Notation: Let $f_X(x)$ denote the probability density of X and $f_{XY}(x, y)$ denote the joint probability density of X and Y.

We are given $f_X(x) = 1$ and $f_{Y|X=x}(y, x) = 1/x$. Our goal is (1) to evaluate $f_{X|Y=y}(x, y)$ and (2) to evaluate the expected value of that conditional probability distribution.

In general, we have

$$f_{XY}(x, y) = f_{Y|X=x}(y, x) f_X(x) = f_{X|Y=y}(x, y) f_Y(y).$$

As such, we can calculate

$$f_{XY}(x, y) = 1 \cdot (1/x) = 1/x.$$

In general, it is true that

$$f_Y(y) = \int f_{XY}(x, y)\,\mathrm{d}x.$$

For some given value of $X = x$, Y is drawn from the range $[0, x]$. We can conversely say that for some given value of $Y = y$, X is drawn from the range $[y, 1]$. As such, we integrate x from y to 1, and we have

$$f_Y(y) = \int_y^1 \frac{1}{x}\,\mathrm{d}x = \left.(\ln x)\right|_y^1 = -\ln y.$$

Now, we can calculate

$$f_{X|Y=y}(x, y) = \frac{f_{Y|X=x}(y, x) f_X(x)}{f_Y(y)} = \frac{(1/x) \cdot 1}{-\ln y} = -\frac{1}{x \ln y}.$$

Finally, we calculate the expected value by evaluating another integral.

$$\mathbb{E}(X \mid Y = y) = \int_y^1 x \cdot f_{Y|X=x}(y, x)\,\mathrm{d}x = \int_y^1 -\frac{1}{\ln y}\,\mathrm{d}x = \frac{y-1}{\ln y}.$$