Introducing Correlation amongs subcarriers

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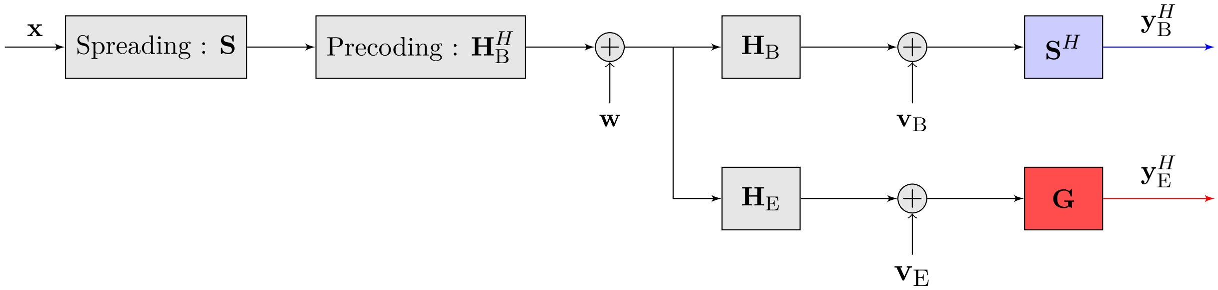
1 Short Reminder

We consider that we have Q channel subcarriers, that we spread (via $\mathbf{S}$) our symbols $\mathbf{x}$ by a factor U called back-off rate (BOR). We send $N = Q/U$ symbols per OFDM block. At the transmitter side, we precode the data by the complex conjugate version of Bob's channel frequency response ($\mathbf{H}_{\mathrm{B}}^{H}$). We add an artifical noise signal ($\mathbf{w}$) which lies in Bob null space, i.e., $\mathbf{S}^H \mathbf{H}_{\mathrm{B}} \mathbf{w} = \mathbf{0}$. At the receiver side, we despread the received sequence at Bob via $\mathbf{S}^H$, and we decode the received sequence with a particular decoding structure $\mathbf{G}$ at Eve. The communication scheme is presented below:



2 Problem Statement

We want to analyze the effect of introducing correlation among channel subcarriers, i.e., the effect of frequency correlation in our system.
In the time-domain, we modelize the channel thanks to L taps:

$$h = [h_1 ... h_L]^T \tag{1}$$

The time-domain correlation function is given by:

$$\mathbb{E}\left[hh^H\right] := C_h(\tau) = \mathrm{diag}(p_1 ... p_L) \tag{2}$$

where $(p_1 ... p_L) := P(\tau)$ is the power delay profile (PDP). It gives the power density (per delay unit) incident onto a local area as a function of the propagation delay τ. The Fourrier transform of the PDP is the correlation function in the frequency domain:

$$C_H(\Delta f) = \int_{-\infty}^{\infty} P(\tau) e^{-j2\pi \Delta f \tau} d\tau \tag{3}$$

where Δf is the frequency difference between two particular subcarriers. The PDP $P(\tau)$ is often assumed to follow an exponential model:

$$P(\tau) = P_0 \, e^{-\tau/\sigma_\tau} \quad , \; \tau \geq 0 \tag{4}$$

where σ_τ is the delay spread which characterizes the PDP duration and depends on the physical environment.
From (3) and (4), we obtain the frequency-domain correlation function as:

$$C_H(\Delta f) = \frac{P_0 \sigma_\tau}{1 + 2\pi j \sigma_\tau \Delta f} \tag{5}$$

Another parameter that characterizes the environment is the coherence bandwidth (Δf_C) which is defined as the bandwidth over which the correlation is above a certain threshold (typically 0.7):

$$\Delta f_C \approx \frac{1}{2\pi\sigma_\tau} \tag{6}$$

In the simulation, we consider a normalized frequency-domain correlation function, i.e. $P_0\sigma_\tau = 1$. If we have Q channel subcarriers, we create a correlation matrix given by:

$$\mathbf{C}_H = \begin{bmatrix} 1 & c_{1,2} & \dots & c_{1,Q} \\ c_{1,2}^* & 1 & \dots & c_{2,Q} \\ \vdots & & \ddots & \vdots \\ c_{1,Q}^* & c_{2,Q}^* & \dots & 1 \end{bmatrix} \tag{7}$$

Due to the spreading, there is a spacing of N subcarriers between two symbol components. If we denote by Δf_k the frequency spacing between k subcarriers, we have $\Delta f_k = k\Delta f_N$, where Δf_N is the subcarrier bandwidth. The subcarrier bandwidth is a fraction, depending on N, of the channel coherence bandwidth:

$$\Delta f_N = \frac{\beta}{N} \Delta f_C \quad , \; \beta \in \mathbb{R}^+ \tag{8}$$

From this, if $\beta/N << 1$, the channel is highly correlated, i.e., the frequency spacing between two subsequent symbol components is smaller than the channel coherence bandwidth. If $\beta/N >> 1$, the channel is strongly uncorrelated, i.e., the frequency spacing between two subsequent symbol components is larger than the channel coherence bandwidth.

