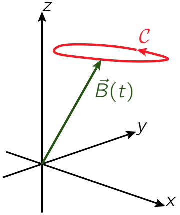
Floquet theory

Sigmund Kohler

Instituto de Ciencia de Materiales de Madrid, CSIC



I. Schrödinger equation

II. Quantum dissipation

III. Application: Landau-Zener-Stückelberg-Majorana interference

IV. Miscellaneous: transport, time-dep. Liouvillians, bichromatric, . . .

https://sigmundkohler.github.io/download/FloquetTutorial.pdf

1. **Schrödinger equation**
   - Geometric phases
   - Time-periodicity, Floquet ansatz, and all that
2. **Quantum dissipation**
   - System-bath model
   - Floquet-Bloch-Redfield formalism
3. **Application: LZSM Interference**
4. **Miscellaneous**
   - Quantum transport
   - Time-periodic Liouvillians
   - Bichromatic driving

- Time evolution of an eigenstate:

$$|\psi(t)\rangle = e^{-iE_n t}|\phi_n\rangle$$

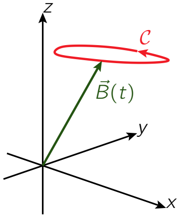
Notation:
ψ: solution of Schrödinger equation
ϕ: other state vector, e.g., eigenstate

- energy $\leftrightarrow$ phase

for (periodically) time-dependent system ?

- Spin in magnetic field $B(t) = B(t+T)$:

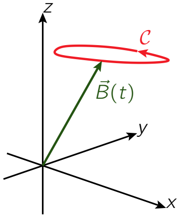
$$H(t) = \frac{1}{2}\vec{B}(t) \cdot \vec{\sigma}$$



Adiabatic time-dependence — Berry phase

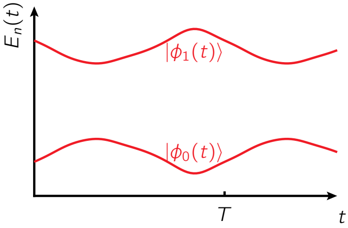
- Spin in magnetic field
  $B(t) = B(t+T)$:

$$H(t) = \frac{1}{2}\vec{B}(t)\cdot\vec{\sigma}$$



- Quantum dynamics for $\dot{B} \ll B^2$:
  state follows the eigenstate adiabatically

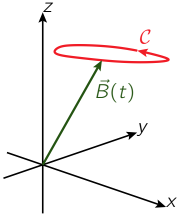
$$|\psi(t)\rangle \propto |\phi_n(t)\rangle$$



➔ $|\psi(t)\rangle$ determined up to phase factor

After one period: $|\psi(T)\rangle = e^{i\varphi}|\psi(0)\rangle$

$$\varphi = -\int_0^T dt\, E_n(t) + \gamma_{\mathcal{C}}$$

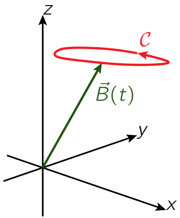


- dynamical phase $\leftrightarrow$ mean energy
- Berry phase $\gamma_{\mathcal{C}}$
  - depends only on closed curve $\mathcal{C}$ in parameter space

M. Berry, Proc. Roy. Soc. London, Ser. A **392**, 45 (1984)

After one period: $|\psi(T)\rangle = e^{i\varphi}|\psi(0)\rangle$

$$\varphi = -\int_0^T dt\, E_n(t) + \gamma_{\mathcal{C}}$$



- dynamical phase $\leftrightarrow$ mean energy
- Berry phase $\gamma_{\mathcal{C}}$
  - depends only on closed curve $\mathcal{C}$ in parameter space

M. Berry, Proc. Roy. Soc. London, Ser. A **392**, 45 (1984)

- Assumptions:
  1. $\vec{B}(t)$ changes adiabatically slowly
  2. initial state: eigenstate $|\phi_n(0)\rangle$

Different perspective:

State vector undergoes periodic time-evolution

- $|\psi(T)\rangle = e^{i\varphi}|\psi(0)\rangle$
- dynamics $|\psi(t)\rangle$ induced by some Hamiltonian $H(t)$

Remarks:

- no adiabatic condition
- $|\psi(t)\rangle$ need not be an eigenstate of $H(t)$
- $H(t)$ is not unique

- only condition: cyclic time-evolution in Hilbert space

$$|\psi(t)\rangle = e^{if(t)}|\phi(t)\rangle, \qquad \phi(t) = \phi(t+T)$$

Non-adiabatic generalization

➔ Phase acquired during cyclic evolution: $\varphi = f(T) - f(0)$

$$\frac{df}{dt} = -\langle\phi|H|\phi\rangle + \langle\phi|i\frac{d}{dt}|\phi\rangle \quad \Longrightarrow \quad \boxed{\varphi = \gamma_{\mathrm{dyn}} + \gamma}$$

Non-adiabatic generalization

➔ Phase acquired during cyclic evolution: $\varphi = f(T) - f(0)$

$$\frac{df}{dt} = -\langle\phi|H|\phi\rangle + \langle\phi|i\frac{d}{dt}|\phi\rangle \quad \Longrightarrow \quad \boxed{\varphi = \gamma_{\text{dyn}} + \gamma}$$

Dynamical phase $\gamma_{\text{dyn}} = -\int_0^T dt\, \langle\phi(t)|H(t)|\phi(t)\rangle$

- depends on choice of $H(t)$
- reflects mean energy

Non-adiabatic generalization

➔ Phase acquired during cyclic evolution: $\varphi = f(T) - f(0)$

$$\frac{df}{dt} = -\langle\phi|H|\phi\rangle + \langle\phi|i\frac{d}{dt}|\phi\rangle \implies \boxed{\varphi = \gamma_{\text{dyn}} + \gamma}$$

Dynamical phase $\gamma_{\text{dyn}} = -\int_0^T dt\, \langle\phi(t)|H(t)|\phi(t)\rangle$

- depends on choice of $H(t)$
- reflects mean energy

Aharonov-Anandan phase ("non-adiabatic Berry phase")

$$\gamma = \int_0^T dt\, \langle\phi|i\frac{d}{dt}|\phi\rangle$$

- depends only on trajectory in Hilbert space — not in parameter space!
- adiabatic limit: $\gamma = \gamma_C$

Aharonov & Anandan, PRL 1987

1. **Schrödinger equation**
   - Geometric phases
   - Time-periodicity, Floquet ansatz, and all that
2. **Quantum dissipation**
   - System-bath model
   - Floquet-Bloch-Redfield formalism
3. **Application: LZSM Interference**
4. **Miscellaneous**
   - Quantum transport
   - Time-periodic Liouvillians
   - Bichromatic driving

Goal: propagator $U(t, t')$

- Time-independent system: diagonalize Hamiltonian ➔ $|\phi_n\rangle$, E_n

$$U(t, t') = U(t - t') = \sum_n e^{-iE_n(t-t')} |\phi_n\rangle\langle\phi_n|$$

Goal: propagator $U(t, t')$

- Time-independent system: diagonalize Hamiltonian ➔ $|\phi_n\rangle$, E_n

$$U(t, t') = U(t - t') = \sum_n e^{-iE_n(t-t')} |\phi_n\rangle\langle\phi_n|$$

- Driven system:

$$i\frac{d}{dt}|\psi\rangle = H(t)|\psi\rangle$$ ➔ numerical integration

problem 1: time-integration not efficient for long times

problem 2: no information about structure of U

Goal: propagator $U(t, t')$

- Time-independent system: diagonalize Hamiltonian ➔ $|\phi_n\rangle$, E_n

$$U(t, t') = U(t - t') = \sum_n e^{-iE_n(t-t')} |\phi_n\rangle\langle\phi_n|$$

- Driven system:

$$i\frac{d}{dt}|\psi\rangle = H(t)|\psi\rangle \quad \text{➔ numerical integration}$$

problem 1: time-integration not efficient for long times

problem 2: no information about structure of U

- Solution for $H(t) = H(t + T)$: "Bloch theory in time"
  cf. $H(x)|\phi\rangle = \epsilon|\phi\rangle$ with $H(x) = H(x + a)$
  ➔ Bloch waves $\phi(x) = e^{iqx}\varphi(x)$, where $\varphi(x)$ is a-periodic

F. Bloch, Z. Phys. A **52**, 555 (1928)

Gaston Floquet (1883):

Ann. de l'Ecole Norm. Sup. **12**, 47 (1883)

Parametric oscillator (cf. Paul trap)

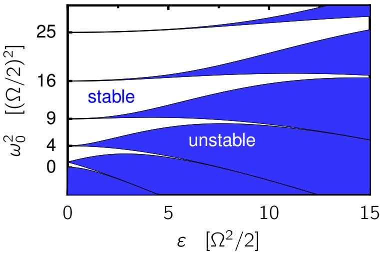
$$\ddot{x} + (\omega_0^2 + \epsilon \cos \Omega t)x = 0$$

Floquet theorem:
solutions have the structure

$$x(t) = e^{\pm i\mu t}\xi(t)$$

where $\xi(t) = \xi(t + 2\pi/\Omega)$

(undriven limit: $\mu = \omega_0$, $\xi = \text{const}$)



