Norges teknisk-naturvitenskapelige universitet
Institutt for teknisk kybernetikk

LØSNINGSFORSLAG
EKSAMEN I TTK4130
MODELLERING OG SIMULERING
25. mai 2007
Tid: 09:00-13:00

Hjelpemidler:
A: Alle kalkulatorer, trykte og håndskrevne hjelpemidler tillatt.

Sensur:
Sensuren vil bli avsluttet i henhold til gjeldende regelverk.

$\boxed{\text{Problem 1}}$

$\boxed{\text{a}}$
Given the matrix

$$\mathbf{R} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & * \\ * & * & * \\ * & * & 1 \end{bmatrix} = \{r_{i,j}\}$$

Since $\mathbf{R} \in SO(3)$. Thus,

**C.1** the row- and column-vectors of $\mathbf{R}$ are orthonormal vectors

**C.2** $\det \mathbf{R} = 1$.

By **C.1**, we have

$$\begin{aligned} r_{11}^2 + r_{12}^2 + r_{13}^2 &= 1 \\ \frac{3}{4} + \frac{1}{4} + r_{13}^2 &= 1 \\ - &> r_{13} = 0 \end{aligned}$$

$$\begin{aligned} r_{31}^2 + r_{32}^2 + r_{33}^2 &= 1 \\ 0 + r_{32}^2 + 1 &= 1 \\ - &> r_{32} = 0 \end{aligned}$$

$$\begin{aligned} r_{13}^2 + r_{23}^2 + r_{33}^2 &= 1 \\ 0 + r_{23} + 1 &= 1 \\ - &> r_{23} = 0 \end{aligned}$$

$$\begin{aligned} r_{11}^2 + r_{21}^2 + r_{31}^2 &= 1 \\ \frac{3}{4} + r_{21}^2 + 0 &= 1 \\ - &> r_{21} = \pm\frac{1}{2} \end{aligned}$$

$$\begin{aligned} r_{12}^2 + r_{22}^2 + r_{32}^2 &= 1 \\ \frac{1}{4} + r_{22}^2 + 0 &= 1 \\ - &> r_{22} = \pm\frac{\sqrt{3}}{2} \end{aligned}$$

Thus,

$$\mathbf{R} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \pm\frac{1}{2} & \pm\frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Due to **C.2**, we have

$$\mathbf{R} = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$\boxed{\text{b}}$

Since

$$\vec{r} = \frac{1}{2}\vec{a}_1 + \frac{1}{3}\vec{a}_2$$

Thus,

$$\mathbf{r}^a = \left[\frac{1}{2}, \frac{1}{3}, 0\right]^\top$$

To find $\mathbf{r}^b$, we apply results from Ch. 6.4.6. We have

$$\mathbf{p}^b = \mathbf{T}_a^b \mathbf{p}^a = \left(\mathbf{T}_b^a\right)^{-1} \mathbf{p}^a$$

where

$$\begin{aligned} \mathbf{p}^a &= \begin{bmatrix} \mathbf{r}^a \\ 1 \end{bmatrix} \\ \mathbf{p}^b &= \begin{bmatrix} \mathbf{r}^b \\ 1 \end{bmatrix} \\ \mathbf{T}_a^b &= \begin{bmatrix} \left(\mathbf{R}_b^a\right)^\top & -\left(\mathbf{R}_b^a\right)^\top \mathbf{r}_{ab}^a \\ \mathbf{0}^\top & 1 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} \mathbf{r}_{ab}^a &= [4, 3, 0]^\top \\ \mathbf{R}_b^a &= \mathbf{R}_z\left(30^0\right) = \begin{bmatrix} \frac{\sqrt{3}}{2} & -\frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{aligned}$$

Hence,

$$\mathbf{p}^b = \begin{bmatrix} \mathbf{r}^b \\ 1 \end{bmatrix} = \begin{bmatrix} -4.3644 \\ -0.5594 \\ 0 \\ 1 \end{bmatrix}$$

i.e.

$$\mathbf{r}^b = [-4.3644, -0.5594, 0]^\top$$

$\boxed{\text{c}}$

$\mathbf{R}_{\mathbf{k},\theta}\mathbf{d} = \mathbf{d} \;\Rightarrow \mathbf{d}$ is a rotation axis.

1st step: find the unit vector $\mathbf{k}$, by normalizing $\mathbf{d}$.

$$\mathbf{k} = \frac{\mathbf{d}}{\|\mathbf{d}\|} = \frac{[1, 0, \sqrt{3}]^T}{\sqrt{(1+3)}} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{\sqrt{3}}{2} \end{bmatrix}^T$$

Using eq. (6.142) from the book,

$$\mathbf{R}_{\mathbf{k},\theta} = \begin{bmatrix} k_x^2 v_\theta + c_\theta & k_x k_y v_\theta - k_z s_\theta & k_x k_z v_\theta + k_y s_\theta \\ k_x k_y v_\theta + k_z s_\theta & k_y^2 v_\theta + c_\theta & k_y k_z v_\theta - k_x s_\theta \\ k_x k_z v_\theta - k_y s_\theta & k_y k_z v_\theta + k_x s_\theta & k_z^2 v_\theta + c_\theta \end{bmatrix}$$

where $s_\theta = \sin\theta$, $c_\theta = \cos\theta$, and $v_\theta = 1 - c_\theta$ $\mathbf{R}_{\mathbf{k},\theta}$ is then given by

$$\mathbf{R}_{\mathbf{k},\theta} = \begin{bmatrix} \frac{5}{8} & -\frac{3}{4} & \frac{\sqrt{3}}{8} \\ \frac{3}{4} & \frac{1}{2} & -\frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{8} & \frac{\sqrt{3}}{4} & \frac{7}{8} \end{bmatrix}$$

The vector $\mathbf{k}$ is not unique though, as $\mathbf{k}_2 = -\mathbf{k}$ also satisfies the condition $\mathbf{R}_{\mathbf{k}_2,\theta}.\mathbf{d} = \mathbf{d}$. However, in this case, the matrix $\mathbf{R}_2$ would be given by $\mathbf{R}_2 = \mathbf{R}^T$.

