On Multivariate Rayleigh and Exponential Distributions

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***Abstract*—In this paper, expressions for multivariate Rayleigh and exponential probability density functions (pdfs) generated from correlated Gaussian random variables are presented. We first obtain a general integral form of the pdfs, and then study the case when the complex Gaussian generating vector is circular. We consider two specific circular cases: the exchangeable case when the variates are evenly correlated, and the exponentially correlated case. Expressions for the multivariate pdf in these cases are obtained in integral form as well as in the form of a series of products of univariate pdfs. We also derive a general expression for the multivariate exponential characteristic function (cf) in terms of determinants. In the exchangeable and exponentially correlated cases, cf expressions are obtained in the form of a series of products of univariate gamma cfs. The cf of the sum of exponential variates in these cases is obtained in closed form. Finally, the bivariate case is presented mentioning its main features. While the integral forms of the multivariate pdfs provide a general analytical framework, the series and determinant expressions for the exponential cfs and the series expressions for the pdfs can serve as a useful tool in the performance analysis of digital modulation over correlated Rayleigh-fading channels using diversity combining.**

***Index Terms*—Characteristic function, correlated Rayleigh fading, exchangeable variates, exponential correlation, multivariate exponential distribution, multivariate Rayleigh distribution.**

I. Introduction

MULTIVARIATE Rayleigh and exponential distributions [1]–[3] play an important role in electrical engineering. Two important applications of these distributions are the analysis of Gaussian noise [4], [5] and the study of performance [6] of digital modulation schemes over correlated fading channels. The Rayleigh distribution is a special case of the Nakagami distribution [7], while the exponential distribution is a special case of the gamma distribution [8]–[11]. A bivariate Rayleigh case has been studied in detail [12]–[14] and applied to fading channels using dual-diversity [15]. A multivariate gamma case has been dealt with in situations in which the characteristic function (cf) has a specific form [7], [16]–[18]. When the gamma parameter is unity, the resulting multivariate exponential cf arising from this form is not a general one.

In this paper, we first obtain a general integral form of the multivariate Rayleigh and exponential probability density functions (pdfs) when these are generated by correlated Gaussian random variables, and then study the case when the complex Gaussian generating vector is circular. We consider two specific circular cases: the exchangeable case, when the variates are evenly correlated; and the exponentially correlated case. Expressions for the multivariate pdf in these cases are obtained in integral form as well as in the form of a series of products of univariate pdfs. We also derive a general expression for the multivariate exponential cf in terms of determinants, which is found to take the form in [7], [16] under certain conditions. In the exchangeable and exponentially correlated cases, cf expressions are obtained in the form of a series of products of univariate gamma cfs. The cf of the sum of exponential variates in these cases is obtained in closed form. Finally, the bivariate case is presented mentioning its main features.

It is to be noted that there may be many multivariate distributions whose univariate marginals are Rayleigh or exponential [1]. Only the Gaussian-generated distributions are dealt with in this paper.

The paper is organized as follows. Expressions for the pdfs are derived in Section II. The multivariate exponential cf is presented in Section III. Section IV deals with the bivariate case. Section V contains two application examples. Section VI gives the conclusions drawn from the results.

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II. Probability Density Functions

Consider zero-mean real Gaussian $L \times 1$ random vectors $\underline{X}_c$ and $\underline{X}_s$ given by

$$\underline{X}_c \triangleq \begin{bmatrix} X_{c_1} \\ \vdots \\ X_{c_L} \end{bmatrix}, \qquad \underline{X}_s \triangleq \begin{bmatrix} X_{s_1} \\ \vdots \\ X_{s_L} \end{bmatrix} \tag{1}$$

with covariance matrices $\underline{K}_{cc}$ and $\underline{K}_{ss}$ and cross-covariance matrix $\underline{K}_{cs}$, that is,

$$\begin{aligned} &\boldsymbol{E}\left[\underline{X}_c \underline{X}_c^T\right] = \underline{K}_{cc}, \qquad \boldsymbol{E}\left[\underline{X}_s \underline{X}_s^T\right] = \underline{K}_{ss}, \\ &\boldsymbol{E}\left[\underline{X}_c \underline{X}_s^T\right] = \underline{K}_{cs}, \end{aligned} \tag{2a}$$

such that

$$\begin{aligned} \boldsymbol{E}\left[X_{c_i}^2\right] &= (\underline{K}_{cc})_{ii} = (\underline{K}_{ss})_{ii} = \boldsymbol{E}\left[X_{s_i}^2\right] \\ \boldsymbol{E}\left[X_{c_i} X_{s_i}\right] &= (\underline{K}_{cs})_{ii} = 0, \qquad i = 1, \ldots, L \end{aligned} \tag{2b}$$

where $\boldsymbol{E}[\cdot]$ denotes the expectation operator and $(\cdot)^T$ denotes the transpose operator. In other words, X_{c_i} and X_{s_i} are independent and identically distributed (i.i.d.) Gaussian random variables for each i. We shall call $\underline{X}_c$ the *in-phase component*, and $\underline{X}_s$ the *quadrature component* of the complex *generating vector* $\underline{X}_c + \jmath \underline{X}_s$, where $\jmath = \sqrt{-1}$.

Define random variables $\alpha_1, \ldots, \alpha_L, \Phi_1, \ldots, \Phi_L$ in terms of $\underline{X}_c, \underline{X}_s$ as

$$\alpha_i \triangleq \left(X_{c_i}^2 + X_{s_i}^2\right)^{\frac{1}{2}}, \qquad \Phi_i \triangleq \tan^{-1}\left(\frac{X_{s_i}}{X_{c_i}}\right), \qquad i = 1, \ldots, L. \tag{3}$$

It is clear that each α_i is *marginally Rayleigh* distributed with second moment $2\left(\underline{K}_{cc}\right)_{ii}$ and each Φ_i, which is independent of α_i, is *marginally uniform* over $[-\pi, \pi)$. Note that the pdf of a sum of random vectors whose amplitudes have arbitrary correlated distributions and phases, that are independent of the amplitudes, are i.i.d. uniform random variables has been obtained in [19]. However, correlated Rayleigh-distributed amplitudes with i.i.d. uniformly distributed phases that are independent of all the amplitudes cannot be generated from correlated Gaussian random variables. Hence, our model differs from that considered in [19].

Let

$$\underline{\alpha} = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_L \end{bmatrix}, \qquad \underline{\Phi} = \begin{bmatrix} \Phi_1 \\ \vdots \\ \Phi_L \end{bmatrix}. \tag{4}$$

Thus, $\underline{\alpha}$ is a *Rayleigh random vector*. We have from (3)

$$X_{c_i} = \alpha_i \cos \Phi_i, \quad X_{s_i} = \alpha_i \sin \Phi_i, \qquad i = 1, \ldots, L. \tag{5}$$

The joint pdf of $\underline{X}_c$ and $\underline{X}_s$ can be written as

$$f_{\underline{X}_c, \underline{X}_s}(\underline{x}_c, \underline{x}_s) = \frac{1}{(2\pi)^L \left(\det(\underline{K})\right)^{\frac{1}{2}}} \times \exp\left(-\frac{1}{2}\left[\underline{x}_c^T \quad \underline{x}_s^T\right] \underline{K}^{-1} \begin{bmatrix} \underline{x}_c \\ \underline{x}_s \end{bmatrix}\right) \tag{6a}$$

where

$$\underline{K} = \begin{bmatrix} \underline{K}_{cc} & \underline{K}_{cs} \\ \underline{K}_{cs}^T & \underline{K}_{ss} \end{bmatrix} \tag{6b}$$

is a *nonsingular symmetric* matrix. It can be easily shown that

$$\underline{K}^{-1} = \begin{bmatrix} \underline{A} & \underline{B} \\ \underline{B}^T & \underline{D} \end{bmatrix} \tag{7a}$$

in which $\underline{A}$ and $\underline{D}$ are nonsingular symmetric $L \times L$ matrices given by

$$\begin{aligned} \underline{A} &= \left(\underline{K}_{cc} - \underline{K}_{cs}\underline{K}_{ss}^{-1}\underline{K}_{cs}^T\right)^{-1} \\ \underline{D} &= \left(\underline{K}_{ss} - \underline{K}_{cs}^T\underline{K}_{cc}^{-1}\underline{K}_{cs}\right)^{-1} \end{aligned} \tag{7b}$$

and

$$\begin{aligned} \underline{B} &= -\underline{K}_{cc}^{-1}\underline{K}_{cs}\left(\underline{K}_{ss} - \underline{K}_{cs}^T\underline{K}_{cc}^{-1}\underline{K}_{cs}\right)^{-1} \\ &= -\left(\underline{K}_{cc} - \underline{K}_{cs}\underline{K}_{ss}^{-1}\underline{K}_{cs}^T\right)^{-1}\underline{K}_{cs}\underline{K}_{ss}^{-1}. \end{aligned} \tag{7c}$$

By matrix row or column operations on $\underline{K}$ it can also be shown that

$$\begin{aligned} \det\left(\underline{K}\right) &= \det\left(\underline{K}_{cc}\right)\det\left(\underline{K}_{ss} - \underline{K}_{cs}^T\underline{K}_{cc}^{-1}\underline{K}_{cs}\right) \\ &= \det\left(\underline{K}_{ss}\right)\det\left(\underline{K}_{cc} - \underline{K}_{cs}\underline{K}_{ss}^{-1}\underline{K}_{cs}^T\right). \end{aligned} \tag{8}$$

We can rewrite (6a) as

$$f_{\underline{X}_c, \underline{X}_s}(\underline{x}_c, \underline{x}_s) = \frac{\exp\left(-\frac{1}{2}\left\{\underline{x}_c^T\underline{A}\,\underline{x}_c + \underline{x}_s^T\underline{D}\,\underline{x}_s + 2\underline{x}_c^T\underline{B}\,\underline{x}_s\right\}\right)}{(2\pi)^L\left(\det(\underline{K})\right)^{\frac{1}{2}}}. \tag{9}$$

Let

$$\underline{A} = \left[A_{ij}\right]_{i,j=1}^L, \quad \underline{B} = \left[B_{ij}\right]_{i,j=1}^L, \quad \underline{D} = \left[D_{ij}\right]_{i,j=1}^L. \tag{10}$$

The Jacobian of $(\underline{X}_c, \underline{X}_s)$ with respect to $(\underline{\alpha}, \underline{\Phi})$ is $\prod_{i=1}^L \alpha_i$. The joint pdf of $\underline{\alpha}, \underline{\Phi}$, obtained from (5), (9), and (10), is

$$f_{\underline{\alpha}, \underline{\Phi}}(\underline{u}, \underline{\phi}) = \frac{\prod_{i=1}^L u_i}{(2\pi)^L\left(\det(\underline{K})\right)^{\frac{1}{2}}} \exp\left(-\frac{1}{2}g(\underline{u}, \underline{\phi})\right) \tag{11a}$$

$$\underline{u} = [u_1, \ldots, u_L]^T \in [0, \infty)^L$$
$$\underline{\phi} = [\phi_1, \ldots, \phi_L]^T \in [-\pi, \pi)^L$$

where

$$\begin{aligned} g(\underline{u}, \underline{\phi}) = \sum_{i=1}^L &\left(A_{ii}\cos^2\phi_i + D_{ii}\sin^2\phi_i + 2B_{ii}\cos\phi_i\sin\phi_i\right)u_i^2 \\ &+ \sum_{\substack{i,j=1 \\ i\neq j}}^L \left(A_{ij}\cos\phi_i\cos\phi_j + D_{ij}\sin\phi_i\sin\phi_j + 2B_{ij}\cos\phi_i\sin\phi_j\right)u_iu_j. \end{aligned} \tag{11b}$$

From (11), the pdf of $\underline{\alpha}$, which is *multivariate Rayleigh*, is, therefore, given by

$$f_{\underline{\alpha}}(\underline{u}) = \frac{\prod_{i=1}^L u_i}{(2\pi)^L\left(\det(\underline{K})\right)^{\frac{1}{2}}} \times \int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\exp\left(-\tfrac{1}{2}\,g(\underline{u}, \underline{\phi})\right)d\phi_1\cdots d\phi_L, \quad \underline{u} \geq \underline{0}. \tag{12}$$

Let an *exponential random vector* $\underline{\gamma}$ be given by

$$\underline{\gamma} = \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_L \end{bmatrix} = \begin{bmatrix} \alpha_1^2 \\ \vdots \\ \alpha_L^2 \end{bmatrix}. \tag{13}$$

From (11)–(13), the joint pdf of $\underline{\gamma}, \underline{\Phi}$ is expressed as

$$f_{\underline{\gamma}, \underline{\Phi}}(\underline{v}, \underline{\phi}) = \frac{1}{(4\pi)^L\left(\det(\underline{K})\right)^{\frac{1}{2}}} \exp\left(-\frac{1}{2}h(\underline{v}, \underline{\phi})\right) \tag{14a}$$

$$\underline{v} = [v_1, \ldots, v_L]^T \in [0, \infty)^L$$
$$\underline{\phi} = [\phi_1, \ldots, \phi_L]^T \in [-\pi, \pi)^L$$

where

$$h(\underline{v},\,\underline{\phi})=\sum_{i=1}^{L}\left(A_{ii}\cos^2\phi_i+D_{ii}\sin^2\phi_i\right.$$
$$\left.+2B_{ii}\cos\phi_i\sin\phi_i\right)v_i$$
$$+\sum_{\substack{i,\,j=1\\ i\neq j}}^{L}\left(A_{ij}\cos\phi_i\cos\phi_j+D_{ij}\sin\phi_i\sin\phi_j\right.$$
$$\left.+2B_{ij}\cos\phi_i\sin\phi_j\right)(v_iv_j)^{\frac{1}{2}} \tag{14b}$$

and the pdf of $\underline{\gamma}$, which is *multivariate exponential*, as

$$f_{\underline{\gamma}}(\underline{v})=\frac{1}{(4\pi)^L\,(\det(\underline{K}))^{\frac{1}{2}}}\times\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\exp\left(-\tfrac{1}{2}\,h(\underline{v},\,\underline{\phi})\right)d\phi_1\cdots d\phi_L,\quad \underline{v}\geq\underline{0}. \tag{15}$$

Equations (12) and (15) provide a *general integral form* of multivariate Rayleigh and exponential pdfs, respectively, when they are generated by correlated Gaussian random variables which satisfy conditions (2).

