“简单力学题”(50 分)(命题人：SCK)

由能动量定理

$$\begin{cases} Mv = m\left(L\dot{\theta}\sin\theta - v\right) \\ \dfrac{1}{2}mv^2 + \dfrac{1}{2}m\left(2\dot{\theta}\sin\theta - v\right)^2 + \dfrac{1}{2}m\left(L\dot{\theta}\cos\theta\right)^2 = mgL\sin\theta \end{cases} \tag{1}$$

解得

$$\dot{\theta} = \sqrt{\frac{2\left(M+m\right)g\sin\theta}{L\left(M+m\cos^2\theta\right)}} \tag{2}$$

$$v = \frac{mL\sin\theta}{M+m}\sqrt{\frac{2\left(M+m\right)g\sin\theta}{L\left(M+m\cos^2\theta\right)}} \tag{3}$$

$$a = \frac{\mathrm{d}v}{\mathrm{d}t} = \frac{\mathrm{d}v}{\mathrm{d}\theta}\dot{\theta} = \frac{m\left[(3M+2m)\sin\theta\cos\theta + m\sin\theta\cos^3\theta\right]}{\left(M+m\cos^2\theta\right)^2}g \tag{4}$$

在木块系中 (F 为绳子拉力)

$$F\cos\theta = Ma \tag{5}$$

考虑临界情况

$$-Mg\frac{L}{2} - Ma\frac{h}{2} + F\cos\theta\left(L + \frac{h}{2}\right) - F\sin\theta L \geq 0 \tag{6}$$

化简得

$$a \geq \frac{g}{2(1-\tan\theta)} \tag{7}$$

即

$$\frac{2m\left[(3M+2m)\left(\sin\theta\cos\theta\right) + msin\theta\cos^3\theta\right]\left(1-\tan\theta\right)}{\left(M+m\cos^2\theta\right)^2} \geq 1 \tag{8}$$

令 $\frac{M}{m} = k$,$\tan\theta = t$

$$\alpha = \frac{2\left[\left(3k+2\right)\left(1+t^2\right)t + t\right]\left(1-t\right)}{\left[k\left(1+t^2\right)+1\right]^2} \geq 1 \tag{9}$$

对 (6) 左式求导取极值

$$\left[k\left(1+t^2\right)+1\right]\left\{\left[(3k+2)\left(1+3t^2\right)+1\right](1-t) - \left[(3k+2)\left(1+t^2\right)t+t\right]\right\} = 4kt\left(1-t\right)\left[(3k+2)\left(1+t^2\right)t+t\right] \tag{10}$$

$$\left(3k^2+2k\right)t^4 + \left(6k^2+14k+8\right)t^3 - \left(6k+b\right)t^2 + \left(6k^2+12k+6\right)t - \left(3k^2+6k+3\right) = 0 \tag{11}$$

$$\text{代入}k=1\text{解得}t = 0.4765453307 \tag{12}$$

$$\text{代入(11)解得}\alpha = 0.7177259631 < 1,\text{故不会翻起} \tag{13}$$

$$\text{代入}k=\frac{1}{2}\text{解得}t = 0.5043814407 \tag{14}$$

$$\text{代入(11)解得}\alpha = 1.017831264 > 1,\text{故会翻起，可得}\theta = 23.688^\circ \tag{15}$$

评分标准:

共 50 分

(1), (2), (3), (4), (5), (7), (8), (9), (10), (11) 各 3 分,(12), (13), (14), (15) 各 5 分