Lecture 5: Antenna Diversity and MIMO Capacity
Theoretical Foundations of Wireless Communications[1]

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9:30-12:00, Kansliet plan 3

[1]Textbook: D. Tse and P. Viswanath, *Fundamentals of Wireless Communication*

Overview

Lecture 1-4: Channel capacity

- Gaussian channels
- Fading Gaussian channels
- Multiuser Gaussian channels
- Multiuser diversity

Lecture 5: Antenna diversity and MIMO capacity

1. Diversity
2. Antenna/Spatial Diversity
   - Receive Diversity (SIMO)
   - Transmit Diversity (MISO), Space-Time Coding
   - 2×2 MIMO Example
3. MIMO Capacity

Diversity

Multiuser diversity (lecture 4)

- Transmissions over independent fading channels.
- Sum capacity increases with the number of users.

$\rightarrow$ High probability that at least one user will have a strong channel.

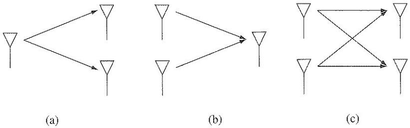
Fading channels (point-to-point links)

- Use diversity to mitigate the effect of (deep) fading.
- Diversity: let symbols pass through multiple paths.
- Time diversity: interleaving and coding, repetition coding.
- Frequency diversity: for example OFDM.
- Antenna Diversity.

Antenna/Spatial Diversity

Motivation: For narrowband channels with large coherence time or delay constraints, time diversity and frequency diversity cannot be exploited!

Figure 3.11 (a) Receive diversity; (b) transmit diversity; (c) transmit and receive diversity.



(D. Tse and P. Viswanath, *Fundamentals of Wireless Communications*.)

Antenna diversity

- Multiple transmit/receive antennas with sufficiently large spacing:
  - Mobiles: rich scattering $\rightarrow$ $1/2 \ldots 1$ carrier wavelength.
  - Base stations on high towers: tens of carrier wavelength.
- Receive diversity: multiple receive antennas,
  $\rightarrow$ single-input/multiple-output (SIMO) systems.
- Transmit diversity: multiple transmit antennas,
  $\rightarrow$ multiple-input/single-output (MISO) systems.
- Multiple transmit and receive antennas,
  $\rightarrow$ multiple-input/multiple-output (MIMO) systems.

Antenna/Spatial Diversity
– Receive Diversity (SIMO)

- Channel model: flat fading channel, 1 transmit antenna, L receive antennas:

$$\begin{aligned}\mathbf{y}[m] &= \mathbf{h}[m]\cdot x[m] + \mathbf{w}[m]\\ y_l[m] &= h_l[m]\cdot x[m] + w_l[m], \quad l = 1,\ldots,L\end{aligned}$$

  with
  - additive noise $w_l[m] \sim \mathcal{CN}(0, N_0)$, independent across antennas,
  - Rayleigh fading coefficients $h_l[m]$.

- Optimal diversity combining: maximum-ratio combining (MRC)

$$r[m] = \mathbf{h}[m]^* \cdot \mathbf{y}[m] = \|\mathbf{h}[m]\|^2 \cdot x[m] + \mathbf{h}^*[m]\mathbf{w}[m]$$

- Error probability for BPSK (conditioned on $\mathbf{h}$)

$$\Pr(x[m] \neq \mathrm{sign}(r[m])) = Q(\sqrt{2\|\mathbf{h}\|^2\mathrm{SNR}})$$

  with the (instantaneous) SNR

$$\gamma = \|\mathbf{h}\|^2\mathrm{SNR} = \|\mathbf{h}\|^2\mathrm{E}\{|x|^2\}/N_0 = L\mathrm{SNR}\cdot\frac{1}{L}\|\mathbf{h}\|^2$$

  $\rightarrow$ Diversity gain due to $\frac{1}{L}\|\mathbf{h}\|^2$ and power/array gain LSNR.

$\rightarrow$ 3 dB gain by doubling the number of antennas.

Antenna/Spatial Diversity
– Transmit Diversity (MISO), Space-Time Coding

Channel model

Flat fading channel, L transmit antennas, 1 receive antenna:

$$y[m] \quad = \quad \mathbf{h}^T[m] \cdot \mathbf{x}[m] + w[m], \quad \text{with}$$

- additive noise $w[m] \sim \mathcal{CN}(0, N_0)$,
- vector $\mathbf{h}[m]$ of Rayleigh fading coefficients $h_l[m]$.