Note that $f_{\underline{\alpha}}(\cdot)$ can be expressed in terms of $f_{\underline{\gamma}}(\cdot)$ as

$$f_{\underline{\alpha}}(\underline{u})=2^L\left(\prod_{i=1}^{L}u_i\right)f_{\underline{\gamma}}\left(u_1^2,\,\ldots,\,u_L^2\right). \tag{16}$$

A. Case When $(\underline{X}_c,\,\underline{X}_s)$ Form a Circular Pair

Now consider the case when the generating vector $\underline{X}_c+\jmath\underline{X}_s$ is a proper [20] random vector, that is, it has a vanishing pseudocovariance matrix. Thus,

$$\boldsymbol{E}\left[(\underline{X}_c+\jmath\underline{X}_s)(\underline{X}_c+\jmath\underline{X}_s)^T\right]=\underline{0}$$

which implies

$$\underline{K}_{cc}=\underline{K}_{ss},\qquad \underline{K}_{cs}^T=-\underline{K}_{cs}. \tag{17}$$

In other words, the generating vector is a circularly symmetric random vector, the in-phase and quadrature components $(\underline{X}_c,\,\underline{X}_s)$ forming a circular pair. Under the condition (17), we obtain from (7)

$$(\underline{K}_{cc}+\jmath\underline{K}_{cs})^{-1}=\underline{A}+\jmath\underline{B} \tag{18a}$$

where

$$\underline{A}=\underline{D}=\left(\underline{K}_{cc}+\underline{K}_{cs}\,\underline{K}_{cc}^{-1}\,\underline{K}_{cs}\right)^{-1} \tag{18b}$$

$$\begin{aligned}\underline{B}&=-\underline{B}^T\\&=-\underline{K}_{cc}^{-1}\,\underline{K}_{cs}\left(\underline{K}_{cc}+\underline{K}_{cs}\,\underline{K}_{cc}^{-1}\,\underline{K}_{cs}\right)^{-1}\\&=-\left(\underline{K}_{cc}+\underline{K}_{cs}\,\underline{K}_{cc}^{-1}\,\underline{K}_{cs}\right)^{-1}\underline{K}_{cs}\,\underline{K}_{cc}^{-1}\end{aligned} \tag{18c}$$

and, from (8)

$$\begin{aligned}\det(\underline{K})&=\det(\underline{K}_{cc})\det\left(\underline{K}_{cc}+\underline{K}_{cs}\,\underline{K}_{cc}^{-1}\,\underline{K}_{cs}\right)\\&=\{\det(\underline{K}_{cc})\}^2\det\left(\underline{I}_L+\{\underline{K}_{cs}\underline{K}_{cc}^{-1}\}^2\right)\end{aligned} \tag{19}$$

where $\underline{I}_L$ is the $L\times L$ identity matrix. Note that $\underline{K}_{cc}+\jmath\underline{K}_{cs}$ and $\underline{K}_{cc}-\jmath\underline{K}_{cs}$ are Hermitian matrices having the same positive determinant. We can then rewrite (19) as

$$\begin{aligned}\det(\underline{K})&=\{\det(\underline{K}_{cc})\}^2\det\left(\underline{I}_L+\jmath\underline{K}_{cs}\underline{K}_{cc}^{-1}\right)\times\det\left(\underline{I}_L-\jmath\underline{K}_{cs}\underline{K}_{cc}^{-1}\right)\\&=\det(\underline{K}_{cc}+\jmath\underline{K}_{cs})\det(\underline{K}_{cc}-\jmath\underline{K}_{cs})\\&=\{\det(\underline{K}_{cc}+\jmath\underline{K}_{cs})\}^2\\&=\{\det(\underline{A}+\jmath\underline{B})\}^{-2}.\end{aligned} \tag{20}$$

For $i,\,j=1,\,\ldots,\,L$, we have from (18)

$$\begin{aligned}&A_{ij}=A_{ji}=D_{ji}=D_{ij}\\&B_{ij}=-B_{ji}\qquad \text{when } i\neq j,\quad B_{ii}=0\end{aligned} \tag{21}$$

which implies (from (11b))

$$\begin{aligned}g(\underline{u},\,\underline{\phi})&=\sum_{i=1}^{L}A_{ii}u_i^2+2\sum_{\substack{i,\,j=1\\ i<j}}^{L}\left(A_{ij}\cos(\phi_i-\phi_j)-B_{ij}\sin(\phi_i-\phi_j)\right)u_iu_j\\&=\sum_{i=1}^{L}A_{ii}u_i^2+2\sum_{\substack{i,\,j=1\\ i<j}}^{L}\left(A_{ij}^2+B_{ij}^2\right)^{\frac{1}{2}}u_iu_j\times\cos(\phi_i-\phi_j+\theta_{ij})\end{aligned} \tag{22a}$$

and (from (14b))

$$h(\underline{v},\,\underline{\phi})=\sum_{i=1}^{L}A_{ii}v_i+2\sum_{\substack{i,\,j=1\\ i<j}}^{L}\left(A_{ij}^2+B_{ij}^2\right)^{\frac{1}{2}}(v_iv_j)^{\frac{1}{2}}\times\cos(\phi_i-\phi_j+\theta_{ij}) \tag{22b}$$

where

$$\theta_{ij}=\tan^{-1}\left(\frac{B_{ij}}{A_{ij}}\right). \tag{22c}$$

Hence, the multivariate Rayleigh pdf of $\underline{\alpha}=[\alpha_1,\,\ldots,\,\alpha_L]^T$ can be expressed as (using (12) and (22a))

$$\begin{aligned}f_{\underline{\alpha}}(\underline{u})=&\left(\prod_{i=1}^{L}u_i\right)\det(\underline{A}+\jmath\underline{B})\exp\left(-\frac{1}{2}\sum_{i=1}^{L}A_{ii}u_i^2\right)\\&\times\frac{1}{(2\pi)^L}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\exp\left(-\sum_{\substack{i,\,j=1\\ i<j}}^{L}\left(A_{ij}^2+B_{ij}^2\right)^{\frac{1}{2}}\times u_iu_j\cos(\phi_i-\phi_j+\theta_{ij})\right)d\phi_1\cdots d\phi_L,\\&\underline{u}=[u_1,\,\ldots,\,u_L]^T\geq\underline{0}\end{aligned} \tag{23}$$

and the multivariate exponential pdf of

$$\underline{\gamma}=[\gamma_1,\,\ldots,\,\gamma_L]^T,\qquad \text{where } \gamma_i=\alpha_i^2,\ i=1,\,\ldots,\,L$$

as (using (15) and (22b))

$$f_{\underline{\gamma}}(\underline{v}) = \det(\underline{A}+\jmath\underline{B})\exp\left(-\tfrac{1}{2}\sum_{i=1}^{L}A_{ii}v_i\right) \times \frac{1}{(4\pi)^L}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\exp\left(-\sum_{\substack{i,j=1\\ i<j}}^{L}\left(A_{ij}^2+B_{ij}^2\right)^{\frac{1}{2}} \times (v_iv_j)^{\frac{1}{2}}\cos(\phi_i-\phi_j+\theta_{ij})\right)d\phi_1\cdots d\phi_L,$$
$$\underline{v}=[v_1,\,\ldots,\,v_L]^T \geq \underline{0} \tag{24}$$

where $\underline{A}+\jmath\underline{B}$ is a nonsingular Hermitian matrix satisfying

$$\underline{A}^T=\underline{A}=[A_{ij}]_{i,j=1}^L, \qquad -\underline{B}^T=\underline{B}=[B_{ij}]_{i,j=1}^L \tag{25}$$

and θ_{ij} is given by (22c).

Note that $f_{\underline{\alpha}}(\underline{u})$ in (23) can be expressed as a function of the $(L(L-1))/2$ real nonnegative parameters $(A_{ij}^2+B_{ij}^2)^{\frac{1}{2}}, i<j$. Denoting

$$\underline{G}=\left[G_1,\,\ldots,\,G_{\frac{L(L-1)}{2}}\right]^T$$

as the $(L(L-1))/2\times 1$ vector of these parameters arranged in some order, we can rewrite $f_{\underline{\alpha}}(\underline{u})$ as $f_{\underline{\alpha}}(\underline{u},\,\underline{G})$. Now

$$f_{\underline{\alpha}}(\underline{u},\underline{0})=\prod_{i=1}^{L}A_{ii}u_i\exp\left(-\frac{1}{2}\sum_{i=1}^{L}A_{ii}u_i^2\right),\ u_1,\,\ldots,\,u_L\geq 0$$

implying that $f_{\underline{\alpha}}(\underline{u},\underline{0})$ is the joint pdf of L independent Rayleigh random variables of second moments $1/A_{11},\ldots,1/A_{LL}$. A multiple power series expansion for $f_{\underline{\alpha}}(\underline{u},\underline{G})$ about $\underline{G}=\underline{0}$ can be obtained. This results in the series

$$f_{\underline{\alpha}}(\underline{u},\,\underline{G})=f_{\underline{\alpha}}(\underline{u},\,\underline{0})+\sum_{k=1}^{\frac{L(L-1)}{2}}G_k\left[\frac{\partial}{\partial G_k}f_{\underline{\alpha}}(\underline{u},\,\underline{G})\right]\Bigg|_{\underline{G}=\underline{0}} + \sum_{n=2}^{\infty}\sum_{\substack{(k_1,\ldots,k_n)\\ 1\leq k_1,\ldots,k_n\leq\frac{L(L-1)}{2}}}G_{k_1}\cdots G_{k_n} \times\left[\frac{\partial^n}{\partial G_{k_1}\cdots\partial G_{k_n}}f_{\underline{\alpha}}(\underline{u},\,\underline{G})\right]\Bigg|_{\underline{G}=\underline{0}}.$$

The first term of the series is the joint pdf when $\alpha_1,\,\ldots,\,\alpha_L$ are independent, while the remaining terms show the effect of correlation on the pdf. This power series approach has been used in [21, eq. (17)] resulting in the series expression given by [21, eq. (29)]. For correlated Rician random variables generated from nonzero-mean Gaussian random vectors, the same approach has also been used in [22, eq. (7)] resulting in the series expression given by [22, eq. (8)].

The case when the elements of $\underline{X}_c$ and $\underline{X}_s$ are samples of a stationary Gaussian process is considered in [23]. The assumption of stationarity results in a symmetric Toeplitz $\underline{K}_{cc}$ and a skew-symmetric Toeplitz $\underline{K}_{cs}$. However, while obtaining the pdf of $\underline{\alpha}$, the Toeplitz property is not utilized, and this results in [23, eq. (3.2)] which is the same as (23). By successive applications of the formula

$$e^{-z\cos\phi}=\sum_{n=-\infty}^{\infty}(-1)^nI_n(z)e^{\jmath n\phi},$$

where $I_n(\cdot)$ denotes the nth-order modified Bessel function of the first kind, in the multiple integral of [23, eq. (3.2)], a series expression [23, eq. (3.3)] for the pdf is obtained.

The bivariate and trivariate Rayleigh pdfs have been presented in [24, eq. (3.11), p. 89] and [24, eq. (3.13), p. 91], respectively. The Neumann addition formula is applied to the multiple integral in [24, eq. (3.13), p. 91] to obtain a series expression [24, eq. (3.15), p. 92] for the trivariate pdf.

Further, if $(\underline{X}_c,\,\underline{X}_s)$ form an i.i.d. pair, that is, $\underline{X}_c$ and $\underline{X}_s$ are i.i.d. zero-mean Gaussian random vectors with covariance matrix $\underline{K}_{cc}$, then

$$\underline{K}_{cc}=\underline{K}_{ss}, \qquad \underline{K}_{cs}=\underline{0} \tag{26a}$$

implying

$$\underline{K}^{-1}=\begin{bmatrix}\underline{K}_{cc}^{-1} & \underline{0}\\ \underline{0} & \underline{K}_{cc}^{-1}\end{bmatrix}=\begin{bmatrix}\underline{A} & \underline{0}\\ \underline{0} & \underline{A}\end{bmatrix}, \qquad \underline{B}=\underline{0}. \tag{26b}$$

Under condition (26a), the multivariate exponential pdf of $\underline{\gamma}$ in (24) becomes

$$f_{\underline{\gamma}}(\underline{v})=\det(\underline{A})\exp\left(-\frac{1}{2}\sum_{i=1}^{L}A_{ii}v_i\right) \times\frac{1}{(4\pi)^L}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\exp\left(-\sum_{\substack{i,j=1\\ i<j}}^{L}A_{ij} \times(v_iv_j)^{\frac{1}{2}}\cos(\phi_i-\phi_j)\right)d\phi_1\cdots d\phi_L,$$
$$\underline{v}=[v_1,\,\ldots,\,v_L]^T\geq\underline{0} \tag{27}$$

where $\underline{A}$ is a nonsingular symmetric matrix. Note that *(27) has the same structure as (24) with* $\left(A_{ij}^2+B_{ij}^2\right)^{\frac{1}{2}}$ *replaced by* A_{ij}, *and* θ_{ij} *by* 0. The multivariate Rayleigh pdf of $\underline{\alpha}$ can be easily obtained from (27) using (16).

Changing the variable of integration in (27) to

$$\phi_i'=\phi_i-\phi_L, \qquad i=1,\,\ldots,\,L-1,\ \phi_L'=\phi_L \tag{28}$$

readjusting the limits of the integration to $-\pi,\,\pi$ due to the periodicity of the integrand in each of the variables $\phi_1',\,\ldots,\,\phi_L'$, and integrating with respect to ϕ_L', we can rewrite the multivariate Rayleigh pdf corresponding to the exponential pdf (27) as an $(L-1)$-fold integral resulting in

$$f_{\underline{\alpha}}(\underline{u})=\left(\prod_{i=1}^{L}u_i\right)\det(\underline{A})\exp\left(-\frac{1}{2}\sum_{i=1}^{L}A_{ii}u_i^2\right) \times\frac{1}{(2\pi)^{L-1}}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\exp\left(-\sum_{i=1}^{L-1}A_{iL}u_iu_L\cos\phi_i' -\sum_{\substack{i,j=1\\ i<j}}^{L-1}A_{ij}u_iu_j\cos(\phi_i'-\phi_j')\right)d\phi_1'\cdots d\phi_{L-1}',$$
$$\underline{u}=[u_1,\,\ldots,\,u_L]^T\geq\underline{0}. \tag{29}$$

The trivariate Toeplitz case, where the elements of $\underline{X}_c$ and $\underline{X}_s$ are samples of a stationary Gaussian process and $\underline{K}_{cs} = \underline{0}$ is considered in [25], [26]. Using the resulting 3×3 symmetric Toeplitz $\underline{K}_{cc}$, a series expression [25, eq. (18)] for the trivariate Rayleigh pdf is obtained. A bivariate Rayleigh pdf and a trivariate Rayleigh pdf corresponding to narrow-band Gaussian noise with

$$\underline{K}_{cc} = \sigma^2 \begin{bmatrix} 1 & \rho \\ \rho & 1 \end{bmatrix}, \qquad \underline{K}_{cc} = \sigma^2 \begin{bmatrix} 1 & \rho_1 & \rho_2 \\ \rho_1 & 1 & \rho_3 \\ \rho_2 & \rho_3 & 1 \end{bmatrix}$$

respectively, are presented in [27, eq. (9)] in the same form as (29). Series expressions for the distributions of quadratic forms in Gaussian random variables, which are related to the distribution of the sum of exponential random variables generated from a Gaussian ensemble when $\underline{K}_{cs} = \underline{0}$, have been derived in [28, Ch. 4, pp. 91-132].

We consider next two particular structures of $\underline{K}_{cc}$ when $\underline{X}_c$ and $\underline{X}_s$ satisfy the circularity condition (17).

1) Case When $\alpha_1, \ldots, \alpha_L$ are Exchangeable: The condition for exchangeability [29] is that the joint pdf $f_{\alpha_{i_1}, \ldots, \alpha_{i_L}}(u_1, \ldots, u_L)$ is the same for all $L!$ permutations $i_1, \ldots, i_L$ of $1, \ldots, L$. Under this condition, $\gamma_1, \ldots, \gamma_L$, where $\gamma_i = \alpha_i^2$, $i = 1, \ldots, L$, are also exchangeable. In our framework, this is equivalent to the situation where the elements of the generating vector $\underline{X}_c + \jmath \underline{X}_s$ are identically distributed and evenly correlated zero-mean complex Gaussian random variables. This results in the set of conditions

$$\underline{K}_{cc} = \underline{K}_{ss}, \qquad \underline{K}_{cs} = -\underline{K}_{cs}^T \tag{30a}$$

$$(\underline{K}_{cc})_{ij} = \begin{cases} \sigma^2, & \text{if } i = j \\ \rho_{cc}\sigma^2, & \text{if } i \neq j, \quad i, j = 1, \ldots, L \end{cases} \tag{30b}$$

$$(\underline{K}_{cs})_{ij} = \begin{cases} 0, & \text{if } i = j \\ \rho_{cs}\sigma^2, & \text{if } i < j \\ -\rho_{cs}\sigma^2, & \text{if } i > j, \quad i, j = 1, \ldots, L \end{cases} \tag{30c}$$

where $\sigma > 0$ and $0 \leq \rho_{cc}^2 + \rho_{cs}^2 < 1$. This can be rewritten in complex notation as

$$(\underline{K}_{cc} + \jmath \underline{K}_{cs})_{ij} = \begin{cases} \sigma^2, & \text{if } i = j \\ \rho_z \sigma^2, & \text{if } i < j \\ \rho_z^* \sigma^2, & \text{if } i > j, \quad i, j = 1, \ldots, L \end{cases} \tag{31}$$

where $(\cdot)^*$ denotes the complex conjugate, and the complex correlation coefficient ρ_z is given by

$$\rho_z = \rho_{cc} + \jmath \rho_{cs}, \qquad |\rho_z| < 1. \tag{32}$$

In (31), $\underline{K}_{cc} + \jmath \underline{K}_{cs}$ is a matrix of even correlations. Such a correlation model is appropriate when, for example, we use space diversity with closely packed diversity antennas [30].

