TTK4190 Guidance and Control of Vehicles

Assignment 2

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Problem 1 Open loop analysis

a) Ground speed

Since the wind speed $\mathbf{V}_w$ is assumed zero, we have that the ground speed is

$$\mathbf{V}_g = \mathbf{V}_a - \mathbf{V}_w = \mathbf{V}_a. \tag{1}$$

Since the airspeed is $V_a = 580\text{km/h}$, the ground speed is also $V_g = 580\text{km/h} \approx 161\text{m/s}$.

b) Sideslip

The definition of crab angle from [1] is

$$\chi_c \triangleq \chi - \psi \tag{2}$$

where χ is the course and ψ is the heading. Since the wind speed is assumed zero, the sideslip $\beta = \chi_c$ and so

$$\beta = \chi - \psi. \tag{3}$$

We can also express this using the velocity of the aircraft. From equation (2.8) in [1] we have that

$$\beta = \sin^{-1}\left(\frac{v_r}{\sqrt{u_r^2 + v_r^2 + w_r^2}}\right) = \sin^{-1}\left(\frac{v_r}{V_a}\right), \tag{4}$$

or since $\mathbf{V}_g = \mathbf{V}_a$,

$$\beta = \sin^{-1}\left(\frac{v_r}{V_g}\right). \tag{5}$$

c) Dutch Roll Mode

From [1], we have that the characteristic function of the dutch-roll mode is

$$s^2 + (-Y_v - N_r)s + (Y_v N_r - N_v Y_r) = 0. \tag{6}$$

With that, we have that the natural frequency ω_0 and relative damping ratio ζ, as in

$$s^2 + 2\zeta\omega_0 s + \omega_0^2 = 0, \tag{7}$$

are respectively

$$\omega_0 = \sqrt{Y_v N_r - N_v Y_r} \quad \text{and} \quad \zeta = -2\frac{Y_v + N_r}{\omega_0}. \tag{8}$$

Following the same derivation as in [1] using the numerical values from the $\mathbf{A}$ matrix, we can find the expresion

$$\begin{bmatrix}\dot{\beta} \\ \dot{r}\end{bmatrix} = \begin{bmatrix}-0.322 & -1.12 \\ 6.87 & -0.32\end{bmatrix}\begin{bmatrix}\beta \\ r\end{bmatrix} \tag{9}$$

which has the eigenvalues $\lambda_1 = -0.321 + 2.77i$ and $\lambda_2 = -0.321, -2.77i$. This yields the natural frequency and relative damping ratio

$$\omega_0 = \sqrt{\lambda_1\lambda_2} \approx 2.79, \quad \text{and} \quad \zeta = -\frac{\lambda_1 + \lambda_2}{2} \approx 0.321. \tag{10}$$

During a Dutch roll, both yaw and roll oscillate like a mass spring damper but the oscillations are not necessarily in phase. Increasing the damping ζ would cause the Dutch roll mode to die off more quickly.

d) Spiral Divergence Mode

In a spiral divergence mode, we let $\dot{p} = p = 0$ and assume the rudder action to be negligable. With that, we get

$$0 = -10.6\beta + 0.46r - 0.65\delta_a \tag{11}$$

$$\dot{r} = 6.87\beta - 0.32r - 0.02\delta_a. \tag{12}$$

We combine this into

$$\dot{r} = \left(6.87\frac{0.46}{10.6} - 0.32\right) r + \left(-0.87\frac{0.65}{10.6} - 0.02\right) \delta_a, \tag{13}$$

which in the Laplace domain becomes

$$r = \frac{-6.87\frac{0.65}{10.6} - 0.02}{s - (6.87\frac{0.46}{10.6} - 0.32)}\delta_a \approx -\frac{0.4}{s + 0.022}\delta_a \tag{14}$$

meaning that the spiral-mode pole is in $\lambda = -0.022$ and hence the mode is actually stable! It should be noted however that this is only by a slight margin. Modifying the airplane ever so slightly can therefore make the spiral mode unstable. In addition, the mode is very slow. [2] seems to argue that the spiral divergence mode appears in aircraft with strong directional (longitudal) stability but weak lateral stability. As such, this might be the case for this specific aircraft.

e) Roll Mode

The dynamics of p are

$$\dot{p} = -10.6\beta - 2.87p + 0.46r - 0.65\delta_a. \tag{15}$$

During the roll mode, we assume $\beta = r = 0$, which yields

$$\dot{p} = -2.87p - 0.65\delta_a. \tag{16}$$

This in turn yields the Laplace-domain dynamics

$$p = -\frac{0.65}{s + 2.87}\delta_a. \tag{17}$$

As such, the pole in this case is in $\lambda = -2.87$ i.e. the roll mode is also stable, and it is *much* faster than the spiral divergence mode. It makes sense for this mode to be faster as the roll dynamics for most airplanes are in general faster than the yaw dynamics (the yaw dynamics are what causes the spiral divergence mode).

