x'=f(t, x, u^0(t, x)),\ x(t_0)=x_0, \tag{4.4.1}$$

admits unique solutions existing for $t\geq t_0$ and for $(t, x)\in\mathbf{R}_+\times S(A)$,

$$b(|\,x\,|)\leq Q(V(t, x))\leq a(|\,x\,|),\ a, b\in K;$$

(iv) $B[V,t,x,u^0,g]\equiv V_t(t,x)+V_x(t,x)f(t,x,u^0(t,x))+g(t,V(t,x), x,u^0)\leq 0$;

(v) $B[V, t, x, u, g]\geq 0$ for any $u\in\Omega$;

(vi) $a(\lambda)<b(A)$ holds;

(vii) any solution $w(t, t_0, w_0)$ of

(4.4.2) $$w' = -g(t, w, x^0(t), u^0(t, x^0(t))),\ w(t_0) = w_0 \geq 0$$

exists on $[t_0, \infty)$ and satisfies

$$Q(w_0) < a(\lambda) \text{ implies } Q(w(t, t_0, w_0)) < b(A),\ t \geq t_0$$

and $\lim\limits_{t \to \infty} w(t, t_0, w_0) = 0$.

Then the control system (4.4.1) is practically asymptotically stable and the inequality

(4.4.3) $$\begin{aligned} &\int_{t_0}^{\infty} g(s, V(s, x^0(s)), x^0(s), u^0(s, x^0(s)))ds \\ &= \min_u \int_{t_0}^{\infty} g(s, V(s, x(s)), x(s), u(s, x(s)))ds \\ &\leq V(t_0, x_0) \end{aligned}$$

holds, that is, $u^0 \in \Omega$ assures optimal stabilization.

Proof

To prove the theorem, we have to verify two facts: (1) the control $u^0(t,x) \in \Omega$ assures practical asymptotic stability and (2) the relation (4.4.3) holds. Let $x^0(t) = x(t, t_0, x_0, u^0)$ be the solution of (4.4.1) corresponding to the control $u^0(t, x) \in \Omega$. Then setting $m(t) = V(t, x^0(t))$ and using the assumptions (i) - (iv), (vi) and (vii), we can prove that the system (4.4.2) is practically stable following the arguments of Theorem 4.2.1. Then we also have

(4.4.4) $$V(t, x^0(t)) \leq w(t, t_0, w_0),\ t \geq t_0,$$

and consequently the assumption $\lim_{t\to\infty} w(t, t_0, w_0)=0$ implies that $\lim_{t\to\infty} x^0(t)=0$ which proves practical asymptotic stability.

Now, to prove the relation (4.4.3), suppose that another control $u^*=u^*(t, x)\in\Omega$ also assures practical asymptotic stability of (4.4.1) with $u=u^*$. The corresponding solution $x^*(t)$ also satisfies

$$|x^*(t)|<A,\ t\geq t_0 \text{ provided } |x_0|<\lambda$$

and $\lim_{t\to\infty} x^*(t)=0$. This implies that

$$\lim_{t\to\infty} V(t, x^*(t))=0 \tag{4.4.5}$$

and we also have from (4.4.4),

$$\lim_{t\to\infty} V(t, x^0(t))=0. \tag{4.4.6}$$

By (iv), we then get the estimate

$$\int_{t_0}^{\infty} g(s, V(s, x^0(s)), x^0(s), u^0(s, x^0(s)))ds\leq V(t_0, x_0) \tag{4.4.7}$$

using (4.4.6). But condition (v) yields, using (4.4.5), the relation

$$V(t_0, x_0)\leq \int_{t_0}^{\infty} g(s, V(s, x^*(s)), x^*(s), u^*(s, x^*(s)))ds. \tag{4.4.8}$$

The inequalities (4.4.7) and (4.4.8) prove the desired relation (4.4.3) and the proof is complete.

As an example, consider the simpler case, when

$$x'=f(t, x, u)\equiv f(t, x)+R(t, x)u,\ x(t_0)=x_0, \tag{4.4.9}$$

where $F \in C[\mathbf{R}_+ \times \mathbf{R}^n, \mathbf{R}^n]$, $R(t, x)$ is an $n \times m$ continuous matrix and u is a control. Suppose that we have

$$V_t(t, x) + V_x(t, x)F(t, x) \equiv w(t, x) \leq 0$$

for $(t, x) \in \mathbf{R}_+ \times S(A)$ where $0 < \lambda < A$ are given. Define the expression

