Comments conference paper

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In my simulation, I consider that: $\mathbb{E}\left(\mathbf{w}\mathbf{w}^H\right) = \sigma_{an}^2 \mathbf{I}_Q$, where $\sigma_{an}^2 = 1/U$ is the AN autocorrelation. Could you simulate for $\mathbb{E}\left(\mathbf{w}'(\mathbf{w}')^H\right) = \mathbf{I}$ please? so that it is in accordance with my derivations

Then, I simulate the energy of the AN at Eve for a particular symbol n, i.e., one of the component of the signal $\mathbf{v}$. Each component of $\mathbf{v}$ is made from a summation of U subcarriers thanks to the despreading matrix $\mathbf{S}^H$

The AN at Eve is

$$\begin{aligned}
\mathbf{v} &= \mathbf{S}^H \mathbf{H}_B |\mathbf{H}_E|^2 \mathbf{w} \\
&= \mathbf{A} |\mathbf{H}_E|^2 \mathbf{V}_2 \mathbf{w}' \\
&= \mathbf{U} \begin{pmatrix} \mathbf{\Sigma} & \mathbf{0}_{N-Q\times N} \end{pmatrix} \begin{pmatrix} \mathbf{V}_1^H \\ \mathbf{V}_2^H \end{pmatrix} |\mathbf{H}_E|^2 \mathbf{V}_2 \mathbf{w}' \\
&= \mathbf{U}\mathbf{\Sigma}\mathbf{V}_1^H |\mathbf{H}_E|^2 \mathbf{V}_2 \mathbf{w}'
\end{aligned}$$

Therefore, since $\mathbf{w} = \mathbf{V}_2 \mathbf{w}'$, we have: $\mathbb{E}\left(\mathbf{w}\mathbf{w}^H\right) = \mathbf{V}_2 \mathbf{V}_2^H = \sigma_{an}^2 \mathbf{I}_Q$ this is wrong! $\mathbf{V}_2 \mathbf{V}_2^H \neq \sigma_{an}^2 \mathbf{I}_Q$ because $\mathbf{V}_2$ is tall

Let us now look at the covariance matrix

$$\mathbb{E}\left(\mathbf{v}\mathbf{v}^H\right) = \mathbb{E}\left(\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}_1^H|\mathbf{H}_E|^2\mathbf{V}_2\mathbf{w}'\left(\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}_1^H|\mathbf{H}_E|^2\mathbf{V}_2\mathbf{w}'\right)^H\right)$$
$$= \mathbb{E}\left(\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}_1^H|\mathbf{H}_E|^2\mathbf{V}_2\mathbf{w}'\mathbf{w}'^H\mathbf{V}_2^H|\mathbf{H}_E|^2\mathbf{V}_1\boldsymbol{\Sigma}^H\mathbf{U}^H\right)$$

Note that $\mathbf{w}'$ is independent of other random variable and has a unit covariance matrix. We can thus put the expectation inside to get

$$\mathbb{E}\left(\mathbf{v}\mathbf{v}^H\right) = \mathbb{E}\left(\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}_1^H|\mathbf{H}_E|^2\mathbf{V}_2\mathbf{V}_2^H|\mathbf{H}_E|^2\mathbf{V}_1\boldsymbol{\Sigma}^H\mathbf{U}^H\right)$$

We rewrite $|\mathbf{H}_E|^2 = \sum_{q=1}^{Q}|H_{E,q}|^2\mathbf{e}_q\mathbf{e}_q^T$ where $\mathbf{e}_q$ is an all zero vector except a 1 at row q to isolate the independent random variable H_E