μ real
➔ oscillating solutions

μ imaginary
➔ one solution unstable

Discrete time translation and Floquet ansatz

- $H(t) = H(t+T)$

  ➔ $t \to t + T$ is symmetry operation

  ➔ solutions of Schrödinger equation obey $|\psi(t+T)\rangle = e^{i\varphi}|\psi(t)\rangle$

- Floquet ansatz

$$|\psi(t)\rangle = e^{-i\epsilon t}|\phi(t)\rangle = e^{-i\epsilon t}\sum_k e^{-ik\Omega t}|c_k\rangle$$

  - ϵ quasienergy (cf. quasi momentum) ➔ long-time dynamics
  - $|\phi(t)\rangle = |\phi(t+T)\rangle$, Floquet state ➔ within driving period

- Floquet theorem: $H(t)$ has a complete set of Floquet solutions

- Schrödinger equation $i\partial_t|\psi\rangle = H(t)|\psi\rangle$ yields

$$\boxed{\left(H(t) - i\partial_t\right)|\phi(t)\rangle = \epsilon|\phi(t)\rangle}$$

Brillouin zone structure

- $|\phi(t)\rangle$ Floquet state with quasienergy ϵ

➔ $e^{ik\Omega t}|\phi(t)\rangle$ Floquet state with $\epsilon + k\Omega$

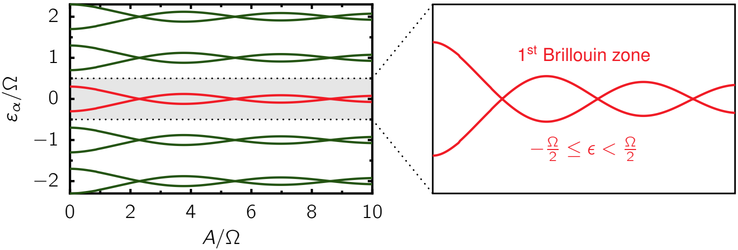
proof: insert into $(H - i\partial_t)|\phi\rangle = \epsilon|\phi\rangle$

- $|\phi(t)\rangle$ Floquet state with quasienergy ϵ

➔ $e^{ik\Omega t}|\phi(t)\rangle$ Floquet state with $\epsilon + k\Omega$

proof: insert into $(H - i\partial_t)|\phi\rangle = \epsilon|\phi\rangle$

e.g. for two-level system



- all Brillouin zones equivalent, choice arbitrary

➔ quasienergies cannot serve for ordering!

Mean energies

- Physical quantity / observable: mean energy

$$E = \frac{1}{T}\int_0^T dt\ \langle\psi(t)|H(t)|\psi(t)\rangle = \frac{1}{T}\int_0^T dt\ \langle\phi(t)|H(t)|\phi(t)\rangle$$

- All equivalent states have the same mean energy
  [proof: insert $e^{-ik\Omega t}|\phi(t)\rangle$]

➔ Floquet states can be ordered by their mean energy

- Mean energy

$$E = \frac{1}{T}\int_0^T dt\,\langle\phi(t)|\{H(t) - i\partial_t + i\partial_t\}|\phi(t)\rangle$$

where $(H - i\partial_t)|\phi(t)\rangle = \epsilon|\phi(t)\rangle$

$$E = \epsilon + \frac{1}{T}\int_0^T dt\,\langle\phi(t)|i\partial_t|\phi(t)\rangle$$

- Mean energy

$$E = \frac{1}{T}\int_0^T dt\, \langle\phi(t)|\left\{H(t) - i\partial_t + i\partial_t\right\}|\phi(t)\rangle$$

where $(H - i\partial_t)|\phi(t)\rangle = \epsilon|\phi(t)\rangle$

$$-\epsilon = -E + \frac{1}{T}\int_0^T dt\, \langle\phi(t)|\, i\partial_t\, |\phi(t)\rangle$$

- Compare to $\varphi = \gamma_{\text{dyn}} + \gamma$

- Mean energy

$$E = \frac{1}{T}\int_0^T dt\, \langle\phi(t)|\big\{H(t) - i\partial_t + i\partial_t\big\}|\phi(t)\rangle$$

where $(H - i\partial_t)|\phi(t)\rangle = \epsilon|\phi(t)\rangle$

$$-\epsilon = -E + \frac{1}{T}\int_0^T dt\, \langle\phi(t)|\, i\partial_t\, |\phi(t)\rangle$$

- Compare to $\varphi = \gamma_{\text{dyn}} + \gamma$

$$\boxed{(E-\epsilon)T \text{ is a geometric phase}}$$

Mean energies — Rapoport relation

Floquet equation as function of Ω and Ωt:

$$\epsilon(\Omega)\phi(\Omega t) = \left[H(\Omega t) - i\Omega\frac{\partial}{\partial\Omega t}\right]\phi(\Omega t)$$

Floquet equation as function of Ω and Ωt:

$$\epsilon(\Omega)\phi(\Omega t) = \left[H(\Omega t) - i\Omega\frac{\partial}{\partial\Omega t}\right]\phi(\Omega t)$$

Compute derivative $(\partial/\partial\Omega)\big|_{\Omega t}$ and apply $\displaystyle\int \frac{dt}{T}\,\phi^{+}$ to obtain

$$\frac{\partial\epsilon}{\partial\Omega} = -\frac{\gamma}{2\pi} = \frac{\epsilon - E}{\Omega} \qquad \Rightarrow \qquad E = \epsilon - \Omega\frac{\partial\epsilon}{\partial\Omega}$$

Floquet equation as function of Ω and Ωt:

$$\epsilon(\Omega)\phi(\Omega t) = \left[H(\Omega t) - i\Omega\frac{\partial}{\partial\Omega t}\right]\phi(\Omega t)$$

Compute derivative $(\partial/\partial\Omega)\big|_{\Omega t}$ and apply $\displaystyle\int \frac{dt}{T}\,\phi^+$ to obtain

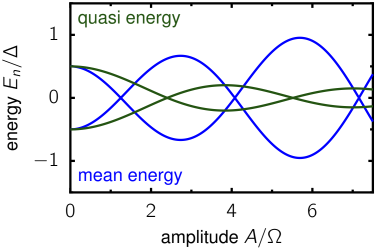
$$\frac{\partial\epsilon}{\partial\Omega} = -\frac{\gamma}{2\pi} = \frac{\epsilon - E}{\Omega} \quad \rightarrow \quad E = \epsilon - \Omega\frac{\partial\epsilon}{\partial\Omega}$$

E.g. for $\gamma = \gamma_{\mathrm{ad}} + \mathcal{O}(\Omega)$

$\rightarrow$ $\epsilon = \mathrm{const} - (\gamma_{\mathrm{ad}}/2\pi)\Omega + \ldots \rightarrow E = E_{\mathrm{ad}} + \mathcal{O}(\Omega^2)$

Fainshtein, Manakov, Rapoport, J. Phys. B (1978)

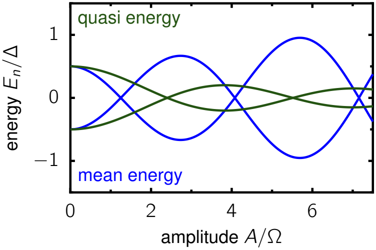
Mean energies — two-level system



Driven undetuned two-level system

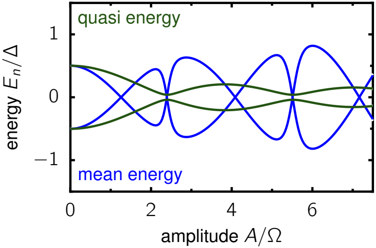
- exact crossings
  (consequence of symmetry)

Mean energies — two-level system



Driven undetuned two-level system

- exact crossings (consequence of symmetry)

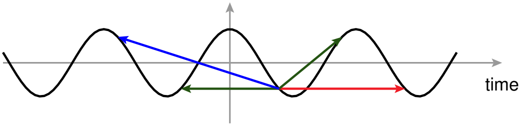


. . . with small detuning

- quasi energies
  - avoided crossings
- mean energies
  - exact crossings remain
  - additional crossings
  - ➔ do not follow from any eigenvalue equation

Symmetries of dipole driving

$H_{\text{dipole}} \propto x \cos(\Omega t)$



1 time periodicity $t \longrightarrow t + T$ ➔ Floquet theory applicable

2 time reversal $t \longrightarrow -t$ ➔ Floquet states real

3 generalized parity $(x, t) \longrightarrow (-x, t + T/2)$
➔ Floquet states even/odd
e.g. symmetric potential with dipole driving

4 time-reversal parity $(x, t - T/4) \longrightarrow (-x, T/4 - t)$

- combination of the other three
- relevant for Floquet scattering theory

Goal: more formal treatment of $H(t) - i\partial_t$

- $|\phi(t)\rangle \in \mathcal{R} \otimes \mathcal{T}$ composite Hilbert space / Sambe space

  Shirley, PR **138**, B979 (1965), Sambe, PRA **7**, 2203 (1973)

  $\mathcal{T}$: Hilbert space of T-periodic functions with inner product

$$\langle f|g\rangle = \int_0^T f(t)^* \, g(t) \frac{dt}{T} = \sum_k f_k^* g_k$$

- extended Dirac notation:
  - $|\phi(t)\rangle = \langle t|\phi\rangle\rangle$
  - Fourier coefficient $|\phi_k\rangle = \langle k|\phi\rangle\rangle$

e.g.: $|\phi(t)\rangle = \langle t|\phi\rangle\rangle = \sum_k \langle t|k\rangle\langle k|\phi\rangle\rangle = \sum_k e^{-ik\Omega t}|\phi_k\rangle$

Completeness and Orthogonality

- $H - i\partial_t$ is hermitian

→ Floquet states $|\phi_\alpha\rangle$ orthonormal and complete in $\mathcal{R} \otimes \mathcal{T}$

$$\langle\langle \phi_\alpha^{(k)} | \phi_\beta^{(k')} \rangle\rangle = \delta_{\alpha\beta}\delta_{kk'}$$

? but in $\mathcal{R}$?

