Statistical Response of Antennas under Uncorrelated Plane Wave Spectrum Illumination

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Abstract - A general formalism is proposed to study the statistical behavior of antennas under uncorrelated plane wave spectrum illumination. It is shown that field statistics are not adequate to understand antenna responses in such cases, and that radiation pattern and polarization must be taken into account. New closed form expressions for correlation functions are derived for the induced voltage and the available power at antenna feeds.

1 Introduction

The increasing demand for better understanding of device response in complex electromagnetic environment shifts nowadays the attention from deterministic to statistical models. This trend exists for a long time in wireless communication, but it also imposes itself in other fields of applied electromagnetics such as electromagnetic compatibility of devices immersed in diffused communication systems. The statistical approach importance is best illustrated by the now commonly accepted use of reverberation chambers to test electronic component susceptibility to electromagnetic fields. In these installations, the perfectly conducting walls produce multiple reflections of an excitation field which illuminate the device under test, giving rise to a kind of worst case scenario from the electromagnetic environment point of view. Reverberation chambers are of course an extreme case, but the same kind of illumination can be expected for instance in metallic enclosures, or, at a larger scale, in confined indoor networks e.g. in industrial environment. Basically, all those situations, from the wireless communication optimization to the reverberation chamber analysis, are theoretically very similar, and the statistical behavior of devices under complex illumination becomes an increasingly important topic. In this paper, we propose to investigate the statistical response of a very general class of devices, antennas, under isotropic uncorrelated plane wave spectrum illumination. The term antenna refers here to any linear device with terminals that can be identified, and the formalism depicted in the next sections has to be understood in its most general acceptation.

Historically, the statistical study of antennas subject to complex illumination began in the framework of wireless communication optimization. The goal was mainly to find the necessary conditions to achieve diversity either at the base station or at the mobile equipment. Related works always dealt with two-dimensional illumination of isotropic antennas [1-4], an assumption acceptable for outdoor communication channels, but too restrictive for a full understanding of responses in complex environments. To take into account polarization effects, Vaughan *et al.* developed a general formalism [5-9], but they restricted their investigations to particular problems relevant to communication channel studies, mainly two-dimensional in nature. Their results are now summarized in a book [10]. This formalism was adapted to the reverberation chamber analysis by Hill [11,12], showing that closed form expressions could be found for the mean power received by antennas and certain correlation functions. We propose here to generalize the same approach, to derive new closed form expressions, and to clearly indicate the effects of radiation patterns and polarization on the statistical response of antennas.

In section II the formalism will be introduced. It will be applied in section III first to the scalar case (polarization not taken into account). Polarization effects will then be evaluated. In the next sections, at time dependence $e^{j\omega t}$ is assumed.

2 Integral representation of antenna response

Let us consider a source-free region typically comprising several free-space wavelengths wherein the mean electric field level can be supposed constant. This region represents for instance the test volume of a reverberation chamber, a metallic enclosure equipped with electronic components, or a small geographical area in a confined wireless network. At any point, the electric field is made up of different waves coming from the scattering and reflection of an excitation field on existing obstacles and boundaries. On physical ground, if plane wave approximation can be applied to each partial wave, it can thus be written as

$$\vec{E}(\vec{r}) = \int_{\Omega} \vec{F}(\Omega)\, e^{-j\vec{k}\cdot\vec{r}}\, d\Omega \tag{1}$$

where the integration is performed over all real angles $\Omega = (\theta, \phi)$, and where $\vec{F}(\Omega)$ is the complex plane wave spectrum of $\vec{E}$. In a spherical coordinate system defined at $\vec{r}$, $\vec{F}(\Omega)$ is written as

$$\vec{F}(\Omega) = F_\theta(\Omega)\,\vec{1}_\theta + F_\phi(\Omega)\,\vec{1}_\phi \tag{2}$$

where both F_θ and F_ϕ are complex.

In real situations, $\vec{E}$ is only made up of a finite number of partial waves, and the continuous limit considered in (1) has to be understood as an approximation made for sake of analytical simplicity, as often in wireless communication studies [10] or reverberation chamber analysis [11].

The open-circuit voltage $V(\vec{r})$ induced by the electric field (1) at the terminals of an antenna located at $\vec{r}$ can be written as

$$V(\vec{r}) = -\int_\Omega \vec{L}(\Omega)\cdot\vec{F}(\Omega)\,e^{-j\vec{k}\cdot\vec{r}}\,d\Omega \tag{3}$$

where $\vec{L}(\Omega)$ is the antenna complex equivalent length [13] defined in the same spherical coordinate system as $\vec{F}(\Omega)$: $\vec{L}(\Omega) = L_\theta(\Omega)\,\vec{1}_\theta + L_\phi(\Omega)\,\vec{1}_\phi$. For an antenna impedance $Z_A = R_A + j\,X_A$, the maximum available power at the antenna feed is

$$P(\vec{r}) = \frac{1}{2}\,\frac{|V(\vec{r})|^2}{4R_A} \tag{4}$$

It is still worth noting here that the antenna radiation resistance R_{AR} is obtained by integration of the complex equivalent length according to [13]

