NTNU – Trondheim
Norwegian University of
Science and Technology

For questions during the exam:
Lars Imsland, tel. 47 23 19 49.

Exam in TTK4130 Modeling and Simulation

Saturday, June 9th 2012

09:00 – 13:00

Permitted aids (code A): All written and handwritten examination support materials are permitted.

Note: A Norwegian text is appended.

Answers in English, Norwegian, or a mixture of the two accepted.

Grades available: As specified by regulations.

Problem 1 (25 %)

Consider the second-order system

$$\ddot{y} + y = 0$$

with initial values $y(0) = 1$, $\dot{y}(0) = 0$. The exact solution to this initial value problem is $y(t) = \cos t$, such that the solution in the phase-plane $(y, \dot{y})$ is a circle.

(10 %) (a) Write the system as an ordinary differential equation (ODE), that is, a first-order system. What are the initial conditions? Is this system linear or nonlinear? What are the system eigenvalues?

Solution: Let $x = (y(t), \dot{y}(t))^\mathsf{T}$, then

$$\dot{x} = Ax, \quad A = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad x(0) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

This is a linear system. The eigenvalues are $\lambda_{1,2} = \pm j$.

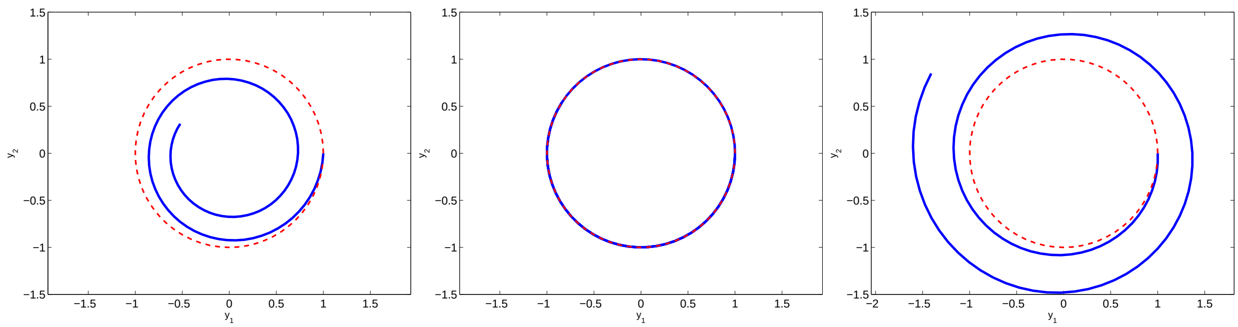
In Figure 1, we have plotted the numerical solution to this system using three Runge-Kutta methods with fixed steplength $h = 0.1\,\mathrm{s}$, namely

I) Explicit Euler,

II) Implicit Euler, and

III) Implicit midpoint rule.

The numerical solution are plotted with whole (thick) line, and the exact solution is plotted with dashed (thin) line.



(a) Method 1 (b) Method 2 (c) Method 3

Figure 1: Numerical solutions using three different methods. Dotted line is exact solution.

(10 %) (b) Which method is used to produce which figure? Explain how you reason.

Solution:

- Explicit Euler is unstable for all purely imaginary eigenvalues, thus Method 3 must be Explicit Euler.
- Implicit Euler dampens out high frequencies (is L-stable), thus Method 1 must be Implicit Euler.
- Implicit Midpoint Rule has stability function that fulfills $|R(j\omega)| = 1$, thus does not dampen out any frequencies (but may suffer from aliasing, see next question).

This can also be deduced by putting the eigenvalues into the stability function for each method.

(5 %) (c) Which of the methods may suffer from aliasing? Is there aliasing in this case? Explain.

Solution: All methods may suffer from aliasing, but the "correct" answer in this case is the implicit midpoint rule. (An answer explaining why all methods suffer will also get full score).

The implicit midpoint rule retains all frequencies, and will therefore potentially suffer from aliasing. The "Nyquist frequency" for the choice $h = 0.1$ is $\omega = \pi/h \approx 30$, which is much larger than the system eigenvalues, and therefore there will be no aliasing in this case.

Implicit Euler dampens out high frequencies, and will therefore not suffer from aliasing if h is chosen correctly (and not in this case, for the same reason as above).

Explicit Euler could also be said to alias high frequencies, but that is not relevant here since it is unstable.

Problem 2 (25 %)

In this problem we will derive the equations of motion for a satellite in an orbit around a planet. We will assume that the mass of the planet, M, is so much larger than the mass of the satellite, m, that we can place an inertial reference frame in the center of the planet (the satellite does not affect the planet motion significantly). Let the position of the satellite in the inertial frame be denoted $\vec{r}$. For simplicity, we assume all motion happen in a plane (we view things in two dimensions). Treat both the planet and satellite as point masses.

