

重力弹球 (80 分)

PART.A 一些准备工作

(A.1)

$$\vec{n}|_p = \frac{\vec{\nabla} \cdot M|_p}{|\vec{\nabla} \cdot M|_p} = \frac{(-\frac{x}{2f}, -\frac{y}{2f}, 1)}{\sqrt{1 + \frac{x^2 + y^2}{4f^2}}} \quad (1)$$

(A.2)

$$v_{\parallel} = \sqrt{v_x^2 + v_y^2}, z = v_z t - \frac{1}{2}gt^2, x' = v_{\parallel}t$$

$$z = \frac{v_z}{v_{\parallel}}x' - \frac{1}{2}g\frac{x'^2}{v_{\parallel}^2} \quad (2)$$

$$p = \frac{v_{\parallel}^2}{g}$$

$$F = \frac{v_x^2 + v_y^2}{2g} \quad (3)$$

(A.3)

$$\vec{v}' = \vec{v} - 2(\vec{n}|_P \cdot \vec{v}) \cdot \vec{n}|_P \quad (4)$$

(A.4)

$$\frac{L_z}{m} = (\vec{r} \times \vec{v})_z = (\vec{r} \times \vec{v}')_z - 2(\vec{n}|_P \cdot \vec{v})(\vec{r} \times \vec{n})_z = (\vec{r} \times \vec{z})_z \quad (5)$$

利用了 $(\vec{r} \times \vec{n}|_P)_z = 0$

PART.B 一种特殊情况

(B.1)

$$\tan \frac{\theta}{2} = \frac{v_r}{v_{\phi}} \quad (6)$$

(B.2) 由于碰撞点在同一水平圆上, 有

$$\begin{cases} 2r_0 \sin \frac{\theta}{2} = \sqrt{v_r^2 + v_{\phi}^2}t \\ v_z t - \frac{1}{2}gt^2 = 0 \end{cases}$$

$$z_z = \frac{1}{2}g \frac{2r_0 \sin \frac{\theta}{2}}{\sqrt{v_r^2 + v_{\phi}^2}} = \frac{gr_0 \sin \frac{\theta}{2}}{\sqrt{v_r^2 + v_{\phi}^2}} \quad (7)$$

由于碰撞点要一直在这个圆上, 所以入射速度与出射速度与水平方向夹角不变

$$\frac{v_r}{v_z} = \frac{r_0}{2f} \quad (8)$$

由 (7), (8) 式

$$v_z = \frac{zf}{r_0}v_r = \frac{gr_0 \sin \frac{\theta}{2}}{v_r \sqrt{1 + \cot^2 \frac{\theta}{2}}} = \frac{gr_0 \sin^2 \frac{\theta}{2}}{v_r}$$

解得

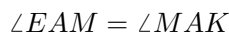
(B.3) 离开圆心 r 处有

化简后可得与 θ 无关

PART.C 二维抛物线内的弹球

(C.1)

(C.2) 由抛物线的光学性质得



$$\angle DAL = \angle LAK$$

$$\angle EAO = -\angle EAM + \angle DAM = -\angle MAK + \angle OAM$$

$$\angle DAO = \angle DAL - \angle DOL = \angle LAK - \angle DAL$$

$$\angle EAO = -(\angle MAK + \angle MAK) + \angle DAM + \angle OAL = -\angle MAL + \angle MAL = 0$$

(C.3)

$$H = z_0 + \frac{\frac{1}{2}mv_z^2}{mg} + \frac{v^2 \cos^2 \alpha}{2g} = z_0 + \frac{v^2}{2g}$$

与碰撞前后无关

(C.4) 由于

$$\angle EAO = \angle OAD, AE = AO$$

可得

$$\triangle OAE \cong \triangle OAD$$

故

$$OE = OD$$

(C.5) 以轨迹焦点为原点

$$\begin{cases} z' = z - R \sin \alpha \\ x' = x - R \cos \alpha \end{cases}$$

而

$$f = \frac{1}{2}(H - R \sin \alpha)$$

$$2(H - R \sin \alpha)(z - R \sin \alpha) = -(x - R \cos \alpha)^2 + (H - R \sin \alpha)^2$$

设

$$S(x, z, \alpha) = 2R(z \sin \alpha + x \cos \alpha) + H^2 - x^2 - R^2 - 2Hz = 0$$

$$\frac{\partial S}{\partial \alpha} = 2R(z \cos \alpha - x \sin \alpha) = 0$$

1)

$$\begin{aligned} \sin \alpha &= \frac{z}{\sqrt{x^2 + z^2}} & \cos \alpha &= \frac{x}{\sqrt{x^2 + z^2}} \\ -2Hz + 2R\sqrt{x^2 + z^2} &+ H^2 - x^2 + R^2 &= 0 \end{aligned}$$

2)

$$\begin{aligned} \sin \alpha &= -\frac{z}{\sqrt{x^2 + z^2}} & \cos \alpha &= -\frac{x}{\sqrt{x^2 + z^2}} \\ -2Hz - 2R\sqrt{x^2 + z^2} &+ H^2 - x^2 - R^2 &= 0 \end{aligned}$$

(C.6) 过 B, C 作竖直线, 由抛物线的第二定义

$$\overline{BA} + \overline{BO} = \overline{BM} + \overline{BD} = \overline{EG}$$

$$\overline{CA} + \overline{CO} = \overline{CE} + \overline{CG} = \overline{DM}$$

到 O, A 距离为定值

参考文献