$$R_{AR} = \frac{Z_0}{4\,\lambda^2}\int_\Omega |\,\vec{L}(\Omega)\,|^2\,d\Omega \tag{5}$$

where Z_0 is the free-space impedance, and λ the free-space wavelength. And, finally, the radiated electric field is related to the complex equivalent length by [13]

$$\vec{E}(\vec{r},\Omega) = -j\omega\,\frac{\mu_0}{4\pi}\,\frac{e^{-jkr}}{r}\,I_a\,\vec{L}(\Omega) \tag{6}$$

where I_a is the current at the antenna feed. The complex equivalent length thus contains all information about the antenna, including its polarization characteristics.

3 Statistical properties of antenna response

Equation (3) gives one realization of the induced voltage, at a given location, and for a given plane wave spectrum. To study its statistical properties, a set of realizations is necessary, for instance by rotating the stirrer in a reverberation chamber, or by recording several data sets in the communication channel case. The ensemble average over these realizations will be noted $< \cdot >$. Basically, from (3), the statistical results depends on both the antenna properties, via the complex equivalent length, and the plane wave spectrum. The former is of course a deterministic entity, and no statistical assumption has to be made on it. On the other hand, the plane wave spectrum statistical properties have to be fixed. We will restrict ourselves here to the case of uncorrelated plane wave spectrum (referred to as uncorrelated fading in wireless communication textbooks [10]), with no dominant partial wave (Rayleigh channel). Those postulates can be translated into mathematical relations satisfied by $\vec{F}(\Omega)$ as shown e.g. in [14]. Since these relations are important for understanding next developments, they are summarized in the Appendix.

In the next sections, we will focus on the spatial correlation function which is defined for two complex random variables u and v by [15]

$$\rho(\vec{r}_1, \vec{r}_2) = \frac{< (u(\vec{r}_1) - < u(\vec{r}_1) >)(v^*(\vec{r}_2) - < v^*(\vec{r}_2) >) >}{\sqrt{< |u - < u >|^2 >< |v - < v >|^2 >}} \tag{7}$$

where * denotes complex conjugates.

3.1 Mean values

In a first step, let us consider the open-circuit voltage and available power mean values. Using (44) and (3), clearly

$$< V(\vec{r}) > = 0 \tag{8}$$

so that the mean voltage received by the antenna is null. To define the available power mean value, let us compute

$$
\begin{aligned}
< | V(\vec{r}) |^2 > = < \int_{\Omega_1} \int_{\Omega_2} & \left(L_\theta(\Omega_1) F_\theta(\Omega_1) + L_\phi(\Omega_1) F_\phi(\Omega_1) \right) \\
& \left(L_\theta^*(\Omega_2) F_\theta^*(\Omega_2) + L_\phi^*(\Omega_2) F_\phi^*(\Omega_2) \right) e^{-j(\vec{k}_1 - \vec{k}_2)\cdot\vec{r}} \, d\Omega_1 d\Omega_2 >
\end{aligned}
\tag{9}
$$

Using (48) and (49), this relation can be written in the final form

$$
\begin{aligned}
< | V(\vec{r}) |^2 > &= \frac{E_0^2}{8\pi} \int_\Omega |\vec{L}(\Omega)|^2 \, d\Omega \\
&= \frac{E_0^2}{2\pi} \frac{\lambda^2}{Z_0} R_{AR}
\end{aligned}
\tag{10}
$$

where E_0^2 is the electric field mean power $< |\vec{E}|^2 >= E_0^2$. With (4), the mean available power at the antenna feed is thus given by

$$
< P(\vec{r}) > = \frac{E_0^2}{16\pi} \frac{\lambda^2}{Z_0} \eta \tag{11}
$$

where the antenna efficiency $\eta = R_{AR}/R_A$ has been introduced. This result has already been obtained by Hill [11] and Warne and Lee [16] in the reverberation chamber framework. It is actually a general result for any uncorrelated plane wave spectrum illumination, including Rayleigh channels. The antenna properties only appear in this relation through the efficiency. This holds for the mean power received by an antenna, or by any linear device for which (3) is valid, and it is expected that the same result would be obtained in susceptibility studies of equipments immersed in diffused communication systems.