$$\mathbb{E}\left(\mathbf{v}\mathbf{v}^H\right) = \sum_{q=1}^{Q}\sum_{q'=1}^{Q}\mathbb{E}(|H_{E,q}|^2|H_{E,q'}|^2)\mathbb{E}\left(\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}_1^H\mathbf{e}_q\mathbf{e}_q^T\mathbf{V}_2\mathbf{V}_2^H\mathbf{e}_{q'}\mathbf{e}_{q'}^T\mathbf{V}_1\boldsymbol{\Sigma}^H\mathbf{U}^H\right)$$
$$= \sum_{q=1}^{Q}\mathbb{E}(|H_{E,q}|^4)\mathbb{E}\left(\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}_1^H\mathbf{e}_q\mathbf{e}_q^T\mathbf{V}_2\mathbf{V}_2^H\mathbf{e}_q\mathbf{e}_q^T\mathbf{V}_1\boldsymbol{\Sigma}^H\mathbf{U}^H\right)$$
$$+ \sum_{q=1}^{Q}\sum_{q'\neq q}^{Q}\mathbb{E}(|H_{E,q}|^2|H_{E,q'}|^2)\mathbb{E}\left(\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}_1^H\mathbf{e}_q\mathbf{e}_q^T\mathbf{V}_2\mathbf{V}_2^H\mathbf{e}_{q'}\mathbf{e}_{q'}^T\mathbf{V}_1\boldsymbol{\Sigma}^H\mathbf{U}^H\right)$$
$$= 2\sum_{q=1}^{Q}\mathbb{E}\left(\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}_1^H\mathbf{e}_q\mathbf{e}_q^T\mathbf{V}_2\mathbf{V}_2^H\mathbf{e}_q\mathbf{e}_q^T\mathbf{V}_1\boldsymbol{\Sigma}^H\mathbf{U}^H\right)$$
$$+ \sum_{q=1}^{Q}\sum_{q'\neq q}^{Q}\mathbb{E}\left(\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}_1^H\mathbf{e}_q\mathbf{e}_q^T\mathbf{V}_2\mathbf{V}_2^H\mathbf{e}_{q'}\mathbf{e}_{q'}^T\mathbf{V}_1\boldsymbol{\Sigma}^H\mathbf{U}^H\right)$$
$$= \sum_{q=1}^{Q}\mathbb{E}\left(\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}_1^H\mathbf{e}_q\mathbf{e}_q^T\mathbf{V}_2\mathbf{V}_2^H\mathbf{e}_q\mathbf{e}_q^T\mathbf{V}_1\boldsymbol{\Sigma}^H\mathbf{U}^H\right)$$
$$+ \mathbb{E}\left(\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}_1^H\sum_{q=1}^{Q}\mathbf{e}_q\mathbf{e}_q^T\mathbf{V}_2\mathbf{V}_2^H\sum_{q'=1}^{Q}\mathbf{e}_{q'}\mathbf{e}_{q'}^T\mathbf{V}_1\boldsymbol{\Sigma}^H\mathbf{U}^H\right)$$
$$= \sum_{q=1}^{Q}\mathbb{E}\left(\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}_1^H\mathbf{e}_q\mathbf{e}_q^T\mathbf{V}_2\mathbf{V}_2^H\mathbf{e}_q\mathbf{e}_q^T\mathbf{V}_1\boldsymbol{\Sigma}^H\mathbf{U}^H\right) + \mathbb{E}\left(\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}_1^H\mathbf{V}_2\mathbf{V}_2^H\mathbf{V}_1\boldsymbol{\Sigma}^H\mathbf{U}^H\right)$$

Using the fact that $\mathbf{V}_2^H\mathbf{V}_1 = \mathbf{0}$, the second term cancels and

$$\mathbb{E}\left(\mathbf{v}\mathbf{v}^H\right) = \mathbb{E}\left(\mathbf{U}\boldsymbol{\Sigma}\mathbf{V}_1^H \sum_{q=1}^{Q} \left(\mathbf{e}_q\mathbf{e}_q^T\mathbf{V}_2\mathbf{V}_2^H\mathbf{e}_q\mathbf{e}_q^T\right)\mathbf{V}_1\boldsymbol{\Sigma}^H\mathbf{U}^H\right)$$

If we assume (to be proven) that all elements of $\mathbf{v}$ have same variance, we can compute it as

$$\begin{aligned}
\frac{1}{N}\mathbb{E}\left(\|\mathbf{v}\|^2\right) &= \frac{1}{N}\mathbb{E}\ \mathrm{tr}\left(\mathbf{v}\mathbf{v}^H\right) \\
&= \frac{1}{N}\mathbb{E}\ \mathrm{tr}\left(\boldsymbol{\Sigma}^2\mathbf{V}_1^H \sum_{q=1}^{Q}\left(\mathbf{e}_q\mathbf{e}_q^T\mathbf{V}_2\mathbf{V}_2^H\mathbf{e}_q\mathbf{e}_q^T\right)\mathbf{V}_1\right)
\end{aligned}$$

