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For questions during the exam:
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Exam in TTK4130 Modeling and Simulation

Thursday, June 6th 2013
09:00 – 13:00

Permitted aids (code A): All written and handwritten examination support materials are permitted.

Note: A Norwegian text is appended.

Answers in English, Norwegian, or a mixture of the two accepted.

Grades available: As specified by regulations.

Problem 1 (15 %)

Consider the open tank system in Figure 1. Assume that the volumetric flow into the tank, q_i, is known, that the fluid is incompressible with (constant) density ρ, and that the flow exiting the tank is given by $q_o = C\sqrt{p - p_0}$, where C is a constant, p is the pressure in the bottom of the tank, and p_0 is the pressure outside the tank.

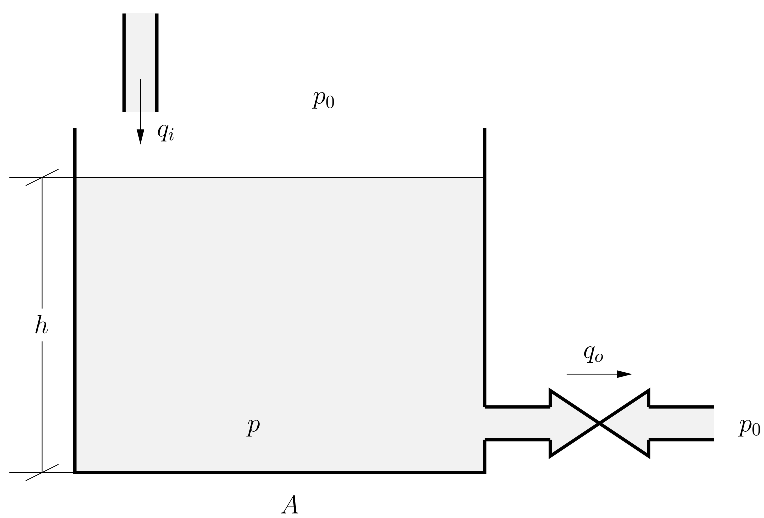

Figure 1: Open tank system

(5 %) (a) Set up a model for the level in the tank, h.

Solution: Mass balance:

$$\frac{\mathrm{d}m}{\mathrm{d}t} = w_i - w_0$$

$$\rho A \frac{\mathrm{d}h}{\mathrm{d}t} = \rho q_i - \rho q_0$$

Inserting $p = p_0 + \rho g h$ and the expression for q_o, we get

$$\frac{\mathrm{d}h}{\mathrm{d}t} = \frac{1}{A} q_i - \frac{C}{A}\sqrt{\rho g h}$$

(10 %) (b) We now want to establish a passivity result for the tank model. Let the "liquid pressure" $y = p - p_0 = \rho g h$ be an output, and choose (half) the potential energy in the tank as a storage function,

$$V(h) = \frac{1}{2} mgh.$$

Choose an appropriate input u, and show that the tank model is passive with this input and $y = p - p_0$ as output.

Solution: We first note that using $m = \rho V = \rho A h$, the storage function can be written

$$V(h) = \frac{1}{2} mgh = \frac{1}{2} \rho A g h^2,$$

and we see that $V(h) \geq 0$. Differentiating the storage function, we get

$$\dot{V} = \rho A g h \dot{h} = \rho g h q_i - \rho g h C \sqrt{\rho g h}$$

Since $y = p - p_0 = \rho g h$, and $g(h) = \rho g h C \sqrt{\rho g h} > 0$, we choose $u = q_i$ such that

$$\dot{V} = yu - g(h)$$

and the system is therefore passive with input $u = q_i$ and output $y = p - p_0 = \rho g h$ (p. 63 in the book).