Completeness and Orthogonality

- $H - i\partial_t$ is hermitian

→ Floquet states $|\phi_\alpha\rangle$ orthonormal and complete in $\mathcal{R} \otimes \mathcal{T}$

$$\langle\langle \phi_\alpha^{(k)} | \phi_\beta^{(k')} \rangle\rangle = \delta_{\alpha\beta}\delta_{kk'}$$

? but in $\mathcal{R}$?

- Consider $\langle \phi_\alpha(t) | \phi_\beta(t) \rangle = \sum_k \lambda_k e^{-ik\Omega t}$ since T-periodic
  with the Fourier coefficient

$$\lambda_k = \frac{1}{T}\int_0^T dt\, e^{ik\Omega t} \langle \phi_\alpha(t) | \phi_\beta(t) \rangle = \langle\langle \phi_\alpha | \phi_\beta^{(k)} \rangle\rangle = \delta_{\alpha\beta}\delta_{k,0}$$

→ Floquet states orthogonal at equal times

Propagator

- propagator in terms of Floquet states

$$U(t,t') = \sum_\alpha |\psi_\alpha(t)\rangle\langle\psi_\alpha(t')| = \sum_\alpha e^{-i\epsilon_\alpha(t-t')}|\phi_\alpha(t)\rangle\langle\phi_\alpha(t')|$$

  - long-time dynamics (depends on $t - t'$)
  - dynamics within driving period (depends on t and t')

Propagator

- propagator in terms of Floquet states

$$U(t,t') = \sum_\alpha |\psi_\alpha(t)\rangle\langle\psi_\alpha(t')| = \sum_\alpha e^{-i\epsilon_\alpha(t-t')}|\phi_\alpha(t)\rangle\langle\phi_\alpha(t')|$$

  - long-time dynamics (depends on $t - t'$)
  - dynamics within driving period (depends on t and t')

- one-period propagator for kicked systems

$$H(t) = H_0 + K\sum_n \delta(t - nT)$$

➔ $U(T) = e^{-iH_0T}e^{-iK}$

✓ easy to compute
✓ provides quasienergies
✗ only long-time dynamics (stroboscopic)

Computation of Floquet states

Solve eigenvalue problem

$$\left\{H(t) - i\partial_t\right\}|\phi\rangle\rangle = \epsilon|\phi\rangle\rangle$$

icmm

Solve eigenvalue problem

$$\left\{H(t) - i\partial_t\right\}|\phi\rangle\rangle = \epsilon|\phi\rangle\rangle$$

Straightforward in Fourier representation (“Floquet matrix”)

$$H_0 + H_1\cos(\Omega t) - i\frac{d}{dt} \leftrightarrow \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & H_0 + 2\Omega & \frac{1}{2}H_1 & 0 & 0 & 0 & \cdots \\ \cdots & \frac{1}{2}H_1 & H_0 + \Omega & \frac{1}{2}H_1 & 0 & 0 & \cdots \\ \cdots & 0 & \frac{1}{2}H_1 & H_0 & \frac{1}{2}H_1 & 0 & \cdots \\ \cdots & 0 & 0 & \frac{1}{2}H_1 & H_0 - \Omega & \frac{1}{2}H_1 & \cdots \\ \cdots & 0 & 0 & 0 & \frac{1}{2}H_1 & H_0 - 2\Omega & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

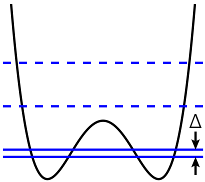
Computation of Floquet states

1. direct diagonalization of $H(t) - i\partial_t$
   - conceptually simple ➔ first choice
   - increasingly difficult with smaller frequency
   - often more efficient after unitary transformation
2. analytical tool: perturbation theory
   strong driving: $H_1 \cos(\Omega t) - i\partial_t$ as zeroth order
3. diagonalization of $U(T, 0)$ ➔ $e^{-i\epsilon T}$, $|\phi(0)\rangle$
4. matrix-continued fraction
5. (t, t') formalism

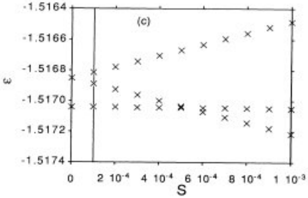
Example I: Coherent destruction of tunneling

1. role of quasienergy crossings
2. perturbation theory (two-level approximation)
3. convenient route to mean energy

Driven double-well potential $H(t) = H_{\rm DW} + Sx\cos(\Omega t)$



quasienergies $\epsilon_\alpha(S)$

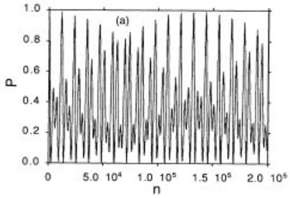


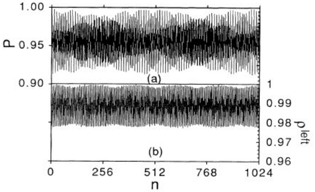
? tunnel oscillations influenced by driving

? dynamics at quasienergy crossing

Example I: Coherent destruction of tunneling

Occupation $P_{\text{left}}(nT)$





far from crossing:

- tunnel oscillations

at crossing:

- particle stays in left well

➔ "coherent destruction of tunneling" by ac field

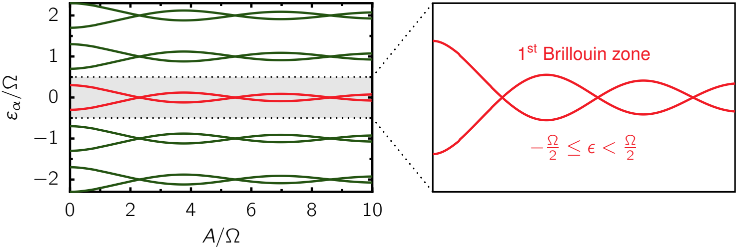
Grossmann *et al.*, PRL 1991

Analytical understanding ➔ two-level approximation

Driven two-level system

$$H(t) = -\frac{\Delta}{2}\sigma_x + \frac{A}{2}\cos(\Omega t)\sigma_z$$

quasienergy spectrum



Analytical approach for $\Delta \ll \Omega$: high-frequency limit

$$H(t) = -\frac{\Delta}{2}\sigma_x + \frac{A}{2}\cos(\Omega t)\sigma_z$$

- zeroth order Floquet equation

$$\left(\frac{A}{2}\cos(\Omega t)\sigma_z - i\frac{d}{dt}\right)|\phi(t)\rangle = \epsilon^{(0)}|\phi(t)\rangle$$

with the Floquet states and quasienergies

$$|\phi_{\mathrm{L/R}}(t)\rangle = e^{\pm i(A/2\Omega)\sin(\Omega t)}|\mathrm{L/R}\rangle, \quad \epsilon^{(0)} = 0 \quad \text{(degenerate!)}$$

→ degenerate perturbation theory

Example I: CDT — perturbation theory

Diagonalize $H_0 = -\frac{\Delta}{2}\sigma_x$ in degenerate subspace

- compute all matrix elements ($\ell = L, R$)

$$\langle\langle\phi_\ell|H_0|\phi_{\ell'}\rangle\rangle = \frac{1}{T}\int_0^T dt\, \langle\phi_\ell(t)|H_0|\phi_{\ell'}(t)\rangle = \begin{cases} 0 & \text{for } \ell = \ell' \\ -\frac{\Delta}{2}J_0(A/\Omega) & \text{for } \ell \neq \ell' \end{cases}$$

  Bessel function $J_n(x)$: nth Fourier coefficent of $e^{-ix\sin(\Omega t)}$

- diagonalize the resulting matrix

$$-\frac{\Delta}{2}J_0(A/\Omega)\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \equiv -\frac{\tilde{\Delta}}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{with } \tilde{\Delta} = \Delta J_0(A/\Omega)$$

➔ eigenvalues $\pm\frac{\tilde{\Delta}}{2}$ and eigenvectors $|\phi_L\rangle\rangle \pm |\phi_R\rangle\rangle$