Problem 2 Autopilot for course hold using aileron and successive loop closure

a)

The transfer function relates δ_a and p through an integrator. Therefore, we extract $\dot{p}$ from the state-space vector.

$$\begin{aligned}
\dot{p} &= -10.6\beta - 2.87p + 0.46r - 0.65\delta_a \\
(s + 2.87)p &= -0.65\delta_a + \frac{(-10.6\beta + 0.46r)\delta_a}{\delta_a} \\
\frac{p}{\delta_a} &= \frac{-0.65}{s + 2.87} + \frac{-10.6\beta + 0.46r}{(s + 2.87)\delta_a}
\end{aligned}$$

Then, as this controller is concerned with the roll, we may assume:

$$\beta \approx 0 \qquad\qquad r \approx 0$$

And may conclude:

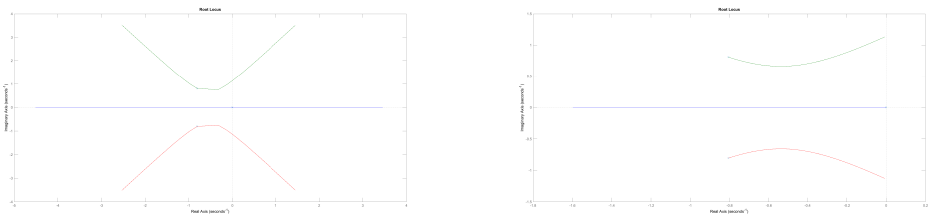
$$a_{\phi_1} = 2.87 \qquad\qquad a_{\phi_2} = -0.65$$

b)

Using equations 6.7, 6.8 and 6.9 from [1, page 100], we get:

$$\begin{aligned}
k_{p_\phi} &= \frac{\delta_a^{\max}}{e_\phi^{\max}} sign(a_{\phi_2}) = -\frac{30^\circ}{15^\circ} sign(-0.65) = -2 \\
\omega_{n_\phi} &= \sqrt{|a_{\phi_2}|\frac{\delta_a^{\max}}{e_\phi^{\max}}} = \sqrt{0.65 * \frac{30^\circ}{15^\circ}} \approx 1.140 \\
k_{d_\phi} &= \frac{2\zeta_\phi \omega_{n_\phi} - a_{\phi_1}}{a_{\phi_2}} = \frac{2 * 0.707 * 1.140 - 2.87}{-0.65} \approx 1.935
\end{aligned}$$

Then using the Evans form from [1, page 102] and the *rlocus* function from Matlab, we get find Figure 1a and Figure 1b.



(a) Root locus with range k_{i_ϕ} from -100 to 100 (b) Root locus with range k_{i_ϕ} from -π to 0

From this figure, we can conclude that the range seems to be slightly smaller than $-\pi$, but not by much, so that the range seems to be around $[-3.2, 0]$, but we are using $-\pi$ as it's a more interesting number.

Further, choosing a lower ω_{n_χ} by using $W_\chi = 10$ we may set $\zeta_\chi = 2$ a little higher without issue.

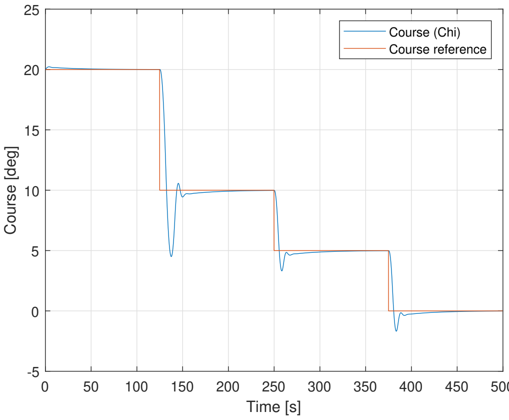
Then we can find:

$$
\begin{aligned}
\omega_{n_\chi} &= \frac{1}{W_\chi}\omega_{n_\phi} = \frac{1}{10}1.140 \approx 0.114 \\
k_{p_\chi} &= \frac{2\zeta_\chi \omega_{n_\chi} V_g}{g} = \frac{2 * 2 * 0.114 * 580}{9.81 * 3.6} \approx 7.49 \\
k_{i_\chi} &= \frac{\omega_{n_\chi}^2 V_g}{g} = \frac{0.114^2 * 580}{9.81 * 3.6} \approx 0.214
\end{aligned}
$$

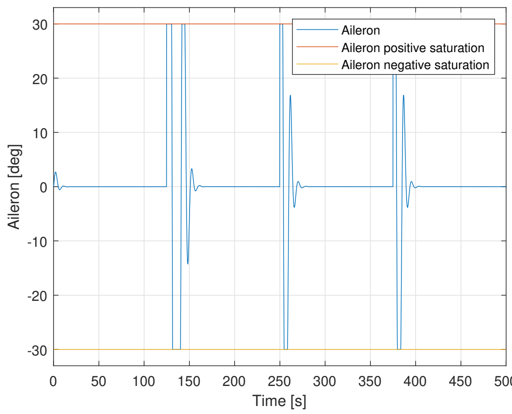
c)