Problem 2 (40 %)

(10 %) (a) A closed tank with constant volume V is completely filled with a *compressible* fluid with bulk modulus β. The fluid exits the tank through a valve, where we assume turbulent flow through the valve. Assume that the pressure p and density ρ are spatially constant in the tank, and that the density is related to the pressure as $\rho = ap + b$, where we assume a and b constant.
A model for the pressure is given as

$$\dot{p} = -C \sqrt{\frac{p - p_0}{ap + b}}.$$

Explain how this model is deduced from relevant balance laws, definitions and valve equations, and suggest an expression for the constant C.

Solution: Mass balance (m mass in tank, w mass flow out):

$$\frac{\mathrm{d}m}{\mathrm{d}t} = -w.$$

Using $m = \rho V$ and $w = \rho q$ we get

$$V \frac{\mathrm{d}\rho}{\mathrm{d}t} = -\rho q.$$

From (4.61),

$$\frac{\mathrm{d}p}{\beta} = \frac{\mathrm{d}\rho}{\rho},$$

which gives

$$\frac{V}{\beta} \frac{\mathrm{d}p}{\mathrm{d}t} = -q. \tag{1}$$

Assuming turbulent exit flow, we can use (4.1) in the book:

$$q = C_d A \sqrt{\frac{2}{\rho}(p - p_0)} = C_d A \sqrt{\frac{2(p - p_0)}{ap + b}}. \tag{2}$$

Putting (1) and (2) together, we get

$$\begin{aligned} \frac{\mathrm{d}p}{\mathrm{d}t} &= -\frac{\beta C_d A}{V} \sqrt{\frac{2(p - p_0)}{ap + b}} \\ &= -C \sqrt{\frac{p - p_0}{ap + b}}, \end{aligned}$$

where

$$C = \frac{\beta C_d A \sqrt{2}}{V}.$$

Side-remark: $\rho = ap + b$ can be deduced from writing (4.61) as

$$\frac{p - p_0}{\beta} = \frac{\rho - \rho_0}{\rho_0},$$

(valid at least for small deviations from ρ_0 and p_0) which gives $a = \frac{\rho_0}{\beta}$ and $b = \rho_0 \left(1 - \frac{p_0}{\beta}\right)$.

(8 %) (b) Linearize the model around a pressure p^*. Explain how this model can give problems when using explicit solvers with automatic adjustment of step size.

Solution: The linearization is

$$\Delta \dot{p} = -\frac{C}{2\sqrt{\frac{p^* - p_0}{ap^* + b}}} \frac{ap_o + b}{(ap^* + b)^2} \Delta p$$

When $p^* \to p_0$, the eigenvalue goes to infinity, which means that the step size of explicit solvers must approach zero to maintain stability. This again implies that the computational effort goes to infinity (the solver will "freeze").

(7 %) (c) Despite the potential problems, we want to use a method described by the following Butcher array to simulate the system:

$$\begin{array}{c|ccc} 0 & & & \\ 1/2 & 1/2 & & \\ 1 & -1 & 2 & \\ \hline & 1/6 & 2/3 & 1/6 \end{array}$$

Write "pseudo-code" for integrating the system in (a) one time-step using this method. Is this an implicit or explicit method?

Solution: This is an explicit Runge-Kutta method with 3 stages (and of order 3, but this was not asked for here – see below).

Given p_n (pressure at time t_n), we calculate p_{n+1} by

1. $k_1 := -C\sqrt{\frac{p_n - p_0}{ap_n + b}}$

2. $k_2 := -C\sqrt{\frac{(p_n+\frac{1}{2}k_1)-p_0}{a(p_n+\frac{1}{2}k_1)+b}}$
3. $k_3 := -C\sqrt{\frac{(p_n-k_1+2k_2)-p_0}{a(p_n-k_1+2k_2)+b}}$
4. $p_{n+1} = p_n + \frac{1}{6}(k_1 + 4k_2 + k_3)$

(All steps are explicit.)

(10 %) (d) Calculate the stability function for the Runge-Kutta method in (c).