3.2 Fundamental relation

Let two antennas be located at $\vec{r}_1$ and $\vec{r}_2$ respectively. The mean product of their induced voltages is given by

$$
\begin{aligned}
< V_1(\vec{r}_1\,)V_2^*(\vec{r}_2\,) >=< \int_{\Omega_1}\int_{\Omega_2} & \left(L_{\theta_1}(\Omega_1)F_\theta(\Omega_1) + L_{\phi_1}(\Omega_1)F_\phi(\Omega_1)\right) \\
& \left(L^*_{\theta_2}(\Omega_2)F^*_\theta(\Omega_2) + L^*_{\phi_2}(\Omega_2)F^*_\phi(\Omega_2)\right)\, e^{-j(\vec{k}_1\cdot\vec{r}_1-\vec{k}_2\cdot\vec{r}_2)}\, d\Omega_1 d\Omega_2 >
\end{aligned}
\tag{12}
$$

Using (48) and (49), this relation becomes

$$
< V_1(\vec{r}_1\,)V_2^*(\vec{r}_2\,) >= \frac{E_0^2}{8\pi}\int_\Omega \vec{L}_1(\Omega)\cdot\vec{L}_2^*(\Omega)\, e^{-j\vec{k}\cdot(\vec{r}_1-\vec{r}_2)}\, d\Omega \tag{13}
$$

So that, using (7), (8) and (10), the fundamental relation for the correlation between the voltages of two antennas is obtained:

$$
\rho(\vec{r}_1,\vec{r}_2) \;=\; \frac{Z_0}{4\lambda^2}\,\frac{1}{\sqrt{R_{AR_1}R_{AR_2}}}\int_\Omega \vec{L}_1(\Omega)\cdot\vec{L}_2^*(\Omega)\, e^{-j\vec{k}\cdot(\vec{r}_1-\vec{r}_2)}\, d\Omega \tag{14}
$$

3.3 Radiation pattern effects

To evaluate radiation pattern effects on the statistical properties of antenna responses, let us consider the correlation between induced voltages on antennas with the same radiation pattern, but at different locations. The fundamental relation (14) then becomes:

$$
\rho(\vec{r}_1,\vec{r}_2) \;=\; \frac{Z_0}{4\lambda^2 R_{AR}}\int_\Omega |\vec{L}(\Omega)|^2\, e^{-j\vec{k}\cdot(\vec{r}_1-\vec{r}_2)}\, d\Omega \tag{15}
$$

For a reference omnidirectional antenna, $\vec{L}(\Omega) \;=\; L\,\vec{1}_L$, where L is a complex constant independent of Ω, and $\vec{1}_L$ is the unit vector parallel to the equivalent length. Without loss of generality, $\vec{r}_1 - \vec{r}_2 = d\,\vec{1}_z$, and (15) can be written as

$$\begin{aligned}\rho(\vec{r}_1, \vec{r}_2) &= \frac{Z_0}{4\lambda^2 R_{AR}}\,|L|^2 \int_\Omega e^{j\,kd\cos\theta}\, d\Omega \\ &= \frac{\pi\, Z_0}{\lambda^2 R_{AR}}\,|L|^2\, \frac{\sin kd}{kd} \\ &= \frac{\sin kd}{kd}\end{aligned} \tag{16}$$

so that the same result as the correlation function for the electric field $\vec{E}$ is obtained [14]. Deriving statistical properties for the fields thus corresponds to an idealized situation, and it is inadequate for full understanding of antenna responses. Conversely, field statistics measurement must take into account this discrepancy to be coherent.

3.3.1 Axisymmetric pattern

In a first step, let the equivalent length of the antenna have a rotational symmetry around z axis:

$$\vec{L}(\Omega) = L(\theta)\,\vec{1}_L \tag{17}$$

To obtain closed form expressions for the correlation function, let us furthermore consider the particular case proposed by Lee [1] of a trigonometric radiation pattern

$$L(\theta) = L\,\sin^n\theta \tag{18}$$

where n defines the half power beamwidth (HPBW) by [10]:

$$\mathrm{HPBW} = 4\ \mathrm{Acos}\left(2^{-1/2n}\right) \tag{19}$$

The fundamental relation (14) then becomes

$$\rho(\vec{r}_1, \vec{r}_2) = \frac{Z_0}{4\lambda^2 R_{AR}}\,|L|^2 \int_\Omega \sin^{2n}\theta\, e^{-j\,\vec{k}\cdot(\vec{r}_1-\vec{r}_2)}\, d\Omega \tag{20}$$

Of course, the particular symmetry axis breaks down the global system symmetry, and the correlation along z axis must be considered independently of the correlation in the xy plane.

a. Correlation along z axis

In this case $\vec{r}_1 - \vec{r}_2 = d\,\vec{1}_z$ and by (20)

$$\rho(\vec{r}_1, \vec{r}_2) = \frac{Z_0}{4\lambda^2 R_{AR}} |L|^2 \int_\Omega \sin^{2n}\theta\, e^{j\,kd\cos\theta}\, d\Omega \tag{21}$$

which gives, after integration and normalization

$$\rho(d) = 2^{n+1/2}\,\Gamma(n + \frac{3}{2})\, \frac{J_{n+1/2}(kd)}{(kd)^{n+1/2}} \tag{22}$$

Where $J_{n+1/2}$ is the Bessel function of order $n + 1/2$, and Γ is the Euler gamma function. When $n = 0$, the antenna is omnidirectional and this results is equivalent to (16). It is worth noting that for an infinitesimal electric dipole, $n = 1$, and (22) gives the same result as the correlation function for E_z with an omnidirectional antenna [14]. When n is higher, the first zero of $\rho(d)$ rapidly increases with n as shown on figure 1: when the antenna directivity increases, both antennas at $\vec{r}_1$ and $\vec{r}_2$ gather more and more rays arriving from broadside with the same phase at both locations, and the correlation is enhanced.

