匀强磁场中带电小球在圆盘上的运动 (50 分)(命题：LJT)

(1) 解：按定义

$$\vec{\mu} = \frac{1}{2} \int \rho \vec{r} \times \vec{v} dV \tag{1}$$

$$Q = \frac{4}{3}\pi R^3 \rho \tag{2}$$

得

$$\vec{\mu} = \frac{1}{5} Q\vec{w}r^2 \tag{3}$$

(2) 解：由无相对滑动，故 $\omega_z = 0$

$$\vec{v_c} + (-r\hat{z}) \times \vec{w} = (x\hat{x} + y\hat{y}) \times (\Omega\hat{z}) \tag{4}$$

$$\begin{cases} \dot{x} - \omega_y r = -\Omega y \\ \dot{y} + \omega_x r = \Omega x \end{cases} \tag{5}$$

质心运动定理

$$F_x\hat{x} + F_y\hat{y} + Q\vec{v_c} \times (B\hat{z}) = m\left(\ddot{x}\hat{x} + \ddot{y}\hat{y}\right) \tag{7}$$

质心转动定理

$$\frac{2}{5}mr^2\frac{\mathrm{d}\vec{\omega}}{\mathrm{d}t} = (-r)\,\hat{z} \times (F_x\hat{x} + F_y\hat{y}) + \vec{\mu} \times \vec{B} \tag{8}$$

$$\begin{cases} rF_y + \mu_y B = \frac{2}{5}mr^2\dot{\omega}_x \\ -(rF_x + \mu_x B) = \frac{2}{5}mr^2\dot{\omega}_y \end{cases} \tag{9}$$

消去 $F_z, F_y, \omega_x, \omega_y$，得二阶二元线性微分方程组

$$\begin{cases} \frac{7}{5}mr\ddot{y} + \left(\frac{6}{5}QBr - \frac{2}{5}mr\Omega\right)\dot{x} + \frac{1}{5}QrB\Omega y = 0 \\ \frac{7}{5}mr\ddot{x} - \left(\frac{6}{5}QBr - \frac{2}{5}mr\Omega\right)\dot{y} + \frac{1}{5}QrB\Omega x = 0 \end{cases} \tag{10}$$

令 $\tilde{\xi} = x + y$，将 (10) 式中两式相加，得

$$\frac{7}{5}mr\ddot{\tilde{\xi}} + \left(\frac{6}{5}QBr - \frac{2}{5}mr\Omega\right)\dot{\tilde{\xi}} + \frac{1}{5}QB\Omega r\tilde{\xi} = 0 \tag{11}$$

令 $\tilde{\xi} = e^{\omega t}$，带入 (11) 式

$$-\frac{7}{5}mr\omega^2 - \left(\frac{6}{5}QBr - \frac{2}{5}mr\Omega\right)\omega + \frac{1}{5}QB\Omega r = 0 \tag{12}$$

得

$$\omega_1 = \frac{3QBr - mr\Omega + \sqrt{(3QBr - mr\Omega)^2 + 7mr^2\omega^2 QB\Omega}}{-7mr} \tag{13}$$

$$\omega_2 = \frac{3QBr - mr\Omega - \sqrt{(3QBr - mr\Omega)^2 + 7mr^2\omega^2 QB\Omega}}{-7mr} \tag{14}$$

将两解线性叠加，并重新拆分为 x, y, 有

$$x = A\cos(\omega_1 t) + B\cos(\omega_2 t)$$

$$y = A\sin(\omega_1 t) + B\sin(\omega_2 t)$$

带入初值

$$\begin{cases} A + B = r_0 \\ A\omega_1 + A\omega_w = \Omega r_0 \end{cases} \tag{15}$$

解得

$$\begin{cases} A = -\dfrac{r_0(\omega_2 - \Omega)}{\omega_1 - \omega_2} \\ B = \dfrac{r_0(\omega_1 - \Omega)}{\omega_1 - \omega_1} \end{cases} \tag{16}$$

评分标准:

共 50 分

(1) 共 5 分 (2) 各 1 分,(1),(3) 各 2 分

(2) 共 45 分 (4),(6) 各 2 分,(8),(12),(13),(14),(15),(16) 各 3 分,(5),(7),(9),(10),(11)4 分, 指出 $\omega_z = 0$ 给 3 分