Alamouti scheme

- Rate-1 space-time block code (STBC) for transmitting two data symbols u_1, u_2 over two symbol times with $L = 2$ transmit antennas.
- Transmitted symbols: $\mathbf{x}[1] = [u_1, u_2]^T$ and $\mathbf{x}[2] = [-u_2^*, u_1^*]^T$.
- Channel observations at the receiver (with channel coefficients h_1, h_2):

$$[y[1], y[2]] = [h_1, h_2] \begin{bmatrix} u_1 & -u_2^* \\ u_2 & u_1^* \end{bmatrix} + [w[1], w[2]].$$

Antenna/Spatial Diversity
– Transmit Diversity (MISO), Space-Time Coding

- Alternative formulation

$$\underbrace{\begin{bmatrix} y[1] \\ y[2]^* \end{bmatrix}}_{=\mathbf{y}} = \begin{bmatrix} h_1 & h_2 \\ h_2^* & -h_1^* \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} + \begin{bmatrix} w[1] \\ w[2]^* \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} h_1 \\ h_2^* \end{bmatrix}}_{=\mathbf{v}_1} u_1 + \underbrace{\begin{bmatrix} h_2 \\ -h_1^* \end{bmatrix}}_{=\mathbf{v}_2} u_2 + \begin{bmatrix} w[1] \\ w[2]^* \end{bmatrix}$$

$\rightarrow$ $\mathbf{v}_1$ and $\mathbf{v_2}$ are orthogonal; i.e., the AS spreads the information onto two dimensions of the received signal space.

- Matched-filter receiver[2]: correlate with $\mathbf{v}_1$ and $\mathbf{v_2}$

$$r_i = \mathbf{v_i}^H \mathbf{y} = \|\mathbf{h}\|^2 u_i + \tilde{w}_i, \quad \text{for } i = 1, 2,$$

with independent $\tilde{w}_i \sim \mathcal{CN}(0, \|\mathbf{h}\|^2 N_0)$.

- SNR (under power constraint $\mathrm{E}\{\|\mathbf{x}\|^2\} = P_0$):

$$\mathrm{SNR} = \frac{\|\mathbf{h}\|^2}{2} \frac{P_0}{N_0} \quad \rightarrow \quad \text{diversity gain of 2!}$$

[2] The textbook uses a projection on the orthonormal basis $\mathbf{v}_1/\|\mathbf{v}_1\|, \mathbf{v}_2/\|\mathbf{v}_2\|$.

Antenna/Spatial Diversity
– Transmit Diversity (MISO), Space-Time Coding

Determinant criterion for space-time code design

- Model: codewords of a space-time code with L transmit antennas and N time slots: $\mathbf{X}_i$, $(L \times N)$ matrix.

$$\mathbf{y}^T = \mathbf{h}^* \mathbf{X}_i + \mathbf{w}^T \quad \text{with} \quad \begin{cases} \mathbf{y}^T & = & [\ y[1], \ldots, y[N]\ ], \\ \mathbf{h}^* & = & [\ h_1, \ldots, h_L\ ], \\ \mathbf{w}^T & = & [\ w_1, \ldots, w_L\ ]. \end{cases}$$

Example: Alamouti scheme: Repetition coding:

$$\mathbf{X}_i = \begin{bmatrix} u_1 & -u_2^* \\ u_2 & u_1^* \end{bmatrix} \qquad \mathbf{X}_i = \begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}$$

- Pairwise error probability of confusing $\mathbf{X}_A$ with $\mathbf{X}_B$ given $\mathbf{h}$

$$\begin{aligned} \Pr(\mathbf{X}_A \to \mathbf{X}_B | \mathbf{h}) &= Q\left(\sqrt{\frac{\|\mathbf{h}^*(\mathbf{X}_A - \mathbf{X}_B)\|^2}{2N_0}}\right) \\ &= Q\left(\sqrt{\frac{\mathrm{SNR}\,\mathbf{h}^*(\mathbf{X}_A - \mathbf{X}_B)(\mathbf{X}_A - \mathbf{X}_B)^*\mathbf{h}}{2}}\right) \end{aligned}$$

