Unit - V 21/06/2023

# Joint Probability and Stochastic Distribution

①. A fare coin is tossed 3 times $X = 0$ or $1$ ~~occured in~~ ~~to~~ according to tail or head occuring on the first toss. and $Y =$ no. of tails. Determine the following.

(a) Marginal distributions of $X$ and $Y$

(b) Joint PDF of $X$ and $Y$

(c) $\mu$ of $X$ ($\mu_x$) and $\mu$ of $Y$ ($\mu_y$), $E(XY)$, variance of $X$
($E(X)$) ($E(Y)$)

variance of $Y$, covariance of $XY$ and correlation of $XY$

$S = \{HHH, HHT, HTH, THH, HTT, THT, TTH, TTT\}$

X: 1, 1, 1, 0, 1, 0, 0, 0

Y: 0, 1, 1, 1, 2, 2, 2, 3

$X = \{0, 1\}$ [If 1st tail is occuring the 0)

$Y = \{0, 1, 2, 3\}$ // No. of tails ~~or head~~

Now ~~Join~~ marginal distribution of $X$

| $X$ | 0 | 1 |
|---|---|---|
| $P(X)$ | 1/2 | 1/2 |

Marginal distribution of $Y$

| $y$ | 0 | 1 | 2 | 3 |
|---|---|---|---|---|
| $P(y)$ | 1/8 | 3/8 | 3/8 | 1/8 |

Joint probability distribution

| $X \backslash Y$ | 0 | 1 | 2 | 3 | |
|---|---|---|---|---|---|
| 0 | 0 $p_{11}$ | 1/8 $p_{12}$ | 2/8 $p_{13}$ | 1/8 | = 1/2 |
| 1 | 1/8 | 2/8 | 1/8 | 0 | = 1/2 |
| | = 1/8 | = 3/8 | = 3/8 | = 1/8 | |

Mean of $x = E(x) = M_x = \Sigma x_i P(x_i)$

$= 0(4/8) + 1(4/8)$

$= 1/2$

Mean of $y = E(y) = M_y = 0(1/8) + 1(3/8) + 2(3/8) + 3(1/8)$

$= 3/8 + 3/8 + 6/8$

$= 12/8$

$= 3/2$

$E(XY) = M_{xy} = \Sigma x_i y_j P(x_i y_j)$ (JPD table)

