Lecture 4
Capacity of Wireless Channels

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What we have learned

- So far: looked at specific schemes and techniques

- Lecture 2: point-to-point wireless channel
  - Diversity: combat fading by exploiting inherent diversity
  - Coding: combat noise, and further exploits degrees of freedom

- Lecture 3: cellular system
  - Multiple access: TDMA, CDMA, OFDMA
  - Interference management: orthogonalization (partial frequency reuse), treat-interference-as-noise (interference averaging)

Information Theory

- Is there a framework to …
  - Compare all schemes and techniques fairly?
  - Assert what is the fundamental limit on how much rate can be reliably delivered over a wireless channel?

- Information theory!
  - Provides a fundamental limit to (coded) performance
  - Identifies the impact of channel resources on performance
  - Suggest novel techniques to communicate over wireless channels

- Information theory provides the basis for the modern development of wireless communication

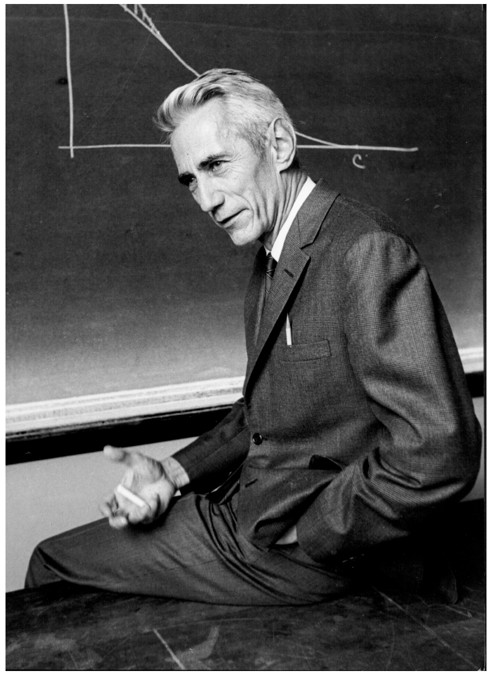
Historical Perspective

G. Marconi

1901

- First radio built 100+ years ago
- Great stride in technology
- But design was somewhat ad-hoc

C. Shannon

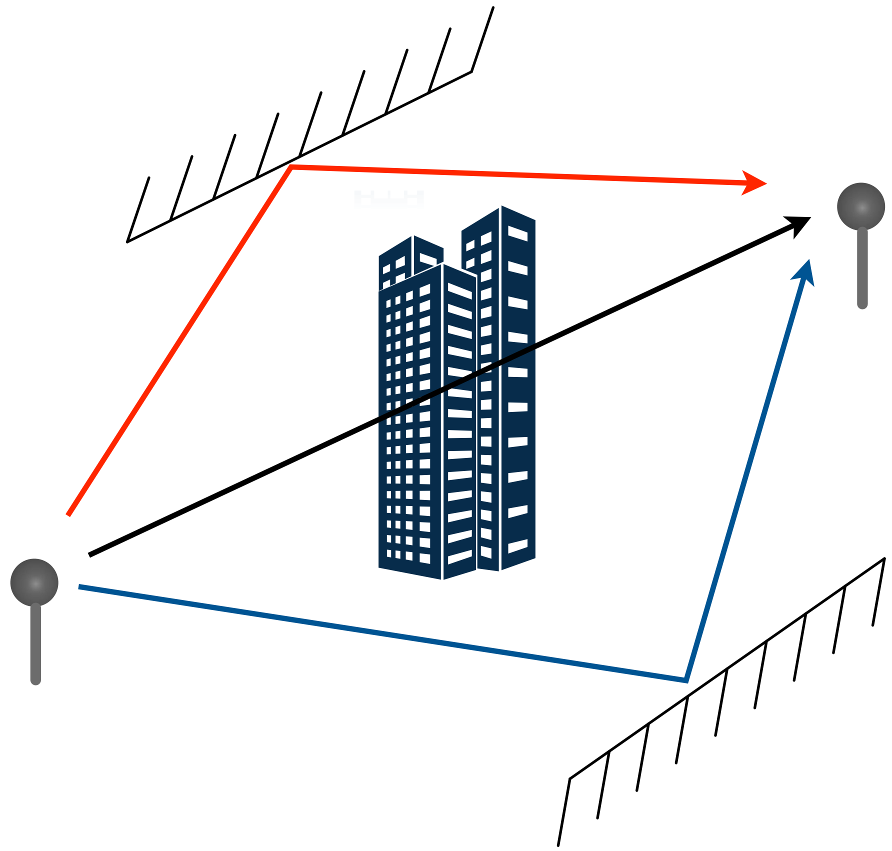


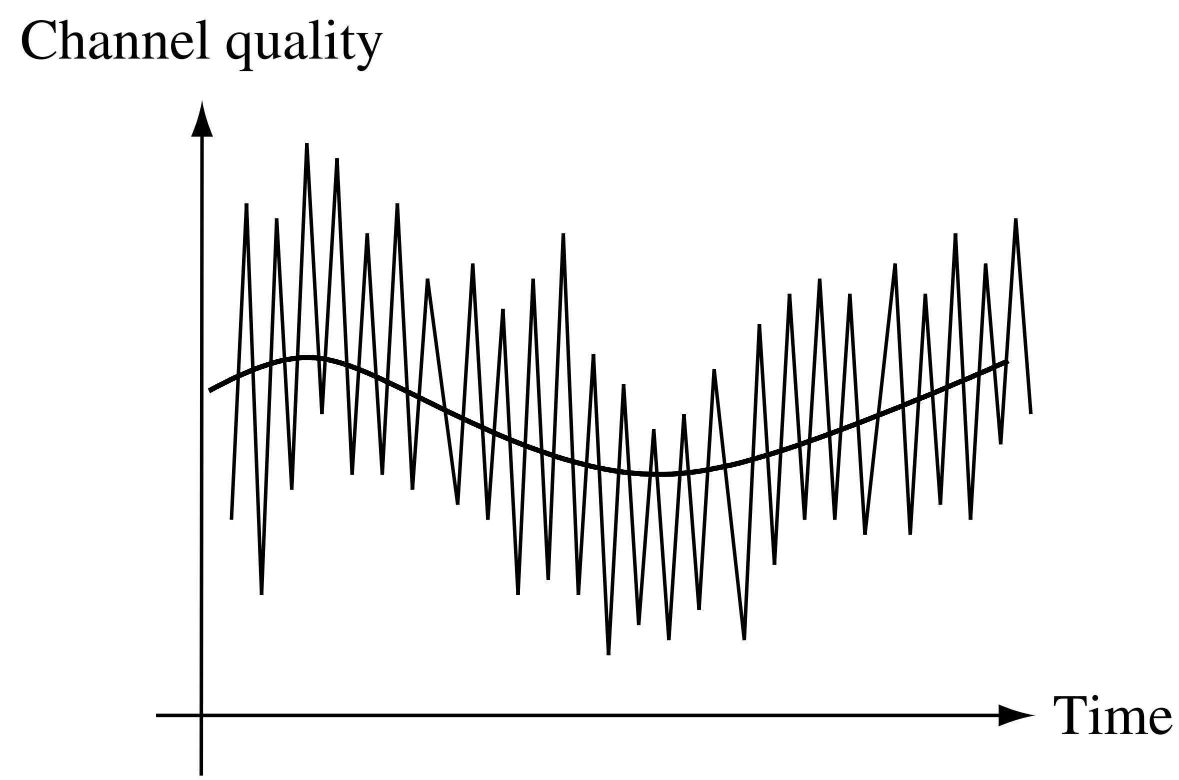
1948

- Information theory: every channel has a capacity
- Provides a systematic view of all communication problems

Engineering meets science
New points of view arise

Modern View on Multipath Fading





- Classical view: fading channels are unreliable
  - Diversity techniques: average out the variation
- Modern view: exploit fading to gain spectral efficiency
  - Thanks to the study on fading channel through the lens of information theory!

