Momentum of complex normal random variables

Sidney Golstein

May 26, 2020

1 Real normal random variables

For real-valued random variable, the moment-generating function is an alternative specification of its probability distribution. In particular, it allows to compute the moments of the probability distribution as:

$$m_n = \mathbb{E}\left[X^n\right] = M_X^{(n)}(0) = \left.\frac{d^n M_X}{dt^n}\right|_{t=0} \tag{1}$$

For a real normal random variable $\mathcal{N}(\mu, \sigma^2)$, the moment-generating function is given by:

$$M_X = e^{t\mu + \frac{1}{2}\sigma^2 t^2} \tag{2}$$

From that, we have:

$$\begin{aligned} \mathbb{E}\left[|X|^2\right] &= \sigma^2 + \mu^2 \\ \mathbb{E}\left[|X|^4\right] &= 3(\sigma^2)^2 + 6\sigma^2\mu^2 + \mu^4 \end{aligned} \tag{3}$$

Example

If we generate $X \sim \mathcal{N}(2, 1/2)$, i.e. $\mu = 2$ and $\sigma^2 = 1/2$, we should obtain (cf. fig.1):

$$\begin{aligned} \mathbb{E}\left[|X|^2\right] &= 9/2 = 4.5 \\ \mathbb{E}\left[|X|^4\right] &= 115/4 = 28.75 \end{aligned} \tag{4}$$

```
>> x = 1/sqrt(2)*randn(1,1e6)+2;
>> mean(abs(x).^2)

ans =

    4.4972

>> mean(abs(x).^4)

ans =

   28.7106
```

Figure 1: Real-valued random normal variable

2 Complex normal random variable

A complex normal random variable is defined as $Z = X + iY$ where $X \sim \mathcal{N}(\mu_x, \sigma_x^2)$ and $Y \sim \mathcal{N}(\mu_y, \sigma_y^2)$. We also have:

$$\begin{aligned} |Z|^2 &= X^2 + Y^2 \\ |Z|^4 &= X^4 + 2X^2Y^2 + Y^4 \end{aligned} \tag{5}$$

Since X and Y are independent and taking into account eq.3, the moments of Z are easy to compute:

$$\begin{aligned} \mathbb{E}\left[|Z|^2\right] &= \sigma_x^2 + \mu_x^2 + \sigma_y^2 + \mu_y^2 \\ \mathbb{E}\left[|Z|^4\right] &= 3(\sigma_x^2)^2 + 6\sigma_x^2\mu_x^2 + \mu_x^4 + 2\left[(\sigma_x^2 + \mu_x^2)(\sigma_y^2 + \mu_y^2)\right] + 3(\sigma_y^2)^2 + 6\sigma_y^2\mu_y^2 + \mu_y^4 \end{aligned} \tag{6}$$

Example

If we generate $X \sim \mathcal{N}(1, 1/2)$, $Y \sim \mathcal{N}(0, 1)$, i.e., $\mu_x = 1$, $\sigma_x^2 = 1/2$, $\mu_y = 0$ and $\sigma_y^2 = 1$, we should obtain (cf. fig.2):

$$\begin{aligned} \mathbb{E}\left[|Z|^2\right] &= 5/2 = 2.5 \\ \mathbb{E}\left[|Z|^4\right] &= 43/4 = 10.75 \end{aligned} \tag{7}$$

Note for Julien:

On retrouve bien $\mathbb{E}\left[|H_e|^4\right] = 2$ *car* $H_e = H_{ex} + iH_{ey}$ *avec* $\mu_{H_{ex}} = \mu_{H_{ey}} = 0$ *et* $\sigma_{H_{ex}}^2 = \sigma_{H_{ey}}^2 = 1/2$.
Pour ton exemple, on retrouve bien 8 car tu avais $\mu_x = \mu_y = 0$ *et* $\sigma_x^2 = \sigma_y^2 = 1$

```
>> x = 1/sqrt(2)*randn(1,1e6)+1;
>> y = randn(1,1e6);
>> z = x+1j*y;
>> mean(abs(z).^2)

ans =

    2.4986

>> mean(abs(z).^4)

ans =

   10.7356
```

Figure 2: Complex-valued random normal variable