The inverse of $\underline{K}_{cc} + \jmath \underline{K}_{cs}$, given by (18), is found to have the structure

$$\begin{aligned} (\underline{A} + \jmath \underline{B})_{ij} &= A_{ij} + \jmath B_{ij} \\ &= \begin{cases} a, & \text{if } i = j \\ b e^{\jmath(\theta_0 + (j-i)\delta)}, & \text{if } i < j \\ b e^{-\jmath(\theta_0 + (i-j)\delta)}, & \text{if } i > j \end{cases} \\ & \qquad i, j = 1, \ldots, L \end{aligned} \tag{33}$$

where $a > 0$, $b \geq 0$, and θ_0, δ are real numbers. Let

$$b_z = b e^{-\jmath(\theta_0 + \delta)}, \qquad \delta_z = e^{-\jmath\delta}. \tag{34}$$

To obtain $\underline{A}$ and $\underline{B}$, we need to solve for a, b_z, δ_z in terms of ρ_z, σ^2, L. The relation

$$[(\underline{K}_{cc} + \jmath \underline{K}_{cs})(\underline{A} + \jmath \underline{B})]_{ij} = \begin{cases} 1, & \text{if } i = j \\ 0, & \text{if } i \neq j \end{cases} \tag{35}$$

gives rise to the following set of three equations:

$$a + \rho_z b_z \frac{(1 - \delta_z^{L-1})}{(1 - \delta_z)} = \sigma^{-2} \tag{36a}$$

$$\rho_z^* a + b_z + \rho_z b_z \frac{(\delta_z - \delta_z^{L-1})}{(1 - \delta_z)} = 0 \tag{36b}$$

$$\rho_z a + b_z^* + \rho_z b_z \frac{(1 - \delta_z^{L-2})}{(1 - \delta_z)} = 0. \tag{36c}$$

It is clear from (33), (34), and (36) that when $L = 2$, δ_z plays no role. Therefore, in the bivariate case, we have

$$\underline{A} + \jmath \underline{B} = \begin{bmatrix} a & b_z \\ b_z^* & a \end{bmatrix} = \frac{\sigma^{-2}}{(1 - |\rho_z|^2)} \begin{bmatrix} 1 & -\rho_z \\ -\rho_z^* & 1 \end{bmatrix} \tag{37}$$

implying

$$\begin{aligned} \underline{A} &= \frac{\sigma^{-2}}{(1 - [\rho_{cc}^2 + \rho_{cs}^2])} \begin{bmatrix} 1 & -\rho_{cc} \\ -\rho_{cc} & 1 \end{bmatrix} \\ \underline{B} &= \frac{\sigma^{-2}}{(1 - [\rho_{cc}^2 + \rho_{cs}^2])} \begin{bmatrix} 0 & -\rho_{cs} \\ \rho_{cs} & 0 \end{bmatrix}. \end{aligned} \tag{38}$$

In the multivariate case, when $L \geq 3$, solving the set (36) gives

$$\begin{aligned} a &= \frac{\left[\rho_z^*(1 - \rho_z)^{L-1} - \rho_z(1 - \rho_z^*)^{L-1}\right]}{\left[\rho_z^*(1 - \rho_z)^L - \rho_z(1 - \rho_z^*)^L\right]} \sigma^{-2} \\ b_z &= -\frac{\rho_z^*(\rho_z^* - \rho_z)(1 - \rho_z)^{L-2}}{\left[\rho_z^*(1 - \rho_z)^L - \rho_z(1 - \rho_z^*)^L\right]} \sigma^{-2} \\ \delta_z &= \frac{(1 - \rho_z^*)}{(1 - \rho_z)}. \end{aligned} \tag{39}$$

Combining (39) and (34), we get

$$\begin{aligned} a &= \frac{\sigma^{-2}}{((1 - \rho_{cc})^2 + \rho_{cs}^2)^{\frac{1}{2}}} \\ &\quad \times \frac{\sin\left((L-1)\tan^{-1}\left(\frac{\rho_{cs}}{1-\rho_{cc}}\right) + \tan^{-1}\left(\frac{\rho_{cs}}{\rho_{cc}}\right)\right)}{\sin\left(L\tan^{-1}\left(\frac{\rho_{cs}}{1-\rho_{cc}}\right) + \tan^{-1}\left(\frac{\rho_{cs}}{\rho_{cc}}\right)\right)} \\ b_z &= b e^{-\jmath(\theta_0 + \delta)} \\ &= -\frac{\rho_{cs}\sigma^{-2}}{((1 - \rho_{cc})^2 + \rho_{cs}^2)} \\ &\quad \times \frac{e^{-\jmath((L-2)\tan^{-1}(\frac{\rho_{cs}}{1-\rho_{cc}}) + \tan^{-1}(\frac{\rho_{cs}}{\rho_{cc}}))}}{\sin\left(L\tan^{-1}\left(\frac{\rho_{cs}}{1-\rho_{cc}}\right) + \tan^{-1}\left(\frac{\rho_{cs}}{\rho_{cc}}\right)\right)} \\ \delta &= -2\tan^{-1}\left(\frac{\rho_{cs}}{1 - \rho_{cc}}\right) \\ b e^{-\jmath\theta_0} &= -\frac{\rho_{cs}\sigma^{-2}}{((1 - \rho_{cc})^2 + \rho_{cs}^2)} \\ &\quad \times \frac{e^{-\jmath(L\tan^{-1}(\frac{\rho_{cs}}{1-\rho_{cc}}) + \tan^{-1}(\frac{\rho_{cs}}{\rho_{cc}}))}}{\sin\left(L\tan^{-1}\left(\frac{\rho_{cs}}{1-\rho_{cc}}\right) + \tan^{-1}\left(\frac{\rho_{cs}}{\rho_{cc}}\right)\right)}. \end{aligned} \tag{40}$$

It is clear from (33) that, in this case, θ_{ij} is given by

$$\theta_{ij} = \begin{cases} (j-i)\delta + \theta_0, & \text{if } i < j \\ (j-i)\delta - \theta_0, & \text{if } i > j, \quad i, j = 1, \ldots, L. \end{cases} \tag{41}$$

Changing the variables of integration in (24) to

$$\epsilon_i = \phi_i + i\delta, \qquad i = 1, \ldots, L \tag{42}$$

the pdf of $\underline{\gamma}$ can be expressed as

$$\begin{aligned} f_{\underline{\gamma}}(\underline{v}) &= \det(\underline{A} + \jmath\underline{B}) \exp\left(-\frac{a}{2}\sum_{i=1}^{L} v_i\right) \\ &\quad \times \frac{1}{(4\pi)^L}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} \exp\left(-b\sum_{\substack{i,j=1\\i<j}}^{L}(v_i v_j)^{\frac{1}{2}} \right.\\ &\quad \left. \times \cos(\epsilon_i - \epsilon_j + \theta_0)\right) d\epsilon_1 \cdots d\epsilon_L, \\ \underline{v} &= [v_1, \ldots, v_L]^T \geq \underline{0} \end{aligned} \tag{43}$$

where a, b, and θ_0 are given by (40). The corresponding expression for $f_{\underline{\alpha}}(\underline{u})$ can be easily obtained from (43) using (16).

When ρ_z is real, that is,

$$\rho_{cs} = 0, \qquad \rho_{cc} = \rho, \qquad -1 < \rho < 1 \tag{44}$$

we have the i.i.d. condition (26a). This implies $\underline{B} = \underline{0}$, and, from (40), by taking the limit as ρ_{cs} goes to zero, we obtain

$$A_{ij} = \begin{cases} a = \dfrac{(1+[L-2]\rho)}{(1-\rho)(1+[L-1]\rho)\sigma^2}, & \text{if } i = j \\ b_z = -\dfrac{\rho}{(1-\rho)(1+[L-1]\rho)\sigma^2}, & \text{if } i \neq j \end{cases} \quad i, j = 1, \ldots, L. \tag{45}$$

The determinant of $\underline{A}$ can be shown to be given by

$$\begin{aligned} \det(\underline{A}) &= \{\det(\underline{K}_{cc})\}^{-1} \\ &= \left\{\sigma^{2L}(1-\rho)^{L-1}(1+[L-1]\rho)\right\}^{-1}. \end{aligned} \tag{46}$$

From (43) and (45), the multivariate exponential pdf in the i.i.d. case is

$$\begin{aligned} f_{\underline{\gamma}}(\underline{v}) &= \frac{1}{\sigma^{2L}(1-\rho)^{L-1}(1+[L-1]\rho)} \\ &\quad \times \exp\left(-\frac{(1+[L-2]\rho)}{2(1-\rho)(1+[L-1]\rho)\sigma^2}\sum_{i=1}^{L} v_i\right) \\ &\quad \times \frac{1}{(4\pi)^L}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi}\exp\left(\frac{\rho}{(1-\rho)(1+[L-1]\rho)\sigma^2} \right.\\ &\quad \left. \times \sum_{\substack{i,j=1\\i<j}}^{L}(v_i v_j)^{\frac{1}{2}}\cos(\phi_i - \phi_j)\right) d\phi_1\cdots d\phi_L, \\ \underline{v} &= [v_1, \ldots, v_L]^T \geq \underline{0}. \end{aligned} \tag{47}$$

The Rayleigh pdf can be obtained from (47) using (16).

2) Case When $\underline{K}_{cc} + \jmath\underline{K}_{cs}$ is a Matrix of Exponential Correlations: Here we consider the situation when

$$\underline{K}_{cc} = \underline{K}_{ss}, \qquad \underline{K}_{cs} = -\underline{K}_{cs}^T \tag{48a}$$

$$(\underline{K}_{cc} + \jmath\underline{K}_{cs})_{ij} = \begin{cases} \sigma^2, & \text{if } i = j \\ \rho_z^{j-i}\sigma^2, & \text{if } i < j \\ (\rho_z^*)^{i-j}\sigma^2, & \text{if } i > j \end{cases} \quad i, j = 1, \ldots, L \tag{48b}$$

where $\sigma > 0$ and $|\rho_z| < 1$. This is equivalent to the Markov property [4], [5] for the Rayleigh random variables $\alpha_1, \ldots, \alpha_L$. For diversity in space or time, this translates to the case where the correlation between the diversity branches decreases with increase of branch spacing, but the fading process remains stationary in the respective domain of space or time [30].

Since the *complex covariance matrix* $\underline{K}_{cc} + \jmath\underline{K}_{cs}$ has the structure

$$(\underline{K}_{cc} + \jmath\underline{K}_{cs})_{ij} = \begin{cases} s_j t_i, & \text{if } i \leq j \\ s_i^* t_j^*, & \text{if } i \geq j \end{cases} \tag{49a}$$

where

$$s_i = \rho_z^i \sigma, \qquad t_i = \rho_z^{-i}\sigma, \qquad i = 1, \ldots, L \tag{49b}$$

its inverse is a Hermitian tridiagonal matrix [33]–[36]. We can show by direct matrix multiplication that

$$A_{ij} + \jmath B_{ij} = \begin{cases} \dfrac{1}{(1-|\rho_z|^2)\sigma^2}, & \text{if } i = j = 1, L \\ \dfrac{(1+|\rho_z|^2)}{(1-|\rho_z|^2)\sigma^2}, & \text{if } i = j = 2, \ldots, L-1 \\ -\dfrac{\rho_z}{(1-|\rho_z|^2)\sigma^2}, & \text{if } j - i = 1, \\ & \quad i = 1, \ldots, L-1 \\ -\dfrac{\rho_z^*}{(1-|\rho_z|^2)\sigma^2}, & \text{if } i - j = 1, \\ & \quad j = 1, \ldots, L-1 \\ 0, & \text{otherwise.} \end{cases} \tag{50}$$

In other words

$$\begin{aligned} &(\underline{K}_{cc} + \jmath\underline{K}_{cs})^{-1} \\ &= \underline{A} + \jmath\underline{B} \\ &= \frac{1}{(1-|\rho_z|^2)\sigma^2} \\ &\quad \times \begin{bmatrix} 1 & -\rho_z & & & \\ -\rho_z^* & (1+|\rho_z|^2) & -\rho_z & & \mathbf{0} \\ & \ddots & \ddots & \ddots & \\ \mathbf{0} & & -\rho_z^* & (1+|\rho_z|^2) & -\rho_z \\ & & & -\rho_z^* & 1 \end{bmatrix}. \end{aligned} \tag{51}$$

Let

$$\rho_z = \rho_r e^{\jmath\psi}, \qquad 0 \leq \rho_r < 1 \tag{52}$$

be the polar representation of ρ_z. Using result (140) in the Appendix, we have

$$\begin{aligned}\det(\underline{A} + \jmath\underline{B}) &= \{\det(\underline{K}_{cc} + \jmath\underline{K}_{cs})\}^{-1} \\ &= \left\{\sigma^{2L}(1-\rho_r^2)^{L-1}\right\}. \end{aligned} \tag{53}$$

From (27), (16), (50), and (53), the multivariate Rayleigh pdf is

$$\begin{aligned} f_{\underline{\alpha}}(\underline{u}) = \frac{\left(\prod_{i=1}^{L} u_i\right)}{\sigma^{2L}(1-\rho_r^2)^{L-1}} \exp\Bigg(&-\frac{1}{2(1-\rho_r^2)\sigma^2} \\ &\times \left[u_1^2 + u_L^2 + (1+\rho_r^2)\sum_{i=2}^{L-1} u_i^2\right]\Bigg) \\ &\times \frac{1}{(2\pi)^L}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} \exp\Bigg(\frac{\rho_r}{(1-\rho_r^2)\sigma^2} \\ &\times \sum_{i=1}^{L-1} u_i u_{i+1}\cos(\phi_i - \phi_{i+1} + \delta)\Bigg) d\phi_1\cdots d\phi_L, \\ &\underline{u} = [u_1, \ldots, u_L]^T \geq \underline{0}. \end{aligned} \tag{54}$$

Let the integral in (54) be denoted as $\mathcal{I}(u_1, \ldots, u_L)$, that is,

$$\begin{aligned} &\mathcal{I}(u_1, \ldots, u_L) \\ &\quad = \int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} \exp\Bigg(\frac{\rho_r}{(1-\rho_r^2)\sigma^2} \\ &\qquad \times \sum_{i=1}^{L-1} u_i u_{i+1}\cos(\phi_i - \phi_{i+1} + \delta)\Bigg) d\phi_1\cdots d\phi_L. \end{aligned} \tag{55}$$