Plot

- Use a heuristic argument (geometric) to introduce the capacity of the AWGN channel

- Discuss the two key resources in the AWGN channel:
  - Power
  - Bandwidth

- The AWGN channel capacity serves as a building block towards fading channel capacity:
  - Slow fading channel: outage capacity
  - Fast fading channel: ergodic capacity

Outline

- AWGN Channel Capacity
- Resources of the AWGN Channel
- Capacity of some LTI Gaussian Channels
- Capacity of Fading Channels

AWGN Channel Capacity

Channel Capacity

- Capacity := the highest data rate can be delivered reliably over a channel
  - Reliably ≡ Vanishing error probability
- Before Shannon, it was widely believed that:
  - to communicate with error probability → 0
    $\Longrightarrow$ data rate must also → 0
- Repetition coding (with M-level PAM) over N time slots on AWGN channel:
  - Error probability $\sim 2Q\left(\sqrt{\frac{6N}{M^3}\mathsf{SNR}}\right)$
  - Data rate $= \frac{\log_2 M}{N}$
  - As long as $M \leq N^{1/3}$, the error probability → 0 as $N \to \infty$
  - But, the data rate $= \frac{\log_2 N}{3N}$ still → 0 as $N \to \infty$

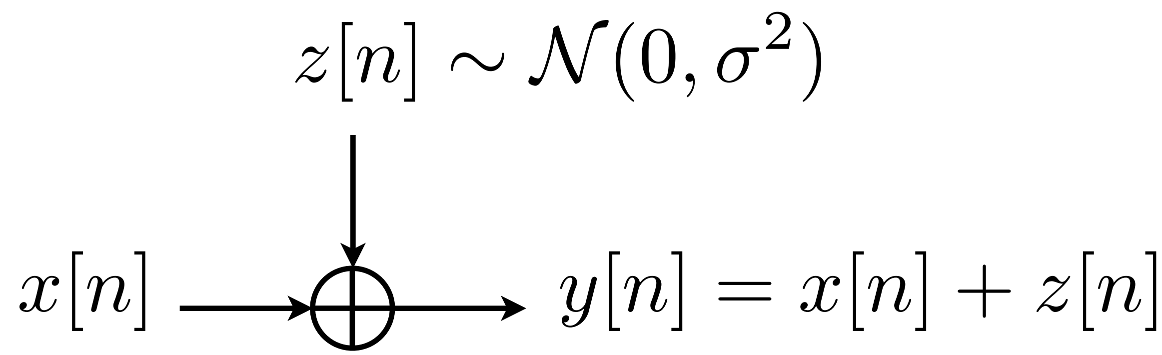
Channel Coding Theorem

- For every memoryless channel, there is a definite number C that is computable such that:
  - If the data rate $R < C$, then there exists a coding scheme that can deliver rate R data over the channel with error probability → 0 as the coding block length $N \to \infty$
  - Conversely, if the data rate $R > C$, then no matter what coding scheme is used, the error probability → 1 as $N \to \infty$

- We shall focus on the additive white Gaussian noise (AWGN) channel
  - Give a heuristic argument to derive the AWGN channel capacity

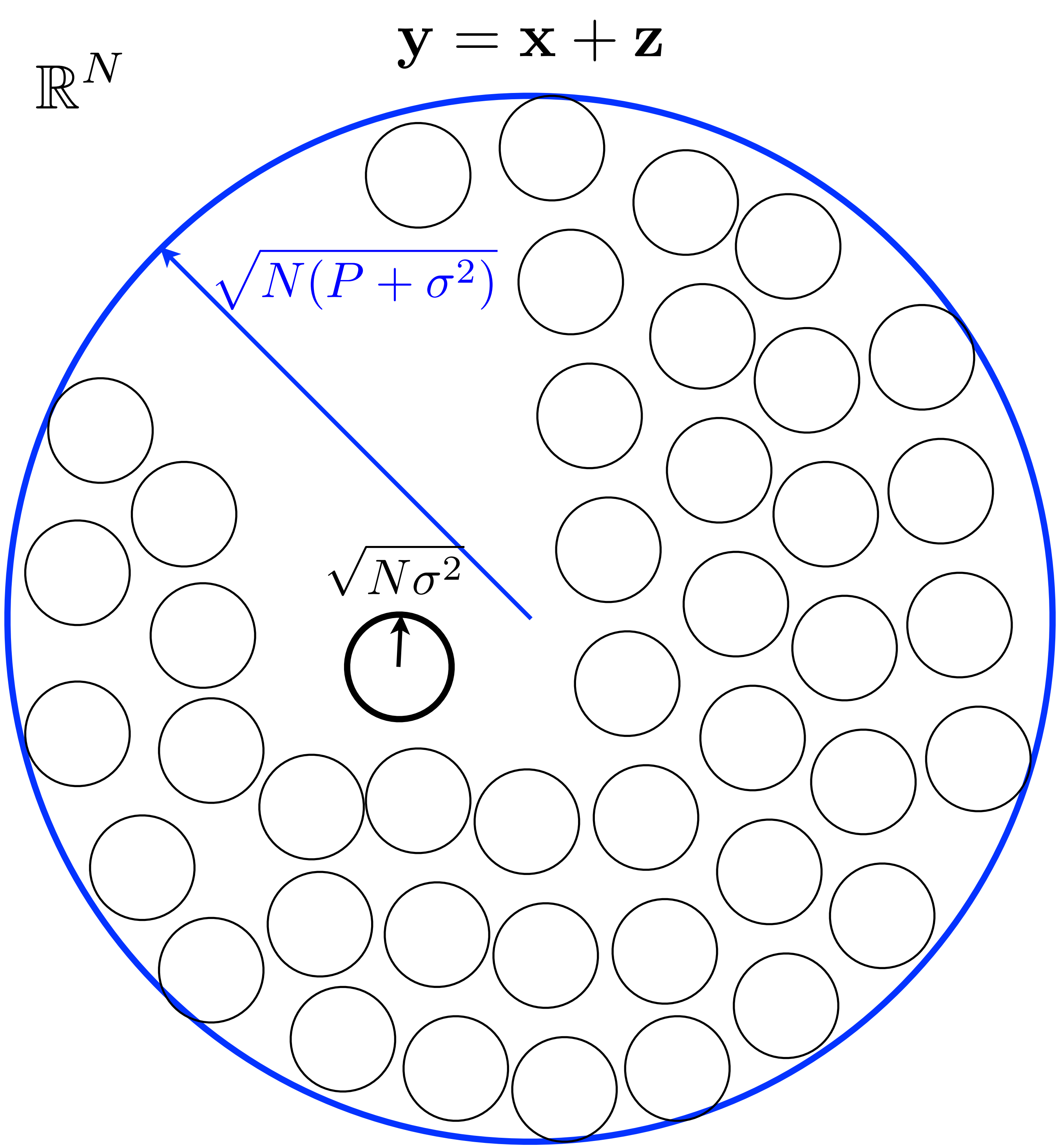
AWGN Channel

Power constraint:
$$\sum_{n=1}^{N} |x[n]|^2 \leq NP$$



- We consider real-valued Gaussian channel
- As mentioned earlier, repetition coding yield zero rate if the error probability is required to vanish as $N \to \infty$
- Because all codewords are spread on a single dimension in an N-dimensional space
- How to do better?

