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For questions during the exam:
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Exam in TTK4130 Modeling and Simulation

Thursday, May 24 2018
09:00 – 13:00

Permitted aids (code A): All written and handwritten examination support materials are permitted.

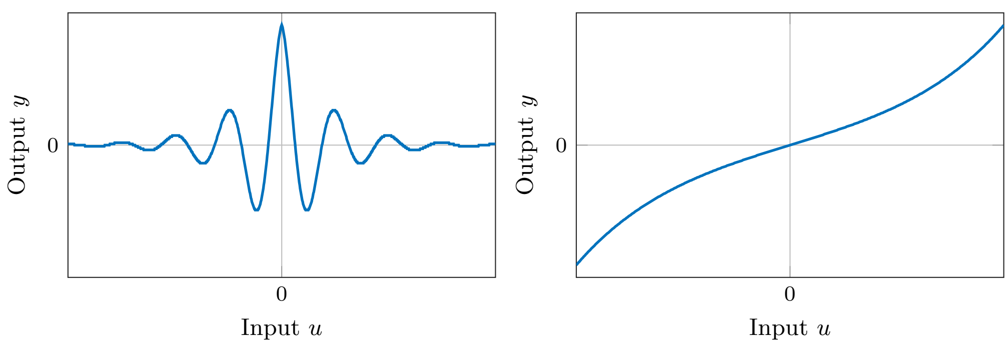
Answers in English, Norwegian, or a mixture of the two are accepted.

Grades available: As specified by regulations.

Problem 1 (16 %)

A memoryless system defines a simple system, that cannot store energy and where the output signal at each time depends only on the input at that time. Consequently, it cannot remember previous values of the input in order to determine the current value of the output. An example of such system is a electrical circuit with only an ideal resistor, where the relationship between current and voltage is given by Ohm's law.

(4 %) (a) In Figure 1 the relationship between input u and output y of two different memoryless systems are shown. Please explain briefly, if the systems are passive or not.



(a) System 1.

(b) System 2.

Figure 1: Relationship between input u and output y of system 1 and system 2.

Solution: A memoryless system is said to be passive if

$$\int_0^T y(t)u(t) \geq 0, \tag{1}$$

since it cannot store energy. Expecting the two systems it can easily be seen that system 1 is not passive, while system 2 is passive. The product $y \cdot u$ of system 2 is always positive. Therefore, the time integral (1) will be positive. This is not the case for system 1. Therefore, it is not passive.

(4 %) (b) The model *ModSim* was implemented previously in Modelica. Now you would like to create a new model *ModSimExpert* that adds the following equation to the model *ModSim*

$$\frac{dp}{dt} = \frac{c^2}{V}(\omega_1 - \omega_2).$$

The variables ω_1 and ω_2 and the parameter V were already defined in the model *ModSim*. Write the Modelica code of the new model *ModSimExpert*. You can choose yourself values for possibly needed parameter and initial values.
Hint: You do not have to implement the model ModSim to solve the task.

Solution:

```
model ModSimExpert

  extends ModSim;

  parameter Real c = 100;
  Real p( start=0);

equation
  der(p) = c^2/V *(omega_1 - omega_2);

  a
end ModSimExpert;
```

(8 %) (c) The following Modelica code has unfortunately several errors. Find the errors in the implementation of the model and explain briefly what is wrong

```
model PredatorPrey
 type birthrate = Real (unit="years-1");
 type deathrate = Real (unit="years-1");
 parameter deathrate b2 = 0.5 "deathrate [years-1]";
 parameter deathrate c = 0.3 "deathrate [years-1]";
 parameter birthrate b1 = 2.0;
 parameter Real p "PredatorInfluence [year-1.predator-1]";
 parameter Real r = 0.004 "PreyInfluence [year-1.predator-1]"

 Real S(initial=2000) "PreyPopulation";
 Real L(initial=80) "PredatorPopulation";

 d_dt(S) = (b1-b2-p*L); // Change of PreyPopulation
 d_dt(L) = (r*S-c)*L;
 0 = S - b1*L;
end PredatorPray;
```

Solution: Errors in the PredatorPrey model

- The equation system is over-determined since it has three equation but only two variables (2%).
- The parameter p has no parameter value.
- Semicolon missing at the end of the line that declares parameter r.
- The keyword "equation" is missing at the beginning of the equation section.

- "$d\_dt(m)$" has to be "$der(m)$".

- The model name is PredatorPrey (it is wrong in the last line).
- The initial values of variables of the keyword "start" and not "initial".

