EKSAMEN

MATEM. 4K (TMA4120)

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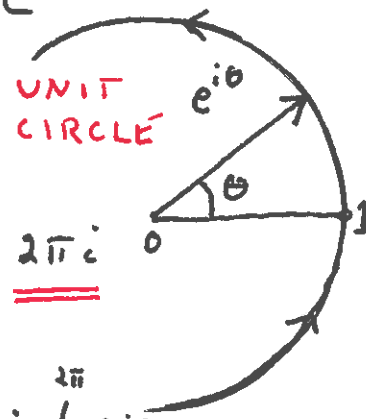
① $z = 1 \cdot e^{i\theta}, \quad 0 \le \theta \le 2\pi, \quad \bar{z} = e^{-i\theta}$

$dz = i e^{i\theta} d\theta = i z\, d\theta$



$$\oint_{|z|=1} \bar{z}\, dz = \int_0^{2\pi} e^{-i\theta} i e^{i\theta} d\theta = i\int_0^{2\pi} d\theta = \underline{\underline{2\pi i}}$$

$$\oint_{|z|=1} z\, dz = \int_0^{2\pi} e^{i\theta} i e^{i\theta} d\theta = i\int_0^{2\pi} e^{2i\theta} d\theta = \frac{i}{2i}\Big/_0^{2\pi} e^{2i\theta} = \underline{\underline{0}}$$

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② $y''(t) + y(t) = (1 - u(t-\pi))\, 2\sin(2t)$

The Laplace transform is:

$$s^2 Y(s) + Y(s) = \mathcal{L}\{2\sin(2t)\} - \mathcal{L}\{2u(t-\pi)\sin(2t)\}$$

$\boxed{\sin(2t) = \sin(2t - 2\pi)}$ period

$$= 2\,\frac{2}{4+s^2} - \underbrace{\mathcal{L}\{2u(t-\pi)\sin[2(t-\pi)]\}}_{2e^{-\pi s}\frac{2}{4+s^2}}.$$

t-shift

$$Y(s) = 4(1-e^{-\pi s})\,\frac{1}{4+s^2}\cdot\frac{1}{1+s^2}$$

partial fractions

$$= \frac{4}{3}(1-e^{-\pi s})\left\{\underbrace{\frac{1}{1+s^2}}_{\sin t} - \underbrace{\frac{1}{4+s^2}}_{\frac{1}{2}\sin(2t)}\right\}.$$

$$y(t) = \frac{4}{3}\sin(t) - \frac{2}{3}\sin(2t)$$

$$- \frac{4}{3}u(t-\pi)\cdot\sin(t-\pi) + \frac{2}{3}u(t-\pi)\cdot\sin(2(t-\pi))$$

$$= \begin{cases} \underline{\frac{4}{3}\sin(t) - \frac{2}{3}\sin(2t)}, & t \leq \pi \\ \underline{\frac{8}{3}\sin(t)}, & t \geq \pi \end{cases}$$

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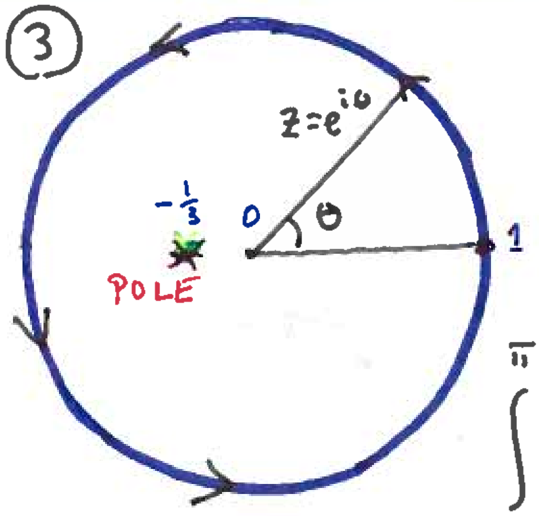
(3)



$$z = e^{i\theta} \qquad \frac{dz}{iz} = d\theta$$

$$\cos\theta = \frac{e^{i\theta}+e^{-i\theta}}{2} = \frac{1}{2}\left(z+\frac{1}{z}\right)$$