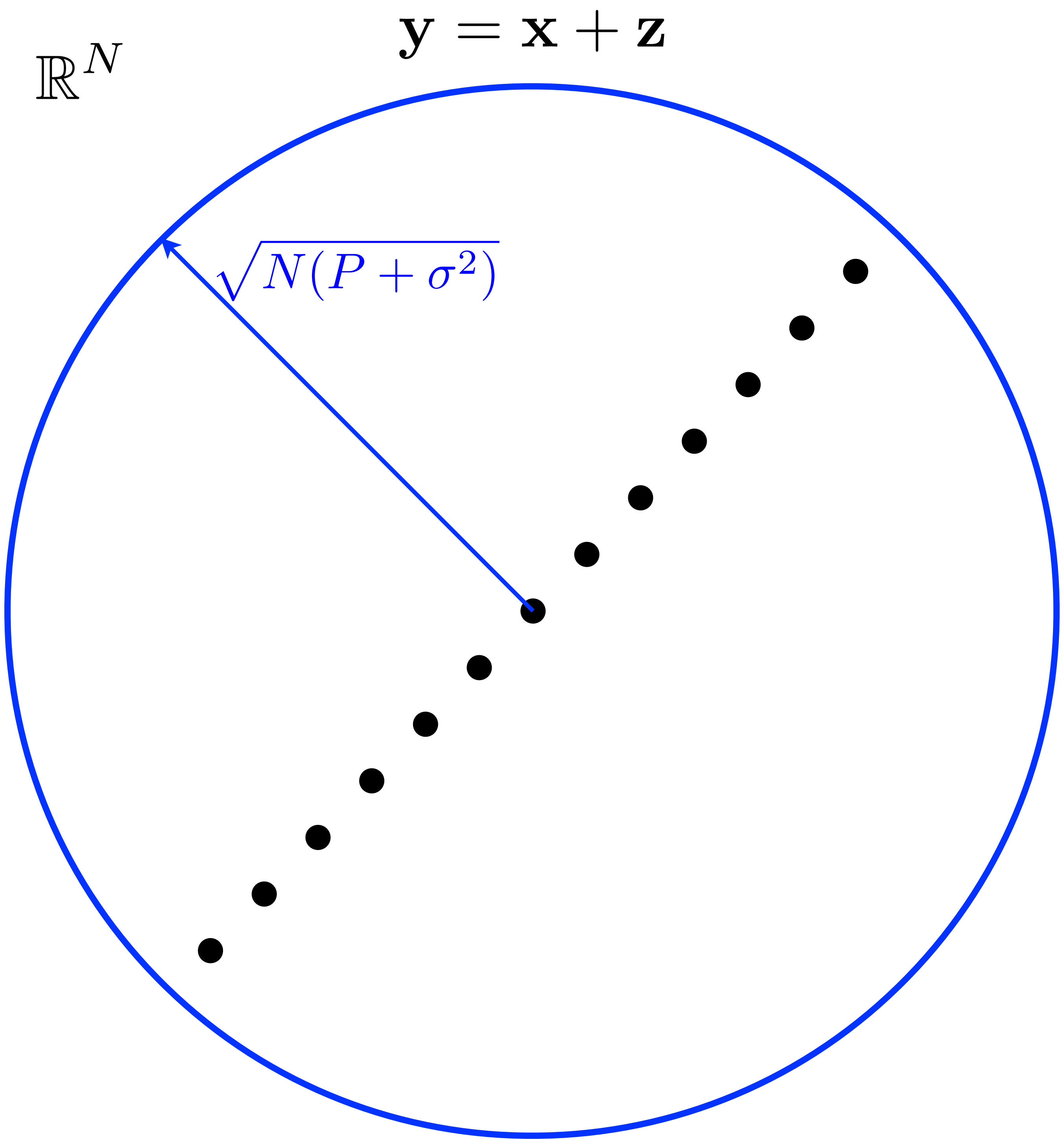
Sphere Packing Interpretation



- By the law of large numbers, as $N \to \infty$, most $\mathbf{y}$ will lie inside the N-dimensional sphere of radius $\sqrt{N(P+\sigma^2)}$
- Also by the LLN, as $N \to \infty$, $\mathbf{y}$ will lie near the surface of the N-dimensional sphere centered at $\mathbf{x}$ with radius $\sqrt{N\sigma^2}$
- Vanishing error probability $\Longrightarrow$ non-overlapping spheres

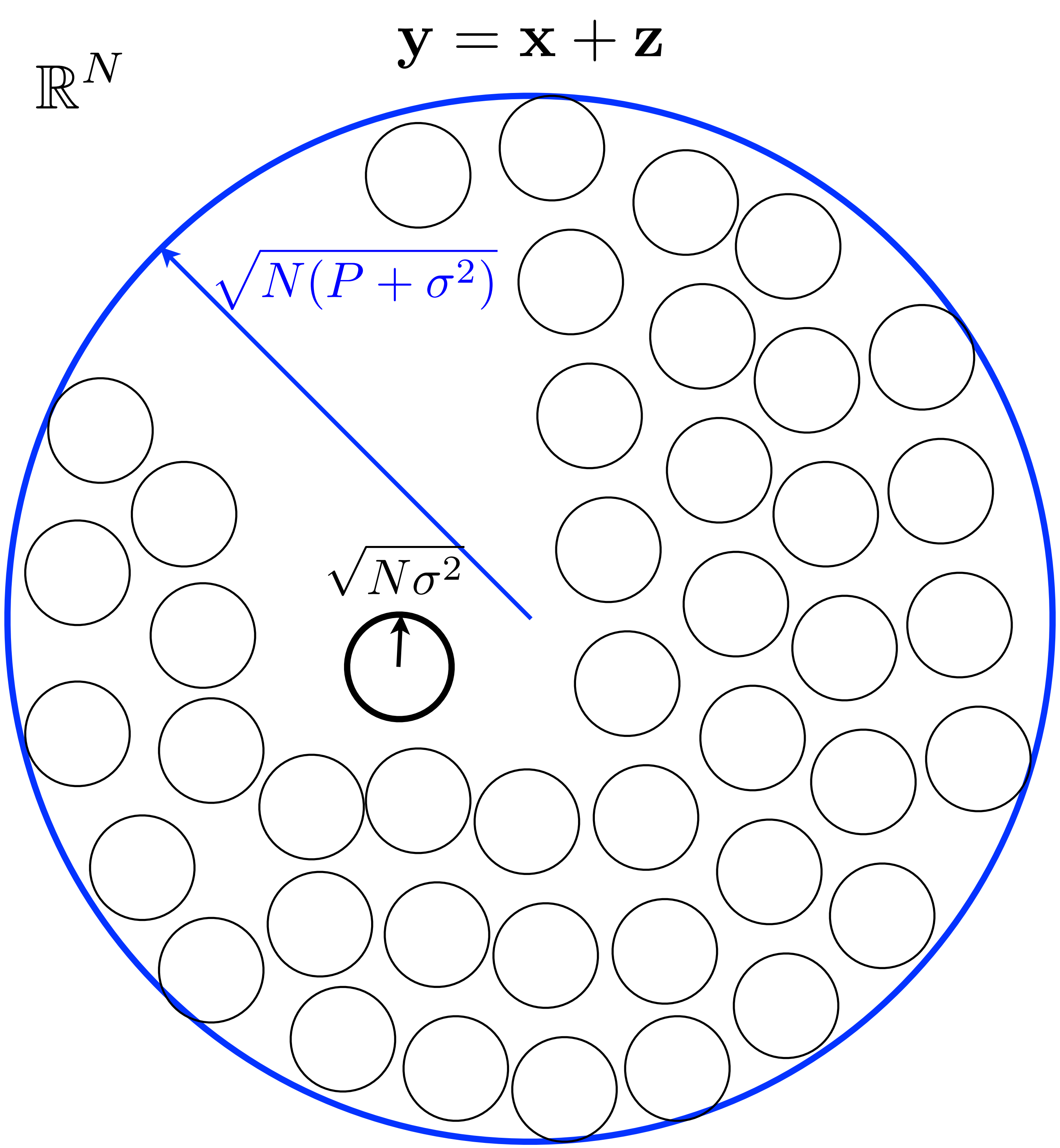
How many non-overlapping spheres can be packed into the large sphere?

