Opinion Dynamics on Signed Graphs

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Abstract—A signed graph offers richer information than an unsigned graph, since it describes both collaborative and competitive relationships in social networks. In this paper, we study the opinion dynamics on a signed graph, based on the Friedkin-Johnsen model. We first interpret the equilibrium opinion in terms of a defined random walk on an augmented signed graph, by representing the equilibrium opinion of every node as a combination of all nodes' internal opinions, with the coefficient of the internal opinion for each node being the difference of two absorbing probabilities. We then quantify some relevant social phenomena and express them in terms of the ℓ_2 norms of vectors. We also design a nearly-linear time signed Laplacian solver for assessing these quantities, by establishing a connection between the absorbing probability of random walks on a signed graph and that on an associated unsigned graph. We further study the opinion optimization problem by changing the initial opinions of a fixed number of nodes. We show that it can be optimally solved in cubic time, and then provide a nearly-linear time algorithm with error guarantee to approximately solve the problem. Finally, we execute extensive experiments on sixteen real-life signed networks, which show that both of our algorithms are effective and efficient, and are scalable to massive graphs with over 20 million nodes.

Index Terms—Opinion dynamics, signed graph, social network, graph algorithm, polarization, opinion optimization.

1 Introduction

Due to the ever-increasing availability of the power of computing, storage, and manipulation, in the past years, social media and online social networks have experienced explosive growth [1], which have constituted an important part of people's lives [2], leading to a substantial change of the ways that people exchange and form opinions [3], [4], [5] regarding voter, product marketing, social hotspots, and so on. Numerous recent studies have shown that online social networks and social media play a vital role during the whole process of opinion dynamics, including opinion diffusion, evolution, as well as formation [6], [7], [8]. In view of the practical relevance, opinion dynamics in social networks has received considerable recent attention from the scientific community, spanning various aspects of this dynamical process, such as modelling opinion formation, quantifying some resultant social phenomena (e.g., polarization [9], [10], disagreement [10], and conflict [11]), and optimizing opinion [12], [13], [14], [15].

The most important step in the study of opinion dynamics is probably the establishment of a mathematical model. Over the past years, a rich variety of models have been proposed for modelling opinion dynamics. Among these models, the Friedkin-Johnsen (FJ) model [16] is a popular one, which has found practical applications. For example, the concatenated FJ model has been applied to explain the Paris Agreement negotiation process [17] and the shift in public opinion on the US-Iraq War [18]. The FJ model has received much recent interest. It has been used to quantify various social phenomena such as disagreement [10], [19], polarization [9], [10], [19], and conflict [11], for which some nearly-linear time approximation algorithms were designed [20], [21]. Moreover, optimization of the overall opinion for the FJ model was also heavily studied [12], [13], [14], [15], [22], [23], [24]. Finally, diverse variants of this well-known model have been introduced [3], [20], by considering different factors affecting opinion formation, such as susceptibility to persuasion [13], peer pressure [25], stubbornness [21], and algorithmic filtering [26].

In addition to the aforementioned aspects, opinion shaping is also significantly affected by the interactions among individuals. Most previous studies for the FJ model only capture positive or cooperative interactions among agents described by an unsigned graph, neglecting those negative or competitive interactions, in spite of the fact that both the friendly and hostile relationships often exist simultaneously in many realistic scenarios, especially in social networks [27], [28]. In view of the ubiquity of competitive interactions in real systems, the FJ model on signed graphs has been built and studied [29], [30], [31], [32], which takes into account both collaborative and antagonistic relationships, providing a comprehensive understanding of human relationships in opinion dynamics [33]. However, the inclusion of antagonistic relationships presents more challenges to analyze relevant properties and develop good algorithms for the signed FJ model. For example, there is still no explanation for the equilibrium opinions. Also, measures and their expressions for some social phenomena (polarization, disagreement, and conflict) are not fully explored. Finally, existing fast approaches for opinion optimization on unsigned graphs are not applicable any longer [20], [21], while the prior algorithm for signed graphs has cubic complexity [29] and is thus not applicable to large signed graphs. These enlighten us to solve the challenges for the FJ model on signed graphs.

In order to fill the aforementioned gap, in this paper, we present an extensive study for the FJ model on a signed

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graph. Our main works and contributions are summarized as follows.

- We interpret the equilibrium expressed opinion in terms of a defined random walk on an augmented signed graph. We represent the expressed opinion of each node as a combination of the internal opinions of all nodes, where the coefficient of every node's internal opinion is the difference of two absorbing probabilities.
- We provide measures and expressions for some relevant social phenomena, including conflict, polarization, and disagreement. Particularly, we express each quantity in terms of the ℓ_2 norm of a vector.
- We construct an unsigned graph for any signed graph, and prove the equivalence of the absorbing probabilities for random walks on the augmented signed graph and its associated augmented unsigned graph. Utilizing this connection, we develop a nearly-linear time signed Laplacian solver approximating the equilibrium expressed opinion and relevant social phenomena.
- We address the problem of maximizing (or minimizing) the overall opinion by selecting a small fixed number of nodes and changing their initial opinions. We show that this problem can be optimally solved in cubic time. We also propose an approximation algorithm for solving the problem, which has nearly-linear time complexity and error guarantee, based on our proposed signed Laplacian solver.
- We demonstrate the performance of our two efficient algorithms by performing extensive experimentation on realistic signed graphs, which indicate that both algorithms are efficient and effective, scalable to graphs with over 20 million nodes.

2 Preliminaries

In this section, we provide a general overview of the notations, signed graphs and their related matrices, and the FJ model.

2.1 Notations

For a vector $\boldsymbol{a}$, the i-th element is denoted as $\boldsymbol{a}_i$. For a matrix $\boldsymbol{A}$, the element at the i-th row and j-th column is denoted as $\boldsymbol{A}_{i,j}$; the i-th row and j-th column are represented as $\boldsymbol{A}_{i,:}$ and $\boldsymbol{A}_{:,j}$, respectively. For matrix $\boldsymbol{A}$ and vector $\boldsymbol{a}$, $\boldsymbol{A}^\top$ and $\boldsymbol{a}^\top$ denote, respectively, their transpose. $\boldsymbol{I}$ and $\boldsymbol{O}$ denote, respectively, the identity matrix and the zero matrix of the appropriate dimension. The vector $\boldsymbol{e}_i$ is a vector of appropriate dimension, where the i-th element is 1 and all other elements are 0. The vector $\mathbf{0}$ (or $\mathbf{1}$) is a vector of appropriate dimension with all entries equal to 0 (or 1). For a vector $\boldsymbol{a}$, its ℓ_2 norm is $\|\boldsymbol{a}\|_2 = \sqrt{\sum_i \boldsymbol{a}_i^2}$, its ℓ_0 norm is denoted as $\|\boldsymbol{a}\|_0$ defined as the number of nonzero elements in $\boldsymbol{a}$, and its norm with respect to a matrix $\boldsymbol{A}$ is $\|\boldsymbol{a}\|_{\boldsymbol{A}} = \sqrt{\boldsymbol{a}^\top \boldsymbol{A}\boldsymbol{a}}$.

For two nonnegative scalars a and b and $0 < \epsilon < 1/2$, we say that a is an ϵ-approximation of b, denoted by $a \approx_\epsilon b$, if $(1-\epsilon)a \le b \le (1+\epsilon)a$. For two positive semidefinite matrices $\boldsymbol{X}$ and $\boldsymbol{Y}$, we say that $\boldsymbol{X} \preceq \boldsymbol{Y}$ if $\boldsymbol{Y} - \boldsymbol{X}$ is positive semidefinite, meaning that $\boldsymbol{x}^\top \boldsymbol{X}\boldsymbol{x} \le \boldsymbol{x}^\top \boldsymbol{Y}\boldsymbol{x}$ holds for any real vector $\boldsymbol{x}$.

2.2 Signed Graph and Their Related Matrices

Let $\mathcal{G} = (V, E, w)$ denote a connected undirected signed graph with $n = |V|$ nodes, $m = |E|$ edges, and edge sign function $w : E \to \{-1, +1\}$. We call an edge $e = (i,j) \in E$ a positive (or negative) edge if $w(i,j)$ is 1 (or -1). The edge sign $w(i,j)$ of edge $e = (i,j) \in E$ represents the relationship between node i and node j. If $w(i,j) = +1$, nodes i and j are cooperative; and if $w(i,j) = -1$, nodes i and j are competitive. Let N_i be the set of neighbours of node i, which can be classified into two disjoint subsets: the friend set N_i^F and the enemy (or opponent) set N_i^E. Any node in N_i^F is directly connected to i by a positive edge, while any node in N_i^E is linked to i by a negative edge. The degree d_i of node i is defined as $d_i = \sum_{j \in N_i} |w(i,j)|$.

For an undirected signed graph $\mathcal{G}$, let $\boldsymbol{D}$ be its degree matrix, which is a diagonal matrix defined as $\boldsymbol{D} = \mathrm{diag}\{d_1, d_2, \dots, d_n\}$. Let $\boldsymbol{A} \in \mathcal{R}^{n\times n}$ be its signed adjacency matrix, with the entry $\boldsymbol{A}_{i,j}$ defined as follows: $\boldsymbol{A}_{i,j} = w(i,j)$ for any edge $(i,j) \in E$, and $\boldsymbol{A}_{i,j} = 0$ otherwise. The signed Laplacian matrix $\boldsymbol{L}$ of $\mathcal{G}$ is defined as $\boldsymbol{L} = \boldsymbol{D} - \boldsymbol{A}$. For any edge in $\mathcal{G}$, we fix an arbitrary orientation. Then we can define the edge-node incidence matrix $\boldsymbol{B}_{m\times n}$ of $\mathcal{G}$, whose entries are defined as follows. For any edge $e = (u,v)$, $\boldsymbol{B}_{e,u} = 1$ if node u is the head of edge e; $\boldsymbol{B}_{e,v} = -w(u,v)$ if node v is the tail of e; and $\boldsymbol{B}_{e,x} = 0$ for $x \neq u, v$. Then, matrix $\boldsymbol{L}$ can be written as $\boldsymbol{L} = \boldsymbol{B}^\top \boldsymbol{B}$, implying that $\boldsymbol{L}$ is symmetric and positive semidefinite.

According to the sign of edges, the signed graph $\mathcal{G}$ can be divided into two spanning subgraphs: the positive graph $\mathcal{G}^+ = (V, E^+, w^+)$ and the negative graph $\mathcal{G}^- = (V, E^-, w^-)$. Both $\mathcal{G}^+$ and $\mathcal{G}^-$ share the same node set V as graph $\mathcal{G}$. $\mathcal{G}^+$ contains all positive edges in E, while $\mathcal{G}^-$ includes all negative edges in E. In other words, $E = E^+ \cup E^-$, and for any edge $e \in E$ with end nodes i and j, $w^+(i,j) = 1$ if $e \in E^+$, $w^-(i,j) = -1$ if $e \in E^-$, and $w^+(i,j) = w^-(i,j) = 0$ otherwise. Let $\boldsymbol{A}^+$ and $\boldsymbol{A}^-$ be the adjacency matrices of $\mathcal{G}^+$ and $\mathcal{G}^-$, respectively. Then, $\boldsymbol{A}^+$ is a non-negative matrix, while $\boldsymbol{A}^-$ is a non-positive matrix. Similarly, for graphs $\mathcal{G}^+$ and $\mathcal{G}^-$, we use $\boldsymbol{D}^+$ and $\boldsymbol{D}^-$ to denote their degree matrices, use $\boldsymbol{B}^+$ and $\boldsymbol{B}^-$ to represent their incidence matrices, and use $\boldsymbol{L}^+$ and $\boldsymbol{L}^-$ to denote their Laplacian matrices.