$$B[V, t, x, u] = w(t, x) + \omega(t, x) + uBu + V_x(t, x)R(t, x)u \tag{4.4.10}$$

where B is a $m \times m$ non-singular matrix and find the control $u^0 = u^0(t,x) \in \Omega$ from the condition of minimum of B, namely,

$$B[V, t, x, u] = 0 \text{ at } u = u^0,$$

$$\frac{\partial B}{\partial u}[V, t, x, u] = 0 \text{ at } u = u^0.$$

We obtain the relation

$$R(t, x)V_x(t, x) + 2Bu^0 = 0$$

from which it follows that

$$u^0(t, x) = -\tfrac{1}{2}\, B^{-1}R(t, x)V_x(t, x). \tag{4.4.11}$$

Substituting (4.4.11) into (4.4.9) we get the system

$$\begin{aligned} x' &= F(t, x) - \frac{1}{2}R(t, x)B^{-1}R(t, x)V_x(t, x) \\ &\equiv f(t, x, u^0(t, x)). \end{aligned} \tag{4.4.12}$$

The practical stability of (4.4.12) can be proved on the basis of the earlier arguments.

To discuss the problem of minimization of the functional $\int_{t_0}^{\infty} g(s, V(s,x(s)), x(s), u(s, x(s)))ds$, we note that

$$V_x(t, x)\ R(t, x)u + uBu = -2u^0Bu^0 + uBu \tag{4.4.13}$$

$$= (u - u^0)\ B(u - u^0) - u^0Bu^0,$$

and therefore, we obtain, because of (4.4.11) and (4.4.13) the relation

$$w(t, x) + \omega(t, x) - u^0Bu^0 = 0,$$

which yields

$$\omega(t, x) = -w(t, x) + u^0Bu^0.$$

Thus $g(t, V, x, u) = -w(t, x) + u^0Bu^0 + uBu$ and now we can repeat the arguments of Theorem 4.4.1 to obtain the desired conclusion. We omit the details.

4.5. SET-VALUED DIFFERENTIAL INEQUALITIES.

Consider the control system

$$x' = f(t, x, u),\ x(t_0) = x_0$$

where $f \in C[\mathbf{R}_+ \times \mathbf{R}^n \times \mathbf{R}^m, \mathbf{R}^n]$ and $u \in U \subset \mathbf{R}^n$ is a control vector, where U is a bounded set. In general, the optimal control u(t) has many jumps and so one needs to work with measureable functions u(t). One of the ways to study the existence problem is to consider the set-valued differential equation

$$x' \in F(t, x),\ x(t_0) = x_0.$$

Here $F(t, x)=f(t, x, U)=\{f(t, x, u): u\in U\}$. Also, in the study of differential equations

$$x'=f(t, x),\ x(t_0)=x_0,$$

where f is discontinuous in x, it is easier to say something about solutions if we consider the set-valued differential equations of the foregoing type where

$$F(t, x)=\bigcap_{\xi>0} \overline{\text{Conv}}\, f(t, x+\overline{B}_\xi(0)),$$
$$\overline{B}_\xi(0)=\{x\in \mathbf{R}^n: |x|\leq\xi\}$$

and $\overline{\text{Conv}}$ denotes the closed convex hull. We note that if f is continuous in x, then $F(t, x)=\{f(t, x)\}$. Thus, investigating set-valued differential equations is of interest in itself. We shall discuss, in this section, the theory of set-valued differential inequalities which we need to analyze practical stability of such systems. Let $E=\mathbf{R}^n$, $D\subset E$ be closed, $J=[t_0,t_0+a]$, $t_0\geq 0$, $a>0$ and G: $J\times D\to 2^E\backslash\phi$ be a set-valued map. We consider the initial value problem

(4.5.1) $$u'\in G(t, u) \text{ a·e· on } J,\ u(t_0)=u_0\in D,$$

where solutions u(t) of (4.5.1) are understood to be absolutely continuous on J.

Let F: $D\to 2^E\backslash\phi$. Then we say that F is upper semi-continuous (usc) if $F^{-1}(A)=\{x\in D: F(x)\cap A\neq\phi\}$ is closed for A closed, lower semicontinuous (lsc) if $F^{-1}(A)$ is open for A open and continuous if F is continuous with respect to Hausdorff metric, namely,

$$d_H(A, B) = \max[\sup_{x \in A} d(x, B), \sup_{x \in B} d(x, A)],$$

where d(x, A) is the distance function. We shall only consider usc maps.