Problem 2 (26 %)

We have an electrical circuit with two resistors, an inductor and a capacitor (Fig. 2). The resistors follow Ohm's law ($V = IR$). The voltage source provides a voltage V_i.
Hint: $exp(x) = \lim_{n\to\infty}(1 + {}^{x}/{}_{n})^n$

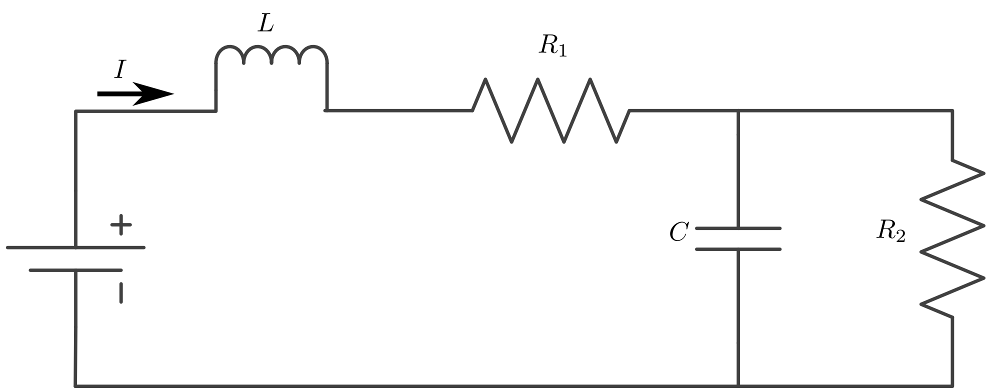


Figure 2: Electrical Circuit.

(4 %) (a) Derive an ODE model of the circuit for the current I and voltage in the capacitor V_c.

Solution: The model of the circuit can be derived by Kirchhoff's first and second law. Kirchhoff's second law results in

$$L\frac{dI}{dt} = V_i - V_c - R_1 I, \tag{2}$$

and Kirchhoff's first law result in

$$C\frac{dV_c}{dt} = I - \frac{1}{R_2}V_c. \tag{3}$$

If you were not able to find an ODE model in a) continue with the following model [which is not necessarily the correct answer to a)]:

$$\begin{aligned}\frac{dI}{dt} &= \frac{1}{R_1^2}V_i + \frac{1}{CR_2}V_c + \frac{R_1}{C}I,\\ \frac{dV_c}{dt} &= -\frac{R_1^2}{L}I + \frac{1}{L}V_c - \frac{C}{LR_2}V_i.\end{aligned}$$

(2 %) (b) Transfer your model into a state-space form.

Solution: With the state vector $\mathbf{x} = [I, V_c]^T$ and the input vector $\mathbf{u} = [V_i]$ the state space model is

$$\dot{\mathbf{x}} = \begin{pmatrix} -\frac{R_1}{L} & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{R_2 C} \end{pmatrix} \mathbf{x} + \begin{pmatrix} \frac{1}{L} \\ 0 \end{pmatrix} \mathbf{u}. \tag{5}$$

Assume the resistor has the resistance $R = R_1 = R_2 = 1000\,\Omega$, the inductor the inductance $L = 10^{-2}$ H, and the capacitor the capacitance $C = 10^{-4}$ F.

(4 %) (c) Explain why it may be problematic to use an explicit Runge-Kutta method to simulate the system.

Solution: The eigenvalues of the system can be calculated with

$$det\left[\begin{pmatrix} -\frac{R}{L} & -\frac{1}{L} \\ \frac{1}{C} & -\frac{1}{RC} \end{pmatrix} - \lambda\mathbf{I}\right] = 0. \tag{6}$$

This results in

$$\lambda_{1,2} = -\frac{1}{2}\left(\frac{R}{L} + \frac{1}{RC}\right) \pm \sqrt{\frac{1}{4}\left(\frac{R}{L}\right)^2 - \frac{3}{2}\frac{1}{LC} + \frac{1}{4}\left(\frac{1}{RC}\right)^2} \tag{7}$$

With the values for R, L and C this gives

$$\lambda_1 \approx -20.002, \qquad \lambda_2 \approx -99989.998, \tag{8}$$

which shows that the system is stiff. An explicit method needs to adjust its step-size to the largest absolute eigenvalue. Therefore, to simulate this system an explicit Runge-Kutta method needs a small step size, which may result in a slow simulation.
System from hint results in $\lambda_1 \approx 200.002, \qquad \lambda_2 \approx 9999899.998$.

(4 %) (d) Use the modified Euler method to simulate the system (Tab. 1). What is the largest step-size you can use for a stable simulation?