$\boxed{\text{d}}$

Let $\mathbf{q_1} = \begin{bmatrix} \eta_1 & \epsilon_{1,1} & \epsilon_{1,2} & \epsilon_{1,3} \end{bmatrix} =$ and $\mathbf{q_2} = \begin{bmatrix} \eta_2 & \epsilon_{2,1} & \epsilon_{2,2} & \epsilon_{2,3} \end{bmatrix}$

The quaternion product is defined as

$$\mathbf{q_1} \otimes \mathbf{q_2} = \begin{bmatrix} \eta_1 \eta_2 - \epsilon_1^T \epsilon_2 \\ \eta_1 \epsilon_2 + \eta_2 \epsilon_1 + \epsilon_1^\times \epsilon_2 \end{bmatrix}$$

Which can be written as

$$\mathbf{q_1} \otimes \mathbf{q_2} = \begin{bmatrix} \eta_1 & -\epsilon_{1,1} & -\epsilon_{1,2} & -\epsilon_{1,3} \\ \epsilon_{1,1} & \eta_1 & -\epsilon_{1,3} & \epsilon_{1,2} \\ \epsilon_{1,2} & \epsilon_{1,3} & \eta_1 & -\epsilon_{1,1} \\ \epsilon_{1,3} & -\epsilon_{1,2} & \epsilon_{1,1} & \eta_1 \end{bmatrix} \begin{bmatrix} \eta_2 \\ \epsilon_{2,1} \\ \epsilon_{2,2} \\ \epsilon_{2,3} \end{bmatrix} = \mathbf{A}(\mathbf{q_1})\mathbf{q_2}$$

$\boxed{\text{e}}$

To find the Euler parameters, we apply Shepperd's algorithm.

- $T :=$Trace$\mathbf{R} = 1$.
- $r_{00} := T = 1$.
- $\max\{r_{00}, r_{11}, r_{22}, r_{33}\} = r_{00} = 1$
- $|z_0| = \sqrt{1 + 2r_{00} - T} = \sqrt{2}$. Thus, $z_0 = \pm\sqrt{2}$.

- From eqs. (6.227)-(6.229)

$$
\begin{aligned}
z_1 &= \frac{r_{32} - r_{23}}{z_0} = 0 \\
z_2 &= \frac{r_{13} - r_{31}}{z_0} = \mp \frac{2}{\sqrt{2}} \\
z_3 &= \frac{r_{21} - r_{12}}{z_0} = 0
\end{aligned}
$$

Thus,

$$
\begin{aligned}
\eta &= \frac{z_0}{2} = \pm \frac{\sqrt{2}}{2} \\
\boldsymbol{\epsilon} &= \frac{1}{2}\{z_i\} = \begin{bmatrix} 0 \\ \mp\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}
\end{aligned}
$$

Knowing that $\eta > 0$ gives

$$
\begin{aligned}
\eta &= \frac{z_0}{2} = \frac{\sqrt{2}}{2} \\
\boldsymbol{\epsilon} &= \frac{1}{2}\{z_i\} = \begin{bmatrix} 0 \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}
\end{aligned}
$$

$\boxed{\text{Problem 2}}$
$\boxed{\text{a}}$
We apply results in Ch. 2.4.6.
**1.** Given

$$H_1(s) = \frac{1}{s+1}$$

Pole: $p_1 \in \{-1\}$. Now, let check the conditions:

- $\text{Re}\{p_1\} = -1 < 0$. OK
- Let $s = j\omega$, where $s \neq p_1$. Consider the frequency response

$$\begin{aligned} H_1(j\omega) &= \frac{1}{j\omega + 1} \\ &= \frac{1}{1+\omega^2} - j\frac{\omega}{1+\omega^2} \end{aligned}$$

  Thus, $\text{Re}\{H_1(j\omega)\} > 0$, $\forall \omega \in R$. OK
- no poles at $s = j\omega$ or $s = \pm\infty$. OK.

Hence, $H_1(s)$ is positive real.
**2.** Given

$$H_2(j\omega) = \frac{1}{s^2 + s + 1}$$

Poles: $\{p_1, p_2\} = \left\{-\frac{1}{2} - j\frac{\sqrt{3}}{2}, -\frac{1}{2} + j\frac{\sqrt{3}}{2}\right\}$. Check the conditions:

- $\text{Re}\{p_i\}_{i=1,2} = -\frac{1}{2} < 0$. OK
- Let $s = j\omega$. Consider the frequency response

$$\begin{aligned} H_2(j\omega) &= \frac{1}{-\omega^2 + j\omega + 1} \\ &= \frac{1-\omega^2}{(1-\omega^2)^2 + \omega} - j\frac{\omega}{(1-\omega^2)^2 + \omega} \end{aligned}$$

  Thus, $\text{Re}\{H_2(j\omega)\} < 0$, $\forall \omega > 1$. Hence, the condition $\text{Re}\{H_2(j\omega)\} > 0$ is not satisfied.

The transferfunction $H_2(s)$ is *not* positive real.

Alternative:

1. Since $H_1(s)$ is passive ($|\measuredangle H_1(j\omega)| \leq 90^0$). Thus, $H_1(s)$ is positive real.
2. Since $H_2(s)$ is not passive ($\measuredangle H_2(j\omega) \to -180$, as $\omega \to \infty$). Thus, $H_2(s)$ is *not* positive real.