For any closed set $D \subset E$ and $u \in D$, we let

$$T_D(u) = \{v: \lim_{h \to 0^+} \inf \frac{1}{h}[d(u+hv, D)] = 0\},$$

which becomes

$$T_D(u) = \{\overline{\lambda(v-u): \lambda \geq 0, v \in D\}},$$

if D is also convex. Let $K \subset E$ be a cone, that is, closed, convex set with $\lambda K \subset K$ for $\lambda \geq 0$ and $K \cap \{-K\} = \{0\}$. In this case, we have, in particular,

$$T_K(0) = K \subset T_K(u).$$

Let us list the following conditions for convenience:

(A_0) for $(t, u) \in J \times D$, $G(t, u)$ is compact convex;

(A_1) $G(t, u)$ is Lebesque measureable in $t \in J$ for each $u \in D$ and usc in $u \in D$ for each $t \in J$;

(A_2) $| G(t, u) | = \max[| v |: v \in G(t, u) \backslash \leq \lambda(t)[1 + | u |]$ on $J \times D$ with $\lambda \in L^1(J)$;

(A_3) $G(t, u) \cap T_D(u) \neq \phi$ on $J \times D$;

(A_4) $g(t, u) \in G(t, u) \subset g(t, u) - K$ on $J \times D$, where $g(t, u)$ is quasimontone in u for each t, that is,

$$g(t, u+v) \in g(t, u) + T_K(v), \tag{4.5.2}$$

for all $(t, u) \in J \times D$ and $v \in K$.

We can now state the know existence result concerning the problem (4.5.1).

Theorem 4.5.1

Assume that $E=\mathbf{R}^n$, $D\subset E$ is closed, $J=[t_0, t_0+a]$, $t_0\geq 0$, $a>0$ and $G: J\times D\to 2^E\backslash\phi$ is such that the assumptions (A_0) - (A_3) are satisfied. Then the problem (4.5.1) has a solution $u(t)$ on J. If, in particular, $K=\mathbf{R}^n_+$ and (A_4) holds, then the problem (4.5.1) has the maximal solution $r(t)$ on J.

For a proof of Theorem 4.5.1, see Deimling [1-3]. Using Theorem 4.5.1, we can prove the following comparison results which we require for our later discussion.

Theorem 4.5.2

Supose that the assumptions (A_0) - (A_4) hold with $D=K=\mathbf{R}^n_+$. Let $m: J\to K$ be absolutely continuous on J and satisfy

$$m'(t)\in G(t, m(t))-K, \text{ a.e. on } J.$$

Then

$$m(t)\leq r(t) \text{ on } J,$$

where $r(t)$ is the maximal solution of (4.5.1) on J provided $m(t_0)\leq u_0$.

Proof

Consider

$$G_0(t, u)=G(t, m(t)+u)-m'(t), \text{ a.e. on } J.$$

Then G_0 is measureable in t and usc in u. Also G_0 satisfies the growth condition (A_2) and G_0 is compact, convex. We also have

$$G_0(t, u) \cap T_K(u) \neq \phi \text{ on } J \times K.$$

To see this, notice first that

$$m'(t) \in G(t, m(t)) - y(t)$$

for some measurable y: $J \to K$. Also (A_4) implies measurability of $g(t,m(t))$ and consequently,

$$m'(t) = g(t, m(t)) + z(t) - y(t),$$

for some measurable z: $J \to K$. Hence

$$g(t, m(t)) + z(t) - y(t) + g(t, m(t) + u) \in G_0(t, u)$$

and

$$g(t, u + m(t)) - g(t, m(t)) \in T_K(u).$$

Since $K \subset T_K(u)$, this proves that $G_0(t, u) \cap T_K(u) \neq \phi$ on $J \times K$. Thus, by the first part of Theorem 4.5.1, we have the existence of a solution w: $J \to K$ of

$$w' \in G_0(t, w),\ w(t_0) = u_0 - m(t_0) \in K.$$

As a result, $u(t) = m(t) + w(t) \geq m(t)$ on J is a solution of (4.5.1). Since, by the second part of Theorem 4.5.1, the maximal solution r(t) of (4.5.1) exists on J, we get immediately

$$m(t) \leq u(t) \leq r(t) \text{ on } J$$

and the proof is complete.

The next comparison result is a weaker form of Theorem 4.5.2.

Theorem 4.5.3

Assume that $G(t, u)$ satisfies (A_0) - (A_2) with $D=K$. Let m: $J \rightarrow K$ be absolutely continuous and satisfy

(4.5.3) $$m'(t) \in G(t, u+m(t)) - T_K(u) \text{ on } J \times K.$$

Then (4.5.1) has a solution $u(t)$ on J such that

$$m(t) \leq u(t),\ t \in J,$$

provided $m(t_0 \leq u_0)$.