0		
1/2	1/2	
	0	1

Table 1: Butcher array of modified Euler method

Solution: The stability function of the modified Euler method is $R(h\lambda) = 1 + \lambda h + \frac{1}{2}\lambda^2 h^2$ (This can also be calculated with $R(h\lambda) = det\left[\mathbf{I} - \lambda h\left(\mathbf{A} - \mathbf{1}\mathbf{b}^T\right)\right]$). Stability of the method requires $|R(h\lambda)| \leq 1$. Therefore, the step-size must be $0 \leq h \leq 2/\lambda_2$. This gives a maximum step-size of $h \approx 2 \times 10^{-5}$.
System from hint results in unstable system that cannot be calculated with the explicit method

(4 %) (e) In order to choose a larger step-size the system should be simulated with the Gauss method of order four (Tab. 2). What is the maximum step-size you can choose to guarantee a stable simulation of the system if you use this method? What kind of other problem can you encounter in the simulations?

$$
\begin{array}{c|cc}
\frac{1}{2}-\frac{\sqrt{3}}{6} & \frac{1}{4} & \frac{1}{4}-\frac{\sqrt{3}}{6} \\
\frac{1}{2}+\frac{\sqrt{3}}{6} & \frac{1}{4}+\frac{\sqrt{3}}{6} & \frac{1}{4} \\
\hline
 & \frac{1}{2} & \frac{1}{2}
\end{array}.
$$

Table 2: Butcher array of Gauss method of order four.

Solution: The Gauss method of order four is A-stable (check book p.549). For an A-stable method it holds $|R(h\lambda) \leq 1|$ for all $Re\lambda \leq 0$. Our system has only negative eigenvalues, therefore, the step-size is not limited ($0 \leq h \leq \infty$). The method is only A-stable, but not L-stable. Therefore, you may observe aliasing in your simulations.
System from hint results in a unstable method with large positive eigenvalues that cannot be stabilized with the Gauss method of order four. This can be proven with the stability function (here the quickest way is to use the stability function given in the Pade-approximation table of the Gauss 4 method).

The exact solution of the differential equation

$$\frac{dI}{dt} = \frac{U}{L} - \frac{R}{L}I,$$

is given by

$$I(t) = I_0 e^{-\frac{R}{L}t} + \frac{U}{R}\left(1 - e^{-\frac{R}{L}t}\right).$$

(5 %) (f) Show that the explicit Euler method applied to the differential equation of I with the initial value $x_0 = I_0$ and the step-size h results in

$$I_n = \left(1 - h\frac{R}{L}\right)^n I_0 + \left(1 - \left(1 - h\frac{R}{L}\right)^n\right)\frac{U}{R},$$

where I_n is the approximation at time $t_n = nh$.

Solution: The differential equation is

$$\frac{dI}{dt} = \frac{U}{L} - \frac{R}{L}I. \tag{9}$$

Using the explicit Euler method

$$y_n = y_{n-1} + hf(y_{n-1}, t_{n-1}), \tag{10}$$

the first step can be written as

$$
\begin{aligned}
I_1 &= I_0 + h\left(U/L - R/L I_0\right), && (11a)\\
&= \left(1 - hR/L\right)^1 I_0 + \left(1 - \left(1 - hR/L\right)^1\right) U/R. && (11b)
\end{aligned}
$$

The second step is

$$
\begin{aligned}
I_2 &= \left(1 - hR/L\right)^1 I_1 + \left(1 - \left(1 - hR/L\right)^1\right) U/R, && (12a)\\
&= \left(1 - hR/L\right)^1 \left[\left(1 - hR/L\right)^1 I_0 + \left(1 - \left(1 - hR/L\right)^1\right) U/R\right] + \left(1 - \left(1 - hR/L\right)^1\right) U/R, && (12b)\\
&= \left(1 - hR/L\right)^2 I_0 + \left[\left(1 - hR/L\right)^1 - \left(1 - hR/L\right)^2\right] U/R + hR/L U/R, && (12c)\\
&= \left(1 - hR/L\right)^2 I_0 + \left(1 - \left(1 - hR/L\right)^2\right) U/R, && (12d)
\end{aligned}
$$

which has the same structure as (11b), therefore

$$I_n = \left(1 - h{}^{R}/{}_{L}\right)^n I_0 + \left(1 - \left(1 - h{}^{R}/{}_{L}\right)^n\right) {}^{U}/{}_{R}, \tag{13}$$

which is what we wanted to proof.
Mathematically a better way to proof this by induction is to show that n and $n+1$ have the same structure.

(3 %) (g) Can you find a guideline to choose h such that the approximation I_n at least qualitatively represent the behaviour of the exact solution $I(t_n)$?

Solution: We have to compare (e) and (f). From the hint we know that $exp(x) = \lim_{n\to\infty} \left(1 + {}^{x}/{}_{n}\right)^n$. We see from (e) that $x = -{}^{R}/{}_{L}t_n = -{}^{R}/{}_{L}nh$, and $exp(-{}^{R}/{}_{L}nh) = \lim_{n\to\infty} \left(1 - {}^{R}/{}_{L}h\right)^n$. Consequently, we have to choose the step-size $h \approx {}^{1}/{}_{n}$.

