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For questions during the exam:
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Exam in TTK4130 Modeling and Simulation

Wednesday, June 2nd 2010
09:00 – 13:00

Permitted aids (code A): All written and handwritten examination support materials are permitted.

Note: A Norwegian text is appended.

Answers in English, Norwegian, or a mixture of the two accepted.

Grades available: As specified by regulations.

Problem 1 (32 %)

Consider a stick of length ℓ with uniformly distributed mass m. It has center of mass/gravity C, about which it has a moment of inertia I_z. The stick is in contact with a frictionless horizontal surface, and moves due to the influence of gravity. See Figure 1.

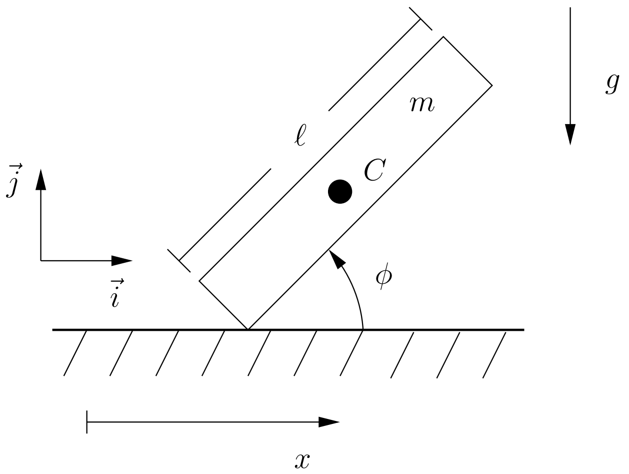

Figure 1: Stick sliding on frictionless surface

(4 %) (a) Choose appropriate generalized coordinates (the figure should give you some hints). What are the corresponding generalized (actuator) forces?

> **Solution:** A natural choice for generalized coordinates are the horizontal position of the center of mass/gravity (denoted x), and ϕ, the angle between the stick and the surface. An alternative to x could be the contact point between the stick and the surface. There are no (generalized) actuator forces corresponding to these coordinates. (A candidate answering 'gravity' might get full score if he/she uses it correctly in the rest of the Problem.)

(6 %) (b) What are the position, velocity, and angular velocity of the center of mass, as function of your chosen generalized coordinates (and/or their derivatives)?

Solution:

$$\vec{r}_c = x\vec{i} + \frac{\ell}{2}\sin\phi\vec{j}$$
$$\vec{v}_c = \dot{x}\vec{i} + \frac{\ell}{2}\dot{\phi}\cos\phi\vec{j}$$
$$\vec{w_{ib}} = \dot{\phi}\vec{k}$$

(Coordinate vectors also accepted for the position and velocity, scalar accepted for angular velocity.)

(6 %) (c) Write up the kinetic and potential energy of the stick, as function of your chosen generalized coordinates (and/or their derivatives).

Solution: The kinetic energy for the rigid body is

$$\begin{aligned} T &= \frac{1}{2}m\vec{v}_c \cdot \vec{v}_c + \frac{1}{2}\vec{w}_{ib} \cdot \vec{M}_{b/c} \cdot \vec{w}_{ib} \\ &= \frac{1}{2}m\left(\dot{x}^2 + \frac{\ell^2}{4}\dot{\phi}^2\cos^2\phi\right) + \frac{1}{2}I_z\dot{\phi}^2. \end{aligned}$$

The potential energy due to gravity is

$$U = mg\frac{\ell}{2}\sin\phi.$$

(16 %) (d) Derive the equations of motion for the stick.