$= 0(0)(0) + 0(1)(1/8) + 0(2)(2/8) + 0(3)(1/8) +$

$1(0)(1/8) + 1(1)(2/8) + 1(2)(1/8 + 1(3)(0)$

$= 0+0+0+0+0+2/8+2/8+0$

$= 4/8$

$= 1/2$

$\sigma_x^2 = V_x = \Sigma x_i^2 P(x_i) - M_x^2$

$= 0(4/8) + 1(4/8) - (1/2)^2$

$\sigma_x^2 = 1/4 = 0.25$

$\sigma_y^2 = V_y = \Sigma y_i^2 P(y_i) - M_y^2$

$= 0(1/8) + 1(3/8) + 4(3/8) + 9(1/8)$

$- 9/4$

$= 3/8 + 12/8 + 9/8 - 9/4$

$= \frac{24}{8} - \frac{9^{\times 2}}{4}$

$= \frac{24-18}{8}$

$= 6/8$

$= 3/4$

$= 0.75$

Covariance of XY $= \cancel{\sum x_i y_i}\ E(xy) - E(x)E(y)$

$= \cancel{1/2 - 1/2 \times 3/2}$

$= \cancel{1/2 - 3/4}$

$= \cancel{\frac{2-3}{4}}$

variance is +ve by covariance i.e variance b/n 2 values it may be negative

$= 1/2 - 1/2 \times 3/2$

$= 1/2 - 3/4$

$= \frac{2-3}{4}$

$= -1/4$

Correlation ($\rho$) of XY $= \frac{\text{Covariance XY}}{\sigma_x \sigma_y}$

$= \frac{-1/4}{\sqrt{1/4}\sqrt{3/4}}$

$= \frac{-1/4}{\sqrt{3/4}}$ $-0.577$

Q. A joint PDF is given by

| X \ Y | -3 | 2 | 4 |
|---|---|---|---|
| 1 | 0.1 | 0.2 | 0.2 |
| 3 | 0.3 | 0.1 | 0.1 |

Find (a) mean of X

(b) mean of Y

(c) variance of X

(d) Variance of Y

(e) Covariance of XY

(f) Correlation

marginal distribution of x

| x | 1 | 3 |
|---|---|---|
| P(x) | 0.5 | 0.5 |

marginal distribution of Y

| y | -3 | 2 | 4 |
|---|---|---|---|
| P(y) | 0.4 | 0.3 | 0.3 |

(a) Mean of X

$\mu_x = \Sigma x_i P(x_i)$

$= 1(0.5) + 3(0.5)$

$= 0.5 + 1.5$

$= 2$

(b) Mean of Y

$\mu_y = \Sigma y_i P(y_i)$

$= -3(0.4) + 2(0.3) + 4(0.3)$

$= -\cancel{1.2} + 0.6 + \cancel{1.2}$

$\mu_y = 0.6$

(c) variance of X

$\sigma_x^2 = \Sigma x_i^2 P(x_i) - \mu_x^2$

$= 1(0.5) + 9(0.5) - 2^2$

$= 0.5 + 4.5 - 4$

$= 5 - 4$

$\sigma_x^2 = 1$

(d) Variance of Y

$\sigma_y^2 = \Sigma y_i^2 P(y_i) - \mu_y^2$

$= 9(0.4) + 4(0.3) + 16(0.3) - (0.6)^2$

$= 3.6 + 1.2 + 4.8 - 0.36$

$= 9.24$

(e) Covariance of (XY)

$= E(XY) - E(X)E(Y)$

$= \Sigma x_i y_j P_{ij}(XY) - E(X)E(Y)$

$= 1(-3)(0.1) + 1(2)(0.2) + 4(0.2)$

$+ 3(-3)(0.3) + 3(2)(0.1)$

$+ 3(4)(0.1) - 2 \times 0.6$

$= -0.3 + 0.4 + 0.8$

$- 2.7 + 0.6 + 1.2 - 1.2$

$= 0 - 1.2$

$= -1.2$

$$\text{Correlation } (\rho) = \frac{\text{cov}(XY)}{\sigma_x \sigma_y}$$

$$= \frac{-1.2}{\sqrt{1}\sqrt{9.24}}$$

$$\rho = -0.394$$

<u>Def</u>: Joint Probability distribution function of 2 variables X and Y is given by

| X \ Y | $y_1$ | $y_2$ | - - | $y_n$ |
|---|---|---|---|---|
| $x_1$ | $P_{11}$ | $P_{12}$ | - - - | $P_{1n}$ |
| $x_2$ | $P_{21}$ | $P_{22}$ | - - - - | $P_{2n}$ |
| ⋮ | ⋮ | ⋮ | | ⋮ |
| $x_n$ | $x_{n1}$ | $x_{n2}$ | - - - | $x_{nn}$ |

Such that (i) $0 \le P_{ij} \le 1$

(ii) $\Sigma P_{ij} = 1$

mean of X → $E(X) = \mu_x = \Sigma x_i P(x_i)$

mean of Y → $E(Y) = \mu_y = \Sigma y_i P(y_i)$

mean of XY → $E(XY) = \mu_{xy} = \Sigma x_i y_j P_{ij}$

Variance of X → $\sigma_x^2 = (\Sigma x_i^2 P(x_i)) - \mu_x^2$

Variance of Y → $\sigma_y^2 = (\Sigma y_i^2 P(y_i)) - \mu_y^2$

Covariance of XY → $\text{Cov}(XY) = E(XY) - E(X)E(Y)$

Correlation → $\text{Correlation } (\rho) = \dfrac{\text{Cov}(XY)}{\sigma_x \sigma_y}$

Imp

Q. The joint PDF of 2 random variables X and Y is

| X\Y | 2 | 3 | 4 |
|---|---|---|---|
| 1 | 0.06 | 0.15 | 0.09 |
| 2 | 0.14 | 0.35 | 0.21 |

(a) Determine the individual distribution of X and Y

(b) Verify whether X and Y are independent

(a) Marginal distribution of X

| X | 1 | 2 |
|---|---|---|
| P(x) | 0.30 | 0.7 |
| | $p_1$ | $p_2$ |

Marginal distribution of Y

| Y | 2 | 3 | 4 |
|---|---|---|---|
| P(y) | 0.2 | 0.5 | 0.3 |
| | $q_1$ | $q_2$ | $q_3$ |