2.3 Friedkin-Johnsen Model on Signed Graphs

The Friedkin-Johnsen (FJ) model is a popular model for analysing opinion evolution and formation on graphs. In the FJ model [16], each node or agent $i \in V$ is associated with two opinions. One is the internal opinion $\boldsymbol{s}_i$, which is a constant value in the interval $[-1, 1]$, reflecting the intrinsic position of node i on a certain topic. The other is the expressed opinion $\boldsymbol{z}_i(t)$ at time t. At each time step, node i updates its expressed opinion according to the rule of minimizing its psycho-social cost $(\boldsymbol{z}_i(t) - \boldsymbol{s}_i)^2 + \sum_{j \in N_i} (\boldsymbol{z}_i(t) - \boldsymbol{A}_{i,j}\boldsymbol{z}_j(t))^2$, which is a function of its expressed opinion, its neighbours' expressed opinions, and its internal opinion [30], [31], [34]. To minimize this social cost function, each agent updates its expressed opinion as follows

$$\boldsymbol{z}_i(t+1) = \frac{\boldsymbol{s}_i + \sum_{j \in N_i} \boldsymbol{A}_{i,j}\boldsymbol{z}_j(t)}{1 + d_i}. \tag{1}$$

That is to say, the expressed opinion $\boldsymbol{z}_i(t+1)$ for node i at time $t+1$ is updated by averaging its internal opinion $\boldsymbol{s}_i$, the expressed opinions of its friends at time t, and the opposite expressed opinions of its opponents at time t. We define the initial opinion vector as $\boldsymbol{s} = (\boldsymbol{s}_1, \boldsymbol{s}_2, \ldots, \boldsymbol{s}_n)^\top$, and define the vector of expressed opinions at time t as $\boldsymbol{z}(t) = (\boldsymbol{z}_1(t), \boldsymbol{z}_2(t), \ldots, \boldsymbol{z}_n(t))^\top$. After long-time evolution, the expressed opinion vector converges to an equilibrium vector $\boldsymbol{z} = (\boldsymbol{z}_1, \boldsymbol{z}_2, \ldots, \boldsymbol{z}_n)^\top$ satisfying

$$\boldsymbol{z} = \lim_{t\to\infty} \boldsymbol{z}(t) = (\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}. \tag{2}$$

Although on both signed and unsigned graphs, the dynamics equation (1) and the equilibrium opinion (2) for the FJ model have the same form of mathematical expressions. There are some notable differences between the unsigned FJ model [16] and the signed FJ model [29], [31]. For example, their fundamental matrix [12] $(\boldsymbol{I}+\boldsymbol{L})^{-1}$ have obviously different properties. For an unsigned graph, matrix $(\boldsymbol{I}+\boldsymbol{L})^{-1}$ is a doubly stochastic matrix, with every element being positive [9], [12]. As a result, the expressed opinion of each node is a convex combination of the internal opinions of all nodes. In contrast, for a signed graph, $(\boldsymbol{I}+\boldsymbol{L})^{-1}$ is generally not a positive matrix and is thus not doubly stochastic. Thus, the expressed opinion of each node is not a convex combination of the internal opinions of all nodes. This poses an important challenge for computing expressed opinion vector and analyzing and optimizing relevant problems for opinion dynamics on signed graphs since existing algorithms for unsigned graphs are no longer applicable.

3 Interpretation of Expressed Opinion for the Signed FJ Model

The unsigned FJ model has been interpreted from various perspectives. For example, the opinion evolution process has been explained according to Nash equilibrium [34], and the equilibrium opinion has been accounted for based on absorbing random walks [12] or spanning diverging forests [20], [21], [24]. However, these interpretations do not hold for signed graphs anymore. In this section, we introduce absorbing random walks on signed graphs, based on which we provide an interpretation of the equilibrium expressed opinion of each node in terms of the absorbing probabilities, and highlight the parallels and distinctions between the FJ model on signed and unsigned graphs.

We begin by extending the signed graph $\mathcal{G} = (V, E, w)$ to an augmented graph $\mathcal{H} = (X, R, y)$ with absorbing states, defined as follows:

1) The node set X is defined as $X = V \cup \bar{V}$, where $\bar{V}$ is a set of n nodes such that for each node $i \in V$, there is a copy $\sigma(i) \in \bar{V}$;
2) The edge set R includes all the edges E of $\mathcal{G}$, plus a new set of edges between each node $i \in V$ and its copy $\sigma(i) \in \bar{V}$. That is, $R = E \cup \bar{E}$, where $\bar{E} = \{(i, \sigma(i)) | i \in V\}$;
3) The edge sign of each new edge $e = (i, \sigma(i)) \in \bar{E}$ is set to 1. For edge $e = (i, j) \in E$, its edge sign is equal to the sign of the corresponding edge in $\mathcal{G}$.

Neglecting the sign of each edge, we can define an absorbing random walk on this augmented graph $\mathcal{H}$, where nodes in V are transient nodes and nodes in $\bar{V}$ are absorbing nodes. Then, the transition matrix $\boldsymbol{P}$ for this absorbing random walk has the form

$$\boldsymbol{P} = \begin{bmatrix} (\boldsymbol{I}+\boldsymbol{D})^{-1}(\boldsymbol{A}^+ - \boldsymbol{A}^-) & (\boldsymbol{I}+\boldsymbol{D})^{-1} \\ \boldsymbol{O} & \boldsymbol{I} \end{bmatrix}$$

For a random walk $r = (u_0, u_1, \ldots, u_t)$ on signed graphs, we introduce a sign $l(r)$ as [35], [36]

$$l(r) = \text{sign}\left(\prod_{x=1}^{t} w(u_{x-1}, u_x)\right),$$

where $\text{sign}(x)$ denotes the sign of nonzero x defined as: $\text{sign}(x) = +1$ if $x > 0$, $\text{sign}(x) = -1$ otherwise. Thus, the sign $l(r)$ of the walk r is $+1$ if the number of its negative edges is even, and $l(r) = -1$ otherwise. For a random walk r on signed graphs, if $l(r) = +1$ we call it a positive random walk; if $l(r) = -1$ we call it a negative random walk.

Based on the above-defined two types of random walks, we define two absorbing probabilities for an absorbing random walk on a signed graph. For each node $i \in V$ and any absorbing state $\sigma(j) \in \bar{V}$, we define two probabilities as follows:

- $p_{i,\sigma(j)}$: the probability that a positive random walk starting from transient state i is absorbed by node $\sigma(j)$.
- $q_{i,\sigma(j)}$: the probability that a negative random walk starting from transient state i is absorbed by node $\sigma(j)$.

Using these two absorbing probabilities, we provide an interpretation of the equilibrium opinion for each node.

Theorem 3.1. *For the FJ model of opinion dynamics on a signed graph $\mathcal{G} = (V, E, w)$, let $\boldsymbol{s}$ be the initial opinion vector. Then the equilibrium expressed opinion of node $i \in V$ is expressed by*

$$\boldsymbol{z}_i = \sum_{j\in V} (p_{i,\sigma(j)} - q_{i,\sigma(j)})\boldsymbol{s}_j.$$

Proof. First, we use a recursive method to derive the expressions for the two absorbing probabilities $p_{i,\sigma(j)}$ and $q_{i,\sigma(j)}$ for any $i \in V$ and $\sigma(j) \in \bar{V}$. To this end, for an absorbing random walk on graph $\mathcal{H}$, we define two matrices, $\boldsymbol{X}(t)$ and $\boldsymbol{Y}(t)$ for $t \geq 0$. The element $\boldsymbol{X}(t)_{i,j}$ of $\boldsymbol{X}(t)$ represents the probability that a positive random walk starting from a transient state i arrives at a transient state j at the t-th step. The element $\boldsymbol{Y}(t)_{i,j}$ of $\boldsymbol{Y}(t)$ denotes the probability that a negative random walk starting from a transient state i reaches a transient state j at the t-th step.

By definition of the signed random walk, we obtain the following recursive relations

$$\boldsymbol{X}(t+1)_{i,:} = \boldsymbol{X}(t)_{i,:}(\boldsymbol{I}+\boldsymbol{D})^{-1}\boldsymbol{A}^+ - \boldsymbol{Y}(t)_{i,:}(\boldsymbol{I}+\boldsymbol{D})^{-1}\boldsymbol{A}^-,$$
$$\boldsymbol{Y}(t+1)_{i,:} = \boldsymbol{Y}(t)_{i,:}(\boldsymbol{I}+\boldsymbol{D})^{-1}\boldsymbol{A}^+ - \boldsymbol{X}(t)_{i,:}(\boldsymbol{I}+\boldsymbol{D})^{-1}\boldsymbol{A}^-.$$

Based on these two equations, we have

$$\begin{aligned} &\boldsymbol{X}(t+1)_{i,:} + \boldsymbol{Y}(t+1)_{i,:} \\ =&(\boldsymbol{X}(t)_{i,:} + \boldsymbol{Y}(t)_{i,:})(\boldsymbol{I}+\boldsymbol{D})^{-1}(\boldsymbol{A}^+ - \boldsymbol{A}^-), \\ &\boldsymbol{X}(t+1)_{i,:} - \boldsymbol{Y}(t+1)_{i,:} \\ =&(\boldsymbol{X}(t)_{i,:} - \boldsymbol{Y}(t)_{i,:})(\boldsymbol{I}+\boldsymbol{D})^{-1}(\boldsymbol{A}^+ + \boldsymbol{A}^-). \end{aligned}$$

Considering $\boldsymbol{X}(0) = \boldsymbol{I}$ and $\boldsymbol{Y}(0) = \boldsymbol{O}$, we obtain

$$\boldsymbol{X}(t)_{i,:} + \boldsymbol{Y}(t)_{i,:} = \boldsymbol{e}_i^\top \left((\boldsymbol{I}+\boldsymbol{D})^{-1}(\boldsymbol{A}^+ - \boldsymbol{A}^-)\right)^t,$$
$$\boldsymbol{X}(t)_{i,:} - \boldsymbol{Y}(t)_{i,:} = \boldsymbol{e}_i^\top \left((\boldsymbol{I}+\boldsymbol{D})^{-1}(\boldsymbol{A}^+ + \boldsymbol{A}^-)\right)^t.$$

Solving the above two equations leads to

$$\begin{aligned}\boldsymbol{X}(t)_{i,:} =&\frac{\boldsymbol{e}_i^\top}{2}(((\boldsymbol{I}+\boldsymbol{D})^{-1}(\boldsymbol{A}^+-\boldsymbol{A}^-))^t\\&+((\boldsymbol{I}+\boldsymbol{D})^{-1}(\boldsymbol{A}^++\boldsymbol{A}^-))^t),\\\boldsymbol{Y}(t)_{i,:} =&\frac{\boldsymbol{e}_i^\top}{2}(((\boldsymbol{I}+\boldsymbol{D})^{-1}(\boldsymbol{A}^+-\boldsymbol{A}^-))^t\\&-((\boldsymbol{I}+\boldsymbol{D})^{-1}(\boldsymbol{A}^++\boldsymbol{A}^-))^t).\end{aligned}$$

Note that if a walk arrives at $j \in V$ at a certain step, it will move to $\sigma(j)$ with probability $1/(1+d_j)$. Then the absorbing probability $p_{i,\sigma(j)}$ is exactly the sum of the probabilities of a positive random walk starting at node i and being absorbed by node $\sigma(j)$ in $t = 1, 2, \ldots, \infty$ steps. Thus, we have

$$\begin{aligned}p_{i,\sigma(j)} =&\sum_{t=0}^{\infty}\frac{1}{1+d_j}\boldsymbol{X}(t)_{i,j}\\=&\frac{1}{2}\boldsymbol{e}_i^\top((\boldsymbol{I}+\boldsymbol{D}-\boldsymbol{A}^++\boldsymbol{A}^-)^{-1}\\&+(\boldsymbol{I}+\boldsymbol{D}-\boldsymbol{A}^+-\boldsymbol{A}^-)^{-1})\boldsymbol{e}_j.\end{aligned}$$

Similarly, we obtain

$$\begin{aligned}q_{i,\sigma(j)} =&\sum_{t=0}^{\infty}\frac{1}{1+d_j}\boldsymbol{Y}(t)_{i,j}\\=&\frac{1}{2}\boldsymbol{e}_i^\top((\boldsymbol{I}+\boldsymbol{D}-\boldsymbol{A}^++\boldsymbol{A}^-)^{-1}\\&-(\boldsymbol{I}+\boldsymbol{D}-\boldsymbol{A}^+-\boldsymbol{A}^-)^{-1})\boldsymbol{e}_j.\end{aligned}$$

Using the obtained expressions for $p_{i,\sigma(j)}$ and $q_{i,\sigma(j)}$, we obtain

$$\sum_{j\in V}(p_{i,\sigma(j)}-q_{i,\sigma(j)})\boldsymbol{s}_j = \boldsymbol{e}_i^\top(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s} = \boldsymbol{z}_i,$$

which completes the proof. □

Theorem 3.1 shows that for each $i \in V$, its equilibrium expressed opinion $\boldsymbol{z}_i$ is a weighted average of internal opinions of all nodes, with the weight of internal opinion s_j being $p_{i,\sigma(j)} - q_{i,\sigma(j)}$, where $j = 1, 2 \ldots, n$. When there are no negative edges, the absorbing probability by negative random walks of each node is 0, and Theorem 3.1 is consistent with the previous result obtained for unsigned graphs [12]. However, $p_{i,\sigma(j)} - q_{i,\sigma(j)}$ may be negative in the presence of negative edges, which is in sharp contrast to that for unsigned graphs, where the absorbing probability $p_{i,\sigma(j)}$ is always positive and $q_{i,\sigma(j)}$ is zero.

4 Measures and Fast Evaluation of Some Social Phenomena

Although some social phenomena have been quantified and evaluated for the unsigned FJ model [10], [11], [19], [20], the expressions and algorithms for these quantities on the signed graphs are still lacking. In this section, we first introduce the measures for some social phenomena based on the signed FJ model and express them in terms of quadratic forms and the ℓ_2 norms of vectors. Then, we design a signed Laplacian solver and use it to propose an efficient algorithm to approximate these social phenomena on signed graphs.

4.1 Measures for Social Phenomena

We focus on some common phenomena in social networks, such as conflict, polarization, and disagreement. Here we introduce measures to quantify these phenomena based on the FJ model on signed graphs.

By the definition of the FJ model, although individuals have internal opinions on a certain topic, their expressed opinions often differ from their internal opinions. The difference between them is called internal conflict [11].