$\boxed{\text{b}}$
Given the system

$$m\ddot{x} + d\dot{x} + kx + f(\dot{x}) = F \tag{1}$$

We apply results in Ch. 2.4.14 (eq. (2.149)). Define the storage function

$$V(x, \dot{x}) = \frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2 \geq 0, \quad \forall (x, \dot{x}) \in \mathbb{R}^2$$

The time derivative of $V$ along the solutions of the system (1) can be written as

$$\dot{V} = -g(\dot{x}) + u\, y$$

where

$$\begin{aligned} u &= F \\ y &= \dot{x} \\ g(v) &= dv^2 + vf(v) > 0, \quad \forall v \in \mathbb{R} \end{aligned}$$

Hence, eq. (2.149) is satisfied. The system (1) is thus passive with input $F$ and output $\dot{x}$.

$\boxed{\text{c}}$
Rewriting the system, one has

$$\frac{d}{dt}\begin{bmatrix} \phi \\ \theta \\ \psi \end{bmatrix} = \begin{bmatrix} p + \sin(\phi)\tan(\theta)q + \cos(\phi)\tan(\theta)r \\ \cos(\phi)q - \sin(\phi)r \\ \sin(\phi)/\cos(\theta)q + \cos(\phi)/\cos(\theta)r \end{bmatrix} \tag{2}$$

Linearizing 2 with small perturbations around $(\phi^*, \theta^*) = (0, 0)$,

$$\begin{aligned} \Delta\dot{\phi} &= \frac{\partial}{\partial\phi}\left[p + \sin(\phi)\tan(\theta)q + \cos(\phi)\tan(\theta)r\right]\Big|_{(\phi,\theta)=(0,0)}\Delta\phi + \\ &\quad + \frac{\partial}{\partial\theta}\left[p + \sin(\phi)\tan(\theta)q + \cos(\phi)\tan(\theta)r\right]\Big|_{(\phi,\theta)=(0,0)}\Delta\theta \\ \Delta\dot{\theta} &= \frac{\partial}{\partial\phi}\left[\cos(\phi)q - \sin(\phi)r\right]\Big|_{(\phi,\theta)=(0,0)}\Delta\phi + \\ &\quad + \frac{\partial}{\partial\theta}\left[\cos(\phi)q - \sin(\phi)r\right]\Big|_{(\phi,\theta)=(0,0)}\Delta\theta \\ \Delta\dot{\psi} &= \frac{\partial}{\partial\phi}\left[\sin(\phi)/\cos(\theta)q + \cos(\phi)/\cos(\theta)r\right]\Big|_{(\phi,\theta)=(0,0)}\Delta\phi + \\ &\quad + \frac{\partial}{\partial\theta}\left[\sin(\phi)/\cos(\theta)q + \cos(\phi)/\cos(\theta)r\right]\Big|_{(\phi,\theta)=(0,0)}\Delta\theta \end{aligned}$$

$\boxed{\text{Problem 3}}$
Given the system

$$\frac{d}{dt}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}, t > 0 \tag{3}$$

with initial conditions

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix} \tag{4}$$

$\boxed{\text{a}}$
Consider

$$\begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} = \begin{bmatrix} -\sin(2t) \\ -2\cos(2t) \end{bmatrix}, t \geq 0 \tag{5}$$

First,

$$\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ -2 \end{bmatrix}$$

Secondly, the time derivative of (5) is

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} -2\cos(2t) \\ 4\sin(2t) \end{bmatrix} \\ &= \begin{bmatrix} x_2(t) \\ -4x_1(t) \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \end{aligned}$$

which shows that (5) is the exact solution of the problem (3)-(4). Q.E.D.
$\boxed{\text{b}}$
Explicit Euler's method (p. 528)

$$\begin{aligned} \mathbf{k}_1 &= \mathbf{f}(\mathbf{x}_n, t_n) \\ &= \mathbf{A}\mathbf{x}_n \\ \mathbf{x}_{n+1} &= \mathbf{x}_n + h\mathbf{k}_1 \\ &= \mathbf{x}_n + h\mathbf{A}\mathbf{x}_n \\ &= (\mathbf{I} + h\mathbf{A})\,\mathbf{x}_n \end{aligned} \tag{6}$$

Thus,

$$\begin{bmatrix} x_{1,n+1} \\ x_{2,n+1} \end{bmatrix} = \begin{bmatrix} x_{1,n} \\ x_{2,n} \end{bmatrix} + h\begin{bmatrix} x_{2,n} \\ -4x_{1,n} \end{bmatrix} \tag{7}$$

Implicit Euler method (p. 535)

$$\begin{aligned}
\mathbf{k}_1 &= \mathbf{f}\left(\mathbf{x}_n + h\mathbf{k}_1, t_n + h\right) \\
&= \mathbf{A}\left(\mathbf{x}_n + h\mathbf{k}_1\right) \\
\rightarrow \mathbf{k}_1 &= \left(\mathbf{I} - h\mathbf{A}\right)^{-1}\mathbf{A}\mathbf{x}_n \\
\\
\mathbf{x}_{n+1} &= \mathbf{x}_n + h\mathbf{k}_1 \\
&= \mathbf{x}_n + h\left(\mathbf{I} - h\mathbf{A}\right)^{-1}\mathbf{A}\mathbf{x}_n \\
&= \left(\mathbf{I} - h\mathbf{A}\right)^{-1}\mathbf{x}_n
\end{aligned} \tag{8}$$

Thus,

$$\left[\begin{array}{c} x_{1,n+1} \\ x_{2,n+1} \end{array}\right] = \frac{1}{1+4h^2}\left[\begin{array}{c} x_{1,n} + hx_{2,n} \\ -4hx_{1,n} + x_{2,n} \end{array}\right] \tag{9}$$