(b) $P_1 = 0.3$ $P_2 = 0.7$ $q_1 = 0.2$ $q_2 = 0.5$ $q_3 = 0.3$

$P_1 q_1 = 0.3 \times 0.2 = 0.06 = P_{11}$

$P_1 q_2 = 0.3 \times 0.5 = 0.15 = P_{12}$

$P_1 q_3 = 0.3 \times 0.3 = 0.09 = P_{13}$

$P_2 q_1 = 0.7 \times 0.2 = 0.14 = P_{21}$

$P_2 q_2 = 0.7 \times 0.5 = 0.35 = P_{22}$

$P_2 q_3 = 0.7 \times 0.3 = 0.21 = P_{23}$

Hence X and Y are independent.

Q. The JPD is given as along.

| X\Y | 0 | 1 |
|---|---|---|
| 0 | 0.1 | 0.2 |
| 1 | 0.4 | 0.2 |
| 2 | 0.1 | 0 |

(a) Determine the marginal distribution of X and Y

(b) Are X and Y independent

(c) find $P(X+Y>1)$

(a) Marginal distribution of X

| X | 0 | 1 | 2 |
|---|---|---|---|
| P(X) | 0.3 | 0.6 | 0.1 |

Marginal distribution of Y

| Y | 0 | 1 |
|---|---|---|
| P(y) | 0.6 | 0.4 |

(b) $P_1 = 0.3$ $P_2 = 0.6$ $P_3(0.1)$ $q_1 = 0.6$ $q_2 = 0.4$

$P_1 q_1 = 0.3 \times 0.6 = 0.18 \neq P_{11}$

X and Y are not independent.

(c) $P(X+Y>1)$ [ Probability where addition of X and Y is greater than 1, such combinations are [(1,1) (2,0) (2,1)] ]

$= 0.2 + 0.1 + 0$

$= 0.3$

$P(X+Y>1) = 0.3$

Q. The distributions of two stochastically independent variables X and Y are

| X | 0 | 1 |
|---|---|---|
| P(X) | 0.2 | 0.8 |

$p_1$ $p_2$

| Y | 1 | 2 | 3 |
|---|---|---|---|
| P(y) | 0.1 | 0.4 | 0.5 |

$q_1$ $q_2$ $q_3$

Find Joint PDF

since X and Y are independent so

$P_1 q_1 = P_{11}$

$P_1 q_2 = P_{12}$

$P_1 q_3 = P_{13}$

$P_2 q_1 = P_{21}$

$P_2 q_2 = P_{22}$

$P_2 q_3 = P_{23}$

| X \ Y | 1 | 2 | 3 |
|---|---|---|---|
| 0 | 0.02 | 0.08 | 0.1 |
| 1 | 0.08 | 0.32 | 0.4 |

Q. Find the unique fixed probability vector for the following stochastic matrix.

$$A = \begin{bmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix}$$

$x+y=1 \Rightarrow$ probability vectors whose row sum

We have to find unique fixed probability vector i.e

$VA = V$ [ VA should be equal to v]

$$[x \quad y]\begin{bmatrix} 3/4 & 1/4 \\ 1/2 & 1/2 \end{bmatrix} = [x \quad y]$$

$$[3/4x + 1/2y \qquad 1/4x + 1/2y] = [x, y]$$

$$3/4x + 1/2y = x$$

$$\Rightarrow -1/4x + 1/2y = 0$$

$$1/4x + 1/2y = y$$

$$\Rightarrow 1/4x - 1/2y = 0$$

and we know that $x+y=1$

so $x = 2/3$ $y = 1/3$.

Q. $$A = \begin{bmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{bmatrix}$$

$VA = V$

$$[x \quad y \quad z]\begin{bmatrix} 0 & 1 & 0 \\ 1/6 & 1/2 & 1/3 \\ 0 & 2/3 & 1/3 \end{bmatrix} = [x, y, z]$$

$$[1/6y, \quad x + 1/2y + 2/3z \quad 1/3y + 1/3z] = [x, y, z]$$

$$1/6y = x$$

$$\Rightarrow 1/6y - x = 0$$

$x + 1/2\, y + 2/3\, z = y$

$\Rightarrow x - 1/2\, y + 2/3\, z = 0$

$1/2\, y + 1/3\, z = z$

$\Rightarrow 1/3\, y - 2/3\, z = 0$

and we know that $x + y + z = 1$

$x = 1/10 \quad y = \frac{3}{5} \quad z = 3/10$

{In obtained 3 equations we should choose any 2 equation and 3rd one should be $x+y+z=1$ because if we choose all three equations we get x, y, z values as '0'.}

Q. Show that $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}$ is a regular stochastic matrix, also find the associated unique fixed probability vector.