Definition 4.1. *For the FJ model on a signed graph $\mathcal{G} = (V, E, w)$, the internal conflict $I(\mathcal{G})$ is the sum of squares of the differences between the internal and expressed opinions over all nodes:*

$$I(\mathcal{G}) = \sum_{i\in V}(\boldsymbol{z}_i - \boldsymbol{s}_i)^2.$$

Besides internal conflict, individuals also suffer from the social cost incurred by their neighbours, as they prefer to express similar opinions as their friends or opposite opinions to their opponents. This social cost is called external conflict or disagreement on the unsigned graphs [10], [11], [19]. For the signed graphs, the disagreement has not been defined yet, since the interaction between individuals is more complex than in unsigned graphs. Below we provide a measure of the disagreement on the signed FJ model.

Definition 4.2. *For the FJ model on a signed graph $\mathcal{G} = (V, E, w)$, its disagreement (or external conflict) $D(\mathcal{G})$ is defined as*

$$D(\mathcal{G}) = \sum_{(i,j)\in E}(\boldsymbol{z}_i - \boldsymbol{A}_{i,j}\boldsymbol{z}_j)^2.$$

Note that at time t the stress or psycho-social cost function of node i is $(\boldsymbol{z}_i(t) - \boldsymbol{s}_i)^2 + \sum_{j\in N_i}(\boldsymbol{z}_i(t) - \boldsymbol{A}_{i,j}\boldsymbol{z}_j(t))^2$. At every time step, each node updates its expressed opinion with an aim to minimize its stress [30], [31], [34]. At the equilibrium, the total social cost, which is the sum of the cost function over all nodes, is $I(\mathcal{G}) + D(\mathcal{G})$.

Since for a node i, each neighbour is either a friend or an enemy, its external social cost can be decomposed into two components: the cost $\sum_{j\in N_i^F}(\boldsymbol{z}_i(t) - \boldsymbol{z}_j(t))^2$ incurred from disagreeing with friends, and the cost $\sum_{j\in N_i^E}(\boldsymbol{z}_i(t)+\boldsymbol{z}_j(t))^2$ incurred from agreeing with opponents. Hence, we give two variants of the disagreement for the FJ model on a signed graph $\mathcal{G}$: disagreement with friends denoted by $F(\mathcal{G})$, and agreement with opponents denoted by $E(\mathcal{G})$.

Definition 4.3. *For the FJ model on a singed graph $\mathcal{G} = (V, E, w)$, the disagreement with friends $F(\mathcal{G})$ is the sum of squares of the differences between expressed opinions over all pairs of friends:*

$$F(\mathcal{G}) = \sum_{(i,j)\in E^+}(\boldsymbol{z}_i - \boldsymbol{z}_j)^2.$$

Definition 4.4. *For the FJ model on a singed graph $\mathcal{G} = (V, E, w)$, the agreement with opponents $E(\mathcal{G})$ is the sum of squares of the sums of expressed opinions over all pairs of opponent nodes:*

$$E(\mathcal{G}) = \sum_{(i,j)\in E^-}(\boldsymbol{z}_i + \boldsymbol{z}_j)^2.$$

In online social networks or social media, users often seek to connect with individuals with similar opinions, which can lead to echo chambers and filter bubbles, and thus reinforce individuals' preexisting opinions and produce polarization [37]. Polarization can be thought of as the degree to which expressed opinions deviate from neutral opinions, i.e., opinion value 0. In the following, we introduce the metric of polarization [9], [30].

Definition 4.5. *For the FJ model on a signed graph $\mathcal{G}=(V,E)$, the polarization is defined as:*

$$P(\mathcal{G})=\frac{1}{n}\sum_{i\in V}\boldsymbol{z}_i^2.$$

According to their definitions, we can explicitly represent the aforementioned social phenomena in terms of quadratic forms and the ℓ_2 norms of vectors, as summarized in Lemma 4.6.

Lemma 4.6. *For the FJ model on a signed graph $\mathcal{G}=(V,E,w)$ with the initial opinion vector $\boldsymbol{s}$, the internal conflict $I(\mathcal{G})$, disagreement $D(\mathcal{G})$, disagreement with friends $F(\mathcal{G})$, agreement with opponents $E(\mathcal{G})$, and polarization $P(\mathcal{G})$ can be conveniently expressed in terms of quadratic forms and the ℓ_2 norms as:*

$$\begin{aligned}
I(\mathcal{G}) &= \boldsymbol{z}^\top\boldsymbol{L}^2\boldsymbol{z}=\boldsymbol{s}^\top(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{L}^2(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\\
&=\left\|\boldsymbol{L}(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_2^2,\\
F(\mathcal{G}) &= \boldsymbol{z}^\top\boldsymbol{L}^+\boldsymbol{z}=\boldsymbol{s}^\top(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{L}^+(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\\
&=\left\|\boldsymbol{B}^+(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_2^2,\\
E(\mathcal{G}) &= \boldsymbol{z}^\top\boldsymbol{L}^-\boldsymbol{z}=\boldsymbol{s}^\top(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{L}^-(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\\
&=\left\|\boldsymbol{B}^-(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_2^2,\\
D(\mathcal{G}) &= \boldsymbol{z}^\top\boldsymbol{L}\boldsymbol{z}=\boldsymbol{s}^\top(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{L}(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\\
&=\left\|\boldsymbol{B}(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_2^2,\\
P(\mathcal{G}) &= \frac{1}{n}\boldsymbol{z}^\top\boldsymbol{z}=\frac{1}{n}\boldsymbol{s}(\boldsymbol{I}+\boldsymbol{L})^{-2}\boldsymbol{s}=\frac{1}{n}\left\|(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_2^2.
\end{aligned}$$

It is easy to verify that these quantities satisfy the following conservation law:

$$I(\mathcal{G})+2D(\mathcal{G})+nP(\mathcal{G})=\sum_{i=1}^{n}\boldsymbol{s}_i^2,\quad F(\mathcal{G})+E(\mathcal{G})=D(\mathcal{G}),$$

which extends the result in [10] for unsigned graphs to signed graphs.

4.2 Signed Laplacian Solver

As shown in Lemma 4.6, directly calculating the social phenomena concerned involves inverting matrix $\boldsymbol{I}+\boldsymbol{L}$, which takes $O(n^3)$ time and is impractical for large graphs. Although some efficient techniques have been proposed for unsigned graphs, they cannot apply to signed graphs in a straightforward way. Actually, by definitions and Lemma 4.6, evaluating these quantities involves the equilibrium opinion vector $\boldsymbol{z}$. Theorem 3.1 reduces the computation of $\boldsymbol{z}_i$ of every node i to evaluate the absorbing probabilities of signed random walks.

In this subsection, we first establish a connection between the absorbing probabilities for random walks on a signed graph and the absorbing probabilities for random walks on an unsigned graph, which is associated with the signed graph. Then, based on this connection we present a signed Laplacian solver, which allows for a fast approximation of relevant social phenomena with proven error guarantees.

For a signed graph $\mathcal{G}=(V,E,w)$ with n nodes and m edges, we can define an unsigned graph $\hat{\mathcal{G}}=(\hat{V},\hat{E})$ with $2n$ nodes and $2m$ edges. For the associated unsigned graph $\hat{\mathcal{G}}$, we label its $2n$ nodes as $1,2,\ldots,2n$. The edge set $\hat{E}$ of $\hat{\mathcal{G}}$ is constructed as follows. If there is a positive edge $(i,j)\in E$, then there are two edges (i,j) and $(i+n,j+n)$ in $\hat{E}$. If there is a negative edge $(i,j)\in E$, then there are two edges $(i,j+n)$ and $(i+n,j)$ in $\hat{E}$.

For the unsigned graph $\hat{\mathcal{G}}=(\hat{V},\hat{E})$, we can define its augmented unsigned graph $\hat{\mathcal{H}}=(\hat{X},\hat{R})$ with absorbing states. The construction details of graph $\hat{\mathcal{H}}=(\hat{X},\hat{R})$ are as follows.

1) The node set $\hat{X}$ is defined as $\hat{X}=\hat{V}\cup\tilde{V}$, where $\tilde{V}$ is a set of $2n$ nodes such that for each node $i\in\hat{V}$, there is a copy $\eta(i)\in\tilde{V}$;
2) The edge set $\hat{R}$ includes all the edges $\hat{E}$ of $\hat{\mathcal{G}}$, plus a new set of edges between each node $i\in\hat{V}$ and its copy $\eta(i)\in\tilde{V}$. That is, $R=\hat{E}\cup\tilde{E}$, where $\tilde{E}=\{(i,\eta(i))|i\in\hat{V}\}$.

We now define an absorbing random walk on an augmented graph $\hat{\mathcal{H}}$ with $4n$ nodes, where the $2n$ nodes in $\hat{V}$ are transient nodes and the $2n$ nodes in $\tilde{V}$ are absorbing nodes. Then, the transition matrix $\boldsymbol{P}'$ of this absorbing random walk is written as $\boldsymbol{P}'=$

$$\begin{bmatrix}(\boldsymbol{I}+\boldsymbol{D})^{-1}\boldsymbol{A}^+ & -(\boldsymbol{I}+\boldsymbol{D})^{-1}\boldsymbol{A}^- & (\boldsymbol{I}+\boldsymbol{D})^{-1} & \boldsymbol{O}\\ -(\boldsymbol{I}+\boldsymbol{D})^{-1}\boldsymbol{A}^- & (\boldsymbol{I}+\boldsymbol{D})^{-1}\boldsymbol{A}^+ & \boldsymbol{O} & (\boldsymbol{I}+\boldsymbol{D})^{-1}\\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{I} & \boldsymbol{O}\\ \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{O} & \boldsymbol{I}\end{bmatrix},$$

Let matrix $\boldsymbol{Q}=\begin{bmatrix}(\boldsymbol{I}+\boldsymbol{D})^{-1}\boldsymbol{A}^+ & -(\boldsymbol{I}+\boldsymbol{D})^{-1}\boldsymbol{A}^-\\ -(\boldsymbol{I}+\boldsymbol{D})^{-1}\boldsymbol{A}^- & (\boldsymbol{I}+\boldsymbol{D})^{-1}\boldsymbol{A}^+\end{bmatrix}$, and let matrix $\boldsymbol{T}=\begin{bmatrix}(\boldsymbol{I}+\boldsymbol{D})^{-1} & \boldsymbol{O}\\ \boldsymbol{O} & (\boldsymbol{I}+\boldsymbol{D})^{-1}\end{bmatrix}$. For a transient state $i\in\hat{V}$ and an absorbing state $\eta(j)\in\tilde{V}$, let $p'_{i,\eta(j)}$ be the absorbing probability of a random walk starting from i is absorbed by $\eta(j)$. Define $2n\times 2n$ matrix $\boldsymbol{S}$ as

$$\boldsymbol{S}=\begin{bmatrix}\boldsymbol{I}+\boldsymbol{D}-\boldsymbol{A}^+ & \boldsymbol{A}^-\\ \boldsymbol{A}^- & \boldsymbol{I}+\boldsymbol{D}-\boldsymbol{A}^+\end{bmatrix}. \tag{3}$$

Clearly, $\boldsymbol{S}$ is symmetric, diagonally-dominant (SDD), since it can be expressed as $\boldsymbol{S}=\boldsymbol{I}+\boldsymbol{L}(\hat{\mathcal{G}})$, where $\boldsymbol{L}(\hat{\mathcal{G}})$ is the Laplacian matrix of the unsigned graph $\hat{\mathcal{G}}$. Then, $p'_{i,\eta(j)}$ can be represented as $p'_{i,\eta(j)}=\boldsymbol{e}_i^\top(\boldsymbol{I}-\boldsymbol{Q})^{-1}\boldsymbol{T}\boldsymbol{e}_j=\boldsymbol{e}_i^\top\boldsymbol{S}^{-1}\boldsymbol{e}_j$.

Since $\boldsymbol{S}$ is a block matrix, according to the matrix inversion in block form, we can further derive an expression of $p'_{i,\eta(j)}$ by distinguishing four cases: (i) $i\in\{1,2,\ldots,n\}$ and $j\in\{1,2,\ldots,n\}$; (ii) $i\in\{1,2,\ldots,n\}$ and $j\in\{n+1,n+2,\ldots,2n\}$; (iii) $i\in\{n+1,n+2,\ldots,2n\}$ and $j\in\{1,2,\ldots,n\}$; and (iv) $i\in\{n+1,n+2,\ldots,2n\}$ and $j\in\{n+1,n+2,\ldots,2n\}$. For the first case $i,j\in\{1,2,\ldots,n\}$, we have

$$\begin{aligned}p'_{i,\eta(j)}=\frac{1}{2}\boldsymbol{e}_i^\top((&\boldsymbol{I}+\boldsymbol{D}-\boldsymbol{A}^++\boldsymbol{A}^-)^{-1}\\ &+(\boldsymbol{I}+\boldsymbol{D}-\boldsymbol{A}^+-\boldsymbol{A}^-)^{-1})\boldsymbol{e}_j.\end{aligned}$$

For the second case $i \in \{1, 2, \dots, n\}$ and $j \in \{n+1, n+2, \dots, 2n\}$, we have

$$p'_{i,\eta(j)} = \frac{1}{2} \boldsymbol{e}_i^\top ((\boldsymbol{I} + \boldsymbol{D} - \boldsymbol{A}^+ + \boldsymbol{A}^-)^{-1} - (\boldsymbol{I} + \boldsymbol{D} - \boldsymbol{A}^+ - \boldsymbol{A}^-)^{-1}) \boldsymbol{e}_{j-n}.$$

For the remaining two cases, considering the symmetry of the matrix $\boldsymbol{S}$, it is easy to verify that $p'_{i+n,\eta(j)} = p'_{i,\eta(j+n)}$ and $p'_{i+n,\eta(j+n)} = p'_{i,\eta(j)}$ hold for $i, j \in \{1, 2, \dots, n\}$.

