NTNU – Trondheim
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For questions during the exam:
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# Exam in TTK4130 Modeling and Simulation

Friday, May 27th 2011
09:00 – 13:00

Permitted aids (code A): All written and handwritten examination support materials are permitted.

**Note**: A Norwegian text is appended.

Answers in English, Norwegian, or a mixture of the two accepted.

Grades available: As specified by regulations.

**Problem 1 (26 %)**

The gyroscopic pendulum consists of a physical pendulum with a rotating symmetric disc at the end, spinning about an axis parallel to the axis of rotation of the pendulum. See Figure 1. The stiff rod has mass $m_1$, length $\ell_1$ and moment of inertia $I_1$. The position of the rod's center of gravity is given by $\ell_{c1}$ (cf. figure). The disc has mass $m_2$ and moment of inertia $I_2$. The pendulum is attached to a fixed coordinate system (axis $x$ and $y$).

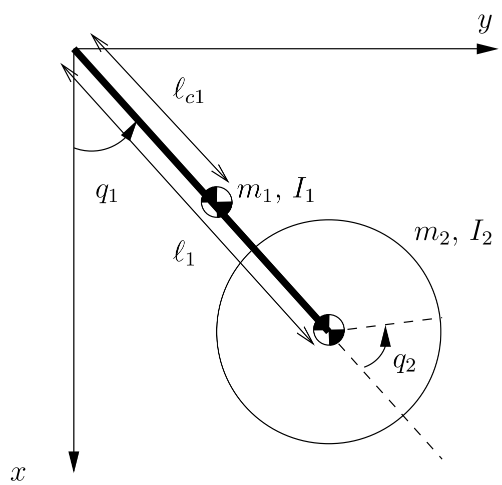


Figure 1: Gyroscopic pendulum

The rotating disc is actuated by a torque $\tau$ (which could be generated e.g. by a DC-motor). The gyroscopic pendulum is sometimes used as an experiment to illustrate nonlinear control theory.

We will develop the equations of motion for the gyroscopic pendulum.

(4 %) (a) Choose appropriate generalized coordinates for this system. The figure should give you some hints. What are the corresponding generalized forces?

(6 %) (b) What is the angular velocity of the disc (that is, of a coordinate system fixed in the disc) in the earth-fixed coordinate system?

(10 %) (c) Find the kinetic and potential energy for the system as functions of the generalized coordinates.

(6 %) (d) Derive the equations of motion for the system.

**Problem 2 (28 %)**

In this problem, we will consider four coordinate systems,

- coordinate system $a$ with axes $\vec{a}_1$, $\vec{a}_2$, and $\vec{a}_3$,
- coordinate system $b$ with axes $\vec{b}_1$, $\vec{b}_2$, and $\vec{b}_3$,
- coordinate system $c$ with axes $\vec{c}_1$, $\vec{c}_2$, and $\vec{c}_3$,
- coordinate system $d$ with axes $\vec{d}_1$, $\vec{d}_2$, and $\vec{d}_3$.

The rotation from $a$ to $b$ is described by a rotation $\alpha$ about $\vec{a}_3$, from $b$ to $c$ by a rotation $\beta$ about $\vec{b}_2$, and from $c$ to $d$ by a rotation $\gamma$ about $\vec{c}_1$. (Here, $\alpha$, $\beta$, and $\gamma$ are angles.)

(8 %) (a) Find the rotation matrix $\mathbf{R}_d^a$. The answer should contain the elements of this rotation matrix.

(8 %) (b) The angle/axis parameters $\vec{k}$ and $\theta$ correspond to $\mathbf{R}_d^a$. Show that

$$\mathbf{R}_d^a - (\mathbf{R}_d^a)^\mathsf{T} = 2\mathbf{k}^\times \sin\theta.$$

(4 %) (c) In which of the coordinate systems $a$, $b$, $c$, and $d$ is the coordinate vector $\mathbf{k}$ specified?

(8 %) (d) Let $\alpha = \frac{\pi}{2}$, $\beta = 0$, and $\gamma = -\frac{\pi}{2}$. Make a sketch of coordinate system $a$ and $d$, and find the parameters $k$ and $\theta$ that correspond to $\mathbf{R}_d^a$ for these values.

**Problem 3 (26 %)**

Given the following Butcher array:

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1-\alpha & \alpha \\ \hline & 1-\alpha & \alpha \end{array}$$

where $\alpha \in [0, 1]$ is a parameter.

(6 %) (a) Write up the equations for this method, for a system $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, t)$. Is the method explicit or implicit? Why?

(6 %) (b) Derive the stability function for this method as a function of $s = h\lambda$ (correct answer without calculations give 50% score).

(8 %) (c) For which $\alpha$ is the method A-stable? Substantiate your answer.

(6 %) (d) For which $\alpha$ is the method L-stable? Substantiate your answer.

**Problem 4 (20 %)**

When ice forms on water, the "rate of freezing" is mainly a function of air temperature (but also other factors, such as water temperature, water salinity, wind, etc.). Assume that this rate of freezing (per unit volume) is known, and denote it $S_h = S_h(\mathbf{x}, t)$.

Let $h = h(\mathbf{x}, t)$ denote the thickness of ice, and assume that ice density is constant.

(4 %) (a) Explain *briefly* what the equation

$$\frac{\mathrm{D}}{\mathrm{D}t} \iiint_{V_m} h \mathrm{d}V = \iiint_{V_m} S_h \mathrm{d}V$$

express.

(6 %) (b) Set up a partial differential equation (on divergence form) for the dynamics of ice thickness. Assume that the velocity $\vec{v}$ of ice is known,

$$\mathbf{v} = \begin{pmatrix} u \\ v \\ w \end{pmatrix},$$

where the velocity in $z$-direction is zero, $w = 0$.

Hint: Reynolds theorem on divergence form is

$$\frac{\mathrm{D}}{\mathrm{D}t}\iiint_V \phi \mathrm{d}V = \iiint_V \frac{\partial \phi}{\partial t} + \vec{\nabla}\cdot[\phi\vec{v}]\,\mathrm{d}V$$

(4 %) (c) If the velocity was not known, where should we start if we were to set up a model for the velocity? The answer should be a single sentence, and not contain mathematics.

(6 %) (d) Outline (briefly, in words or simple mathematics) a way to simulate the model found in (b).