Problem 1

a) Find the inverse Laplace transform of

$$F(s) = \frac{s(s+2)}{s^3 + s^2 + s + 1}.$$

(Hint: $s^3 + s^2 + s + 1 = (s^2 + 1)(s + 1)$.)

b) Solve the integral equation $f(t) = \cos t + e^{-2t} \int_0^t f(\tau)e^{2\tau}d\tau$.

Problem 2   Let $f(x)$ be the 2-periodic function such that $f(x) = 1 - |x|$ for $|x| < 1$.

a) Find the Fourier series of $f(x)$.

b) Find a particular solution of the differential equation $y'' + 9y = f(x)$.

Problem 3   Compute the Fourier transform of the function

$$f(x) = \begin{cases} e^{-|x|} - e^{-1}, & |x| < 1 \\ 0, & |x| \geq 1 \end{cases}$$

and write down the solution of the initial value problem for the heat equation $u_t = u_{xx}$ for $-\infty < x < \infty$, $t > 0$ $u(x, 0) = f(x)$ in integral form.

Problem 4   Find the image of the half-plane $\{\mathrm{Re}\, z > 0\}$ under the mapping $w = e^z$.

Problem 5   Consider the series $\sum_{n=1}^{\infty} \frac{3^n}{2n} z^{2n}$.

a) Find the radius of convergence of this series.

b) Let $f(z)$ be the sum of the series, write down the series expansion of $f'(z)$ and find $f'(z)$.

c) Show that $f(z) = -\frac{1}{2}\mathrm{Ln}(1 - 3z^2)$ in a disk around the origin.

Problem 6   Evaluate the integral $\int_{-\pi}^{\pi} \frac{d\theta}{1 + \sin^2 \theta}$.