Theorem 3.1 shows that in order to determine the expressed opinion $\boldsymbol{z}_i$ of node i, we can alternatively compute the absorbing probabilities $p_{i,\sigma(j)}$ and $q_{i,\sigma(j)}$ of positive and negative random walk on the augmented signed graph $\mathcal{H}$ of the signed graph $\mathcal{G}$. According to the above arguments, we establish a direct relationship between the absorbing probabilities for absorbing random walks on signed graph $\mathcal{H}$ and unsigned graph $\hat{\mathcal{H}}$. Specifically, for any $i, j \in \{1, 2, \dots, n\}$, $p'_{i,\eta(j)} = p'_{i+n,\eta(j+n)} = p_{i,\sigma(j)}$ and $p'_{i,\eta(j+n)} = p'_{i+n,\eta(j)} = q_{i,\sigma(j)}$. These relations allow us to provide an alternative expression for the equilibrium expressed opinion for the FJ model on a signed graph, by using a matrix associated with an unsigned graph.

Theorem 4.7. *For the FJ model of opinion dynamics on a signed graph $\mathcal{G}$ with initial opinion vector $\boldsymbol{s}$, the equilibrium opinion vector can be expressed as*

$$\boldsymbol{z} = \frac{1}{2} \begin{bmatrix} \boldsymbol{I} & -\boldsymbol{I} \end{bmatrix} \boldsymbol{S}^{-1} \begin{bmatrix} \boldsymbol{I} \\ -\boldsymbol{I} \end{bmatrix} \boldsymbol{s}, \tag{4}$$

where matrix $\boldsymbol{S}$ is defined by (3)*, which is the fundamental matrix for an unsigned graph $\hat{\mathcal{G}}$ associated with the signed graph $\mathcal{G}$.*

Theorem 4.7 shows that to determine the equilibrium opinion $\boldsymbol{z}$ for the FJ model on a signed graph $\mathcal{G}$, we can alternatively evaluate $\boldsymbol{S}^{-1} \begin{bmatrix} \boldsymbol{s} \\ -\boldsymbol{s} \end{bmatrix}$ denoted by $\boldsymbol{s}'$, where $\boldsymbol{S}$ is an SDD matrix, which in fact corresponds to a fundamental matrix for the FJ model on an unsigned graph $\hat{\mathcal{G}}$, expanded from $\mathcal{G}$. In order to compute $\boldsymbol{s}'$, we resort to the fast SDD linear system solver [38], [39]. Specifically, we propose a signed solver by extending the SDD solver to the FJ model on signed graphs, which avoids computing the inverse of $\boldsymbol{S}$ or $\boldsymbol{I} + \boldsymbol{L}$, but has proven error guarantees for various problems defined on signed graphs. Before doing so, we first introduce the SDD solver.

Lemma 4.8. *[38], [39] Given a symmetric positive semi-definite matrix $\boldsymbol{T} \in \mathbb{R}^{n \times n}$ with m nonzero entries, a vector $\boldsymbol{b} \in \mathbb{R}^n$, and an error parameter $\delta > 0$, there exists a solver, denoted as $\boldsymbol{a} = \text{SOLVER}(\boldsymbol{T}, \boldsymbol{b}, \delta)$, which returns a vector $\boldsymbol{a} \in \mathbb{R}^n$ such that $\left\| \boldsymbol{a} - \boldsymbol{T}^{-1}\boldsymbol{b} \right\|_{\boldsymbol{T}} \le \delta \left\| \boldsymbol{T}^{-1}\boldsymbol{b} \right\|_{\boldsymbol{T}}$. The runtime of this solver is expected to be $\tilde{O}(m)$, where $\tilde{O}(\cdot)$ notation suppresses* $\text{poly}(\log n)$ *factors.*

Based on Lemma 4.8, we propose a signed Laplacian solver, which returns a vector $\boldsymbol{f}$ as an approximation of the expressed opinion vector $\boldsymbol{z} = (\boldsymbol{I} + \boldsymbol{L})^{-1}\boldsymbol{s}$ for the FJ model in the signed graph $\mathcal{G} = (V, E, w)$.

Lemma 4.9. *Given a signed graph $\mathcal{G} = (V, E, w)$ with Laplacian matrix $\boldsymbol{L}$, a matrix $\boldsymbol{S}$ defined in* (3)*, a vector $\boldsymbol{y} \in \mathbb{R}^n$, and an error parameter $\delta > 0$, there is a signed Laplacian solver $\boldsymbol{f} = \text{SIGNEDSOLVER}(\boldsymbol{I} + \boldsymbol{L}, \boldsymbol{y}, \delta) = \frac{1}{2} \begin{bmatrix} \boldsymbol{I} & -\boldsymbol{I} \end{bmatrix} \cdot \text{SOLVER}\left(\boldsymbol{S}, \begin{bmatrix} \boldsymbol{I} \\ -\boldsymbol{I} \end{bmatrix} \boldsymbol{y}, \delta\right)$, which returns a vector $\boldsymbol{f}$ satisfying*

$$\left\| \boldsymbol{f} - (\boldsymbol{I} + \boldsymbol{L})^{-1}\boldsymbol{y} \right\|_{\boldsymbol{I}+\boldsymbol{L}} \le \delta \left\| (\boldsymbol{I} + \boldsymbol{L})^{-1}\boldsymbol{y} \right\|_{\boldsymbol{I}+\boldsymbol{L}}. \tag{5}$$

This signed Laplacian solver runs in expected time $\tilde{O}(m)$, where $\tilde{O}(\cdot)$ notation suppresses the $\text{poly}(\log n)$ *factors.*

Proof. In order to prove (5), it is equivalent to prove

$$\boldsymbol{f}^\top (\boldsymbol{I} + \boldsymbol{L}) \boldsymbol{f} + \boldsymbol{y}^\top (\boldsymbol{I} + \boldsymbol{L})^{-1} \boldsymbol{y} - 2\boldsymbol{f}^\top \boldsymbol{y} \le \delta^2 \boldsymbol{y}^\top (\boldsymbol{I} + \boldsymbol{L})^{-1} \boldsymbol{y}.$$

Let $\boldsymbol{c} = \text{SOLVER}\left(\boldsymbol{S}, \begin{bmatrix} \boldsymbol{I} \\ -\boldsymbol{I} \end{bmatrix} \boldsymbol{y}, \delta\right)$. Eliminating $\boldsymbol{f}$ and $\boldsymbol{I} + \boldsymbol{L}$ by $\boldsymbol{c}$ and $\boldsymbol{S}$ yields

$$\frac{1}{8} \boldsymbol{c}^\top \begin{bmatrix} \boldsymbol{I} \\ -\boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{I} & -\boldsymbol{I} \end{bmatrix} \boldsymbol{S} \begin{bmatrix} \boldsymbol{I} \\ -\boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{I} & -\boldsymbol{I} \end{bmatrix} \boldsymbol{c} + \frac{1-\delta^2}{2} \boldsymbol{y}^\top \begin{bmatrix} \boldsymbol{I} & -\boldsymbol{I} \end{bmatrix} \boldsymbol{S}^{-1} \begin{bmatrix} \boldsymbol{I} \\ -\boldsymbol{I} \end{bmatrix} \boldsymbol{y} - \boldsymbol{c}^\top \begin{bmatrix} \boldsymbol{I} \\ -\boldsymbol{I} \end{bmatrix} \boldsymbol{y} \le 0. \tag{6}$$

By definition of $\boldsymbol{c}$, we have

$$\left\| \boldsymbol{c} - \boldsymbol{S}^{-1} \begin{bmatrix} \boldsymbol{I} \\ -\boldsymbol{I} \end{bmatrix} \boldsymbol{y} \right\|_{\boldsymbol{S}} \le \delta \left\| \boldsymbol{S}^{-1} \begin{bmatrix} \boldsymbol{I} \\ -\boldsymbol{I} \end{bmatrix} \boldsymbol{y} \right\|_{\boldsymbol{S}},$$

which is equivalent to

$$\boldsymbol{c}^\top \boldsymbol{S} \boldsymbol{c} + (1-\delta^2) \boldsymbol{y}^\top \begin{bmatrix} \boldsymbol{I} & -\boldsymbol{I} \end{bmatrix} \boldsymbol{S}^{-1} \begin{bmatrix} \boldsymbol{I} \\ -\boldsymbol{I} \end{bmatrix} \boldsymbol{y} - 2\boldsymbol{c}^\top \begin{bmatrix} \boldsymbol{I} \\ -\boldsymbol{I} \end{bmatrix} \boldsymbol{y} \le 0. \tag{7}$$

Thus, to prove (6), we only need to prove

$$\frac{1}{8} \boldsymbol{c}^\top \begin{bmatrix} \boldsymbol{I} \\ -\boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{I} & -\boldsymbol{I} \end{bmatrix} \boldsymbol{S} \begin{bmatrix} \boldsymbol{I} \\ -\boldsymbol{I} \end{bmatrix} \begin{bmatrix} \boldsymbol{I} & -\boldsymbol{I} \end{bmatrix} \boldsymbol{c} - \frac{1}{2} \boldsymbol{c}^\top \boldsymbol{S} \boldsymbol{c} \le 0,$$

which can be recast as

$$\boldsymbol{c}^\top \left(4\boldsymbol{S} - \begin{bmatrix} \boldsymbol{I} & -\boldsymbol{I} \\ -\boldsymbol{I} & \boldsymbol{I} \end{bmatrix} \boldsymbol{S} \begin{bmatrix} \boldsymbol{I} & -\boldsymbol{I} \\ -\boldsymbol{I} & \boldsymbol{I} \end{bmatrix} \right) \boldsymbol{c} \ge 0. \tag{8}$$

Plugging the expression for matrix $\boldsymbol{S}$ in (3) into (8) gives the following equation:

$$4\boldsymbol{S} - \begin{bmatrix} \boldsymbol{I} & -\boldsymbol{I} \\ -\boldsymbol{I} & \boldsymbol{I} \end{bmatrix} \boldsymbol{S} \begin{bmatrix} \boldsymbol{I} & -\boldsymbol{I} \\ -\boldsymbol{I} & \boldsymbol{I} \end{bmatrix} = \begin{bmatrix} 2\boldsymbol{M} & 2\boldsymbol{M} \\ 2\boldsymbol{M} & 2\boldsymbol{M} \end{bmatrix},$$

where $\boldsymbol{M} = \boldsymbol{I} + \boldsymbol{D} - \boldsymbol{A}^+ + \boldsymbol{A}^-$ is an SDDM matrix. Rewrite $\boldsymbol{c}$ as $\boldsymbol{c} = \begin{bmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{bmatrix}$. Then (8) is expressed as

$$\boldsymbol{u}^\top \boldsymbol{M} \boldsymbol{u} + \boldsymbol{v}^\top \boldsymbol{M} \boldsymbol{v} + 2\boldsymbol{u}^\top \boldsymbol{M} \boldsymbol{v} = (\boldsymbol{u} + \boldsymbol{v})^\top \boldsymbol{M} (\boldsymbol{u} + \boldsymbol{v}) \ge 0,$$

which is true since matrix $\boldsymbol{M}$ is a positive definite matrix. Combining the above analyses completes the proof. □

4.3 Fast Evaluation Algorithm

Lemma 4.9 indicates that for those quantities on signed graphs having form $(\boldsymbol{I} + \boldsymbol{L})^{-1}\boldsymbol{s}$, we can exploit the signed solver to significantly reduce the computational time. We next apply Lemma 4.9 to obtain approximations for the quantities defined in 4.1.