By changing the variables of integration to

$$\phi_i' = \phi_i - \phi_{i+1} + \delta, \qquad i = 1, \ldots, L-1, \qquad \phi_L' = \phi_L - L\delta$$

we obtain from (55)

$$\begin{aligned} \mathcal{I}(u_1, \ldots, u_L) &= \int_{\phi_L'=-\pi-L\delta}^{\pi-L\delta} \int_{\phi_{L-1}'=-\pi-\phi_L'-(L-1)\delta}^{\pi-\phi_L'-(L-1)\delta} \\ &\quad \cdots \int_{\phi_1'=-\pi-\phi_2'-\cdots-\phi_L'-\delta}^{\pi-\phi_2'-\cdots-\phi_L'-\delta} \\ &\quad \prod_{i=1}^{L-1} \exp\left(\frac{\rho_r}{(1-\rho_r^2)\sigma^2}\, u_i u_{i+1}\cos\phi_i'\right) \\ &\quad d\phi_1'\cdots d\phi_{L-1}' d\phi_L' \\ &= (2\pi)^L \prod_{i=1}^{L-1} I_0\left(\frac{\rho_r}{(1-\rho_r^2)\sigma^2}\, u_i u_{i+1}\right) \end{aligned} \tag{56}$$

where $I_0(\cdot)$ is the zeroth-order modified Bessel function of the first kind, and is an even function of its argument. From (54)–(56), the multivariate Rayleigh pdf simplifies to

$$\begin{aligned} f_{\underline{\alpha}}(\underline{u}) = \frac{\left(\prod_{i=1}^{L} u_i\right)}{\sigma^{2L}(1-\rho_r^2)^{L-1}} \exp\Bigg(&-\frac{1}{2(1-\rho_r^2)\sigma^2} \\ &\times \left[u_1^2 + u_L^2 + (1+\rho_r^2)\sum_{i=2}^{L-1} u_i^2\right]\Bigg) \\ &\times \prod_{i=1}^{L-1} I_0\left(\frac{\rho_r}{(1-\rho_r^2)\sigma^2}\, u_i u_{i+1}\right), \\ &\underline{u} = [u_1, \ldots, u_L]^T \geq \underline{0}. \end{aligned} \tag{57}$$

Similarly, the multivariate exponential pdf simplifies to

$$\begin{aligned} f_{\underline{\gamma}}(\underline{v}) = \frac{1}{2^L\sigma^{2L}(1-\rho_r^2)^{L-1}} \exp\Bigg(&-\frac{1}{2(1-\rho_r^2)\sigma^2} \\ &\times \left[v_1 + v_L + (1+\rho_r^2)\sum_{i=2}^{L-1} v_i\right]\Bigg) \\ &\times \prod_{i=1}^{L-1} I_0\left(\frac{\rho_r}{(1-\rho_r^2)\sigma^2}\, (v_i v_{i+1})^{\frac{1}{2}}\right), \\ &\underline{v} = [v_1, \ldots, v_L]^T \geq \underline{0}. \end{aligned} \tag{58}$$

If we take L samples of the envelope of a narrow-band Gaussian noise, then the samples form a first-order Markov chain, and the joint pdf of the samples has the "product of modified Bessel functions" form as in (57). This has been discussed in [4], [31], [32] and derived in [4], where the i.i.d. case (26a) with

$$(\underline{K}_{cc})_{ij} = \begin{cases} \sigma^2, & \text{if } i = j \\ \sigma^2 \prod_{i=1}^{j-1} \rho_{i,\, i+1}, & \text{if } i < j \\ \sigma^2 \prod_{j=1}^{i-1} \rho_{j,\, j+1}, & \text{if } i > j, \quad i, j = 1, \ldots, L \end{cases} \tag{59}$$

is considered. Thus, $\underline{K}_{cc}$ is characterized by the marginal variance σ^2 and the correlation coefficients

$$\rho_{i,\, i+1}, \qquad -1 < \rho_{i,\, i+1} < 1, \qquad i = 1, \ldots, L-1.$$

It can be shown that $\underline{K}_{cc}^{-1}$ is a symmetric tridiagonal matrix [33]–[36]; hence, the multivariate Rayleigh pdf of the envelope samples of narrow-band noise has the same form as in (57).

The generalized multivariate Rayleigh pdf has been considered in [37], [38, pp. 37–39], [39, pp. 160–162]. This is the same as the joint pdf of the square roots of correlated $\chi^2(n)$ random variables with correlation matrix $\underline{K}_{cc}$. Assuming $\underline{K}_{cc}^{-1}$ to be a Jacobi (symmetric tridiagonal) matrix, a "product of modified Bessel functions" form of the generalized multivariate Rayleigh pdf is given by [37, eq. (2.1)], [38, eq. (34), p. 37], [39, eq. (9), p. 162]. When $n = 2$ in [37, eq. (2.1)], [38, eq. (34), p. 37], [39, eq. (9), p. 162], we get a multivariate Rayleigh pdf with $\underline{K}_{cs} = \underline{0}$ which has the same form as in (57). The multivariate $\chi^2(n)$ pdf can be obtained by squaring the generalized multivariate Rayleigh random variables. Such a pdf with $\underline{K}_{cc}$ given by (59) has been presented in [40, eq. (48.69), p. 473]. When the order of the χ^2 distribution is 2 in [40, eq. (48.69), p. 473], we get a multivariate exponential pdf with $\underline{K}_{cs} = \underline{0}$ which has the same form as in (58).

Using the expansion

$$I_0(z) = \sum_{k=0}^{\infty} \frac{\left(\frac{1}{4}z^2\right)^k}{(k!)^2} \tag{60}$$

in (58), we get

$$f_{\underline{\gamma}}(\underline{v}) = \frac{1}{2^L\sigma^{2L}(1-\rho_r^2)^{L-1}} \exp\left(-\frac{1}{2(1-\rho_r^2)\sigma^2} \times \left[v_1 + v_L + (1+\rho_r^2)\sum_{i=2}^{L-1} v_i\right]\right) \times \sum_{n=0}^{\infty} \sum_{\substack{(l_1,\ldots,l_{L-1}) \\ 0\le l_1,\ldots,l_{L-1}\le n \\ l_1+l_2+\cdots+l_{L-1}=n}} \left(\frac{\rho_r^2}{4(1-\rho_r^2)\sigma^4}\right)^n \times \frac{v_1^{l_1} v_2^{l_1+l_2}\cdots v_{L-1}^{l_{L-2}+l_{L-1}} v_L^{l_{L-1}}}{(l_1!)^2\cdots(l_{L-1}!)^2}, \quad \underline{v} = [v_1,\ldots,v_L]^T \ge \underline{0}. \tag{61}$$

Let

$$g_{gam}(v;k,c) \triangleq \begin{cases} \dfrac{v^{k-1}e^{-\frac{v}{c}}}{(k-1)!c^k}, & \text{if } v \ge 0 \\ 0, & \text{if } v < 0 \end{cases} \tag{62}$$

where $c > 0$, $k = 1, 2, \ldots$, denote a *univariate gamma pdf* with parameter k and mean kc. We can rewrite (61) in terms of $g_{gam}(\cdot;\cdot,\cdot)$ as

$$f_{\underline{\gamma}}(\underline{v}) = (1-\rho_r^2)\sum_{n=0}^{\infty}\rho_r^{2n} \times \sum_{\substack{(l_1,\ldots,l_{L-1}) \\ 0\le l_1,\ldots,l_{L-1}\le n \\ l_1+l_2+\cdots+l_{L-1}=n}} g_{gam}\left(v_1; l_1+1, 2(1-\rho_r^2)\sigma^2\right) \times g_{gam}\left(v_L; l_{L-1}+1, 2(1-\rho_r^2)\sigma^2\right) \times \left[\prod_{i=2}^{L-1}\binom{l_{i-1}+l_i}{l_i} \times \frac{g_{gam}\left(v_i; l_{i-1}+l_i+1, \frac{2(1-\rho_r^2)\sigma^2}{(1+\rho_r^2)}\right)}{(1+\rho_r^2)^{l_{i-1}+l_i+1}}\right]. \tag{63}$$

Thus, (63) is *a series of products of univariate gamma pdfs*, and is an alternative form of (58). Defining the *univariate Nakagami pdf* as

$$g_{Nak}(u;k,c) \triangleq 2u\,g_{gam}(u^2;k,c) \tag{64}$$

we obtain from (64) the following expression for the pdf of $\underline{\alpha}$:

$$f_{\underline{\alpha}}(\underline{u}) = (1-\rho_r^2)\sum_{n=0}^{\infty}\rho_r^{2n} \times \sum_{\substack{(l_1,\ldots,l_{L-1}) \\ 0\le l_1,\ldots,l_{L-1}\le n \\ l_1+l_2+\cdots+l_{L-1}=n}} g_{Nak}\left(u_1; l_1+1, 2(1-\rho_r^2)\sigma^2\right) \times g_{Nak}\left(u_L; l_{L-1}+1, 2(1-\rho_r^2)\sigma^2\right) \times \left[\prod_{i=2}^{L-1}\binom{l_{i-1}+l_i}{l_i} \times \frac{g_{Nak}\left(u_i; l_{i-1}+l_i+1, \frac{2(1-\rho_r^2)\sigma^2}{(1+\rho_r^2)}\right)}{(1+\rho_r^2)^{l_{i-1}+l_i+1}}\right]. \tag{65}$$

This is a series of products of univariate Nakagami pdfs, and is an alternative form of (57).

B. Covariance Matrix of the Exponential Random Vector

We express the elements of the covariance matrix $\underline{K}_{\underline{\gamma}}$ of the exponential random vector $\underline{\gamma}$ in terms of the elements of $\underline{K}_{cc}$, $\underline{K}_{ss}$, $\underline{K}_{cs}$. Observe that

$$\boldsymbol{E}[\gamma_i] = \boldsymbol{E}\left[X_{c_i}^2 + X_{s_i}^2\right] = 2\left(\underline{K}_{cc}\right)_{ii} \tag{66a}$$

$$\begin{aligned}\boldsymbol{E}\left[\gamma_i^2\right] &= \boldsymbol{E}\left[X_{c_i}^4 + X_{s_i}^4 + 2X_{c_i}^2X_{s_i}^2\right] \\ &= 3\left\{\boldsymbol{E}\left[X_{c_i}^2\right]\right\}^2 + 3\left\{\boldsymbol{E}\left[X_{s_i}^2\right]\right\}^2 + 2\boldsymbol{E}\left[X_{c_i}^2\right]\boldsymbol{E}\left[X_{s_i}^2\right] \\ &= 8\left\{\left(\underline{K}_{cc}\right)_{ii}\right\}^2 \end{aligned} \tag{66b}$$

and, for $i \ne j$

$$\begin{aligned}\boldsymbol{E}[\gamma_i\gamma_j] &= \boldsymbol{E}\left[X_{c_i}^2X_{c_j}^2 + X_{s_i}^2X_{s_j}^2 + X_{c_i}^2X_{s_j}^2 + X_{s_i}^2X_{c_j}^2\right] \\ &= \boldsymbol{E}\left[X_{c_i}^2\right]\boldsymbol{E}\left[X_{c_j}^2\right] + 2\left\{\boldsymbol{E}\left[X_{c_i}X_{c_j}\right]\right\}^2 \\ &\quad + \boldsymbol{E}\left[X_{s_i}^2\right]\boldsymbol{E}\left[X_{s_j}^2\right] + 2\left\{\boldsymbol{E}\left[X_{s_i}X_{s_j}\right]\right\}^2 \\ &\quad + \boldsymbol{E}\left[X_{c_i}^2\right]\boldsymbol{E}\left[X_{s_j}^2\right] + 2\left\{\boldsymbol{E}\left[X_{c_i}X_{s_j}\right]\right\}^2 \\ &\quad + \boldsymbol{E}\left[X_{s_i}^2\right]\boldsymbol{E}\left[X_{c_j}^2\right] + 2\left\{\boldsymbol{E}\left[X_{s_i}X_{c_j}\right]\right\}^2 \\ &= 4\left(\underline{K}_{cc}\right)_{ii}\left(\underline{K}_{cc}\right)_{jj} \\ &\quad + 2\left\{\left(\underline{K}_{cc}\right)_{ij}\right\}^2 + 2\left\{\left(\underline{K}_{ss}\right)_{ij}\right\}^2 \\ &\quad + 2\left\{\left(\underline{K}_{cs}\right)_{ij}\right\}^2 + 2\left\{\left(\underline{K}_{cs}\right)_{ji}\right\}^2. \end{aligned} \tag{66c}$$

The variance of γ_i and the covariance of $\gamma_i\gamma_j$, $i \ne j$, obtained from (66), are

$$\operatorname{var}[\gamma_i] = 4\left\{\left(\underline{K}_{cc}\right)_{ii}\right\}^2, \tag{67a}$$

$$\begin{aligned}\operatorname{cov}[\gamma_i,\gamma_j] &= 2\left\{\left(\underline{K}_{cc}\right)_{ij}\right\}^2 + 2\left\{\left(\underline{K}_{ss}\right)_{ij}\right\}^2 \\ &\quad + 2\left\{\left(\underline{K}_{cs}\right)_{ij}\right\}^2 + 2\left\{\left(\underline{K}_{cs}\right)_{ji}\right\}^2, \qquad i \ne j. \end{aligned} \tag{67b}$$

Note that all entries of $\underline{K}_{\underline{\gamma}}$ are nonnegative.

It is clear from (67) that *in general, it is not possible to specify the multivariate exponential distribution, which depends on* $\underline{K}_{cc}$, $\underline{K}_{ss}$, $\underline{K}_{cs}$, *in terms of* $\underline{K}_{\underline{\gamma}}$ *alone.*

In the special case when the circularity condition (17) holds, we get from (67)

$$\left(\underline{K}_{\underline{\gamma}}\right)_{ij} = 4\left\{\left(\underline{K}_{cc}\right)_{ij}\right\}^2 + 4\left\{\left(\underline{K}_{cs}\right)_{ij}\right\}^2, \quad i,j = 1,\ldots,L \tag{68}$$

and when the i.i.d. condition (26a) holds, we get

$$\left(\underline{K}_{\underline{\gamma}}\right)_{ij} = 4\left\{\left(\underline{K}_{cc}\right)_{ij}\right\}^2, \qquad i,j = 1,\ldots,L. \tag{69}$$

We conclude from (69) that it is not possible to establish a one-to-one map between $\underline{K}_{cc}$ and $\underline{K}_{\underline{\gamma}}$ even when $\underline{K}_{cs} = \underline{0}$ since the off-diagonal entries of $\underline{K}_{cc}$ can be negative.

In the circular exchangeable case, with $\underline{K}_{cc} + \jmath\underline{K}_{cs}$ as in (31), $\underline{K}_{\underline{\gamma}}$ is given by

$$\left(\underline{K}_{\underline{\gamma}}\right)_{ij} = \begin{cases} 4\sigma^4, & \text{if } i = j \\ \left(\rho_{cc}^2 + \rho_{cs}^2\right)(4\sigma^4), & \text{if } i \ne j \end{cases} \quad i,j = 1,\ldots,L. \tag{70}$$

In the case when $\underline{K}_{cc} + \jmath \underline{K}_{cs}$ is a matrix of exponential correlations as in (48), $\underline{K}_{\underline{\gamma}}$ is expressed as

$$\left(\underline{K}_{\underline{\gamma}}\right)_{ij} = \begin{cases} 4\sigma^4, & \text{if } i = j \\ |\rho_z|^{2|i-j|}(4\sigma^4), & \text{if } i \neq j \end{cases} \qquad i, j = 1, \ldots, L. \tag{71}$$

III. Characteristic Functions

Expressions for the univariate and bivariate Rayleigh cfs are given in [12, p. 409]. Although it is difficult to obtain a closed-form expression for the multivariate Rayleigh cf, the multivariate exponential cf can be expressed in closed form.

