Floquet theory for open quantum systems

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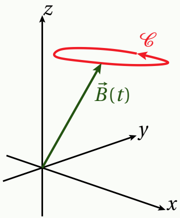
I. Driven quantum systems and Floquet theory

II. Floquet theory and quantum dissipation

III. Fermionic environments & miscellaneous



http://www.icmm.csic.es/sigmundkohler/Download/FloquetTutorial.pdf

1. **Geometric phases**
2. **Floquet theory**
3. **Quantum dissipation**
4. **Floquet-Bloch-Redfield formalism**
5. **Dissipative phenomena in driven systems**
   - The driven double-well potential
   - Influence of the system–bath coupling
   - Coherence stabilization by ac fields
6. **Floquet transport theory**
   - scattering theory
   - master equation
7. **Miscellaneous — time-dependent Liouvillians**
   - Matrix continued fractions
   - Bichromatic driving

- Time evolution of an eigenstate:

$$|\psi(t)\rangle = \mathrm{e}^{-\mathrm{i}E_n t}|\phi_n\rangle$$

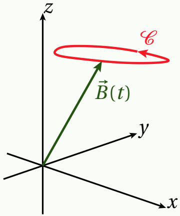
Notation:
ψ: solution of Schrödinger equation
ϕ: other state vector, e.g., eigenstate

- energy $\leftrightarrow$ phase

$\boxed{\text{for (periodically) time-dependent system ?}}$

- Spin in magnetic field
  $B(t) = B(t+T)$:

$$H(t) = \frac{1}{2}\vec{B}(t)\cdot\vec{\sigma}$$



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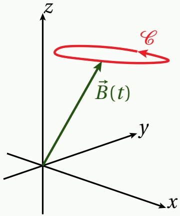
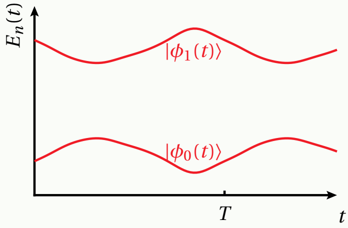


- Quantum dynamics for $\dot{B} \ll B^2$:
  state follows the eigenstate adiabatically

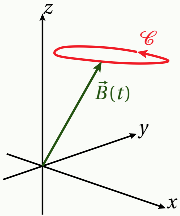
$$|\psi(t)\rangle \propto |\phi_n(t)\rangle$$



➔ $|\psi(t)\rangle$ determined up to phase factor

After one period: $|\psi(T)\rangle = \mathrm{e}^{\mathrm{i}\varphi}|\psi(0)\rangle$

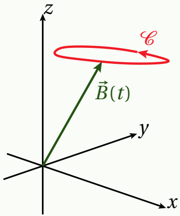
$$\varphi = -\int_0^T \mathrm{d}t\, E_n(t) + \gamma_{\mathscr{C}}$$



- dynamical phase ↔ mean energy
- Berry phase $\gamma_{\mathscr{C}}$
  - depends only on closed curve $\mathscr{C}$ in parameter space

M. Berry, Proc. Roy. Soc. London, Ser. A **392**, 45 (1984)

Berry phase



After one period: $|\psi(T)\rangle = e^{i\varphi}|\psi(0)\rangle$

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- dynamical phase ↔ mean energy
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M. Berry, Proc. Roy. Soc. London, Ser. A **392**, 45 (1984)

- Assumptions:
  1. $\vec{B}(t)$ changes adiabatically slowly
  2. initial state: eigenstate $|\phi_n(0)\rangle$

Non-adiabatic generalization

Different perspective:

State vector undergoes periodic time-evolution

- $|\psi(T)\rangle = \mathrm{e}^{\mathrm{i}\varphi}|\psi(0)\rangle$
- dynamics $|\psi(t)\rangle$ induced by some Hamiltonian $H(t)$

Remarks:

- no adiabatic condition
- $|\psi(t)\rangle$ need not be an eigenstate of $H(t)$
- $H(t)$ is not unique

- only condition: cyclic time-evolution in Hilbert space

Remove phase factor by projection $\Pi : \mathscr{H} \to \mathscr{P}$ where

- $\Pi|\psi_1\rangle = \Pi|\psi_2\rangle$ if $|\psi_1\rangle = c|\psi_2\rangle$ for any $c \in \mathbb{C}$
- all parallel vectors are projected to the same vector

Projective Hilbert space

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- all parallel vectors are projected to the same vector

Cyclic time-evolution: $|\psi(t)\rangle = \mathrm{e}^{\mathrm{i}f(t)}|\phi(t)\rangle$

- $|\psi\rangle \in \mathscr{H}$ (Hilbert space)
- $|\phi\rangle \in \mathscr{P}$ (projective Hilbert space)

➔ $|\phi(t+T)\rangle = |\phi(t)\rangle$ ➔ image $|\phi(t)\rangle = \Pi|\psi(t)\rangle$ is T-periodic

From Schödinger equation follows

$$\frac{\mathrm{d}f}{\mathrm{d}t} = -\langle\phi(t)|H(t)|\phi(t)\rangle + \langle\phi(t)|\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}|\phi(t)\rangle$$

➔ Phase acquired during cyclic evolution: $\varphi = f(T) - f(0)$

$$\frac{\mathrm{d}f}{\mathrm{d}t} = -\langle\phi|H|\phi\rangle + \langle\phi|\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}|\phi\rangle \quad \Longrightarrow \quad \boxed{\varphi = \gamma_{\mathrm{dyn}} + \gamma}$$

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Dynamical phase $\gamma_{\mathrm{dyn}} = -\int_0^T \mathrm{d}t\, \langle\phi(t)|H(t)|\phi(t)\rangle$

- depends on choice of $H(t)$
- reflects mean energy

Non-adiabatic generalization

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$$\frac{\mathrm{d}f}{\mathrm{d}t} = -\langle\phi|H|\phi\rangle + \langle\phi|\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}|\phi\rangle \quad\Longrightarrow\quad \boxed{\varphi = \gamma_{\mathrm{dyn}} + \gamma}$$

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Aharonov-Anandan phase ("non-adiabatic Berry phase")

$$\gamma = \int_0^T \mathrm{d}t\,\langle\phi|\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}|\phi\rangle$$

- depends only on trajectory in Hilbert space — not in parameter space!
- adiabatic limit: $\gamma = \gamma_{\mathscr{C}}$

Aharonov & Anandan, PRL 1987

Some standard references

- Classic work:
  - Shirley, Phys. Rev. 138, B979 (1965)
  - Sambe, Phys. Rev. A 7, 2203 (1973)
- Reviews:
  - Grifoni, Hänggi, Phys. Rep. 304, 229 (1998)
  - Hänggi, Chap.5 of “Quantum transport and dissipation” (1998)
    http://www.physik.uni-augsburg.de/theo1/hanggi/Papers/Chapter5.pdf

Time-dependent Schrödinger equation

Goal: propagator $U(t, t')$

- Time-independent system: diagonalize Hamiltonian ➔ $|\phi_n\rangle$, E_n

$$U(t, t') = U(t - t') = \sum_n e^{-iE_n(t-t')} |\phi_n\rangle\langle\phi_n|$$

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$$U(t,t') = U(t-t') = \sum_n \mathrm{e}^{-\mathrm{i}E_n(t-t')}|\phi_n\rangle\langle\phi_n|$$

- Driven system:

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}|\psi\rangle = H(t)|\psi\rangle$$ ➔ numerical integration

problem 1: time-integration not efficient for long times

problem 2: no information about structure of U

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- Driven system:

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problem 1: time-integration not efficient for long times

problem 2: no information about structure of U

- Solution for $H(t) = H(t+T)$: "Bloch theory in time"
  cf. $H(x)|\phi\rangle = \epsilon|\phi\rangle$ with $H(x) = H(x+a)$
  ➔ Bloch waves $\phi(x) = \mathrm{e}^{\mathrm{i}qx}\varphi(x)$, where $\varphi(x)$ is a-periodic

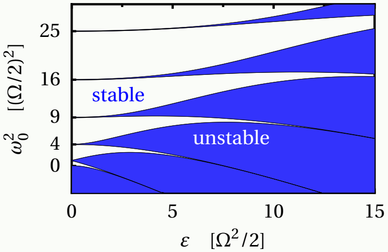
F. Bloch, Z. Phys. A, **52**, 555 (1928)

Floquet (1883):

Ann. de l'Ecole Norm. Sup. **12**, 47 (1883)

Parametric oscillator (cf. Paul trap)

$$\ddot{x} + (\omega_0^2 + \epsilon \cos \Omega t) x = 0$$



Floquet theorem:
solutions have the structure

$$x(t) = e^{\pm i\mu t} \xi(t)$$

where $\xi(t) = \xi(t + 2\pi/\Omega)$

(undriven limit: $\mu = \omega_0$, $\xi = \text{const}$)

μ real
➔ oscillating solutions

μ imaginary
➔ one solution unstable

Discrete time translation and Floquet ansatz

- $H(t) = H(t+T)$

  ➔ $t \to t+T$ is symmetry operation

  ➔ solutions of Schrödinger equation obey $|\psi(t+T)\rangle = e^{i\varphi}|\psi(t)\rangle$

- Floquet ansatz

$$|\psi(t)\rangle = e^{-i\epsilon t}|\phi(t)\rangle = e^{-i\epsilon t}\sum_k e^{-ik\Omega t}|c_k\rangle$$

  - ϵ quasienergy (cf. quasi momentum) ➔ long-time dynamics
  - $|\phi(t)\rangle = |\phi(t+T)\rangle$, Floquet state ➔ within driving period

- Floquet theorem: $H(t)$ has a complete set of Floquet solutions

- Schrödinger equation $i\partial_t|\psi\rangle = H(t)|\psi\rangle$ yields

$$\boxed{\big(H(t) - i\partial_t\big)|\phi(t)\rangle = \epsilon|\phi(t)\rangle}$$

Brillouin zone structure

- $|\phi(t)\rangle$ Floquet state with quasienergy ϵ

➔ $e^{ik\Omega t}|\phi(t)\rangle$ Floquet state with $\epsilon + k\Omega$

proof: insert into $(H - i\partial_t)|\phi\rangle = \epsilon|\phi\rangle$

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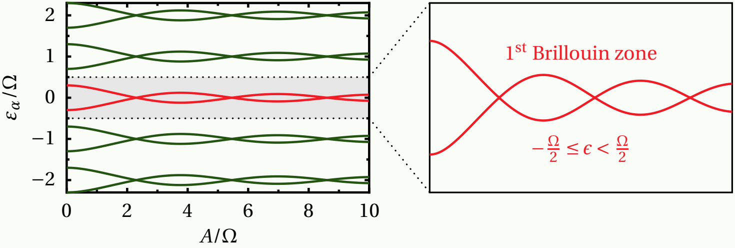
e.g. for two-level system



- all Brillouin zones equivalent, choice arbitrary

➔ quasienergies cannot serve for ordering!