Lemma 4.10. *Given a signed graph $\mathcal{G} = (V, E, w)$ with Laplacian matrix $\boldsymbol{L}$, incident matrix $\boldsymbol{B}$, positive incident matrix $\boldsymbol{B}^+$, negative incident matrix $\boldsymbol{B}^-$, and a parameter $\epsilon \in (0, \frac{1}{2})$, consider* *the FJ model of opinion dynamics on $\mathcal{G} = (V, E, w)$ with the internal opinion vector $\boldsymbol{s}$ and let $\boldsymbol{q} = \text{SIGNEDSOLVER}(\boldsymbol{I} + \boldsymbol{L}, \boldsymbol{s}, \delta)$, where*

$$\delta \leq \min\left\{ \frac{\epsilon}{3\sqrt{2n}}, \frac{\|\boldsymbol{s}\|_{\boldsymbol{L}}}{6\sqrt{2}n^2}\epsilon, \frac{\|\boldsymbol{L}\boldsymbol{s}\|_2}{12\sqrt{2}n^3}\epsilon, \frac{\|\boldsymbol{s}\|_2}{2\sqrt{2n}}\epsilon \right\}.$$

Then, the following relations hold:

$$\left\|(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_2^2 \approx_\epsilon \|\boldsymbol{q}\|_2^2, \tag{9}$$

$$\left\|\boldsymbol{B}(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_2^2 \approx_\epsilon \|\boldsymbol{B}\boldsymbol{q}\|_2^2, \tag{10}$$

$$\left\|\boldsymbol{L}(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_2^2 \approx_\epsilon \|\boldsymbol{L}\boldsymbol{q}\|_2^2, \tag{11}$$

$$\left|\left\|\boldsymbol{B}^+\boldsymbol{q}\right\|_2^2 - \left\|\boldsymbol{B}^+(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_2^2\right| \leq \epsilon, \tag{12}$$

$$\left|\left\|\boldsymbol{B}^-\boldsymbol{q}\right\|_2^2 - \left\|\boldsymbol{B}^-(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_2^2\right| \leq \epsilon. \tag{13}$$

Proof. We prove this lemma in turn. We first prove (9). By Lemma 4.9, we obtain

$$\left\|\boldsymbol{q} - (\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_{\boldsymbol{I}+\boldsymbol{L}}^2 \leq \delta^2 \left\|(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_{\boldsymbol{I}+\boldsymbol{L}}^2.$$

The term on the left-hand side (lhs) is bounded by

$$\begin{aligned}\left\|\boldsymbol{q} - (\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_{\boldsymbol{I}+\boldsymbol{L}}^2 &\geq \left\|\boldsymbol{q} - (\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_2^2 \\ &\geq \left|\|\boldsymbol{q}\|_2 - \left\|(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_2\right|^2,\end{aligned}$$

while the term on the right-hand side (rhs) is bounded by

$$\left\|(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_{\boldsymbol{I}+\boldsymbol{L}}^2 \leq 2n \left\|(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_2^2.$$

These two bounds together lead to

$$\left|\|\boldsymbol{q}\|_2 - \left\|(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_2\right|^2 \leq 2n\delta^2 \left\|(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_2^2.$$

Considering $\delta \leq \frac{\epsilon}{3\sqrt{2n}}$, we get

$$\frac{\left|\|\boldsymbol{q}\|_2 - \left\|(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_2\right|}{\left\|(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_2} \leq \sqrt{2n\delta^2} \leq \frac{\epsilon}{3}$$

and

$$\begin{aligned}(1-\epsilon/3)^2 \left\|(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_2^2 &\leq \|\boldsymbol{q}\|_2^2 \\ &\leq (1+\epsilon/3)^2 \left\|(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_2^2.\end{aligned}$$

Using the condition $0 < \epsilon < \frac{1}{2}$, we have

$$(1-\epsilon) \left\|(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_2^2 \leq \|\boldsymbol{q}\|_2^2 \leq (1+\epsilon) \left\|(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_2^2,$$

which completes the proof of (9).

Then, we prove (10). By Lemma 4.9, we have

$$\left\|\boldsymbol{q} - (\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_{\boldsymbol{I}+\boldsymbol{L}}^2 \leq \delta^2 \left\|(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_{\boldsymbol{I}+\boldsymbol{L}}^2.$$

The lhs can be bounded as

$$\begin{aligned}\left\|\boldsymbol{q} - (\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_{\boldsymbol{I}+\boldsymbol{L}}^2 &\geq \left\|\boldsymbol{q} - (\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_{\boldsymbol{L}}^2 \\ &= \left\|\boldsymbol{B}\boldsymbol{q} - \boldsymbol{B}(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_2^2 \\ &\geq \left|\|\boldsymbol{B}\boldsymbol{q}\|_2 - \left\|\boldsymbol{B}(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_2\right|^2,\end{aligned}$$

while rhs is bounded as

$$\left\|(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_{\boldsymbol{I}+\boldsymbol{L}}^2 \leq 2n \left\|(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_2^2 \leq 2n^2,$$

where the relation $|\boldsymbol{s}_i| \leq 1$ is used. These two obtained bounds gives

$$\left|\|\boldsymbol{B}\boldsymbol{q}\|_2 - \left\|\boldsymbol{B}(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_2\right|^2 \leq 2\delta^2 n^2.$$

On the other hand,

$$\left\|\boldsymbol{B}(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_2^2 \geq \frac{1}{4n^2} \|\boldsymbol{s}\|_{\boldsymbol{L}}^2.$$

Considering $\delta \leq \frac{\|\boldsymbol{s}\|_{\boldsymbol{L}}}{6\sqrt{2}n^2}\epsilon$, one obtains

$$\frac{\left|\|\boldsymbol{B}\boldsymbol{q}\|_2 - \left\|\boldsymbol{B}(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_2\right|}{\left\|\boldsymbol{B}(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_2} \leq \sqrt{\frac{8\delta^2 n^4}{\|\boldsymbol{s}\|_{\boldsymbol{L}}^2}} \leq \frac{\epsilon}{3},$$

which can be rewritten as

$$\begin{aligned}(1-\epsilon/3)^2 \left\|\boldsymbol{B}(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_2^2 &\leq \|\boldsymbol{B}\boldsymbol{q}\|_2^2 \\ &\leq (1+\epsilon/3)^2 \left\|\boldsymbol{B}(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_2^2.\end{aligned}$$

Using $0 < \epsilon < \frac{1}{2}$, we obtain

$$\begin{aligned}(1-\epsilon) \left\|\boldsymbol{B}(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_2^2 &\leq \|\boldsymbol{B}\boldsymbol{q}\|_2^2 \\ &\leq (1+\epsilon) \left\|\boldsymbol{B}(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_2^2,\end{aligned}$$

completing the proof of (10).

Next, we prove (11). Applying Lemma 4.9, we obtain

$$\left\|\boldsymbol{q} - (\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_{\boldsymbol{I}+\boldsymbol{L}}^2 \leq \delta^2 \left\|(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_{\boldsymbol{I}+\boldsymbol{L}}^2.$$

The term on the lhs is bounded by

$$\begin{aligned}&\left\|\boldsymbol{q} - (\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_{\boldsymbol{I}+\boldsymbol{L}}^2 \geq \left\|\boldsymbol{q} - (\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_2^2 \\ \geq& \frac{1}{4n^2} \left\|\boldsymbol{L}\boldsymbol{q} - \boldsymbol{L}(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_2^2 \\ \geq& \frac{1}{4n^2} \left|\|\boldsymbol{L}\boldsymbol{q}\|_2 - \left\|\boldsymbol{L}(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_2\right|^2.\end{aligned}$$

Using the above-proved relation $\left\|(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_{\boldsymbol{I}+\boldsymbol{L}}^2 \leq 2n^2$, we have

$$\left|\|\boldsymbol{L}\boldsymbol{q}\|_2 - \left\|\boldsymbol{L}(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_2\right|^2 \leq 8\delta^2 n^4.$$

On the other hand,

$$\left\|\boldsymbol{L}(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_2^2 \geq \frac{1}{4n^2} \|\boldsymbol{L}\boldsymbol{s}\|_2^2.$$

Combining the above-obtained inequalities and $\delta \leq \frac{\|\boldsymbol{L}\boldsymbol{s}\|_2}{12\sqrt{2}n^3}\epsilon$ yields

$$\frac{\left|\|\boldsymbol{L}\boldsymbol{q}\|_2 - \left\|\boldsymbol{L}(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_2\right|}{\left\|\boldsymbol{L}(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_2} \leq \sqrt{\frac{32\delta^2 n^6}{\|\boldsymbol{L}\boldsymbol{s}\|_2^2}} \leq \frac{\epsilon}{3},$$

which can be rewritten as

$$\begin{aligned}(1-\epsilon/3)^2 \left\|\boldsymbol{L}(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_2^2 &\leq \|\boldsymbol{L}\boldsymbol{q}\|_2^2 \\ &\leq (1+\epsilon/3)^2 \left\|\boldsymbol{L}(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_2^2.\end{aligned}$$

Considering $0 < \epsilon < \frac{1}{2}$, we derive

$$\begin{aligned}(1-\epsilon) \left\|\boldsymbol{L}(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_2^2 &\leq \|\boldsymbol{L}\boldsymbol{q}\|_2^2 \\ &\leq (1+\epsilon) \left\|\boldsymbol{L}(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_2^2,\end{aligned}$$

which finishes the proof of (11).

Finally, we prove the last two inequalities (12) and (13). By Lemma 4.9, we have

$$\left\|\boldsymbol{q} - (\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_{\boldsymbol{I}+\boldsymbol{L}}^2 \leq \delta^2 \left\|(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_{\boldsymbol{I}+\boldsymbol{L}}^2.$$

The term on the lhs is bounded as

$$\begin{aligned}&\left\|\boldsymbol{q}-(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_{\boldsymbol{I}+\boldsymbol{L}}^{2}\geq\left\|\boldsymbol{q}-(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_{\boldsymbol{L}^{+}}^{2}\\ =&\left\|\boldsymbol{B}^{+}\boldsymbol{q}-\boldsymbol{B}^{+}(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_{2}^{2}\\ \geq&\left|\left\|\boldsymbol{B}^{+}\boldsymbol{q}\right\|_{2}-\left\|\boldsymbol{B}^{+}(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_{2}\right|^{2},\end{aligned}$$

while the term of the rhs is bounded as

$$\left\|(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_{\boldsymbol{I}+\boldsymbol{L}}^{2}\leq 2n\left\|(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_{2}^{2}\leq 2n^{2}.$$

Combining these two bounds gives

$$\left|\left\|\boldsymbol{B}^{+}\boldsymbol{q}\right\|_{2}-\left\|\boldsymbol{B}^{+}(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_{2}\right|^{2}\leq 2\delta^{2}n^{2}.$$

Considering the fact that $\delta\leq\frac{\|\boldsymbol{s}\|_2}{2\sqrt{2}n}\epsilon$, we have

$$\begin{aligned}&\left|\left\|\boldsymbol{B}^{+}\boldsymbol{q}\right\|_{2}^{2}-\left\|\boldsymbol{B}^{+}(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_{2}^{2}\right|\\ =&\left|\left(\left\|\boldsymbol{B}^{+}\boldsymbol{q}\right\|_{2}+\left\|\boldsymbol{B}^{+}(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_{2}\right)\right.\\ &\left.\cdot\left(\left\|\boldsymbol{B}^{+}\boldsymbol{q}\right\|_{2}-\left\|\boldsymbol{B}^{+}(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_{2}\right)\right|\\ \leq&2\sqrt{2}\delta n\left\|\boldsymbol{B}^{+}(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}\right\|_{2}\leq 2\sqrt{2}\delta n\left\|\boldsymbol{s}\right\|_{2}\leq\epsilon,\end{aligned}$$

which completes the proof of (12).

By replacing $\boldsymbol{B}^{+}$ in the proof process of (12) by $\boldsymbol{B}^{-}$, we can prove (13) in a similar way. □

Based on Lemmas 4.9 and 4.10, we propose a nearly-linear time algorithm called APPROXQUAN, which approximates the internal conflict $I(\mathcal{G})$, disagreement $D(\mathcal{G})$, disagreement with friends $F(\mathcal{G})$, agreement with opponents $E(\mathcal{G})$, and polarization $P(\mathcal{G})$ for the FJ model on a signed graph. Algorithm 1 provides the details of algorithm APPROXQUAN, the performance of which is summarized in Theorem 4.11.

Algorithm 1: APPROXQUAN $(\mathcal{G},\boldsymbol{s},\epsilon)$

Input : $\mathcal{G}=(V,E,w)$: a signed graph; $\boldsymbol{s}$: initial opinion vector; ϵ: the error parameter in $(0,\frac{1}{2})$

Output : $\{\tilde{I}(\mathcal{G}),\tilde{D}(\mathcal{G}),\tilde{F}(\mathcal{G}),\tilde{E}(\mathcal{G}),\tilde{P}(\mathcal{G})\}$

1 $\delta=\min\left\{\frac{\epsilon}{3\sqrt{2n}},\frac{\|\boldsymbol{s}\|_{\boldsymbol{L}}}{6\sqrt{2}n^{2}}\epsilon,\frac{\|\boldsymbol{L}\boldsymbol{s}\|_{2}}{12\sqrt{2}n^{3}}\epsilon,\frac{\|\boldsymbol{s}\|_{2}}{2\sqrt{2}n}\epsilon\right\}$
2 $\boldsymbol{q}=$ SIGNEDSOLVER$(\boldsymbol{I}+\boldsymbol{L},\boldsymbol{s},\delta)$
3 $\tilde{I}(\mathcal{G})=\|\boldsymbol{L}\boldsymbol{q}\|_2^2$
4 $\tilde{D}(\mathcal{G})=\|\boldsymbol{B}\boldsymbol{q}\|_2^2$
5 $\tilde{F}(\mathcal{G})=\left\|\boldsymbol{B}^{+}\boldsymbol{q}\right\|_2^2$
6 $\tilde{E}(\mathcal{G})=\left\|\boldsymbol{B}^{-}\boldsymbol{q}\right\|_2^2$
7 $\tilde{P}(\mathcal{G})=\|\boldsymbol{q}\|_2^2/n$
8 **return** $\{\tilde{I}(\mathcal{G}),\tilde{D}(\mathcal{G}),\tilde{F}(\mathcal{G}),\tilde{E}(\mathcal{G}),\tilde{P}(\mathcal{G})\}$

Theorem 4.11. *Given a signed undirected graph $\mathcal{G}$, an error parameter $\epsilon\in(0,\frac{1}{2})$, and the internal opinion vector $\boldsymbol{s}$, the algorithm* APPROXQUAN $(\mathcal{G},\boldsymbol{s},\epsilon)$ *runs in expected time $\tilde{O}(m)$, where $\tilde{O}(\cdot)$ notation suppresses the* $\mathrm{poly}(\log n)$ *factors, and returns approximations $\tilde{I}(\mathcal{G})$, $\tilde{D}(\mathcal{G})$, $\tilde{F}(\mathcal{G})$, $\tilde{E}(\mathcal{G})$, $\tilde{P}(\mathcal{G})$ for the internal conflict $I(\mathcal{G})$, disagreement $D(\mathcal{G})$, disagreement with friends $F(\mathcal{G})$, agreement with opponents $E(\mathcal{G})$, and polarization $P(\mathcal{G})$, satisfying $\tilde{I}(\mathcal{G})\approx_\epsilon I(\mathcal{G})$, $\tilde{D}(\mathcal{G})\approx_\epsilon D(\mathcal{G})$, $|\tilde{F}(\mathcal{G})-F(\mathcal{G})|\leq\epsilon$, $|\tilde{E}(\mathcal{G})-E(\mathcal{G})|\leq\epsilon$, and $\tilde{P}(\mathcal{G})\approx_\epsilon P(\mathcal{G})$.*

5 OVERALL OPINION OPTIMIZATION

In this section, we propose a problem of optimizing the overall expressed opinion for the signed FJ model by changing the initial opinions of a fixed number of nodes. We then provide an algorithm optimally solve the problem in $O(n^3)$ time. To reduce the running time, we also design an efficient algorithm to approximately solve the problem in nearly-linear time.