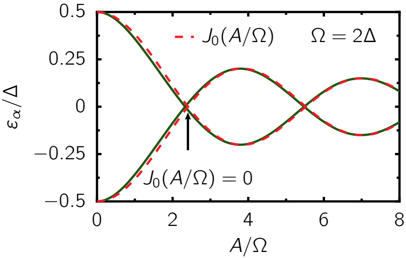
Example I: CDT — perturbation theory

Floquet states

$$|\phi_\pm\rangle\rangle = \frac{|\phi_L\rangle\rangle \pm |\phi_R\rangle\rangle}{\sqrt{2}}$$

quasienergies

$$\pm\frac{\Delta}{2}J_0(A/\Omega)$$



for $J_0(A/\Omega) = 0$: tunneling matrix element vanishes

Grossmann *et al.*, EPL 1992

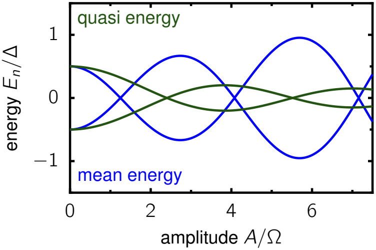
Example I: CDT — mean energies

- Rapoport relation

$$E = \epsilon - \Omega \frac{\partial \epsilon}{\partial \Omega}$$

- Bessel function $J_1 = -J_0'$:

$$E_\pm = \pm \frac{\Delta}{2} \left[J_0(A/\Omega) - \frac{A}{\Omega} J_1(A/\Omega) \right]$$



Example II: Rabi problem beyond RWA

$$H(t) = -\frac{\Delta}{2}\sigma_z + \frac{A}{2}\cos(\Omega t)\sigma_x$$

- close to resonance: $\delta = \Delta - \Omega \ll \Delta$, small amplitude: $A \ll \Delta$

Example II: Rabi problem beyond RWA

$$H(t) = -\frac{\Delta}{2}\sigma_z + \frac{A}{2}\cos(\Omega t)\sigma_x$$

- close to resonance: $\delta = \Delta - \Omega \ll \Delta$, small amplitude: $A \ll \Delta$

$$\mathcal{H}_0 = \frac{\Omega}{2}\sigma_z - i\frac{\partial}{\partial t} \qquad \mathcal{H}_1 = \frac{\delta}{2}\sigma_z + \frac{A}{2}\cos(\Omega t)\sigma_x$$

- degenerate perturbation theory ➔ two-level Floquet Hamiltonian

$$\mathcal{H} \approx \frac{1}{2}\begin{pmatrix} \delta + A^2/4\Omega & A \\ A & -\delta - A^2/4\Omega \end{pmatrix}$$

Rabi Hamiltonian
beyond RWA

- quasienergy splitting: $\epsilon_2 - \epsilon_1 = \bar{\omega}$, where $\bar{\omega}^2 = \delta^2 + A^2 + \frac{A^2\delta}{8\Omega}$
- absorption maximum at

$$\Omega_{\text{res}} \approx \Delta + \frac{A^2}{16\Delta}$$

Bloch-Siegert shift

Some standard references

- Classic work:
  - Shirley, Phys. Rev. 138, B979 (1965)
  - Sambe, Phys. Rev. A 7, 2203 (1973)
- Reviews:
  - Grifoni, Hänggi, Phys. Rep. 304, 229 (1998)
  - Hänggi, Chap.5 of “Quantum transport and dissipation” (1998)
    http://www.physik.uni-augsburg.de/theo1/hanggi/Papers/Chapter5.pdf
  - Eckardt, Rev. Mod. Phys. 89, 011004 (2017)

1. **Schrödinger equation**
   - Geometric phases
   - Time-periodicity, Floquet ansatz, and all that
2. **Quantum dissipation**
   - System-bath model
   - Floquet-Bloch-Redfield formalism
3. **Application: LZSM Interference**
4. **Miscellaneous**
   - Quantum transport
   - Time-periodic Liouvillians
   - Bichromatic driving

Heuristic approach

coupling of qubit to electromagnetic environment ➔ sponaneous decay

$$|\psi\rangle \longrightarrow \begin{cases} \sigma_-|\psi\rangle & \text{decay with probability } \alpha \ll 1 \\ |\psi\rangle + |\delta\psi\rangle & \text{no decay, probability } 1-\alpha \end{cases}$$

- normalization requires $|\delta\psi\rangle = \frac{\alpha}{2}\sigma_+\sigma_-|\psi\rangle$

Heuristic approach

coupling of qubit to electromagnetic environment ➔ sponaneous decay

$$|\psi\rangle \longrightarrow \begin{cases} \sigma_-|\psi\rangle & \text{decay with probability } \alpha \ll 1 \\ |\psi\rangle + |\delta\psi\rangle & \text{no decay, probability } 1-\alpha \end{cases}$$

- normalization requires $|\delta\psi\rangle = \frac{\alpha}{2}\sigma_+\sigma_-|\psi\rangle$
- corresponding density operator

$$\rho \longrightarrow \rho + \frac{\alpha}{2}\Big(2\sigma_-\rho\sigma_+ - \sigma_+\sigma_-\rho - \rho\sigma_+\sigma_-\Big)$$

- add continuous time-evolution ➔ master equation

$$\frac{d}{dt}\rho = -i[H,\rho] + \frac{\gamma}{2}\Big(2\sigma_-\rho\sigma_+ - \sigma_+\sigma_-\rho - \rho\sigma_+\sigma_-\Big)$$

Lindblad form

Time evolution must conserve

- hermiticity and trace of ρ
- positivity (all eigenvalues of $\rho \geq 0$)

Lindblad form

Time evolution must conserve

- hermiticity and trace of ρ
- positivity (all eigenvalues of $\rho \geq 0$)

Fulfilled by a Markovian master equation iff of "Lindblad form"

$$\frac{d}{dt}\rho = -i[H, \rho] + \sum_n \gamma_n \left(2Q_n \rho Q_n^\dagger - Q_n^\dagger Q_n \rho - \rho Q_n^\dagger Q_n \right)$$

G. Lindblad, Comm. Math. Phys. **48**, 119 (1976)

V. Gorini, J. Math. Phys. **17**, 821 (1976)

- Interpretation: incoherent transitions $|\psi\rangle \rightarrow Q_n|\psi\rangle$

Lindblad form

Time evolution must conserve

- hermiticity and trace of ρ
- positivity (all eigenvalues of $\rho \geq 0$)

Fulfilled by a Markovian master equation iff of “Lindblad form”

$$\frac{d}{dt}\rho = -i[H,\rho] + \sum_n \gamma_n \left(2Q_n \rho Q_n^\dagger - Q_n^\dagger Q_n \rho - \rho Q_n^\dagger Q_n \right)$$

G. Lindblad, Comm. Math. Phys. **48**, 119 (1976)

V. Gorini, J. Math. Phys. **17**, 821 (1976)

- Interpretation: incoherent transitions $|\psi\rangle \to Q_n|\psi\rangle$

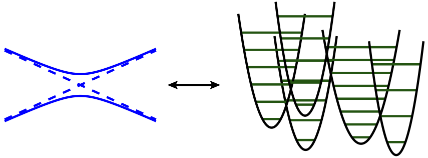
✗ Critique

- request for Markovian evolution unphysical
- axiomatic, not based on physical model
- high-temperature limit typically wrong
  i.e. not the Klein-Kramers or the Smoluchowski equation

Caldeira-Leggett model

Magalinskii 1959; Caldeira, Leggett 1981

Coupling of a system to bath of harmonic oscillators



$$H = H_{\text{system}}(t) + X \sum_{\nu} \gamma_\nu (b_\nu^\dagger + b_\nu) + \sum_\nu \omega_\nu b_\nu^\dagger b_\nu$$

→ eliminate bath

→ equation of motion for reduced density operator

- interpretation: bath “measures” system operator X

Bloch-Redfield theory

Total density operator $R \approx \rho \otimes \rho_{\text{bath,eq}}$

$$\dot{R} = -i[H_{\text{total}}, R]$$

2nd order perturbation theory in system-bath coupling

$$\frac{d}{dt}\rho = -i[H_{\text{sys}}, \rho] - i\int_0^{(t-t_0)\to\infty} d\tau \mathcal{A}(\tau)[X, [\tilde{X}(-\tau), \rho(t-\tau)]_+] - \int_0^{(t-t_0)\to\infty} d\tau \mathcal{S}(\tau)[X, [\tilde{X}(-\tau), \rho(t-\tau)]]$$

- Heisenberg operator $\tilde{X}(-\tau) = U(\tau)\, X\, U^\dagger(\tau)$
- bath correlation functions $\mathcal{A}$, $\mathcal{S}$
- non-Markovian
- short system-bath correlation time: Markov approximation

Bath correlation functions

- anti-symmetric correlation function

$$\mathcal{A}(\tau) = -i\langle[\xi(\tau), \xi(0)]\rangle$$

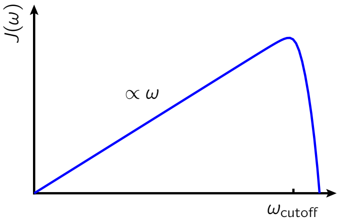
- Fourier transformed: spectral density $\longrightarrow$ continuum limit

$$\mathcal{A}(\omega) = \pi \sum_{\nu} |\gamma_\nu|^2 \delta(\omega - \omega_\nu) \longrightarrow J(\omega)$$