Proof

Consider the problem

$$w' \in G_0(t, w) \text{ a·e· on } J,\ w(t_0) = u_0 - m(t) \in K,$$

where $G_0(t, w) = G(t, w+m(t)) - m'(t)$ a·e· on J. Since, in view of (4.5.3), G_0 satisfies the assumptions (A_0) - (A_3), the proof follows readily from Theorem 4.5.1.

4.6. PRACTICAL STABILITY CRITERIA.

We shall employ the method of vector Lyapunov functions to investigate practical stability of

(4.6.1) $$x' \in F(t, x) \text{ a·e· on } \mathbf{R}_+,\ x(t_0) = x_0 \in D,\ t_0 \geq 0$$

where F: $\mathbf{R}_+ \times D \rightarrow 2^E \backslash \phi$ is a set-valued map, $E = \mathbf{R}^n$ and $D \subset E$ is a closed set. As before, solutions x(t) of (4.6.1) are understood as absolutely continuous functions on $\mathbf{R}_+$. We need the following definition.

Definition 4.6.1

Given $0 < \lambda < A$ and $t_0 \in \mathbf{R}_+$, the system (4.6.1) is said to be
(i) weakly practically stable, if there exists a solution x(t) of (4.6.1) such that

$$|x_0| < \lambda \text{ implies } |x(t)| < A, \ t \geq t_0$$

for some $t_0 \in \mathbf{R}_+$; (ii) practically stable if all solutions x(t) of (4.6.1) satisfy $|x_0| < \lambda$ implies $|x(t)| < A$, $t \geq t_0$.

Concerning the practical stability of the comparison system (4.5.1) we require the following: Let $Q \in C[K, \mathbf{R}_+]$, Q(u) is nondecreasing in u with $Q(0)=0$. Then the system (4.5.1) practically stable if given $0 < \lambda < A$ and $t_0 \in \mathbf{R}_+$ we have

$$Q(u_0) < \lambda \text{ implies } Q(u(t)) < A, \ t \geq t_0$$

for all solutions u(t) of (4.5.1).

Other definitions of practical stability may be formulated on the basis of the foregoing definitions.

We shall suppose that for $(t, x) \in \mathbf{R}_+ \times D$, the mapping F satisfies the following conditions:

(C_0) F(t, x) is compact, convex;

(C_1) F(t, x) is measurable in t for each x and usc in u for each t;

(C_2) $|F(t, x)| \leq \lambda_0(t)[1+|x|]$ with $\lambda_0 \in L^1(J)$;

(C_3) $F(t, x) \cap T_D(x) \neq \phi$.

In view of (C_0) - (C_3), there exists a solutions $x(t)$ of (4.6.1) on every interval $[t_0, t_0+a]$, $a>0$, by Theorem 4.5.1.

To employ the method of vector Lyapunov functions, we suppose that $K=\mathbf{R}_+^N$ and utilize comparison theorems discussed in Section 4.5.

Theorem 4.6.1

Assume that

(i) $0<\lambda<A$ and $\overline{S(A)} \subset D$;

(ii) V: $\mathbf{R}_+ \times D \to K$ is absolutely continuous in t, continuously differentiable in x and

$$b(|x|) \leq Q(V(t, x)) \leq a(|x|), \quad a, b \in K,$$

where $Q \in C[K, \mathbf{R}_+]$, $Q(u)$ is nondecreasing in u with $Q(0)=0$;

(iii) $V'(t, x)(y) = V_t(t, x) + V_x(t, x)y \in G(t, V(t, x)+u) - T_K(u)$ for $u \in K$, $x \in S(A)$ and $y \in F(t, x) \cap T_K(x)$ and F satisfies (C_0) - (C_3);

(iv) G: $\mathbf{R}_+ \times K \to 2^{R^N} \backslash \phi$ and the set-valued map G satisfies the assumptions (A_0) - (A_2);

(v) $a(\lambda) < b(A)$ holds.

Then the practical stability properties of (4.5.1) implies the corresponding weak practical stability properties of (4.6.1).

Proof

We shall only prove weak practical stability since other notions can be proved similarly. Suppose that the system (4.5.1) is practically stable.