b. Correlation in the xy plane

The correlation must be isotropic in this plane. Without loss of generality, $\vec{r}_1 - \vec{r}_2 = d\vec{1}_y$, and by (20)

$$\rho(\vec{r}_1, \vec{r}_2) = \frac{Z_0}{4\lambda^2 R_{AR}} |L|^2 \int_\Omega \sin^{2n}\theta\, e^{j\,kd\sin\theta\,\sin\phi}\, d\Omega \tag{23}$$

After integration and normalization

$$\rho(d) = {}_1F_2\left(n + 1\,;\,(1, n + 3/2)\,;\, -\frac{(kd)^2}{4}\right) \tag{24}$$

with ${}_1F_2$ a generalized hypergeometric function [17].

In this case, the first zero of $\rho(d)$ is less than the first zero of the omnidirectional reference antenna as shown on figure 2. Indeed, when the antenna directivity increases, both antennas

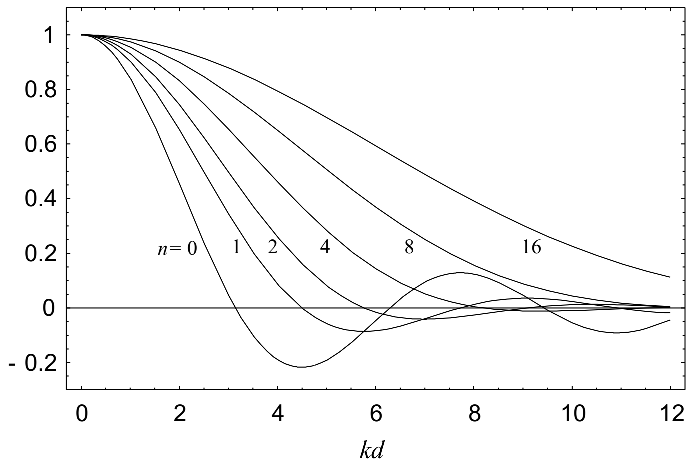


Figure 1: Correlation functions for the induced voltage along z axis (22).

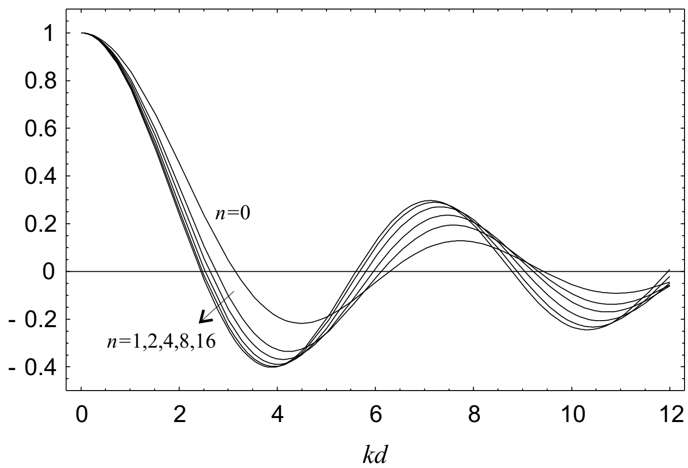


Figure 2: Correlation functions for the induced voltage in the xy plane (24).

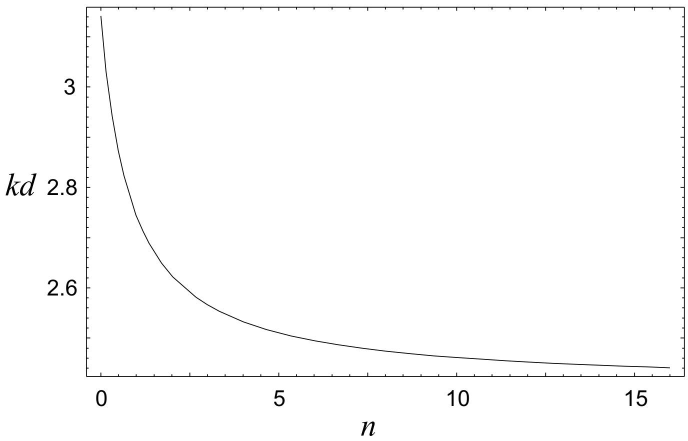


Figure 3: First correlation zero in the xy plane.

gather more and more rays propagating in the xy plane, each arriving at $\vec{r}_1$ and $\vec{r}_2$ with different phases, and the correlation decreases. Unfortunately from the diversity point of view, the correlation cannot be made as low as desired by increasing n, as shown on figure 3. The minimal correlation seems to be almost reached for $n = 4$, the first correlation zero being shifted to around $kd = 2.5$, compared to $kd = \pi$ in the omnidirectional case. For low correlation, the geometric configuration in the xy plane is however by far better than along z axis. For an infinitesimal electric dipole for instance, the first correlation zero appears at $d = 0.71\,\lambda$ along z, and at $d = 0.436\,\lambda$ in the xy plane.