5.1 Problem Statement

The overall expressed opinion is defined as the sum of expressed opinions $\boldsymbol{z}_i$ of nodes $i\in V$ at equilibrium, which can be expressed as $\sum_{i=1}^{n}\boldsymbol{z}_i=\mathbf{1}^\top(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}$. This expression shows that the overall expressed opinion is influenced by both the internal opinion $\boldsymbol{s}_i$ of each node and the network structure characterized by matrix $(\boldsymbol{I}+\boldsymbol{L})^{-1}$. These two factors determine together the opinion dynamics in the signed FJ model. Define vector $\boldsymbol{h}=(\boldsymbol{I}+\boldsymbol{L})^{-1}\mathbf{1}$. Then the overall opinion is rewritten as $\mathbf{1}^\top(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}=\boldsymbol{h}^\top\boldsymbol{s}=\sum_{i=1}^{n}\boldsymbol{h}_i\boldsymbol{s}_i$, where $\boldsymbol{h}_i$ determines the extent to which the internal opinion $\boldsymbol{s}_i$ of node i contributes to the overall opinion. Note that $\boldsymbol{h}_i$ is determined by the network structure, which is thus called the structure centrality of node i in the FJ model [40].

As shown above, the overall opinion is a function $g(\cdot)$ of the initial opinion $\boldsymbol{s}$ and structure centrality $\boldsymbol{h}$. Then, it can be expressed as $g(\boldsymbol{s})=\mathbf{1}^\top(\boldsymbol{I}+\boldsymbol{L})^{-1}\boldsymbol{s}=\boldsymbol{h}^\top\boldsymbol{s}=\sum_{i=1}^{n}\boldsymbol{h}_i\boldsymbol{s}_i$, when the network structure is fixed. In this paper, we study the influence of initial opinions on the overall opinion, while keeping the network structure unchanged. Then, a natural problem arises, how to maximize the multi-variable objective function $g(\boldsymbol{s})$ by changing the initial opinions of a fixed number of nodes. Mathematically, the opinion maximization problem is formally stated as follows.

Problem 1 (OpinionMax)**.** *Given a signed graph $\mathcal{G}=(V,E,w)$, an initial opinion vector $\boldsymbol{s}$, and an integer $k\ll n$, suppose that for each $i\in V$, its internal opinion $\boldsymbol{s}_i$ is in the interval $[-1,1]$. The problem is how to optimally choose k nodes and change their internal opinions, leading to a new initial opinion vector $\boldsymbol{y}\in[-1,1]^n$, such that the overall opinion $g(\boldsymbol{y})$ is maximized under the constraint $\|\boldsymbol{y}-\boldsymbol{s}\|_0\leq k$.*

In a similar way, we can minimize the overall opinion by optimally changing the initial opinions of k nodes, which is called the problem OPINIONMIN. Note that both problem OPINIONMAX and problem OPINIONMIN are equivalent to each other. One can invert positive and negative signs of the initial opinions to ascertain this equivalence. Thus, in what follows, we focus on problem OPINIONMAX.

It should be mentioned that a similar opinion maximization problem has been proposed in [29] by changing the initial opinions of an unfixed number of nodes. In the problem, the total amount of modification of the initial opinions has an upper bound [29]. In contrast, we focus on selecting a fixed number of nodes to change their initial opinions, with no constraints on the change of initial opinions, as long as they lie in the interval $[-1,1]$.

It is easy to show that for unsigned undirected graphs, increasing the internal opinion $\boldsymbol{s}_i$ of any node i leads to the increase of the overall equilibrium opinion. However, for signed graphs, increasing $\boldsymbol{s}_i$ of node i not necessarily results

Algorithm 2: OPTIMAL($\mathcal{G}, \boldsymbol{s}, k$)

Input : A graph $\mathcal{G} = (V, E)$; an internal opinion vector $\boldsymbol{s}$; an integer k obeying relation $1 \leq k \ll n$
Output : $\boldsymbol{y}$: A modified internal opinion vector with $\|\boldsymbol{y} - \boldsymbol{s}\|_0 \leq k$

Initialize solution $\boldsymbol{y} = \boldsymbol{s}$
Compute $\boldsymbol{h} = (\boldsymbol{I} + \boldsymbol{L})^{-1}\mathbf{1}$
for $i \in V$ **do**
  if $\boldsymbol{h}_i = 0$ **then**
    $c_i \leftarrow 0$
  else
    $c_i \leftarrow |\boldsymbol{h}_i|(1 - \boldsymbol{s}_i|\boldsymbol{h}_i|/\boldsymbol{h}_i)$
for $t = 1$ *to* k **do**
  Select i s. t. $i \leftarrow \arg\max_{i \in V} c_i$
  if $\boldsymbol{h}_i = 0$ **then**
    break
  Update $c_i \leftarrow 0$
  Update solution $\boldsymbol{y}_i \leftarrow |\boldsymbol{h}_i|/\boldsymbol{h}_i$
return $\boldsymbol{y}$.

Algorithm 3: APPROXOPIN($\mathcal{G}, \boldsymbol{s}, k$)

Input : A signed $\mathcal{G} = (V, E, w)$; an internal opinion vector $\boldsymbol{s}$; an integer k obeying relation $1 \leq k \ll n$
Output : $\boldsymbol{y}$: A modified internal opinion vector with $\|\boldsymbol{y} - \boldsymbol{s}\|_0 \leq k$

Set $\delta = \frac{\epsilon}{2k\sqrt{n+4m}}$
Initialize solution $\boldsymbol{y} = \boldsymbol{s}$
Compute $\bar{\boldsymbol{h}} = \text{SIGNEDSOLVER}(\boldsymbol{I} + \boldsymbol{L}, \mathbf{1}, \delta)$
for $i \in V$ **do**
  if $\bar{\boldsymbol{h}}_i = 0$ **then**
    $\bar{c}_i \leftarrow 0$
  else
    $\bar{c}_i \leftarrow |\bar{\boldsymbol{h}}_i|(1 - \boldsymbol{s}_i|\bar{\boldsymbol{h}}_i|/\bar{\boldsymbol{h}}_i)$
for $t = 1$ *to* k **do**
  Select i s. t. $i \leftarrow \arg\max_{i \in V} \bar{c}_i$
  if $\bar{\boldsymbol{h}}_i = 0$ **then**
    break
  Update $\bar{c}_i \leftarrow 0$
  Update solution $\boldsymbol{y}_i \leftarrow |\bar{\boldsymbol{h}}_i|/\bar{\boldsymbol{h}}_i$
return $\boldsymbol{y}$.

in an increase in the overall equilibrium opinion, as node i many have negative structure centrality $\boldsymbol{h}_i$. According to the expression $g(\boldsymbol{s}) = \sum_{i=1}^{n} \boldsymbol{h}_i \boldsymbol{s}_i$, we can draw the following conclusion for a node i. If $\boldsymbol{h}_i > 0$, increasing $\boldsymbol{s}_i$ implies increasing the overall opinion; If $\boldsymbol{h}_i < 0$, decreasing $\boldsymbol{s}_i$ leads to an increase of the overall opinion; If $\boldsymbol{h}_i = 0$, changing $\boldsymbol{s}_i$ has no influence on the overall opinion. Moreover, it is not difficult to derive that for any node $i \in V$ with $\boldsymbol{h}_i \neq 0$, changing $\boldsymbol{s}_i$ in a proper way can result in an increase of the overall opinion, with the maximum increment being $c_i = |\boldsymbol{h}_i|(1 - \boldsymbol{s}_i|\boldsymbol{h}_i|/\boldsymbol{h}_i)$ if $\boldsymbol{h}_i \neq 0$ for any $i \in V$. Below we leverage this property to develop two algorithms solving the problem OPINIONMAX.

5.2 Optimal Solution

Despite the combinatorial nature, the OPINIONMAX problem can be optimally solved as follows. We first compute $c_i = |\boldsymbol{h}_i|(1 - \boldsymbol{s}_i|\boldsymbol{h}_i|/\boldsymbol{h}_i)$ for each node i with nonzero $\boldsymbol{h}_i$. Since c_i is the largest marginal gain for node $i \in V$, we then select the k nodes with the maximum value of c_i. Finally, we change the initial opinions of these k selected nodes in the following way. If nodes have positive structure centrality, change their internal opinions to 1; otherwise change their internal opinions to -1.

Based on the above three operations, we design an algorithm to optimally solve the problem OPINIONMAX, which is outlined in Algorithm 2. This algorithm first computes the inverse of matrix $\boldsymbol{I} + \boldsymbol{L}$ in $O(n^3)$ time. It then computes the vector $\boldsymbol{h}$ in $O(n^2)$ time, and calculates c_i for each $i \in V$ in $O(n)$ time. Finally, Algorithm 2 chooses k nodes and modifies their internal opinions according to the structure centrality $\boldsymbol{h}_i$ and the value c_i for each candidate node i, which takes $O(n)$ time. Therefore, the overall time complexity of Algorithm 2 is $O(n^3)$.

Since computing the vector $\boldsymbol{h}$ for structure centrality takes much time, Algorithm 2 is computationally unacceptable for large graphs. In the next subsection, we will design an efficient algorithm based on the signed Laplacian solver SIGNEDSOLVER.

5.3 Fast Algorithm for Opinion Optimization

To solve problem OPINIONMAX efficiently, using the signed Laplacian solver SIGNEDSOLVER we propose a fast algorithm to approximate the structure centrality vector $\boldsymbol{h} = (\boldsymbol{I}+\boldsymbol{L})^{-1}\mathbf{1}$ and solve the problem in nearly-linear time with respect to m, the number of edges. Let vector $\bar{\boldsymbol{h}}$ be the approximation of $\boldsymbol{h}$ returned by SIGNEDSOLVER. The following lemma shows the relationship between elements in $\bar{\boldsymbol{h}}$ and $\boldsymbol{h}$.

Lemma 5.1. *Given a signed graph $\mathcal{G}$ with Laplacian matrix $\boldsymbol{L}$, a positive integer k, and a parameter $\epsilon > 0$, let $\bar{\boldsymbol{h}} = \text{SIGNEDSOLVER}(\boldsymbol{I} + \boldsymbol{L}, \mathbf{1}, \delta)$ be the approximation of $\boldsymbol{h} = (\boldsymbol{I} + \boldsymbol{L})^{-1}\mathbf{1}$. Then the following inequality holds:*

$$|\boldsymbol{h}_i - \bar{\boldsymbol{h}}_i| \leq \epsilon/(4k),$$

for any $\delta = \frac{\epsilon}{4k\sqrt{n+4m}}$.

Proof. Let $\tilde{\boldsymbol{h}} = \boldsymbol{h} - \bar{\boldsymbol{h}}$, then we have $\left\|\tilde{\boldsymbol{h}}\right\|_{\boldsymbol{I}+\boldsymbol{L}}^2 \leq \delta^2 \left\|\boldsymbol{h}\right\|_{\boldsymbol{I}+\boldsymbol{L}}^2$. Thus, we obtain that for any $i \in V$,

$$\tilde{\boldsymbol{h}}_i^2 \leq \tilde{\boldsymbol{h}}_i^\top(\boldsymbol{I}+\boldsymbol{L})\tilde{\boldsymbol{h}}_i \leq \delta^2 \boldsymbol{h}^\top(\boldsymbol{I}+\boldsymbol{L})\boldsymbol{h} \leq (n+4m)\delta^2 \leq \frac{\epsilon^2}{16k^2},$$

which completes the proof. □

Based on Lemma 5.1, we can approximate each element of $\boldsymbol{h}$ with an absolute error guarantee. Exploiting this lemma, we propose a fast algorithm to approximately solve the problem OPINIONMAX, which is outlined in Algorithm 3. The performance of this fast algorithm is stated in the following theorem.