Then, because of (v), given $0<a(\lambda)<b(A)$, it follows that

(4.6.2) $$Q(u_0)<a(\lambda) \text{ implies } Q(u(t))<b(A),\ t\geq t_0,$$

where $u(t)=u(t,\ t_0,\ u_0)$ is any solution of (4.5.1) on $[t_0,\ \infty)$. We claim that the system (4.6.1) is weakly practically stable for $(\lambda,\ A)$. If this is not true, then there exists a solution $x(t)=x(t,\ t_0,\ x_0)$ of (4.6.1) by Theorem 4.5.1 on $t_0\leq t\leq t_1$ for some $t_1>t_0$ such that

(4.6.3) $$|\ x_0\ |<\lambda,\ |\ x(t_1)\ |=A \text{ and } |\ x(t)\ |<A,\ t_0\leq t\leq t_1.$$

Setting $m(t)=V(t,\ x(t))$ for $t_0\leq t\leq t_1$ and choosing $V(t_0,\ x_0)=u_0$, we get, in view of (iii) and the facts $x(t)\in S(A)$, $x'(t)\in F(t,x(t))\cap T_K(x(t))$,

(4.6.4) $$m'(t)=V_t(t,\ x(t))+V_x(t,\ x(t))x'(t)\in G(t,\ m(t)+u)-T_K(u),$$

for $u\in K$. Since G satisfies the conditions of Theorem 4.5.3, we obtain the estimate

(4.6.5) $$m(t)\leq u(t),\ t_0\leq t\leq t_1$$

where $u(t)$ is a solution of (4.5.1) on $[t_0,\ t_1]$. It then follows, using (4.6.2), (4.6.3), (4.6.5) and assumption (ii),

$$b(A)=b(|\ x(t_1)\ |)\leq Q(V(t_1,\ x(t_1)))\leq Q(u(t_1))<b(A),$$

Since $Q(u_0)=Q(V(t_0,\ x_0))\leq a(|\ x_0|)<a(\lambda)$. This contradiction proves that the system (4.6.1) is weakly practically stable and the proof is complete.

The next result deals with practical stability of (4.6.1) under stronger assumptions.

Theorem 4.6.2

Assume that (i), (ii) and (v) of Theorem 4.6.1 hold. Further, suppose that

$$V'(t, x)(y) \in G(t, V(t, x)) - K \tag{4.6.6}$$

for every $y \in F(t, x)$, $x \in S(A)$, $F(t, x) \subset T_D(x)$ and F satisfies (C_0) - (C_3), where G: $\mathbf{R}_+ \times K \to 2^{R^N} \backslash \phi$ satisfies the assumptions (A_0) - (A_4) on $\mathbf{R}_+ \times K$. Then the practical stability properties of (4.5.1) imply the corresponding practical stability properties of (4.6.1).

Proof

The proof is almost similar to the proof of Theorem 4.6.1 except that for every solution $x(t) = x(t, t_0, x_0)$ of (4.6.1) we have, instead of (4.6.4), the relation

$$m'(t) \in G(t, m(t)) - K, \; t_0 \le t \le t_1,$$

Consequently, we get by Theorem 4.5.2, the estimate

$$m(t) \le r(t), \; t_0 \le t \le t_1,$$

where r(t) is the maximal solution of (4.5.1) on $[t_0, t_1]$. The rest of the proof is similar and hence we omit the details.

4.7. NOTES.

The contents of Section 4.1 are modelled on the work of Lakshmikantham, Leela and Tsokos [1], and Lakshmikantham and Tsokos [1]. The results of Section 4.2 and 4.3 are adapted from the work of Bitsoris [1, 2], where as the general example is taken from Martynyuk [3, 4, 17]. Section 4.4 contains the results taken from Martynyuk [4, 17]. See also Krasovski [2].

See Deimling [1 - 3], and Deimling and Lakshmikantham [1] for the results given in Section 4.5. All the results of Section 4.6 are new. For further information on multivalued differential equations see Aubin [1], Aubin and Cellina [1], Filipov [1] and Deimling [1 - 3].

Appendix

A.1. DIFFERENTIAL INEQUALITIES.

An important method in the theory of differential equations is concerned with estimating a function satisfying a differential or an integral inequality by the maximal solution of the corresponding differential or integral equation. One of the results that is widely employed is the following comparison theorem.

Theorem A.1.1

Assume that

(i) $m \in C[\mathbf{R}_+, \mathbf{R}_+]$ and $D_-m(t) = \liminf_{h \to 0^-} \frac{1}{h}[m(t+h) - m(t)] \leq g(t, m(t))$, $t \in \mathbf{R}_+$, where $g \in C[\mathbf{R}_+ \times \mathbf{R}_+, \mathbf{R}]$;

(ii) $r(t)$ is the maximal solution of

$$u' = g(t, u), \; u(t_0) = u_0 \geq 0, \; t_0 \geq 0, \tag{A.1.1}$$

existing on $[t_0, \infty)$.