Let us rewrite $\mathbf{V}_1 = \sum_l \mathbf{e}_l\mathbf{v}_{1,l}^H$ where $\mathbf{v}_{1,l}^H$ is the l-th row of $\mathbf{V}_1$ (of dimension $N \times 1$)

$$\begin{aligned}
\frac{1}{N}\mathbb{E}\left(\|\mathbf{v}\|^2\right) &= \frac{1}{N}\sum_{q=1}^{Q}\sum_{l}\sum_{l'}\mathbb{E}\ \mathrm{tr}\left(\boldsymbol{\Sigma}^2\mathbf{v}_{1,l}\mathbf{e}_{l'}^T\mathbf{e}_q\mathbf{e}_q^T\mathbf{V}_2\mathbf{V}_2^H\mathbf{e}_q\mathbf{e}_q^T\mathbf{e}_l\mathbf{v}_{1,l}^H\right) \\
&= \frac{1}{N}\sum_{q=1}^{Q}\sum_{l}\sum_{l'}\delta_{l'-q}\delta_{l-q}\mathbb{E}\ \mathrm{tr}\left(\boldsymbol{\Sigma}^2\mathbf{v}_{1,l}\mathbf{e}_q^T\mathbf{V}_2\mathbf{V}_2^H\mathbf{e}_q\mathbf{v}_{1,l}^H\right) \\
&= \frac{1}{N}\sum_{q=1}^{Q}\mathbb{E}\ \mathrm{tr}\left(\boldsymbol{\Sigma}^2\mathbf{v}_{1,q}\mathbf{e}_q^T\mathbf{V}_2\mathbf{V}_2^H\mathbf{e}_q\mathbf{v}_{1,q}^H\right)
\end{aligned}$$

Let us rewrite $\mathbf{V}_2 = \sum_l \mathbf{e}_l\mathbf{v}_{2,l}^H$ where $\mathbf{v}_{2,l}^H$ is the l-th row of $\mathbf{V}_2$ (of dimension

$Q - N \times 1)$

$$
\begin{aligned}
\frac{1}{N}\mathbb{E}\left(\|\mathbf{v}\|^2\right) &= \frac{1}{N}\sum_{q=1}^{Q}\sum_{l}\sum_{l'}\mathbb{E}\ \mathrm{tr}\left(\boldsymbol{\Sigma}^2\mathbf{v}_{1,q}\mathbf{e}_q^T\mathbf{e}_l\mathbf{v}_{2,l}^H\mathbf{v}_{2,l'}\mathbf{e}_{l'}^T\mathbf{e}_q\mathbf{v}_{1,q}^H\right) \\
&= \frac{1}{N}\sum_{q=1}^{Q}\mathbb{E}\ \mathrm{tr}\left(\boldsymbol{\Sigma}^2\mathbf{v}_{1,q}\mathbf{v}_{2,q}^H\mathbf{v}_{2,q}\mathbf{v}_{1,q}^H\right) \\
&= \frac{1}{N}\sum_{q=1}^{Q}\mathbb{E}\left(\|\mathbf{v}_{2,q}\|^2\mathbf{v}_{1,q}^H\boldsymbol{\Sigma}^2\mathbf{v}_{1,q}\right) \\
&= \frac{1}{N}\sum_{q=1}^{Q}\mathbb{E}\left(\|\mathbf{v}_{2,q}\|^2\|\mathbf{v}_{1,q}\|^2\sum_{n}\sigma_n^2\right)
\end{aligned}
$$

What I am not sure of (to be proven) is that $\sum_n \sigma_n^2$ will go to N, $\|\mathbf{v}_{1,q}\|^2$ will go to $\frac{N}{Q}$ and $\|\mathbf{v}_{2,q}\|^2$ will go to $\frac{Q-N}{Q}$ so that

$$
\begin{aligned}
\frac{1}{N}\mathbb{E}\left(\|\mathbf{v}\|^2\right) &= \frac{1}{N}Q\frac{Q-N}{Q}\frac{N}{Q}N \\
&= \frac{Q-N}{Q}N
\end{aligned}
$$

Here, we should obtain, i.e., it converges to my simulation results:

$$
\begin{aligned}
\frac{1}{N}\mathbb{E}\left(\|\mathbf{v}\|^2\right) &= \frac{1}{Q}\left[\frac{1}{N}Q\frac{Q-N}{Q}\frac{N}{Q}N\right] \\
&= \frac{Q-N}{Q^2}N \\
&= \frac{U-1}{U^2}
\end{aligned}
$$

However, this is not equal to $\frac{1}{U+1}$, which is the real expression where my simulation converges. But, for high values of U, we have that $\frac{U-1}{U^2} \to \frac{1}{U+1}$.

Typically, for $U = 4$, we already observe a good match between the expressions.