TABLE 1: Statistics for networks and performance of algorithms ApproxQuan and ApproxOpin. The initial opinions obey a uniform distribution.

Type	Networks	Nodes	Edges	Quantification of Social Phenomena							Opinion Optimization		
				Time (seconds)		Relative Error ($\times 10^{-8}$)					Time (seconds)		Relative Error ($\times 10^{-8}$)
				Exact	ApproxQuan	$I(\mathcal{G})$	$D(\mathcal{G})$	$F(\mathcal{G})$	$E(\mathcal{G})$	$P(\mathcal{G})$	Optimal	ApproxOpin	
Original Signed Graphs	Bitcoinalpha	3,783	24,186	1.92	0.38	0.26	0.01	0.02	0.37	0.02	1.78	0.21	2.47
	Bitcoinotc	5,881	35,592	4.40	0.40	1.70	0.08	0.14	0.95	0.15	4.16	0.07	0.98
	Wikielections	7,118	103,675	9.66	0.18	0.42	0.03	0.03	0.04	0.04	8.94	0.02	0.37
	WikiS	9,211	646,316	19.11	1.41	3.54	0.59	0.67	1.01	1.56	19.51	0.12	0.15
	WikiM	34,404	904,768	1276	2.42	7.15	0.18	0.02	1.36	0.23	1091	1.70	0.77
	SlashdotZoo	79,120	515,397	–	1.32	–	–	–	–	–	–	1.60	–
	WikiSigned	138,592	740,397	–	1.70	–	–	–	–	–	–	1.66	–
	Epinions	131,828	841,372	–	1.89	–	–	–	–	–	–	1.06	–
	WikiL	258,259	3,187,096	–	7.03	–	–	–	–	–	–	6.59	–
Modified Signed Graphs	PagesGovernment	7,057	89,455	9.31	0.72	1.88	6.94	8.41	0.04	0.91	8.33	0.40	1.57
	Anybeat	12,645	49,132	87.3	0.52	0.23	1.94	0.01	1.23	0.06	85.9	0.53	0.87
	Google	875,713	5,105,040	–	16.72	–	–	–	–	–	–	16.03	–
	YoutubeSnap	1,134,890	2,987,624	–	10.13	–	–	–	–	–	–	9.27	–
	Pokec	1,632,803	30,622,564	–	108.03	–	–	–	–	–	–	97.57	–
	DBpediaLinks	18,268,992	172,183,984	–	732.75	–	–	–	–	–	–	703.17	–
	FullUSA	23,947,300	57,708,600	–	186.72	–	–	–	–	–	–	170.69	–

Theorem 5.2. *For given parameters k and ϵ, algorithm* ApproxOpin *runs in time $\tilde{O}(m)$, and outputs a solution vector $\boldsymbol{y}$ satisfying $|g(\boldsymbol{y}^*) - g(\boldsymbol{y})| \leq \epsilon$, where $\boldsymbol{y}^*$ is the optimal solution to problem* OpinionMax.

Proof. Define $\bar{c}_i = |\bar{\boldsymbol{h}}_i|(1-\boldsymbol{s}_i|\bar{\boldsymbol{h}}_i|/\bar{\boldsymbol{h}}_i)$. Using Lemma 5.1 and the relation $c_i = |\boldsymbol{h}_i|(1 - |\boldsymbol{h}_i|/\boldsymbol{h}_i\boldsymbol{s}_i)$, we suppose that inequality $|c_i - \bar{c}_i| < \epsilon/(2k)$ holds for any $i \in V$. Assume that sets T_1 and T_2 are returned by algorithms Optimal and ApproxOpin, respectively. Then, we have

$$g(\boldsymbol{y}^*) - g(\boldsymbol{y}) = \sum_{i \in T_1} c_i - \sum_{j \in T_2} c_j \geq 0.$$

On the other hand, using $|c_i - \bar{c}_i| < \epsilon/(2k)$, we obtain

$$g(\boldsymbol{y}^*) - g(\boldsymbol{y}) = \sum_{i \in T_1} c_i - \sum_{j \in T_2} c_j \leq \sum_{i \in T_1} \bar{c}_i - \sum_{j \in T_2} c_j + \epsilon/2 \leq \epsilon,$$

which completes the proof. □

6 Experiments

To evaluate the accuracy and efficiency of our algorithms ApproxQuan and ApproxOpin for two different tasks, we conduct extensive experiments on sixteen signed networks of different sizes.

6.1 Setup

Datasets. We use sixteen datasets of two types of signed graphs: real-world original signed graphs and artificially modified real-world graphs. The real-world original signed graphs are from actual networks, for which the original signs are kept unchanged. The artificially modified graphs are generated from real unsigned graphs, by randomly assigning a negative sign to each edge in unsigned graphs with a probability of 0.3. These network datasets are publicly available in KONECT [41] and SNAP [42], and their statistic is presented in Table 1, where networks are listed in increasing order of the number of nodes.

Environment and repeatability. We conduct all experiments using a single thread on a machine with a 2.4 GHz Intel i5-9300 CPU and 128GB of RAM. All algorithms are realized using the programming language *Julia*. The parameter ϵ is set to be 10^{-5} for all experiments. Our code is publicly available at https://github.com/signFJ/signFJ.

Internal opinion distributions. In our experiments, we use three different distributions of initial opinions: uniform, exponential, and power-law, which are generated as follows. For the uniform distribution, the initial opinion of every node is generated uniformly in the range of $[-1, 1]$. For the exponential and power-law distributions, we first generate the initial opinions of all nodes in the range of $[0, 1]$ as in [20]. Then for every node, we change its initial opinion to its opposite number with probability of 0.5. Note that for all these three distributions of initial opinions, the experiment results are similar. We here only present the results for the uniform initial opinions due to space constraints but report the results for the other two distributions of initial opinions in Appendix at https://github.com/signFJ/signFJ.

6.2 Performance of Algorithm ApproxQuan

We first evaluate the efficiency of our fast algorithm ApproxQuan for quantifying various social phenomena. For this purpose, we compare it with the exact algorithm, called Exact, which computes all relevant quantities by inverting the matrix $\boldsymbol{I} + \boldsymbol{L}$. Table 1 reports the running time of ApproxQuan and Exact on different networks. As shown in Table 1, the running time of ApproxQuan is always less than that of Exact for each of the considered networks with relatively small sizes. For networks with more than 40,000 nodes, Exact fails to run due to the high memory and time requirements. In contrast, ApproxQuan is able to approximate all the quantities in less than one thousand seconds. Moreover, ApproxQuan is scalable to massive networks with over 20 million nodes.

In addition to being highly efficient, algorithm ApproxQuan is also very accurate, compared with algorithm Exact. To show this, in Table 1, we compare the approximate results for ApproxQuan with the exact results for Exact. For each network, we compute the relative error $|\theta - \tilde{\theta}|/\theta$ for each quantity θ and its approximation $\tilde{\theta}$ returned by ApproxQuan. Table 1 gives the relative errors for the five estimated quantities, including internal conflict $I(\mathcal{G})$, disagreement $D(\mathcal{G})$, disagreement with friends $F(\mathcal{G})$, agreement with opponents $E(\mathcal{G})$, and polarization $P(\mathcal{G})$. The results

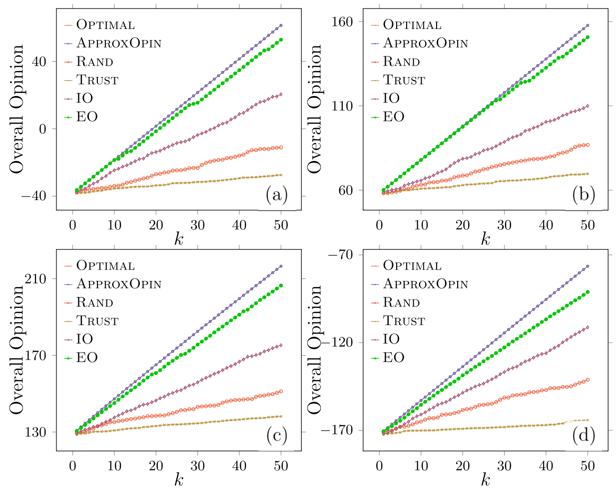

Fig. 1: Overall opinions for algorithms ApproxOpin, Optimal, and four baselines on four real networks: (a) Bitcoinalpha, (b) Wikielections, (c) WikiM, and (d) Anybeat.

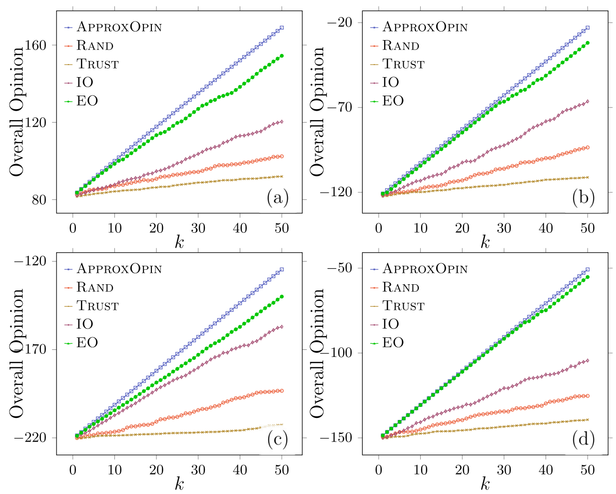

Fig. 2: Overall opinions for algorithms ApproxOpin and four baselines on four real networks: (a) Epinions, (b) WikiL, (c) Pokec, and (d) FullUSA.

indicate that the actual relative errors for all quantities and networks are negligible, with all errors less than 10^{-7}. Thus, ApproxQuan is not only highly fast but also highly effective in practice.

6.3 Performance of Algorithm ApproxOpin

We continue to evaluate the performance of ApproxOpin. To achieve our goal, we compare algorithm ApproxOpin with the optimal algorithm Optimal and four baselines [29], including Rand [35], Trust [43], IO [10], and EO [11]. Rand [35] randomly selects k nodes and changes their initial opinions to 1. Trust [43] selects k nodes with the largest differences between the numbers of friends and opponents, and changes their initial opinions to 1. IO [10] selects k nodes with the lowest internal opinions and changes their initial opinions to 1. EO [11] selects k nodes with the lowest expressed opinions, and changes their initial opinions to 1.

We first assess the effectiveness of ApproxOpin. We change the initial opinions of $k = 1, 2, \dots, 50$ nodes by using ApproxOpin, Optimal, and the four baseline approaches. Figure 1 illustrates the comparison of overall opinion for these methods on four small networks with less than 40,000 nodes, since for networks with over 40,000 nodes, Optimal fails to run. We observe that for these small networks, ApproxOpin consistently returns results that are close to the optimal solutions, both of which outperform the other four baselines. To further demonstrate the accuracy of ApproxOpin, in Table 1, we compare the relative error for the gain of the overall opinion for ApproxOpin with respect to that for Optimal on seven small networks with $k = 50$. As shown in Table 1, the relative errors are all less than 10^{-7}, indicating the high similarity of the results obtained by ApproxOpin and Optimal. We also compare ApproxOpin with the baseline strategies on four relatively large networks with over 40,000 nodes, and report the results in Figure 2, which again indicates that ApproxOpin is much better than the four baselines.

With regard to the efficiency, in Table 1, we compare the running time of ApproxOpin and Optimal on different networks for $k = 50$. As shown in Table 1, ApproxOpin is significantly faster than Optimal, especially when networks become larger. Particularly, Optimal fails to run on networks with more than 40,000 nodes, while ApproxOpin can still run efficiently, which is even scalable to massive networks with more than twenty million nodes.

7 Related Work

In this section, we briefly review some existing studies related to ours.

FJ model for opinion dynamics. In the study of opinion dynamics, a crucial step for understanding this dynamical process is the establishment of mathematical models. In past decades, several models have been proposed to better understand opinion propagation and formulation [3], [4], [5], [44]. A popular one is the FJ model [16], which has been extensively studied on unsigned graphs. For example, the sufficient condition for stability was studied in [45], the formula for the equilibrium expressed opinion was derived in [34], [46], and the interpretations were provided from different aspects in [12], [34], [47]. Besides, by incorporating different aspects affecting opinion evolution and formulation, many variants of the FJ model have been proposed, including peer pressure [25], stubbornness [21], interactions among higher-order neighbours [48], and so on. Most prior works for the FJ model are based on unsigned graphs, which capture only the positive or cooperative relationships between individuals, ignoring the antagonistic or competitive relationships. Very recently, the FJ model was extended to the signed graphs, which incorporate both cooperative and competitive relationships [29], [30], [31], [32]. For the FJ model on signed graphs, some relevant problems have been addressed, including the convergence criteria [31], explanation of opinion update [30], and opinion maximization by changing initial opinion [29]. However, the interpretation for expressed opinions is still lacking.

