INTERCHAPTER IX

The Method of Least Squares

As we have just seen, the period of unexpected productivity and fruitful cooperation with Wilhelm Weber came to a close with the end of the year 1837. Gauss and Weber continued their joint work as the editors of their geomagnetism journal but their separation did not permit any genuine exchange of ideas. In 1849, Weber returned to Göttingen, after yet another political upheaval, but by then Gauss was very old and no longer active in research. The last 18 years of Gauss's life were free of the painful commotions which had accompanied and marred his earlier years, but they were also no longer productive in a strict sense, though Gauss was far from inactive. Below, we shall give an account of Gauss's scientific work during this period, but it seems appropriate to pause for a moment and discuss in what light Gauss viewed his own research and the philosophical notions which accompanied it. They assumed their final shape during this period of Gauss's life and were now much clearer than in his earlier approaches.

One of Gauss's most efficient tools in his research was the method of least squares. When he first developed it, shortly before the turn of the century, he did not consider it very important; Gauss remarked later that he was certain that his predecessor at the astronomical observatory in Göttingen, the elder of the Tobias Mayers, had known the method. After looking through Mayer's papers Gauss saw that this was not the case, but he still could not take credit for the invention of the method. Formal priority belongs to Legendre, who first published it in 1806, i.e., clearly before Gauss made the method public, though he had clearly and frequently used it much earlier.[1]

Gauss motivated and derived the method of least squares in several substantially different ways. Here, we summarize his most "mature"[2] approach, as developed in the two papers "Theoria combinationis observationum erroribus minimis obnoxiae" I and II (1821, 1823). Part I is devoted to a systematic and theoretical investigation of the theory of errors, presented as a part of probability theory. Of the two, essentially different, types of error, systematic and accidental errors (the latter are the *Zufallsfehler*), only the accidental errors are relevant; for certain domains, a certain probability can be assigned to them. Formally, Gauss defines the function $\varphi(x)$ as the relative error in the observation x. Then $\varphi(x)dx$ expresses the probability of an error between x and $x + dx$. φ is normalized by the condition

$$\int_{-\infty}^{\infty} \varphi(x)\,dx = 1.$$

The decisive requirement is that the integral

$$\int x^2 \varphi\,dx$$

attain a minimum. This condition expresses the idea that the square of the error is its most suitable weight. This is where Gauss's approach is different from that of Laplace, who earlier had tried to use the absolute value of an error for its weight. This is why Gauss's method is called the method of least squares: computationally, it is clearly superior to Laplace's original method.

After developing the theoretical basis of his theory, a suitable function φ had to be found. In general, the distribution of the errors will not be known in advance, and one has to choose from arbitrary functions ψ with the single requirement that $(*)$ be satisfied. At this point, Gauss introduces, after some heuristic preparations, the exponential e^{-x^2} ("normal distribution") as a particularly natural way in which the errors of observation occur. Gauss concludes Part I of the paper with some complicated considerations which are motivated by astronomical questions and are of no interest here.

The second part of "Theoria combinationis . . ." contains applications of the method of least squares, mostly to problems from astronomy. Gauss also develops a complicated, but not difficult, elimination procedure to determine the best observations. Another problem concerns the inclusion of new experimental data at an advanced stage of the evaluation. Gauss shows how to make use of the new data without having to discard any earlier calculations. The paper ends with discussions of the relation between the observed and the calculated errors and of the accuracy with which the average error can be determined.

A supplement is devoted to a problem from geodesy. It deals with a situation in which it is not known how the observed parameters depend on certain other, equally unknown, elements; only their mutual dependence is given explicitly. Gauss solves this problem by deriving once again the method of least squares; the approach is basically the same as his earlier derivation. The supplement also contains two worked out examples from geodesy in which Gauss uses his own data and data from the Dutch triangulation; he does not omit to comment how important the use of real data is. The direct measurement of the three angles of a triangle will usually not add up to 180°.

Gauss never mentioned, in any of his papers, the possibility of statistical distributions other than the normal one. The very satisfactory results which he obtained by using the exponential distribution did not prompt him to look for other approaches.

Gauss gave altogether three different derivations of the method of least squares, the first of them in *Th. mot.* Legendre, in his brochure of 1806 on the orbits of comets, seems to have developed it from a strictly computational motivation, and this appears to have also been Gauss's initial approach. Soon after Legendre had published the method, Laplace succeeded in connecting the least squares with probabilistic considerations, but we do not know whether Gauss used Laplace's work or whether he developed the foundations independently. There are some reasons which make the latter assumption more likely; in any event, Gauss's work went beyond Laplace's, and for Gauss the method gained in value by its probabilistic justification.