Problem 3 (14 %)
The thickness of a homogenous right triangular plate of mass m varies linearly with the perpendicular distance from the vertex at $z = h_0$ toward the base (Fig. 3). The thickness at the base of the plate is l_0, the length of the side in x-direction is b_0 and in z-direction h_0. The mass m of the triangular plate is

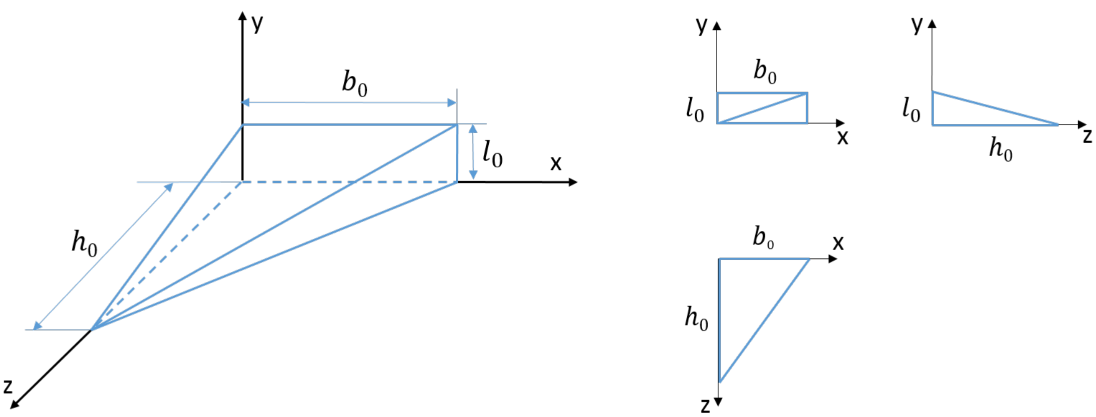


Figure 3: 3D drawing of triangular plate and projections from each side.

$$m = \frac{1}{3}\rho l_0 b_0 h_0,$$

where ρ is the density of the plate. The center of mass in x-direction is at $x_c = {}^{3}/{}_{8}b_0$ and in y-direction at $y_c = {}^{3}/{}_{8}l_0$.

(2 %) (a) Find a dependency between the thickness l of the triangular plate and the coordinate axes.

Solution: The thickness l varies only with the z-coordinate

$$l(z) = \frac{l_0}{h_0}(h_0 - z). \tag{14}$$

(2 %) (b) Find an equation that describes how the length $b(z)$ varies in z-direction.

Solution: The length b varies also only with the z-coordinate

$$b(z) = \frac{b_0}{h_0}(h_0 - z). \tag{15}$$

(6 %) (c) Find the position of the center of mass z_c along the z-axis of the plate (calculation required).

Solution: The center of mass z_c along the z-axis is given by

$$z_c = \frac{1}{m}\int z dm. \tag{16}$$

The mass element dm is

$$\begin{aligned}
dm &= \rho \cdot dV, && \text{(17a)}\\
&= \rho l(z) \cdot dA, && \text{(17b)}\\
&= \rho \frac{l_0}{h_0}(h_0 - z)b(z)dz, && \text{(17c)}\\
&= \rho \frac{l_0 b_0}{h_0^2}(h_0 - z)^2 dz, && \text{(17d)}
\end{aligned}$$

We used that the height $l(z)$ and width $b(z)$ of the plate only varies in z-direction, but are independent of the y-direction. The center of mass can be calculated solving

$$\begin{aligned}
z_c &= \frac{1}{m}\int_0^{h_0} z\rho \frac{l_0 b_0}{h_0^2}(h_0 - z)^2 dz, && \text{(18a)}\\
&= \frac{1}{m}\int_0^{h_0} z\rho \frac{l_0 b_0}{h_0^2}(h_0 - z)^2 dz, && \text{(18b)}\\
&= \frac{1}{12}\frac{1}{m}\rho l_0 b_0 h_0^2, && \text{(18c)}\\
&= \frac{1}{4}h_0, && \text{(18d)}
\end{aligned}$$

where we used in the last step our knowledge about the mass m.
Small comment: *One can also write out all three integrals, which results in the same center of mass.*

$$\begin{aligned}
z_c &= \frac{\rho}{m}\int_0^{h_0}\int_0^{b(z)}\int_0^{l(z)} z dy dx dz, && \text{(19a)}\\
&= \frac{\rho}{m}\int_0^{h_0}\int_0^{b(z)} z l(z) dx dz, && \text{(19b)}\\
&= \frac{\rho}{m}\int_0^{h_0} z l(z) b(z) dz, && \text{(19c)}\\
&= \frac{1}{4}h_0, && \text{(19d)}
\end{aligned}$$

(4 %) (d) The moment of inertia I_x with respect to the x-axis in the given coordinate system with the origin

in the vertex of the triangular plate is

$$I_{x,0} = \int z^2 dm = \frac{1}{10} m h_0^2.$$

What is the moment of inertia $I_{x,c}$ about the x-axis with respect to the center of mass? Assume that $l_0 << b_0$ and $l_0 << h_0$.