Quantification and algorithms for social phenomena. The explosive growth of social media and online social networks produces diverse social phenomena, such as polarization [9], [10], [49], disagreement [10], filter bubbles [50],

[51], conflict [11], and controversy [11], to name a few. In addressing these challenges, research has evolved in different directions. Some studies aimed to efficiently compute these quantities [20], [21]. Later, research aimed to find user groups open to "counter-information" [52], [53] and tried to connect users with opposing views, hoping to lessen filter bubble effects [10], [54], [55], [56]. Recently, the exploration of using influence models in social media to combat filter bubbles has gained traction [57], [58], [59], [60]. These measures and algorithms for social phenomena tend not to apply to signed graphs. To make up for this deficiency, we extend these measures for the FJ model to signed graphs. Due to the incorporation of competitive relationships, previous approximation algorithms [20], [21] are not suitable for signed graphs anymore. This motivates us to present a nearly linear time algorithm for estimating these quantities on signed graphs.

Optimization of overall opinion. Various schemes have been proposed to maximize or minimize the overall opinion based on different models for opinion dynamics. On the basis of the DeGroot model, many groups have addressed the problem of maximizing the overall opinion by leader selection [61], [62], [63], [64] or link suggestion [65], [66]. Based on independent cascade and linear threshold models, a similar problem, called the influence maximization problem, has also been studied [67], [68], [69], [70]. The opinion optimization problem has also attracted extensive attention for the FJ model. In the past decade, different node-based strategies have been applied to optimize the overall opinion of the FJ model on unsigned graphs, including modifying initial opinions [22], expressed opinions [12], and susceptibility to persuasion [13], [14], [15], [23]. Most previous research focused on opinion optimization on unsigned graphs, with the exception of a few work [29]. In [29], the problem of opinion optimization on signed graphs was studied by changing initial and external opinions, and two algorithms were developed with complexity $O(n^3)$, which are computationally infeasible for large graphs. Although we address a similar problem, our algorithm is efficient and effective, with nearly-linear time complexity and proven error guarantee compared to the optimal solution.

In the realm of signed graph research, a variety of studies have extensively delved into diverse aspects of the topic, including finding conflicting groups [71], exploring polarization [72], detecting communities [73], [74], and so on. On the topic of signed cliques, different algorithms have been proposed for computing and enumerating [75], [76], [77]. Furthermore, the influence diffusion process and influence maximization problem have also been studied on signed networks [78], [79], [80]. However, the aforementioned studies on signed graphs are not applicable to the FJ model within signed networks.

8 Conclusion

In this paper, we studied the Friedkin-Johnsen (FJ) model for opinion dynamic on a signed graph. We first interpreted the equilibrium opinion of every node by expressing it in terms of the absorbing probabilities of a defined absorbing random walk on an augmented signed graph. We then quantified some relevant social phenomena and represented them as the ℓ_2 norms of vectors. Moreover, we proposed a signed Laplacian solver, which approximately evaluating these quantities in nearly-linear time but has error guarantees. We also considered the problem of opinion optimization by modifying the initial opinions of a fixed number of nodes, and presented two algorithms to solve this problem. The first algorithm optimally solves the problem in cubic time, while the second algorithm provides an approximation solution with an error guarantee in nearly-linear time. Extensive experiments on real singed graphs demonstrate the effectiveness and efficiency of our approximation algorithms.

Future work includes the applications of our signed Laplacian solver to other problems for FJ model on signed graph, such as optimizing disagreement, conflict, and polarization under different constraints, or maximizing (or minimizing) the overall opinion by using other strategies different that using in this paper.

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Appendix

In this section, we provide additional results of Section 6.

.1 Experiments for Exponential and Power-Law distributions of Initial Opinions

In this subsection, we present the experimental results for the other two distributions of initial opinions, i.e., exponential distribution and power-law distribution.

Tables 2 and 3 present the results for our algorithms when the initial opinions are distributed according to exponential distribution and power-law distribution, respectively. As in Table 1, Tables 2 and 3 display the running time and relative error of our algorithms.

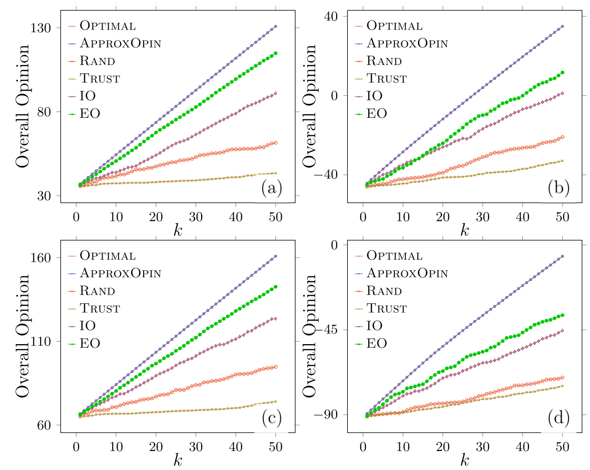

Fig. 3: Overall opinions for algorithms ApproxOpin, Optimal, and four baselines on four real networks: (a) Bitcoinalpha, (b) Wikielections, (c) WikiM, and (d) Anybeat.

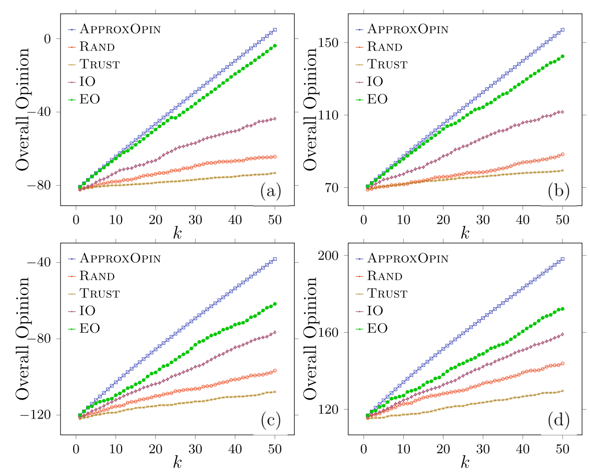

Fig. 4: Overall opinions for algorithms ApproxOpin and four baselines on four real networks: (a) Epinions, (b) WikiL, (c) Pokec, and (d) FullUSA.

With respect to the opinion optimization problem, Figures 3 and 4 display the results obtained by the algorithm ApproxOpin and other baseline methods for the case when the initial opinions follow an exponential distribution. Figures 5 and 6 present the results when the initial opinions obey a power-law distribution.

TABLE 2: Statistics for networks and performance of algorithms ApproxQuan and ApproxOpin. The initial opinions obey an exponential distribution.

Type	Networks	Nodes	Edges	Quantification of Social Phenomena							Opinion Optimization		
				Time $(seconds)$		Relative Error $(\times 10^{-8})$					Time $(seconds)$		Relative Error $(\times 10^{-8})$
				Exact	ApproxQuan	$I(\mathcal{G})$	$D(\mathcal{G})$	$F(\mathcal{G})$	$E(\mathcal{G})$	$P(\mathcal{G})$	Optimal	ApproxOpin	
Original Signed Graphs	Bitcoinalpha	3,783	24,186	1.90	0.37	1.58	0.93	0.01	0.71	0.12	1.77	0.20	1.55
	Bitcoinotc	5,881	35,592	4.33	0.40	1.65	0.12	3.07	1.99	0.03	4.08	0.10	0.68
	Wikielections	7,118	103,675	9.56	0.38	0.41	0.01	0.06	0.17	0.01	8.88	0.15	0.22
	WikiS	9,211	646,316	19.30	1.36	0.64	0.99	1.91	2.54	0.82	19.11	0.18	1.97
	WikiM	34,404	904,768	1283	2.61	2.08	0.03	0.27	0.82	3.67	1071	1.68	1.08
	SlashdotZoo	79,120	515,397	–	1.33	–	–	–	–	–	–	1.56	–
	WikiSigned	138,592	740,397	–	1.70	–	–	–	–	–	–	1.81	–
	Epinions	131,828	841,372	–	1.98	–	–	–	–	–	–	1.23	–
	WikiL	258,259	3,187,096	–	7.11	–	–	–	–	–	–	6.72	–
Modified Signed Graphs	PagesGovernment	7,057	89,455	9.34	0.70	5.73	3.24	1.18	0.48	1.38	8.52	0.43	0.67
	Anybeat	12,645	49,132	86.4	0.55	0.28	0.64	0.48	3.21	2.08	85.1	0.50	1.37
	Google	875,713	5,105,040	–	16.92	–	–	–	–	–	–	16.22	–
	YoutubeSnap	1,134,890	2,987,624	–	10.12	–	–	–	–	–	–	9.28	–
	Pokec	1,632,803	30,622,564	–	108.34	–	–	–	–	–	–	97.67	–
	DBpediaLinks	18,268,992	172,183,984	–	733.41	–	–	–	–	–	–	702.48	–
	FullUSA	23,947,300	57,708,600	–	185.34	–	–	–	–	–	–	170.15	–

TABLE 3: Statistics for networks and performance of algorithms ApproxQuan and ApproxOpin. The initial opinions obey a power-law distribution.

Type	Networks	Nodes	Edges	Quantification of Social Phenomena							Opinion Optimization		
				Time $(seconds)$		Relative Error $(\times 10^{-8})$					Time $(seconds)$		Relative Error $(\times 10^{-8})$
				Exact	ApproxQuan	$I(\mathcal{G})$	$D(\mathcal{G})$	$F(\mathcal{G})$	$E(\mathcal{G})$	$P(\mathcal{G})$	Optimal	ApproxOpin	
Original Signed Graphs	Bitcoinalpha	3,783	24,186	1.80	0.23	0.22	0.23	0.08	1.73	1.68	1.70	0.31	1.98
	Bitcoinotc	5,881	35,592	4.35	0.61	1.82	0.68	3.21	1.37	0.30	4.37	0.12	3.84
	Wikielections	7,118	103,675	9.82	0.49	1.35	1.68	0.86	0.03	0.01	8.74	0.33	0.07
	WikiS	9,211	646,316	19.03	1.21	1.66	2.58	0.74	1.68	2.07	19.41	0.84	0.13
	WikiM	34,404	904,768	1289	2.70	3.28	3.89	0.78	0.57	1.39	1086	1.61	1.12
	SlashdotZoo	79,120	515,397	–	1.36	–	–	–	–	–	–	1.70	–
	WikiSigned	138,592	740,397	–	1.85	–	–	–	–	–	–	1.73	–
	Epinions	131,828	841,372	–	1.90	–	–	–	–	–	–	1.12	–
	WikiL	258,259	3,187,096	–	7.04	–	–	–	–	–	–	6.64	–
Modified Signed Graphs	PagesGovernment	7,057	89,455	9.33	0.89	0.46	0.07	1.62	0.34	0.71	8.36	0.50	0.90
	Anybeat	12,645	49,132	86.3	0.49	2.37	1.71	0.43	0.03	0.07	85.5	0.67	0.07
	Google	875,713	5,105,040	–	16.74	–	–	–	–	–	–	15.90	–
	YoutubeSnap	1,134,890	2,987,624	–	10.10	–	–	–	–	–	–	9.35	–
	Pokec	1,632,803	30,622,564	–	107.12	–	–	–	–	–	–	96.73	–
	DBpediaLinks	18,268,992	172,183,984	–	735.37	–	–	–	–	–	–	704.97	–
	FullUSA	23,947,300	57,708,600	–	186.14	–	–	–	–	–	–	172.88	–

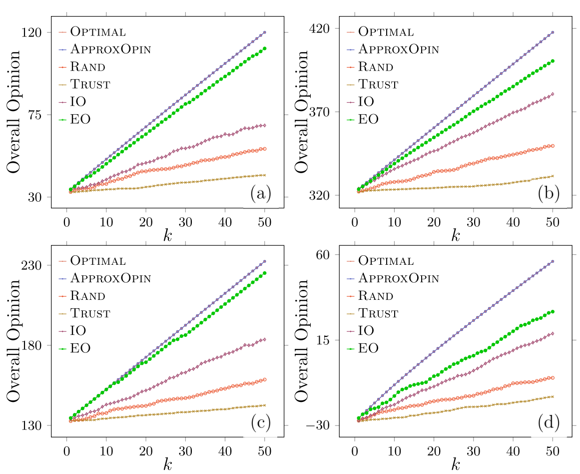

Fig. 5: Overall opinions for algorithms ApproxOpin, Optimal, and four baselines on four real networks: (a) Bitcoinalpha, (b) Wikielections, (c) WikiM, and (d) Anybeat.

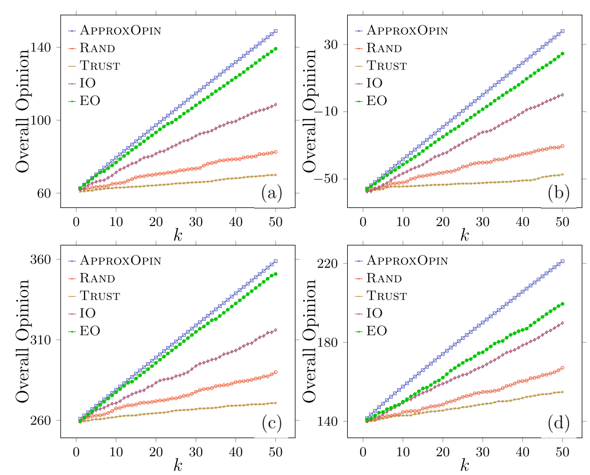

Fig. 6: Overall opinions for algorithms ApproxOpin and four baselines on four real networks: (a) Epinions, (b) WikiL, (c) Pokec, and (d) FullUSA.