In order to generate a correlated channel, we will perform a Choleski decomposition of the correlation matrix. It decomposes a square, symmetric and positive semi-definite matrix into an unique product of a lower triangular matrix and its hermitian, such that:

$$\mathbf{C}_H = \mathbf{T} \; \mathbf{T}^H \tag{9}$$

where

$$\mathbf{T} = \begin{bmatrix} t_{1,1} & 0 & \dots & 0 \\ t_{2,1} & t_{2,2} & \dots & 0 \\ \vdots & & \ddots & \vdots \\ t_{Q,1} & t_{Q,2} & \dots & t_{Q,Q} \end{bmatrix} \tag{10}$$

The Choleski coefficients can be computed in a recursive way thanks to the Cholski algorithm. So, for a given PDP model and delay spread, we can determine the coefficients of the Choleski matrix

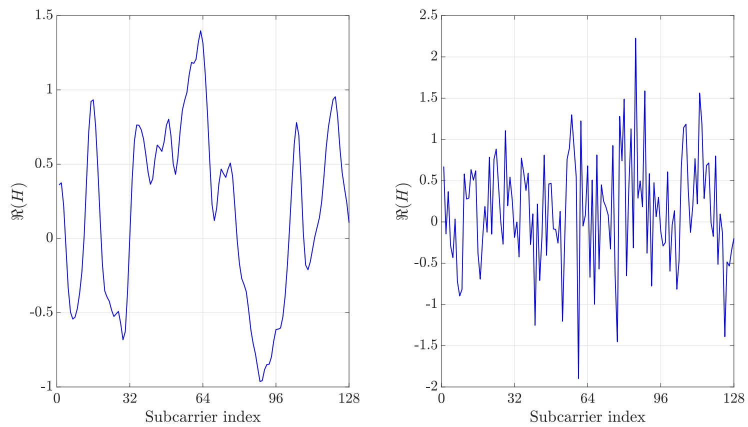


Figure 1: Correlated channel ($\beta/N = 1/6$) ; Uncorrelated channel ($\beta/N = 100$)

The diagonal elements are also positive. Bob correlated channel ($\mathbf{H}_\text{B}$) will then be the product of the Choleski matrix $\mathbf{T}$ with an uncorrelated channel ($\mathbf{H}$), i.e., $\mathbf{H}_\text{B} = \mathbf{T}\ \mathbf{H}$. A a consequence, a particular channel component will be a weighted sum depending on the previous ones:

$$h_{\text{B},j} = \sum_{k=1}^{j} t_{j,k} h_k \tag{11}$$

Note that for an uncorrelated channel, $\mathbf{T} = \mathbf{I}_Q$ is the identity matrix. In doing so, each subcarrier is independant from the others. We also have:

$$\sum_{k=1}^{j} \left|t_{j,k}\right|^2 = 1 \tag{12}$$

3 SINR Modeling

In this section, we introduce correlation among Bob's subcarriers only. We consider that Eve channel is uncorrelated. Au début, je pensais intuitivement que ne pas mettre de la corrélation chez Eve était le worst case scénario en terme de sécurité (que ça maximisait la capacité à Eve). Ca s'est avéré ne pas être vrai (cfr plus bas). Mais du coup, je n'ai fait les calculs qu'en considérant de la corrélation à Bob.. As a reminder, the ergodic SINR is modelized in order to find an approximation of the ergodic capacity using the Jensen's innequality:

$$\mathbb{E}\left[\log_2(1+X)\right] \leq \log_2(1+\mathbb{E}\left[X\right]) \tag{13}$$

where X is Bob/Eve SINR. When no correlation was introduced, the innequality bound was tight such that it was used as an approximation.
As a reminder, Bob and Eve are spatially uncorrelated.