From (3) and (13), we can express the cf of $\underline{\gamma}$ as

$$\begin{aligned}\Psi_{\underline{\gamma}}(\jmath\underline{\omega}) &= \boldsymbol{E}\left[\exp\left\{\jmath\underline{\omega}^T\underline{\gamma}\right\}\right] = \boldsymbol{E}\left[\exp\left\{\jmath\sum_{i=1}^{L}\omega_i\gamma_i\right\}\right] \\ &= \boldsymbol{E}\left[\exp\left\{\jmath\sum_{i=1}^{L}\omega_i\left(X_{c_i}^2 + X_{s_i}^2\right)\right\}\right] \\ &= \boldsymbol{E}\left[\exp\left\{\jmath\left[\underline{X}_c^T \quad \underline{X}_s^T\right]\right.\right. \\ &\qquad \left.\left.\times\begin{bmatrix}\mathrm{diag}(\underline{\omega}) & \underline{0} \\ \underline{0} & \mathrm{diag}(\underline{\omega})\end{bmatrix}\begin{bmatrix}\underline{X}_c \\ \underline{X}_s\end{bmatrix}\right\}\right]\end{aligned} \tag{72a}$$

where

$$\underline{\omega} = \begin{bmatrix}\omega_1 \\ \vdots \\ \omega_L\end{bmatrix}, \quad \mathrm{diag}(\underline{\omega}) = \begin{bmatrix}\omega_1 & & \underline{0} \\ & \ddots & \\ \underline{0} & & \omega_L\end{bmatrix}. \tag{72b}$$

Let

$$\underline{X} \triangleq \begin{bmatrix}\underline{X}_c \\ \underline{X}_s\end{bmatrix}. \tag{73}$$

It is clear that $\underline{X}$ is a zero-mean real Gaussian $2L \times 1$ random vector with covariance matrix $\underline{K}$ given by (6b). Then (72) can be rewritten as

$$\begin{aligned}&\Psi_{\underline{\gamma}}(\jmath\underline{\omega}) \\ &= \int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\left\{\jmath\underline{x}^T\begin{bmatrix}\mathrm{diag}(\underline{\omega}) & \underline{0} \\ \underline{0} & \mathrm{diag}(\underline{\omega})\end{bmatrix}\underline{x}\right\} \\ &\quad\times\frac{1}{(2\pi)^L\left(\det(\underline{K})\right)^{\frac{1}{2}}}\exp\left\{-\frac{1}{2}\,\underline{x}^T\underline{K}^{-1}\underline{x}\right\}d\underline{x} \\ &= \frac{1}{(2\pi)^L\left(\det(\underline{K})\right)^{\frac{1}{2}}}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \\ &\quad\exp\left\{-\frac{1}{2}\,\underline{x}^T\left(\underline{K}^{-1} - 2\jmath\begin{bmatrix}\mathrm{diag}(\underline{\omega}) & \underline{0} \\ \underline{0} & \mathrm{diag}(\underline{\omega})\end{bmatrix}\right)\underline{x}\right\}d\underline{x} \\ &= \frac{\left\{\det\left(\underline{K}^{-1} - 2\jmath\begin{bmatrix}\mathrm{diag}(\underline{\omega}) & \underline{0} \\ \underline{0} & \mathrm{diag}(\underline{\omega})\end{bmatrix}\right)\right\}^{-\frac{1}{2}}}{\left\{\det(\underline{K})\right\}^{\frac{1}{2}}}\end{aligned} \tag{74}$$

using the fact that

$$\frac{1}{(2\pi)^L}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty}\exp\left\{-\frac{1}{2}\,\underline{x}^T\underline{C}\,\underline{x}\right\}d\underline{x} = \left\{\det(\underline{C})\right\}^{-\frac{1}{2}}.$$

This implies

$$\Psi_{\underline{\gamma}}(\jmath\underline{\omega}) = \left\{\det\left(\underline{I}_{2L} - 2\jmath\begin{bmatrix}\mathrm{diag}(\underline{\omega}) & \underline{0} \\ \underline{0} & \mathrm{diag}(\underline{\omega})\end{bmatrix}\times\begin{bmatrix}\underline{K}_{cc} & \underline{K}_{cs} \\ \underline{K}_{cs}^T & \underline{K}_{ss}\end{bmatrix}\right)\right\}^{-\frac{1}{2}} \tag{75a}$$

$$= \left\{\det\left(\begin{bmatrix}\underline{I}_L - 2\jmath\,\mathrm{diag}(\underline{\omega})\underline{K}_{cc} & -2\jmath\,\mathrm{diag}(\underline{\omega})\underline{K}_{cs} \\ -2\jmath\,\mathrm{diag}(\underline{\omega})\underline{K}_{cs}^T & \underline{I}_L - 2\jmath\,\mathrm{diag}(\underline{\omega})\underline{K}_{ss}\end{bmatrix}\right)\right\}^{-\frac{1}{2}} \tag{75b}$$

$$\begin{aligned}&= \left\{\det\left(\underline{I}_L - 2\jmath\,\mathrm{diag}(\underline{\omega})\underline{K}_{cc}\right)\right\}^{-\frac{1}{2}} \\ &\quad\times\left\{\det\left(\left[\underline{I}_L - 2\jmath\,\mathrm{diag}(\underline{\omega})\underline{K}_{ss}\right]\right.\right. \\ &\qquad\quad + 4\,\mathrm{diag}(\underline{\omega})\underline{K}_{cs}^T\left[\underline{I}_L - 2\jmath\,\mathrm{diag}(\underline{\omega})\underline{K}_{cc}\right]^{-1} \\ &\qquad\quad \left.\left.\times\mathrm{diag}(\underline{\omega})\underline{K}_{cs}\right)\right\}^{-\frac{1}{2}}\end{aligned} \tag{75c}$$

$$\begin{aligned}&= \left\{\det\left(\underline{I}_L - 2\jmath\,\mathrm{diag}(\underline{\omega})\underline{K}_{ss}\right)\right\}^{-\frac{1}{2}} \\ &\quad\times\left\{\det\left(\left[\underline{I}_L - 2\jmath\,\mathrm{diag}(\underline{\omega})\underline{K}_{cc}\right]\right.\right. \\ &\qquad\quad + 4\,\mathrm{diag}(\underline{\omega})\underline{K}_{cs}\left[\underline{I}_L - 2\jmath\,\mathrm{diag}(\underline{\omega})\underline{K}_{ss}\right]^{-1} \\ &\qquad\quad \left.\left.\times\mathrm{diag}(\underline{\omega})\underline{K}_{cs}^T\right)\right\}^{-\frac{1}{2}}\end{aligned} \tag{75d}$$

where $\underline{K}_{cc}$, $\underline{K}_{ss}$ are real-symmetric matrices, and (see (2b))

$$(\underline{K}_{cc})_{ii} = (\underline{K}_{ss})_{ii}, \qquad (\underline{K}_{cs})_{ii} = 0, \qquad i = 1, \ldots, L. \tag{75e}$$

Note that (75c) and (75d) can be obtained from (75b) by matrix row or column operations as in (8). Thus, (75) represents a *general expression of the cf* corresponding to the pdf (15).

A. Case When $(\underline{X}_c, \underline{X}_s)$ Form a Circular Pair

In the special case when the generating Gaussian random vectors $(\underline{X}_c, \underline{X}_s)$ form a circular pair satisfying (17), the cf (75) can be rewritten as

$$\begin{aligned}\Psi_{\underline{\gamma}}(\jmath\underline{\omega}) &= \left\{\det\left(\underline{I}_L - 2\jmath\,\mathrm{diag}(\underline{\omega})\underline{K}_{cc}\right)\right\}^{-1} \\ &\quad\times\left\{\det\left(\underline{I}_L - 4\left\{\mathrm{diag}(\underline{\omega})\underline{K}_{cs}\right.\right.\right. \\ &\qquad\left.\left.\left.\times\left[\underline{I}_L - 2\jmath\,\mathrm{diag}(\underline{\omega})\underline{K}_{cc}\right]^{-1}\right\}^2\right)\right\}^{-\frac{1}{2}} \\ &= \left\{\det\left(\underline{I}_L - 2\jmath\,\mathrm{diag}(\underline{\omega})\left[\underline{K}_{cc} + \jmath\underline{K}_{cs}\right]\right)\right. \\ &\quad\left.\times\det\left(\underline{I}_L - 2\jmath\,\mathrm{diag}(\underline{\omega})\left[\underline{K}_{cc} - \jmath\underline{K}_{cs}\right]\right)\right\}^{-\frac{1}{2}}.\end{aligned} \tag{76}$$

This corresponds to the pdf (24).

If we have the stronger condition that $\underline{X}_c$ and $\underline{X}_s$ are i.i.d. vectors satisfying (26a), the cf (76) simplifies to

$$\begin{aligned}\Psi_{\underline{\gamma}}(\jmath\underline{\omega}) &= \left\{\det\left(\underline{I}_L - 2\jmath\,\mathrm{diag}(\underline{\omega})\underline{K}_{cc}\right)\right\}^{-1} \\ &= \left\{\det\left(\underline{I}_L - 2\jmath\,\mathrm{diag}(\underline{\omega})\underline{A}^{-1}\right)\right\}^{-1}\end{aligned} \tag{77}$$

which corresponds to the pdf (27). The multivariate gamma cfs in [7], [16], [17, p. 43] reduce to (77) when the gamma parameter equals unity. Therefore, *the widely used form (77) of the multivariate exponential cf is not a general one but a special case of (75) under the i.i.d. condition (26a).*

1) Case When $\underline{K}_{cc} + \jmath \underline{K}_{cs}$ Is a Matrix of Exponential Correlations: In the case of an exponential correlation matrix $\underline{K}_{cc}$ as in (48), we can apply the result [41]

$$\int_{-\infty}^{\infty} g_{gam}(v;\, k,\, c) e^{j\omega v}\, du = \int_0^{\infty} \frac{v^{k-1} e^{-\frac{v}{c}}}{(k-1)! c^k}\, e^{j\omega v}\, dv = \frac{1}{(1-j\omega c)^k} \tag{78}$$

on (63) so as to obtain

$$\begin{aligned}\Psi_{\underline{\gamma}}(\jmath\underline{\omega}) = &(1-\rho_r^2)\sum_{n=0}^{\infty}\rho_r^{2n} \\ &\times \sum_{\substack{(l_1,\,\ldots,\,l_{L-1}) \\ 0\le l_1,\,\ldots,\,l_{L-1}\le n \\ l_1+l_2+\cdots+l_{L-1}=n}} \frac{1}{(1-2\jmath\omega_1(1-\rho_r^2)\sigma^2)^{l_1+1}} \\ &\times \frac{1}{(1-2\jmath\omega_L(1-\rho_r^2)\sigma^2)^{l_{L-1}+1}} \\ &\times \left[\prod_{i=2}^{L-1}\binom{l_{i-1}+l_i}{l_i}\right. \\ &\left.\times \frac{(1+\rho_r^2)^{-(l_{i-1}+l_i+1)}}{\left(1-2\jmath\omega_i\frac{(1-\rho_r^2)\sigma^2}{(1+\rho_r^2)}\right)^{l_{i-1}+l_i+1}}\right].\end{aligned} \tag{79}$$

This represents an alternative form of $\Psi_{\underline{\gamma}}(\jmath\underline{\omega})$ as a series of products of univariate gamma cfs, which does not require evaluation of the determinant in (76).

2) Case When $\alpha_1, \ldots, \alpha_L$ Are Exchangeable: In the case when $\underline{K}_{cc}$ is a matrix of even correlations as in (30b) and the i.i.d. condition (44) holds, we can evaluate the determinant of (77) in closed form. We rewrite (77) as

$$\Psi_{\underline{\gamma}}(\jmath\underline{\omega}) = \{\det(\underline{K}_{cc})\}^{-1}\left\{\det\left(\underline{K}_{cc}^{-1} - 2\jmath\,\mathrm{diag}(\underline{\omega})\right)\right\}^{-1} \tag{80}$$

where $\underline{K}_{cc}^{-1}$ is given by (45). Now consider the $L \times L$ symmetric matrix $\underline{Z}$, whose element in the kth row and lth column is given by

$$(\underline{Z})_{kl} = \begin{cases} 1+z_k, & \text{if } k=l \\ 1, & \text{if } k\ne l \end{cases} \tag{81}$$

where $z_1, \ldots, z_L$ are complex quantities. Let the function $G_L(z_1, \ldots, z_L)$ denote the determinant of $\underline{Z}$. It can be shown by row and column operations that

$$G_L(z_1,\,\ldots,\,z_L) = \left(\prod_{k=1}^{L} z_k\right)\left(1+\sum_{l=1}^{L}\frac{1}{z_l}\right). \tag{82}$$

Denoting

$$\begin{aligned} a &= \frac{(1+[L-2]\rho)}{\sigma^2(1-\rho)(1+[L-1]\rho)}, \\ b_z &= -\frac{\rho}{\sigma^2(1-\rho)(1+[L-1]\rho)} \end{aligned} \tag{83}$$

as in (45), we obtain from (82)

$$\begin{aligned}\det&\left(\underline{K}_{cc}^{-1} - 2\jmath\,\mathrm{diag}(\underline{\omega})\right) \\ &= b_z^L\left\{ G_L(z_1,\,\ldots,\,z_L)\big|_{\substack{z_k=\frac{[a-b_z]-2\jmath\omega_k}{b_z}, \\ k=1,\,\ldots,\,L}} \right\} \\ &= b_z^L\left(\prod_{k=1}^{L}\frac{[a-b_z]-2\jmath\omega_k}{b_z}\right) \\ &\quad\times\left(1+\sum_{l=1}^{L}\frac{b_z}{[a-b_z]-2\jmath\omega_l}\right).\end{aligned} \tag{84}$$

By putting $\omega_1 = \cdots = \omega_L = 0$ in (84), we also get

$$\begin{aligned}\{\det(\underline{K}_{cc})\}^{-1} &= \det\left(\underline{K}_{cc}^{-1}\right) \\ &= (a-b_z)^{L-1}(a+[L-1]b_z).\end{aligned} \tag{85}$$

A combination of (80), (84), and (85) yields

$$\Psi_{\underline{\gamma}}(\jmath\underline{\omega}) = \frac{\left(1+L\frac{b_z}{[a-b_z]}\right)}{\prod_{k=1}^{L}\left(1-\frac{2\jmath\omega_k}{[a-b_z]}\right)} \times\left(1+\frac{b_z}{[a-b_z]}\sum_{l=1}^{L}\frac{1}{1-\frac{2\jmath\omega_l}{[a-b_z]}}\right)^{-1}. \tag{86}$$

Applying the formal power series expansion

$$(1+x)^{-1} = \sum_{n=0}^{\infty}(-x)^n \tag{87}$$

in (86), we get

$$\Psi_{\underline{\gamma}}(\jmath\underline{\omega}) = \frac{\left(1+L\frac{b_z}{[a-b_z]}\right)}{\prod_{k=1}^{L}\left(1-\frac{2\jmath\omega_k}{[a-b_z]}\right)} \times\left\{\sum_{n=0}^{\infty}\left(-\frac{b_z}{[a-b_z]}\right)^n\left(\sum_{l=1}^{L}\frac{1}{1-\frac{2\jmath\omega_l}{[a-b_z]}}\right)^n\right\}. \tag{88}$$

Using the multinomial expansion formula

$$\left(\sum_{l=1}^{L}x_l\right)^n = \sum_{\substack{(l_1,\,\ldots,\,l_L) \\ 0\le l_1,\,\ldots,\,l_L\le n \\ l_1+l_2+\cdots+l_L=n}}\binom{n}{l_1,\,\ldots,\,l_L}\prod_{k=1}^{L}x_k^{l_k} \tag{89a}$$

where

$$\binom{n}{l_1,\,\ldots,\,l_L} = \frac{n!}{l_1!\cdots l_L!} \tag{89b}$$

we obtain from (88) and (83) the expression

$$\begin{aligned}\Psi_{\underline{\gamma}}(\jmath\underline{\omega}) = &\left(\frac{1-\rho}{1+[L-1]\rho}\right)\sum_{n=0}^{\infty}\left(\frac{\rho}{1+[L-1]\rho}\right)^n \\ &\times\sum_{\substack{(l_1,\,\ldots,\,l_L) \\ 0\le l_1,\,\ldots,\,l_L\le n \\ l_1+l_2+\cdots+l_L=n}}\binom{n}{l_1,\,\ldots,\,l_L} \\ &\times\prod_{i=1}^{L}\frac{1}{(1-2\jmath\omega_i\sigma^2(1-\rho))^{l_i+1}}.\end{aligned} \tag{90}$$

Applying the result (78) in (90), we obtain the pdf expression

$$f_{\underline{\gamma}}(\underline{v}) = \left(\frac{1-\rho}{1+[L-1]\rho}\right) \sum_{n=0}^{\infty} \left(\frac{\rho}{1+[L-1]\rho}\right)^{n} \times \sum_{\substack{(l_1,\ldots,l_L)\\ 0\le l_1,\ldots,l_L\le n\\ l_1+l_2+\cdots+l_L=n}} \binom{n}{l_1,\ldots,l_L} \times \prod_{i=1}^{L} g_{gam}\left(v_i;\, l_i+1,\, 2\sigma^2(1-\rho)\right) \tag{91}$$

where $g_{gam}(\cdot;\,\cdot,\,\cdot)$ is given by (62). Like (63), this expression is also a series of products of univariate gamma pdfs, and is an alternative form of (47).

