The Art of Linear Algebra

– Graphic Notes on “Linear Algebra for Everyone” –

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with the kindest help of Lucius Annaeus Seneca †

November 1, 2023/updated November 13, 2023

Abstract

I try to intuitively visualize some important concepts introduced in “Linear Algebra for Everyone”,[1] which include Column-Row (CR), Gaussian Elimination (LU), Gram-Schmidt Orthogonalization (QR), Eigenvalues and Diagonalization ($Q\Lambda Q$), and Singular Value Decomposition ($U\Sigma V$). This paper aims at promoting the understanding of vector/matrix calculations and algorithms from the perspective of matrix factorization.

Foreword

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– Ipsum Lorem
Lucius Annaeus Seneca

Contents

1 Viewing a Matrix – 4 Ways

A matrix ($m \times n$) can be viewed as 1 matrix, mn numbers, n columns and m rows.

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[1] “Linear Algebra for Everyone”: `http://math.mit.edu/everyone/` with Japanese translation from Kindai Kagaku.

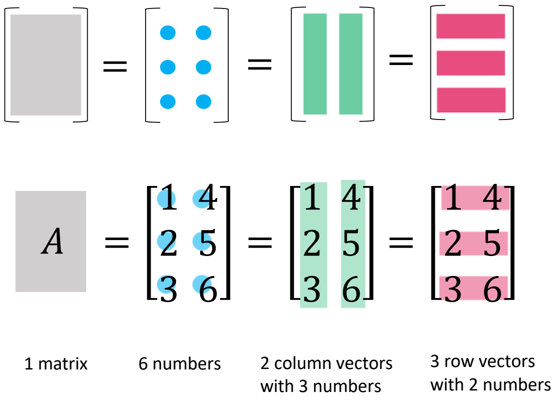


Figure 1: Viewing a Matrix in 4 Ways

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \\ a_{31} & a_{32} \end{bmatrix} = \begin{bmatrix} | & | \\ a_1 & a_2 \\ | & | \end{bmatrix} = \begin{bmatrix} -a_{1*}- \\ -a_{2*}- \\ -a_{3*}- \end{bmatrix}$$

Here, the column vectors are in bold as a_1. Row vectors include $*$ as in a_{1*}. Transposed vectors and matrices are indicated by T as in a and A.

2 Vector times Vector – 2 Ways

Hereafter I point to specific sections of "Linear Algebra for Everyone" and present graphics which illustrate the concepts with short names in gray circles.

- Sec. 1.1 (p.2) Linear combination and dot products
- Sec. 1.3 (p.25) Matrix of Rank One
- Sec. 1.4 (p.29) Row way and column way

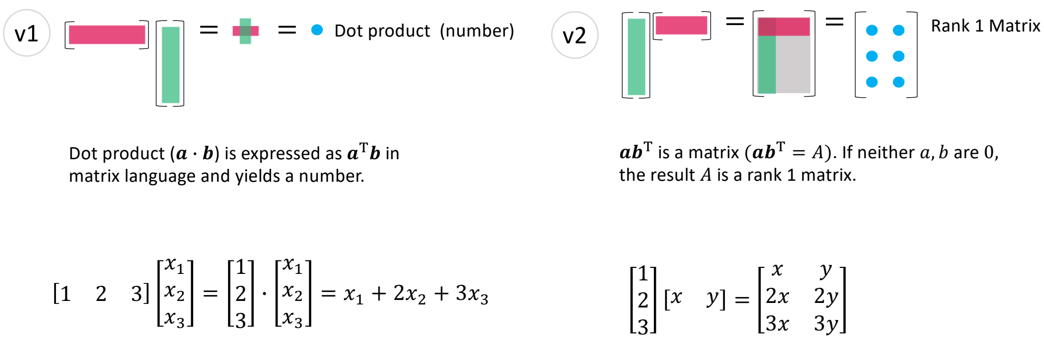


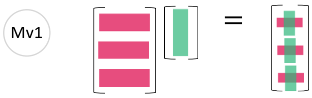
Figure 2: Vector times Vector - (v1), (v2)

(v1) is an elementary operation of two vectors, but (v2) multiplies the column to the row and produces a rank 1 matrix. Knowing this outer product (v2) is the key to the following sections.

3 Matrix times Vector – 2 Ways

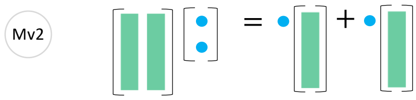
A matrix times a vector creates a vector of three dot products (Mv1) as well as a linear combination (Mv2) of the column vectors of A.

- Sec. 1.1 (p.3) Linear combinations
- Sec. 1.3 (p.21) Matrices and Column Spaces



The row vectors of A are multiplied by a vector $\boldsymbol{x}$ and become the three dot-product elements of $A\boldsymbol{x}$.

$$A\boldsymbol{x} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (x_1+2x_2) \\ (3x_1+4x_2) \\ (5x_1+6x_2) \end{bmatrix}$$



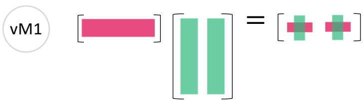
The product $A\boldsymbol{x}$ is a linear combination of the column vectors of A.

$$A\boldsymbol{x} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$$

Figure 3: Matrix times Vector - (Mv1), (Mv2)

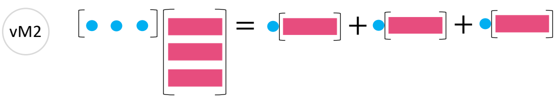
At first, you learn (Mv1). But when you get used to viewing it as (Mv2), you can understand Ax as a linear combination of the columns of A. Those products fill the column space of A denoted as $\mathbf{C}(A)$. The solution space of $Ax = 0$ is the nullspace of A denoted as $\mathbf{N}(A)$. To understand the nullspace, let the right-hand side of (Mv1) be 0 and see all the dot products are zero.