Do not write $\bar{z}$!

$$\int_{-\pi}^{\pi} \frac{d\theta}{5+3\cos(\theta)} =$$

$$= \oint_{|z|=1} \frac{dz}{iz\left(5+\frac{3}{2}\left(z+\frac{1}{z}\right)\right)} = \frac{2}{i}\oint \frac{dz}{\underbrace{3z^2+10z+3}_{3(z+3)\left(z+\frac{1}{3}\right)}}$$

$$= \frac{2}{i}\cdot 2\pi i \operatorname*{Res}_{z=-\frac{1}{3}}\left[\frac{1}{3(z+3)\left(z+\frac{1}{3}\right)}\right]$$

$$= 4\pi\,\frac{1}{3\left(-\frac{1}{3}+3\right)} = \underline{\underline{\frac{\pi}{2}}}$$

The pole $z=-3$ is outside the unit circle, only $z=-\frac{1}{3}$ is included.

(4a) The sum

$$S(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos(nx) + b_n \sin(nx) \right)$$

is

$$S(0) = \frac{f(0-0) + f(0+0)}{2} = \frac{2\pi - \pi}{2} = \frac{\pi}{2}$$

$$S(\pi) = f(\pi) = \pi$$

since the function $f(x)$ is piecewise smooth.

(4b) $$a_0 = \frac{\text{area}}{2\pi} = \frac{\frac{3}{2}\pi^2}{2\pi} = \frac{3\pi}{4}$$

$$f(x) = \begin{cases} 2\pi - x, & \text{when } 0 < x \leq \pi \\ 3\pi - 2x, & \text{when } \pi \leq x < 2\pi \end{cases}$$

and $f(x + 2k\pi) = f(x)$; period $= 2\pi$.

$$a_n = \frac{1}{\pi} \int_0^{2\pi} f(x) \cos(nx)\, dx = \frac{1}{\pi} \int_0^{\pi} (2\pi - x) \cos(nx)\, dx$$

Integrate by parts.

$$+ \frac{1}{\pi} \int_{\pi}^{2\pi} (3\pi - 2x) \cos(nx)\, dx = \dots$$

$$= \frac{\cos(n\pi) - 1}{\pi n^2} = \begin{cases} 0, & n = 2, 4, 6, \dots \\ -\frac{2}{\pi n^2}, & n = 1, 3, 5, \dots \end{cases}$$

$$b_n = \frac{1}{\pi}\int_0^{\pi}(2\pi - x)\sin(nx)\,dx + \frac{1}{\pi}\int_{\pi}^{2\pi}(3\pi - 2x)\sin(nx)\,dx$$

Integr. by parts

$$= \cdots\cdots = \underline{\underline{\frac{3}{n}}}, \quad n = 1, 2, 3, \ldots$$

The Fourier series is

$$\frac{3\pi}{4} - \frac{2}{\pi}\sum_{k=1}^{\infty}\frac{\cos[(2k-1)x]}{(2k-1)^2} + 3\sum_{n=1}^{\infty}\frac{\sin(nx)}{n}.$$

odd indices

5a $\quad u(x,t) = X(x)\,T(t)$

$$\frac{\partial u}{\partial x} = X'(x)\,T(t)$$

Endpoint condition:

$$\begin{cases} X'(0) = 0 \\ X'(\pi) = 0 \end{cases}$$

$$\boxed{\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + 2u, \quad u = u(x,t)}$$

The equation becomes

$$\boxed{XT' = X''T + 2XT}$$

Separation of variables:

$$\frac{T'}{T} = \frac{X''}{X} + 2$$

Easy to deal with in the form

$$\frac{X''}{X} = \frac{T'}{T} - 2 = \lambda$$

CONSTANT OF SEPARATION

- $\underline{X'' = \lambda X}$. Three cases:

1) $\underline{\lambda = 0}$ yields only $X = \text{const.}$

2) $\underline{\lambda > 0}$ $X(x) = a e^{\sqrt{\lambda}x} + b e^{-\sqrt{\lambda}x}$

Only $X \equiv 0$ satisfies the end-point conditions.