Then $m(t) \leq u_0$ implies $m(t) \leq r(t)$, $t \geq t_0$.

A useful corollary which deals with an integral inequality is as follows

Corollary A.1.1

Assume that $m \in C[\mathbf{R}_+, \mathbf{R}_+]$ and

$$m(t) \leq m(t_0) + \int_{t_0}^{t} g(s, m(s))ds, \ t \geq t_0,$$

where $g \in C[\mathbf{R}_+ \times \mathbf{R}_+, \mathbf{R}]$ and $g(t, u)$ is nondecreasing in u for each $t \in \mathbf{R}_+$. If assumption (ii) of Theorem A.1.1 holds, then the conclusion of Theorem A.1.1 remains valid.

The next result is concerned with an integral inequality which is a special case of Bihari and Gronwall inequalities.

Lemma A.1.1

Let m, $\lambda \in C[\mathbf{R}_+, \mathbf{R}_+]$ and

$$m(t) \leq m(t_0) + \int_{t_0}^{t} \lambda(s) m^{\alpha}(s)ds, \ t \geq t_0, \ \alpha > 0.$$

Then

(a) if $\alpha\in(0, 1)$,

$$m(t)\leq[m(t_0)+(1-\alpha)\int_{t_0}^{t}\lambda(s)ds]^{\frac{1}{1-\alpha}},\ t\geq t_0;$$

(b) if $\alpha\in(1, \infty)$,

$$m(t)\leq m(t_0)[1-(\alpha-1)m^{\alpha-1}(t_0)\int_{t_0}^{t}\lambda(s)ds]^{\frac{1}{1-\alpha}},\ t\in[t_0, T],$$

where $T=\sup[t\geq t_0:\ \int_{t_0}^{t}\lambda(s)ds<\frac{1}{(\alpha-1)[m(t_0)]^{\alpha-1}}$;

(c) if $\alpha=1$,

$$m(t)\leq m(t_0)\exp[\int_{t_0}^{t}\lambda(s)ds],\ t\geq t_0.$$

A result which connects the left and right maximal solutions is the following.

Lemma A.1.2

Let $g, g_0\in C[\mathbf{R}_+\times\mathbf{R}_+, \mathbf{R}]$ satisfy

$$g_0(t, u)\leq g(t, u),\ (t, u)\in\mathbf{R}_+\times\mathbf{R}_+.$$

Then, the right maximal solution $r(t, t_0, u_0)$ of (A.1.1) and the left maximal solution $\eta(t, T, v_0)$ of

$$u'=g_0(t, u),\ u(T)=v_0\geq 0,$$

satisfy the relation

$$r(t, t_0, u_0)\leq\eta(t, T, v_0),\ t_0\leq t\leq T,$$

whenever $r(T, t_0, u_0) \le v_0$.

It is well known that the method of variation of parameters is an important tool in the investigation of the properties of solutions of nonlinear systems. The result that we give below discusses different forms of the variation of parameters formula which offer flexibility in applications.

Theorem A.1.2

Assume that

(i) the system $x' = f(t, x)$, $x(t_0) = x_0$, where $f \in C[\mathbf{R}_+ \times \mathbf{R}^n, \mathbf{R}^n]$, admits unique solutions $x(t, t_0, x_0)$ for $t \ge t_0$;

(ii) $\Phi(t, t_0, x_0) = \dfrac{\partial x}{\partial x_0}(t, t_0, x_0)$ exists, is continuous and $\Phi^{-1}(t, t_0, x_0)$ exists for $t \ge t_0$, and $v(t)$ is any solution of

$$v' = \Phi^{-1}(t, t_0, x_0) R(t, x(t, v_0, v)) \quad v(t_0) = v_0,$$

existing for $t \ge t_0$.

Then any solution $y(t) = y(t, t_0, x_0)$ of

$$y' = f(t, y) + R(t, y), \ t(t_0) = x_0,$$

where $R \in C[\mathbf{R}_+ \times \mathbf{R}^n, \mathbf{R}^n]$ satisfies the relations

$$\text{(A.1.2)} \quad y(t) = x[t, t_0, x_0 + \int_{t_0}^{t} \Phi^{-1}(s, t_0, v(s)) R(s, y(s, t_0, x_0)) ds,$$

$$\text{(A.1.3)} \quad y(t) = x(t, t_0, x_0) + \int_{t_0}^{t} \Phi(t, t_0, v(s)) \Phi^{-1}(s, t_0, v(s)) R(s, y(s, t_0, x_0)) ds,$$

for $t \geq t_0$.