c. Correlation for the available power

As shown by Hill in the reverberation chamber framework [11], the induced voltage amplitude satisfies a Rayleigh distribution in case of uncorrelated plane wave spectrum incidence. The correlation function for the available power ρ_P is then given by [10]:

$$\rho_P = |\rho|^2 \tag{25}$$

This function is drawn on figure 4 for various values of n, for correlation in the xy plane and along z axis. When n is high, the lower correlation in the xy plane is evident, at the price of

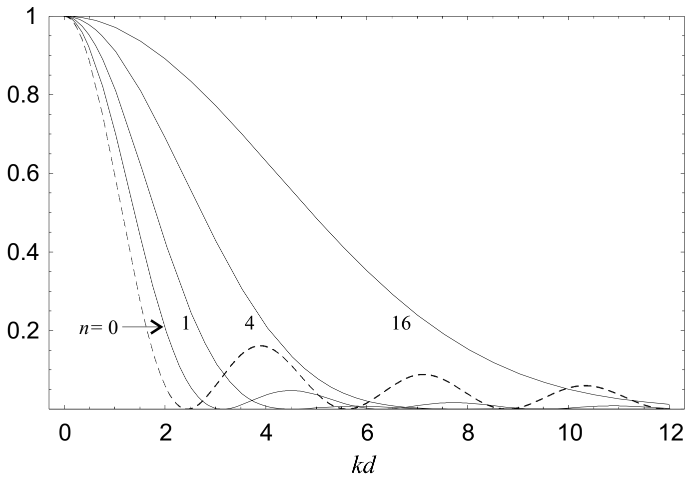


Figure 4: Correlation functions for the available power along z axis (solid lines) and in the xy plane for n=16 (dashed line). The correlation function for $n = 0$ is common to both cases.

small oscillations. As will be shown, those oscillations will become more dramatic when the rotational symmetry is broken. Basically, correlation functions are thus strongly anisotropic, with a relative independence with respect to radiation pattern in the plane transverse to the symmetry axis. The correlation function obtained for $\vec{E}$ [14], or with an omnidirectional antenna (16) can be considered as a good first approximation of the actual correlation function of axisymmetric antennas in this plane, but not along the symmetry axis.

3.3.2 General pattern

Closed form expressions for the correlation function can be found in the general case if trigonometric radiation patterns are chosen:

$$\vec{L}(\Omega) = L \sin^m \phi \cos^n \theta \, \vec{1}_L \tag{26}$$

Under this approximation, any sidelobe structure of a real pattern is supposed to have a secondary effect. Along z axis, the ϕ dependence has no influence on the correlation. But

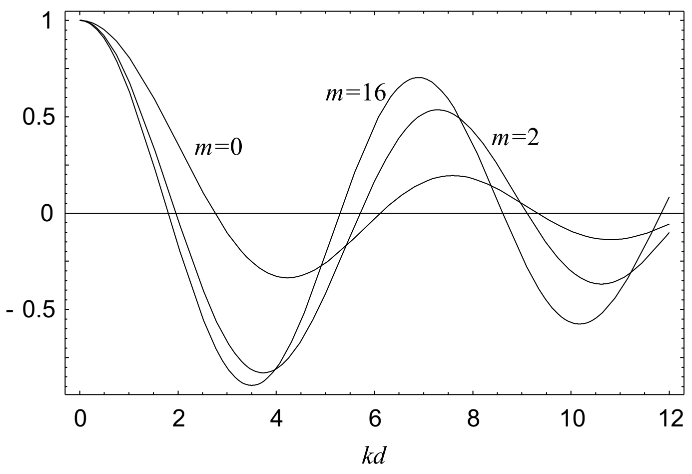


Figure 5: Correlation functions for the induced voltage along y axis (n=1).

when $\vec{r}_1 - \vec{r}_2 = d\vec{1}_y$, i.e. for correlation in the maximal radiation direction, (14) becomes

$$\rho(\vec{r}_1, \vec{r}_2) = \frac{Z_0}{4\lambda^2 R_{AR}} |L|^2 \int_\Omega \sin^{2n}\theta \, \sin^{2m}\phi \, e^{-jkd\sin\theta\,\sin\phi} \, d\Omega \tag{27}$$

which can be evaluated in closed form:

$$\rho(d) = {}_2F_3\left((m+1/2, n+1)\,; (1/2, m+1, n+3/2)\,; -\frac{(kd)^2}{4} \right) \tag{28}$$

This function is drawn on figure 5, for $n = 1$ and various values of m. When m is large, $\rho(d)$ begins to oscillate, and any decorrelation distance is difficult to define. In this case, the double lobe structure focus on partial waves propagating in opposite directions along the axis joining the antennas. When the distance between antennas differs by wavelength multiples, these waves arrive with the same phase at $\vec{r}_1$ and $\vec{r}_2$, and the correlation is high and positive.