Implicit midpoint method (p. 537)

$$\begin{aligned}
\mathbf{k}_1 &= \mathbf{f}\left(\mathbf{x}_n + \frac{h}{2}\mathbf{k}_1, t_n + \frac{h}{2}\right) \\
&= \mathbf{A}\left(\mathbf{x}_n + \frac{h}{2}\mathbf{k}_1\right) \\
\rightarrow \mathbf{k}_1 &= \left(\mathbf{I} - \frac{h}{2}\mathbf{A}\right)^{-1}\mathbf{A}\mathbf{x}_n \\
\\
\mathbf{x}_{n+1} &= \mathbf{x}_n + h\mathbf{k}_1 \\
&= \mathbf{x}_n + h\left(\mathbf{I} - \frac{h}{2}\mathbf{A}\right)^{-1}\mathbf{A}\mathbf{x}_n \\
&= \left(\mathbf{I} - \frac{h}{2}\mathbf{A}\right)^{-1}\left(\mathbf{I} - \frac{h}{2}\mathbf{A} + h\mathbf{A}\right)\mathbf{x}_n \\
&= \left(\mathbf{I} - \mathbf{A}\frac{h}{2}\right)^{-1}\left(\mathbf{I} + \frac{h}{2}\mathbf{A}\right)\mathbf{x}_n
\end{aligned} \tag{10}$$

Thus,

$$\left[\begin{array}{c} x_{1,n+1} \\ x_{2,n+1} \end{array}\right] = \frac{1}{1+h^2}\left[\begin{array}{c} \left(1-h^2\right)x_{1,n} + hx_{2,n} \\ -4hx_{1,n} + \left(1-h^2\right)x_{2,n} \end{array}\right] \tag{11}$$

$\boxed{\text{c}}$

Explicit Euler's method (p. 521)

Applying the scalar test system $\dot{x} = \lambda x$ to (6) gives

$$x_{n+1} = \left(1 + h\lambda\right)x_n = R\left(h\lambda\right)x_n$$

Thus,

$$R(h\lambda) = 1 + h\lambda \tag{12}$$

Implicit Euler method (p. 535)
Applying the scalar test system $\dot{x} = \lambda x$ to (8) gives

$$x_{n+1} = (1 - h\lambda)^{-1} x_n = R(h\lambda) x_n$$

where

$$R(h\lambda) = \frac{1}{1 - h\lambda} \tag{13}$$

Implicit midpoint method (p. 538)

Applying the scalar test system $\dot{x} = \lambda x$ to (10) gives

$$x_{n+1} = \left(1 - \lambda\frac{h}{2}\right)^{-1} \left(1 + \frac{h}{2}\lambda\right) x_n = R(\lambda h) x_n$$

Thus,

$$R(h\lambda) = \frac{1 + \frac{h\lambda}{2}}{1 - \frac{h\lambda}{2}} \tag{14}$$

$\boxed{\text{d}}$
The eigenvalues of the system matrix $\mathbf{A}$ are

$$\lambda_{1,2} = \pm i2$$

Thus, we have

- Explicit Euler's method: From (12)

$$\begin{aligned} |R(\pm i2h)| &= |1 \pm i2h| \\ &= \sqrt{1 + 4h^2} > 1, \quad \forall h \neq 0 \end{aligned}$$

  i.e. $\mathbf{x}_{n+1} > \mathbf{x}_n$, $\forall n$.

- Implicit Euler method: From (13)

$$\begin{aligned} |R(\pm i2h)| &= \frac{1}{|1 \mp i2h|} \\ &= \frac{1}{\sqrt{1 + 4h^2}} < 1, \quad \forall h \neq 0 \end{aligned}$$

  i.e. $\mathbf{x}_{n+1} < \mathbf{x}_n$, $\forall n$.

- Implicit Midpoint method: From (14)

$$|R(\pm i2h)| = \frac{|1 \pm ih|}{|1 \mp ih|} = 1, \quad \forall h \neq 0$$

i.e. $\mathbf{x}_{n+1} = \mathbf{x}_n$, $\forall n$.

Hence,

- Figure 3: Explicit Euler's method (since $\mathbf{x}_{n+1} > \mathbf{x}_n$, $\forall n$).
- Figure 4: Implicit midpoint method (since $\mathbf{x}_{n+1} = \mathbf{x}_n$, $\forall n$).

$\boxed{\text{e}}$

Alternative 1

The local error $\mathbf{e}_{n+1}$ is defined as the difference of the computed solution $\mathbf{x}_{n+1}$ from the local solution $\mathbf{x}_L(t_n; t_{n+1})$ at time $t_{n+1}$ *given* that $\mathbf{x}_n = \mathbf{x}(t_n)$, i.e.

$$\mathbf{e}_{n+1} = \mathbf{x}_{n+1} - \mathbf{x}_L(t_n; t_{n+1}), \text{ given that } \mathbf{x}_n = \mathbf{x}(t_n)$$

Here,

$$\begin{array}{c} t_{n+1} = 1\,\text{sec} \\ h = 0.1\,\text{sec} \end{array} \Rightarrow n = 10$$

Thus,

$$\mathbf{e}_{10} = \mathbf{x}_{10} - \mathbf{x}_L(0.9; 1)$$

given that

$$\mathbf{x}_9 = \mathbf{x}(.9) = \begin{bmatrix} -\sin 1.8 \\ -2\cos 1.8 \end{bmatrix}$$