[Regular stochastic matrix is nothing but a matrix which has each entry positive and ~~non-zero~~]

To get that we shall find $P^2$, $P^3$ and $P^n$ so on, until we get regular stochastic matrix.

$3 \times 4$

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$$P^2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix}$$

$$P^3 = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

$$P^4 = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \end{bmatrix}$$

$$P^5 = \begin{bmatrix} 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \\ 1/4 & 1/2 & 1/4 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} = \begin{bmatrix} 1/4 & 1/4 & 1/2 \\ 1/4 & 2/2 & 1/4 \\ 1/8 & 3/8 & 1/2 \end{bmatrix}$$

Hence for $p^5$, all zeros are eliminated

Hence matrix p is regular stochastic matrix.

Now to find unique probability vector.

$$VP = V$$

$$[x \quad y \quad z] \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} = [x, y, z]$$

$$\left[\tfrac{1}{2}z \quad x + \tfrac{1}{2}z \quad y\right] = [x, y, z]$$

$$x = \tfrac{1}{2}z$$

$$\Rightarrow x - \tfrac{1}{2}z = 0$$

$$x + \tfrac{1}{2}z = y$$

$$\Rightarrow x - y + \tfrac{1}{2}z = 0$$

$$y = z$$

$$\Rightarrow y - z = 0$$

so we know that

$$x + y + z = 1$$

$x = 1/5$ $\quad y = 2/5$ $\quad z = 2/5$

Q. Prove that the marchochain whose transition probability matrix is $P = \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix}$ is irredusable.

$$P^2 = \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2/3 & 1/3 \\ 1/2 & 0 & 1/2 \\ 1/2 & 1/2 & 0 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/6 & 1/3 \\ 1/4 & 7/12 & 1/6 \\ 1/4 & 2/3 & 5/12 \end{bmatrix}$$

All the entries are positive hence it is irredusable.

Q. A students study habits are as follows, If he studies one night he is 70% sure not to study the next night. On the other hand if he does not one night he is 60% sure not to study the next night. In the long run how often does he study.

$$P = \begin{array}{ccc} & \text{Studying} & \text{not-study} \\ \text{Studying} & 0.3 & 0.7 \\ \text{Not studying} & 0.4 & 0.6 \end{array}$$

In the long run

$$VP = V$$

$$[x, y] \begin{bmatrix} 0.3 & 0.7 \\ 0.4 & 0.6 \end{bmatrix} = [x \;\; y]$$

$$[0.3x + 0.4y \quad 0.7x + 0.6y] = [x, y]$$

$$0.3x + 0.4y = x$$

$$\Rightarrow -0.7 + 0.4y = 0$$

$$0.7x + 0.6y = y$$

$$\Rightarrow 0.7x - 0.4y = 0$$

and we know that

$$x + y = 1$$

If the question says that long run, then we need to find unique fixed probability vector.

$x = 4/11 \quad y = 7/11$

In the long run the probability of he studying is 36.36% and not studying is 63.63%.

Q. A salesman teretory consists of 3 cities A, B, C. He never sells in same city on successive days. If he sells in city A then the next day sails in city B. However if he sells in either B or C then the next day he is twice as likely to sell in city A as in other city. In the long run how often does he sell in each of the cities.

<u>Transition matrix</u>

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 2/3 & 0 & 1/3 \\ 2/3 & 1/3 & 0 \end{bmatrix} \end{array}$$

(no successive sells in same city → diag zero)

(twise as likey so 2/3, 1/3)

in the long run

$$VA = V$$

$$[x \ \ y \ \ z] \begin{bmatrix} 0 & 1 & 0 \\ 2/3 & 0 & 1/3 \\ 2/3 & 1/3 & 0 \end{bmatrix} = [x \ \ y \ \ z]$$

$$\left[2/3\,y + 2/3\,z \quad x + 1/3\,z \quad 1/3\,y\right] = [x \ \ y \ \ z]$$

$$-x + 2/3\,y + 2/3\,z = 0$$

$$x - y + 1/3\,z = 0$$

$$1/3\,y - z = 0$$

and we know that

$$x + y + z = 1$$

$$x = 2/5 \quad y = 9/20 \quad z = 3/20$$

He sell in city A 2/5 and B 9/20 and C 3/20.