Solution: We use (14.112), which says

$$\begin{aligned}
R_E(h\lambda) &= \det\left[I - h\lambda\left(A - eb^{\mathsf{T}}\right)\right] \\
&= \det\left[\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} - h\lambda\left(\begin{pmatrix}0&0&0\\1/2&0&0\\-1&2&0\end{pmatrix} - \begin{pmatrix}1\\1\\1\end{pmatrix}\begin{pmatrix}1/6 & 2/3 & 1/6\end{pmatrix}\right)\right] \\
&= \det\left[\begin{pmatrix}1&0&0\\0&1&0\\0&0&1\end{pmatrix} - \frac{h\lambda}{6}\begin{pmatrix}-1&-4&-1\\2&-4&-1\\-7&8&-1\end{pmatrix}\right] \\
&= \frac{1}{6}(h\lambda)^3 + \frac{1}{2}(h\lambda)^2 + h\lambda + 1
\end{aligned}$$

(5 %) (e) What is the order of the Runge-Kutta method in (c)? Explain how you reason.

Solution: The stability function coincides with the three first terms in the Taylor series expansion of $e^{h\lambda}$, so the order of the method is 3 (see book p. 533).

Problem 3 (20 %)

Consider a system consisting of three identical beads of mass m. The beads are sliding without friction on a fixed, horizontal circular hoop, with radius r, and are connected by four identical, massless springs with spring constant k. The springs are attached to the beads and to a fixed point as shown in Figure 2. Assume that the springs are unstretched/uncompressed in the positions in the figure, and that they stretch/compress along the hoop/circle.

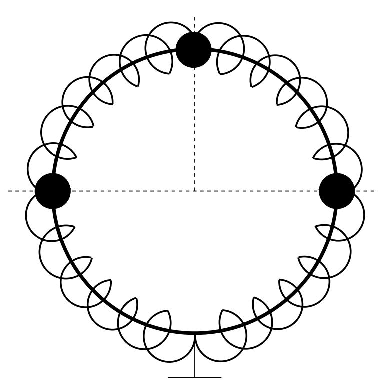

Figure 2: Three spring-connected beads on a hoop. The beads move along the horizontal, circular hoop. Two of the springs have one end attached to a fixed point.

(10 %) (a) Choose appropriate generalized coordinates. What are the kinetic and potential energy as function of these coordinates?

> **Solution:** We choose angles θ_1, θ_2 and θ_3, deviations from the resting position indicated in the figure, as generalized coordinates (other angles are possible). The kinetic and potential energy are
>
> $$T = \frac{1}{2} m r^2 \left(\dot{\theta}_1^2 + \dot{\theta}_2^2 + \dot{\theta}_3^2)\right)$$
> $$U = \frac{1}{2} k r^2 \left(\theta_1^2 + (\theta_2 - \theta_1)^2 + (\theta_3 - \theta_2)^2 + \theta_3^2\right)$$

(10 %) (b) What are the equations of motion for this system?

> **Solution:** We use Lagrange mechanics: Let $\mathcal{L} = T - U$, for θ_1 we then get
>
> $$\frac{\mathrm{d}}{\mathrm{d}t} \frac{\partial \mathcal{L}}{\partial \dot{\theta}_1} - \frac{\partial \mathcal{L}}{\partial \theta_1} = 0$$
> $$m r^2 \ddot{\theta}_1 + k r^2 (2\theta_1 - \theta_2) = 0$$
>
> Doing this for all three generalized coordinates, and defining $\omega = \sqrt{\frac{k}{m}}$, we get
>
> $$\ddot{\theta}_1 = -\omega^2 (2\theta_1 - \theta_2)$$
> $$\ddot{\theta}_2 = -\omega^2 (-\theta_1 + 2\theta_2 - \theta_3)$$
> $$\ddot{\theta}_3 = -\omega^2 (-\theta_2 + 2\theta_3)$$

Problem 4 (25 %)

(8 %) (a) Given the matrix

$$\mathbf{R} = \frac{\sqrt{2}}{2} \begin{pmatrix} 1 & 0 & 1 \\ 0 & \sqrt{2} & 0 \\ -1 & 0 & 1 \end{pmatrix}.$$

Show that this is a rotation matrix. What simple rotation does this matrix correspond to?