Solution: The parallel axes theorem states that the inertia I_0 through some other axes is equal to the inertia through the center of mass plus the mass multiplied with the distance of the new axes to the center of mass squared

$$I_0 = I_c + md^2, \tag{20}$$

where d is the distance of the new axes to the center of mass. The distance of the rotational axis x to the center of mass is $d = \sqrt{(1/4 h_0)^2 + (3/8 l_0)^2}$. Consequently, the moment of inertia with respect to the center of mass is

$$I_{x,c} = I_{x,0} - md^2 = \frac{1}{10} m h_0^2 - \frac{1}{16} m h_0^2 - \frac{9}{64} l_0^2 = \frac{3}{80} m h_0^2 - \frac{9}{64} l_0^2 \approx \frac{3}{80} m h_0^2. \tag{21}$$

Problem 4 (24 %)

A particle with mass m rolls frictionless down the inner surface of a circular cone (Fig. 4). The cones angle is denoted by α.

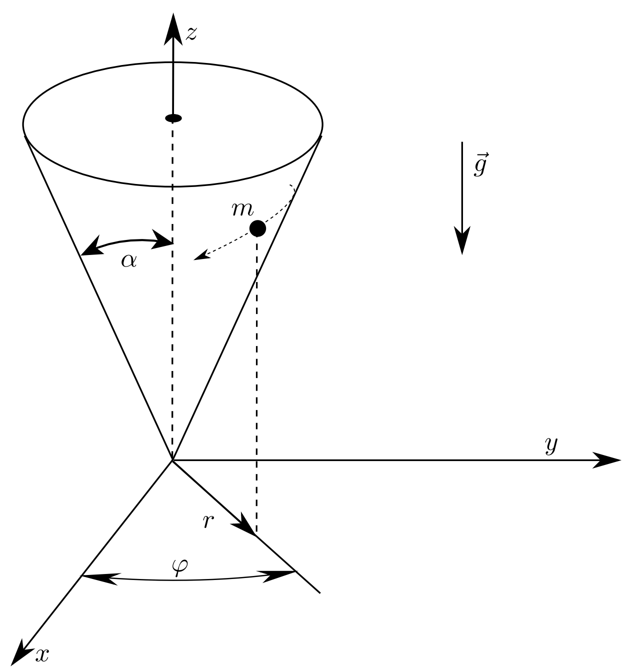


Figure 4: Circular cone with mass particle.

(4 %) (a) The system is best described using the cylindrical coordinates. Write down expressions for Cartesian coordinates (x, y, z) as function of the cylindrical coordinates. How many degrees of freedom does the system have?

Solution: The mass particle m can move in $r-$, $\varphi-$ and $z-$direction. Moreover, the geometry reduces the degree of freedom to two, since z and r are connected,

$$x = r\cos\varphi, \quad y = r\sin\varphi, \quad z = r\tan^{-1}\alpha. \tag{22}$$

(10 %) (b) Derive the equations of motion of the mass particle m in the coordinates r and φ with the Lagrange approach.

Solution: We begin with deriving the kinematic relationship. We express the position of the particle in the cylindrical coordinate frame. Therefore, we have two coordinate systems: the inertial coordinate frame and the cylindrical coordinate frame, which rotates around the z-axis with the angular velocity $\dot{\varphi}$ and has the coordinate axes

$$\vec{b}_1 = \vec{e}_r \quad \vec{b}_2 = \vec{e}_\varphi \quad \vec{b}_3 = \vec{e}_\varphi. \tag{23}$$

In order to derive the velocity of the particle we differentiate the position vector r_m

$$\vec{r}_m = r\vec{e}_r + z\vec{e}_z \tag{24}$$

of the mass particle m. The velocity vector $\vec{v}_m$ is (in book: Sec. 6.12.2)

$$\vec{v}_m = \dot{r}\vec{e}_r + \vec{\omega}_{ib} \times r\vec{e}_r + \dot{z}\vec{e}_z, \tag{25a}$$
$$= \dot{r}\vec{e}_r + r\dot{\varphi}\vec{e}_\varphi + \dot{z}\vec{e}_z. \tag{25b}$$