Comparing (47) with (91), and substituting

$$y_i = \left(-\frac{\rho}{(1-\rho)(1+[L-1]\rho)\sigma^2}\right)^{\frac{1}{2}} v_i^{\frac{1}{2}}, \qquad i=1,\ldots,L \tag{92}$$

we obtain the *general relation*

$$\frac{1}{(2\pi)^L}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} \exp\left(-\frac{1}{2}\sum_{\substack{i,j=1\\ i\ne j}}^{L} y_i y_j e^{\jmath(\phi_i-\phi_j)}\right) d\phi_1\cdots d\phi_L = \exp\left(\frac{1}{2}\sum_{i=1}^{L} y_i^2\right)\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n \times \sum_{\substack{(l_1,\ldots,l_L)\\ 0\le l_1,\ldots,l_L\le n\\ l_1+l_2+\cdots+l_L=n}} \binom{n}{l_1,\ldots,l_L} \prod_{i=1}^{L}\frac{y_i^{2l_i}}{l_i!} \tag{93}$$

which is valid when $y_1,\ldots,y_L$ satisfies one of the following four conditions:

1) $y_1,\ldots,y_L$ are real and $y_1,\ldots,y_L \ge 0$,
2) $y_1,\ldots,y_L$ are real and $y_1,\ldots,y_L \le 0$,
3) $y_1,\ldots,y_L$ are imaginary and $y_1/\jmath,\ldots,y_L/\jmath \ge 0$,
4) $y_1,\ldots,y_L$ are imaginary and $y_1/\jmath,\ldots,y_L/\jmath \le 0$. (94)

3) Another Special Case: Now consider the pdf of $\underline{\gamma}$ given by (24) under the circularity condition (17). We can rewrite (24) as

$$f_{\underline{\gamma}}(\underline{v}) = \frac{\det(\underline{A}+\jmath\underline{B})}{2^L}\exp\left(-\frac{1}{2}\sum_{i=1}^{L} A_{ii}v_i\right) \times \frac{1}{(2\pi)^L}\int_{-\pi}^{\pi}\cdots\int_{-\pi}^{\pi} \exp\left(-\frac{1}{2}\sum_{\substack{i,j=1\\ i\ne j}}^{L}(A_{ij}+\jmath B_{ij})(v_iv_j)^{\frac{1}{2}}\left(e^{\jmath(\phi_i-\phi_j)}\right)\right) d\phi_1\cdots d\phi_L,\quad \underline{v}=[v_1,\ldots,v_L]^T \ge \underline{0}. \tag{95}$$

If there exist complex numbers $c_1,\ldots,c_L$ such that $A_{ij}+\jmath B_{ij}$ can be expressed as

$$A_{ij}+\jmath B_{ij} = c_i c_j^*, \qquad i\ne j,\ \ i,j=1,\ldots,L \tag{96}$$

then, by changing the variables of integration in (95) to

$$\epsilon_i = \phi_i + \arg(c_i), \qquad i=1,\ldots,L \tag{97}$$

as in (42), and using the relation (93) under condition 1) of (94), we obtain the pdf expression

$$f_{\underline{\gamma}}(\underline{v}) = \frac{\det(\underline{A}+\jmath\underline{B})}{2^L}\exp\left(-\frac{1}{2}\sum_{i=1}^{L}\left(A_{ii}-|c_i|^2\right)v_i\right) \times\sum_{n=0}^{\infty}\left(-\frac{1}{2}\right)^n \sum_{\substack{(l_1,\ldots,l_L)\\ 0\le l_1,\ldots,l_L\le n\\ l_1+l_2+\cdots+l_L=n}} \binom{n}{l_1,\ldots,l_L}\prod_{i=1}^{L}\frac{|c_i|^{2l_i}v_i^{l_i}}{l_i!},\quad \underline{v}=[v_1,\ldots,v_L]^T\ge\underline{0}. \tag{98}$$

This can be rewritten as

$$f_{\underline{\gamma}}(\underline{v}) = \det(\underline{A}+\jmath\underline{B})\sum_{n=0}^{\infty}(-1)^n \sum_{\substack{(l_1,\ldots,l_L)\\ 0\le l_1,\ldots,l_L\le n\\ l_1+l_2+\cdots+l_L=n}} \binom{n}{l_1,\ldots,l_L} \times\prod_{i=1}^{L}\frac{|c_i|^{2l_i}}{\left(A_{ii}-|c_i|^2\right)^{l_i+1}} \times g_{gam}\left(v_i;\, l_i+1,\, \frac{2}{\left(A_{ii}-|c_i|^2\right)}\right) \tag{99}$$

where $g_{gam}(\cdot;\,\cdot,\,\cdot)$ is given by (62). Like (63) and (91), this expression too is a series of products of univariate gamma pdfs, and is *an alternative form of (24) when condition (96) is satisfied, that is, the off-diagonal entries of the inverse of the complex covariance matrix have a product form*. We can now apply the result (78) on (99) so as to obtain

$$\Psi_{\underline{\gamma}}(\jmath\underline{\omega}) = \det(\underline{A}+\jmath\underline{B})\sum_{n=0}^{\infty}(-1)^n \sum_{\substack{(l_1,\ldots,l_L)\\ 0\le l_1,\ldots,l_L\le n\\ l_1+l_2+\cdots+l_L=n}} \binom{n}{l_1,\ldots,l_L} \times\prod_{i=1}^{L}\frac{|c_i|^{2l_i}\left(A_{ii}-|c_i|^2\right)^{-(l_i+1)}}{\left(1-\frac{2\jmath\omega_i}{(A_{ii}-|c_i|^2)}\right)^{l_i+1}}. \tag{100}$$

This is an alternative form of $\Psi_{\underline{\gamma}}(\jmath\underline{\omega})$ under condition (96) as a series of products of univariate gamma cfs, which does not require evaluation of the determinant in (76).

B. Characteristic Function (cf) of $\gamma_1+\cdots+\gamma_L$

The cf of $\gamma_{\rm tot}=\gamma_1+\cdots+\gamma_L$, which is often used in the performance analysis of digital modulation over Rayleigh-fading channels with predetection maximal-ratio combining (MRC) or postdetection equal-gain combining (EGC) can be obtained from $\Psi_{\underline{\gamma}}(\jmath\omega)$. In the general case, we get from (75)

$$\Psi_{\gamma_{\rm tot}}(\jmath\omega) = \left\{\det\left(\underline{I}_L-2\jmath\omega\underline{K}_{cc}\right)\right\}^{-\frac{1}{2}} \times\left\{\det\left(\left[\underline{I}_L-2\jmath\omega\underline{K}_{ss}\right]+4\omega^2\underline{K}_{cs}^T \times\left[\underline{I}_L-2\jmath\omega\underline{K}_{cc}\right]^{-1}\underline{K}_{cs}\right)\right\}^{-\frac{1}{2}}$$

$$= \{\det(\underline{I}_L - 2\jmath\omega\underline{K}_{ss})\}^{-\frac{1}{2}} \times \left\{\det\left([\underline{I}_L - 2\jmath\omega\underline{K}_{cc}] + 4\omega^2\underline{K}_{cs} \times [\underline{I}_L - 2\jmath\omega\underline{K}_{ss}]^{-1}\underline{K}_{cs}^T\right)\right\}^{-\frac{1}{2}}. \quad (101)$$

Under the circularity condition (17), we obtain from (76)

$$\begin{aligned}\Psi_{\gamma_{\rm tot}}(\jmath\omega) &= \{\det(\underline{I}_L - 2\jmath\omega\underline{K}_{cc})\}^{-1} \times \left\{\det\left(\underline{I}_L - 4\omega^2\left\{\underline{K}_{cs}[\underline{I}_L - 2\jmath\omega\underline{K}_{cc}]^{-1}\right\}^2\right)\right\}^{-\frac{1}{2}} \\ &= \{\det(\underline{I}_L - 2\jmath\omega[\underline{K}_{cc} + \jmath\underline{K}_{cs}]) \times \det(\underline{I}_L - 2\jmath\omega[\underline{K}_{cc} - \jmath\underline{K}_{cs}])\}^{-\frac{1}{2}}. \quad (102)\end{aligned}$$

Since $\underline{K}_{cc} + \jmath\underline{K}_{cs}$ is a Hermitian matrix under the circularity condition, both this matrix and its complex conjugate have the same characteristic polynomial. As a result, we can rewrite (102) as

$$\begin{aligned}\Psi_{\gamma_{\rm tot}}(\jmath\omega) &= \{\det(\underline{I}_L - 2\jmath\omega[\underline{K}_{cc} + \jmath\underline{K}_{cs}])\}^{-1} \\ &= \{\det(\underline{I}_L - 2\jmath\omega[\underline{K}_{cc} - \jmath\underline{K}_{cs}])\}^{-1}. \quad (103)\end{aligned}$$

Under the i.i.d. condition (26a), we get

$$\Psi_{\gamma_{\rm tot}}(\jmath\omega) = \{\det(\underline{I}_L - 2\jmath\omega\underline{K}_{cc})\}^{-1}. \quad (104)$$

When the circularity condition (17) holds, the random variable $\gamma_{\rm tot}$ can be written as a quadratic form of zero-mean complex circular Gaussian random variables $X_{c_i} + \jmath X_{s_i}$, $i = 1, \ldots, L$, as

$$\gamma_{\rm tot} = (\underline{X}_c + \jmath\underline{X}_s)^H(\underline{X}_c + \jmath\underline{X}_s)$$

where $(\cdot)^H$ denotes the Hermitian (conjugate transpose) operator, with covariance matrix

$$\boldsymbol{E}\left[(\underline{X}_c + \jmath\underline{X}_s)(\underline{X}_c + \jmath\underline{X}_s)^H\right] = 2(\underline{K}_{cc} - \jmath\underline{K}_{cs}).$$

Using the result of [42] for the cf of Hermitian quadratic forms in complex circular Gaussian variables, we find that (103) is obtained by putting in [42, eq. (4a)] the means of the variables as zero and the matrix of the quadratic form as an identity matrix. The cf of the combined received power of a space diversity system, given by [43, eq. (14a)], also has the same form as (103). The cf (103) is used in [44] to analyze the performance of binary phase-shift keying (BPSK) with MRC in correlated Rayleigh fading. The cf of a Hermitian quadratic form of a zero-mean complex circular Gaussian random vector is also used in [45] for the performance analysis of transmit diversity schemes in correlated Rayleigh fading.

The cf of the sum of correlated gamma variates given in [46] has the same form as the cf (104) under the i.i.d. condition (26a) when the gamma parameter is unity.

The cf of a symmetric quadratic form in real Gaussian random variables with nonzero means is presented in [28, Ch. 3, pp. 38–41]. When the means of the variables are zero and the matrix of the quadratic form is an identity matrix, the case is similar to the cf of $\underline{X}_c^T\underline{X}_c$, which is given by

$$\Psi_{\underline{X}_c^T\underline{X}_c}(\jmath\omega) = \{\det(\underline{I}_L - 2\jmath\omega\underline{K}_{cc})\}^{-\frac{1}{2}}. \quad (105)$$

It can easily be seen that (105) can be obtained by taking the square root of the right-hand side of (104).

In the case of exchangeable variates with $\underline{K}_{cc}$ given by (30b) and the i.i.d. condition (44) (which results from (26a)), we get

$$\Psi_{\gamma_{\rm tot}}(\jmath\omega) = \frac{1}{(-2\jmath\rho\sigma^2\omega)^L} \times \frac{1}{\left\{G_L(z_1, \ldots, z_L)\big|_{\substack{z_k = \frac{(1-\rho)}{\rho} - \frac{1}{2\jmath\rho\sigma^2\omega},\\ k=1,\ldots,L}}\right\}} \quad (106)$$

where $G_L(\cdot)$ is given by (82). Simplification of (106) yields

$$\Psi_{\gamma_{\rm tot}}(\jmath\omega) = \frac{1}{(1 - 2\jmath(1-\rho)\sigma^2\omega)^{L-1}} \times \frac{1}{(1 - 2\jmath(1 + [L-1]\rho)\sigma^2\omega)}. \quad (107)$$

On the other hand, for the circular exponentially correlated case with $\underline{K}_{cc} + \jmath\underline{K}_{cs}$ given by (48), we obtain, from (142) of the Appendix

$$\begin{aligned}\Psi_{\gamma_{\rm tot}}(\jmath\omega) &= (-2\jmath\sigma^2\omega)^{-L}\left|\underline{M}^{-1}(\rho_z) - \frac{1}{2\jmath\sigma^2\omega}\underline{I}_L\right|^{-1} \\ &= \frac{(-1)^L}{(1 - |\rho_z|^2)^{L-1}H_{L,\rho_z}(2\jmath\sigma^2\omega)} \quad (108)\end{aligned}$$

where $H_{L,\rho_z}(\cdot)$ is given by (134) or (137) in the Appendix. An alternative expression for (108) in terms of the eigenvalues of $\underline{K}_{cc} + \jmath\underline{K}_{cs}$ is given by [43, eqs. (44), (48a), (48b)].

IV. Bivariate Case

Now consider the bivariate case. When $L = 2$, we denote, without loss of generality

$$\underline{K}_{cc} = \begin{bmatrix} \sigma_1^2 & \rho_{cc}\sigma_1\sigma_2 \\ \rho_{cc}\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$$
$$\underline{K}_{ss} = \begin{bmatrix} \sigma_1^2 & \rho_{ss}\sigma_1\sigma_2 \\ \rho_{ss}\sigma_1\sigma_2 & \sigma_2^2 \end{bmatrix}$$
$$\underline{K}_{cs} = \begin{bmatrix} 0 & \rho_{cs}\sigma_1\sigma_2 \\ \rho_{sc}\sigma_1\sigma_2 & 0 \end{bmatrix} \quad (109)$$

where $\sigma_1, \sigma_2 > 0$, and $\rho_{cc}, \rho_{ss}, \rho_{cs}, \rho_{sc}$, the correlation coefficients of (X_{c1}, X_{c2}), (X_{s1}, X_{s2}), (X_{c_1}, X_{s2}), (X_{s_1}, X_{c_2}), respectively, are in $(-1, 1)$. Note that X_{c_i} and X_{s_i} are i.i.d. Gaussian with mean zero and variance σ_i^2 for $i = 1, 2$.