From (5) and (6) we have

$$\begin{aligned} \mathbf{x}_{10} &= (\mathbf{I} + h\mathbf{A})\,\mathbf{x}_9 \\ &= \left(\mathbf{I} + .1\begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}\right)\begin{bmatrix} -\sin 1.8 \\ -2\cos 1.8 \end{bmatrix} \\ \mathbf{x}_L(0.9; 1) &= \begin{bmatrix} -\sin 2 \\ -2\cos 2 \end{bmatrix} \end{aligned}$$

Thus,

$$\begin{aligned} \mathbf{e}_{10} &= \mathbf{x}_{10} - \mathbf{x}_L(0.9; 1) \\ &\approx \begin{bmatrix} -0.0191 \\ 0.0116 \end{bmatrix} \end{aligned}$$

Alternative 2:

The local error $\mathbf{e}_{n+1}$ is defined as the difference of the computed solution $\mathbf{x}_{n+1}$ from the local solution $\mathbf{x}_L(t_n; t_{n+1})$ at time $t_{n+1}$ *given* that $\mathbf{x}_L(t_n; t_n) = \mathbf{x}_n$, i.e.

$$\mathbf{e}_{n+1} = \mathbf{x}_{n+1} - \mathbf{x}_L(t_n; t_{n+1}), \text{ given that } \mathbf{x}_L(t_n; t_n) = \mathbf{x}_n$$

From (6),

$$
\begin{aligned}
\mathbf{x}_9 &= (\mathbf{I} + h\mathbf{A})^9 \, \mathbf{x}_0 \\
&= \left(\mathbf{I} + h \begin{bmatrix} 0 & 1 \\ -4 & 0 \end{bmatrix}\right)^9 \begin{bmatrix} 0 \\ -2 \end{bmatrix} \\
&\approx \begin{bmatrix} -1.1679 \\ 0.4875 \end{bmatrix}
\end{aligned}
$$

and $\mathbf{x}_L\,(t_9; t)$ can be shown to be

$$
\mathbf{x}_L\,(t_9; t) = \begin{bmatrix} c_1 \cos(2t) + c_2 \sin(2t) \\ -2c_1 \sin(2t) + 2c_2 \cos(2t) \end{bmatrix}, \quad t > t_9
$$

where

$$
\begin{aligned}
\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} &= \begin{bmatrix} \cos 1.8 & \sin 1.8 \\ -2\sin 1.8 & 2\cos 1.8 \end{bmatrix}^{-1} \mathbf{x}_9 \\
&\approx \begin{bmatrix} 0.0280 \\ -1.1927 \end{bmatrix}
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\mathbf{e}_{10} &= \mathbf{x}_{10} - \mathbf{x}_L\,(t_9; t_{10}) \\
&= (\mathbf{I} + h\mathbf{A})\,\mathbf{x}_9 - \mathbf{x}_L\,(0.9; 1) \\
&\approx \begin{bmatrix} -0.0230 \\ 0.0128 \end{bmatrix}
\end{aligned}
$$

$\boxed{\text{Problem 4}}$

Define the coordinate systems (Figure 1),

$$\begin{aligned} i &: \left\{ \vec{i}_1, \vec{i}_2, \vec{i}_3 \right\} \\ a &: \{ \vec{a}_1, \vec{a}_2, \vec{a}_3 \} \\ b &: \left\{ \vec{b}_1, \vec{b}_2, \vec{b}_3 \right\} \end{aligned}$$

where

$$\begin{aligned} \vec{a}_1 &= \sin\theta_1 \vec{i}_1 + \cos\theta_1 \vec{i}_2 \\ \vec{a}_2 &= -\cos\theta_1 \vec{i}_1 + \sin\theta_1 \vec{i}_2 \\ \vec{a}_3 &= \vec{i}_3 \end{aligned}$$

and

$$\begin{aligned} \vec{b}_1 &= \sin\theta_2 \vec{i}_1 + \cos\theta_2 \vec{i}_2 \\ \vec{b}_2 &= -\cos\theta_2 \vec{i}_1 + \sin\theta_2 \vec{i}_2 \\ \vec{b}_3 &= \vec{i}_3 \end{aligned}$$

The position and velocity of

- Wagon:

$$\begin{aligned} \vec{r}_v &= x\vec{i}_1 \\ \vec{v}_v &= \frac{{}^i d}{dt}\vec{r}_v = \dot{x}\vec{i}_1 \end{aligned}$$

- Pendulum 1:

$$\begin{aligned} \vec{r}_{p1} &= \vec{r}_v + L_1\vec{a}_1 \\ \vec{v}_{p1} &= \frac{{}^i d}{dt}\vec{r}_{p1} \\ &= \frac{{}^i d}{dt}\left(\vec{r}_v + L_1\vec{a}_1\right) \\ &= \dot{x}\vec{i}_1 + \frac{{}^a d}{dt}\left(L_1\vec{a}_1\right) + \vec{\omega}_{ia} \times \left(L_1\vec{a}_1\right) \\ &= \dot{x}\vec{i}_1 + \left(-\dot{\theta}_1\vec{a}_3\right) \times \left(L_1\vec{a}_1\right) \\ &= \dot{x}\vec{i}_1 - L_1\dot{\theta}_1\vec{a}_2 \end{aligned}$$

