重力弹球 (80 分)

PART.A 一些准备工作

(A.1)

$$\vec{n}|_p = \frac{\vec{\nabla} \cdot M|_p}{\left|\vec{\nabla} \cdot M|_p\right|} = \frac{(-\frac{x}{2f}, -\frac{y}{2f}, 1)}{\sqrt{1 + \frac{x^2+y^2}{4f^2}}} \tag{1}$$

(A.2)

$$v_\parallel = \sqrt{v_x^2 + v_y^2}, z = v_z t - \frac{1}{2}gt^2, x' = v_\parallel t \tag{2}$$

$$z = \frac{v_z}{v_\parallel}x' - \frac{1}{2}g\frac{x'^2}{v_{\parallel^2}} \tag{3}$$

$$p = \frac{v_\parallel^2}{g} \tag{4}$$

$$F = \frac{v_x^2 + v_y^2}{2g} \tag{5}$$

(A.3)

$$\vec{v'} = \vec{v} - 2(\vec{n}|_P \cdot \vec{v}) \cdot \vec{n}|_P \tag{6}$$

(A.4)

$$\frac{L_z}{m} = (\vec{r} \times \vec{v})z = (\vec{r} \times \vec{v'})_z - 2(\vec{n}|_P \cdot \vec{v})(\vec{r} \times \vec{n})_z = (\vec{r} \times \vec{z})_z \tag{7}$$

利用了 $(\vec{r} \times \vec{n}|_P)_z = 0$

PART.B 一种特殊情况

(B.1)

$$\tan\frac{\theta}{2} = \frac{v_r}{v_\phi} \tag{8}$$

(B.2) 由于碰撞点在同一水平圆上，有

$$\begin{cases} 2r_0 \sin\dfrac{\theta}{2} = \sqrt{v_r^2 + v_\phi^2}t \\ v_z t - \dfrac{1}{2}gt^2 = 0 \end{cases} \tag{9}$$

$$z_z = \frac{1}{2}g\frac{2r_0\sin\frac{\theta}{2}}{\sqrt{v_r^2+v_\phi^2}} = \frac{gr_0\sin\frac{\theta}{2}}{\sqrt{v_r^2+v_\phi^2}} \tag{10}$$

由于碰撞撞点要一直在这个圆上, 所以入射速度与出射速度与水平方向夹角不变

$$\frac{v_r}{v_z} = \frac{r_0}{2f} \tag{11}$$

由 (7), (8) 式

$$v_z = \frac{zf}{r_0}v_r = \frac{gr_0\sin\frac{\theta}{2}}{v_r\sqrt{1+\cot^2\frac{\theta}{2}}} = \frac{gr_0\sin^2\frac{\theta}{2}}{v_r} \tag{12}$$

$$\Rightarrow \quad v_r^2 = \frac{gr_0^2}{2f}\sin^2\frac{\theta}{2} = \frac{gr_0^2}{2f}\frac{1-\cos\theta}{2} \tag{13}$$

解得

$$\begin{cases} v_r = \dfrac{r_0}{2}\sqrt{\dfrac{g}{f}(1-\cos\theta)} \\ v_z = \sqrt{gf(1-\cos\theta)} \\ v_\phi = \dfrac{r_0}{2}\sqrt{\dfrac{g}{f}(1+\cos\theta)} \end{cases} \tag{14}$$

(B.3) 离开圆心 r 处有

$$\begin{cases} x' = r_0\sin\dfrac{\theta}{2} - \sqrt{r^2 - r_0^2\cot^2\dfrac{\theta}{2}} = \sqrt{v_r^2+v_\phi^2}t \\ z' = v_z t - \dfrac{1}{2}gt^2 = \dfrac{v_z x}{\sqrt{v_r^2+v_\psi^2}} - \dfrac{1}{2}g\dfrac{x^2}{v_r^2+v_\psi^2} \end{cases} \tag{15}$$

化简后可得与 θ 无关

$$z = z' + \frac{r_0^2}{4y} - f = \frac{r_0^2}{4y} - \frac{fr^2}{r_0^2} \qquad r \in [v_0\cos\frac{\theta}{2}, v_0] \tag{16}$$