If instead of assumption (ii), we suppose that f possesses continuous partial derivatives $\frac{\partial f}{\partial x}$ on $\mathbf{R}_+ \times \mathbf{R}^n$, then $y(t, t_0, x_0)$ satisfies

$$\text{(A.1.4)} \quad y(t,t_0,x_0) = x(t,t_0,x_0) + \int_{t_0}^{t} \Phi(t,s,y(s,t_0,x_0))R(s,y(s,t_0,x_0))ds,$$

for $t \geq t_0$ and the relations (A.1.3) and (A.1.4) are equivalent.

Corollary A.1.2

If $f(t, x) = A(t)x$, where $A(t)$ is an $n \times n$ continuous matrix, then $x(t, t_0, x_0) = \Phi(t, t_0)x_0$, $\Phi(t, t_0)$ being the fundamental matrix solution of $x' = A(t)x$, with $\Phi(t_0, t_0) = I$, the identity matrix. Then the relations (A.1.2) - (A.1.3) yield the well known formula

$$\text{(A.1.5)} \quad y(t,t_0,x_0) = \Phi(t,t_0)x_0 + \int_{t_0}^{t} \Phi(t,t_0)\Phi^{-1}(s,t_0)R(s,y(s,t_0,x_0))ds,$$

since $\Phi(t, t_0)$ is nonsingular for $t \geq t_0$ in this case. If $A(t) \equiv A$, then (A.1.5) reduces to

$$y(t, t_0, x_0) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, s)\, R(s, y(s, t_0, x_0))ds.$$

Formula (A.1.4) is known as Alekseev's formula.

An extension of Theorem A.1.1 to systems requires $g(t, u)$ to be quasimonotone nondecreasing in u, which is also a necessary condition for the existence of extremal solutions of

(A.1.6) $$u'=g(t, u),\ u(t_0)=u_0\geq 0,\ t_0\geq 0,$$

where $g\in C[\mathbf{R}_+\times\mathbf{R}^N, \mathbf{R}^N]$. A function $g(t, u)$ is said to be quasimontone nondecreasing in u, if $u\leq v$ and $u_i=v_i$ for $1\leq i\leq N$ implies $g_i(t, u)\leq g_i(t, v)$ for all i. Of course, the inequalities between vectors are understood to be componentwise inequalities. We can now state the extension.

*Theorem A.*1.3

Suppose that

(i) $m\in C[\mathbf{R}_+, \mathbf{R}_+^N]$ and $D_-m(t)\leq g(t, m(t))$, $t\geq t_0$;

(ii) $g\in C[\mathbf{R}_+\times\mathbf{R}_+^N, \mathbf{R}^N]$ and $g(t, u)$ is quasimonotone nondecreasing in u for each $t\in\mathbf{R}_+$;

(iii) $r(t)$ is the maximal solution of (A.1.6) existing on $[t_0, \infty)$.

Then $m(t_0)\leq r(t)$, $t\geq t_0$.

If, on the other hand, $D_-m(t)\geq g(t, m(t))$, $t\geq t_0$ and $\rho(t)$ is the minimal solution of (A.1.6) existing on $[t_0, \infty)$, then $m(t_0)\geq u_0$ implies $m(t)\geq\rho(t)$, $t\geq t_0$.

Another comparison result, which is more useful in certain situations, is the following.

*Theorem A.*1.4.

Assume that

(i) $m, v\in C[\mathbf{R}_+, \mathbf{R}_+^N]$ and $D_-m(t)\leq g(t, m(t), v(t)), t\geq t_0$;

(ii) $g\in C[\mathbf{R}_+\times\mathbf{R}_+^N\times\mathbf{R}_+^N, \mathbf{R}^N]$ and $g(t, u, v)$ is quasimonotone nondecreasing in u and monotone nondecreasing in v;

(iii) $r(t)$ is the maximal solution of

$$u' = g(t, u, u), \; u(t_0) = u_0 \geq 0, \; t_0 \geq 0,$$

existing on$[t_0, \infty)$. Then $v(t) \leq r(t)$ and $m(t_0) \leq u_0$ implies $m(t) \leq r(t)$, $t \geq t_0$.

A.2. INTEGRO-DIFFERENTIAL EQUATIONS.