Mean energies

- Physical quantity: mean energy

$$E = \frac{1}{T}\int_0^T \mathrm{d}t\, \langle\psi(t)|H(t)|\psi(t)\rangle = \frac{1}{T}\int_0^T \mathrm{d}t\, \langle\phi(t)|H(t)|\phi(t)\rangle$$

- All equivalent states have the same mean energy
  [proof: insert $\mathrm{e}^{-\mathrm{i}k\Omega t}|\phi(t)\rangle$]

➔ Floquet states can be ordered by their mean energy

- Mean energy

$$E = \frac{1}{T}\int_0^T \mathrm{d}t\, \langle\phi(t)|\left\{H(t) - \mathrm{i}\partial_t + \mathrm{i}\partial_t\right\}|\phi(t)\rangle$$

where $(H - \mathrm{i}\partial_t)|\phi(t)\rangle = \epsilon|\phi(t)\rangle$

$$E = \epsilon + \frac{1}{T}\int_0^T \mathrm{d}t\, \langle\phi(t)|\mathrm{i}\partial_t|\phi(t)\rangle$$

- Mean energy

$$E = \frac{1}{T}\int_0^T \mathrm{d}t \, \langle \phi(t) | \{ H(t) - \mathrm{i}\partial_t + \mathrm{i}\partial_t \} | \phi(t) \rangle$$

where $(H - \mathrm{i}\partial_t)|\phi(t)\rangle = \epsilon |\phi(t)\rangle$

$$-\epsilon = -E + \frac{1}{T}\int_0^T \mathrm{d}t \, \langle \phi(t) | \mathrm{i}\partial_t | \phi(t) \rangle$$

- Compare to $\varphi = \gamma_{\mathrm{dyn}} + \gamma$

- Mean energy

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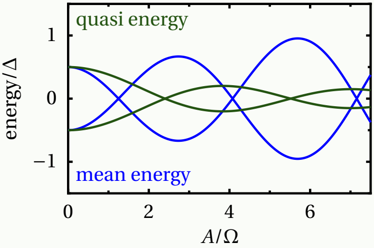
- Compare to $\varphi = \gamma_{\mathrm{dyn}} + \gamma$

$\boxed{(E-\epsilon)\,T \text{ is a geometric phase}}$

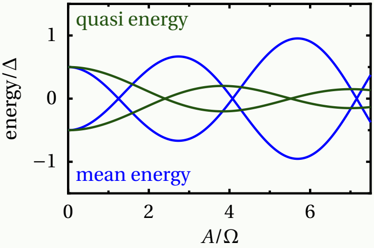
Mean energies — two-level system



Driven undetuned two-level system

- exact crossings
  (consequence of symmetry)

Mean energies — two-level system



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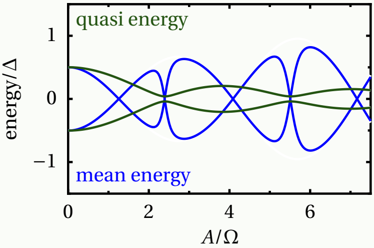
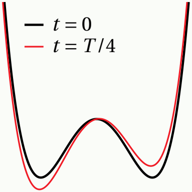


… with small detuning

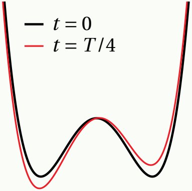
- quasi energies
  - avoided crossings
- mean energies
  - exact crossings remain
  - additional crossings
  - → do not follow from any eigenvalue equation

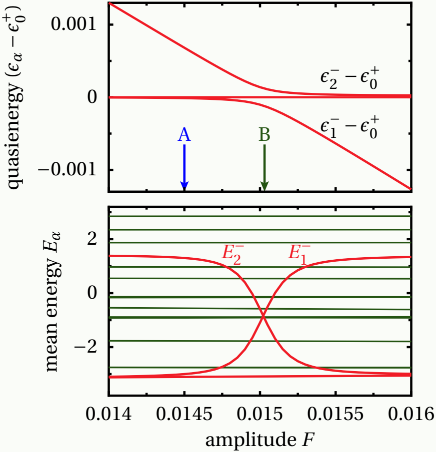
Mean energies — driven double well



- $H(t) = H_{\mathrm{DW}} + Fx\sin(\Omega t)$

Mean energies — driven double well





- $H(t) = H_{\mathrm{DW}} + Fx\sin(\Omega t)$

➔ doublet structure

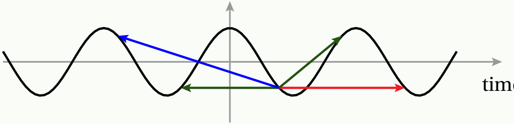
➔ states interchange their morphology at avoided quasienergy crossing

➔ mean energies interchanged

SK *et al.*, PRE 1998

Symmetries of dipole driving

$H_{\text{dipole}} \propto x\cos(\Omega t)$



1. time periodicity $t \longrightarrow t + T$ ➔ Floquet theory applicable
2. time reversal $t \longrightarrow -t$ ➔ Floquet states real
3. generalized parity $(x, t) \longrightarrow (-x, t + T/2)$ ➔ Floquet states even/odd
   e.g. symmetric potential with dipole driving

4. time-reversal parity $(x, t - T/4) \longrightarrow (-x, T/4 - t)$
   - combination of the other three
   - relevant for Floquet scattering theory (Lecture III on fermionic environments)

Sambe space

Goal: more formal treatment of $H(t) - \mathrm{i}\partial_t$

- $|\phi(t)\rangle \in \mathscr{R} \otimes \mathscr{T}$ composite Hilbert space / Sambe space

  Shirley, PR **138**, B979 (1965), Sambe, PRA **7**, 2203 (1973)

  $\mathscr{T}$: Hilbert space of T-periodic functions with inner product

$$\langle f|g\rangle = \int_0^T f(t)^* \, g(t) \frac{\mathrm{d}t}{T} = \sum_k f_k^* g_k$$

- extended Dirac notation:
  - $|\phi(t)\rangle = \langle t|\phi\rangle\rangle$
  - Fourier coefficient $|\phi_k\rangle = \langle k|\phi\rangle\rangle$

e.g.: $|\phi(t)\rangle = \langle t|\phi\rangle\rangle = \sum_k \langle t|k\rangle \langle k|\phi\rangle\rangle = \sum_k \mathrm{e}^{-\mathrm{i}k\Omega t} |\phi_k\rangle$

Completeness and Orthogonality

- $H - \mathrm{i}\partial_t$ is hermitian

➔ Floquet states $|\phi_\alpha\rangle$ orthonormal and complete in $\mathscr{R} \otimes \mathscr{T}$

$$\langle\langle\phi_\alpha^{(k)}|\phi_\beta^{(k')}\rangle\rangle = \delta_{\alpha\beta}\delta_{kk'}$$

? but in $\mathscr{R}$?

Completeness and Orthogonality

- $H - \mathrm{i}\partial_t$ is hermitian

➔ Floquet states $|\phi_\alpha\rangle$ orthonormal and complete in $\mathscr{R} \otimes \mathscr{T}$

$$\langle\langle\phi_\alpha^{(k)}|\phi_\beta^{(k')}\rangle\rangle = \delta_{\alpha\beta}\delta_{kk'}$$

? but in $\mathscr{R}$?

- Consider $\langle\phi_\alpha(t)|\phi_\beta(t)\rangle = \sum_k \lambda_k \mathrm{e}^{-\mathrm{i}k\Omega t}$ since T-periodic

  with the Fourier coefficient

$$\lambda_k = \frac{1}{T}\int_0^T \mathrm{d}t\, \mathrm{e}^{\mathrm{i}k\Omega t}\langle\phi_\alpha(t)|\phi_\beta(t)\rangle = \langle\langle\phi_\alpha|\phi_\beta^{(k)}\rangle\rangle = \delta_{\alpha\beta}\delta_{k,0}$$

➔ Floquet states orthogonal at equal times

Propagator

- propagator in terms of Floquet states

$$U(t,t') = \sum_\alpha |\psi_\alpha(t)\rangle\langle\psi_\alpha(t')| = \sum_\alpha \mathrm{e}^{-\mathrm{i}\epsilon_\alpha(t-t')} |\phi_\alpha(t)\rangle\langle\phi_\alpha(t')|$$

  - long-time dynamics (depends on $t-t'$)
  - dynamics within driving period (depends on t and t')

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  - long-time dynamics (depends on $t-t'$)
  - dynamics within driving period (depends on t and t')

- one-period propagator for kicked systems

$$H(t) = H_0 + K \sum_n \delta(t-nT)$$

➔ $U(T) = \mathrm{e}^{-\mathrm{i}H_0 T}\mathrm{e}^{-\mathrm{i}K}$

✓ easy to compute
✓ provides quasienergies
✗ only long-time dynamics (stroboscopic)

Solve eigenvalue problem

$$\left\{H(t) - \mathrm{i}\partial_t\right\}|\phi\rangle\rangle = \epsilon|\phi\rangle\rangle$$

Solve eigenvalue problem

$$\left\{H(t) - \mathrm{i}\partial_t\right\}|\phi\rangle\rangle = \epsilon|\phi\rangle\rangle$$

Straightforward in Fourier representation ("Floquet matrix")

$$H_0 + H_1\cos(\Omega t) - \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t} \leftrightarrow \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & H_0 + 2\Omega & \frac{1}{2}H_1 & 0 & 0 & 0 & \cdots \\ \cdots & \frac{1}{2}H_1 & H_0 + \Omega & \frac{1}{2}H_1 & 0 & 0 & \cdots \\ \cdots & 0 & \frac{1}{2}H_1 & H_0 & \frac{1}{2}H_1 & 0 & \cdots \\ \cdots & 0 & 0 & \frac{1}{2}H_1 & H_0 - \Omega & \frac{1}{2}H_1 & \cdots \\ \cdots & 0 & 0 & 0 & \frac{1}{2}H_1 & H_0 - 2\Omega & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

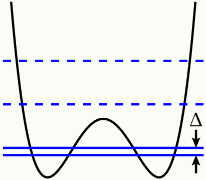
1. direct diagonalization of $H(t) - \mathrm{i}\partial_t$
   - conceptually simple ➔ first choice
   - increasingly difficult with smaller frequency
   - often more efficient after unitary transformation
2. analytical tool: perturbation theory
   strong driving: $H_1 \cos(\Omega t) - \mathrm{i}\partial_t$ as zeroth order

3. diagonalization of $U(T, 0)$ ➔ $\mathrm{e}^{-\mathrm{i}\epsilon T}$, $|\phi(0)\rangle$
4. matrix-continued fraction
5. (t, t') formalism