(Normalization: unit energy per symbol $\to$ SNR $= 1/N_0$)

Antenna/Spatial Diversity
– Transmit Diversity (MISO), Space-Time Coding

- Average pairwise error probability

$$\Pr(\mathbf{X}_A \to \mathbf{X}_B) = \mathrm{E}\{\Pr(\mathbf{X}_A \to \mathbf{X}_B | \mathbf{h})\}$$

- Some useful facts...
  - $(\mathbf{X}_A - \mathbf{X}_B)(\mathbf{X}_A - \mathbf{X}_B)^*$ is Hermitian (i.e., $\mathbf{Z}^* = \mathbf{Z}$).
  - $(\mathbf{X}_A - \mathbf{X}_B)(\mathbf{X}_A - \mathbf{X}_B)^*$ can be diagonalized by an unitary transform,

$$(\mathbf{X}_A - \mathbf{X}_B)(\mathbf{X}_A - \mathbf{X}_B)^* = \mathbf{U}\boldsymbol{\Lambda}\mathbf{U}^*,$$

    where $\mathbf{U}$ is unitary (i.e., $\mathbf{U}^*\mathbf{U} = \mathbf{U}\mathbf{U}^* = \mathbf{I}$) and $\boldsymbol{\Lambda} = \mathrm{diag}\{\lambda_1^2, \ldots, \lambda_L^2\}$, with the singular values λ_l of $\mathbf{X}_A - \mathbf{X}_B$.

- And we get (with $\tilde{\mathbf{h}} = \mathbf{U}^*\mathbf{h}$)

$$\begin{aligned}\Pr(\mathbf{X}_A \to \mathbf{X}_B) &= \mathrm{E}\left\{ Q\left(\sqrt{\frac{\mathrm{SNR} \sum_{l=1}^{L} |\tilde{h}_l|^2 \lambda_l^2}{2}} \right) \right\}, \\ &\le \prod_{l=1}^{L} \frac{1}{1 + \mathrm{SNR}\, \lambda_l^2/4}\end{aligned}$$

Antenna/Spatial Diversity
– Transmit Diversity (MISO), Space-Time Coding

- If all $\lambda_l^2 > 0$ (only possible if $N \geq L$), we get

$$
\begin{aligned}
\Pr(\mathbf{X}_A \to \mathbf{X}_B) &\leq \prod_{l=1}^{L} \frac{1}{1 + \mathrm{SNR}\,\lambda_l^2/4} \leq \frac{4^L}{\mathrm{SNR}^L \prod_{l=1}^{L} \lambda_l^2} \\
&= \frac{1}{\mathrm{SNR}^L} \cdot \frac{4^L}{\det[(\mathbf{X}_A - \mathbf{X}_B)(\mathbf{X}_A - \mathbf{X}_B)^*]}
\end{aligned}
$$

→ Diversity gain of L is achieved.

→ Coding gain is determined by the determinant

$$
\det[(\mathbf{X}_A - \mathbf{X}_B)(\mathbf{X}_A - \mathbf{X}_B)^*] \quad \text{(determinant criterion)}.
$$

Antenna/Spatial Diversity
– 2 × 2 MIMO Example

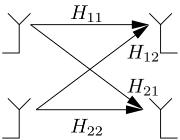
Channel Model

- 2 transmit antennas, 2 receive antennas:

$$\underbrace{\begin{bmatrix} y_1 \\ y_2 \end{bmatrix}}_{\mathbf{y}} = \underbrace{\begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}}_{\mathbf{H}} \cdot \underbrace{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}}_{\mathbf{x}} + \underbrace{\begin{bmatrix} w_1 \\ w_2 \end{bmatrix}}_{\mathbf{w}}$$

- Rayleigh distributed channel gains h_{ij} from transmit antenna j to receive antenna i.
- Additive white complex Gaussian noise $w_i \sim \mathcal{CN}(0, N_0)$.