The following result helps in bringing out the inherent rich behaviour of linear integro-differential systems by finding an equivalent linear differential system.

Theorem A.2.1

Assume that there exists an $n \times n$ continuous matrix function $L(t, x)$ on $\mathbf{R}_+^2$ such that $\frac{\partial L}{\partial s}(t, s)$ exists, is continuous and satisfies the relation

$$K(t, x) + \frac{\partial L}{\partial s}(t, s) + L(t, s)A(s) + \int_s^t L(t, \sigma)K(\sigma, s)d\sigma = 0$$

where $K(t, s)$, $A(s)$ are continuous $n \times n$ matrices on $\mathbf{R}_+^2$ and $\mathbf{R}_+$ respectively. Then the IVP

$$x'(t) = A(t)x(t) + \int_{t_0}^t K(t, s)x(s)ds + F(t), \; x(t_0) = x_0,$$

where $F \in C[\mathbf{R}_+, \mathbf{R}^n]$, is equivalent to the IVP

$$y'(t) = B(t)y(t) + L(t, t_0)x_0 + H(t), \; y(t_0 = x_0,$$

where $B(t) = A(t) - L(t, t)$ and $H(t) = F(t) + \int_{t_0}^t L(t, s)F(S)ds$.

A.3. IMPULSIVE DIFFERENTIAL INEQUALITIES.

A general comparison result that corresponds to Theorem A.1.1 is the following result. We denote by PC the class of piecewise continuous functions on $\mathbf{R}_+$ to $\mathbf{R}$ with discontinuities of the first kind at $t=t_k$ only.

Theorem A.3.1

Assume that

(i) the sequence $\{t_k\}$ satisfies $0 \le t_0 < t_1 < t_2 < \ldots$, with $t_k \to \infty$ as $k \to \infty$;

(ii) $m \in PC[\mathbf{R}_+, \mathbf{R}]$ and $m(t)$ is left continuous at $t=t_k$;

(iii) $g \in C[\mathbf{R}_+ \times \mathbf{R}, \mathbf{R}]$, ψ_k: $\mathbf{R} \to \mathbf{R}$, $\psi_k(u)$ is nondecreasing in u and for each k,

$$D_- m(t) \le g(t, m(t)),\ t \ne t_k,\ m(t_0) \le u_0,$$

$$m(t_k^+) \le \psi_k(m(t_k));$$

(iv) $r(t)$ is the maximal solution of the impulsive differential equation

$$u' = g(t, u),\ t \ne t_k$$

$$u(t_k^+) = \psi_k(u(t_k)),$$

$$u(t_0) = u_0.$$

Then $m(t) \le r(t,)$, $t \ge t_0$.

We shall next consider Bihari type of integral inequality with impulse effects.

Theorem A.3.2

Assume that (i) and (ii) of Theorem A.3.1 hold. Suppose further that $p \in C[\mathbf{R}_+, \mathbf{R}_+]$, $g \in C[\mathbf{R}, (0, \infty)]$, $g(u)$ is nondecreasing in u, $g(\lambda u) \geq \mu(\lambda) g(u)$ for $\lambda > 0$, $u \in \mathbf{R}$, where $\mu(\lambda) > 0$ for $\lambda > 0$, and

$$m(t) \leq C + \int_{t_0}^{t} p(s)\, g(m(s))ds + \sum_{t_0 < t_k < t} \beta_k m(t_k),$$

with $\beta_k \geq 0$, $C \geq 0$. Then, ,for $\lambda_0 \leq t < T$,

$$m(t) \leq G^{-1}[G(C \prod_{t_0 < t_k < t} (1+\beta_k)) + \int_{t_0}^{t} \sum_{s < t_k < t} \frac{1+\beta_k}{\mu(1+\beta_k)} p(s)ds],$$

where $G'(u) = \frac{1}{g(u)}$, G^{-1} is the inverse of G and

$$T = \sup[t \geq t_0 \colon\ G(C \sum_{t_0 < t_k < t} (1+\beta_k) + \int_{t_0}^{t} \sum_{s < t_k < t} \frac{1+\beta_k}{\mu(1+\beta_k)} p(s)ds) \xi \mathrm{dom} G^{-1}].$$

For Theorems A.1.1, A.1.3, A.1.4 and Corollary A.1.1, see Lakshmikantham and Leela [1]. The remaining results of Section A.1 and Theorem A.2.1 are taken from Lakshmikantham, Leela and Martynyuk [1], Martynyuk [1]. See Lakshmikantham, Bainov and Semeonov [1] for Theorems A.3.1 and A.3.2.

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