From (7), we get

$$\underline{A} = \frac{1}{\Delta}\begin{bmatrix} \frac{1}{\sigma_1^2}(1 - \rho_{ss}^2 - \rho_{sc}^2) & -\frac{1}{\sigma_1\sigma_2}(\rho_{cc} - \rho_{ss}[\rho_{cc}\rho_{ss} - \rho_{cs}\rho_{sc}]) \\ -\frac{1}{\sigma_1\sigma_2}(\rho_{cc} - \rho_{ss}[\rho_{cc}\rho_{ss} - \rho_{cs}\rho_{sc}]) & \frac{1}{\sigma_2^2}(1 - \rho_{ss}^2 - \rho_{cs}^2) \end{bmatrix} \quad (110a)$$

$$\underline{D} = \frac{1}{\Delta}\begin{bmatrix} \frac{1}{\sigma_1^2}\left(1-\rho_{cc}^2-\rho_{cs}^2\right) & -\frac{1}{\sigma_1\sigma_2}\left(\rho_{ss}-\rho_{cc}[\rho_{cc}\rho_{ss}-\rho_{cs}\rho_{sc}]\right) \\ -\frac{1}{\sigma_1\sigma_2}\left(\rho_{ss}-\rho_{cc}[\rho_{cc}\rho_{ss}-\rho_{cs}\rho_{sc}]\right) & \frac{1}{\sigma_2^2}\left(1-\rho_{cc}^2-\rho_{sc}^2\right) \end{bmatrix} \tag{110b}$$

$$\underline{B} = \frac{1}{\Delta}\begin{bmatrix} \frac{1}{\sigma_1^2}\left(\rho_{ss}\rho_{cs}+\rho_{cc}\rho_{sc}\right) & -\frac{1}{\sigma_1\sigma_2}\left(\rho_{sc}+\rho_{cs}[\rho_{cc}\rho_{ss}-\rho_{cs}\rho_{sc}]\right) \\ -\frac{1}{\sigma_1\sigma_2}\left(\rho_{cs}+\rho_{sc}[\rho_{cc}\rho_{ss}-\rho_{cs}\rho_{sc}]\right) & \frac{1}{\sigma_2^2}\left(\rho_{cc}\rho_{cs}+\rho_{ss}\rho_{sc}\right) \end{bmatrix} \tag{110c}$$

where

$$\begin{aligned}\Delta &= 1-\left[\rho_{cc}^2+\rho_{ss}^2+\rho_{cs}^2+\rho_{sc}^2\right]+\left[\rho_{cc}\rho_{ss}-\rho_{cs}\rho_{sc}\right]^2\\ &= \frac{\det(\underline{K})}{\sigma_1^4\sigma_2^4}.\end{aligned} \tag{110d}$$

From (12) and (110), the *general integral form of the bivariate Rayleigh pdf* can be expressed as

$$\begin{aligned}f_{\alpha_1,\alpha_2}(u_1, u_2) &= \frac{u_1u_2}{4\pi^2\sigma_1^2\sigma_2^2\Delta^{\frac{1}{2}}}\\ &\quad\times\int_{-\pi}^{\pi}\int_{-\pi}^{\pi}\exp\left(-\frac{1}{2}\,g(u_1, u_2, \phi_1, \phi_2)\right)d\phi_1 d\phi_2,\\ &\quad u_1, u_2 \geq 0\end{aligned} \tag{111a}$$

where

$$\begin{aligned}g(u_1, u_2, \phi_1, \phi_2) &= \frac{u_1^2}{\sigma_1^2\Delta}\left(1-\left[\rho_{ss}^2+\rho_{sc}^2\right]\cos^2\phi_1-\left[\rho_{cc}^2+\rho_{cs}^2\right]\sin^2\phi_1\right.\\ &\qquad\left.+\,2\left[\rho_{ss}\rho_{cs}+\rho_{cc}\rho_{sc}\right]\cos\phi_1\sin\phi_1\right)\\ &\quad+\frac{u_2^2}{\sigma_2^2\Delta}\left(1-\left[\rho_{ss}^2+\rho_{cs}^2\right]\cos^2\phi_2-\left[\rho_{cc}^2+\rho_{sc}^2\right]\sin^2\phi_2\right.\\ &\qquad\left.+\,2\left[\rho_{cc}\rho_{cs}+\rho_{ss}\rho_{sc}\right]\cos\phi_2\sin\phi_2\right)\\ &\quad-\frac{2u_1u_2}{\sigma_1\sigma_2\Delta}\left(\left[\rho_{cc}-\rho_{ss}\{\rho_{cc}\rho_{ss}-\rho_{cs}\rho_{sc}\}\right]\cos\phi_1\cos\phi_2\right.\\ &\quad+\left[\rho_{ss}-\rho_{cc}\{\rho_{cc}\rho_{ss}-\rho_{cs}\rho_{sc}\}\right]\sin\phi_1\sin\phi_2\\ &\quad+\left[\rho_{cs}+\rho_{sc}\{\rho_{cc}\rho_{ss}-\rho_{cs}\rho_{sc}\}\right]\cos\phi_1\sin\phi_2\\ &\quad\left.+\left[\rho_{sc}+\rho_{cs}\{\rho_{cc}\rho_{ss}-\rho_{cs}\rho_{sc}\}\right]\cos\phi_2\sin\phi_1\right)\end{aligned} \tag{111b}$$

and Δ is given by (110d). From (16), the bivariate exponential pdf of $(\gamma_1, \gamma_2) = (\alpha_1^2, \alpha_2^2)$ in its general integral form is

$$f_{\gamma_1,\gamma_2}(v_1, v_2) = \frac{1}{4(v_1v_2)^{\frac{1}{2}}}\,f_{\alpha_1,\alpha_2}\left(v_1^{\frac{1}{2}}, v_2^{\frac{1}{2}}\right) \tag{112}$$

where $f_{\alpha_1,\alpha_2}(.,.)$ is given by (111). It is clear from (66) and (67) that

$$\boldsymbol{E}\left[\alpha_i^2\right] = 2\sigma_i^2, \qquad \operatorname{var}\left[\alpha_i^2\right] = 4\sigma_i^4, \qquad i = 1, 2$$

$$\operatorname{cov}\left[\alpha_1^2, \alpha_2^2\right] = 2\sigma_1^2\sigma_2^2\left(\rho_{cc}^2+\rho_{ss}^2+\rho_{cs}^2+\rho_{sc}^2\right) \tag{113}$$

and, therefore,

$$\frac{\operatorname{cov}\left[\alpha_1^2, \alpha_2^2\right]}{\left(\operatorname{var}\left[\alpha_1^2\right]\operatorname{var}\left[\alpha_2^2\right]\right)^{\frac{1}{2}}} = \frac{\rho_{cc}^2+\rho_{ss}^2+\rho_{cs}^2+\rho_{sc}^2}{2},\qquad 0 \leq \frac{\rho_{cc}^2+\rho_{ss}^2+\rho_{cs}^2+\rho_{sc}^2}{2} < 1. \tag{114}$$

A. Circular Case

In the special case when the circularity condition (17) holds, we have

$$\rho_{cc} = \rho_{ss}, \qquad \rho_{cs} = -\rho_{sc} \tag{115}$$

which implies from (111b) that

$$\begin{aligned}g(u_1, u_2, \phi_1, \phi_2) &= \frac{2}{\left(1-\left[\rho_{cc}^2+\rho_{cs}^2\right]\right)}\left(\frac{u_1^2}{\sigma_1^2}+\frac{u_2^2}{\sigma_2^2}-\frac{u_1u_2}{\sigma_1\sigma_2}\left[\rho_{cc}^2+\rho_{cs}^2\right]^{\frac{1}{2}}\right.\\ &\qquad\left.\times\cos(\phi_1-\phi_2+\theta_{12})\right)\end{aligned} \tag{116a}$$

where

$$\theta_{12} = \tan^{-1}\left(\frac{\rho_{cs}}{\rho_{cc}}\right). \tag{116b}$$

Using the same approach as in (14) and (15) for the exponentially correlated case, we can simplify (111a) combined with (116) to yield the bivariate Rayleigh pdf expression

$$\begin{aligned}f_{\alpha_1,\alpha_2}(u_1, u_2) &= \frac{u_1u_2}{\sigma_1^2\sigma_2^2\left(1-\left[\rho_{cc}^2+\rho_{cs}^2\right]\right)}\\ &\quad\times\exp\left(-\frac{1}{2\left(1-\left[\rho_{cc}^2+\rho_{cs}^2\right]\right)}\left[\frac{u_1^2}{\sigma_1^2}+\frac{u_2^2}{\sigma_2^2}\right]\right)\\ &\quad\times I_0\left(\frac{\left[\rho_{cc}^2+\rho_{cs}^2\right]^{\frac{1}{2}}}{\left(1-\left[\rho_{cc}^2+\rho_{cs}^2\right]\right)}\frac{u_1u_2}{\sigma_1\sigma_2}\right), \quad u_1, u_2 \geq 0.\end{aligned} \tag{117}$$

From (113)–(115), we have

$$\operatorname{cov}\left[\alpha_1^2, \alpha_2^2\right] = 4\sigma_1^2\sigma_2^2\left(\rho_{cc}^2+\rho_{cs}^2\right), \qquad 0 \leq \rho_{cc}^2+\rho_{cs}^2 < 1. \tag{118}$$

The circular case has been analyzed in [12]–[14], [47], and [48]. For example, the bivariate Rayleigh pdf given by [47, eq. (8-103), p. 163] is the same as (117). A bivariate Rayleigh pdf for the equal-variance (when $\sigma_1^2 = \sigma_2^2$) circular case, which arises in the envelope distribution of quasi-harmonic oscillations, has been presented in [49, eq. (46)].

The bivariate exponential pdf is obtained by applying (112) on (117) resulting in

$$\begin{aligned} f_{\gamma_1,\gamma_2}(v_1, v_2) &= \frac{1}{4\sigma_1^2\sigma_2^2\left(1-[\rho_{cc}^2+\rho_{cs}^2]\right)} \\ &\quad\times \exp\left(-\frac{1}{2\left(1-[\rho_{cc}^2+\rho_{cs}^2]\right)}\left[\frac{v_1}{\sigma_1^2}+\frac{v_2}{\sigma_2^2}\right]\right) \\ &\quad\times I_0\left(\frac{[\rho_{cc}^2+\rho_{cs}^2]^{\frac{1}{2}}}{\left(1-[\rho_{cc}^2+\rho_{cs}^2]\right)}\frac{(v_1v_2)^{\frac{1}{2}}}{\sigma_1\sigma_2}\right), \quad v_1, v_2 \geq 0. \end{aligned} \tag{119}$$

Note that ρ_{cc} and ρ_{cs} occur as $\left(\rho_{cc}^2+\rho_{cs}^2\right)$ in the pdf (119); hence, it is possible to specify the bivariate exponential pdf in the circular case (115) in terms of

$$\underline{K}_{\underline{\gamma}} = 4\begin{bmatrix} \sigma_1^4 & \sigma_1^2\sigma_2^2\left(\rho_{cc}^2+\rho_{cs}^2\right) \\ \sigma_1^2\sigma_2^2\left(\rho_{cc}^2+\rho_{cs}^2\right) & \sigma_2^4 \end{bmatrix}.$$

This will not hold when the number of variates is greater than two, as explained in Section II-B.

Under the stronger i.i.d. condition (26a), that is,

$$\rho_{cc} = \rho_{ss}, \qquad \rho_{cs} = 0 \tag{120}$$

we obtain from (119) the bivariate exponential pdf

$$\begin{aligned} f_{\gamma_1,\gamma_2}(v_1, v_2) &= \frac{1}{4\sigma_1^2\sigma_2^2\left(1-\rho_{cc}^2\right)}\exp\left(-\frac{1}{2\left(1-\rho_{cc}^2\right)}\left[\frac{v_1}{\sigma_1^2}+\frac{v_2}{\sigma_2^2}\right]\right) \\ &\quad\times I_0\left(\frac{|\rho_{cc}|}{\left(1-\rho_{cc}^2\right)}\frac{(v_1v_2)^{\frac{1}{2}}}{\sigma_1\sigma_2}\right), \qquad v_1, v_2 \geq 0 \end{aligned} \tag{121}$$

which has the same form as (119) with $\rho_{cc}^2+\rho_{cs}^2$ replaced by ρ_{cc}^2.

From (75b) and (109), the bivariate exponential cf can be expressed as a fourth-order determinant. Under the i.i.d. condition (120), this determinant simplifies to

$$\psi_{\gamma_1,\gamma_2}(\jmath\omega_1, \jmath\omega_2) = \frac{1}{1-2\jmath\left(\sigma_1^2\omega_1+\sigma_2^2\omega_2\right)-4\sigma_1^2\sigma_2^2\left(1-\rho_{cc}^2\right)\omega_1\omega_2} \tag{122}$$

which corresponds to the pdf (121). Since the pdfs (119) and (121) have the same form, we obtain, for the circularity condition (115), the cf

$$\psi_{\gamma_1,\gamma_2}(\jmath\omega_1, \jmath\omega_2) = \frac{1}{1-2\jmath\left(\sigma_1^2\omega_1+\sigma_2^2\omega_2\right)-4\sigma_1^2\sigma_2^2\left(1-[\rho_{cc}^2+\rho_{cs}^2]\right)\omega_1\omega_2} \tag{123}$$

corresponding to the pdf (119).

V. Application Examples

We will give two examples in which the results obtained in Sections II–IV can be applied.

A. Bit-Error Probability of Coherent Binary Signaling With Dual-Diversity MRC in General Correlated Rayleigh Fading

Consider a coherent dual-diversity reception system over a general correlated flat Rayleigh-fading channel in which the receiver employs matched-filter detection and MRC is used. Signaling is binary with equienergy complex baseband signals $s_1(t)$ and $s_0(t)$ having support $[0, T_b)$, where T_b is the bit duration. Let the real part of the correlation coefficient between the signals be ϵ, where $-1 < \epsilon < 1$, and let $\nu = \frac{1-\epsilon}{2}$. Note that ν is 1 for antipodal signaling and $\frac{1}{2}$ for orthogonal signaling. Here the number of diversity branches is two, implying $L = 2$, and the instantaneous signal-to-noise ratio (SNR) of the combiner output with MRC is $\gamma_{\text{tot}} = \gamma_1 + \gamma_2$, where γ_i denotes the instantaneous SNR of the ith diversity branch. The bit-error probability P_e of such a scheme is given by [6]

$$P_e = Q\left(\sqrt{2\nu\gamma_{\text{tot}}}\right) = \frac{1}{\pi}\int_0^{\frac{\pi}{2}}\Psi_{\gamma_{\text{tot}}}\left(-\frac{\nu}{\sin^2\theta}\right)d\theta \tag{124}$$

where $Q(\cdot)$ denotes the Gaussian-Q function. Since we are considering general correlated Rayleigh fading, the in-phase component of the received signal at each diversity branch is correlated with the quadrature components of the received signals at other branches, and the matrices $\underline{K}_{cc}$, $\underline{K}_{ss}$, and $\underline{K}_{cs}$ are given by (109). Substituting (109) in (101) for the cf of γ_{tot}, we get

$$\Psi_{\gamma_{\text{tot}}}(\jmath\omega) = \left\{1+\jmath\omega C_1-\omega^2C_2-\jmath\omega^3C_3+\omega^4C_4\right\}^{-\frac{1}{2}} \tag{125a}$$

where

$$\begin{aligned} C_1 &= -4\left(\sigma_1^2+\sigma_2^2\right) \\ C_2 &= 4\left[\left(\sigma_1^2+\sigma_2^2\right)^2+\sigma_1^2\sigma_2^2\left(2-\left[\rho_{cc}^2+\rho_{ss}^2+\rho_{cs}^2+\rho_{sc}^2\right]\right)\right] \\ C_3 &= -8\sigma_1^2\sigma_2^2\left(\sigma_1^2+\sigma_2^2\right)\left(2-\left[\rho_{cc}^2+\rho_{ss}^2+\rho_{cs}^2+\rho_{sc}^2\right]\right) \\ C_4 &= 16\sigma_1^4\sigma_2^4\left(1-\left[\rho_{cc}^2+\rho_{ss}^2+\rho_{cs}^2+\rho_{sc}^2\right]+\left[\rho_{cc}\rho_{ss}-\rho_{cs}\rho_{sc}\right]^2\right). \end{aligned} \tag{125b}$$

Substituting (125a) in (124), we get

$$\begin{aligned} P_e &= \frac{1}{\pi}\int_0^{\frac{\pi}{2}}\left\{1-\frac{\nu C_1}{\sin^2\theta}+\frac{\nu^2C_2}{\sin^4\theta}-\frac{\nu^3C_3}{\sin^6\theta}+\frac{\nu^4C_4}{\sin^8\theta}\right\}^{-\frac{1}{2}}d\theta \\ &= \frac{1}{\pi}\int_0^{\frac{\pi}{2}}\frac{\sin^4\theta}{\sqrt{\left(\sin^8\theta-\nu C_1\sin^6\theta+\nu^2C_2\sin^4\theta-\nu^3C_3\sin^2\theta+\nu^4C_4\right)}}d\theta \end{aligned} \tag{126}$$

which is a finite-range single-variable integral, and can be conveniently evaluated numerically.