- here: Ohmic with cutoff

$$J(\omega) = 2\pi\alpha\omega e^{-\omega/\omega_{\mathrm{cutoff}}}$$

- dimensionless dissipation strength α



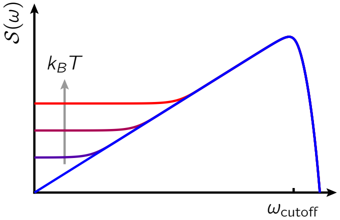
Bath correlation functions

- symmetric bath correlation function

$$\mathcal{S}(\tau) = \frac{1}{2}\langle[\xi(\tau),\xi(0)]_+\rangle$$

$$\mathcal{S}(\omega) = J(\omega)\coth\left(\frac{\omega}{2k_BT}\right)$$

$$= \begin{cases} 4\pi\alpha k_\mathrm{B}T & \text{high } k_BT \\ 2\pi\alpha\omega & \text{low } k_BT \end{cases}$$



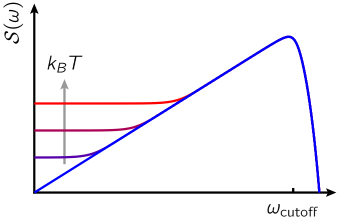
Bath correlation functions

- symmetric bath correlation function

$$\mathcal{S}(\tau) = \frac{1}{2}\langle[\xi(\tau),\xi(0)]_+\rangle$$

$$\mathcal{S}(\omega) = J(\omega)\coth\left(\frac{\omega}{2k_BT}\right)$$

$$= \begin{cases} 4\pi\alpha k_\mathrm{B}T & \text{high } k_BT \\ 2\pi\alpha\omega & \text{low } k_BT \end{cases}$$



- $\mathcal{S}(\omega)$ evaluated at transition frequencies

➔ dissipation strength depends on coherent spectrum/dynamics

Born-Markov approximation

- Ohmic, short memory times (e.g. for $\gamma < k_\mathrm{B} T$)
  ➔ Bloch-Redfield master equation

$$\dot{\rho} = -i[H_\mathrm{S}, \rho] + i\gamma[X, \{[H_\mathrm{S}, X], \rho\}] - [X, [Q, \rho]]$$

coherent dynamics dissipation decoherence

coherent dynamics enters via $Q = \int_0^\infty d\tau\, \mathcal{S}(\tau)\, \tilde{X}(-\tau)$

Born-Markov approximation

- Ohmic, short memory times (e.g. for $\gamma < k_{\mathrm{B}} T$)
  ➔ Bloch-Redfield master equation

$$\dot{\rho} = -i[H_{\mathrm{S}}, \rho] + i\gamma[X, \{[H_{\mathrm{S}}, X], \rho\}] - [X, [Q, \rho]]$$

coherent dynamics   dissipation   decoherence

coherent dynamics enters via $Q = \int_0^\infty d\tau\, \mathcal{S}(\tau)\, \tilde{X}(-\tau)$

- not of Lindblad form
  - ✗ positivity might be violated
  - ✓ happens only on unphysically small time scales
- high-temperature limit: Fokker-Planck equation

Pauli master equation

- Decomposition into energy basis and rotating-wave approximation

➔ rate equation for the populations (Pauli master equation)

$$\frac{d}{dt}\rho_{\alpha\alpha} = \sum_{\alpha'} \left[w_{\alpha\leftarrow\alpha'}\, \rho_{\alpha'\alpha'} - w_{\alpha'\leftarrow\alpha}\, \rho_{\alpha\alpha} \right]$$

with the golden-rule rates

$$w_{\alpha\leftarrow\alpha'} = J(E_\alpha - E_{\alpha'}) \left| \langle \phi_\alpha | X | \phi_{\alpha'} \rangle \right|^2 n_{\text{th}}(E_\alpha - E_{\alpha'})$$

- notice: $-n_{\text{th}}(-\omega) = n_{\text{th}}(\omega) + 1$

Pauli master equation

- Decomposition into energy basis and rotating-wave approximation

➔ rate equation for the populations (Pauli master equation)

$$\frac{d}{dt}\rho_{\alpha\alpha} = \sum_{\alpha'} \left[w_{\alpha\leftarrow\alpha'}\, \rho_{\alpha'\alpha'} - w_{\alpha'\leftarrow\alpha}\, \rho_{\alpha\alpha} \right]$$

with the golden-rule rates

$$w_{\alpha\leftarrow\alpha'} = J(E_\alpha - E_{\alpha'})\, \left|\langle\phi_\alpha|X|\phi_{\alpha'}\rangle\right|^2 n_{\mathrm{th}}(E_\alpha - E_{\alpha'})$$

- notice: $-n_{\mathrm{th}}(-\omega) = n_{\mathrm{th}}(\omega)+1$

✓ fluctuation theorem $\dfrac{w_{\alpha\leftarrow\alpha'}}{w_{\alpha'\leftarrow\alpha}} = e^{-(E_\alpha - E_{\alpha'})/k_B T}$

✓ Lindblad form

✗ high-temperature limit typically wrong

$\boxed{\text{full Bloch-Redfield: golden rule for non-diagonal } \rho_{\alpha\beta}}$

Floquet-Bloch-Redfield master equation

Driven system ➔ decoherence becomes time-dependent

$$\dot{\rho} = \ldots - [X, [Q(t), \rho]], \quad Q(t) = \int_0^\infty d\tau\, \mathcal{S}(\tau)\, \tilde{X}(t-\tau, t)$$

Floquet-Bloch-Redfield master equation

Driven system ➔ decoherence becomes time-dependent

$$\dot{\rho} = \ldots - [X, [Q(t), \rho]], \quad Q(t) = \int_0^\infty d\tau\, \mathcal{S}(\tau)\, \tilde{X}(t-\tau, t)$$

Central idea:

1. adapted basis: Floquet states $|\phi_\alpha(t)\rangle$ ➔ captures coherent dynamics
2. master equation in Floquet basis

$$\frac{d}{dt}\rho_{\alpha\beta} = -i(\epsilon_\alpha - \epsilon_\beta)\rho_{\alpha\beta} + \sum_{\alpha'\beta'} \mathcal{L}_{\alpha\beta,\alpha'\beta'}(t)\, \rho_{\alpha'\beta'}$$

   where $\mathcal{L}(t) = \mathcal{L}(t+T)$

3. moderate rotating-wave approximation:
   time average $\mathcal{L}(t) \to \bar{\mathcal{L}}$, but keep all $\rho_{\alpha\beta}$

Blümel *et al.*, PRA 1991; SK, Dittrich, Hänggi, PRE 1997

- Numerical method: compute $\mathcal{L}$ and solve

$$\dot{\rho}_{\alpha\beta} = -i(\epsilon_\alpha - \epsilon_\beta)\rho_{\alpha\beta} + \sum_{\alpha'\beta'} \bar{\mathcal{L}}_{\alpha\beta,\alpha'\beta'}\, \rho_{\alpha'\beta'}$$

  1. time-independent master equation for driven system
  2. ac driving captured by choice of basis ➔ efficient
  3. includes impact of bath on dissipation strength
     (very relevant for fermionic baths; see Lecture III)

- Analytical tool: find H_{eff} and approx. for $\overline{Q(t)}$

  ➔ effective time-independent Bloch-Redfield equation

➔ full RWA ➔ (Pauli master equation)

$$\frac{d}{dt}\rho_{\alpha\alpha} = \sum_{\alpha'} w_{\alpha\leftarrow\alpha'}\ \rho_{\alpha'\alpha'} - \sum_{\alpha} w_{\alpha'\leftarrow\alpha}\ \rho_{\alpha\alpha}$$

with the golden-rule rates

$$w_{\alpha\leftarrow\alpha'} = \sum_k J(\epsilon_\alpha - \epsilon_{\alpha'} + k\Omega) \left| \sum_{k'} \langle \phi_{\alpha,k+k'} | X | \phi_{\alpha',k} \rangle \right|^2 n_{\mathrm{th}}(\epsilon_\alpha - \epsilon_{\alpha'} + k\Omega)$$

- sidebands contribute to $w_{\alpha\leftarrow\alpha'}$

  ... but NOT as independent states!
- no simple relation between forward/backward rates

Summary: Floquet-Bloch-Redfield equation

- master equation based on Floquet states
  - ✓ efficient basis
  - ✓ captures dissipative phase lag
- driving affects decoherence
- for driven systems, the system-bath coupling operator matters

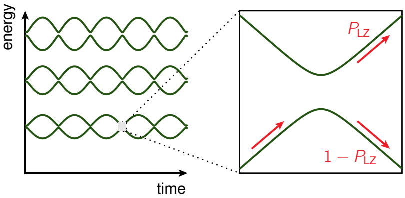
Homework

1. derive the BR equation for the harmonic oscillator
2. two-level system with resonant driving — Blattmann, PRA 91, 042109 (2015)
   1. derive the effective Hamiltonian
   2. derive the equation of motion for the Bloch vector
3. compute the effective spectral density for dynamical decoupling