The kinetic energy is given by

$$\begin{aligned} T = \frac{1}{2}m\vec{v}_m\vec{v}_m &= \frac{1}{2}m(\dot{r}^2 + r^2\dot{\varphi}^2 + \dot{z}^2), \\ &= \frac{1}{2}m\left(\dot{r}^2[1+\cot^2\alpha] + r^2\dot{\varphi}^2\right). \end{aligned}$$

The potential energy of the particle is given by its height

$$U = mgz = mg\frac{r}{\tan\alpha}.$$

The Lagrange equation of motion for this system is

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = 0.$$

where L is the Lagrangian

$$L = T(\mathbf{q}, \dot{\mathbf{q}}, t) - U(\mathbf{q}) = \frac{1}{2}m\left(\dot{r}^2[1+\cot^2\alpha] + r^2\dot{\varphi}^2\right) - mgr\cot\alpha.$$

For r:

$$\frac{\partial L}{\partial r} = mr\dot{\varphi} - mg\cot\alpha, \tag{27a}$$
$$\frac{\partial L}{\partial \dot{r}} = m\dot{r}(1+\cot^2\alpha), \tag{27b}$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{r}} = m\ddot{r}(1+\cot^2\alpha), \tag{27c}$$
$$0 = m\ddot{r}(1+\cot^2\alpha) - mr\dot{\varphi} + mg\cot\alpha, \tag{27d}$$
$$0 = \ddot{r}(\tan\alpha + \cot\alpha) - r\dot{\varphi}\tan\alpha + g. \tag{27e}$$

For φ:

$$\frac{\partial L}{\partial \varphi} = 0, \tag{28a}$$

$$\frac{\partial L}{\partial \dot{\varphi}} = mr^2\dot{\varphi}, \tag{28b}$$

$$\frac{d}{dt}\frac{\partial L}{\partial \dot{\varphi}} = 2mr\dot{r}\dot{\varphi} + mr^2\ddot{\varphi}, \tag{28c}$$

$$0 = 2mr\dot{r}\dot{\varphi} + mr^2\ddot{\varphi}, \tag{28d}$$

$$0 = 2\dot{r}\dot{\varphi} + r\ddot{\varphi}. \tag{28e}$$

The equations of motion are given by (27e) and (28e).

(3 %) (c) Find the sum of forces acting on the mass particle in the cylindrical coordinate system.

Solution: Two forces acting on the mass particle, the contact force $\vec{\delta}$ (normal force) and gravitational force $\vec{G}$ (Fig. 5).

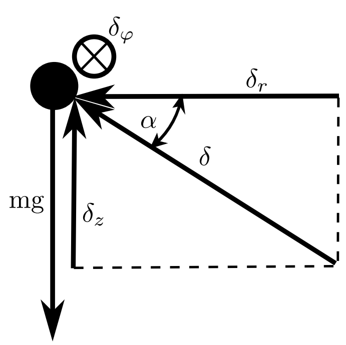


Figure 5: Forces on the mass particle m.

The sum of forces on the particle in the cylindrical coordinate system are

$$\vec{F} = -\delta_r\vec{e}_r + \delta_z\vec{e}_z - mg\vec{e}_z,$$

where $\delta_\varphi = 0$, since the movement of the particle is frictionless. Through the geometry of the circular cylinder we can find

$$\delta_z = \delta_r \tan\alpha.$$

(7 %) (d) Derive the equations of motion of the mass particle m in the coordinates r and φ with the Newton-Euler approach.
The following you may find useful

$$\vec{\omega}_{ib} \times \vec{e}_z = \vec{0}, \quad \vec{\omega}_{ib} \times \vec{e}_\varphi = -\dot{\varphi}\vec{e}_r,$$

where $\vec{\omega}_{ib}, \vec{e}_r, \vec{e}_\varphi$ and $\vec{e}_z$ are the angular velocity and the unit vectors of the coordinate axes, respectively.
Hint: It is not necessary, but it is recommended to operate in the cylindrical coordinate system.

Solution: We derive the acceleration of the particle by using the already derived velocity (25). The acceleration vector $\vec{a}_m$ is (in book: Sec. 6.12.3)

$$\begin{aligned}\vec{a}_m =& \ddot{r}\vec{e}_r + \vec{\omega}_{ib} \times \dot{r}\vec{e}_r + \dot{r}\dot{\varphi}\vec{e}_\varphi + r\ddot{\varphi}\vec{e}_\varphi + \vec{\omega}_{ib} \times r\dot{\varphi}\vec{e}_\varphi + \ddot{z}\vec{e}_z, & \text{(29a)}\\ =& \ddot{r}\vec{e}_r + \dot{r}\dot{\varphi}\vec{e}_\varphi + \dot{r}\dot{\varphi}\vec{e}_\varphi + r\ddot{\varphi}\vec{e}_\varphi - r\dot{\varphi}^2\vec{e}_r + \ddot{z}\vec{e}_z, & \text{(29b)}\\ =& (\ddot{r} - r\dot{\varphi}^2)\vec{e}_r + (2\dot{r}\dot{\varphi} + r\ddot{\varphi})\vec{e}_\varphi + \ddot{z}\vec{e}_z. & \text{(29c)}\end{aligned}$$