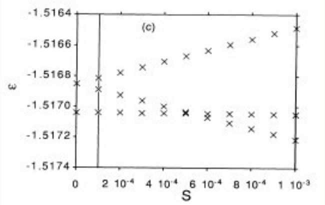
Example I: Coherent destruction of tunneling

1. role of quasienergy crossings
2. perturbation theory (two-level approximation)
3. convenient route to mean energy

Driven double-well potential $H(t) = H_{\mathrm{DW}} + Sx\cos(\Omega t)$



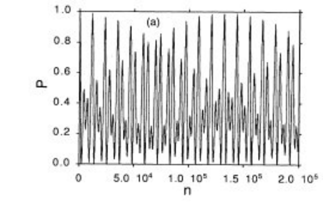
quasienergies $\epsilon_\alpha(S)$

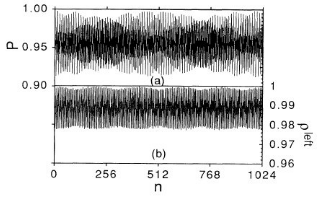


- ? tunnel oscillations influenced by driving
- ? dynamics at quasienergy crossing

Example I: Coherent destruction of tunneling

Occupation $P_{\text{left}}(nT)$





far from crossing:

- tunnel oscillations

at crossing:

- particle stays in left well

➔ “coherent destruction of tunneling” by ac field

Grossmann *et al.*, PRL 1991

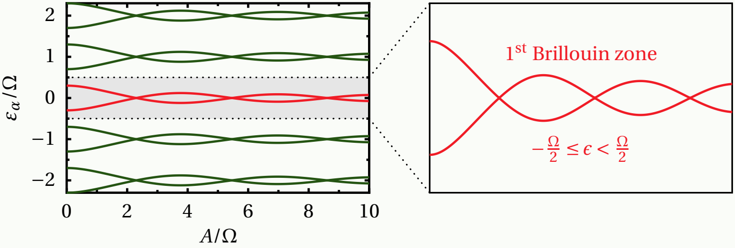
Analytical understanding ➔ two-level approximation

Example I: CDT — two-level approach

Driven two-level system

$$H(t) = -\frac{\Delta}{2}\sigma_x + \frac{A}{2}\cos(\Omega t)\sigma_z$$

quasienergy spectrum



Example I: CDT — perturbation theory

Analytical approach for $\Delta \ll \Omega$: high-frequency limit

$$H(t) = -\frac{\Delta}{2}\sigma_x + \frac{A}{2}\cos(\Omega t)\sigma_z$$

- zeroth order Floquet equation

$$\left(\frac{A}{2}\cos(\Omega t)\sigma_z - \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\right)|\phi(t)\rangle = \epsilon^{(0)}|\phi(t)\rangle$$

with the Floquet states and quasienergies

$$|\phi_{\mathrm{L/R}}(t)\rangle = \mathrm{e}^{\pm\mathrm{i}(A/2\Omega)\sin(\Omega t)}|\mathrm{L/R}\rangle, \quad \epsilon^{(0)} = 0 \quad \text{(degenerate!)}$$

➔ degenerate perturbation theory

Example I: CDT — perturbation theory

Diagonalize $H_0 = -\frac{\Delta}{2}\sigma_x$ in degenerate subspace

- compute all matrix elements ($\ell = L, R$)

$$\langle\langle\phi_\ell|H_0|\phi_{\ell'}\rangle\rangle = \frac{1}{T}\int_0^T \mathrm{d}t\,\langle\phi_\ell(t)|H_0|\phi_{\ell'}(t)\rangle = \begin{cases} 0 & \text{for } \ell = \ell' \\ -\frac{\Delta}{2}J_0(A/\Omega) & \text{for } \ell \neq \ell' \end{cases}$$

  Bessel function $J_n(x)$: nth Fourier coefficent of $\mathrm{e}^{-\mathrm{i}x\sin(\Omega t)}$

- diagonalize the resulting matrix

$$-\frac{\Delta}{2}J_0(A/\Omega)\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \equiv -\frac{\tilde{\Delta}}{2}\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix} \quad \text{with } \tilde{\Delta} = \Delta J_0(A/\Omega)$$

➔ eigenvalues $\pm\frac{\tilde{\Delta}}{2}$ and eigenvectors $|\phi_L\rangle\rangle \pm |\phi_R\rangle\rangle$

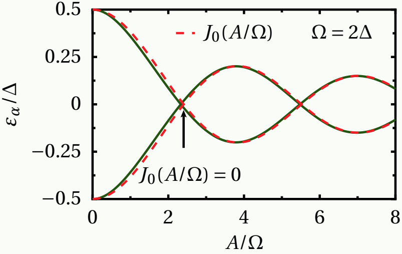
Example I: CDT — perturbation theory

Floquet states

$$|\phi_\pm\rangle\rangle = \frac{|\phi_L\rangle\rangle \pm |\phi_R\rangle\rangle}{\sqrt{2}}$$

quasienergies

$$\pm\frac{\Delta}{2}J_0(A/\Omega)$$



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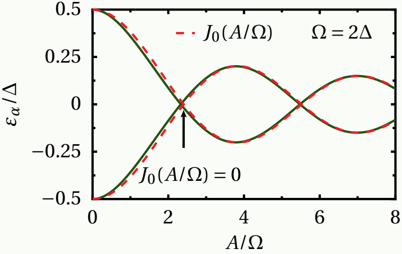

solution for initial state $|L\rangle$

$$|\psi(t)\rangle = \cos\left(\frac{\tilde{\Delta} t}{2}\right)\exp\left[-\mathrm{i}\frac{A}{2\Omega}\sin(\Omega t)\right]|L\rangle + \sin\left(\frac{\tilde{\Delta} t}{2}\right)\exp\left[\mathrm{i}\frac{A}{2\Omega}\sin(\Omega t)\right]|R\rangle$$

➔ for $J_0(A/\Omega) = 0$: $|\psi(t)\rangle \propto |L\rangle$ ➔ tunneling supressed

Grossmann *et al.*, EPL 1992

Perturbation theory for mean energies

goal: $E = \epsilon + \langle\langle \phi | \mathrm{i}\partial_t | \phi \rangle\rangle$

1. compute $\langle\langle \phi | \mathrm{i}\partial_t | \phi \rangle\rangle$ from perturbed Floquet states
2. apply Hellman-Feynman theorem

$$A_\lambda |u_\lambda\rangle = a_\lambda |u_\lambda\rangle \quad \Rightarrow \quad \frac{\partial a_\lambda}{\partial \lambda} = \langle u_\lambda | \frac{\partial A_\lambda}{\partial \lambda} | u_\lambda \rangle$$

problem: $\frac{\partial H(t)}{\partial \Omega}$ not T-periodic [notice: $\frac{\partial}{\partial \Omega} \cos(\Omega t) = -t \sin(\Omega t)$]

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- solution: scaled time $\xi = \Omega t$
  $H(t) - \mathrm{i}\frac{\partial}{\partial t}$ and $\mathscr{H}' = H(\xi/\Omega) - \mathrm{i}\Omega\frac{\partial}{\partial \xi}$ have the same Floquet matrix

$$\rightarrow \frac{\partial \epsilon}{\partial \Omega} = \left\langle\left\langle \frac{\partial \mathscr{H}'}{\partial \Omega} \right\rangle\right\rangle = -\langle\langle \mathrm{i}\partial_\xi \rangle\rangle = -\frac{\langle\langle \mathrm{i}\partial_t \rangle\rangle}{\Omega} = \frac{\epsilon - E}{\Omega} \quad \rightarrow \quad \boxed{E = \epsilon - \Omega \frac{\partial \epsilon}{\partial \Omega}}$$

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- For TLS: $E_\pm = \pm\frac{\Delta}{2}\left(J_0(A/\Omega) - \frac{A}{\Omega}J_1(A/\Omega)\right)$ since $J_1 = -J_0'$

Example II: Rabi problem beyond RWA

$$H(t) = -\frac{\Delta}{2}\sigma_z + \frac{A}{2}\cos(\Omega t)\sigma_x$$

- close to resonance: $\delta = \Delta - \Omega \ll \Delta$, small amplitude: $A \ll \Delta$

Example II: Rabi problem beyond RWA

$$H(t) = -\frac{\Delta}{2}\sigma_z + \frac{A}{2}\cos(\Omega t)\sigma_x$$

- close to resonance: $\delta = \Delta - \Omega \ll \Delta$, small amplitude: $A \ll \Delta$

$$\mathcal{H}_0 = \frac{\Omega}{2}\sigma_z - \mathrm{i}\frac{\partial}{\partial t} \qquad \mathcal{H}_1 = \frac{\delta}{2}\sigma_z + \frac{A}{2}\cos(\Omega t)\sigma_x$$

- degenerate perturbation theory ➔ two-level Floquet Hamiltonian

$$\mathcal{H} \approx \frac{1}{2}\begin{pmatrix} \delta + A^2/4\Omega & A \\ A & -\delta - A^2/4\Omega \end{pmatrix}$$

Rabi Hamiltonian
beyond RWA

- quasienergy splitting: $\epsilon_2 - \epsilon_1 = \bar{\omega}$, where $\bar{\omega}^2 = \delta^2 + A^2 + \frac{A^2\delta}{8\Omega}$
- absorption maximum at

$$\Omega_{\text{res}} \approx \Delta + \frac{A^2}{16\Delta}$$

Bloch-Siegert shift

Summary

- Floquet ansatz
- properties of quasienergies and Floquet states
- composite Hilbert space
- methods for computing Floquet states

Homework

1. Given a time-dependent Hamiltonian $H(t)$ with an eigenstate $|u(t)\rangle$ and energy $E(t)$. Write down the adiabatic solution of the Schrödinger equation and the corresponding Floquet state $|\phi(t)\rangle$
2. Compute numerically the quasienergies of the driven TLS
3. Perform the corresponding perturbation theory for $\Delta \ll \Omega$
4. derive the relation $E = \epsilon - \Omega \frac{\partial \epsilon}{\partial \Omega}$

1. **Geometric phases**
2. **Floquet theory**
3. **Quantum dissipation**
4. **Floquet-Bloch-Redfield formalism**
5. **Dissipative phenomena in driven systems**
   - The driven double-well potential
   - Influence of the system–bath coupling
   - Coherence stabilization by ac fields
6. **Floquet transport theory**
   - scattering theory
   - master equation
7. **Miscellaneous — time-dependent Liouvillians**
   - Matrix continued fractions
   - Bichromatic driving

Heuristic approach

coupling of qubit to electromagnetic environment ➔ sponaneous decay

$$|\psi\rangle \longrightarrow \begin{cases} \sigma_-|\psi\rangle & \text{decay with probability } \alpha \ll 1 \\ |\psi\rangle + |\delta\psi\rangle & \text{no decay, probability } 1-\alpha \end{cases}$$