1. **Schrödinger equation**
   - Geometric phases
   - Time-periodicity, Floquet ansatz, and all that
2. **Quantum dissipation**
   - System-bath model
   - Floquet-Bloch-Redfield formalism
3. **Application: LZSM Interference**
4. **Miscellaneous**
   - Quantum transport
   - Time-periodic Liouvillians
   - Bichromatic driving

AC-driving and Landau-Zener transitions

Quantum system in AC-field, $H(t)$

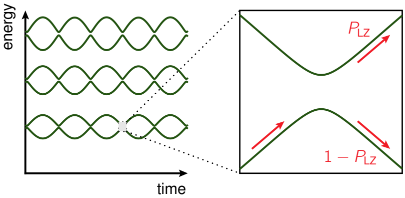


- non-adiabatic transition probability

$$P_{\rm LZ} = e^{-\pi\Delta^2/2\hbar v}$$

Landau, Zener, Stückelberg, Majorana, 1932

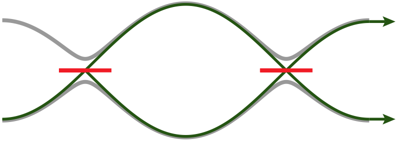
Quantum system in AC-field, $H(t)$



- non-adiabatic transition probability

$$P_{\mathrm{LZ}} = e^{-\pi\Delta^2/2\hbar v}$$

Landau, Zener, Stückelberg, Majorana, 1932



➔ beam splitter, interference

➔ Landau-Zener-(Stückelberg-Majorana) interferometry

Interference condition

$$H(t) = \frac{\Delta}{2}\sigma_x + \frac{g(t)}{2}\sigma_z \qquad g(t) = \epsilon + A\cos(\Omega t)$$

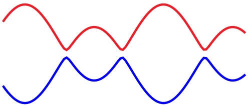
1. (avoided) crossing requires $A > |\epsilon|$

Interference condition

$$H(t) = \frac{\Delta}{2}\sigma_x + \frac{g(t)}{2}\sigma_z \qquad g(t) = \epsilon + A\cos(\Omega t)$$

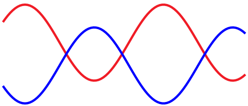
1. (avoided) crossing requires $A > |\epsilon|$
2. relative phase between dominant paths

adiabatic: $P_{\mathrm{LZ}} \ll 1$



$$\varphi(T) = \int_0^T dt\, |g(t)|$$

diabatic: $1 - P_{\mathrm{LZ}} \ll 1$



$$\varphi(T) = \int_0^T dt\, g(t)$$

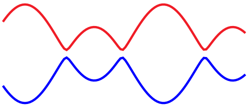
➔ fringes for $\varphi(T) = 2\pi k$

Interference condition

$$H(t) = \frac{\Delta}{2}\sigma_x + \frac{g(t)}{2}\sigma_z \qquad g(t) = \epsilon + A\cos(\Omega t)$$

1. (avoided) crossing requires $A > |\epsilon|$
2. relative phase between dominant paths

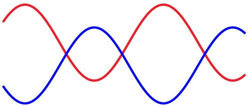
adiabatic: $P_{\mathrm{LZ}} \ll 1$



$$\varphi(T) = \int_0^T dt\, |g(t)|$$

➔ fringes for $\varphi(T) = 2\pi k$

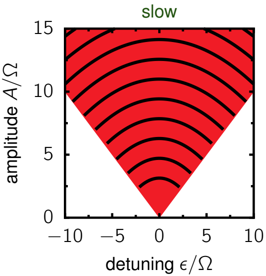
diabatic: $1 - P_{\mathrm{LZ}} \ll 1$

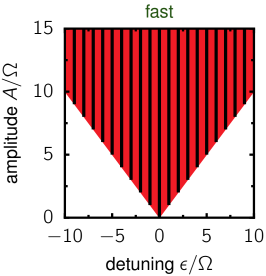


$$\varphi(T) = \int_0^T dt\, g(t) = \epsilon T$$

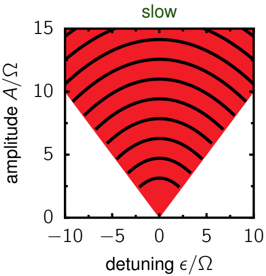
$\epsilon = k\Omega$ “k-photon resonance”

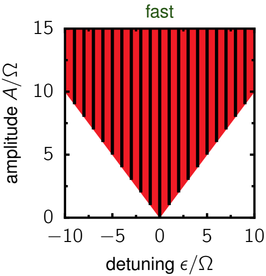
Patterns for two-level systems



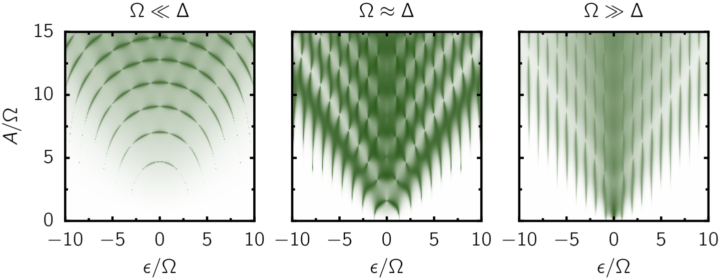


Patterns for two-level systems





E.g. excitation probabilities of a TLS:

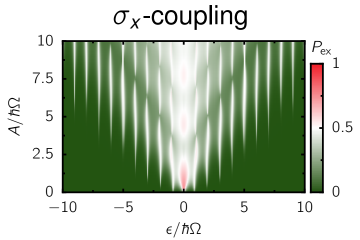


Dependence on system-bath coupling

$$H(t) = \frac{\Delta}{2}\sigma_x + \frac{1}{2}\big(\epsilon + A\cos(\Omega t)\big)\sigma_z \qquad +\sigma_x\xi \quad \text{or} \quad +\sigma_z\xi$$

Dependence on system-bath coupling

$$H(t) = \frac{\Delta}{2}\sigma_x + \frac{1}{2}\big(\epsilon + A\cos(\Omega t)\big)\sigma_z \qquad +\sigma_x\xi \quad \text{or} \quad +\sigma_z\xi$$

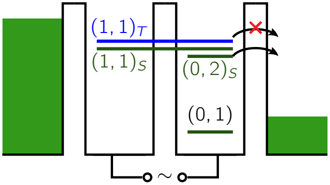


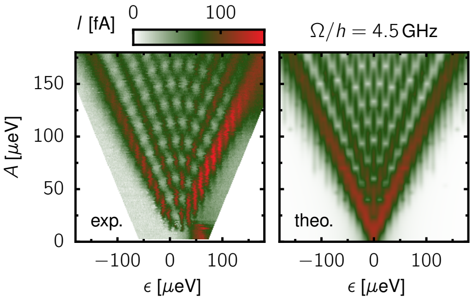
➔ driving & dissipation: not just decay towards ground state

Blattmann, Hänggi, SK, PRA 2015

DC current:

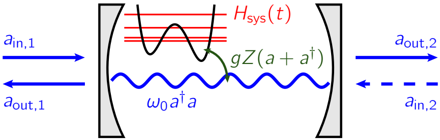
$$I(t) = e_0 \Gamma_R \langle n_R(t) \rangle \quad \rightarrow \quad \overline{\langle n_R(t) \rangle}^T$$





Forster *et al.*, PRL 2014

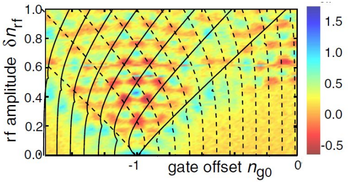
Measurement II: Cavity transmission



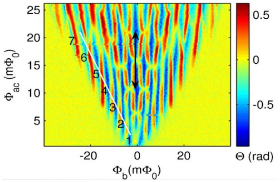
Dispersive frame:
effective cavity frequency

$$\omega_0 \longrightarrow \omega_0 + \frac{g^2}{\epsilon_{qb} - \omega_0}\sigma_z$$

➔ qubit excitation
(under certain conditions)

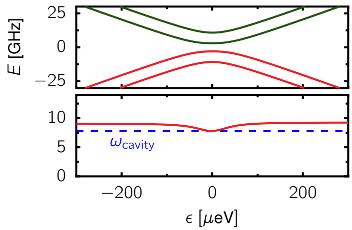


Sillanpää *et al.* PRL 2006



Izmalkov *et al.*, PRL 2008

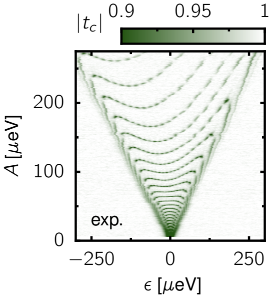
LZSM with silicon double quantum dots



Experiment

- Si DQD: four-level system
- splittings: $\sim$ 10 GHz
  driving: $\sim$ 0.1 GHz

➔ adiabatic, non-eq. population tiny



Heuristic finding

- resonance condition for arcs:

$$\bar{E}_1 - \bar{E}_0 = \omega_0 + k\Omega$$

➔ cavity must play a role

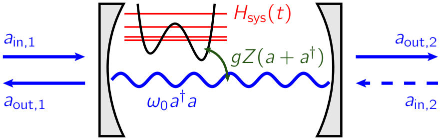
Mi, SK, Petta, PRB 2018

Recent experiments are beyond ...