3.1 Bob SINR

At Bob, the received signal is given by:

$$\mathbf{y}_\text{B}^H = \sqrt{\alpha}\mathbf{S}^H \left|\mathbf{H}_\text{B}\right|^2 \mathbf{S}\ \mathbf{x} + \mathbf{S}^H \mathbf{v}_\text{B} \tag{14}$$

The ergodic SINR is then:

$$\mathbb{E}\left[\gamma_B\right] = \frac{\alpha \mathbb{E}\left[\left|\mathbf{S}^H \left|\mathbf{H}_\mathrm{B}\right|^2 \mathbf{S}\ \right|^2\right]}{\mathbb{E}\left[\left|\mathbf{S}^H \mathbf{v}_\mathrm{B}\right|^2\right]} \tag{15}$$

For a particular symbol n, the expected value of the data component is:

$$\begin{aligned}\mathbb{E}\left[|\mathrm{data}|^2\right] &= \frac{\alpha}{U^2}\mathbb{E}\left[\left|\sum_{i=0}^{U-1}\left|h_{B,n+iN}\right|^2\right|^2\right] \\ &= \frac{\alpha}{U^2}\mathbb{E}\left[\sum_{i=0}^{U-1}\left|h_{B,n+iN}\right|^4 + \sum_{i=0}^{U-1}\sum_{j\neq i}^{U-1}\left|h_{B,n+iN}\right|^2\left|h_{B,n+jN}\right|^2\right]\end{aligned} \tag{16}$$

We can show that:

$$\mathbb{E}\left[\sum_{i=0}^{U-1}\left|h_{B,n+iN}\right|^4\right] = 2U \tag{17}$$

Also:

$$\begin{aligned}\mathbb{E}\left[\sum_{i=0}^{U-1}\sum_{j\neq i}^{U-1}\left|h_{B,n+iN}\right|^2\left|h_{B,n+jN}\right|^2\right] = \sum_{i=0}^{U-1}\sum_{j>i}^{U-1}\Bigg(&\sum_{k=1}^{n+iN} 2\left|t_{n+iN,k}\right|^2\left|t_{n+jN,k}\right|^2 + \\ &\sum_{k=1}^{n+iN}\sum_{k'\neq k}^{n+jN}\left|t_{n+iN,k}\right|^2\left|t_{n+jN,k'}\right|^2 + \\ &\sum_{k=1}^{n+iN}\sum_{k'>k}^{n+jN} 2R[t_{n+iN,k}t^*_{n+iN,k'}t^*_{n+jN,k}t_{n+jN,k'}]\Bigg)\end{aligned} \tag{18}$$

For the noise component:

$$\mathbb{E}\left[|\mathrm{noise}|^2\right] = \sigma_B^2 \tag{19}$$

The SINR is simply the ratio between eq.(18) and (19)

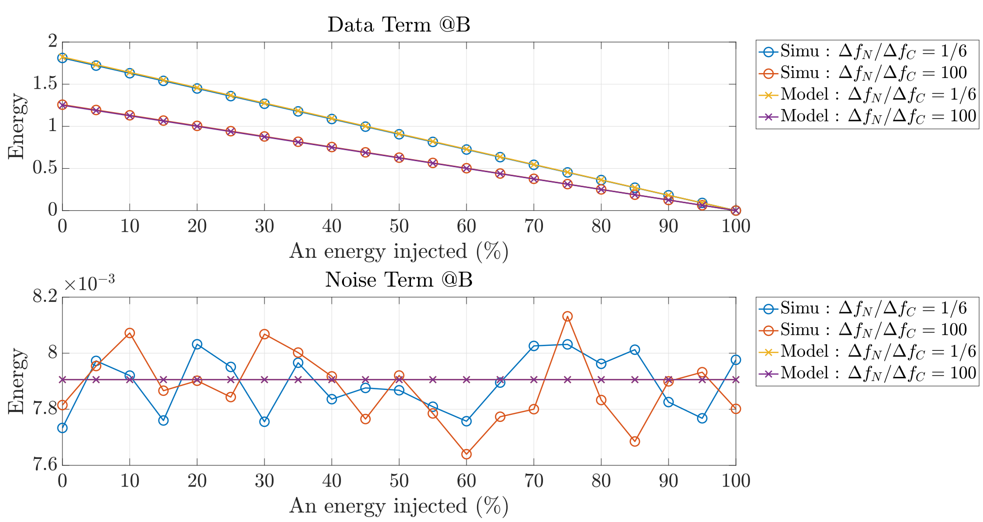


Figure 2: Bob SINR terms fiting, modelization vs simulation

From Fig.2, we observe that the analytic expressions (18) and (19) well fit the simulation results. In particular, we remark that the received data energy increases when the correlation increases.