- normalization requires $|\delta\psi\rangle = \frac{\alpha}{2}\sigma_+\sigma_-|\psi\rangle$

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- normalization requires $|\delta\psi\rangle = \frac{\alpha}{2}\sigma_+\sigma_-|\psi\rangle$
- corresponding density operator

$$\rho \longrightarrow \rho + \frac{\alpha}{2}\Big(2\sigma_-\rho\sigma_+ - \sigma_+\sigma_-\rho - \rho\sigma_+\sigma_-\Big)$$

- add continuous time-evolution ➔ master equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho = -\mathrm{i}[H,\rho] + \frac{\gamma}{2}\Big(2\sigma_-\rho\sigma_+ - \sigma_+\sigma_-\rho - \rho\sigma_+\sigma_-\Big)$$

Lindblad form

Time evolution must conserve

- hermiticity and trace of ρ
- positivity (all eigenvalues of $\rho \geq 0$)

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Fulfilled by a Markovian master equation iff of "Lindblad form"

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho = -\mathrm{i}[H,\rho] + \sum_n \gamma_n \left(2Q_n\rho Q_n^\dagger - Q_n^\dagger Q_n\rho - \rho Q_n^\dagger Q_n\right)$$

G. Lindblad, Comm. Math. Phys. **48**, 119 (1976)

V. Gorini, J. Math. Phys. **17**, 821 (1976)

- Interpretation: incoherent transitions $|\psi\rangle \to Q_n|\psi\rangle$

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- Interpretation: incoherent transitions $|\psi\rangle \to Q_n|\psi\rangle$

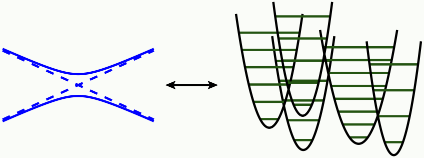
✗ Critique

- request for Markovian evolution unphysical
- axiomatic, not based on physical model
- high-temperature limit typically wrong
  i.e. not the Klein-Kramers or the Smoluchowski equation

Caldeira-Leggett model

Magalinskii 1959; Caldeira, Leggett 1981

Coupling of a system to bath of harmonic oscillators



$$H = H_{\text{system}}(t) + X \sum_{\nu} \gamma_{\nu} (b_{\nu}^{\dagger} + b_{\nu}) + \sum_{\nu} \omega_{\nu} b_{\nu}^{\dagger} b_{\nu}$$

- ➔ eliminate bath
- ➔ equation of motion for reduced density operator
- interpretation: bath “measures” system operator X

Bloch-Redfield theory

Total density operator $R \approx \rho \otimes \rho_{\text{bath,eq}}$

$$\dot{R} = -\mathrm{i}[H_{\text{total}}, R]$$

2nd order perturbation theory in system-bath coupling

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho = -\mathrm{i}[H_{\text{sys}}, \rho] - \mathrm{i}\int_0^{(t-t_0)\to\infty} \mathrm{d}\tau\, \mathscr{A}(\tau)[X, [\tilde{X}(-\tau), \rho(t-\tau)]_+] - \int_0^{(t-t_0)\to\infty} \mathrm{d}\tau\, \mathscr{S}(\tau)[X, [\tilde{X}(-\tau), \rho(t-\tau)]]$$

- Heisenberg operator $\tilde{X}(-\tau) = U(\tau)\, X\, U^\dagger(\tau)$
- bath correlation functions $\mathscr{A}$, $\mathscr{S}$
- non-Markovian
- short system-bath correlation time: Markov approximation

Bath correlation functions

- anti-symmetric correlation function

$$\mathscr{A}(\tau) = -\mathrm{i}\langle[\xi(\tau), \xi(0)]\rangle$$

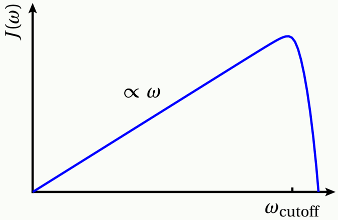
- Fourier transformed: spectral density $\longrightarrow$ continuum limit

$$\mathscr{A}(\omega) = \pi \sum_{\nu} |\gamma_{\nu}|^2 \delta(\omega - \omega_{\nu}) \longrightarrow J(\omega)$$

  - here: Ohmic with cutoff

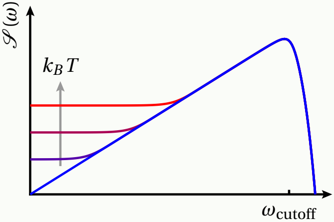
$$J(\omega) = 2\pi\alpha\omega \mathrm{e}^{-\omega/\omega_{\mathrm{cutoff}}}$$

  - dimensionless dissipation strength α



- symmetric bath correlation function

$$\begin{aligned}\mathscr{S}(\tau) &= \frac{1}{2}\langle[\xi(\tau),\xi(0)]_+\rangle \\ \mathscr{S}(\omega) &= J(\omega)\coth\left(\frac{\omega}{2k_BT}\right) \\ &= \begin{cases} 4\pi\alpha k_\mathrm{B}T & \text{high } k_BT \\ 2\pi\alpha\omega & \text{low } k_BT \end{cases}\end{aligned}$$



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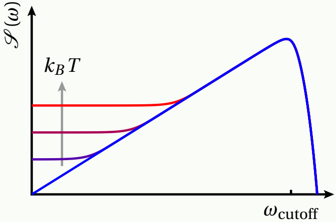

- $\mathscr{S}(\omega)$ evaluated at transition frequencies

➔ dissipation strength depends on coherent spectrum/dynamics

Born-Markov approximation

- Ohmic, short memory times (e.g. for $\gamma < k_B T$)
  ➔ Bloch-Redfield master equation

$$\dot{\rho} = -\mathrm{i}[H_S, \rho] + \mathrm{i}\gamma[X, \{[H_S, X], \rho\}] - [X, [Q, \rho]]$$

coherent dynamics dissipation decoherence

coherent dynamics enters via $Q = \int_0^\infty \mathrm{d}\tau\, \mathscr{S}(\tau)\, \tilde{X}(-\tau)$

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coherent dynamics   dissipation   decoherence

coherent dynamics enters via $Q = \int_0^\infty \mathrm{d}\tau\, \mathscr{S}(\tau)\, \tilde{X}(-\tau)$

- not of Lindblad form
  - ✗ positivity might be violated
  - ✓ happens only on unphysically small time scales
- high-temperature limit: Fokker-Planck equation

Pauli master equation

- Decomposition into energy basis and rotating-wave approximation

➔ rate equation for the populations (Pauli master equation)

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_{\alpha\alpha} = \sum_{\alpha'} \left[w_{\alpha\leftarrow\alpha'}\, \rho_{\alpha'\alpha'} - w_{\alpha'\leftarrow\alpha}\, \rho_{\alpha\alpha} \right]$$

with the golden-rule rates

$$w_{\alpha\leftarrow\alpha'} = J(E_\alpha - E_{\alpha'})\, |\langle\phi_\alpha|X|\phi_{\alpha'}\rangle|^2\, n_{\mathrm{th}}(E_\alpha - E_{\alpha'})$$

- notice: $-n_{\mathrm{th}}(-\omega) = n_{\mathrm{th}}(\omega)+1$

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- notice: $-n_{\mathrm{th}}(-\omega) = n_{\mathrm{th}}(\omega) + 1$

✓ fluctuation theorem $\dfrac{w_{\alpha\leftarrow\alpha'}}{w_{\alpha'\leftarrow\alpha}} = \mathrm{e}^{-(E_\alpha - E_{\alpha'})/k_B T}$

✓ Lindblad form

✗ high-temperature limit typically wrong

full Bloch-Redfield: golden rule for non-diagonal $\rho_{\alpha\beta}$

Floquet-Bloch-Redfield master equation

Driven system ➔ decoherence becomes time-dependent

$$\dot{\rho} = \ldots - [X, [Q(t), \rho]], \qquad Q(t) = \int_0^\infty \mathrm{d}\tau\, \mathscr{S}(\tau)\, \tilde{X}(t-\tau, t)$$

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Central idea:

1. adapted basis: Floquet states $|\phi_\alpha(t)\rangle$ ➔ captures coherent dynamics
2. master equation in Floquet basis

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_{\alpha\beta} = -\mathrm{i}(\epsilon_\alpha - \epsilon_\beta)\rho_{\alpha\beta} + \sum_{\alpha'\beta'} \mathscr{L}_{\alpha\beta,\alpha'\beta'}(t)\, \rho_{\alpha'\beta'}$$

   where $\mathscr{L}(t) = \mathscr{L}(t+T)$

3. moderate rotating-wave approximation:
   time average $\mathscr{L}(t) \to \bar{\mathscr{L}}$, but keep all $\rho_{\alpha\beta}$
   (can sometimes be avoided, see Lecture III)

SK, Dittrich, Hänggi, PRE **55**, 300 (1997)

- Numerical method: compute $\mathscr{L}$ and solve

$$\dot{\rho}_{\alpha\beta} = -\mathrm{i}(\epsilon_\alpha - \epsilon_\beta)\rho_{\alpha\beta} + \sum_{\alpha'\beta'} \bar{\mathscr{L}}_{\alpha\beta,\alpha'\beta'}\, \rho_{\alpha'\beta'}$$

1. time-independent master equation for driven system
2. ac driving captured by choice of basis ➔ efficient
3. includes impact of bath on dissipation strength
   (very relevant for fermionic baths; see Lecture III)

- Analytical tool: find H_{eff} and approx. for $\overline{Q(t)}$
  ➔ effective time-independent Bloch-Redfield equation

Floquet-Bloch-Redfield master equation — RWA

➔ full RWA ➔ (Pauli master equation)

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho_{\alpha\alpha} = \sum_{\alpha'} w_{\alpha\leftarrow\alpha'}\ \rho_{\alpha'\alpha'} - \sum_{\alpha} w_{\alpha'\leftarrow\alpha}\ \rho_{\alpha\alpha}$$

with the golden-rule rates

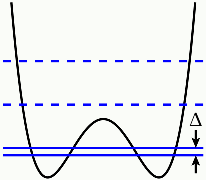
$$w_{\alpha\leftarrow\alpha'} = \sum_{k} J(\epsilon_\alpha - \epsilon_{\alpha'} + k\Omega)\ \left|\sum_{k'} \langle\phi_{\alpha,k+k'}|X|\phi_{\alpha',k}\rangle\right|^2 n_{\mathrm{th}}(\epsilon_\alpha - \epsilon_{\alpha'} + k\Omega)$$

- sidebands contribute to $w_{\alpha\leftarrow\alpha'}$

  … but NOT as independent states!
- no simple relation between forward/backward rates

Example 1: Driven double-well potential

1. long-time solution of a “non-trivial” problem ➔ populations
2. semi-classical limit ➔ capability of the formalism



$$H(x,p,t) = H_{DW}(x,p) + Fx\cos(\Omega t)$$

Symmetries:

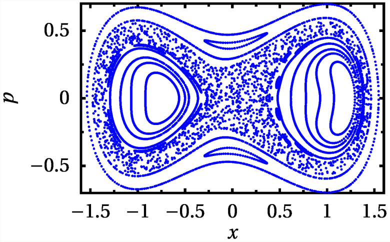
TR: $(x,p,t) \to (x,-p,-t)$

GP: $(x,p,t) \to (-x,-p,t+T/2)$

SK, PhD thesis, 1999

SK, Utermann, Dittrich, Hänggi, PRE 1998

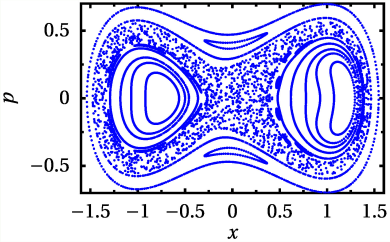
Classical phase space



Stroboscopic map $[x(nT), p(nT)]$

- regular vs. chaotic
- chaos augments with amplitude
- symmetry $p \to -p$
  (consequence of TR)

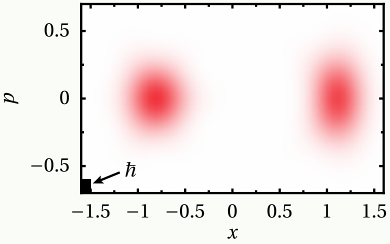
Classical phase space

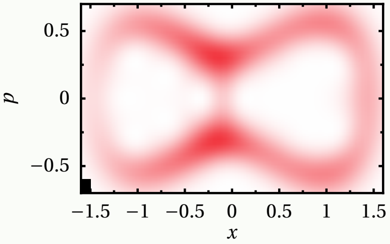


Stroboscopic map $[x(nT), p(nT)]$

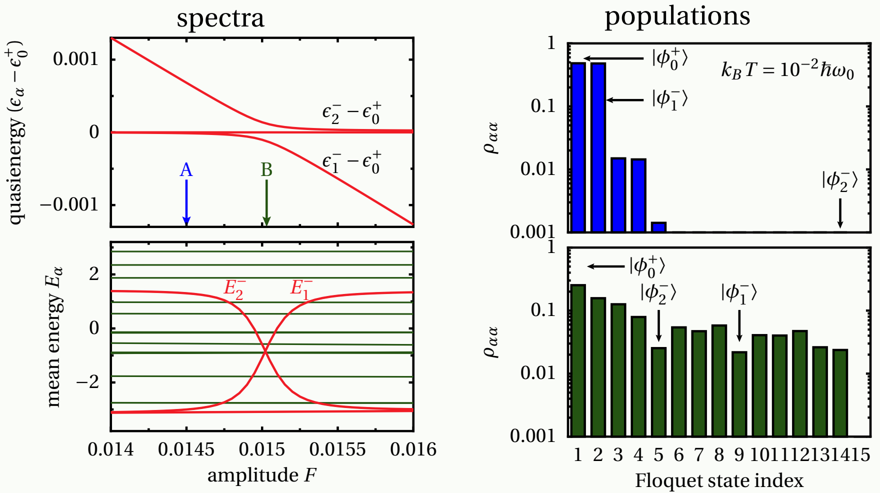
- regular vs. chaotic
- chaos augments with amplitude
- symmetry $p \to -p$ (consequence of TR)

Husimi functions of Floquet states (at times nT)





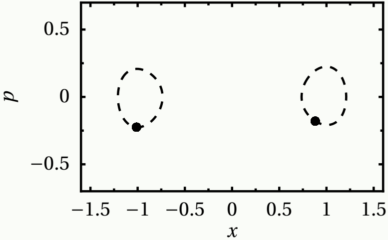
Population of Floquet states



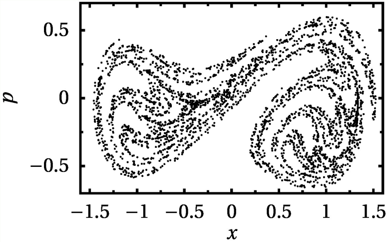
- far from crossing: occupation according to E_α
- at crossing: no general rule

Phase-space structure (dissipative)

large dissipation:
➔ fixed points, limits cycles



weak dissipation:
➔ strange attractor



- dissipative term $-\gamma\dot{x}$ breaks time reversal

  ➔ phase lag due to dissipation ➔ no longer symmetric in p

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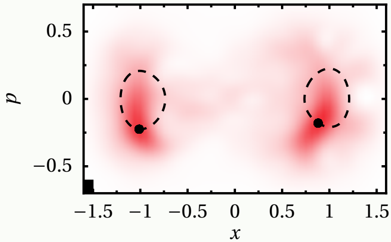

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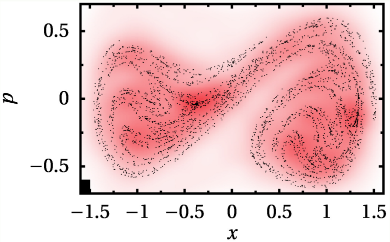

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  ➔ phase lag due to dissipation ➔ no longer symmetric in p

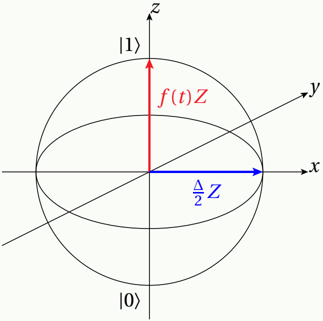
✓ Floquet-Bloch-Redfield capable of phase lag

✗ RWA: $\rho = \sum_\alpha p_\alpha |\phi_\alpha\rangle\langle\phi_\alpha|$ would preserve symmetry

for ratchet potential: Denisov, SK, Hänggi, EPL 2009

Example 2: LZSM Pattern for the Spin-Boson Model

1. influence of qubit-bath coupling on long-time solution
2. (approximation by time-independent Bloch equation)



- driven qubit

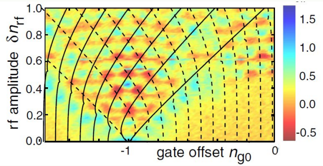
$$H(t) = \frac{\Delta}{2}X + \frac{1}{2}\big(\epsilon + A\cos(\Omega t)\big)Z$$

- qubit-bath coupling

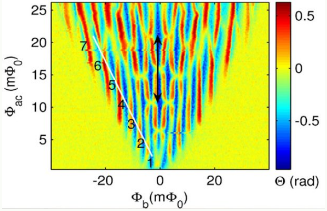
$$H_{\text{qb-bath}} = X\xi \quad \text{or} \quad +Z\xi$$

Blattmann, Hänggi, SK, PRA 2015

Superconducting qubits



Sillanpää *et al.* PRL 2006



Izmalkov *et al.*, PRL 2008

Experiments with driven superconducting qubits

- mean occupation of $|\uparrow\rangle$

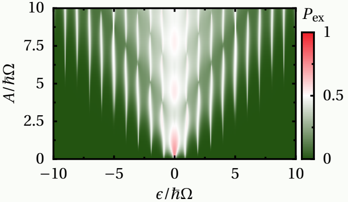
➔ LZSM (Landau Zener Stückelberg Majorana) interference pattern

Shevchenko, Ashhab, Nori, Phys. Rep. 2010

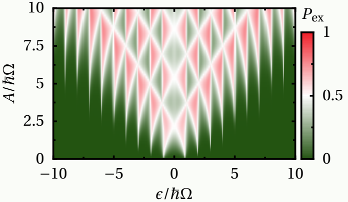
LZSM patterns for X / Z

Numerical solution via Floquet-Bloch-Redfield

X-coupling



Z-coupling



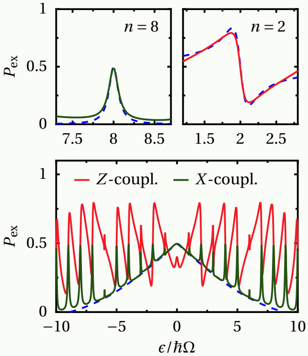
- Lorentz peaks
- $P_{ex} \leq 1/2$

- triangular structure
- population inversion

Blattmann, Hänggi, SK, PRA 2015

Shape of the peaks

$$H(t) = \frac{\Delta}{2}X + \frac{1}{2}\big(\epsilon + A\cos(\Omega t)\big)Z \qquad +X\xi \quad \text{or} \quad +Z\xi$$



Main features

- triangular background
- X-coupling: Lorentzians
- Z-coupling: asymmetric peaks
- both: X dominates

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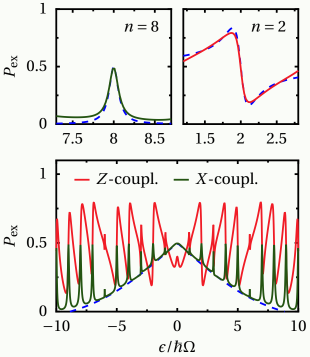

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Bloch equations

X: decay towards $|\downarrow\rangle$ or $|\uparrow\rangle$

Z: decay towards $|\downarrow\rangle \pm |\uparrow\rangle$

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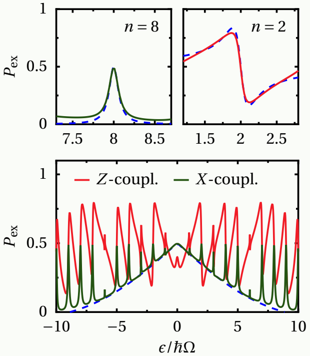
$$H(t) = \frac{\Delta}{2} X + \frac{1}{2}\big(\epsilon + A\cos(\Omega t)\big) Z \qquad + X\xi \quad \text{or} \quad + Z\xi$$



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Bloch equations

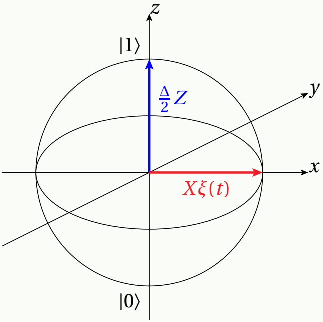
X: decay towards $|\downarrow\rangle$ or $|\uparrow\rangle$

Z: decay towards $|\downarrow\rangle \pm |\uparrow\rangle$

➔ driving & dissipation: bath is more than decay towards ground state

Example III: Qubit with bit-flip noise

1. influence of driving on decoherence ➔ transient dynamics
2. derive effective time-independent master equation



- (undriven) qubit coupled to bath

$$H = -\frac{\Delta}{2}Z + X\xi(t) + H_{\text{bath}}$$

- driving

  $Z\cos(\Omega t)$ or $X\cos(\Omega t)$

? average decay of Bloch vector

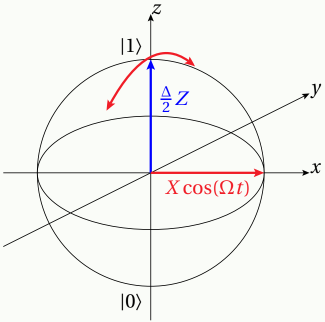
Fonseca Romero, SK, Hänggi, Chem. Phys. 2004