- ... two-level system
- ... dispersive limit
- ... low-frequency cavity

➔ excitation probability $\neq$ cavity signal

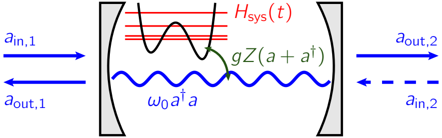
Dispersive readout of driven quantum systems



Mi, SK, Petta, Phys. Rev. B **98**, 161404(R) (2018)

SK, Phys. Rev. A **98**, 023849 (2018)

Qubit-cavity Hamiltonian



$$H = H_{sys}(t) + gZ(a^\dagger + a) + \omega_0 a^\dagger a$$

- Backaction: cavity ➔ qubit ➔ cavity
- Cavity equation (input/output formalism)

$$\frac{d}{dt}a = -i\omega_0 a - \frac{\kappa}{2}a - \sum_{\nu=1,2}\sqrt{\kappa_\nu}\,a_{in,\nu} - igZ$$

Dispersive readout — qubit response function

- (non equilibrium) Kubo formula $Z(t) = g \int dt' \, \chi(t-t') a(t')$ with the response function (may depend on the initial state!)

$$\chi(t) = -i\big\langle [Z(t), Z] \big\rangle \theta(t-t')$$

➔ $$-i\omega a = -i\big(\omega_0 + g^2\chi(\omega)\big) a - \frac{\kappa}{2} a - \sum_{\nu=1,2} \sqrt{\kappa_\nu} a_{\mathrm{in},\nu}$$

➔ measured quantity: (non-equilibrium) susceptibility

Response of periodically driven system

$$\chi(t,t') = -i\big\langle [Z(t), Z(t')] \big\rangle_{\text{non-eq}} = \chi(t+T, t'+T)$$

such that

$$\chi(t, t-\tau) = \sum_k e^{-ik\Omega t} \int d\omega\, e^{-i\omega\tau} \chi^{(k)}(\omega)$$

Response function for driven system

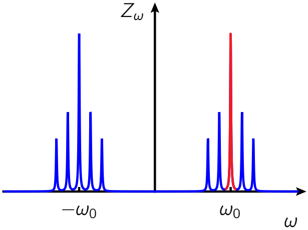
Response of periodically driven system

$$\chi(t,t') = -i\langle [Z(t), Z(t')] \rangle_{\text{non-eq}} = \chi(t+T, t'+T)$$

such that

$$\chi(t, t-\tau) = \sum_k e^{-ik\Omega t} \int d\omega\, e^{-i\omega\tau} \chi^{(k)}(\omega)$$

- resonant cavity driving, $\omega = \omega_0$
- response $Z(t)$ acquires sidebands
- good cavity limit, $\kappa \ll \omega_0, \Omega$



Response function for driven system

Relevant component:

$$\chi^{(0)}(\omega_0) = \sum_{\beta,\alpha,k} \frac{(p_\alpha - p_\beta)|Z_{\beta\alpha,k}|^2}{\epsilon_\alpha - \epsilon_\beta + \omega_0 + k\Omega + i\gamma/2}$$

- Floquet theory ➔ quasi-energies ϵ_α
- Floquet-Bloch-Redfield ➔ populations p_α

Response function for driven system

Relevant component:

$$\chi^{(0)}(\omega_0) = \sum_{\beta,\alpha,k} \frac{(p_\alpha - p_\beta)|Z_{\beta\alpha,k}|^2}{\epsilon_\alpha - \epsilon_\beta + \omega_0 + k\Omega + i\gamma/2}$$

- Floquet theory ➔ quasi-energies ϵ_α
- Floquet-Bloch-Redfield ➔ populations p_α

Resonance conditions

cavity response:

$$\Delta\epsilon = \omega_0 + k\Omega$$

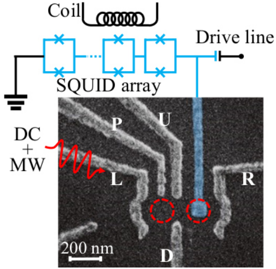
cf. population:

$$\Delta\epsilon = k\Omega$$

e.g. Ivakhnenko *et al.*, Phys.Rep. 2023

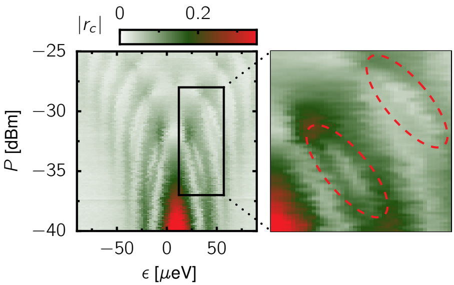
➔ Agree only for low-frequency oscillator!

Readout of Floquet state population



Chen *et al.*, Phys. Rev. B **103**, 205428 (2021)

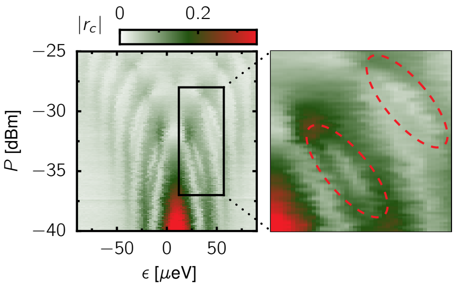
Motivation: Holes in interference fringes



Experiment (Cao & Guo, Hefei)

- holes in LZSM pattern
- GaAs DQD ➔ two-level sys.

Motivation: Holes in interference fringes



Experiment (Cao & Guo, Hefei)

- holes in LZSM pattern
- GaAs DQD ➔ two-level sys.

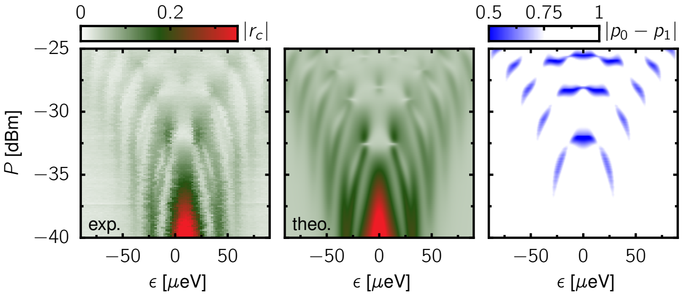
Recap: susceptibility (two-level system)

$$\chi^{(0)}(\omega_0) = (p_0 - p_1) \sum_k \frac{|Z_{10,k}|^2}{\epsilon_1 - \epsilon_0 + \omega_0 + k\Omega + i\gamma/2}$$

response determined by

- resonance condition for cavity signal
- Floquet state population

Experiment *vs.* Theory



- competing resonance conditions
- holes in fringes when $p_0 \approx p_1 \approx 1/2$

➔ cavity response provides information about Floquet state population

Mean energy state *vs.* Floquet-Gibbs state

Steady state of driven dissipative quantum system

quasi energy: $p_\alpha \propto e^{-\epsilon_\alpha/kT}$ mean energy: $p_\alpha \propto e^{-E_\alpha/kT}$

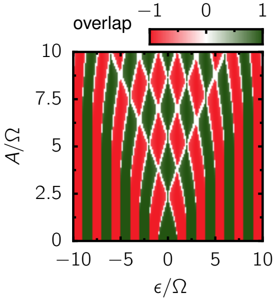
Mean energy state *vs.* Floquet-Gibbs state

Steady state of driven dissipative quantum system

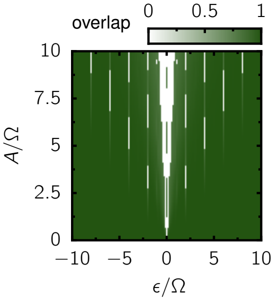
quasi energy: $p_\alpha \propto e^{-\epsilon_\alpha/kT}$

mean energy: $p_\alpha \propto e^{-E_\alpha/kT}$

bath via σ_z



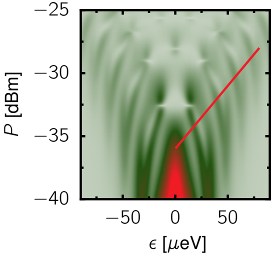
bath via σ_x

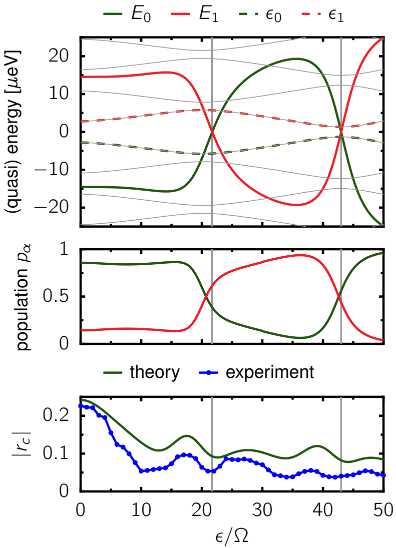


Floquet-Gibbs state
vs. anti Floquet-Gibbs

The present case!