Newton's law for the particle gives

$$m\vec{a}_m = \sum_i^n F_i,$$

where the sum of forces were derived in c). Three equations (one for each coordinate direction) can be derived using Newton's law and the kinematic equations (29c):

$$\begin{aligned}m(\ddot{r} - r\dot{\varphi}^2) &= -\delta_r, & \text{(30a)}\\ m(r\ddot{\varphi} + 2\dot{r}\dot{\varphi}) &= 0 & \text{(30b)}\\ m\ddot{z} &= \delta_r \tan\alpha - mg. & \text{(30c)}\end{aligned}$$

(22) and (30c) give

$$m\frac{\ddot{r}}{\tan\alpha} = \delta_r \tan\alpha - mg,$$

and with (30a) we get

$$\begin{aligned}m\frac{\ddot{r}}{\tan\alpha} + m(\ddot{r} - r\dot{\varphi}^2)\tan\alpha + mg &= 0, & \text{(31a)}\\ \ddot{r}(\tan\alpha + \cot\alpha) - r\dot{\varphi}^2\tan\alpha + g &= 0. & \text{(31b)}\end{aligned}$$

From (30b) we get

$$r\ddot{\varphi} + 2\dot{r}\dot{\varphi} = 0. \tag{32}$$

(31b) and (32) are the equations of motion for the particle m.

Problem 5 (20 %)

The cascade if three tanks is used to mix liquids (Fig. 6). The cross-sectional areas A_1, A_2, and A_3 of the tanks, the resistance of the valves R_1, R_2, the density ρ are parameters and the inflow ω_0 and its derivatives $\dot{\omega}_0, \ddot{\omega}_0, \ldots$ are known inputs. The mass flow between the tanks and the outflow of tank 3 is given by ω_1, ω_2 and ω_3, respectively.

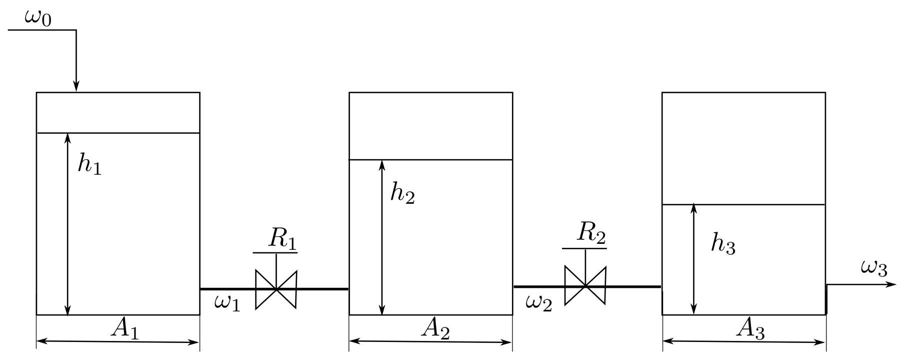


Figure 6: Mixing cascade of three tanks.

(3 %) (a) Derive ordinary differential equations for the change of the fluid height h_i for the tanks $i = 1, 2, 3$.

Solution: The mass balance of each tank is given by

$$\frac{dm_i}{dt} = \omega_{i-1} - \omega_i \tag{33a}$$

$$\rho A_i \frac{dh_i}{dt} = \omega_{i-1} - \omega_i \quad i = 1, 2, 3 \tag{33b}$$

In addition, the following equations are applied

$$\omega_1 R_1 = f_1(h_1, h_2), \tag{34a}$$

$$\omega_2 R_2 = f_2(h_2, h_3), \tag{34b}$$

$$\omega_3 = f_3(h_3), \tag{34c}$$

where f_1, f_2 and f_3 are smooth functions.

(5 %) (b) Give the vectors of differential and algebraic variables of the system. Check if the DAE-system has a differential index of one.