Coherent Destruction of Tunneling

$$H = -\frac{\Delta}{2}Z + X\xi(t) + H_{\mathrm{bath}} + AX\cos(\Omega t)$$

- interaction picture with respect to $X\cos(\Omega t)$
- in rotating frame: time-dependent z-axis
- averaged angular frequency:

$$\Delta_{\mathrm{eff}} = J_0(A/\Omega)\Delta$$



CDT: decoherence

- transformation to rotating frame
  - $\Delta \longrightarrow \Delta_{\text{eff}}$
  - system-bath coupling $X\xi$ unchanged
- modified decoherence rate

$$\Gamma_{\text{CDT}} = \mathscr{S}(\Delta_{\text{eff}}) = 2\pi\alpha\Delta_{\text{eff}} \coth \frac{\Delta_{\text{eff}}}{2k_{\text{B}}T}$$

- at low temperatures

$$\frac{\Gamma_{\text{CDT}}}{\Gamma} = \frac{\Delta_{\text{eff}}}{\Delta} = J_0(A/\Omega)$$

  - under “CDT conditions”: coherence significantly stabilized
  - but: coherent dynamics also slowed down

DD driving: $H_{\mathrm{DD}} = Z\cos(\Omega t)$

$$H = -\frac{\Delta}{2} Z + X\xi(t) + H_{\mathrm{bath}} + AZ\cos(\Omega t)$$

- pulses: Dynamical Decoupling

  Carr, Purcell, Phys. Rev. **94**, 630 (1954)

  Viola, Lloyd, PRA **58**, 2733 (1998)

- central idea: flip coupling operator X to revert influence of noise

DD driving: $H_{\mathrm{DD}} = Z\cos(\Omega t)$

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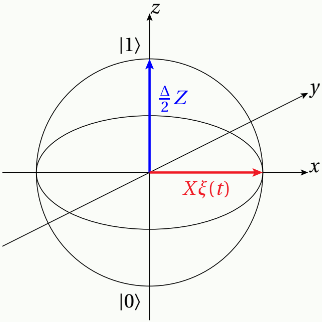
- pulses: Dynamical Decoupling

  Carr, Purcell, Phys. Rev. **94**, 630 (1954)

  Viola, Lloyd, PRA **58**, 2733 (1998)
- central idea: flip coupling operator X to revert influence of noise
- eliminates noise $\perp Z$ with $\omega < \Omega$
- system Hamiltonian unchanged

here: cw driving



Dynamical decoupling: decoherence

- transformation to rotating frame w.r.t. $H_{\mathrm{DD}} = Z\cos(\Omega t)$
  - tunnel Hamiltonian $-\frac{\Delta}{2}Z$ remains
  - coupling $X\xi \longrightarrow X\eta + Y\eta'$ with correlation function

$$\mathscr{S}_\eta(t,\tau) = \langle \eta(t+\tau)\eta(t)\rangle$$

  - effective low-frequency noise

$$\mathscr{S}_{\mathrm{eff}}(\omega) \approx \sum_{k=-\infty}^{\infty} J_k{}^2(A/\Omega)\mathscr{S}(\omega + k\Omega)$$

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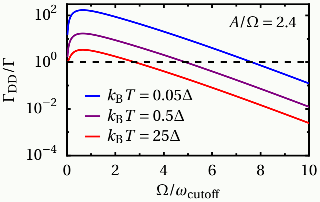
- Thus, for $\Delta \ll \Omega$

$$\frac{\Gamma_{\mathrm{DD}}}{\Gamma} = J_0^2(A/\Omega) + 2\sum_{k=1}^{\infty} J_k^2(A/\Omega)\frac{k\Omega}{\Delta}\frac{\coth(k\Omega/2k_{\mathrm{B}}T)}{\coth(\Delta/2k_{\mathrm{B}}T)}\mathrm{e}^{-k\Omega/\omega_{\mathrm{cutoff}}}$$

- note: $J_0(A/\Omega) = 0$ corresponds to π-pulse

Dynamical decoupling: decoherence

$$\frac{\Gamma_{\mathrm{DD}}}{\Gamma} = J_0^2\,(A/\Omega) + 2\sum_{k=1}^{\infty} J_k^2(A/\Omega)\,\frac{k\Omega}{\Delta}\,\frac{\coth(k\Omega/2k_{\mathrm{B}}T)}{\coth(\Delta/2k_{\mathrm{B}}T)}\,\mathrm{e}^{-k\Omega/\omega_{\mathrm{cutoff}}}$$



$\omega_{\mathrm{cutoff}} = 500\Delta$

- for $\Omega \gg \omega_{\mathrm{cutoff}}$: noise reduction by factor $J_0^2(A/\Omega)$

Summary: Floquet-Bloch-Redfield equation

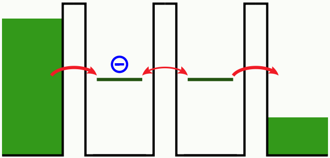
- master equation based on Floquet states
  - ✓ efficient basis
  - ✓ captures dissipative phase lag
- driving affects decoherence
- for driven systems, the system-bath coupling operator matters

Homework

1. derive the BR equation for the harmonic oscillator
2. two-level system with resonant driving   Blattmann, PRA 91, 042109 (2015)
   1. derive the effective Hamiltonian
   2. derive the equation of motion for the Bloch vector
3. compute the effective spectral density for dynamical decoupling

1. **Geometric phases**
2. **Floquet theory**
3. **Quantum dissipation**
4. **Floquet-Bloch-Redfield formalism**
5. **Dissipative phenomena in driven systems**
   - The driven double-well potential
   - Influence of the system–bath coupling
   - Coherence stabilization by ac fields
6. **Floquet transport theory**
   - scattering theory
   - master equation
7. **Miscellaneous — time-dependent Liouvillians**
   - Matrix continued fractions
   - Bichromatic driving

Environment: electron source/drain



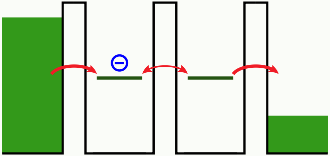
Heuristic Lindblad approach:

- source-to-dot tunneling

$|\psi\rangle \longrightarrow c_1^\dagger|\psi\rangle$

➔ $\dot{\rho} = \ldots + \Gamma\left(c_1^\dagger \rho c_1 - \frac{1}{2} c_1 c_1^\dagger \rho - \frac{1}{2} \rho c_1 c_1^\dagger\right)$

Environment: electron source/drain



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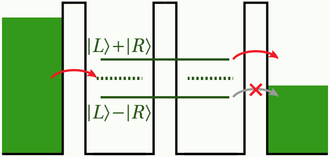


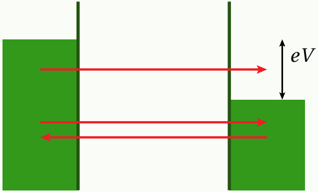
✗ hybridized levels, finite voltage

➔ Lindblad only for large bias

1. scattering formalism
2. Bloch-Redfield master equation
3. Keldysh-Green functions

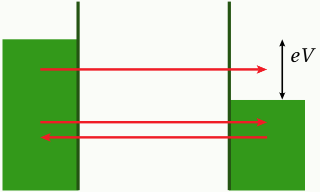
Landauer-Büttiker formula

- Landauer (1957): „conductance is transmission“



Landauer-Büttiker formula

- Landauer (1957): „conductance is transmission“



- current $\boxed{I = \frac{e}{2\pi\hbar} \int \mathrm{d}E\, T(E) \left[f(E+eV) - f(E) \right]}$

- transmission of an electron with energy E

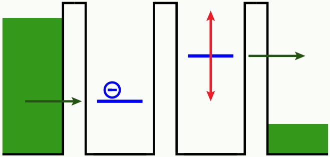
$$T(E) = \Gamma_L \Gamma_R |\langle 1 | G(E) | N \rangle|^2$$

Fischer, Lee, PRB 1981; Meir, Wingreen, PRL 1992

Time-dependent gating *vs.* ac bias

1 ac gate voltage ➔ oscillating levels

2 ac bias voltage ➔ bias: chemical potential difference



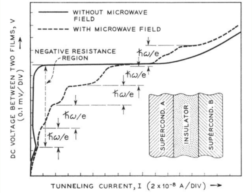


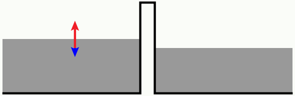
FIG. 1. Bias voltage vs tunneling current of a superconducting Al-Al_2O_3-In diode as measured by Dayem and Martin with and without the microwave field. $\hbar\omega/e=0.16$ mV.

Tien, Gordon, Phys.Rev. 1963

Tien-Gordon theory

- ac bias voltage:

$$V_0 \longrightarrow V_0 + V_{ac} \cos(\Omega t)$$



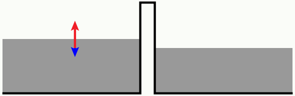
- time-dependent energy shift by $eV_{ac} \cos(\Omega t)$

$$e^{-iEt} \longrightarrow \exp\left(-iEt - i\frac{eV_{ac}}{\Omega}\sin(\Omega t)\right)$$

Tien-Gordon theory

- ac bias voltage:

$$V_0 \longrightarrow V_0 + V_{\mathrm{ac}} \cos(\Omega t)$$



- time-dependent energy shift by $eV_{\mathrm{ac}} \cos(\Omega t)$

$$e^{-\mathrm{i}Et} \longrightarrow \exp\left(-\mathrm{i}Et - \mathrm{i}\frac{eV_{\mathrm{ac}}}{\Omega}\sin(\Omega t)\right) = \sum_k J_k(eV_{\mathrm{ac}}/\Omega)\, e^{-\mathrm{i}(E+k\Omega)t}$$

  - sidebands occupied with probability $J_k^2(\ldots)$
  - energy $k\Omega$ corresponds to additonal DC bias voltage $k\Omega/e$

$$\boxed{I(V_0, V_{\mathrm{ac}}) = \sum_k J_k^2\left(\frac{eV_{\mathrm{ac}}}{\Omega}\right) I_0(V_0 + k\Omega/e)}$$

DC conductivity determines the current !

Tucker, Feldmann, Rev. Mod. Phys. 1985

- Derivation rather heuristic

➔ Rigorous derivation ?

➔ When is Tien-Gordon theory applicable ?