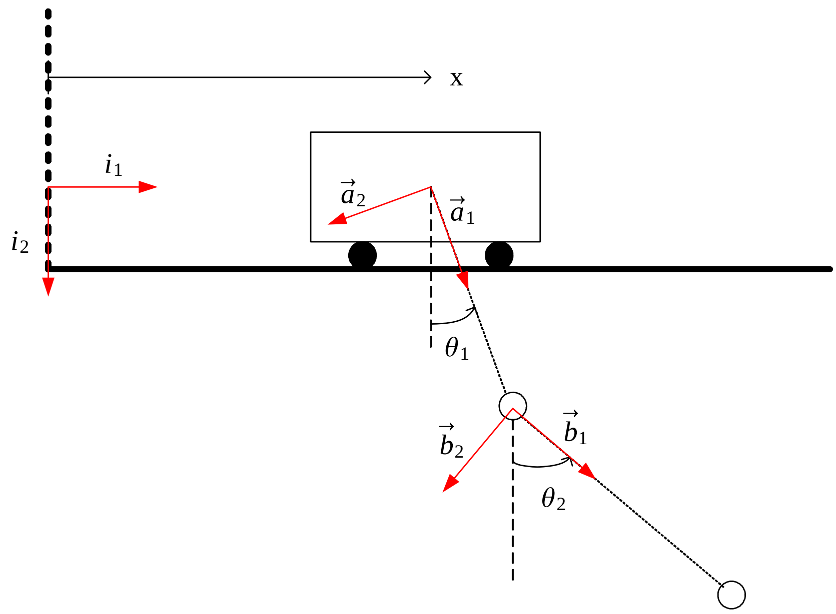


Figure 1:

- Pendulum 2:

$$
\begin{aligned}
\vec{r}_{p2} &= \vec{r}_{p1} + L_2\vec{b}_1 \\
\vec{v}_{p2} &= \frac{{}^i d}{dt}\vec{r}_{p2} \\
&= \frac{{}^i d}{dt}\left(\vec{r}_{p1} + L_2\vec{b}_1\right) \\
&= \vec{v}_{p1} + \frac{{}^b d}{dt}\left(L_2\vec{b}_1\right) + \vec{\omega}_{ib} \times \left(L_2\vec{b}_1\right) \\
&= \vec{v}_{p1} + \left(-\dot{\theta}_2\vec{b}_3\right) \times \left(L_2\vec{b}_1\right) \\
&= \vec{v}_{p1} - L_2\dot{\theta}_2\vec{b}_2
\end{aligned}
$$

<u>The kinetic energy of the system</u>

- Wagon:

$$
\begin{aligned}
T_v &= \frac{1}{2}M\vec{v}_v \cdot \vec{v}_v \\
&= \frac{1}{2}M\dot{x}^2
\end{aligned}
$$

- Pendulum 1:

$$\begin{aligned}
T_{p1} &= \frac{1}{2}m_1\vec{v}_{p1}\cdot\vec{v}_{p1} \\
&= \frac{1}{2}m_1\left[\dot{x}^2 - 2L_1\dot{\theta}_1\dot{x}\ \vec{i}_1\cdot\vec{a}_2 + \left(L_1\dot{\theta}_1\right)^2\right] \\
&= \frac{1}{2}m_1\left[\dot{x}^2 + 2L_1\dot{\theta}_1\dot{x}\ \cos\theta_1 + \left(L_1\dot{\theta}_1\right)^2\right]
\end{aligned}$$

- Pendulum 2:

$$\begin{aligned}
T_{p2} &= \frac{1}{2}m_2\vec{v}_{p2}\cdot\vec{v}_{p2} \\
&= \frac{1}{2}m_2\left[\vec{v}_{p1}\cdot\vec{v}_{p1} - 2L_2\dot{\theta}_2\vec{v}_{p1}\cdot\vec{b}_2 + \left(L_2\dot{\theta}_2\right)^2\right] \\
&= \frac{1}{2}m_2\left[\vec{v}_{p1}\cdot\vec{v}_{p1} - 2L_2\dot{\theta}_2\vec{v}_{p1}\cdot\vec{b}_2 + \left(L_2\dot{\theta}_2\right)^2\right]
\end{aligned}$$

where

$$\begin{aligned}
\vec{v}_{p1}\cdot\vec{b}_2 &= \left[\dot{x}\vec{i}_1 - L_1\dot{\theta}_1\vec{a}_2\right]\cdot\left[-\cos\theta_2\vec{i}_1 + \sin\theta_2\vec{i}_2\right] \\
&= -\dot{x}\cos\theta_2 - L_1\dot{\theta}_1\vec{a}_2\cdot\left[\cos\theta_2\vec{i}_1 + \sin\theta_2\vec{i}_2\right] \\
&= -\dot{x}\cos\theta_2 - L_1\dot{\theta}_1\left[\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\right] \\
&= -\dot{x}\cos\theta_2 - L_1\dot{\theta}_1\cos\left(\theta_1 - \theta_2\right)
\end{aligned}$$

Thus,

$$T_{p2} = \frac{1}{2}m_2\left[\vec{v}_{p1}\cdot\vec{v}_{p1} + 2L_2\dot{x}\dot{\theta}_2\cos\theta_2 + 2L_1L_2\dot{\theta}_1\dot{\theta}_2\cos\left(\theta_1-\theta_2\right) + \left(L_2\dot{\theta}_2\right)^2\right]$$

Hence,

$$\begin{aligned}
T &= T_v + T_{p1} + T_{p2} \\
&= \frac{1}{2}M\dot{x}^2 \\
&\quad + \frac{1}{2}m_1\left[\dot{x}^2 + 2L_1\dot{\theta}_1\dot{x}\ \cos\theta_1 + \left(L_1\dot{\theta}_1\right)^2\right] \\
&\quad + \frac{1}{2}m_2\left[\vec{v}_{p1}\cdot\vec{v}_{p1} + 2L_2\dot{x}\dot{\theta}_2\cos\theta_2 + 2L_1L_2\dot{\theta}_1\dot{\theta}_2\cos\left(\theta_1-\theta_2\right) + \left(L_2\dot{\theta}_2\right)^2\right]
\end{aligned}$$

The potential energy of the system

Let

$$\vec{g} = g\ \vec{i}_2$$

where $g = 9.81\ m/s^2$ is the gravitational constant.