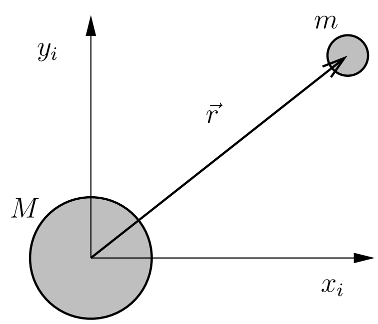


Figure 2: Satellite in orbit around a planet

(10 %) (a) We will first derive the equations of motion for the satellite using Lagrangian dynamics. Let the coordinate description of $\vec{r}$ be

$$\mathbf{r} = \begin{pmatrix} r\cos\phi \\ r\sin\phi \end{pmatrix},$$

and choose r and ϕ as generalized coordinates (define these, for example in a figure). The gravitational field (in this case the potential energy of the satellite) is

$$U(r) = -\frac{GMm}{r}$$

where G is a constant. Derive the differential equations for r and ϕ using Lagrangian mechanics.

Solution: The kinetic energy of the satellite is $T = \frac{1}{2}m\dot{\vec{r}}\cdot\dot{\vec{r}} = \frac{1}{2}m\left(\dot{r}^2 + r^2\dot{\phi}^2\right)$, the potential energy is given above. Let $L = T - U$, then Lagrange EoM gives

$$\ddot{\phi}r^2 + 2\dot{\phi}r\dot{r} = 0,$$
$$\ddot{r} - r\dot{\phi}^2 + \frac{GM}{r^2} = 0.$$

(10 %) (b) Given that

$$\vec{F} = \frac{\partial U}{\partial \vec{r}} = -\frac{GMm}{r^3}\vec{r},$$

use Newton-Euler's equation of motion to derive the equations of motion for x and y, assuming

$$\mathbf{r} = \begin{pmatrix} x \\ y \end{pmatrix}.$$

(You do not have to show that these are equivalent to those found in (a).)

Solution: From Newton's law we get

$$m\ddot{\mathbf{r}} = -\frac{GMm}{r^3}\mathbf{r}.$$

Writing this out for x and y:

$$\ddot{x} = -GM\frac{x}{\left(x^2+y^2\right)^{\frac{3}{2}}}$$
$$\ddot{y} = -GM\frac{y}{\left(x^2+y^2\right)^{\frac{3}{2}}}$$

A remark: There is an error in the text here, since we should have $\vec{F} = -\frac{\partial U}{\partial \vec{r}}$.

(5 %) (c) Was using Lagrange equations of motion simpler than using Newton-Euler in this case? Did it result in fewer equations of motion? Why/why not?

Solution: There are no forces of constraints, so no forces are eliminated in the Lagrange formalism to simplify things. Put another way, there are as many degrees of freedom (generalized coordinates, r and ϕ) as there are "normal" coordinates.

Problem 3 (15 %)

For any $a \in \mathbb{R}^3$, show that

$$R_a = \left(I - a^\times\right)^{-1}\left(I + a^\times\right) \in SO(3)$$

(that is, show that R_a is a rotation matrix).

The following identities may be useful (for matrices A and $B \in \mathbb{R}^{3\times 3}$ and scalar c):

$$\begin{aligned}
&\det(I) = 1, \quad \det(A^\top) = \det(A), \quad \det(A^{-1}) = \frac{1}{\det(A)},\\
&\det(AB) = \det(A)\det(B), \quad \det(cA) = c^3\det(A),\\
&(AB)^\top = B^\top A^\top, \quad (AB)^{-1} = B^{-1}A^{-1}, \quad \left(A^{-1}\right)^\top = \left(A^\top\right)^{-1},\\
&I + A \text{ and } I - A \text{ commute:} \quad (I+A)(I-A) = (I-A)(I+A)
\end{aligned}$$

Solution: Rotation matrices fulfill $R^\top R = I$ and $\det(R) = 1$.

$$\begin{aligned}
R_a^\top R_a &= \left(\left(I - a^\times\right)^{-1}\left(I + a^\times\right)\right)^\top \left(I - a^\times\right)^{-1}\left(I + a^\times\right)\\
&= \left(I + a^\times\right)^\top \left(I - a^\times\right)^{-\top}\left(I - a^\times\right)^{-1}\left(I + a^\times\right)\\
&= \left(I - a^\times\right)\left(I + a^\times\right)^{-1}\left(I - a^\times\right)^{-1}\left(I + a^\times\right)\\
&= \left(I - a^\times\right)\left[\left(I - a^\times\right)\left(I + a^\times\right)\right]^{-1}\left(I + a^\times\right)\\
&= \left(I - a^\times\right)\left[\left(I + a^\times\right)\left(I - a^\times\right)\right]^{-1}\left(I + a^\times\right)\\
&= \left(I - a^\times\right)\left(I - a^\times\right)^{-1}\left(I + a^\times\right)^{-1}\left(I + a^\times\right)\\
&= I
\end{aligned}$$

(This is slightly easier if you show $R_a R_a^\top = 1$, which is the same.)

Using $\det(A^\top) = \det(A)$ and $\left(I - a^\times\right)^\top = I + a^\times$,

$$\det(R_a) = \frac{\det\left(I + a^\times\right)}{\det\left(I - a^\times\right)} = \frac{\det\left(I + a^\times\right)}{\det\left(I + a^\times\right)} = 1.$$

The vector a is sometimes called the *Cayley* parameterization of a rotation matrix.