Solution: It is probably easiest to use Lagrange's equation of motion. Define the Lagrangian $L = T - U$. Then the first equation of motion is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} = 0,$$

which reduces to

$$\ddot{x} = 0.$$

The second equation of motion is

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{\phi}}\right) - \frac{\partial L}{\partial \phi} = 0$$

which gives

$$\left(\frac{m\ell^2}{4}\cos^2\phi + I_z\right)\ddot{\phi} - \frac{m\ell^2}{4}\dot{\phi}^2\cos\phi\sin\phi + mg\frac{\ell}{2}\cos\phi = 0.$$

Problem 2 (16 %)

In this problem, we will look at an alternative representation of homogenous transformation matrices, using the concept of *dual numbers*. A dual number $\hat{g}$ is written

$$\hat{g} = g_r + \epsilon g_d$$

where g_r is the "real" part of the dual number and g_d is the "dual" or "pure" part of the dual number. The dual numbers (which could be in the form of vectors or matrices) have bases 1 and ϵ, which follow the rules:

$$\begin{aligned} 1 \cdot 1 &= 1, \\ 1 \cdot \epsilon = \epsilon \cdot 1 &= \epsilon, \\ \epsilon^2 &= 0. \end{aligned}$$

For a homogenous transformation matrix

$$\mathbf{T}_b^a = \begin{pmatrix} \mathbf{R}_b^a & \mathbf{r}_{ab}^a \\ \mathbf{0}^\top & 1 \end{pmatrix} \in SE(3),$$

the dual representation is

$$\hat{\mathbf{T}}_b^a = \mathbf{R}_b^a + \epsilon \left(\left(\mathbf{r}_{ab}^a\right)^\times \mathbf{R}_b^a \right).$$

(6 %) (a) Show that $\left(\hat{\mathbf{T}}_b^a\right)^\top \hat{\mathbf{T}}_b^a = \mathbf{I}$. (Remember that $(x^\times)^\top = -x^\times$.) Interpret this result.

Solution:

$$\begin{aligned}
\left(\hat{\mathbf{T}}_b^a\right)^\top \hat{\mathbf{T}}_b^a &= \left(\mathbf{R}_b^a + \epsilon\left(\left(\mathbf{r}_{ab}^a\right)^\times \mathbf{R}_b^a\right)\right)^\top \left(\mathbf{R}_b^a + \epsilon\left(\left(\mathbf{r}_{ab}^a\right)^\times \mathbf{R}_b^a\right)\right) \\
&= \left(\left(\mathbf{R}_b^a\right)^\top - \epsilon\left(\left(\mathbf{R}_b^a\right)^\top \left(\mathbf{r}_{ab}^a\right)^\times\right)\right) \left(\mathbf{R}_b^a + \epsilon\left(\left(\mathbf{r}_{ab}^a\right)^\times \mathbf{R}_b^a\right)\right) \\
&= \left(\mathbf{R}_b^a\right)^\top \mathbf{R}_b^a + \epsilon\left(\left(\mathbf{R}_b^a\right)^\top \left(\mathbf{r}_{ab}^a\right)^\times \mathbf{R}_b^a\right) - \epsilon\left(\left(\mathbf{R}_b^a\right)^\top \left(\mathbf{r}_{ab}^a\right)^\times \mathbf{R}_b^a\right) \\
&\qquad - \epsilon^2\left(\left(\mathbf{R}_b^a\right)^\top \left(\mathbf{r}_{ab}^a\right)^\times \left(\mathbf{r}_{ab}^a\right)^\times \mathbf{R}_b^a\right) \\
&= \mathbf{I}.
\end{aligned}$$

This means that the dual form of the homogenous transformation matrix is orthogonal. A consequence of this is that the inverse of such a matrix is its transpose.

(10 %) (b) Let $\mathbf{T}_b^a$ and $\mathbf{T}_c^b$ be homogenous transformation matrices and $\hat{\mathbf{T}}_b^a$ and $\hat{\mathbf{T}}_c^b$ their dual equivalents. Show that the product $\mathbf{T}_b^a\mathbf{T}_c^b$ is equivalent to the product $\hat{\mathbf{T}}_b^a\hat{\mathbf{T}}_c^b$.

