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Matematikk 4K (tma 4120)

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(1) $f(z) = \underbrace{x^2 + y^2}_{u(x,y)} + i\,v(x,y)$

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial^2 u}{\partial x^2} = 2$$

$$\frac{\partial u}{\partial y} = 2y, \quad \frac{\partial^2 u}{\partial y^2} = 2$$

$$\Delta u = \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 2 + 2 = 4 \neq 0$$

Since u does not satisfy the Laplace equation, it cannot be the real part of an analytic function. It follows that $f(z)$ is not analytic. — No v will do!

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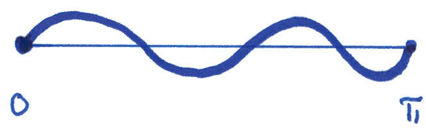
(2) The separation of variables and principle of superposition yields

$$u(x,t) = \sum_{n=1}^{\infty} B_n e^{-4n^2 t} \sin(nx)$$



The initial condition requires

$$u(x,0) = \sum_{n=1}^{\infty} B_n \sin(nx) \stackrel{?}{=} \sum_{n=1}^{\infty} \frac{\sin(nx)}{1+n^4}.$$

Thus $B_n = \dfrac{1}{1+n^4}$ and the desired solution is

$$u(x,t) = \sum_{n=1}^{\infty} \frac{e^{-4n^2t}\sin(nx)}{1+n^4}.$$

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(3) $|z|^2 = z\bar{z} = 1.$

$$\left|\frac{2z-i}{2+iz}\right|^2 = \frac{|2z-i|^2}{|2+iz|^2} = \frac{(2z-i)(2\bar{z}+i)}{(2+iz)(2-i\bar{z})}$$

$$= \frac{\overbrace{4|z|^2+1}^{5} + 2zi - 2\bar{z}i}{\underbrace{4+|z|^2}_{5} + 2iz - 2i\bar{z}} = \frac{5+2i(z-\bar{z})}{5+2i(z-\bar{z})} = 1$$

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(4) $u(x,0) = e^x$

$u_t(x,0) = \dfrac{1}{1+x^2}$

(5) $$\mathcal{L}\{f'(t)\} = \int_0^\infty e^{-st} f'(t)\,dt$$

$$= \int_0^\infty e^{-st}\,d f(t) = \underbrace{\Big/_0^\infty e^{-st} f(t)}_{\substack{0 - 1 f(0) \\ \text{when } s>0}} + \underbrace{s\int_0^\infty e^{-st} f(t)\,dt}_{s\,\mathcal{L}\{f(t)\}}$$

Thus $\mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - f(0)$.

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(6) $$\widehat{e^{-|x-2|}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-i\omega x} e^{-|x-2|}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{2} \underbrace{e^{-i\omega x} e^{x-2}}_{e^{x(1-i\omega)-2}}\,dx + \frac{1}{\sqrt{2\pi}} \int_{2}^{\infty} \underbrace{e^{-i\omega x} e^{2-x}}_{e^{-x(1+i\omega)+2}}\,dx$$

$$= \frac{1}{\sqrt{2\pi}} \Big/_{-\infty}^{2} \frac{e^{x-2-i\omega x}}{1-i\omega} + \frac{1}{\sqrt{2\pi}} \Big/_{2}^{\infty} \frac{e^{2-x-i\omega x}}{-(1+i\omega)}$$

$$= \frac{1}{\sqrt{2\pi}} \frac{e^{-2i\omega}}{1-i\omega} + \frac{1}{\sqrt{2\pi}} \frac{e^{-2i\omega}}{1+i\omega} = \sqrt{\frac{2}{\pi}}\, \frac{e^{-2i\omega}}{1+\omega^2}.$$

⑦ The function

$$f(z) = e^{2+\frac{1}{z}} = e^{2} e^{1/z}$$

$$= e^{2}\left[1 + z^{-1} + \frac{1}{2!} z^{-2} + \cdots\right]$$

$$= e^{2} + \boxed{e^{2}} z^{-1} + \frac{e^{2}}{2!} z^{-2} + \cdots \qquad (z \neq 0)$$

has an <u>essential singularity</u> at the origin and the residue

$$\operatorname*{Res}_{z=1}\left\{e^{2+1/z}\right\} = \boxed{e^{2}}$$

can be read off from the Laurent expansion above. By the residue theorem

$$\oint_{|z|=\frac{1}{10}} e^{2+\frac{1}{z}}\, dz = 2\pi i \cdot e^{2}.$$