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ON THE HISTORY OF THE METHOD OF LEAST SQUARES.

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IT is the object of this article to present some brief notes concerning the various demonstrations of the method of Least Squares, with references to the original works or memoirs in which they were given.

The honor of the first publication of the method belongs to Legendre. It is given in his work *Nouvelles methodes pour la determination des orbites des cometes,* published at Paris in 1805. It is a quarto of viii + 80 pages: some copies have a supplement and a title page dated 1806. In this work the rule that the sum of the squares of the errors should be made a minimum to obtain the adjusted values of observed quantities is proposed as a convenient method only. The rule for the formation of normal equations is deduced and applied in practical examples, the arithmetical mean is shown to be a particular case of the method and the analogy of the formula with those derived in mechanics for finding the centre of gravity of bodies is noticed. Referring to this analogy Legendre says: "la méthode des moindres quarrés fait connoître, en quelque sorte, le centre autour duquel viennent se ranger tous les résultats fournis par l'expérience, de manièr à s'en écarter le moins qu'il possible." Although this can scarcely be called a proof, it shows that Legendre fully appreciated the advantage of the method.

The first proof of the method of Least Squares was given by Dr. Robert Adrain in *The Analyst,* an American Journal edited by him during the early part of this century. The title of his paper is *Research concerning the probabilities of the errors which happen in making observations,* and it is given in *The Analyst* for 1808, No. IV, pp. 93 — 109. The term "Least Squares" is not used and Adrain seems to have been entirely unacquainted with Legendre's researches. The proof consists in showing that the probability y of the error x is given by an equation of the form

$$y = c\,e^{-h^2x^2}$$

in which c and h are constants and e the base of the natural system of logarithms. Hence by well known rules the probability of the system of errors $x_1, x_2, \ldots . x_n$ is

$$Y = c^n e^{-h^2(x_1^2+x_2^2+ \ldots . + x_n^2)}$$

and the most advantageous or most probable system will be that for which Y is a maximum, and this requires that

$$x_1^2+x_2^2+x_3^2+ \ldots . +x_n^2 = \text{a minimum}$$

whence the method of Least Squares. The proof by which the law of facility or probability of error is established cannot be discussed here: it is analysed and shown to be unsatisfactory by Glaisher in the *Memoirs* . . . . . *Astron. Soc. London,* Vol. XXXIX, pp. 75—81. Adrain's paper was unknown to mathematicians until 1871, when it was republished in *Amer. Jour. Sci.,* Vol. I, pp. 412—414.

The second proof was given by Gauss in the *Theoria motus corporum coelestium* . . . . published at Hamburg in 1809. It occupies pages 205—224 of the volume. Assuming that the arithmetical mean is the most probable value of a quantity directly observed, Gauss shows that the law of facility of error must be of the exponential form

$$y = c\,e^{-h^2x^2}$$

from which the rule of Least Squares follows. This is the proof given in the great majority of books. The assumption of the rule of the arithmetical mean impairs greatly its strictness, and as presented in Chauvenet's treatise (Appendix to *Astronomy*) it is particularly illogical, the constant c being given the value $h\pi^{-\frac{1}{2}}$, so that the probability of an error x_1 is a finite quantity. The constant c should be $h\pi^{-\frac{1}{2}}dx$.

The third proof was given by Laplace in 1810 in the *Memoires...Institut France,* for 1809, pp. 383 — 389 and 559 — 565, and is reproduced in his *Theorie analytique des Probabilites,* Chap. IV. It uses only the axiom that positive and negative errors are equally probable and is independent of the particular form of the law of facility of error, provided the number of observations is infinite. It is the most satisfactory of all the demonstrations of the method. For an excellent account and simplification of Laplace's reasoning see Glaisher, *Memoirs* . . . . . . *Astron. Soc. London,* Vol. XXXIX, p. 92, et. sq. The proof is given by De Morgan, Airy and other English writers.

On pages 318—319 of the *Theorie* . . . *des Prob.* is given what is sometimes called Laplace's second demonstration. It depends on the axiom that the mean of all the errors taken positively must be a minimum to give the most advantageous values.

The fifth proof was given by Gauss in a memoir entitled: *Theoria combinationis observationum erroribus minimis obnoxiae* which was published in 1823 in the *Comment . . . Soc . . . Gottingen,* Vol. V, pp. 33—90 and also issued separately. The proof takes for granted that the mean value of the sum of the squares of the errors may be used as a measure of the precision of the observations. It introduces no law of facility of error. The proof is entirely untenable and has we believe only been followed by Helmert in 1872.