Solution: We have that

$$\begin{aligned}
\hat{\mathbf{T}}_b^a\hat{\mathbf{T}}_c^b &= \left(\mathbf{R}_b^a + \epsilon\left(\left(\mathbf{r}_{ab}^a\right)^\times \mathbf{R}_b^a\right)\right)\left(\mathbf{R}_c^b + \epsilon\left(\left(\mathbf{r}_{bc}^b\right)^\times \mathbf{R}_c^b\right)\right) \\
&= \mathbf{R}_b^a\mathbf{R}_c^b + \epsilon\left(\mathbf{R}_b^a\left(\mathbf{r}_{bc}^b\right)^\times \mathbf{R}_c^b\right) + \epsilon\left(\left(\mathbf{r}_{ab}^a\right)^\times \mathbf{R}_b^a\mathbf{R}_c^b\right) \\
&\qquad + \epsilon^2\left(\left(\mathbf{r}_{ab}^a\right)^\times \mathbf{R}_b^a\left(\mathbf{r}_{bc}^b\right)^\times \mathbf{R}_c^b\right).
\end{aligned}$$

Then, using $\epsilon^2 = 0$ and

$$\left(\mathbf{r}_{bc}^a\right)^\times = \mathbf{R}_b^a\left(\mathbf{r}_{bc}^b\right)^\times \mathbf{R}_a^b \qquad \Rightarrow \qquad \mathbf{R}_b^a\left(\mathbf{r}_{bc}^b\right)^\times = \left(\mathbf{r}_{bc}^a\right)^\times \mathbf{R}_b^a$$

(see Example 89 in book), we get

$$\hat{\mathbf{T}}_b^a \hat{\mathbf{T}}_c^b = \mathbf{R}_c^a + \epsilon \left((\mathbf{r}_{bc}^a)^\times \mathbf{R}_c^a \right) + \epsilon \left((\mathbf{r}_{ab}^a)^\times \mathbf{R}_c^a \right)$$
$$= \mathbf{R}_c^a + \epsilon \left((\mathbf{r}_{ac}^a)^\times \mathbf{R}_c^a \right).$$

This is the dual form of

$$\mathbf{T}_b^a \mathbf{T}_c^b = \begin{pmatrix} \mathbf{R}_b^a & \mathbf{r}_{ab}^a \\ \mathbf{0}^\mathsf{T} & 1 \end{pmatrix} \begin{pmatrix} \mathbf{R}_c^b & \mathbf{r}_{bc}^b \\ \mathbf{0}^\mathsf{T} & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{R}_b^a \mathbf{R}_c^b & \mathbf{R}_b^a \mathbf{r}_{bc}^b + \mathbf{r}_{ab}^a \\ \mathbf{0}^\mathsf{T} & 1 \end{pmatrix} = \begin{pmatrix} \mathbf{R}_c^a & \mathbf{r}_{ac}^a \\ \mathbf{0}^\mathsf{T} & 1 \end{pmatrix}.$$

Problem 3 (32 %)

An incompressible liquid of substance C enters a perfectly mixed tank (a continuous stirred tank reactor, CSTR) with mass flow w_C and temperature T_C. In the tank, the substance reacts (e.g. due to the presence of a catalyst) to form the substance D with a rate JV, where J is the reaction rate per unit volume, and $V = Ah$ is the volume of the tank. The tank then consists of a mixture of C and D, which leaves the tank with mass flow w and temperature T. The mass of substance C in the tank is denoted m_C, and the mass of substance D is denoted m_D.

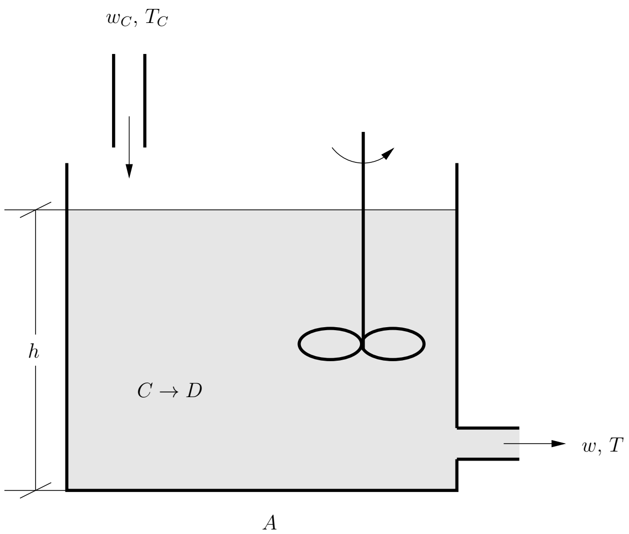

Figure 2: Tank reactor

(5 %) (a) Set up a differential equation for the level of the tank. (Hint: Use the ordinary overall mass balance. Assume that the average density ρ is constant.)