There doesn't seem like there's a need for an integrator for this model. The integrator is usually needed for removing a disturbance that enters before δ_a in Figure 1 of the assignment task. As there is no disturbance here, the only reason we'd need an integral term would be to stabilize the system, but we can see from the root-locus analysis that the system is stable for $k_{i_\phi} = 0$. Therefore, we may choose this value.

d)



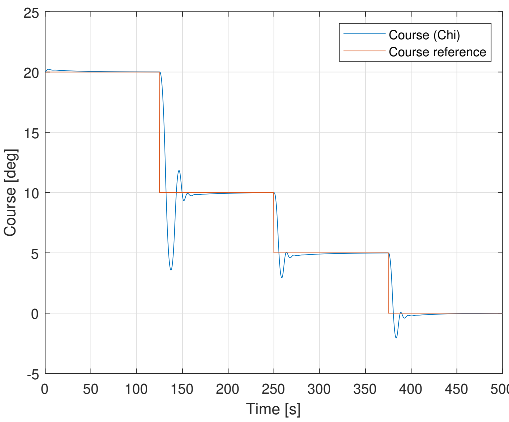
(a) The course and course reference χ



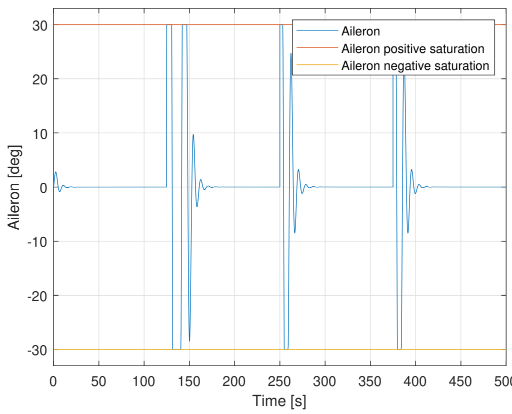
(b) The aileron with saturation δ_a

The results seem good and well controlled, and we can note that the integral term for ϕ was unnecessary, as discussed earlier.

e)



(a) The course and course reference χ



(b) The aileron with saturation δ_a

As we can see when comparing the figures, especially the courses Figure 2a and Figure 3a, the simplified model reproduces the true dynamics very well. The system is still stable, and the values still converge quite quickly towards the reference points.

f)

Integrator windup doesn't seem like a big problem in these simulations, though we can see the overshoot in Figure 3a and Figure 2a. This is due to the integrator windup. This is caused by a large error over time causing the input to saturate.

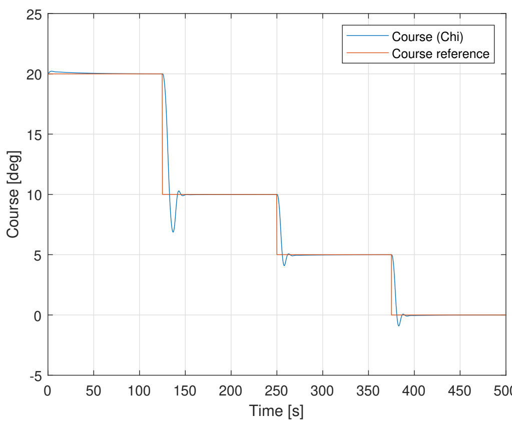
One method for handling this could be to turn the integrator off while the input is saturated. Thereby the integrator will not integrate up any error due to saturation. Another method could be to have a certain cutoff value for the integrator. Using this we may reset (set to zero) any integrator which are too large.

Another method still would be to calculate both the saturated and unsaturated inputs, and then incrementing the integrator by a proportion of the input difference:

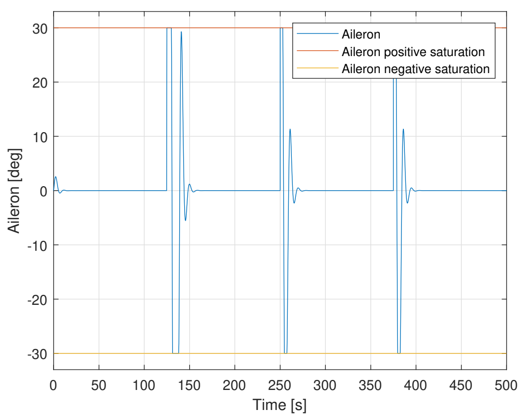
$$\Delta I = \frac{1}{k_i}(u - u_{\text{unsat}})$$

This will end up subtracting the exact amount needed so that the input can be kept at the saturation point, rather than keep increasing the longer the input is saturated.

Implementing the first method (not increasing the integrator while saturated), we can see the results in Figure 4a and Figure 4b. This is not exactly the results we'd expect, as the integrator still seems to jump whenever there's a saturation. We've further tested the other methods, but we still get a slight *dip* in Figure 4a.



(a) The course and course reference χ



(b) The aileron with saturation δ_a

References

[1] R. W. Beard and T. W. McLain, *Small unmanned aircraft: theory and practice.* Princeton University Press, 2012.

[2] Sp-367 introduction to the aerodynamics of flight. [Online]. Available: https://web.archive.org/web/20190714122028/https://history.nasa.gov/SP-367/chapt9.htm