3.2 Eve SINR

SDS Decoding Structure

When Eve simply despread the received signal, we have

$$\mathbf{y}_{\mathrm{E}}^{SDS} = \sqrt{\alpha}\mathbf{S}^H \mathbf{H}_{\mathrm{B}}^H \mathbf{H}_{\mathrm{E}} \mathbf{S}\ \mathbf{x} + \mathbf{S}^H \mathbf{v}_{\mathrm{E}} + \sqrt{1-\alpha}\mathbf{S}^H \mathbf{H}_{\mathrm{E}} \mathbf{w} \tag{20}$$

The ergodic SINR is then:

$$\mathbb{E}\left[\gamma_E^{SDS}\right] = \frac{\alpha \mathbb{E}\left[\left|\mathbf{S}^H \mathbf{H}_{\mathrm{E}} \mathbf{H}_{\mathrm{B}}^H \mathbf{S}\ \right|^2\right]}{\mathbb{E}\left[\left|\mathbf{S}^H \mathbf{v}_{\mathrm{E}}\right|^2 + (1-\alpha)\left|\mathbf{S}^H \mathbf{H}_{\mathrm{E}} \mathbf{w}\right|^2\right]} \tag{21}$$

For a particular symbol n, the ergodic data component energy is:

$$\begin{aligned}
\mathbb{E}\left[|\mathrm{data}|^2\right] &= \frac{\alpha}{U^2}\mathbb{E}\left[\left|\sum_{i=0}^{U-1} h_{B,n+iN}^* h_{E,n+iN}\right|^2\right] \\
&= \frac{\alpha}{U^2}\mathbb{E}\left[\sum_{i=0}^{U-1}\left|h_{B,n+iN}\right|^2\left|h_{E,n+iN}\right|^2 + \sum_{i=0}^{U-1}\sum_{j\neq i}^{U-1} h_{B,n+iN}^* h_{B,n+jN} h_{E,n+iN} h_{E,n+jN}^*\right] \\
&= \frac{\alpha}{U^2}\mathbb{E}\left[\sum_{i=0}^{U-1}\left|h_{B,n+iN}\right|^2 .1\right] = \frac{\alpha}{U}
\end{aligned} \tag{22}$$

Eq.(22) gives the same result as if no correlation was introduced at Bob. Obviously the AN and noise terms will not depend on Bob correlation. For the noise component:

$$\mathbb{E}\left[|\mathrm{noise}|^2\right] = \sigma_E^2 \tag{23}$$

For the AN component:

$$\mathbb{E}\left[|\mathrm{AN}|^2\right] = \frac{(1-\alpha)}{U}\sigma_{AN}^2 = (1-\alpha) \tag{24}$$

The SINR is therefore identical as in the case where Bob channel is uncorrelated.

Je montre pas les calculs pour le matched filter ni le own channel decoder parce que le problème qui suit aparaît déjà pour la capacité à Bob. Mais, les mêmes soucis apparaissent pour toutes les structures de décodage à Eve

4 Jensen's innequality issue

In the past works, the Jensen innequality was used as a good approximation when no correlation was introduced among Bob's subcarriers. When correlation is present, the approximation does not hold anymore, as seen in Fig(3). In particular, we observe that, when $\Delta f_N / \Delta f_C = 100$, i.e., uncorrelated channel, the fit between analytic and simulation curves is almost perfect. However, when $\Delta f_N / \Delta f_C = 1/6$, i.e., highly correlated channel, the ergodic capacity decreases but the approximation suggests that it should increase. In order to understand the behaviour of the analytic model of the capacity derived using the Jensen's innequality as an approximation, we will analyze

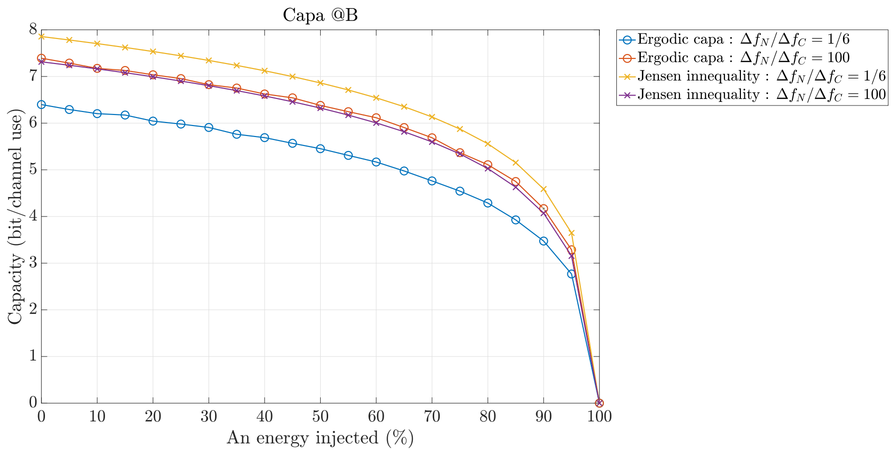


Figure 3: Bob capacity, variable correlation, ergodic vs Jensen's innequality

eq.(16). When Bob's channel is fully correlated, i.e., all subcarriers are an identical complex scalar, eq.(16) becomes:

$$\begin{aligned}\mathbb{E}\left[|\text{data}|^2\right] &= \frac{\alpha}{U^2}\mathbb{E}\left[\left|\sum_{i=0}^{U-1}\left|h_{B,n+iN}\right|^2\right|^2\right] \\ &= \frac{\alpha}{U^2}\mathbb{E}\left[\sum_{i=0}^{U-1}\left|h_{B,n+iN}\right|^4 + \sum_{i=0}^{U-1}\sum_{j\neq i}^{U-1}\left|h_{B,n+iN}\right|^2\left|h_{B,n+jN}\right|^2\right] \\ &= \frac{\alpha}{U^2}\left(2U + 2U(U-1)\right) = 2\alpha\end{aligned} \tag{25}$$

When Bob's channel is totally uncorrelated, eq.(16) becomes:

$$\begin{aligned}\mathbb{E}\left[|\text{data}|^2\right] &= \frac{\alpha}{U^2}\mathbb{E}\left[\left|\sum_{i=0}^{U-1}\left|h_{B,n+iN}\right|^2\right|^2\right] \\ &= \frac{\alpha}{U^2}\mathbb{E}\left[\sum_{i=0}^{U-1}\left|h_{B,n+iN}\right|^4 + \sum_{i=0}^{U-1}\sum_{j\neq i}^{U-1}\left|h_{B,n+iN}\right|^2\left|h_{B,n+jN}\right|^2\right] \\ &= \frac{\alpha}{U^2}\left(2U + U(U-1)\right) = \frac{\alpha(U+1)}{U}\end{aligned} \tag{26}$$

Eq.(25) and (26) suggest that Bob ergodic SINR increases when the correlation increases:

$$\left.\frac{\alpha(U+1)}{U\sigma_B^2}\right|_{\text{no correl}} \leq \mathbb{E}\left[\gamma_B\right] \leq \left.\frac{2\alpha}{\sigma_B^2}\right|_{\text{full correl}} \tag{27}$$

This behaviour can be seen in Fig.(4). This implies that the approximated capacity $\log_2\left(1+\mathbb{E}\left[\gamma_B\right]\right)$ should increase with an increase of the correlation. However, it is observed in Fig.(4) that the ergodic capacity $\mathbb{E}\left[\log_2\left(1+\gamma_B\right)\right]$ decreases with an increase of the correlation.

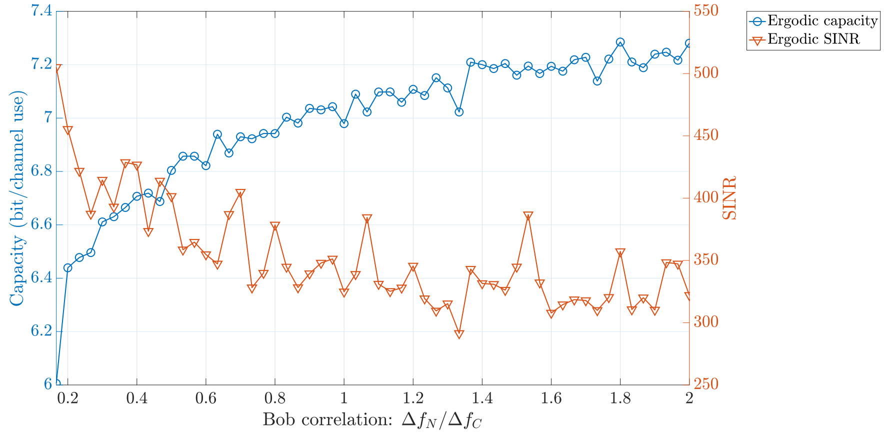


Figure 4: Bob SINR/capacity as a function of the frequency correlation

5 Second order Taylor approximation of the capacity

New approximation cfr FC17 verso

$$\mathbb{E}\left[\log_2(1+X)\right] \approx \log_2(1+\mathbb{E}\left[X\right]) - \frac{\mathrm{var}(X)}{2(1+\mathbb{E}\left[X\right])^2} \tag{28}$$

Results avec 2e ordre approx: SINR modeles 1/2 1er et 2e ordre Capa modeles 1/2 1er et 2e ordre SR pour modele 1/2 1r et 2e ordre

Montrer aussi les matrices d'erreurs

6 Simulation Results