PART.C 二维抛物线内的弹球

(C.1)

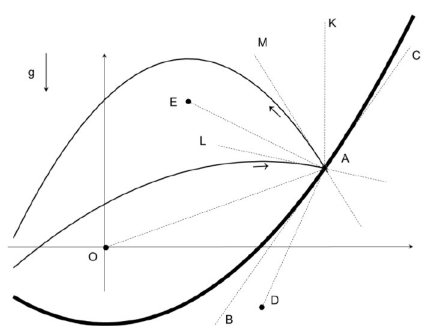
$$x = v\cos\alpha t \tag{17}$$

$$z = v\sin\alpha t - \frac{1}{2}gt^2 = x\tan\alpha - \frac{1}{2}gt^2 = x\tan\alpha - \frac{1}{2}g\frac{x^2}{v^2\cos^2\alpha} \tag{18}$$

$$f = \frac{v^2\cos^2\alpha}{2g} \tag{19}$$

$$x = \frac{\tan\alpha}{-2(-\frac{1}{2}g\frac{1}{v^2\sin\alpha})} = \frac{v^2\sin\alpha\cos\alpha}{g} = \frac{v^2\sin 2\alpha}{2g} \tag{20}$$

(C.2) 由抛物线的光学性质得



$$\angle EAM = \angle MAK \tag{21}$$

$$\angle DAL = \angle LAK \tag{22}$$

$$\angle EAO = -\angle EAM + \angle DAM = -\angle MAK + \angle OAM \tag{23}$$

$$\angle DAO = \angle DAL - \angle DAL = \angle LAK - \angle DAL \tag{24}$$

$$\angle EAO = -(\angle MAK + \angle MAK) + \angle DAM + \angle OAL = -\angle MAL + \angle MAL = 0 \tag{25}$$

(C.3)

$$H = z_0 + \frac{\frac{1}{2} m v_z^2}{mg} + \frac{v^2 \cos^2\alpha}{2g} = z_0 + \frac{v^2}{2g} \tag{26}$$

$$\text{与碰撞前后无关} \tag{27}$$

(C.4) 由于

$$\angle EAO = \angle OAD, AE = AO \tag{28}$$

可得

$$\triangle OAE \cong \triangle OAD \tag{29}$$

故

$$OE = OD \tag{30}$$

(C.5) 以轨迹焦点为原点

$$\begin{cases} z' = z - R\sin\alpha \\ x' = x - R\cos\alpha \end{cases} \tag{31}$$

而

$$f = \frac{1}{2}(H - R\sin\alpha) \tag{32}$$

$$2(H - R\sin\alpha)(z - R\sin\alpha) = -(x - R\cos\alpha)^2 + (H - R\sin\alpha)^2 \tag{33}$$

设

$$S(x, z, \alpha) = 2R(z\sin\alpha + x\cos\alpha) + H^2 - x^2 - R^2 - 2Hz = 0 \tag{34}$$

$$\frac{\partial S}{\partial \alpha} = 2R\left(z\cos\alpha - x\sin\alpha\right) = 0 \tag{35}$$

1)

$$\sin\alpha = \frac{z}{\sqrt{x^2+z^2}} \quad \cos\alpha = \frac{x}{\sqrt{x^2+z^2}} \tag{36}$$

$$-2Hz + 2R\sqrt{x^2+z^2} \quad + H^2 - x^2 + R^2 = 0 \tag{37}$$

2)

$$\sin\alpha = -\frac{z}{\sqrt{x^2+z^2}} \quad \cos\alpha = -\frac{x}{\sqrt{x^2+z^2}} \tag{38}$$

$$-2Hz - 2R\sqrt{x^2+z^2} + H^2 - x^2 - R^2 = 0 \tag{39}$$

(C.6) 过 B, C 作竖直线, 由抛物线的第二定义

$$\overline{BA} + \overline{BO} = \overline{BM} + \overline{BD} = \overline{EG} \tag{40}$$

$$\overline{CA} + \overline{CO} = \overline{CE} + \overline{CG} = \overline{DM} \tag{41}$$

$$\text{到}O, A\text{距离为定值} \tag{42}$$

参考文献

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