Transport and driving:

Green's function and Landauer formula for time-dependent situation

Transport and driving:
Green's function and Landauer formula for time-dependent situation

- Floquet equation
  with self-energy $\Sigma = |1\rangle \frac{\mathrm{i}\Gamma_L}{2} \langle 1| + |N\rangle \frac{\mathrm{i}\Gamma_R}{2} \langle N|$

$$\left(H(t) + \Sigma - \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\right)|\varphi_\alpha(t)\rangle = (\epsilon_\alpha - \mathrm{i}\gamma_\alpha)|\varphi_\alpha(t)\rangle$$

- propagator in the presence of the contacts

$$G(t, t-\tau) = \sum_k \mathrm{e}^{\mathrm{i}k\Omega t} \int \mathrm{d}\epsilon\, \mathrm{e}^{-\mathrm{i}\epsilon\tau} \underbrace{\sum_{\alpha,k'} \frac{|\varphi_{\alpha,k+k'}\rangle\langle\varphi_{\alpha,k'}|}{\epsilon - (\epsilon_\alpha + k'\Omega - \mathrm{i}\gamma_\alpha)}}_{G^{(k)}(\epsilon)}$$

propagation under absorption/emission of $|k|$ photons

Floquet transport theory: current

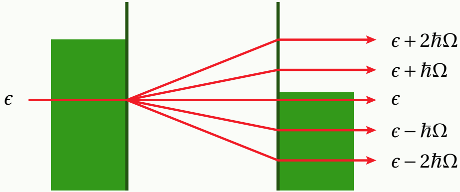
- dc current [note: no blocking factors $(1-f_\ell)$]

$$I = \frac{e}{2\pi\hbar}\sum_k \int d\epsilon \left\{ T_{LR}^{(k)}(\epsilon) f(\epsilon-\mu_L) - T_{RL}^{(k)}(\epsilon) f(\epsilon-\mu_R) \right\}$$

Wagner, Sols, PRL 1999

- transmission under absorption of k photons

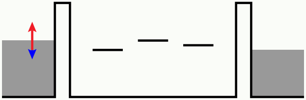
$$T_{LR}^{(k)}(\epsilon) = \Gamma_L \Gamma_R |\langle 1|G^{(k)}(\epsilon)|N\rangle|^2 \not\equiv T_{RL}^{(\pm k)}(\epsilon \pm k\Omega)$$



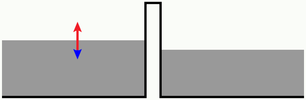
Camalet, Lehmann, SK, Hänggi, PRL 2003

When is Tien-Gordon theory applicable ?

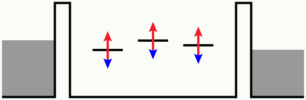
applicable for



- AC bias voltage

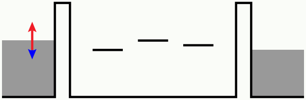


- tunnel barriers
  (studied by Tien & Gordon)



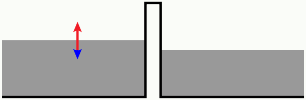
- uniform AC gate voltage

When is Tien-Gordon theory applicable ?



applicable for

- AC bias voltage



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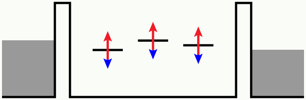
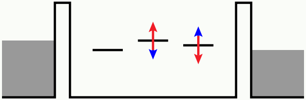


- uniform AC gate voltage

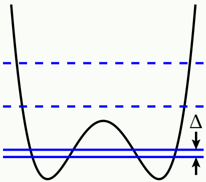


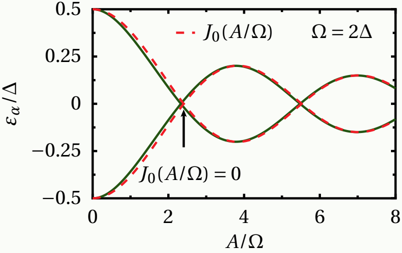
but not for

- non-uniform gating
- dipole force

Camalet, SK, Hänggi, PRB 2004

Example I: Coherent suppression of current





- quasienergies: $\Delta \to \Delta J_0(A/\hbar\Omega)$

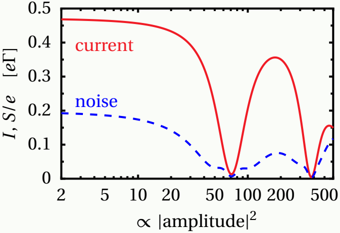
  ➔ coherent destruction of tunnelling

- Electron reservoirs

  ✗ reduce coherence

  ✓ localize electrons

Coherent suppression of current



- current suppression for $J_0(\ldots) = 0$

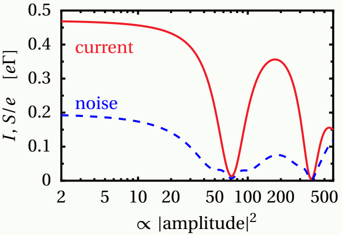
  Lehmann, Camalet, SK, Hänggi, CPL 2003

  arXiv:physics/0205060

- shot noise suppressed as well

  Camalet, Lehmann, SK, Hänggi, PRL 2003

Coherent suppression of current



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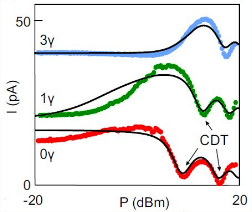


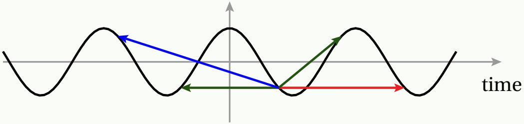
- n-photon resonance: $I \propto J_n^2(\dots)$

  Stehlik *et al.*, PRB 2012

  arXiv:1205.6173

REMINDER: Symmetries of dipole driving

$H_{\mathrm{dipole}} \propto x\cos(\Omega t)$



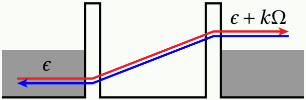
1. time periodicity $t \longrightarrow t+T$ ➔ Floquet theory applicable
2. time reversal $t \longrightarrow -t$ ➔ Floquet states real
3. generalized parity $(x,t) \longrightarrow (-x, t+T/2)$ ➔ Floquet states even/odd
   e.g. symmetric potential with dipole driving
4. time-reversal parity $(x, t-T/4) \longrightarrow (-x, T/4-t)$
   - combination of the other three
   - relevant for Floquet scattering theory

Consequences for scattering probabilities ?

Symmetry-related processes have the same probability

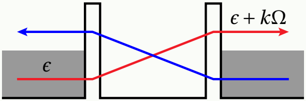
time-reversal

$t \rightarrow -t$



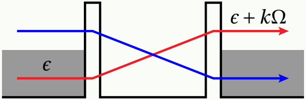
generalized parity

$(x, t \rightarrow -x, t + \frac{T}{2})$

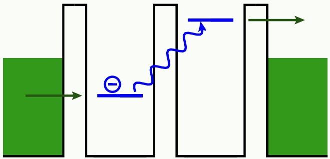


time-reversal parity

$(x, t \rightarrow -x, -t)$



Example II: Non-adiabatic electron pumping



- zero voltage: $\mu_L = \mu_R$
- coupling to rf-field:

$$H_{\mathrm{rf}}(t) \sim (n_L - n_R)\cos(\Omega t)$$

- no generalized parity

SK *et al.*, Phys.Rep. 2005

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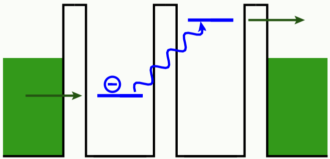

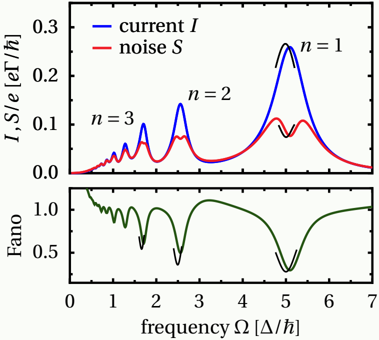

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- no generalized parity

SK *et al.*, Phys.Rep. 2005

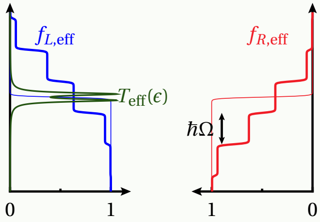
- resonance peaks at $\epsilon \approx k\Omega$
- reduced shot noise

Strass, Hänggi, SK, PRL 2005

- at kth resonance $\epsilon \approx k\Omega$
  - inter-dot tunneling
  - dot-lead tunneling

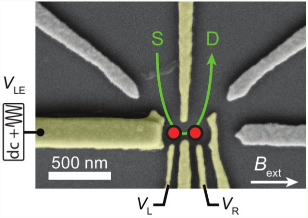
Analytic approach

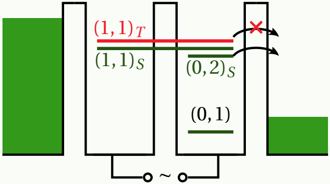
- at kth resonance $\epsilon \approx k\Omega$
  - inter-dot tunneling
  - dot-lead tunneling
- renormalized tunneling: $\Delta \longrightarrow \Delta_k = J_k(A/\Omega)\Delta \rightarrow T_{\text{eff}}$
- effective effective electron distribution



- ac-induced “voltage”: $f_{L,\text{eff}}(0) - f_{R,\text{eff}}(0) = J_0^2(A/2\Omega)$ (while $V_0 = 0$)

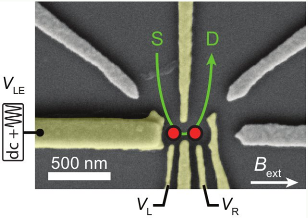
Realistic modelling of quantum dots

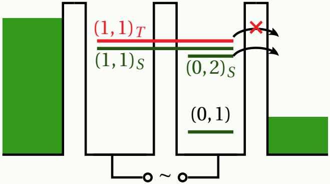




- ✓ dot-lead tunneling
- ✓ detuning
- ✓ AC gate voltage
  $H_{\mathrm{rf}}(t) \propto \cos(\Omega t)$
- ✓ Zeeman splitting
- ➔ scattering theory

Realistic modelling of quantum dots





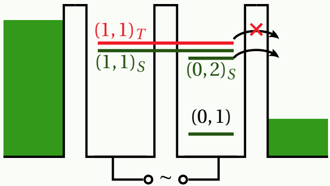
- ✓ dot-lead tunneling
- ✓ detuning
- ✓ AC gate voltage
  $H_{\mathrm{rf}}(t) \propto \cos(\Omega t)$
- ✓ Zeeman splitting
- ➔ scattering theory

- ✗ Coulomb repulsion
- ✗ coupling to phonons
- ✗ spin relaxation
- ➔ master equation