Why Repetition Coding is Bad



It only uses one dimension out of N !

Capacity Upper Bound



Maximum # of non-overlapping spheres
= Maximum # of codewords that can be reliably delivered

$$2^{NR} \leq \frac{\sqrt{N(P+\sigma^2)}^N}{\sqrt{N\sigma^2}^N}$$

$$\Longrightarrow R \leq \frac{1}{N} \log \left(\frac{\sqrt{N(P+\sigma^2)}^N}{\sqrt{N\sigma^2}^N} \right)$$

$$= \boxed{\frac{1}{2} \log \left(1 + \frac{P}{\sigma^2} \right)}$$

This is hence an upper bound of the capacity C.

How to achieve it?

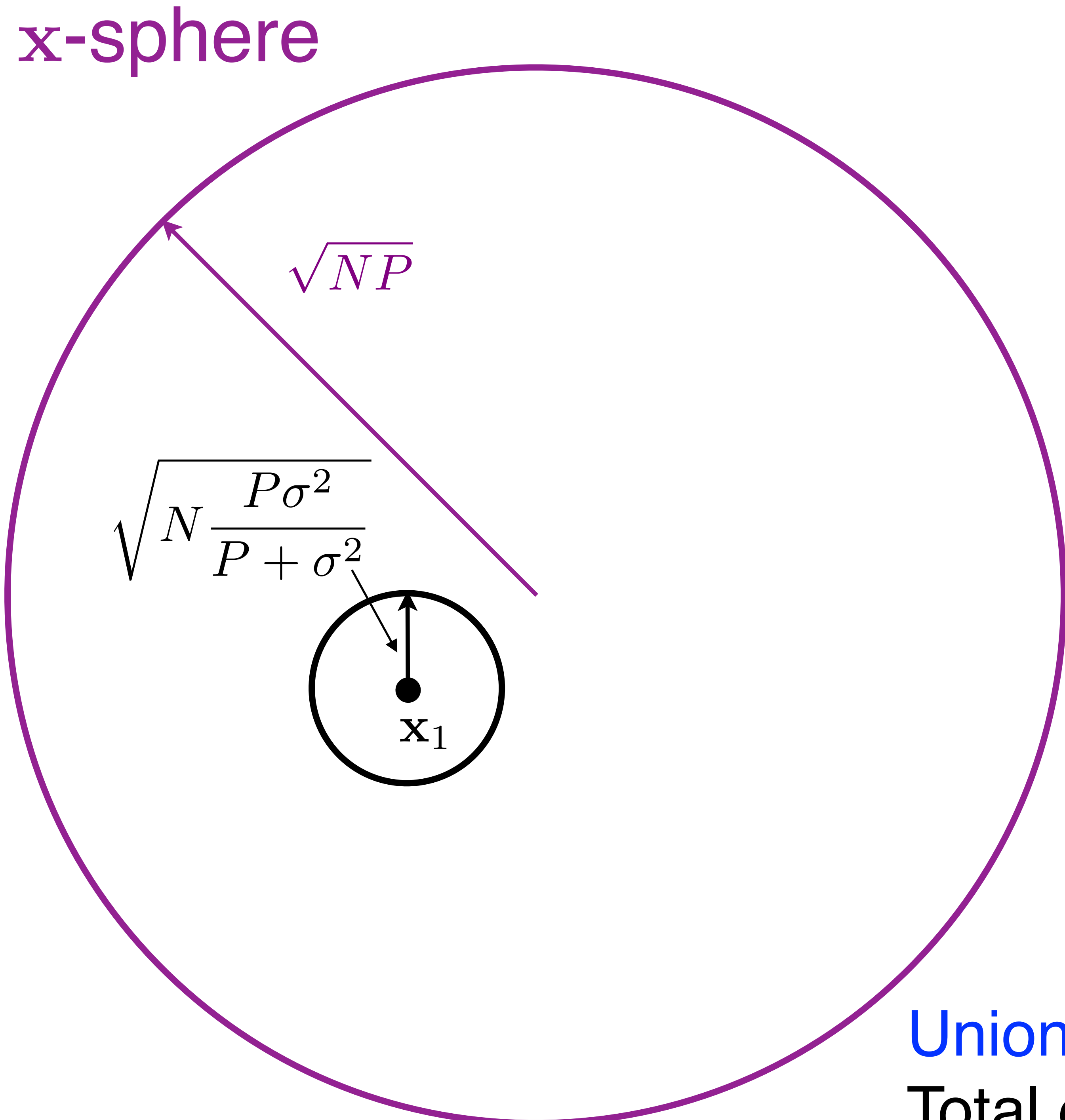
Achieving Capacity (1/3)

- (random) Encoding: randomly generate 2^{NR} codewords $\{\mathbf{x}_1, \mathbf{x}_2, ...\}$ lying inside the "$\mathbf{x}$-sphere" of radius $\sqrt{NP}$
- Decoding:

$$\mathbf{y} \longrightarrow \boxed{\text{MMSE}} \xrightarrow{\alpha := \frac{P}{P+\sigma^2}} \alpha\mathbf{y} \longrightarrow \boxed{\text{Nearest Neighbor}} \longrightarrow \widehat{\mathbf{x}}$$

- Performance analysis: WLOG let $\mathbf{x}_1$ is sent
  - By the LLN,
    $$||\alpha\mathbf{y} - \mathbf{x}_1||^2 = ||\alpha\mathbf{w} + (\alpha-1)\mathbf{x}_1||^2 \approx \alpha^2 N\sigma^2 + (\alpha-1)^2 NP = N\frac{P\sigma^2}{P+\sigma^2}$$
  - As long as $\alpha\mathbf{y}$ lies inside the uncertainty sphere centered at $\mathbf{x}_1$ with radius $\sqrt{N\frac{P\sigma^2}{P+\sigma^2}}$, decoding will be correct
  - Pairwise error probability (see next slide) = $\left(\frac{\sigma^2}{P+\sigma^2}\right)^{N/2}$

Achieving Capacity (2/3)



When does an error occur?

Ans: when another codeword falls inside the uncertainty sphere of $\mathbf{x}_1$

What is that probability (pairwise error probability)?

Ans: the ratio of the volume of the two spheres

$$\Pr\{\mathbf{x}_1 \to \mathbf{x}_2\} = \frac{\sqrt{NP\sigma^2/(P+\sigma^2)}^N}{\sqrt{NP}^N}$$

$$= \left(\frac{\sigma^2}{P+\sigma^2}\right)^{N/2}$$

Union bound:

Total error probability $\leq 2^{NR}\left(\frac{\sigma^2}{P+\sigma^2}\right)^{N/2}$

Achieving Capacity (3/3)

- Total error probability (by union bound)

$$\Pr\{\mathcal{E}\} \leq 2^{NR}\left(\frac{\sigma^2}{P+\sigma^2}\right)^{N/2} = 2^{N\left(R+\frac{1}{2}\log\left(\frac{1}{1+\frac{P}{\sigma^2}}\right)\right)}$$