B. Outage Probability With Three-Branch Selection Combining in Exponentially Correlated Rayleigh Fading

Consider a diversity reception system with three branches over an exponentially correlated flat Rayleigh-fading channel in which selection combining (SC) is used. The instantaneous SNR of the ith diversity branch is denoted as γ_i. The instantaneous SNR of the combiner output with SC is given by $\gamma_{SC} =$

$\max\{\gamma_1, \gamma_2, \gamma_3\}$. Since we are considering exponentially correlated Rayleigh fading with three-branch diversity, the matrices $\underline{K}_{cc}$, $\underline{K}_{ss}$, and $\underline{K}_{cs}$ are given by (48), and the joint pdf of $\gamma_1, \gamma_2, \gamma_3$ is given by (63) with $L = 3$.

Putting $L = 3$ in (63), we get

$$\begin{aligned} &f_{\gamma_1,\gamma_2,\gamma_3}(v_1, v_2, v_3) \\ &= \left(\frac{1-\rho_r^2}{1+\rho_r^2}\right) \sum_{n=0}^{\infty} \left(\frac{\rho_r^2}{1+\rho_r^2}\right)^n \\ &\quad \times g_{gam}\left(v_2; n+1, \frac{2(1-\rho_r^2)\sigma^2}{(1+\rho_r^2)}\right) \\ &\quad \times \sum_{l=0}^{n} \binom{n}{l} g_{gam}\left(v_1; l+1, 2(1-\rho_r^2)\sigma^2\right) \\ &\quad \times g_{gam}\left(v_3; n-l+1, 2(1-\rho_r^2)\sigma^2\right). \end{aligned} \tag{127}$$

The outage probability $P_{\text{out}}(\gamma_{\text{th}})$ corresponding to a threshold SNR γ_{th} is given by

$$\begin{aligned} P_{\text{out}}(\gamma_{\text{th}}) &= \Pr[\gamma_{SC} \le \gamma_{\text{th}}] \\ &= \Pr[\max\{\gamma_1, \gamma_2, \gamma_3\} \le \gamma_{\text{th}}] \\ &= \Pr[\gamma_1 \le \gamma_{\text{th}}, \gamma_2 \le \gamma_{\text{th}}, \gamma_3 \le \gamma_{\text{th}}] \\ &= \int_0^{\gamma_{\text{th}}} \int_0^{\gamma_{\text{th}}} \int_0^{\gamma_{\text{th}}} f_{\gamma_1,\gamma_2,\gamma_3}(v_1, v_2, v_3)\, dv_1\, dv_2\, dv_3. \end{aligned} \tag{128}$$

It can easily be shown from (62) that

$$\int_0^x g_{gam}(v; k, c)\, dv = 1 - e^{-\frac{x}{c}} \sum_{q=0}^{k-1} \frac{x^q}{q!c^q}. \tag{129}$$

Substituting (127) in (128), and applying the result (129), we get

$$\begin{aligned} &P_{\text{out}}(\gamma_{\text{th}}) \\ &= \left(\frac{1-\rho_r^2}{1+\rho_r^2}\right) \sum_{n=0}^{\infty} \left(\frac{\rho_r^2}{1+\rho_r^2}\right)^n \\ &\quad \times \left[1 - e^{-\frac{(1+\rho_r^2)\gamma_{\text{th}}}{2(1-\rho_r^2)\sigma^2}} \sum_{q_2=0}^{n} \frac{\left[(1+\rho_r^2)\gamma_{\text{th}}\right]^{q_2}}{q_2!\,[2(1-\rho_r^2)\sigma^2]^{q_2}}\right] \\ &\quad \times \sum_{l=0}^{n} \binom{n}{l} \left[1 - e^{-\frac{\gamma_{\text{th}}}{2(1-\rho_r^2)\sigma^2}} \sum_{q_1=0}^{l} \frac{\gamma_{\text{th}}^{q_1}}{q_1!\,[2(1-\rho_r^2)\sigma^2]^{q_1}}\right] \\ &\quad \times \left[1 - e^{-\frac{\gamma_{\text{th}}}{2(1-\rho_r^2)\sigma^2}} \sum_{q_3=0}^{n-l} \frac{\gamma_{\text{th}}^{q_3}}{q_3!\,[2(1-\rho_r^2)\sigma^2]^{q_3}}\right]. \end{aligned} \tag{130}$$

This is a series expression for the outage probability, which can be truncated to a certain finite number of terms depending on the value of ρ_r^2, enabling easy computation. For example, if the squared correlation coefficient $\rho_r^2 \ll 1$, we can approximate $P_{\text{out}}(\gamma_{\text{th}})$ using only the $n = 0$ and $n = 1$ terms of (130) resulting in

$$\begin{aligned} &P_{\text{out}}(\gamma_{\text{th}})\big|_{\rho_r^2 \ll 1} \\ &\approx \frac{(1-\rho_r^2)}{(1+\rho_r^2)} \left[1 - e^{-\frac{(1+\rho_r^2)\gamma_{\text{th}}}{2(1-\rho_r^2)\sigma^2}}\right] \\ &\quad \times \left[1 - e^{-\frac{\gamma_{\text{th}}}{2(1-\rho_r^2)\sigma^2}}\right]^2 + \frac{2\rho_r^2(1-\rho_r^2)}{(1+\rho_r^2)^2} \\ &\quad \times \left[1 - e^{-\frac{(1+\rho_r^2)\gamma_{\text{th}}}{2(1-\rho_r^2)\sigma^2}} \left(1 + \frac{(1+\rho_r^2)\gamma_{\text{th}}}{2(1-\rho_r^2)\sigma^2}\right)\right] \\ &\quad \times \left[1 - e^{-\frac{\gamma_{\text{th}}}{2(1-\rho_r^2)\sigma^2}} \left(1 + \frac{\gamma_{\text{th}}}{2(1-\rho_r^2)\sigma^2}\right)\right] \\ &\quad \times \left[1 - e^{-\frac{\gamma_{\text{th}}}{2(1-\rho_r^2)\sigma^2}}\right]. \end{aligned} \tag{131}$$

VI. CONCLUSION

Starting with a general integral form of multivariate Rayleigh and exponential pdfs (12) and (15) generated from correlated Gaussian random variables, we have arrived at two specific pdf forms: one for a circular pair $(\underline{X}_c, \underline{X}_s)$ of generating random vectors (i.e., (23), (24)), and the other for an i.i.d pair (i.e., (27)), which is a specialization of the circular case. We obtained the pdfs for the exchangeable and exponentially correlated cases, both in integral form and as a series of products of univariate pdfs. In the exponentially correlated case, the pdf expression can be simplified in terms of a product of zeroth-order modified Bessel functions of the first kind owing to the fact that the inverse of the complex covariance matrix is tridiagonal.

The multivariate exponential cfs have been expressed in terms of determinants. In the exchangeable and exponentially correlated cases, these have been expressed as a series of products of univariate gamma cfs. We have also obtained a general relation (93) which is used to get a series expression for the multivariate exponential cf when the off-diagonal entries of the inverse of the complex covariance matrix have a product form $(A_{ij} + \jmath B_{ij} = c_i c_j^*)$. In addition, closed-form expressions for the cf of the sum of exponential variates have been obtained in the exchangeable and exponentially correlated cases.

The commonly used exponential cf (77) is the one corresponding to i.i.d. generating random vectors, and is therefore restrictive. The more general forms (76) and (75) may be appropriate in some situations. For example, in analysis of diversity combining when the in-phase component of the received signal at each diversity branch is correlated with the quadrature components of the received signals at other branches, the general model or the circular model may be appropriate. Such a situation with $L = 2$ (dual diversity) has been analyzed in [50] considering a circular model.

Analysis of the bivariate Rayleigh case reveals a general form (111), which reduces to the simpler form (117) in the circular case. The structure of the bivariate Rayleigh or exponential pdf does not simplify further in the case of i.i.d. generating vectors, as shown by a comparison of (119) and (121). We also observe that in case of the bivariate exponential generated by circular or i.i.d. random vectors, the pdf parameters can be completely specified by the covariance matrix. This does not hold when the number of variates is more than two.

While the integral forms of the multivariate pdfs provide a general analytical framework, the series expressions for the pdfs and exponential cfs can be used in the performance analysis of digital modulation over correlated Rayleigh-fading channels using various diversity combining methods. Expressions for the cf of the sum of exponential variates serve as a valuable tool to analyze performance with predetection MRC and postdetection EGC.

APPENDIX

We first evaluate the characteristic polynomial $H_{L,\,\rho_z}(\lambda)$ of the $L \times L$ Hermitian tridiagonal matrix

$$\underline{M}(\rho_z) = \frac{1}{(1-|\rho_z|^2)} \times \begin{bmatrix} 1 & -\rho_z & & & \\ -\rho_z^* & (1+|\rho_z|^2) & -\rho_z & & \mathbf{0} \\ & \ddots & \ddots & \ddots & \\ \mathbf{0} & & -\rho_z^* & (1+|\rho_z|^2) & -\rho_z \\ & & & -\rho_z^* & 1 \end{bmatrix}, \quad |\rho_z| < 1 \tag{132}$$

which has the same structure as in (51) except for a scale factor of σ^2. Let $\underline{I}_L$ denote the $L \times L$ identity matrix. Then we have

$$H_{L,\,\rho_z}(\lambda) = (-1)^L |\underline{M}(\rho_z) - \lambda \underline{I}_L|. \tag{133}$$

The matrix $\underline{M}(\rho_z) - \lambda \underline{I}_L$ is also tridiagonal. Using the result in [36] for the determinant of a tridiagonal matrix, we obtain from (133)

$$\begin{aligned} H_{L,\,\rho_z}(\lambda) = \frac{(-1)^L}{(1-|\rho_z|^2)^L} \sum_{q=0}^{\lfloor \frac{L}{2} \rfloor} (-1)^q |\rho_z|^{2q} \times \Bigg\{ & \binom{L-2-q}{q-2} \times ([1+|\rho_z|^2] - \lambda[1-|\rho_z|^2])^{L-2q} \\ & + 2\binom{L-2-q}{q-1} (1-\lambda[1-|\rho_z|^2]) \times ([1+|\rho_z|^2] - \lambda[1-|\rho_z|^2])^{L-1-2q} \\ & + \binom{L-2-q}{q} (1-\lambda[1-|\rho_z|^2])^2 \times ([1+|\rho_z|^2] - \lambda[1-|\rho_z|^2])^{L-2-2q} \Bigg\} \end{aligned} \tag{134}$$

where

$$\binom{m}{n} = \begin{cases} \dfrac{m!}{(m-n)!n!}, & \text{if } m \geq n \geq 0 \\ 0, & \text{otherwise.} \end{cases} \tag{135}$$

Note that $H_{L,\,\rho_z}(\lambda)$ depends on $|\rho_z|$ but not on the phase of ρ_z.

Let $h(x, n)$, where x is a real number and n an integer, be defined as

$$\begin{aligned} h(x, n) &\triangleq \sum_{q=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^q \binom{n-q}{q} x^{n-2q} \\ &= \begin{cases} \dfrac{\left(x + \jmath\sqrt{4-x^2}\right)^{n+1} - \left(x - \jmath\sqrt{4-x^2}\right)^{n+1}}{\jmath 2^{n+1}\sqrt{4-x^2}}, \\ \qquad \text{if } 0 \leq |x| < 2,\ n \geq 0 \\ \dfrac{\left(x + \sqrt{x^2-4}\right)^{n+1} - \left(x - \sqrt{x^2-4}\right)^{n+1}}{2^{n+1}\sqrt{x^2-4}}, \\ \qquad \text{if } |x| \geq 2,\ n \geq 0 \\ 0, \qquad \text{if } n < 0. \end{cases} \end{aligned} \tag{136}$$

Then $H_{L,\,\rho_z}(\lambda)$ can be expressed in terms of $h(.,\,.)$ as

$$\begin{aligned} H_{L,\,\rho_z}(\lambda) = \frac{(-1)^L |\rho_z|^L}{(1-|\rho_z|^2)^L} \times \Bigg\{ & h\left(\frac{([1+|\rho_z|^2] - \lambda[1-|\rho_z|^2])}{|\rho_z|},\, (L-4)\right) \\ & - \frac{2(1-\lambda[1-|\rho_z|^2])}{|\rho_z|} \times h\left(\frac{([1+|\rho_z|^2] - \lambda[1-|\rho_z|^2])}{|\rho_z|},\, (L-3)\right) \\ & + \frac{(1-\lambda[1-|\rho_z|^2])^2}{|\rho_z|^2} \times h\left(\frac{([1+|\rho_z|^2] - \lambda[1-|\rho_z|^2])}{|\rho_z|},\, (L-2)\right) \Bigg\}. \end{aligned} \tag{137}$$

By putting $\lambda = 0$ in (137), we obtain an expression for the determinant of $\underline{M}(\rho_z)$, which turns out to be

$$|\underline{M}(\rho_z)| = (-1)^L H_{L,\,\rho_z}(0) = \frac{1}{(1-|\rho_z|^2)^{L-1}}. \tag{138}$$

Now $\underline{M}^{-1}(\rho_z)$ is given by

$$\left(\underline{M}^{-1}(\rho_z)\right)_{ij} = \begin{cases} 1, & \text{if } i = j \\ \rho_z^{j-i}, & \text{if } i < j \\ (\rho_z^*)^{i-j}, & \text{if } i > j \end{cases} \qquad i, j = 1, \ldots, L \tag{139}$$

and from (138), its determinant is

$$\left|\underline{M}^{-1}(\rho_z)\right| = (1-|\rho_z|^2)^{L-1}. \tag{140}$$

Let z be a complex variable. The matrix $\underline{M}^{-1}(\rho_z) - (1-z)\underline{I}_L$ is, therefore, expressed as

$$\left(\underline{M}^{-1}(\rho_z) - (1-z)\underline{I}_L\right)_{ij} = \begin{cases} z, & \text{if } i = j \\ \rho_z^{j-i}, & \text{if } i < j \\ (\rho_z^*)^{i-j}, & \text{if } i > j \end{cases} \qquad i, j = 1, \ldots, L \tag{141}$$

and, from (133) and (139), its determinant becomes

$$\begin{aligned}&\left|\underline{M}^{-1}(\rho_z) - (1-z)\underline{I}_L\right| \\ &= (1-z)^L \left|\underline{M}^{-1}(\rho_z)\right| \left|\frac{1}{(1-z)}\,\underline{I}_L - \underline{M}(\rho_z)\right| \\ &= (1-z)^L (1-|\rho_z|^2)^{L-1} H_{L,\,\rho_z}\left(\frac{1}{1-z}\right) \qquad (142)\end{aligned}$$

where $H_{L,\,\rho_z}(\cdot)$ is given by (134) or (137).

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