Finally, let the radiation pattern have a single lobe structure, for instance

$$\vec{L}(\Omega) = \begin{cases} L \sin^m\phi \cos^n\theta \, \vec{1}_L & \phi \in [0, \pi] \\ 0 & \phi \in [\pi, 2\pi] \end{cases} \tag{29}$$

As a consequence, an imaginary part must be added to the previous result

$$\begin{aligned}\mathrm{Im}(\rho(d)) = & kd \frac{\Gamma^2(m+1)\Gamma^2(n+3/2)}{\Gamma(m+3/2)\Gamma(m+1/2)\Gamma(n+1)\Gamma(n+2)} \\ & {}_2F_3\left((m+1, n+3/2)\,;\,(3/2, m+3/2, n+2)\,;\,-\frac{(kd)^2}{4}\right)\end{aligned} \tag{30}$$

3.4 Polarization effects

When the correlation between different antennas is considered, polarization must be taken into account. All information about polarization is contained in the complex equivalent lengths, and (13) can still be used as starting point to obtain closed form expressions. Let (x_1, y_1) and (x_2, y_2) denote the polarization principal axis of the two antennas, and ξ the angle between these two coordinate systems (see figure 6(a)). In these axis, the complex equivalent lengths can be written as

$$\begin{aligned}\vec{L}_1(\Omega) &= |L_{1x}(\Omega)|\, e^{j\zeta_1+\sigma_1\frac{\pi}{4}}\, \vec{1}_{x_1} + |L_{1y}(\Omega)|\, e^{j\zeta_1-\sigma_1\frac{\pi}{4}}\, \vec{1}_{y_1} \\ \vec{L}_2(\Omega) &= |L_{2x}(\Omega)|\, e^{j\zeta_2+\sigma_2\frac{\pi}{4}}\, \vec{1}_{x_2} + |L_{2y}(\Omega)|\, e^{j\zeta_2-\sigma_2\frac{\pi}{4}}\, \vec{1}_{y_2}\end{aligned} \tag{31}$$

where ζ_i (i=1,2) are the antenna phase references, and σ_i define the polarization orientations: σ_i=1 for left-hand polarization, σ_i=-1 for right-hand polarization, and σ_i=0 for linear polarization. It can be easily verified that ζ_1 and ζ_2 can be omitted in (13) without loss of generality, and, assuming ξ being independent of Ω, this relation yields:

$$\begin{aligned}< V_1(\vec{r}_1)V_2^*(\vec{r}_2) >= \frac{E_0^2}{8\pi} \Big[& \cos\xi\, e^{j\frac{\pi}{4}(\sigma_1-\sigma_2)} \int_\Omega |L_{1x}||L_{2x}|\, e^{-j\vec{k}\cdot(\vec{r}_1-\vec{r}_2)}\, d\Omega \\ & - \sin\xi\, e^{j\frac{\pi}{4}(\sigma_1+\sigma_2)} \int_\Omega |L_{1x}||L_{2y}|\, e^{-j\vec{k}\cdot(\vec{r}_1-\vec{r}_2)}\, d\Omega \\ & + \sin\xi\, e^{-j\frac{\pi}{4}(\sigma_1+\sigma_2)} \int_\Omega |L_{1y}||L_{2x}|\, e^{-j\vec{k}\cdot(\vec{r}_1-\vec{r}_2)}\, d\Omega \\ & + \cos\xi\, e^{-j\frac{\pi}{4}(\sigma_1-\sigma_2)} \int_\Omega |L_{1y}||L_{2y}|\, e^{-j\vec{k}\cdot(\vec{r}_1-\vec{r}_2)}\, d\Omega \Big]\end{aligned} \tag{32}$$

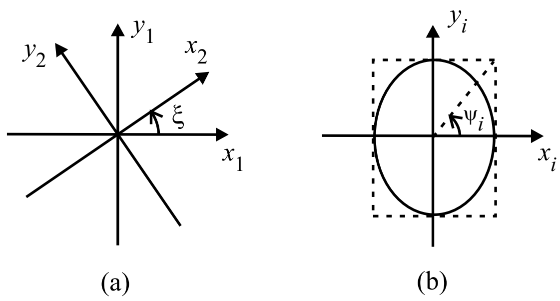


Figure 6: Polarization principal axis.

For sake of simplicity while clearly showing polarization effects, it will be supposed that both antennas are located at the same point. All subsequent results are however valid when the integrals in (32) are real, namely in case of radiation pattern having rotational symmetry.