The sixth proof was given by Ivory in 1825 in an article *On the Method of the Least Squares* in Tìlloch's *Phil. Mag.*, Vol. LXV, pp. 3—10. It rests on a vague analogy with the properties of the lever, and was well refuted by Ellis in *Trans. Camb. Phil. Soc.*, Vol. VIII, pp. 217—219.

The seventh attempted proof was given by Ivory in 1826 in Tìlloch's *Phil. Mag.*, Vol. LXVIII, pp. 161—165. Although quoted by Francoeur in his *Astronomie practique* (Paris 1830), pp. 426—428, as perfectly valid, it is in our opinion the most unsatisfactory of all. See also Ellis' paper quoted above.

The eighth proof was given in 1837 by Hagen in his *Grundzuge der Wahrscheinlichkeitsrechnung* published at Berlin in octavo. The proof rests on the hypothesis that an error is the algebraic sum of an indefinitely large number of small elementary errors, which are all equal and which are equally likely to occur as positive or negative. This postulated, Hagen proves by the law of combinations that the law of facility of error is

$$y = c\,e^{-h^2x^2}$$

whence the rule of Least Squares as shown above. Next to Laplace's demonstration this is to us the most satisfactory proof. It is given by Wittstein in 1849, by Dienger in 1852, by Encke in 1850 (see *Berliner Jahrbuch* for 1853), and in a slightly modified form by Quetelet, Tait and others. The "*New Investigation . . . . .*" in this JOURNAL for Sept. 1876, also gsves Hagen's proof on p. 133 — 135. A second edition of Hagen's book, we may mention, appeared in 1867.

The ninth proof is due to Bessel and was published in an article entitled *Untersuchungen ueber die Wahrscheinlichkeit der Beobachtungsfehler* which appeared in the *Astronomische Nachrichten,* 1838, Vol. XV, col. 369–404. It considers errors as arising from sources of error, which need not as in Hagen's proof be very small. The result is that the law of error approximates closely to the exponential form when many sources of error act together.

The tenth proof was given in 1844 by Donkin in *An essay on the Theory of the Combination of Observations* printed at Oxford, and in 1855 translated in Liouville's *Jour. Math.*, Vol. XV, pp. 297—322. The usual rules for the adjustment of observations are here deduced by a kind of metaphysical statics from the laws of Mechanics. No law of facility is employed.

The eleventh proof was given in 1850 by John Herschel in a review, *Quetelet on Probabilites*, in the *Edinburgh Review*, Vol. XCII, pp. 1—57. It deduces the exponential law of facility by reasoning which is not received by the best mathematicians. See Glaisher's paper quoted above, and Ellis in *Lond. Phil. Mag.*, 1850, Vol. XXXVII, pp. 321—328.

A twelvth proof of the law of facility of error "resting on a natural and obvious assumption" was given by Donkin in the *Quart. Jour. Math.*, 1857, Vol. I, pp. 152—162. The assumption does not appear however to be perfectly obvious.

A thirteenth investigation which may be mentioned is by Crofton, *On the Proof of the Law of Errors of Observations* in the *Phil. Trans....* London for 1870, pp. 175—188, in which the law of facility of error is investigated, on the hypothesis that an error arises from the joint operation of a large number of small errors, positive and negative errors not being equally probable *a priori*. The discussion is not very clear.

Although no one of these thirteen proofs is perfectly satisfactory, yet each is valuable as setting the subject in a new light and illustrating the laws of thought which have led to the universal employment of the arithmetical mean, and other rules for the adjustment of observations. No discussion of their relative values can be attempted here. The critical papers of Ellis and of Glaisher, quoted above, may in this connection be mentioned as of great value to students of the theory of the subject.

ON THE ROTATION OF SATURN.

BY PROF. A. HALL.

ON the night of December 7th while observing Japetus I noticed a bright spot on the ball of Saturn. The spot was 2″ or 3″ in diameter, round and well defined, and of a brilliant white color. It was north of the Ring, nearly midway of the disk in the direction of a circle of declination, and when first seen was about one third the distance from the center of the Ball toward the following edge. The transit of the spot over the center of the disk or rather over the line bisecting the disk and perpendicular to the major