Also, (vM1) and (vM2) show the same pattern for a row vector times a matrix.



$$\boldsymbol{y}A = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} (y_1+3y_2+5y_3) & (2y_1+4y_2+6y_3) \end{bmatrix}$$

A row vector $\boldsymbol{y}$ is multiplied by the two column vectors of A and become the two dot-product elements of $\boldsymbol{y}A$.



$$\boldsymbol{y}A = \begin{bmatrix} y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix} = y_1 \begin{bmatrix} 1 & 2 \end{bmatrix} + y_2 \begin{bmatrix} 3 & 4 \end{bmatrix} + y_3 \begin{bmatrix} 5 & 6 \end{bmatrix}$$

The product $\boldsymbol{y}A$ is a linear combination of the row vectors of A.

Figure 4: Vector times Matrix - (vM1), (vM2)

The products fill the row space of A denoted as $\mathbf{C}(A)$. The solution space of $yA = 0$ is the left-nullspace of A, denoted as $\mathbf{N}(A)$.

The four subspaces consist of $\mathbf{N}(A) + \mathbf{C}(A)$ (which are perpendicular to each other) in $\mathbb{R}^n$ and $\mathbf{N}(A) + \mathbf{C}(A)$ in $\mathbb{R}^m$ (which are perpendicular to each other).

- Sec. 3.5 (p.124) Dimensions of the Four Subspaces

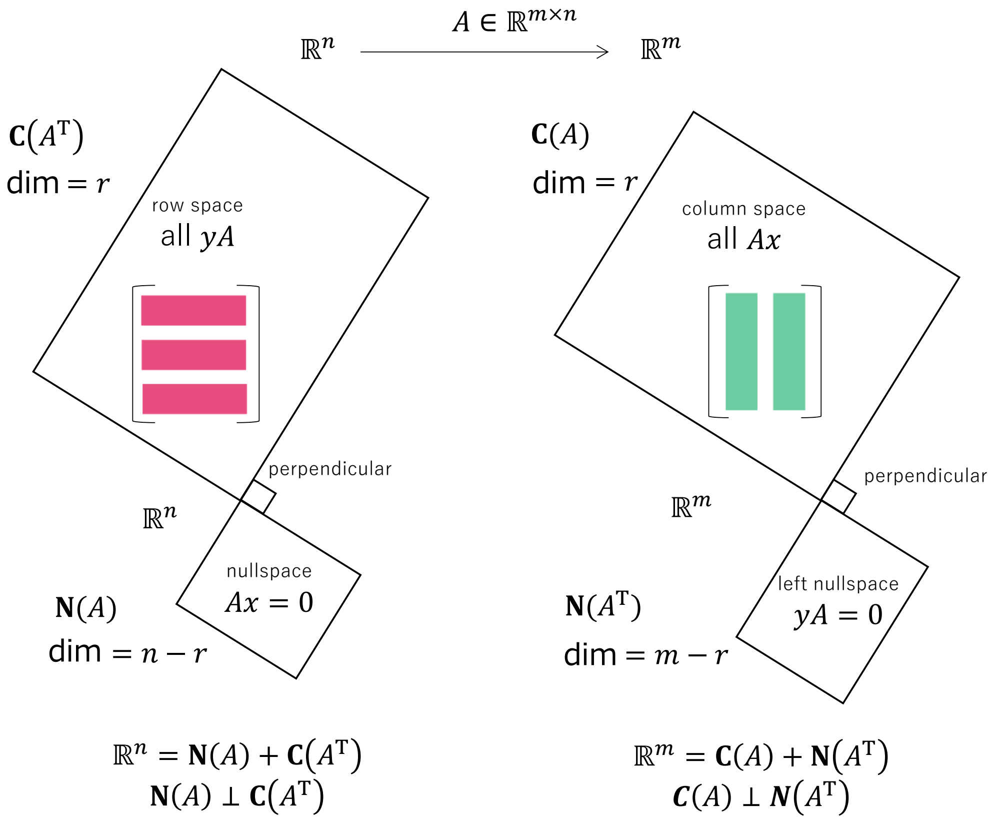


Figure 5: The Four Subspaces

See $A = CR$ (Sec 6.1) for the rank r.

4 Conclusion and Acknowledgements

I have presented a systematic visualization of matrix/vector multiplication and its applications to the Five Matrix Factorizations. I hope you enjoy them and find them useful in understanding Linear Algebra.

Ashley Fernandes helped me with typesetting, which makes this paper much more appealing and professional.

To conclude this paper, I'd like to thank Prof. Gilbert Strang for publishing "Linear Algebra for Everyone". It presents a new pathway to these beautiful landscapes in Linear Algebra. Everyone can reach a fundamental understanding of its underlying ideas in a practical manner that introduces us to contemporary and also traditional data science and machine learning.

References and Related Works

1. Gilbert Strang(2020),*Linear Algebra for Everyone*, Wellesley Cambridge Press., `http://math.mit.edu/everyone`

2. Gilbert Strang(2016), *Introduction to Linear Algebra*,Wellesley Cambridge Press, 6th ed., `http://math.mit.edu/linearalgebra`