3.4.1 Two linearly polarized antennas

In this case $\sigma_1 = \sigma_2 = 0$. Let $|L_{1y}| = |L_{2y}| = 0$, so that (32) becomes

$$< V_1(\vec{r}_1)V_2^*(\vec{r}_2) > = \frac{E_0^2}{8\pi} \cos\xi \int_\Omega |L_{1x}||L_{2x}| \, d\Omega \tag{33}$$

In this equation, the integral corresponds to the scalar case where polarization is not taken into account, and, after normalization, the corresponding correlation function will be referred to as $\tilde{\rho}(d)$. Normalization yields

$$\rho(d) \; = \; \cos\xi \, \tilde{\rho}(d) \tag{34}$$

and the available power correlation function is given by

$$\rho_P(d) \; = \; \cos^2\xi \, \tilde{\rho}_P(d) \tag{35}$$

This result was already obtained in the particular case of elliptical radiation patterns by Vaughan [8], and, interestingly, the $\cos\xi$ factor appears in the reverberation chamber and

Rayleigh channel cases for correlation functions for field components rotated by an angle ξ [11,14]. Compared to the scalar case, $\rho_p(d)$ is thus modulated by an oscillating function, and the correlation always vanishes when the antennas are orthogonal. Of course, this result only holds when the plane wave spectrum is uncorrelated, and, in actual communication channels, a small correlation always exists between orthogonal field components.

3.4.2 Linearly et elliptically polarized antennas

Let $|L_{1y}| = 0$, $\sigma_1 = 0$, the second antenna being elliptically polarized ($\sigma_2 = \pm 1$). Equation (32) then yields:

$$< V_1(\vec{r}_1) V_2^*(\vec{r}_2) > = \frac{E_0^2}{8\pi} e^{-j\frac{\pi}{4}\sigma_2} \left[\cos\xi \int_\Omega |L_{1x}||L_{2x}| \, d\Omega \; - \; j\sigma_2 \sin\xi \int_\Omega |L_{1x}||L_{2y}| \, d\Omega \right] \qquad (36)$$

The introduction of a second component in the polarization adds thus an imaginary part to the correlation function. It is useful here to introduce the axial ratio coefficient, assumed, to foster analytical results, to be independent of Ω

$$\chi_2 \; = \; \frac{|L_{2y}|}{|L_{2x}|} \qquad (37)$$

so that, after normalization, the correlation function for the available power is given by

$$\rho_p(d) \; = \; \tilde{\rho}_p(d) \, \frac{\cos^2\xi \; + \; \chi_2^2 \sin^2\xi}{1 + \chi_2^2} \qquad (38)$$

A simpler form can be obtained by introducing the ellipticity $\psi_2 = \mathrm{tg}^{-1}\,\chi_2$ ($\psi_2 \in [0, \pi/2]$, see figure 6(b))

$$\rho_p(d) \; = \; \tilde{\rho}_p(d) \left(\frac{1}{2} \; + \; \frac{1}{2} \cos 2\psi_2 \cos 2\xi \right) \qquad (39)$$

As a function of ξ, the modulation coefficient oscillates around $1/2$ with an amplitude $1/2 \, \cos 2\psi_2$. In the particular case of circular polarization, $\psi_2 = \pi/4$ and $\rho_p(d) \; = \; 1/2 \, \tilde{\rho}_p(d)$ without oscillations for any ξ: using a circular antenna ensures that the correlation never vanishes if $\tilde{\rho}_p(d)$ is not zero, with an overall correlation divided by a factor two. Basically, it can be seen that the geometric rotation angle ξ play the same role as the ellipticity ψ_2, and, rotating the antenna is equivalent to changing its polarization ellipse aperture.

3.5 Two elliptically polarized antennas

If both antennas are elliptically polarized, the same procedure yields the available power correlation functions. In case of two identically oriented polarizations ($\sigma_1 = \sigma_2$):

$$\begin{aligned}\rho_P(d) = \tilde{\rho}_P(d) \Big[&\frac{1}{2} \Big(\cos^2(\psi_1 - \psi_2) + \sin^2(\psi_1 + \psi_2) \Big) \\ &+ \frac{1}{2} \Big(\cos^2(\psi_1 - \psi_2) - \sin^2(\psi_1 + \psi_2) \Big) \cos 2\xi \Big]\end{aligned} \tag{40}$$

While in case of different orientations ($\sigma_1 = -\sigma_2$):

$$\begin{aligned}\rho_P(d) = \tilde{\rho}_P(d) \Big[&\frac{1}{2} \Big(\cos^2(\psi_1 + \psi_2) + \sin^2(\psi_1 - \psi_2) \Big) \\ &+ \frac{1}{2} \Big(\cos^2(\psi_1 + \psi_2) - \sin^2(\psi_1 - \psi_2) \Big) \cos 2\xi \Big]\end{aligned} \tag{41}$$