- As long as the following holds, $\Pr\{\mathcal{E}\} \to 0$ as $N \to \infty$

$$R < \frac{1}{2}\log\left(1+\frac{P}{\sigma^2}\right)$$

- Hence, indeed the capacity is

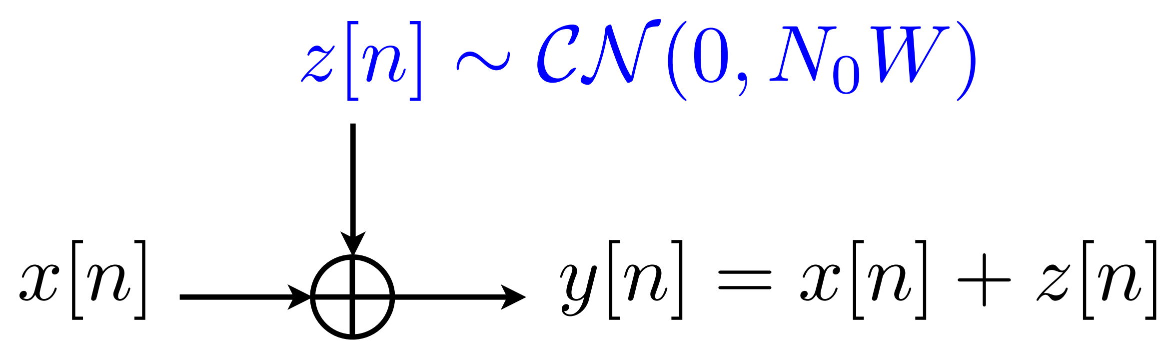
$$\boxed{C = \frac{1}{2}\log\left(1+\frac{P}{\sigma^2}\right) \text{ bits per symbol time}}$$

Resources of AWGN Channel

Continuous-Time AWGN Channel

- System parameters:
  - Power constraint: P watts; Bandwidth: W Hz
  - Spectral density of the white Gaussian noise: $N_0/2$
- Equivalent discrete-time baseband channel (complex)

Power constraint:
$$\sum_{n=1}^{N} |x[n]|^2 \leq NP$$



  - 1 complex symbol = 2 real symbols
- Capacity:

$$C_{\text{AWGN}}(P, W) = 2 \times \frac{1}{2} \log\left(1 + \frac{P/2}{N_0 W/2}\right) \text{ bits per symbol time}$$

SNR $:= P/N_0 W$

SNR per complex symbol

$$= W \log\left(1 + \frac{P}{N_0 W}\right) \text{ bits/s } = \log\left(1 + \mathsf{SNR}\right) \text{ bits/s/Hz}$$

Complex AWGN Channel Capacity

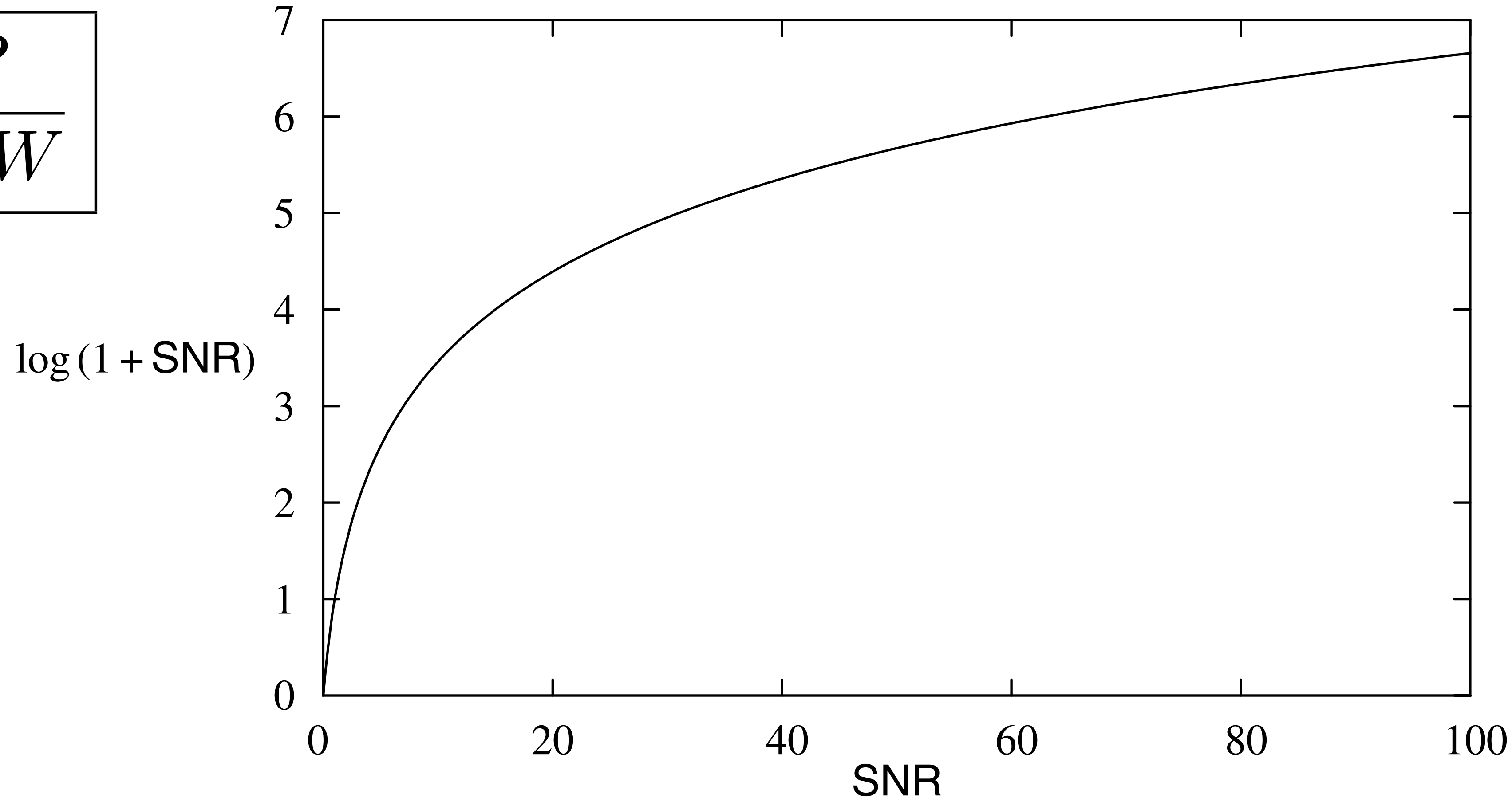
$$C_{\mathrm{AWGN}}(P, W) = W \log\left(1 + \frac{P}{N_0 W}\right) \ \text{bits/s}$$

$$= \log\left(1 + \mathsf{SNR}\right) \ \text{bits/s/Hz} \quad \text{Spectral Efficiency}$$

- The capacity formula provides a high-level way of thinking about how the performance fundamentally depends on the basic resources available in the channel
- No need to go into details of specific coding and modulation schemes
- Basic resources: power P and bandwidth W