> **Solution:** This is a rotational matrix as $\mathbf{R}^\mathsf{T}\mathbf{R} = \mathbf{I}$ and $\det(\mathbf{R}) = 1$. As we can write
>
> $$\mathbf{R} = \begin{pmatrix} \cos 45^\circ & 0 & \sin 45^\circ \\ 0 & 1 & 0 \\ -\sin 45^\circ & 0 & \cos 45^\circ \end{pmatrix}$$
>
> we recognize this as a 45° rotation about the y-axis. See book p. 222.

(8 %) (b) A coordinate system b with orthogonal unit vectors $\vec{b}_1$, $\vec{b}_2$ and $\vec{b}_3$ along the coordinate axes, undergoes a rotation. The resulting coordinate system is called c and has orthogonal unit vectors $\vec{c}_1$, $\vec{c}_2$ and $\vec{c}_3$ along the coordinate axes. Write the rotation matrix as

$$\mathbf{R}_c^b = \begin{pmatrix} \mathbf{n} & \mathbf{s} & \mathbf{a} \end{pmatrix}.$$

Give an interpretation of the column vectors $\mathbf{n}$, $\mathbf{s}$ and $\mathbf{a}$ in terms of the orthogonal unit vectors mentioned above.

Solution: Since

$$\mathbf{u}^b = \mathbf{R}_c^b \mathbf{u}^c,$$

and $\mathbf{c}_1^c = (1,0,0)$, we see that

$$\mathbf{n} = \mathbf{R}_c^b \mathbf{c}_1^c = \mathbf{c}_1^b.$$

That is,

$$\mathbf{n} = \mathbf{c}_1^b, \quad \mathbf{s} = \mathbf{c}_2^b, \quad \text{and} \quad \mathbf{a} = \mathbf{c}_3^b.$$

See also (6.94) in the book.

(9 %) (c) A rigid body rotates with angular velocity $\vec{\omega}_{ib}$ about the origin of an inertial frame (i). Two fixed points in the rigid body has coordinates $\mathbf{r}_1^b = (1,0,0)^\mathsf{T}$ and $\mathbf{r}_2^b = (0,0,1)^\mathsf{T}$ in the body-fixed coordinate system (b). Assume the velocity in the first point is $\mathbf{v}_1^b = (0,3,5)^\mathsf{T}$. What is the x-component of the velocity, $\mathbf{v}_2^b$, in the second point?

Solution: We will use that

$$\vec{v} = \frac{{}^i\mathrm{d}}{\mathrm{d}t}\vec{r} = \frac{{}^b\mathrm{d}}{\mathrm{d}t}\vec{r} + \vec{\omega}_{ib} \times \vec{r} = \vec{\omega}_{ib} \times \vec{r}$$

where the latter equality follows if $\vec{r}$ is fixed in the body (as is the case for the two points in this problem). See (6.408) in the book.
Let $\boldsymbol{\omega}_{ib}^b = (\omega_x, \omega_y, \omega_z)^\mathsf{T}$. From the information given, we find that

$$\begin{pmatrix} 0 \\ 3 \\ 5 \end{pmatrix} = \mathbf{v}_1^b = (\boldsymbol{\omega}_{ib}^b)^\times \mathbf{r}_1^b = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \omega_z \\ -\omega_y \end{pmatrix}$$

That is, we can find two of the components of $\boldsymbol{\omega}_{ib}^b$ as $\omega_y = -5$ and $\omega_z = 3$. Then, from

$$\mathbf{v}_2^b = (\boldsymbol{\omega}_{ib}^b)^\times \mathbf{r}_2^b = \begin{pmatrix} 0 & -\omega_z & \omega_y \\ \omega_z & 0 & -\omega_x \\ -\omega_y & \omega_x & 0 \end{pmatrix} \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix} = \begin{pmatrix} \omega_y \\ -\omega_x \\ 0 \end{pmatrix}$$

we see that the x-component of $\mathbf{v}_2^b$ is $\omega_y = -5$.