Solution: The mass balance equation is

$$\frac{\mathrm{d}}{\mathrm{d}t} \iiint_{V_f} \rho \mathrm{d}V = - \iint_{\partial V_f} \rho \vec{v} \cdot \vec{n} \mathrm{d}A$$
$$\frac{\mathrm{d}}{\mathrm{d}t} (\rho A h) = w_C - w$$
$$\frac{\mathrm{d}}{\mathrm{d}t} h = \frac{w_C - w}{\rho A}$$

(5 %) (b) In a material volume V_m, the following holds:

$$\frac{\mathrm{D}}{\mathrm{D}t}\iiint\limits_{V_m}\rho_C\mathrm{d}V=-\iiint\limits_{V_m}J\mathrm{d}V.$$

Use this together with the appropriate form of the transport theorem to explain that the mass balance for substance C on integral form in a fixed control volume V_f is

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint\limits_{V_f}\rho_C\mathrm{d}V=-\iiint\limits_{V_f}J\mathrm{d}V-\iint\limits_{\partial V_f}\rho_C\vec{v}\cdot\vec{n}\mathrm{d}A.$$

(In this particular case, the natural control volume, the volume of liquid in the tank, is not fixed, but this can be ignored since $\rho_C\vec{v}_c\cdot\vec{n}=0$ – expansion of the volume does not accumulate more of substance C.)

Solution: Take eq. (10.90) in the book, set $\phi=\rho_C$ and insert the first equation to obtain the result.

(5 %) (c) Use this to write up the mass balance for the mass of substance C in the tank ($\frac{\mathrm{d}}{\mathrm{d}t}m_C=\ldots$). Assume here, and for the rest of the problem, that J is proportional to the density of substance C, $J=k\frac{m_C}{V}$, and that the outflow of substance C is proportional to the mass ratio of substance C to the total mass in the tank, and the total outflow, $w_{C,out}=\frac{m_C}{m_C+m_D}w$.

Solution:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(\rho_C Ah)&=w_C-JV-w_{C,out}\\ \frac{\mathrm{d}}{\mathrm{d}t}m_C&=w_C-km_C-\frac{m_C}{m_C+m_D}w\end{aligned}$$

(5 %) (d) What is the mass balance equation on integral form for substance D (in a fixed volume)? Use this to write up the mass balance of substance D.

Solution: For substance D, we have

$$\frac{\mathrm{D}}{\mathrm{D}t}\iiint\limits_{V_m}\rho_D\mathrm{d}V=\iiint\limits_{V_m}J\mathrm{d}V.$$

Insertion into (10.90) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint\limits_{V_f}\rho_D\mathrm{d}V=\iiint\limits_{V_f}J\mathrm{d}V-\iint\limits_{\partial V_f}\rho_D\vec{v}\cdot\vec{n}\mathrm{d}A.$$

Solving the integrals, give

$$\frac{\mathrm{d}}{\mathrm{d}t}m_D=JV-w_D=km_C-\frac{m_D}{m_C+m_D}w.$$

(5 %) (e) Check that the solution in (c) and (d) agrees with the answer in (a).

Solution:

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}m &= \frac{\mathrm{d}}{\mathrm{d}t}m_C + \frac{\mathrm{d}}{\mathrm{d}t}m_D \\ &= w_C - km_C - \frac{m_C}{m_C + m_D}w + km_C - \frac{m_D}{m_C + m_D}w \\ &= w_C - w.\end{aligned}$$

This agrees with the solution to (a).

(7 %) (f) Set up a differential equation for the temperature in the tank. Assume that the heat generated by the reaction is proportional to J, with proportionality constant c. Disregard kinetic energy, potential energy and pressure work. Assume no 'heat flux' (the tank is well insulated). Assume the internal energy is $u = c_pT$.