H.W: A man's smoking habits are as follows. If he smokes filter sigarate one week he switches to non filter sigeratte next week with probability 0.2. On the other hand if he smokes ~~non-sigeratte~~ filter sigeratte one week there is probability of 0.7 that he will ~~non~~-filter sigerate. In the long run how often does he smoke filter cigrates

Transition matrix

$$P = \begin{array}{c|cc} & \text{filter} & \text{nonfilter} \\ \text{filter} & 0.8 & 0.2 \\ \text{nonfilter} & 0.3 & 0.7 \end{array}$$

In the long run

$$VP = V$$

$$[x \;\; y] \begin{bmatrix} 0.8 & 0.2 \\ 0.3 & 0.7 \end{bmatrix} = [x \;\; y]$$

$$0.8x + 0.3y = x$$

$$\Rightarrow -0.2x + 0.3y = 0$$

$$0.2x + 0.7y = y$$

$$\Rightarrow 0.2x - 0.3y = 0$$

and $x + y = 1$

$x = 3/5 \qquad y = 2/5$

Q. A habitual Gambler is a member of 2 Clubs A and B. He visits either of the clubs everyday for playing cards. He never visits club A on 2 consecutive days. But if he visits club B on particular day then the next day he as likey to visit club A and / club B. (i) Find the transition matrix of the Markochain (ii) Show that the matrix is regular stochastic matrix (iii) find the unique fixed probability vector. (iv) If the person had visited club B on monday find the probability that he visits club A on thursday.

(i) Transition matrix

$$P = \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix}$$

(ii) To show matrix is regular stochastic matrix

$$P^2 = \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} = \begin{bmatrix} 1/2 & 1/2 \\ 1/4 & 3/4 \end{bmatrix}$$

Hence the matrix is regular stochastic matrix

(iii) To find unique fixed probability vector

$VP = V$

$$[x \quad y] \begin{bmatrix} 0 & 1 \\ 1/2 & 1/2 \end{bmatrix} = [x \quad y]$$

$$[1/2 y \quad x + 1/2 y] = [x, y]$$

$-x + 1/2 y = 0$

$x + 1/2 y = 0$

and $x + y = 1$

$x = 1/3 \quad y = 2/3$

(iii) He we have to find something called stationary

so

$$p^{(3)} = p^{(0)} p^{3}$$ (Here we are talking about 3 days down to line)

$$= \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} 1/4 & 3/4 \\ 3/8 & 5/8 \end{bmatrix}$$

$$= \begin{bmatrix} 3/8 & 5/8 \end{bmatrix}$$

[ $p^{(0)}$ → initial condition → He visit B clubon monday
$p^{(3)}$ → final condition → He visiting club A in 3 days down to line]

Q. 3 boys A, B and C are throwing ball to each other. A always throws a ball to ~~B~~ B. And B always throws the ball to C. C is just as likely to throw the ball to B as to A. If C was the first person to throw the ball find the probabilities that after 3 throws (or 4th throw)

(i) A has the ball

(ii) is B has the ball

(iii) C has the ball

The transition matrix is

$$P = \begin{array}{c} \\ A \\ B \\ C \end{array} \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \end{array}$$

$$p^{(0)} = \begin{array}{c} \begin{array}{ccc} A & B & C \end{array} \\ \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \end{array}$$ (C has ball)

$$p^{2} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

$$p^{3} = \begin{bmatrix} 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1/2 & 1/2 & 0 \end{bmatrix}$$

$$p^{3} = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

$$p^{(3)} = p^{(0)}\, p^{3}$$

$$= \begin{bmatrix} 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 1/2 & 0 \\ 0 & 1/2 & 1/2 \\ 1/4 & 1/4 & 1/2 \end{bmatrix}$$

$$= \begin{bmatrix} 1/4 & 1/4 & 1/2 \end{bmatrix}$$

Q. If $A = \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$ is stochastic matrix and

$v = [v_1 \quad v_2]$ is probability vector. Show that $VA$ is also a probability vector.