Least squares were Gauss's indispensable theoretical tool in experimental research; increasingly, he came to see it as the most important witness to the connection between mathematics and nature. Its efficiency was the most blatant demonstration of the fact that natural phenomena could efficiently be investigated by mathematical methods. Gauss would have expressed this fact by a much stronger statement—for him, mathematics governed the workings of nature, and the mathematical penetration of the natural sciences showed to what degree they had been understood.

We have seen that there were other mathematical techniques which Gauss found helpful in his desire to understand natural phenomena and processes. They were potential theory, including Coulomb's law, extremal principles, and the calculus of variations. Gauss was even aware of an application of the theory of numbers: he knew that crystalline structures could be described with the help of ternary forms.[3] These examples fortified Gauss's conviction of the mathematical character of nature; this belief had its roots in the 18th century, but Gauss did much to make it more credible.

There are few explicit remarks by Gauss about his understanding of nature and the role of mathematics in the physical sciences. We quote here the last paragraph of his paper about an extremal principle in mechanics:

> It is very remarkable that the free movements, if they cannot coexist with the necessary conditions, are modified by Nature in exactly the same way in which the calculating mathematician, according to the method of least squares, reconciles observations which are connected to each other by necessary dependencies. This analogy could be pursued further, but I do not intend to do so at the moment.[4]

Volume XII of Gauss's works contains a lengthy article with the title "Astronomische Antrittsvorlesung" ("Astronomical Inaugural Lecture"). It cannot be dated exactly, but it is quite early. In it, Gauss discusses a variety of topics which one might mention in such a lecture, most importantly the different motivations for the study of astronomy. Astronomy is a useful science, but Gauss sees this only as a secondary motive. It is primarily the disinterested quest for truth which makes astronomy such a gratifying object of research. The highest reward of the astronomer is the satisfaction of being able to contemplate the wonderful ways in which the world has been organized and to experience the reassurance which comes from the recognition of the harmony of the creation.*

One can find out more about Gauss's opinions and beliefs from remarks in the correspondence. Occasionally, he ventured philosophical statements, expressing a general disdain for the vagueness of most philosophers. There are derogatory remarks about Plato, Wolff, and the contemporaries Schelling and Hegel.[6] Gauss used Hegel's absurd astronomical speculations in his dissertation as an example of the stupidity of his kind. Kant and his work were appreciated, though naturally not his geometrical ideas (Kant proved the "necessity" of Euclidean space) and, more importantly, his a priori classification of mathematics as a synthetic science. We also know that Gauss was quite fond of the work of J. F. Fries (1773–1843), one of the few philosophers of this age with a serious interest in the experimental sciences, particularly astronomy. Philosophically, Fries stood in the tradition of Kant, but tried to integrate the contemporary elements of positivism into his philosophical system. Gauss was particularly fond of a history of philosophy which Fries wrote.†[7] In general, Gauss was not interested in philosophical arguments, and his attitude reflects more the modern scientist than the philosophically inclined mathematician or scientific scholar of the 18th century.

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* In this lecture, Gauss makes a "political" statement which may come as a surprise. He argues against a narrow utilitarian point of view; we quote:

To judge in this way demonstrates not only how poor we are, but also how small, narrow, and indolent our minds are. it shows a disposition always to calculate the payoff before the work, a cold heart and a lack of feeling for everything that is great and honors man. One can unfortunately not deny that such a mode of thinking is not uncommon in our age, and I am convinced that this is closely connected with the catastrophes which have befallen many countries in recent times. do not mistake me, I do not talk of the general lack of concern for science, but of the source from which all this has come, of the tendency to everywhere look out for one's advantage and to relate everything to one's.physical well-being, of the indifference towards great ideas, of the aversion to any effort which derives from pure enthusiasm I believe that such attitudes, if they prevail, can be decisive in catastrophes of the kind we have experienced.[5]

This last allusion seems to refer to the Napoleonic wars and to the defeats of the German states and Prussia. The latest date for the lecture is 1815 (the year of Napoleon's final defeat) but it could have been as early as 1808. If given so early it would have been a bold and explicit political statement

† Fries, never particularly popular, was forgotten soon after his death. In the 1920s, the philosopher Leonard Nelson (1882–1929) started a Fries renaissance in his attempt to find some new direction between Husserl's phenomenalism and the then vigorous neo-Kantian school Nelson, a grandson of Dirichlet, taught at Göttingen and was quite popular among the mathematicians of the Hilbert school.