Solution: The differential variables are

$$\mathbf{x} = [h_1, h_2, h_3]^T, \tag{35}$$

and the algebraic variables are

$$\mathbf{y} = [\omega_1, \omega_2, \omega_3]^T. \tag{36}$$

The algebraic equations $\mathbf{g}(\mathbf{x}, \mathbf{y})$ are

$$\omega_1 R_1 - f_1(h_1, h_2) = 0, \tag{37a}$$

$$\omega_2 R_2 - f_2(h_2, h_3) = 0, \tag{37b}$$

$$\omega_3 - f_3(h_3) = 0. \tag{37c}$$

The derivative of $\mathbf{g}(\mathbf{x}, \mathbf{y})$ with respect to $\mathbf{y}$ is

$$\frac{\partial \mathbf{g}(\mathbf{x}, \mathbf{y})}{\partial \mathbf{y}} = \begin{pmatrix} R_1 & 0 & 0 \\ 0 & R_2 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \tag{38}$$

which is nonsingular. Therefore, the system has the differential index of one.

In the following assume that the tank cross-sections are identical ($A_i = A$ for $i = 1, 2, 3$) and the resistance of the valves can be neglected ($R_1 = R_2 = 0$). Consequently, (34) can be replaced with (39).

$$h_1 = h_2, \tag{39a}$$
$$h_2 = h_3, \tag{39b}$$
$$\omega_3 = A\rho\sqrt{2gh_3}, \tag{39c}$$

(7 %) (c) Determine the differential index of the new DAE-system by transferring the system into an ODE-system.

Solution: The new DAE-system is given by

$$\rho A_i \frac{dh_i}{dt} = \omega_{i-1} - \omega_i \quad i = 1, 2, 3, \tag{40a}$$
$$0 = h_1 - h_2, \tag{40b}$$
$$0 = h_2 - h_3, \tag{40c}$$
$$0 = \omega_3 - A\rho\sqrt{2gh_3}. \tag{40d}$$

The first differentiation of the algebraic equation results in

$$0 = \frac{dh_1}{dt} - \frac{dh_2}{dt}, \tag{41a}$$
$$0 = \frac{dh_2}{dt} - \frac{dh_3}{dt}, \tag{41b}$$
$$0 = \frac{d\omega_3}{dt} - A\rho\sqrt{\frac{g}{2h_3}}\frac{dh_3}{dt}. \tag{41c}$$

Use the differential equations in (41)

$$0 = \frac{\omega_0 - \omega_1}{\rho A} - \frac{\omega_1 - \omega_2}{\rho A}, \tag{42a}$$
$$0 = \frac{\omega_1 - \omega_2}{\rho A} - \frac{\omega_2 - \omega_3}{\rho A}, \tag{42b}$$
$$0 = \frac{d\omega_3}{dt} - \sqrt{\frac{g}{2h_3}}(\omega_2 - \omega_3). \tag{42c}$$

Differential equation for ω_1 and ω_2 are still needed.
First, (42a) and (42b) can be rewritten to

$$\omega_1 = \frac{2\omega_0 + \omega_3}{3}, \tag{43a}$$
$$\omega_2 = \frac{\omega_0 + 2\omega_3}{3}, \tag{43b}$$

The second differentiation will transfer the system into a ODE. The system is of Index-2.
The result of this differentiation is given below. For full score an explanation instead of the calculation is possible.
The second differentiation and the use of (42c), (43a) and (43b) gives

$$\frac{d\omega_1}{dt} = \frac{2}{3}\frac{d\omega_0}{dt} + \frac{1}{9}\sqrt{\frac{g}{2h_3}}(\omega_0 - \omega_3), \tag{44a}$$
$$\frac{d\omega_2}{dt} = \frac{1}{3}\frac{d\omega_0}{dt} + \frac{1}{9}\sqrt{\frac{2g}{h_3}}(\omega_0 - \omega_3), \tag{44b}$$

Two differentiation are necessary to transform the DAE system into an ODE system. Therefore, the differential index is of order 2.

(5 %) (d) Use your result of the previous sub-task and write down a suitable index-1 system. Check your result by examine if your new system is structurally regular.

Solution: Your system has six variable, ergo six equations are needed. Moreover, as many algebraic equations and as few differential equations as possible should be used. *(In the solution of this task any index 1 system will give full points)*. One example for an index-1 system is

$$\rho A_1 \frac{dh_1}{dt} = \omega_0 - \omega_1, \tag{45a}$$
$$0 = h_1 - h_2, \tag{45b}$$
$$0 = h_2 - h_3, \tag{45c}$$
$$0 = \omega_3 - A\rho\sqrt{2gh_3}, \tag{45d}$$
$$0 = \omega_0 - 2\omega_1 + \omega_2, \tag{45e}$$
$$0 = \omega_1 - 2\omega_2 + \omega_3. \tag{45f}$$

Equation ↓ - variable →	h_2	h_3	ω_1	ω_2	ω_3
(45b)	[×]				
(45c)	×	[×]			
(45d)		×			[×]
(45e)			×	[×]	
(45f)			[×]	×	

Table 3: Index-Matrix

The algebraic system is structurally regular. The differential equation (45a) determines h_1.