Floquet-Bloch-Redfield master equation

Perturbation theory in DQD-environment coupling V

$$\frac{\mathrm{d}}{\mathrm{d}t}\rho = -\mathrm{i}\left[H_{\mathrm{DQD}}(t), \rho\right] - \int_0^\infty \mathrm{d}\tau \left\langle [V, [V(t-\tau, t), \rho]] \right\rangle_{\mathrm{env}}$$



Floquet-Bloch-Redfield master equation

Perturbation theory in DQD-environment coupling V

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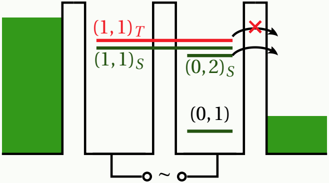
- Floquet theory for QDs ➔ rf-field exact

$$\left(H_{\mathrm{DQD}}(t) - \mathrm{i}\partial_t\right)|\phi_\alpha(t)\rangle = \epsilon_\alpha |\phi_\alpha(t)\rangle$$

- w/o RWA
- in RWA ➔ rate equation

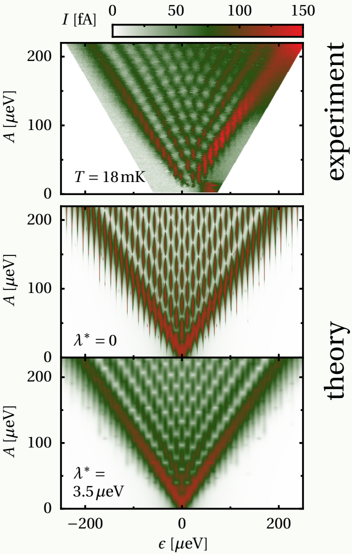
$$\frac{\mathrm{d}}{\mathrm{d}t}P_\alpha = \sum_{\alpha'} w_{\alpha\leftarrow\alpha'}P_{\alpha'} - \sum_{\alpha'} w_{\alpha'\leftarrow\alpha}P_\alpha$$



Evaluation of the rates $w_{\alpha\leftarrow\alpha'}$

	Dissipation	Transport
Environment	harmonic oscillators	electron source/drain
Coupling of mode ν	$X(a_\nu^\dagger + a_\nu)$	$c^\dagger c_\nu + c_\nu^\dagger c$
Absorption / tunnel in	$n_{\mathrm{th}}(\omega)$	$f(\epsilon-\mu)$
Emission / tunnel out	$1+n_{\mathrm{th}}(\omega)$	$1-f(\epsilon-\mu)$
"Ohmic"	$J(\omega)\propto\omega$	$\Gamma(\omega)=\mathrm{const}$

Example: LZSM for current



- resonance peaks
- fades at higher temperature

- inhomogeneous broadening

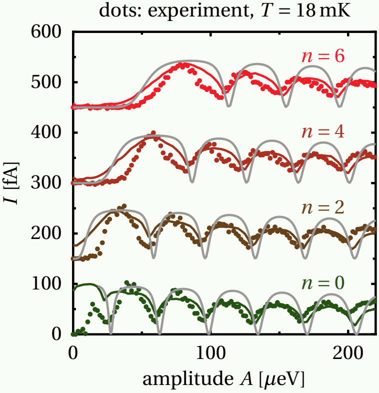
➔ convolute with Gaussian

...

➔ determine system-bath coupling

Forster *et al.*, PRL 2014

Example: LZSM for current



Vertical slices at resonances $\epsilon = n\hbar\Omega$:

- qualitatively: $\Delta \to \Delta J_n(\dots)$

$$I \propto |J_n(\dots)|^2$$

- quantitative agreement depends on
  - phonons
  - Coulomb interaction
  - spin relaxation
  - …

1. **Geometric phases**
2. **Floquet theory**
3. **Quantum dissipation**
4. **Floquet-Bloch-Redfield formalism**
5. **Dissipative phenomena in driven systems**
   - The driven double-well potential
   - Influence of the system–bath coupling
   - Coherence stabilization by ac fields
6. **Floquet transport theory**
   - scattering theory
   - master equation
7. **Miscellaneous — time-dependent Liouvillians**
   - Matrix continued fractions
   - Bichromatic driving

Periodically time-dependent Liouvillians

Master equation of type

$$\frac{\mathrm{d}}{\mathrm{d}t} P = L(t) P$$

- Floquet-Bloch-Redfield beyond moderate RWA
- time-dependent system with Lindblad dissipator
  $\dot{\rho} = -\mathrm{i}[H(t), \rho] + \gamma(2a^\dagger \rho a - a^\dagger a \rho - \rho a^\dagger a)$
  - very weak dissipation
  - transport problem with large bias

➔ long-time solution T-periodic

➔ Floquet ansatz with “quasienergy” zero

$$P(t) = \sum_k \mathrm{e}^{-ik\Omega t} p_k$$

Matrix-continued fractions

$\frac{\mathrm{d}}{\mathrm{d}t} P = L(t) P$ with

$$L(t) = L_0 + 2L_1 \cos(\Omega t)$$

➔ tridiagonal Floquet matrix

$$L_0 + 2L_1 \cos(\Omega t) - \partial_t \leftrightarrow \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & L_0 + 2\mathrm{i}\Omega & L_1 & 0 & 0 & 0 & \cdots \\ \cdots & L_1 & L_0 + \mathrm{i}\Omega & L_1 & 0 & 0 & \cdots \\ \cdots & 0 & L_1 & L_0 & L_1 & 0 & \cdots \\ \cdots & 0 & 0 & L_1 & L_0 - \mathrm{i}\Omega & L_1 & \cdots \\ \cdots & 0 & 0 & 0 & L_1 & L_0 - 2\mathrm{i}\Omega & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Matrix-continued fractions

- ansatz $P(t) = \sum_k \mathrm{e}^{-ik\Omega t} p_k$ yields

$$L_1 p_{k-1} + (L_0 + \mathrm{i}k\Omega) p_k + L_1 p_{k+1} = 0$$

- idea: truncate and iterate $p_{k-1} = -L_1^{-1}\left\{(L_0 - \mathrm{i}k\Omega) p_k + L_1 p_{k+1}\right\}$

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  - ✗ fails, L_1 generally singular
- solution: ansatz $p_k = S_k L_1 p_{k\mp 1}$ $(k \gtrless 0)$ leads to

$$S_k = -(L_0 + \mathrm{i}k\Omega + L_1 S_{k\pm 1} L_1)^{-1} \quad \longrightarrow \quad S_{\pm 1} \tag{1}$$

$$0 = (L_1 S_{-1} L_1 + L_0 + L_1 S_1 L_1) p_0 \tag{2}$$

➔ truncate at $\pm k_0$, iterate (1), and solve (2)

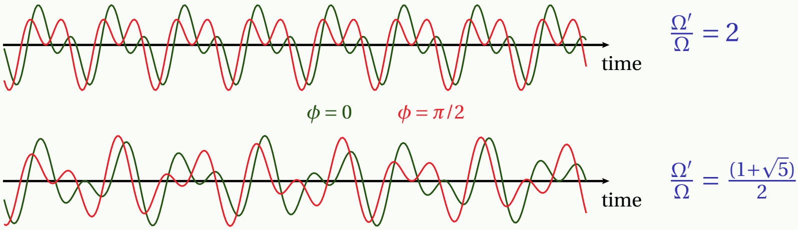
➔ time-averaged $P(t) = p_0$ ➔ time-averaged expectation values

Risken, "The Fokker-Planck Equation"

Appendix of Forster *et al.*, PRB 2015

Bichromatic Driving

- $f(t) = \sin(\Omega t) + \eta \sin(\Omega' t + \phi)$
  Ω', Ω commensurable *vs.* incommensurable



➔ periocdic *vs* quasi-periodic

Two-color Floquet theory (commensurable)

- $\frac{\mathrm{d}}{\mathrm{d}t} P = L(t) P$ with

$$L(t) = L_0 + L_1 \cos(n\Omega t) + L_1' \cos(n'\Omega t)$$

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- Floquet ansatz for long-time solution

$$P(t) = \sum_k \mathrm{e}^{-\mathrm{i}k\Omega t} p_k$$

➔ equations for p_k: Floquet matrix with additional diagonal

Two-color Floquet theory (incommensurable)

- $\frac{\mathrm{d}}{\mathrm{d}t} P = L(t) P$ with

$$L(t) = L_0 + L_1 \cos(\Omega t) + L_1' \cos(\omega t)$$

- auxiliary angular coordinate $\omega t \longrightarrow \theta$

$$\frac{\mathrm{d}}{\mathrm{d}t} \mathscr{P} = \mathscr{L}(t, \theta) \mathscr{P}$$

$$\mathscr{L}(t, \theta) = L_0 + L_1 \cos(\Omega t) + L_1' \cos(\theta) - \omega \frac{\partial}{\partial \theta}$$

cf. t-t' formalism by Peskin, Moiseyev, J.Chem.Phys. 1993

➔ $2\pi/\Omega$-periodic time-dependence

➔ usual Floquet tools, e.g. matrix-continued fractions

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→ $2\pi/\Omega$-periodic time-dependence

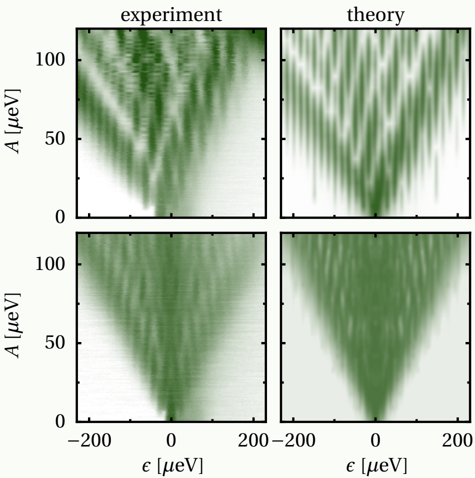
→ usual Floquet tools, e.g. matrix-continued fractions

- connection $P(t) = \mathscr{P}(t, \theta)|_{\theta = \omega t}$

Forster *et al.*, PRB 2015

Examples: LZSM pattern

$$\text{driving: } f(t) = \sin(\Omega t) + \eta \sin(\Omega' t + \phi)$$



$\Omega'/\Omega = 2$

- ϕ-dependent

$\Omega'/\Omega = \frac{1}{2}(1+\sqrt{5})$

- interference despite quasi-random phase factors

Forster *et al.*, PRB 2015

Summary of Floquet methods

Schrödinger equation

- Floquet matrix
- perturbation theory
- … or any other diagonalization technique

Master equations

- Floquet-Bloch-Redfield (BR in Floquet basis)
  - basis adapted to coherent dynamics
  - captures effect of driving on environment
  - dissipative phase shift
  - weak dissipation: RWA ➔ Lindblad
- time-dependent Liouvillian (Lindblad form in “natural” basis)
  - stat. solution via matrix-continied fraction
  - bichromatic, commensurable: extended Floquet matrix
  - bichromatic, incommensurable: Floquet matrix & MCF