Power

$$\boxed{\mathsf{SNR} = \frac{P}{N_0 W}}$$



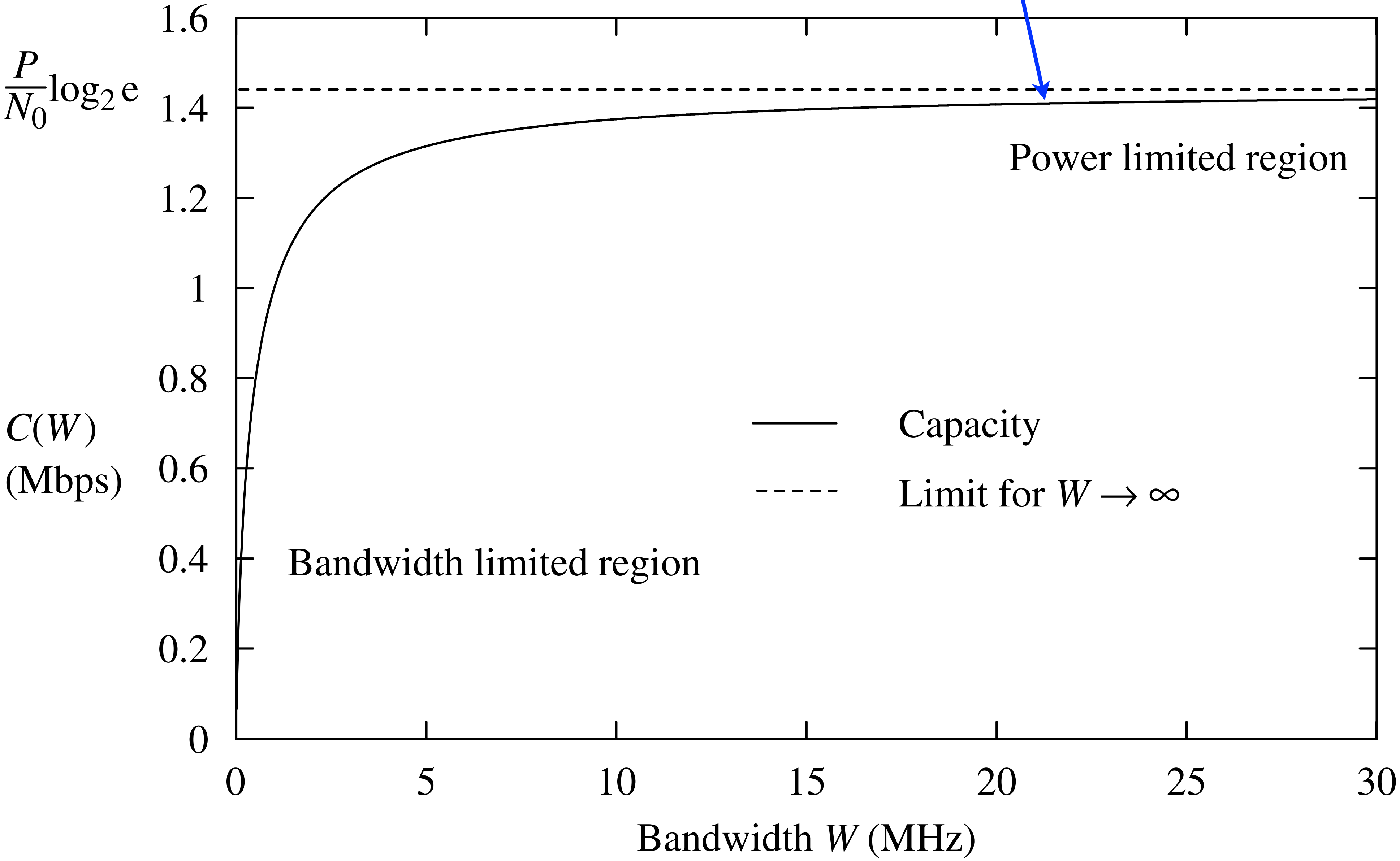
Fix W:

- High SNR: $C = \log(1 + \mathsf{SNR}) \approx \log \mathsf{SNR}$
  - Logarithmic growth with power
- Low SNR: $C = \log(1 + \mathsf{SNR}) \approx \mathsf{SNR} \log_2 e$
  - Linear growth with power

Bandwidth

Fix P

$$C(W) = W \log\left(1 + \frac{P}{N_0 W}\right) \approx W \frac{P}{N_0 W} \log_2 e = \frac{P}{N_0} \log_2 e$$



Bandwidth-limited vs. Power-limited

$$C_{\text{AWGN}}(P, W) = W \log\left(1 + \frac{P}{N_0 W}\right) \text{ bits/s}$$

$$\mathsf{SNR} = \frac{P}{N_0 W}$$

- When SNR ≪ 1: (Power-limited regime)

$$C_{\text{AWGN}}(P, W) \approx W \left(\frac{P}{N_0 W}\right) \log_2 e = \frac{P}{N_0} \log_2 e$$

  - Linear in power; Insensitive to bandwidth

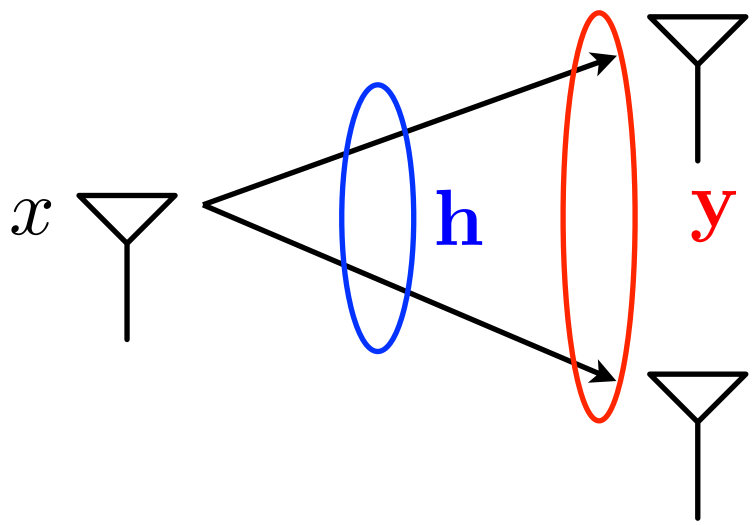
- When SNR ≫ 1: (Bandwidth-limited regime)

$$C_{\text{AWGN}}(P, W) \approx W \log\left(\frac{P}{N_0 W}\right)$$

  - Logarithmic in power; Approximately linear in bandwidth

Capacity of Some LTI Gaussian Channels

SIMO Channel



$$\mathbf{y} = \mathbf{h}x + \mathbf{w} \in \mathbb{C}^L$$
$$\downarrow$$
$$\boxed{\text{MRC}, \quad \frac{\mathbf{h}^*}{||\mathbf{h}||}}$$
$$\downarrow$$
$$\tilde{y} = ||\mathbf{h}||x + \tilde{w}, \quad \tilde{w} \sim \mathcal{CN}\left(0, \sigma^2\right)$$

Power constraint: P

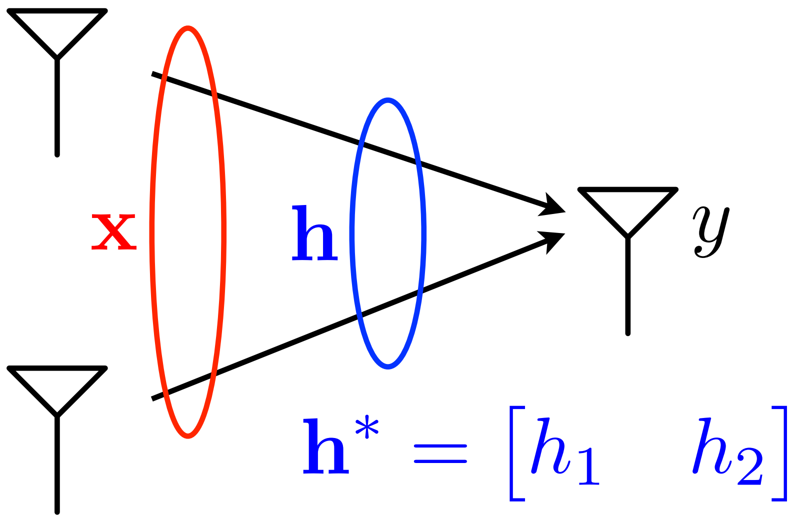
$\mathbf{w} \sim \mathcal{CN}\left(\mathbf{0}, \sigma^2 \mathbf{I}_L\right)$