Floquet state population





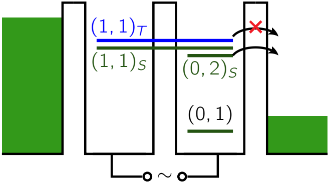
- p_α determined by E_α

→ holes in reflection consistent with mean-energy state

→ bath coupling (predominantly) via σ_x

1. **Schrödinger equation**
   - Geometric phases
   - Time-periodicity, Floquet ansatz, and all that
2. **Quantum dissipation**
   - System-bath model
   - Floquet-Bloch-Redfield formalism
3. **Application: LZSM Interference**
4. **Miscellaneous**
   - Quantum transport
   - Time-periodic Liouvillians
   - Bichromatic driving

Dissipation *vs.* Transport



- Floquet states for central system
- evaluate rates $w_{\alpha \leftarrow \alpha'}$
- ➔ dc current

	Dissipation	Transport
Environment	harmonic oscillators	electron source/drain
Coupling of mode ν	$X(a_\nu^\dagger + a_\nu)$	$c^\dagger c_\nu + c_\nu^\dagger c$
Absorption / tunnel in	$n_{\text{th}}(\omega)$	$f(\epsilon - \mu)$
Emission / tunnel out	$1 + n_{\text{th}}(\omega)$	$1 - f(\epsilon - \mu)$
“Ohmic”	$J(\omega) \propto \omega$	$\Gamma(\omega) = \text{const}$

Periodically time-dependent Liouvillians

Master equation of type

$$\frac{d}{dt}P = L(t)P$$

- Floquet-Bloch-Redfield beyond moderate RWA
- time-dependent system with Lindblad dissipator
  $\dot{\rho} = -i[H(t), \rho] + \gamma(2a^\dagger \rho a - a^\dagger a \rho - \rho a^\dagger a)$
  - very weak dissipation
  - transport problem with large bias

➔ long-time solution T-periodic

➔ Floquet ansatz with “quasienergy” zero

$$P(t) = \sum_k e^{-ik\Omega t} p_k$$

$\frac{d}{dt}P = L(t)P$ with

$$L(t) = L_0 + 2L_1\cos(\Omega t)$$

➔ kernel of tridiagonal Floquet matrix

$$L_0 + 2L_1\cos(\Omega t) - \partial_t \leftrightarrow \begin{pmatrix} \ddots & \vdots & \vdots & \vdots & \vdots & \vdots & \\ \cdots & L_0 + 2i\Omega & L_1 & 0 & 0 & 0 & \cdots \\ \cdots & L_1 & L_0 + i\Omega & L_1 & 0 & 0 & \cdots \\ \cdots & 0 & L_1 & L_0 & L_1 & 0 & \cdots \\ \cdots & 0 & 0 & L_1 & L_0 - i\Omega & L_1 & \cdots \\ \cdots & 0 & 0 & 0 & L_1 & L_0 - 2i\Omega & \cdots \\ & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}$$

Matrix-continued fractions

- ansatz $P(t) = \sum_k e^{-ik\Omega t} p_k$ yields

$$L_1 p_{k-1} + (L_0 + ik\Omega) p_k + L_1 p_{k+1} = 0$$

- idea: truncate and iterate $p_{k-1} = -L_1^{-1} \left\{ (L_0 - ik\Omega) p_k + L_1 p_{k+1} \right\}$

Matrix-continued fractions

- ansatz $P(t) = \sum_k e^{-ik\Omega t} p_k$ yields

$$L_1 p_{k-1} + (L_0 + ik\Omega) p_k + L_1 p_{k+1} = 0$$

- idea: truncate and iterate $p_{k-1} = -L_1^{-1}\left\{(L_0 - ik\Omega)p_k + L_1 p_{k+1}\right\}$
  - ✗ fails, L_1 generally singular

Matrix-continued fractions

- ansatz $P(t) = \sum_k e^{-ik\Omega t} p_k$ yields

$$L_1 p_{k-1} + (L_0 + ik\Omega) p_k + L_1 p_{k+1} = 0$$

- idea: truncate and iterate $p_{k-1} = -L_1^{-1}\left\{(L_0 - ik\Omega) p_k + L_1 p_{k+1}\right\}$
  - ✗ fails, L_1 generally singular
- solution: ansatz $p_k = S_k L_1 p_{k\mp 1}$ ($k \gtrless 0$) leads to

$$S_k = -\left(L_0 + ik\Omega + L_1 S_{k\pm 1} L_1\right)^{-1} \quad \longrightarrow \quad S_{\pm 1} \tag{1}$$

$$0 = \left(L_1 S_{-1} L_1 + L_0 + L_1 S_1 L_1\right) p_0 \tag{2}$$

➔ truncate at $\pm k_0$, iterate (1), and solve (2)

➔ time-averaged $P(t) = p_0$ ➔ time-averaged expectation values

Risken, "The Fokker-Planck Equation"

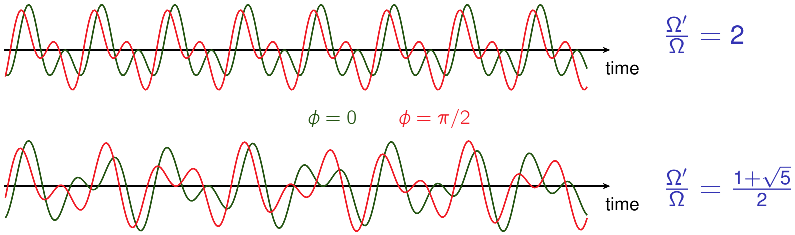
Appendix of Forster *et al.*, PRB 2015

Bichromatic driving

- $g(t) = \cos(\Omega t) + \eta \cos(\Omega' t + \phi)$

  Ω', Ω commensurable *vs.* incommensurable

  ➔ $g(t)$ periodic *vs.* quasi-periodic



Quantum master equation

$$\frac{d}{dt}\rho = L(t)\rho \quad \text{with} \quad L(t) = L_0 + L_1 \cos(n\Omega t) + L_1' \cos(n'\Omega t)$$

Quantum master equation

$$\frac{d}{dt}\rho = L(t)\rho \quad \text{with} \quad L(t) = L_0 + L_1 \cos(n\Omega t) + L_1' \cos(n'\Omega t)$$

- long-time solution periodic, “Floquet solution with eigenvalue 0”

$$\rho(t) = \rho(t + 2\pi/\Omega) = \sum_k e^{-ik\Omega t}\rho_k$$

→ homogeneous set of equations for ρ_k

→ ρ_0, time-averaged expectation values

Two-color Floquet theory (incommensurable)

- $\frac{d}{dt}\rho = L(t)\rho$ with

$$L(t) = L_0 + L_1 \cos(\Omega t) + L_1' \cos(\omega t)$$

- auxiliary angular coordinate $\omega t \longrightarrow \theta$

$$\frac{d}{dt}\mathcal{P} = \mathcal{L}(t,\theta)\mathcal{P}$$

$$\mathcal{L}(t,\theta) = L_0 + L_1 \cos(\Omega t) + L_1' \cos(\theta) - \omega \frac{\partial}{\partial \theta}$$

➔ $2\pi/\Omega$-periodic time-dependence

➔ solve by usual Floquet tools — here: matrix-continued fractions

Two-color Floquet theory (incommensurable)

- $\frac{d}{dt}\rho = L(t)\rho$ with

$$L(t) = L_0 + L_1 \cos(\Omega t) + L_1' \cos(\omega t)$$

- auxiliary angular coordinate $\omega t \longrightarrow \theta$

$$\frac{d}{dt}\mathcal{P} = \mathcal{L}(t, \theta)\mathcal{P}$$

$$\mathcal{L}(t, \theta) = L_0 + L_1 \cos(\Omega t) + L_1' \cos(\theta) - \omega \frac{\partial}{\partial \theta}$$

➔ $2\pi/\Omega$-periodic time-dependence

➔ solve by usual Floquet tools — here: matrix-continued fractions

- connection $\rho(t) = \mathcal{P}(t, \theta)\big|_{\theta=\omega t}$

cf. t-t' formalism, see Peskin & Moiseyev, J.Chem.Phys. 1993

Summary

- Berry phase and its non-adiabatic generalization
- Floquet theory
  - long-time dynamics: quasienergies $\leftrightarrow$ geometric phase
  - within driving period: Floquet states
- Floquet-Bloch-Redfield theory
  - Floquet $+$ Bloch-Redfield
  - time-independent master equation
  - stationary state
  - susceptibility
- Various
  - readout
  - transport
  - matrix-continued fractions
  - bichromatic drive

Thanks to . . .

Experiments by

- Stefan Ludwig (PDI Berlin)
- Jason R. Petta (UCLA)
- Guo-Ping Guo & Gang Cao (Hefei)
- Mark Buitelaar (UC London)

GOBIERNO DE ESPAÑA
MINISTERIO DE ECONOMÍA Y COMPETITIVIDAD
CSIC
CONSEJO SUPERIOR DE INVESTIGACIONES CIENTÍFICAS
QTEP
Quantum Technologies Platform