Solution: The book does not treat energy balances with "internally generated" energy. We must therefore derive the energy balance on integral form for this (as we did for the mass balance above).

Under the assumptions made, (11.164) takes the form

$$\frac{\mathrm{D}}{\mathrm{D}t}\iiint_{V_m} \rho u \mathrm{d}V = \iiint_{V_m} cJ\mathrm{d}V$$

($e = u$, pressure work and heat flux ignored, but heat from reaction added.) Insertion into (11.169) (for a fixed volume) gives

$$\frac{\mathrm{d}}{\mathrm{d}t}\iiint_{V_f} \rho u \mathrm{d}V = \iiint_{V_f} cJ\mathrm{d}V - \iint_{\partial V_f} \rho u\vec{v}\cdot\vec{n}\mathrm{d}A.$$

Inserting $u = c_pT$ and resolving the integrals, we get

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho c_p TV\right) &= cJV + w_C c_p T_C - wc_pT \\ \rho c_p V\frac{\mathrm{d}}{\mathrm{d}t}T + \rho c_p TA\frac{\mathrm{d}}{\mathrm{d}t}h &= cJV + w_C c_p T_C - wc_pT\end{aligned}$$

Insertion of the result in (a), and using $JV = km_C$,

$$\begin{aligned}\rho c_p Ah\frac{\mathrm{d}}{\mathrm{d}t}T + c_pT(w_c - w) &= ckm_C + w_C c_p T_C - wc_pT \\ \frac{\mathrm{d}}{\mathrm{d}t}T &= \frac{ckm_C + c_p w_c(T_C - T)}{\rho c_p Ah}\end{aligned}$$

Correct result (without derivation of the energy balance) will give full score.

Problem 4 (20 %)

Given the two following Butcher arrays:

$$\begin{array}{c|cc} 0 & \frac{1}{2} & 0 \\ 1 & \frac{1}{2} & 0 \\ \hline & 0 & 1 \end{array} \qquad\qquad \begin{array}{c|cc} 0 & & \\ \frac{1}{2} & \frac{1}{2} & \\ \hline & 0 & 1 \end{array}$$

(6 %) (a) Write up the equations for each of these methods, for a system $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y})$. What is the main difference between these two methods (in one brief sentence)?

Solution:

$$\begin{aligned} \mathbf{k}_1 &= \mathbf{f}(\mathbf{y}_n + \frac{h}{2}\mathbf{k}_1) \\ \mathbf{k}_2 &= \mathbf{f}(\mathbf{y}_n + \frac{h}{2}\mathbf{k}_1) \\ \mathbf{y}_{n+1} &= y_n + h\mathbf{k}_2 \end{aligned} \qquad \begin{aligned} \mathbf{k}_1 &= \mathbf{f}(\mathbf{y}_n) \\ \mathbf{k}_2 &= \mathbf{f}(\mathbf{y}_n + \frac{h}{2}\mathbf{k}_1) \\ \mathbf{y}_{n+1} &= y_n + h\mathbf{k}_2 \end{aligned}$$

The main difference is that the first method is implicit (the first equation is an implicit equation), while the second is explicit.

(8 %) (b) Calculate the stability function for each of these methods as function of $s = h\lambda$ (answer without calculations give 75% score).

Solution: The methods can be recognized as Lobatto IIIB order 2 and modified Euler, and with the help of Example 226 and Table 14.2 we can find that the stability functions are

$$\frac{1+\frac{1}{2}s}{1-\frac{1}{2}s}, \qquad 1 + s + \frac{s^2}{2}.$$

For calculations, we can use formula 14.142 (or 14.112 for the explicit method).

(6 %) (c) For each om the methods: Is the method A-stable? L-stable? Substantiate your answer.

Solution: The first method has the same stability function as the trapezoidal rule, which we know is A-stable (see Figure 14.16 in the book). As $R(j\infty) = 1$, it is not L-stable. Method two is an explicit Runge-Kutta method, and therefore not A-stable, and hence not L-stable either.