- MRC is a lossless operation: we can generate $\mathbf{y}$ from $\tilde{y}$: $\mathbf{y} = \tilde{y}\left(\mathbf{h}/||\mathbf{h}||\right)$
- Hence the SIMO channel capacity is equal to the capacity of the equivalent AWGN channel, which is

$$C_{\text{SIMO}} = \log\left(1 + \frac{||\mathbf{h}||^2 P}{\sigma^2}\right)$$

Power gain due to Rx beamforming

MISO Channel



$$y = \mathbf{h}^*\mathbf{x} + w \in \mathbb{C}, \quad \mathbf{x}, \mathbf{h} \in \mathbb{C}^L$$

$$\downarrow$$

$$\boxed{\begin{array}{c}\text{Tx Beamforming} \\ \mathbf{x} = x\mathbf{h}/||\mathbf{h}||\end{array}}$$

$$\downarrow$$

$$y = x||\mathbf{h}|| + w$$

Power constraint:

$$\sum_{n=1}^{N} ||\mathbf{x}||^2 \leq NP$$

- Goal: maximize the received power $||\mathbf{h}^*\mathbf{x}||^2$
  - The answer is $||\mathbf{h}||^2 P$! (check. Hint: Cauchy–Schwarz inequality)
- Achieved by Tx beamforming
  - Send a scalar symbol x on the direction of $\mathbf{h}$
  - Power constraint on x : still P
- Capacity:

$$C_{\text{MISO}} = \log\left(1 + \frac{||\mathbf{h}||^2 P}{\sigma^2}\right)$$

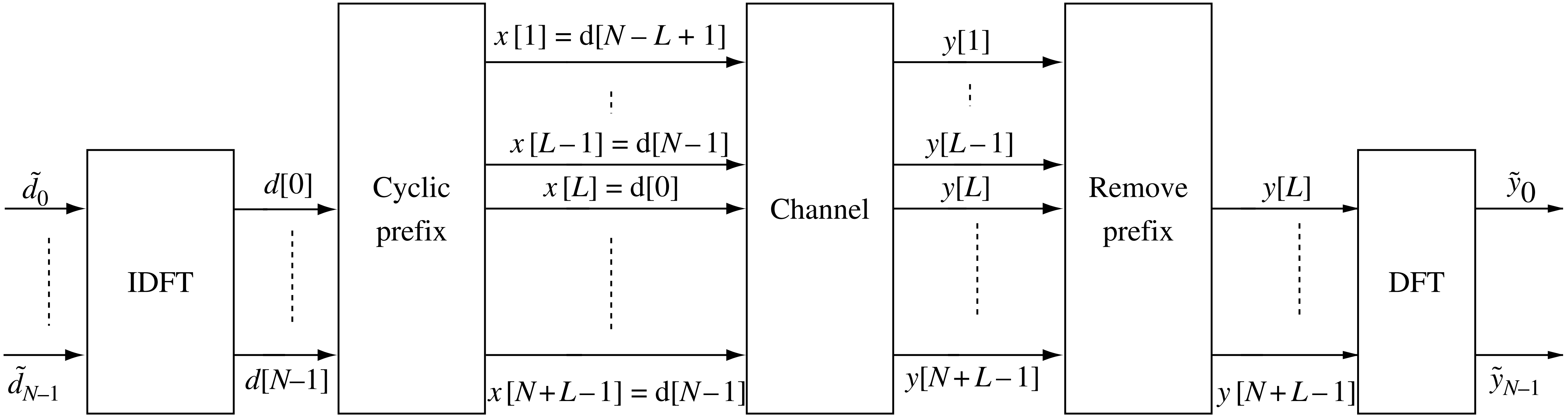
Frequency-Selective Channel

$$y[m] = \sum_{l=0}^{L-1} h_l x[m-l] + w[m]$$

- Key idea 1: use OFDM to convert the channel with ISI into a bunch of parallel AWGN channels
  - But there is loss/overhead due to cyclic prefix
- Key idea 2: CP overhead → 0 as $N_c \to \infty$

- First focus on finding the capacity of parallel AWGN channels of any finite N_c
- Then take $N_c \to \infty$ to find the capacity of the frequency-selective channel

Recap: OFDM



$$\mathbf{y} := y[L : N_c + L - 1], \quad \mathbf{w} := w[L : N_c + L - 1],$$

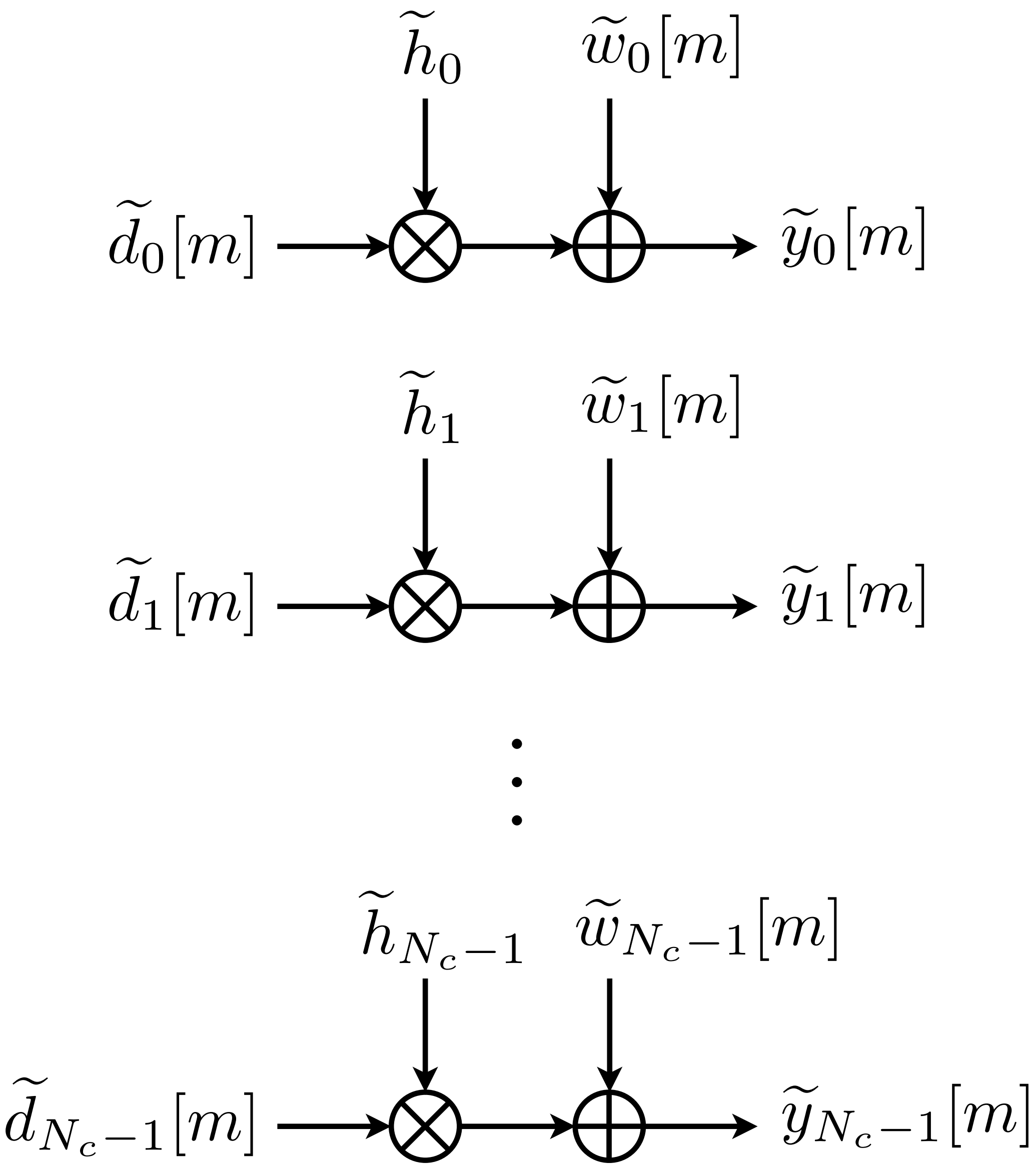
$$\mathbf{h} := \begin{bmatrix} h_0 & h_1 & \cdots & h_{L-1} & 0 & \cdots & 0 \end{bmatrix}^T$$

$$\boxed{\widetilde{y}_n = \widetilde{h}_n \widetilde{d}_n + \widetilde{w}_n, \quad n = 0, 1, \ldots, N_c - 1}$$