→ 4 independently faded signal paths, maximum diversity gain of 4.



$$\boldsymbol{H} = \begin{bmatrix} H_{11} & H_{12} \\ H_{21} & H_{22} \end{bmatrix}$$

Antenna/Spatial Diversity
– 2 × 2 MIMO Example

Degrees of freedom

- Number of dimensions of the received signal space.
- MISO: one degree of freedom for every symbol time.
  - $\rightarrow$ Repetition coding ($L = 2$): 1 dimension over 2 time slots.
  - $\rightarrow$ Alamouti scheme ($L = 2$): 2 dimension over 2 time slots.
- SIMO: one degree of freedom for every symbol time.
  $\rightarrow$ Only one vector is used to transmit the data,

$$\mathbf{y} = \mathbf{h}x + \mathbf{w}.$$

- MIMO: potentially two degrees of freedom for every symbol time.
  $\rightarrow$ Two degrees of freedom if $\mathbf{h}_1$ and $\mathbf{h}_2$ are linearly independent.

$$\mathbf{y} = \mathbf{h}_1 x_1 + \mathbf{h}_2 x_2 + \mathbf{w}.$$

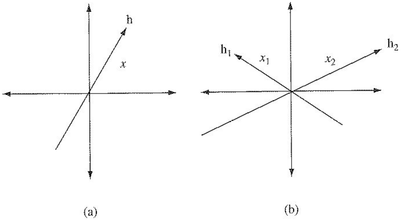


Figure 3.12 (a) In the 1 × 2 channel, the signal space is one-dimensional, spanned by **h**. (b) In the 2 × 2 channel, the signal space is two-dimensional, spanned by $\mathbf{h}_1$ and $\mathbf{h}_2$.

(D. Tse and P. Viswanath, *Fundamentals of Wireless Communications*.)

Antenna/Spatial Diversity
– 2×2 MIMO Example

Spatial multiplexing

- Motivation: Neither repetition coding nor the Alamouti scheme utilize all degrees of freedom of the channel.
- Spatial multiplexing (V-BLAST) utilizes all degrees of freedom.
  $\rightarrow$ Transmit independent uncoded symbols over the different antennas and the different symbol times.
- Pairwise error probability for transmit vectors $\mathbf{x}_1$, $\mathbf{x}_2$

$$\Pr(\mathbf{x}_1 \rightarrow \mathbf{x}_2) \leq \left[\frac{1}{1 + \mathsf{SNR}\,\|\mathbf{x}_1 - \mathbf{x}_2\|^2/4}\right]^2 \leq \frac{16}{\mathsf{SNR}^2\|\mathbf{x}_1 - \mathbf{x}_2\|^4}$$

$\rightarrow$ Diversity gain of 2 (not 4) but higher coding gain as compared to the Alamouti scheme (see example in the book).

$\rightarrow$ Spatial multiplexing is more efficient in exploiting the degrees of freedom.

- Optimal detector, joint ML detection: complexity grows exponentially with the number of antennas.
- Linear detection, e.g., decorrelator (zero forcing): $\tilde{\mathbf{y}} = \mathbf{H}^{-1}\mathbf{y}$

MIMO Capacity

- MIMO channel model with n_t transmit and n_r receive antennas:

$$\mathbf{y} = \mathbf{H}\mathbf{x} + \mathbf{w}, \quad \text{with} \quad \mathbf{w} \sim \mathcal{CN}(\mathbf{0}, N_0\mathbf{I}).$$

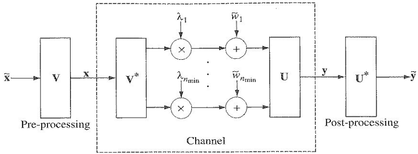
  - $\mathbf{x} \in \mathcal{C}^{n_t}$, $\mathbf{y} \in \mathcal{C}^{n_r}$, and $\mathbf{H} \in \mathcal{C}^{n_r \times n_t}$.
  - Channel matrix $\mathbf{H}$ is known at the transmitter and receiver.
  - Power constraint $\mathrm{E}\{\|\mathbf{x}\|^2\} = P$.

- Singular value decomposition (SVD): $\mathbf{H} = \mathbf{U}\mathbf{\Lambda}\mathbf{V}^*$, where
  - $\mathbf{U} \in \mathcal{C}^{n_r \times n_r}$ and $\mathbf{V} \in \mathcal{C}^{n_t \times n_t}$ are unitary matrices;
  - $\mathbf{\Lambda} \in \mathcal{R}^{n_r \times n_t}$ is a matrix with diagonal elements $\lambda_1, \dots, \lambda_{n_\text{min}}$ and off-diagonal elements equal to zero;
  - $\lambda_1, \dots, \lambda_{n_\text{min}}$, with $n_\text{min} = \min\{n_r, n_t\}$ are the ordered singular values of the matrix $\mathbf{H}$;
  - $\lambda_1^2, \dots, \lambda_{n_\text{min}}^2$ are the eigenvalues of $\mathbf{H}\mathbf{H}^*$ and $\mathbf{H}^*\mathbf{H}$.
  - Alternative formulation: $\mathbf{H} = \sum_{i=1}^{n_\text{min}} \lambda_i \mathbf{u}_i \mathbf{v}_i^*$.

