悬链线 (40 分)(命题人：SCK)

(1)

取微元受力分析

$$\begin{cases} T\left(\theta+\mathrm{d}\theta\right)\sin\left(\theta+\mathrm{d}\theta\right)=\lambda g \mathrm{d}l+T(\theta)\sin\theta \\ T\left(\theta+\mathrm{d}\theta\right)\cos\left(\theta+\mathrm{d}\theta\right)=T(\theta)\cos\theta \end{cases} \tag{1}$$

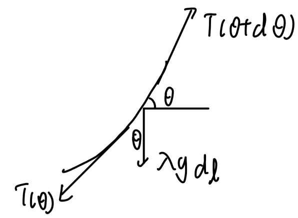

图 1

可得

$$\begin{cases} \mathrm{d}(T\sin(\theta))=\lambda g\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\mathrm{d}x \\ \mathrm{d}(T\cos(\theta))=0 \end{cases} \tag{2}$$

对 (2) 积分得:

$$T\cos(\theta)=T_0 \tag{3}$$

$$T\sin(\theta)=T_0\frac{\mathrm{d}y}{\mathrm{d}x} \tag{4}$$

联立 (2)(4)

$$T_0\frac{\mathrm{d}^2y}{\mathrm{d}x^2}=\lambda g\sqrt{1+\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \tag{5}$$

令 $\frac{\mathrm{d}y}{\mathrm{d}x}=u$, 得

$$T_0\frac{\mathrm{d}u}{\mathrm{d}x}=\lambda g\sqrt{1+u^2} \tag{6}$$

$$T_0\frac{\mathrm{d}u}{\mathrm{d}y}u=\lambda g\sqrt{1+u^2} \tag{7}$$

$$T_0\frac{u\mathrm{d}u}{\sqrt{1+u^2}}=\lambda g\mathrm{d}y \tag{8}$$

注意到:$\mathrm{d}\left(\sqrt{1+u^2}\right)=\frac{u\mathrm{d}u}{\sqrt{1+u^2}}$

$$T_0\mathrm{d}\left(\sqrt{1+u^2}\right)=+\lambda g\mathrm{d}y$$

$$T_0\sqrt{1+u^2}-T_0=\lambda g\mathrm{d}y$$

$$\sqrt{\left(\frac{T_0+\lambda g y}{T_0}\right)^2-1}=\frac{\mathrm{d}y}{\mathrm{d}x}$$

为了解上面的微分方程, 注意到:

$$\cosh^2 x-\sinh^2 x=1 \tag{9}$$

令

$$\frac{T_0+\lambda g y}{T_0}=\cosh\xi \tag{10}$$

则上述方程化为

$$\sinh\xi=\frac{\mathrm{d}y}{\mathrm{d}x} \tag{11}$$

对 (10) 求导

$$\frac{\lambda g}{T_0} = \sinh\xi \frac{\mathrm{d}\xi}{\mathrm{d}y} \tag{12}$$

带入 (11)

$$\frac{\lambda g}{T_0}\frac{\mathrm{d}y}{\mathrm{d}\xi} = \frac{\mathrm{d}y}{\mathrm{d}x} \tag{13}$$

得到

$$\xi = \frac{\lambda g}{T_0}x \tag{14}$$

可得

$$y = \frac{T_0\left(\cosh\left(\frac{\lambda g}{T_0}x\right) - 1\right)}{\lambda g} \tag{15}$$

(2)

取微元受力分析，可得起沿着径向, 有

$$2T_0\frac{1}{2}\mathrm{d} = BI\mathrm{d}l \tag{16}$$

可得

$$R = \frac{T_0}{BI} \tag{17}$$

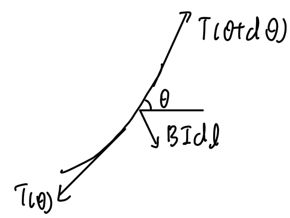

图 2

可得方程

$$x^2 + \left(y - \frac{T_0}{BI}\right)^2 = \left(\frac{T_0}{BI}\right)^2 \tag{18}$$