N_c parallel AWGN channels

$$\widetilde{y}_n := \mathrm{DFT}(\mathbf{y})_n, \quad \widetilde{d}_n := \mathrm{DFT}(\mathbf{d})_n, \quad \widetilde{w}_n := \mathrm{DFT}(\mathbf{w})_n, \quad \widetilde{h}_n := \sqrt{N_c}\,\mathrm{DFT}(\mathbf{h})_n$$

Parallel AWGN Channels



$$m = 1, 2, \ldots, M$$

$$(M \text{ channel uses})$$

Parallel Channels

$$\tilde{y}_n = \tilde{h}_n \tilde{d}_n + \tilde{w}_n, \quad n \in [0 : 1 : N_c - 1]$$

Equivalent Vector Channel

$$\tilde{\mathbf{y}} = \tilde{\mathbf{H}}\tilde{\mathbf{d}} + \tilde{\mathbf{w}} \qquad \tilde{\mathbf{w}} \sim \mathcal{CN}\left(\mathbf{0}, \sigma^2 \mathbf{I}\right)$$

$$\tilde{\mathbf{H}} = \mathrm{diag}\left(\tilde{h}_0, \ldots, \tilde{h}_{N_c-1}\right)$$

Power Constraint

$$\sum_{n=1}^{M} ||\tilde{\mathbf{d}}[n]||^2 \leq M N_c P$$

Due to Parseval theorem of DFT

Independent Uses of Parallel Channels

- One way to code over such parallel channels (a special case of a vector channel): treat each channel separately
  - It turns out that coding across parallel channels does not help!
- Power allocation:
  - Each of the N_c channels get a portion of the total power
  - Channel n gets power P_n , which must satisfy $\sum_{n=0}^{N_c-1} P_n \leq N_c P$
- For a given power allocation $\{P_n\}$, the following rate can be achieved:

$$R = \sum_{n=0}^{N_c-1} \log\left(1 + \frac{|\widetilde{h}_n|^2 P_n}{\sigma^2}\right)$$

Optimal Power Allocation

- Power allocation problem:

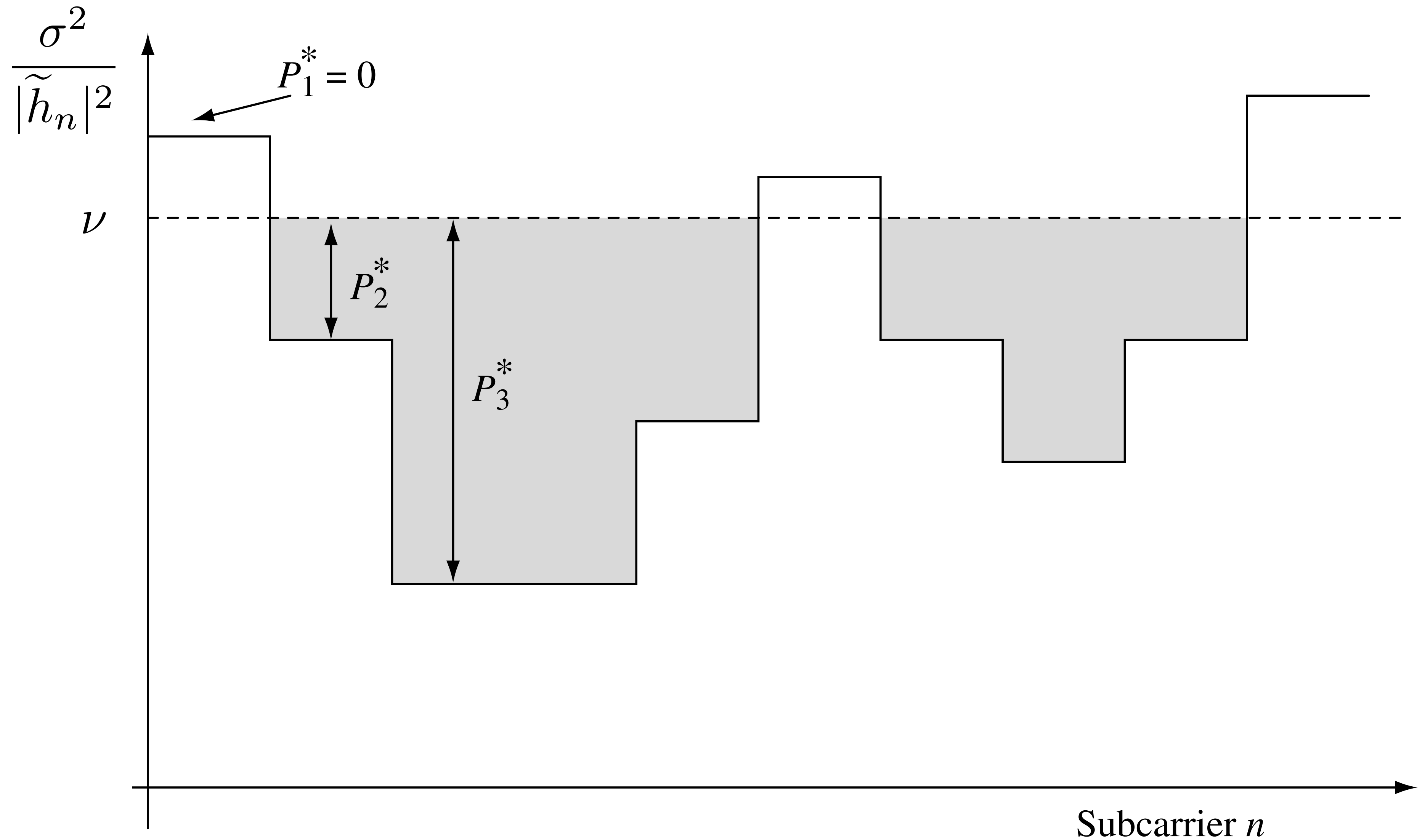
$$\max_{P_0,\ldots,P_{N_c-1}} \sum_{n=0}^{N_c-1} \log\left(1+\frac{|\widetilde{h}_n|^2 P_n}{\sigma^2}\right),$$

$$\text{subject to } \sum_{n=0}^{N_c-1} P_n = N_c P, \quad P_n \geq 0,\ n=0,\ldots,N_c-1$$

- It can be solved explicitly by Lagrangian methods
- Final solution: let $(x)^+ := \max(x\ ,\ 0)$

$$P_n^* = \left(\nu - \frac{\sigma^2}{|\widetilde{h}_n|^2}\right)^+, \quad \nu \text{ satisfies } \sum_{n=0}^{N_c-1}\left(\nu - \frac{\sigma^2}{|\widetilde{h}_n|^2}\right)^+ = N_c P$$

Waterfilling



Note: $\widetilde{h}_n = H_b\left(\frac{nW}{N_c}\right)$

Baseband frequency response at $f = nW/N_c$

Frequency-Selective Channel Capacity

- Final step: making $N_c \to \infty$
  - Replace all $\widetilde{h}_n = H_b\left(\frac{nW}{N_c}\right)$ by $H_b(f)$, summation over $[0 : Nc - 1]$ becomes integration from 0 to W

- Power allocation problem becomes

$$\max_{P(f)} \int_0^W \log\left(1 + \frac{|H(f)|^2 P(f)}{\sigma^2}\right) df,$$

$$\text{subject to } \int_0^W P(f) = P, \quad P(f) \geq 0,\ f \in [0, W]$$

- Optimal solution becomes

$$P(f)^* = \left(\nu - \frac{\sigma^2}{|H(f)|^2}\right)^+, \quad \nu \text{ satisfies } \int_0^W \left(\nu - \frac{\sigma^2}{|H(f)|^2}\right)^+ df = P$$

Waterfilling over the Frequecy Spectrum