    $\rightarrow$ Sum of rank-1 matrices $\lambda_i \mathbf{u}_i \mathbf{v}_i^*$.
    $\rightarrow$ $\mathbf{H}$ has rank n_min.

MIMO Capacity

Figure 7.1 Converting the MIMO channel into a parallel channel through the SVD.



(D. Tse and P. Viswanath, *Fundamentals of Wireless Communications*.)

- SVD can be used to decompose the MIMO channel into $n_{\min}$ parallel SISO channels.

$$\left\{ \begin{array}{ccc} \tilde{\mathbf{x}} & = & \mathbf{V}^*\mathbf{x}, \\ \tilde{\mathbf{y}} & = & \mathbf{U}^*\mathbf{y}, \\ \tilde{\mathbf{w}} & = & \mathbf{U}^*\mathbf{w} \end{array} \right\} \quad \Rightarrow \quad \tilde{\mathbf{y}} = \boldsymbol{\Lambda}\tilde{\mathbf{x}} + \tilde{\mathbf{w}}$$

  with $\tilde{\mathbf{w}} \sim \mathcal{CN}(0, N_0 \mathbf{I}_{n_r})$ and $\|\tilde{\mathbf{x}}\|^2 = \|\mathbf{x}\|^2$; i.e., the energy is preserved.

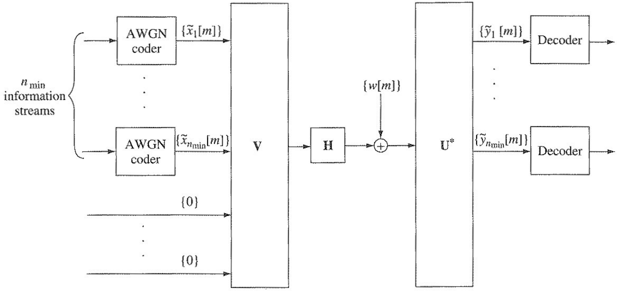
- MIMO capacity (with waterfilling)

$$C = \sum_{i=1}^{n_{\min}} \log\left(1 + \frac{P_i^* \lambda_i^2}{N_0}\right) \quad \text{with} \quad P_i^* = \left[\mu - \frac{N_0}{\lambda_i^2}\right]^+$$

  with μ chosen to satisfy the total power constraint $\sum P_i^* = P$.

MIMO Capacity

SVD architecture for MIMO communications



(D. Tse and P. Viswanath, *Fundamentals of Wireless Communications*.)

MIMO Capacity

Capacity at high SNR

- Uniform power allocation is asymptotically optimal; i.e., $P_i = P/k$.

$$C \approx \sum_{i=1}^{k} \log\left(1 + \frac{P\lambda_i^2}{kN_0}\right) \approx k \log \mathrm{SNR} + \sum_{i=1}^{k} \log\left(\frac{\lambda_i^2}{k}\right)$$

$\rightarrow$ k spatial degrees of freedom; if $\mathbf{H}$ has full rank $k = n_{\min}$.

- With Jensen's inequality

$$C \approx k \cdot \frac{1}{k} \sum_{i=1}^{k} \log\left(1 + \frac{P}{kN_0}\lambda_i^2\right) \leq k \log\left(1 + \frac{P}{kN_0}\left(\frac{1}{k}\sum_{i=1}^{k}\lambda_i^2\right)\right)$$

$\rightarrow$ Maximum capacity in high SNR if all singular values are equal.

- Condition number: $\max_i \lambda_i / \min_i \lambda_i$, $\mathbf{H}$ is well conditioned if CN≈ 1.

Capacity at low SNR

- Allocate power only to the strongest eigenmode

$$C \approx \frac{P}{N_0}(\max_i \lambda_i^2) \log_2 e$$