Since A is stochastic matrix
we have $\left. \begin{array}{l} a_1 + a_2 = 1 \\ b_1 + b_2 = 1 \end{array} \right\} \text{——} ①$

Since B is probability vector
we have $v_1 + v_2 = 1$ —— ②

No we shall find, VA

$$VA = [v_1 \quad v_2] \begin{bmatrix} a_1 & a_2 \\ b_1 & b_2 \end{bmatrix}$$

$$= [a_1v_1 + b_1v_2 \quad a_2v_1 + b_2v_2]$$

Now we

$$a_1v_1 + b_1v_2 + a_2v_1 + b_2v_2 = (a_1 + a_2)v_1 + (b_1 + b_2)v_2 \quad (\text{from } ①)$$

$$= v_1 + v_2 \quad (\text{from } ②)$$

$$= 1$$

Hence VA is also probability vector.

Q. If $P_1 = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix}$ $P_2 = \begin{bmatrix} 1-b & b \\ a & 1-a \end{bmatrix}$ show that

$P_1$, $P_2$ and $P_1P_2$ is stochastic matrix.

For $P_1$ $1-a+a = 1$ and $1-b+b = 1$ hence it is stochastic matrix since addition of rows is 1 and entries are non-negative.

For $P_2$

$1-b+b=1$ and $a+1-a=1$

Since row addition is 1 and entries are non-negative

$P_2$ is also stochastic matrix

$$P_1P_2 = \begin{bmatrix} 1-a & a \\ b & 1-b \end{bmatrix} \begin{bmatrix} 1-b & b \\ a & 1-a \end{bmatrix}$$

$$= \begin{bmatrix} 1-a-b+ab+a^2 & b-ab+a-a^2 \\ b-b^2+a-ab & b^2+1-a-b+ab \end{bmatrix}$$

Here

$1-\cancel{a}-\cancel{b}+\cancel{ab}+\cancel{a^2}+\cancel{b}-ab+\cancel{a}-\cancel{a^2} = 1$

and

$\cancel{b}-\cancel{b^2}+\cancel{a}-\cancel{ab}+\cancel{b^2}+1-\cancel{a}-\cancel{b}+\cancel{ab} = 1$

So $P_1P_2$ is stochastic matrix since all row addition is 1 and entries are non-negative.

⊙ If $t=(t_1, t_2 \ldots\ldots t_m)$ is a vector and $T$ is square matrix whose rows are each the same vector $t$. Then $pT=t$ if $p=(P_1, P_2, \ldots P_n)$ is a probability vector.

$$T = \begin{bmatrix} t_1 & t_2 & \cdots & t_m \\ t_1 & t_2 & \cdots & t_m \\ \vdots & \vdots & \ddots & \vdots \\ t_1 & t_2 & \cdots & t_m \end{bmatrix}$$

∵ $T$ is a square matrix

$t=(t_1\ t_2 \ldots t_m)$

$P = (P_1\ P_2\ P_3 \cdots P_m)$ is a probability vector

$\Rightarrow P_1 + P_2 + \cdots + P_m = 1$

To show that

$PT = t$

$$PT = (P_1\ P_2\ \cdots P_m)\begin{bmatrix} t_1 & t_2 & \cdots & t_m \\ t_1 & t_2 & \cdots & t_m \\ \vdots & \vdots & \ddots & \vdots \\ t_1 & t_2 & \cdots & t_m \end{bmatrix}$$

$$= [P_1 t_1 + P_2 t_1 + \cdots P_m t_1 \quad P_1 t_2 + P_2 t_2 + \cdots + P_m t_2 \quad \cdots\cdots \quad P_1 t_m + P_2 t_m + \cdots P_m t_m]$$

$$= [t_1 (P_1 + P_2 + \cdots P_m) \quad t_2 (P_1 + P_2 + \cdots P_m) \quad \cdots\cdots \quad t_m (P_1 + P_2 + \cdots P_m)]$$

$$= [t_1 \quad t_2 \cdots \quad t_m]$$

$$= t$$

## Conditional Joint Probability Distribution

conditional probability distribution of $Y$ given that $X = x$ is given

$$h(y/x) = \frac{h(x, y)}{f(x)}$$

Q. Given the joint distribution

| Y \ X | 1 | 2 | 3 |
|---|---|---|---|
| 1 | 0.05 | 0.05 | 0.1 |
| 2 | 0.05 | 0.1 | 0.35 |
| 3 | 0 | 0.2 | 0.1 |

(i) Find the marginal distribution
(ii) $P(y=3/x=2)$ find

marginal distribution of - $x$

| | | | |
|---|---|---|---|
| $x$ : | 1 | 2 | 3 |
| $P(x)$ : | 0.1 | 0.35 | 0.55 |