Problem 4 (10 %)

Under the influence of gravity ($g = 9.8m/s^2$), a 10 kg mass B is suspended by a massless cable from a drum A (radius 0.10 m) with moment of inertia 0.50 kg·m^2. If the system is released from rest, determine the angular acceleration of the drum and the tension of the cable.

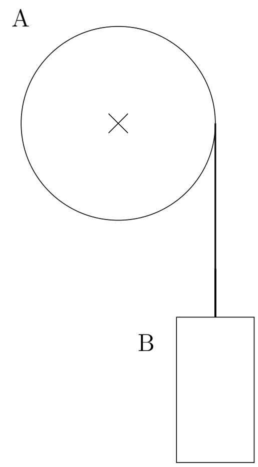


Figure 3: Drum with load

Solution: Moment and force balance (eq. 7.41-7.42 in book) gives

$$I\ddot{\theta} = Tr$$
$$ma = mg - T$$

Together with $a = r\ddot{\theta}$ (from the geometry/kinematics), this gives

$$I\ddot{\theta} = r(mg - mr\ddot{\theta})$$

or

$$\ddot{\theta} = \frac{mgr}{I + mr^2} = 16rad/s^2.$$

The tension of the cable is $T = 82N$.

This problem can also be solved using Lagrange mechanics (gives also full score). One generalized coordinate (say q, position of mass B), positive downwards:

$$K = \frac{1}{2}I\left(\frac{\dot{q}}{r}\right)^2 + \frac{1}{2}m\dot{q}^2, \quad U = -mgq, \quad L = K - U.$$

Solving this gives an expression for $a = \ddot{q}$, from which $\ddot{\theta} = a/r$ can be found. Then the force balance for the mass, $ma = mg - T$ must be used to obtain T.

Problem 5 (25 %)

A *chemostat* (a type of *bioreactor*) is used to grow microorganisms (bacterial culture), for example for use in experiments.

A vessel (the chemostat) is filled with a liquid containing a nutritient and bacteria, and their respective concentrations at time t are denoted $n(t)$ and $b(t)$. The unit of $n(t)$ is mass of nutrition per volume, and the unit of $b(t)$ is mass of bacteria per volume.

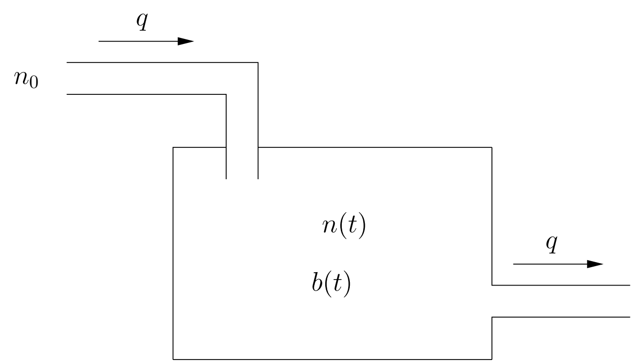


Figure 4: Chemostat

A solution of nutritient, with constant concentration n_0, is pumped into the vessel to replenish the nutritient consumed by the bacteria, with flowrate q [m^3/s]. To keep the volume V of the vessel constant, an outflow valve is controlled such that the mixture in the tank is being drained at the same rate. We assume perfect mixing in the tank.

The growth rate of the bacteria (per volume) is proportional to the concentration of bacteria and to a nonlinear function of the concentration of nutritient,

$$k(n)b,$$

where we assume $k(n)$ is given by the Michaelis-Menten kinetics,

$$k(n) = \frac{\alpha_1 n}{\alpha_2 + n}.$$

where α_1 and α_2 are constants (in words, the rate of growth increases with nutritient availability only up to some limiting value.)

The consumption rate of nutritient is proportional to the growth rate of bacteria,

$$\gamma k(n)b,$$

where γ is a constant of proportionality.

Set up a dynamical model for the concentration of nutritient $n(t)$ and concentration of bacteria $b(t)$ in the chemostat.

Solution: We use mass balances:

- Rate of change of mass of bacteria is given by growth-rate, minus what leaves the vessel.
- Rate of change of mass of nutritient is given of what enters the vessel, minus consumption and the amount leaving the vessel.

Mathematically:

$$\frac{\mathrm{d}bV}{\mathrm{d}t} = \frac{\alpha_1 n}{\alpha_2 + n} bV - bq$$
$$\frac{\mathrm{d}nV}{\mathrm{d}t} = -\gamma \frac{\alpha_1 n}{\alpha_2 + n} bV + n_0 q - nq$$

Divide both equations by the (constant) volume to obtain

$$\frac{\mathrm{d}b}{\mathrm{d}t} = \frac{\alpha_1 n}{\alpha_2 + n} b - \frac{bq}{V}$$
$$\frac{\mathrm{d}n}{\mathrm{d}t} = -\gamma \frac{\alpha_1 n}{\alpha_2 + n} b + \frac{q(n_0 - n)}{V}$$

(Remark: To control the amount of bacteria to desired level, we can use the flowrate q as manipulated variable.)