- Wagon:

$$U_v = -M\ \vec{r}_v \cdot \vec{g} = 0$$

- Pendulum 1:

$$U_{p1} = -m_1\ \vec{r}_{p1} \cdot \vec{g} = -m_1 g L_1 \cos\theta_1$$

- Pendulum 2:

$$U_{p2} = -m_2\ \vec{r}_{p2} \cdot \vec{g} = -m_2 g L_1 \cos\theta_1 - m_2 g L_2 \cos\theta_2$$

Hence,

$$\begin{aligned} U &= U_v + U_{p1} + U_{p2} \\ &= -\left(m_1 + m_2\right) g L_1 \cos\theta_1 - m_2 g L_2 \cos\theta_2 \end{aligned}$$

Define the Lagrange function:

$$L\left(\mathbf{q}, \dot{\mathbf{q}}\right) = T\left(\mathbf{q}, \dot{\mathbf{q}}\right) - U\left(\mathbf{q}\right)$$

where $\mathbf{q} = \left[x, \theta_1, \theta_2\right]^\top$.

Equations of motion:

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} &= F \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right) - \frac{\partial L}{\partial \theta_1} &= 0 \\ \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_2}\right) - \frac{\partial L}{\partial \theta_2} &= 0 \end{aligned}$$

where

- $$\begin{aligned} \frac{\partial L}{\partial \dot{x}} &= \frac{\partial T}{\partial \dot{x}} \\ &= \left(M + m_1 + m_2\right)\dot{x} + \left(m_1 + m_2\right) L_1 \dot{\theta}_1 \cos\theta_1 + m_2 L_2 \dot{\theta}_2 \cos\theta_2 \end{aligned}$$

- $$\begin{aligned}\frac{\partial L}{\partial \dot{\theta}_1} &= \frac{\partial T}{\partial \dot{\theta}_1} \\ &= m_1 L_1 \dot{x}\cos\theta_1 + m_2 L_1 \dot{x}\cos\theta_1 \\ &\quad + L_1^2 m_1 \dot{\theta}_1 + L_1^2 m_2 \dot{\theta}_1 + L_1 L_2 m_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \\ &= (m_1 + m_2) L_1 \dot{x}\cos\theta_1 \\ &\quad + (m_1 + m_2) L_1^2 \dot{\theta}_1 + m_2 L_1 L_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \\ &= L_1 \left[(m_1 + m_2)\dot{x}\cos\theta_1 + (m_1 + m_2) L_1 \dot{\theta}_1 + m_2 L_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2)\right]\end{aligned}$$

- $$\begin{aligned}\frac{\partial L}{\partial \dot{\theta}_2} &= \frac{\partial T}{\partial \dot{\theta}_2} \\ &= m_2 \left[\dot{x} L_2 \cos\theta_2 + L_2^2 \dot{\theta}_2 + L_1 L_2 \dot{\theta}_1 \cos(\theta_1 - \theta_2)\right] \\ &= m_2 L_2 \left[\dot{x}\cos\theta_2 + L_2 \dot{\theta}_2 + L_1 \dot{\theta}_1 \cos(\theta_1 - \theta_2)\right]\end{aligned}$$

- $$\frac{\partial L}{\partial x} = \frac{\partial T}{\partial x} - \frac{\partial U}{\partial x} = 0$$

- $$\frac{\partial L}{\partial \theta_1} = \frac{\partial T}{\partial \theta_1} - \frac{\partial U}{\partial \theta_1}$$

  where

  $$\begin{aligned}\frac{\partial T}{\partial \theta_1} &= -m_1 \dot{x} L_1 \dot{\theta}_1 \sin\theta_1 \\ &\quad - m_2 \dot{x} L_1 \dot{\theta}_1 \sin\theta_1 - L_1 L_2 m_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) \\ &= -(m_1 + m_2) L_1 \dot{x} \dot{\theta}_1 \sin\theta_1 - m_2 L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) \\ \frac{\partial U}{\partial \theta_1} &= (m_1 + m_2) g L_1 \sin\theta_1\end{aligned}$$

- $$\frac{\partial L}{\partial \theta_2} = \frac{\partial T}{\partial \theta_2} - \frac{\partial U}{\partial \theta_2}$$

  where

  $$\begin{aligned}\frac{\partial T}{\partial \theta_2} &= m_2 \left[L_1 L_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) - L_2 \dot{x} \dot{\theta}_2 \sin\theta_2\right] \\ \frac{\partial U}{\partial \theta_2} &= m_2 g L_2 \sin\theta_2\end{aligned}$$

Hence, we get the equations of motion

**Eq. 1**

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right)-\frac{\partial L}{\partial x} &= F \\
\frac{d}{dt}\left[\left(M+m_1+m_2\right)\dot{x}+\left(m_1+m_2\right)L_1\dot{\theta}_1\cos\theta_1+m_2L_2\dot{\theta}_2\cos\theta_2\right] &= F
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\left(M+m_1+m_2\right)\ddot{x} \\
&\qquad+\left(m_1+m_2\right)L_1\left[\ddot{\theta}_1\cos\theta_1-\dot{\theta}_1^2\sin\theta_1\right] \\
&\qquad+m_2L_2\left[\ddot{\theta}_2\cos\theta_2-\dot{\theta}_2^2\sin\theta_2\right] \\
&=F
\end{aligned}
$$