marginal distribution of $Y$

| | | | |
|---|---|---|---|
| $Y$ : | 1 | 2 | 3 |
| $P(Y)$ : | 0.2 | 0.5 | 0.3 |

$P(y=3|x=2) \rightarrow$ probability of $y$ taking the value of 3 when $x$ has already taken 2

$$h(y \backslash x) = \frac{h(x,y)}{f_1(x)}$$

$f(x)$ is probability of $x$ only at 2

$h(2,3)$

$$P(y=3|x=2) = \frac{0.2}{0.35}$$
$$= 0.571$$

Q. Given the joint probability

| $Y \backslash X$ | 0 | 1 | 2 |
|---|---|---|---|
| 0 | 0.1 | 0.4 | 0.1 |
| 1 | 0.2 | 0.2 | 0 |

(i) Find marginal distribution of $X$ and $Y$
(ii) Are $X$ and $Y$ independent
(iii) Find conditional probability distribution $h(x/y=1)$

(i) Marginal distribution of $X$

| | | | |
|---|---|---|---|
| $x$ : | 0 | 1 | 2 |
| $p(x)$ : | 0.3 | 0.6 | 0.1 |
| | $P_1$ | $P_2$ | $P_3$ |

of $Y$

| | | |
|---|---|---|
| $Y$ : | 0 | 1 |
| $P(Y)$ : | 0.6 | 0.4 |
| | $q_1$ | $q_2$ |

ii) $p_1 = 0.3 \quad p_2 = 0.6 \quad p_3 = 0.1$

$q_1 = 0.6 \quad q_2 = 0.4$

$p_1 q_1 = 0.18 \neq p_{11}$

$X$ and $Y$ are not independent.

(iii) $h(x/y=1)$

$h(0/1) = \frac{0.2}{0.4} = 0.5$

$h(1/1) = \frac{0.2}{0.4} = 0.5$

$h(2/1) = \frac{0}{0.4} = 0$

$\Sigma h(x/y=1) = 0.5+0.5+0 = 1$

(this should be 1)

Q If X and Y are independent random variables. X takes the values ~~25~~ 2, 5, 7 with probability 1/2, 1/4, 1/4. and Y takes the values 3, 4, 5 with 1/3, 1/3, 1/3.
(i) Find ~~joint prob~~ JPD (ii) Show that covariance is zero.
(iii) Find the probability distribution $z = x+y$.

| | | | |
|---|---|---|---|
| X | 2 | 5 | 7 |
| p(x) | 1/2 | 1/4 | 1/4 |
| Y | 3 | 4 | 5 |
| P(Y) | 1/3 | 1/3 | 1/3 |

| | | | |
|---|---|---|---|
| X | 2 | 5 | 7 |
| P(x) | 1/2 | 1/4 | 1/4 |
| Y | 3 | 4 | 5 |
| P(y) | 1/3 | 1/3 | 1/3 |

(i)

| X\Y | 3 | 4 | 5 |
|---|---|---|---|
| 2 | 1/6 | 1/6 | 1/6 |
| 5 | 1/12 | 1/12 | 1/12 |
| 7 | 1/12 | 1/12 | 1/12 |

$\mu x = 1 + 5/4 + 7/4 = 16/4 = 4$

$\mu y = 1 + 4/3 + 5/3 = 12/3 = 4$

$E xy = 2\times3\times\frac{1}{6} + 2\times4\times1/6 + 2\times5\times1/6 + 5\times3\times1/12 + 5\times4\times1/12 + 5\times5\times1/12 + 7\times3\times1/12 + 7\times4\times1/12 + 7\times5\times1/12$

$= 1 + \frac{4}{3} + \frac{5}{3} + \frac{5}{4} + \frac{5}{3} + \frac{25}{12} + \frac{7}{4} + \frac{28}{12}$

$= 1 + \frac{14}{3} + \frac{12}{4} + \frac{53}{12}$

$= 16$

(ii) $Cov(XY) = E(X) + (E(Y)) = 16 - 4\times4 = 0$

(iii) $z = x+y \Rightarrow [5, 6, 7, 8, 9, 10, 10, 11, 12]$
Corresponding probabilities = 1/6, 1/6, 1/6, 1/12, 1/12, 1/12, 1/12, 1/12, 1/12