3) $\underline{\lambda < 0}$ Write $\lambda = -\omega^2 < 0$.

$$X(x) = a\cos(\omega x) + b\sin(\omega x)$$

$$X'(x) = -a\omega\sin(\omega x) + b\omega\cos(\omega x)$$

$X'(0) = 0 \iff b = 0$. Then

$X'(\pi) = 0$ requires $\omega = n = \text{integer}$.

Thus $X(x) = A_n \cos(nx)$, where

$n = 0$ (case $\lambda = 0$), $n = 1, 2, 3, \ldots$.

- $\underline{T' - 2T = -n^2 T}$. $T = C e^{(2-n^2)t}$

$n = 0, 1, 2, 3, \ldots$

$$\boxed{u(x,t) = A_n \cos(nx)\, e^{(2-n^2)t}}$$

<u>Remark:</u> The values $n = -1, -2, -3, \ldots$ are absorbed in the expression.

(56) By superposition

$$u(x,t) = \sum_{n=0}^{\infty} A_n \cos(nx)\, e^{(2-n^2)t}$$

$$u(x,0) = \sum_{n=0}^{\infty} A_n \cos(nx)$$

FOURIER COSINE SERIES

$$\stackrel{?}{=} (1+\cos(x))^2$$

$$= 1 + 2\cos(x) + \cos^2(x)$$

$$= 1 + 2\cos(x) + \frac{\cos(2x)+1}{2}$$

THIS IS THE COSINE SERIES!

$$= \underbrace{\frac{3}{2}}_{A_0} + \underbrace{2}_{A_1}\cos(x) + \underbrace{\frac{1}{2}}_{A_2}\cos(2x)$$

$(A_n = 0,\ n \geq 3)$

Answer:

$$u(x,t) = \left(\frac{3}{2} + 2\cos(x)\, e^{-t} + \frac{1}{2}\cos(2x)\cdot e^{-4t}\right) e^{2t}$$

Remark: More difficult to use is

$$A_n = \frac{2}{\pi}\int_0^{\pi} (1+\cos(x))^2 \cos(nx)\, dx = \cdots\cdots\cdots$$

Oppgave 7 Løs integrallikningen

$$f(x) - \int_{-\infty}^{\infty} e^{-3|x-t|} f(t)\, dt = e^{-3|x|}.$$

First,

$$\mathcal{F}(e^{-a|x|}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-a|x|} e^{-i\omega x} dx \qquad (a > 0)$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{0} e^{x(a-i\omega)} dx + \frac{1}{\sqrt{2\pi}} \int_{0}^{\infty} e^{-(a+i\omega)x} dx = \cdots$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ \frac{1}{a-i\omega} + \frac{1}{a+i\omega} \right\} = \frac{2a}{\sqrt{2\pi}(a^2+\omega^2)}.$$

The equation contains the convolution $f * e^{-3|x|}$. Taking the Fourier transform of the equation and using the rule $\mathcal{F}(f*g) = \sqrt{2\pi}\,\hat{f}\,\hat{g}$, we get

$$\hat{f}(\omega) - \cancel{\sqrt{2\pi}} \frac{2\cdot 3}{\cancel{\sqrt{2\pi}}(9+\omega^2)} \hat{f}(\omega) = \frac{2\cdot 3}{\sqrt{2\pi}(9+\omega^2)}$$

$$\frac{3+\omega^2}{\cancel{9+\omega^2}} \hat{f}(\omega) = \frac{6}{\sqrt{2\pi}\cancel{(9+\omega^2)}}$$

$$\hat{f}(\omega) = \frac{6}{\sqrt{2\pi}(3+\omega^2)} = \sqrt{3}\, \frac{2\cdot\sqrt{3}}{\sqrt{2\pi}(\sqrt{3}^2+\omega^2)} \qquad (a = \sqrt{3}\,!)$$

$$\boxed{f(x) = \sqrt{3}\, e^{-\sqrt{3}\, x}}$$