(3)

取微元受力分析

$$\begin{cases} T\left(\theta + \mathrm{d}\theta\right)\sin\left(\theta + \mathrm{d}\theta\right) = BI\mathrm{d}l\cos\theta + \lambda g\mathrm{d}l + T(\theta)\sin\theta \\ T\left(\theta + \mathrm{d}\theta\right)\cos\left(\theta + \mathrm{d}\theta\right) + BI\mathrm{d}l\sin\theta = T(\theta)\cos\theta \end{cases} \tag{19}$$

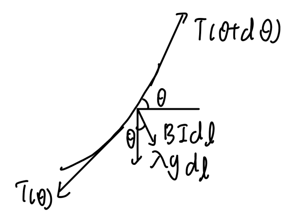

图 3

可得

$$\begin{cases} \mathrm{d}(T\sin(\theta)) = BI\mathrm{d}x + \lambda g\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2}\mathrm{d}x \\ \mathrm{d}(T\cos(\theta)) - BI\mathrm{d}y \end{cases} \tag{20}$$

对 (20) 积分得:

$$T\cos(\theta) = T_0 - BIy \tag{21}$$

$$T\sin(\theta) = (T_0 - BIy)\frac{\mathrm{d}y}{\mathrm{d}x} \tag{22}$$

联立 (20)(22)

$$T_0\frac{\mathrm{d}^2y}{\mathrm{d}x^2} - BI\left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2 - BIy\frac{\mathrm{d}^2y}{\mathrm{d}x^2} = BI + \lambda g\sqrt{1 + \left(\frac{\mathrm{d}y}{\mathrm{d}x}\right)^2} \tag{23}$$

令 $\frac{\mathrm{d}y}{\mathrm{d}x} = u$, 得

$$T_0\frac{\mathrm{d}u}{\mathrm{d}x} - BIu^2 - BIy\frac{\mathrm{d}u}{\mathrm{d}x} = BI + \lambda g\sqrt{1 + u^2} \tag{24}$$

$$T_0\frac{\mathrm{d}u}{\mathrm{d}y}u = BI\left(1 + u^2\right) + BIyu\frac{\mathrm{d}u}{\mathrm{d}x} + \lambda g\sqrt{1 + u^2} \tag{25}$$

$$T_0\frac{u\mathrm{d}u}{\sqrt{1+u^2}}=BI\sqrt{1+u^2}\mathrm{d}y+BIy\frac{u\mathrm{d}u}{\sqrt{1+u^2}}+\lambda g\mathrm{d}y \tag{26}$$

注意到:$\mathrm{d}\left(\sqrt{1+u^2}\right)=\dfrac{u\mathrm{d}u}{\sqrt{1+u^2}}$

$$T_0\mathrm{d}\left(\sqrt{1+u^2}\right)=BI\mathrm{d}\left(y\sqrt{1+u^2}\right)+\lambda g\mathrm{d}y \tag{27}$$

$$T_0\sqrt{1+u^2}-T_0=BI\mathrm{d}\left(y\sqrt{1+n^2}\right)+\lambda g\mathrm{d}y \tag{28}$$

$$\sqrt{\left(\frac{T_0+\lambda gy}{T_0-BIy}\right)^2-1}=\frac{\mathrm{d}y}{\mathrm{d}x} \tag{29}$$

评分标准:

共 40 分

(1) 共 15 分 (1), (2), (3), (4), (5), (6), (7), (8), (9), (10), (11), (12), (13), (14), (15) 各 1 分

(2) 共 5 分 (16) 各 1 分,(17), (18) 各 2 分

(3) 共 20 分 (19), (20), (21), (22), (23), (24), (25), (26), (27), (28), (29), (30), (31), (32), (33) 各 2 分