**Eq. 2**

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}_1}\right)-\frac{\partial L}{\partial \theta_1}=0 \\
&L_1\frac{d}{dt}\left\{\left(m_1+m_2\right)\dot{x}\cos\theta_1+\left(m_1+m_2\right)L_1\dot{\theta}_1+m_2L_2\dot{\theta}_2\cos\left(\theta_1-\theta_2\right)\right\} \\
&\qquad+\left(m_1+m_2\right)L_1\dot{x}\dot{\theta}_1\sin\theta_1+m_2L_1L_2\dot{\theta}_1\dot{\theta}_2\sin\left(\theta_1-\theta_2\right) \\
&\qquad+\left(m_1+m_2\right)gL_1\sin\theta_1 \\
&=0
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&\left(m_1+m_2\right)L_1\left[\ddot{x}\cos\theta_1-\dot{x}\dot{\theta}_1\sin\theta_1\right] \\
&\qquad+\left(m_1+m_2\right)L_1^2\ddot{\theta}_1 \\
&\qquad+m_2L_1L_2\left[\ddot{\theta}_2\cos\left(\theta_1-\theta_2\right)-\dot{\theta}_2\left(\dot{\theta}_1-\dot{\theta}_2\right)\sin\left(\theta_1-\theta_2\right)\right] \\
&\qquad+\left(m_1+m_2\right)L_1\dot{x}\dot{\theta}_1\sin\theta_1+m_2L_1L_2\dot{\theta}_1\dot{\theta}_2\sin\left(\theta_1-\theta_2\right) \\
&\qquad+\left(m_1+m_2\right)gL_1\sin\theta_1 \\
&=0
\end{aligned}
$$

which simplifies as

$$
\begin{aligned}
&(m_1+m_2)\,L_1\ddot{x}\cos\theta_1\\
&\qquad+(m_1+m_2)\,L_1^2\ddot{\theta}_1\\
&\qquad+m_2L_1L_2\ddot{\theta}_2\cos(\theta_1-\theta_2)\\
&\qquad+m_2L_1L_2\dot{\theta}_2^2\sin(\theta_1-\theta_2)\\
&\qquad+(m_1+m_2)\,gL_1\sin\theta_1\\
&=0
\end{aligned}
$$

**Eq. 3:**

$$
\begin{aligned}
&\frac{d}{dt}\left(\frac{\partial L}{\partial\dot{\theta}_2}\right)-\frac{\partial L}{\partial\theta_2}=0\\
&\frac{d}{dt}\left(m_2L_2\left[\dot{x}\cos\theta_2+L_2\dot{\theta}_2+L_1\dot{\theta}_1\cos(\theta_1-\theta_2)\right]\right)\\
&\qquad-m_2\left[L_1L_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2)-L_2\dot{x}\dot{\theta}_2\sin\theta_2\right]\\
&\qquad+m_2gL_2\sin\theta_2\\
&=0
\end{aligned}
$$

Thus,

$$
\begin{aligned}
&m_2L_2\left[\ddot{x}\cos\theta_2-\dot{x}\dot{\theta}_2\sin\theta_2\right]\\
&\qquad+m_2\;L_2^2\ddot{\theta}_2\\
&\qquad+m_2L_1L_2\left[\ddot{\theta}_1\cos(\theta_1-\theta_2)-\dot{\theta}_1\left(\dot{\theta}_1-\dot{\theta}_2\right)\sin(\theta_1-\theta_2)\right]\\
&\qquad-m_2\left[L_1L_2\dot{\theta}_1\dot{\theta}_2\sin(\theta_1-\theta_2)-L_2\dot{x}\dot{\theta}_2\sin\theta_2\right]\\
&\qquad+m_2gL_2\sin\theta_2\\
&=0
\end{aligned}
$$

which simplifies as

$$
\begin{aligned}
&m_2L_2\ddot{x}\cos\theta_2\\
&\qquad+m_2L_2^2\ddot{\theta}_2\\
&\qquad+m_2L_1L_2\ddot{\theta}_1\cos(\theta_1-\theta_2)\\
&\qquad-m_2L_1L_2\dot{\theta}_1^2\sin(\theta_1-\theta_2)\\
&\qquad+m_2gL_2\sin\theta_2\\
&=0
\end{aligned}
$$

or

$$\begin{aligned}
&\ddot{x}\cos\theta_2 \\
&\qquad +L_2\ddot{\theta}_2 \\
&\qquad +L_1\ddot{\theta}_1\cos(\theta_1-\theta_2) \\
&\qquad -L_1\dot{\theta}_1^2\sin(\theta_1-\theta_2) \\
&\qquad +g\sin\theta_2 \\
&=0
\end{aligned}$$

$\boxed{\text{Problem 5}}$

$\boxed{\text{a}}$ The mass-balance gives

$$
\begin{aligned}
\frac{d}{dt}(\rho A h) &= w_1 - w_2 \\
\rho A \frac{d}{dt} h &= w_1 - w_2 \\
\dot{h} &= \frac{1}{\rho A}(w_1 - w_2)
\end{aligned}
\tag{15}
$$

$\boxed{\text{b}}$ The energy-balance gives

$$
\begin{aligned}
\frac{d}{dt}(\rho u A h) &= u_1 w_1 - u_2 w_2 + \frac{P_0 h}{\beta} \\
\frac{d}{dt}(\rho c_p T A h) &= c_p T_1 w_1 - c_p T w_2 + \frac{P_0 h}{\beta}
\end{aligned}
$$

where $u$ denotes the specific internal energy of water,

$$u = c_p T$$

Thus,

$$
\begin{aligned}
\rho A c_p \left(\dot{h} T + h \dot{T}\right) &= c_p T_1 w_1 - c_p T w_2 + \frac{P_0 h}{\beta} \\
c_p (w_1 - w_2) T + \rho A c_p h \dot{T} &= c_p T_1 w_1 - c_p T w_2 + \frac{P_0 h}{\beta} \\
\dot{T} &= \frac{w_1}{\rho A h}(T_1 - T) + \frac{P_0}{\rho A c_p \beta}
\end{aligned}
$$

where (15) has been applied.