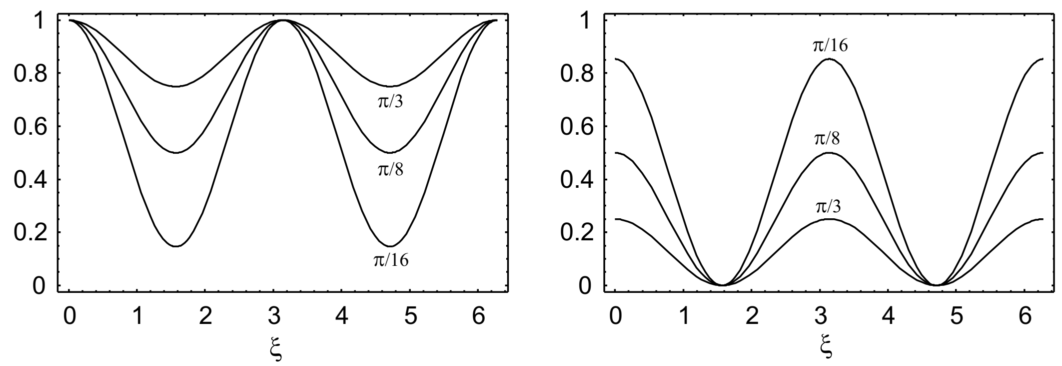
An important situation is obtained when both axial ratios are the same $\psi_1 = \psi_2 \equiv \psi$. If the ellipses are co-polarized:

$$\rho_P(d) = \tilde{\rho}_P(d) \Big(\cos^2 \xi + \sin^2 \xi \sin^2 2\psi \Big) \tag{42}$$

while in the cross-polarized case:

$$\rho_P(d) = \tilde{\rho}_P(d) \cos^2 \xi \cos^2 2\psi \tag{43}$$

Again, the modulation coefficients oscillate as a function of ξ as shown on figure 7. Compared to the linear case (35), the presence of a second component (possibly parasitic) in the polarization always increases the correlation in the co-polar case, while it always decreases it in the cross-polar one. If both antennas are circularly polarized, $\psi = \pi/4$, and the correlation vanishes in the cross-polar case (orthogonal polarizations) while it equals the scalar



(a) (b)

Figure 7: Correlation function for the available power for $\psi = \pi/3, \pi/8, \pi/16$. (a) co-polar case (b) cross-polar case

correlation in the co-polar one. The closer ψ is to $\pi/4$, the smaller the oscillations. It is worth noting on the one hand that the modulation coefficient never vanishes in the co-polar case if $\psi \neq 0$, and on the hand that in the cross-polar situation, a rotation angle $\xi = \pi/2$ ensures zero correlation whatever $\tilde{\rho}(d)$, for any ψ . This last property implies that polarization should be a suitable mean to obtain diversity in communication systems.

4 Conclusion

Statistical approach becomes more and more important nowadays to model propagation channels in modern communication systems, either to improve transmission performances, or to study electromagnetic compatibility issues. In this paper, a general formalism has been proposed to study the statistical behavior of linear devices immersed in uncorrelated plane wave spectrum, and it has been applied to antenna responses. It has been shown that classical approaches dealing with field statistics are not adequate, and correspond to ideal cases obtained with omnidirectional antennas. Radiation patterns and polarization play an important role in available power statistics, and strong anisotropies have been found in correlation functions. The same kind of results should be found concerning other linear devices like transmission lines, a subject currently under investigation.

Appendix: plane wave spectrum statistical properties [14]

Since each partial wave has a random phase, $< \vec{E} >= 0$, implying

$$< F_\theta > = < F_\phi > = 0 \tag{44}$$

Two partial waves arriving from different directions are supposed to be uncorrelated:

$$\begin{aligned} < F_{\theta r}(\Omega_1) F_{\theta r}(\Omega_2) > = & < F_{\theta i}(\Omega_1) F_{\theta i}(\Omega_2) > \\ & = < F_{\phi r}(\Omega_1) F_{\phi r}(\Omega_2) > \\ & = < F_{\phi i}(\Omega_1) F_{\phi i}(\Omega_2) > = C\,\delta(\Omega_1 - \Omega_2) \end{aligned} \tag{45}$$

Where the subscripts r and i denote respectively the real and imaginary parts. Since all partial waves are supposed to carry equal energy, C is independent of Ω. Using (1), it can be linked to the electric field mean power $< |\vec{E}|^2 >= E_0^2$:

$$C = \frac{E_0^2}{16\pi} \tag{46}$$

Finally, the real and imaginary parts of the plane wave spectrum and its θ and ϕ components are supposed to be uncorrelated:

$$\begin{aligned} < F_{\theta r}(\Omega_1) F_{\theta i}(\Omega_2) > = & < F_{\phi r}(\Omega_1) F_{\phi i}(\Omega_2) > \\ & = < F_{\phi r}(\Omega_1) F_{\theta r}(\Omega_2) > \\ & = < F_{\phi i}(\Omega_1) F_{\theta i}(\Omega_2) > \\ & = < F_{\phi r}(\Omega_1) F_{\theta i}(\Omega_2) > \\ & = < F_{\phi i}(\Omega_1) F_{\theta r}(\Omega_2) > = 0 \end{aligned} \tag{47}$$

From these equations, two other useful expressions can be deduced:

$$< F_\theta(\Omega_1)F^*_\phi(\Omega_2) > = 0 \tag{48}$$

and

$$< F_\theta(\Omega_1)F^*_\theta(\Omega_2) > = < F_\phi(\Omega_1)F^*_\phi(\Omega_2) > = 2C\,\delta(\Omega_1 - \Omega_2) \tag{49}$$

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