NTNU – Trondheim
Norwegian University of
Science and Technology

For questions during the exam:
Leif Andersson, tel. 944 80 364.

Exam in TTK4130 Modeling and Simulation

Thursday, June 9th 2016
09:00 – 13:00

Permitted aids (code A): All written and handwritten examination support materials are permitted.

Answers in English, Norwegian, or a mixture of the two accepted.

Grades available: As specified by regulations.

Problem 1 (28 %)

Consider the following simulation method:

$$\begin{aligned}
\mathbf{k}_1 &= \mathbf{f}(\mathbf{y}_n, t_n) \\
\mathbf{k}_2 &= \mathbf{f}(\mathbf{y}_n + \frac{h}{4}\mathbf{k}_1 + \frac{h}{4}\mathbf{k}_2, t_n + \frac{h}{2}) \\
\mathbf{k}_3 &= \mathbf{f}(\mathbf{y}_n + h\mathbf{k}_2, t_n + h) \\
\mathbf{y}_{n+1} &= \mathbf{y}_n + \frac{h}{6}\left(\mathbf{k}_1 + 4\mathbf{k}_2 + \mathbf{k}_3\right)
\end{aligned}$$

(2 %) (a) Is this method explicit or implicit? How many stages does it have?

(3 %) (b) Write up the Butcher array for this method.

(1 %) (c) How can you see from the Butcher array whether this is an implicit or explicit method?

(4 %) (d) Comment on how much work it is to implement and solve this algorithm, compared to a general three-stage implicit Runge-Kutta method.

(8 %) (e) Find the stability function for this method. *Hint*:

$$\begin{pmatrix} a & 0 & 0 \\ d & b & 0 \\ 0 & e & c \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{a} & 0 & 0 \\ -\frac{d}{ab} & \frac{1}{b} & 0 \\ \frac{de}{abc} & -\frac{e}{bc} & \frac{1}{c} \end{pmatrix}$$

(4 %) (f) Is the method A-stable? L-stable? Justify the answer.

(6 %) (g) What is the order of the method? Justify the answer. Use the fact that for a Runge-Kutta method of order p, the stability function $R(s)$ approximates e^s with error $O(s^{p+1})$. (*Hint*: If you do long calculations/derivations, then you are probably attacking this the wrong (or at least not the most straightforward) way.)

Problem 2 (32 %)

The double inverted pendulum on a cart (DIPC) poses a challenging control problem. In a DIPC system, two rods are connected together on a moving cart as shown in Figure 1. The length of the first rod is denoted by l_1 and the length of the second rod by l_2. The mass of the cart is denoted by m_0, its length by l_0 and its width by b_0. The height of the cart is denoted by h_0. Both rods have a mass, which are denoted by m_1 and m_2. All masses are assumed to be concentrated into the centre of mass. The moments of inertia are denoted by I_i. Furthermore, the force τ is acting on the cart.

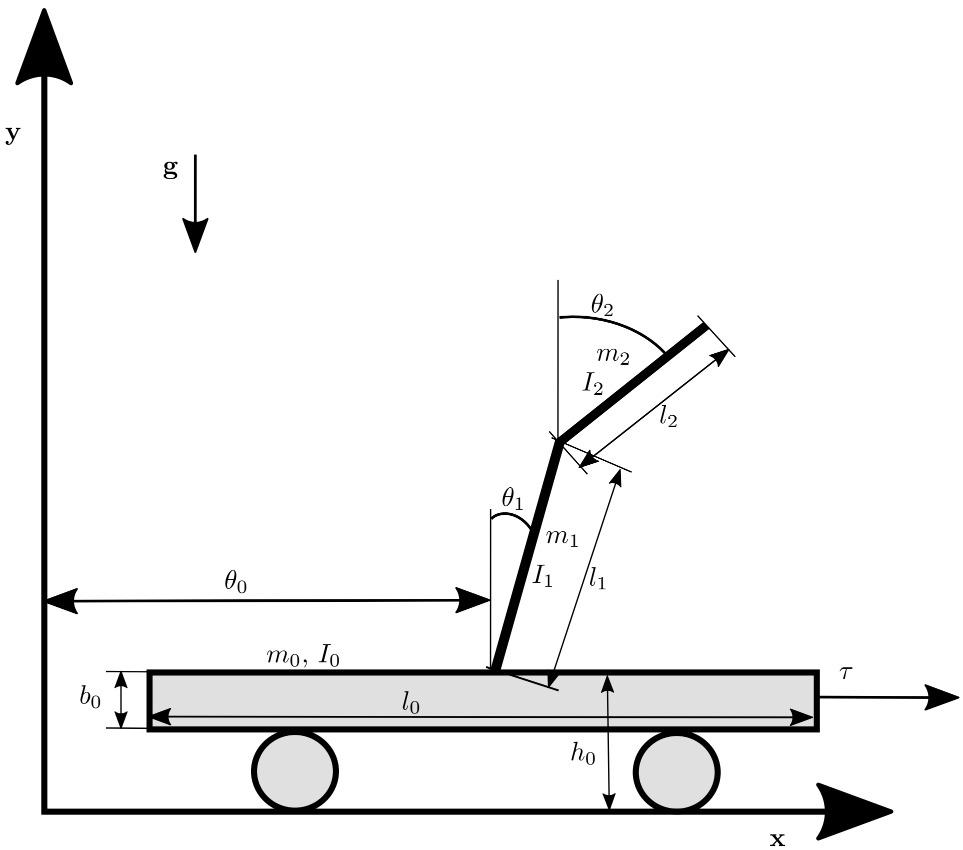

Figure 1: Double inverted pendulum on a cart

(4 %) (a) Choose generalized coordinates, and find the positions of the centers of mass for each of the three bodies (the cart and the two rods).

(10 %) (b) Find the kinetic energy of the system. (*Hint:* the following identity may simplify the expressions: $\cos(x-y) = \cos x \cos y + \sin x \sin y$.)

(4 %) (c) Find the potential energy of the system.

(14 %) (d) Find the equations of motion of the system.

Problem 3 (16 %)

Figure 2 illustrates two coordinate frames in three dimensions. Note that all of the unit vectors shown and the dashed line segment (of 5 cm) are in the same plane, and unit vectors pointing into or out of the paper plane is not shown.

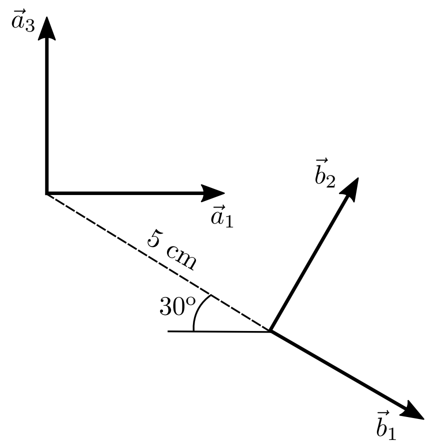

Figure 2: Two coordinate frames, rotated and translated.

(10 %) (a) What is $\mathbf{T}_b^a$, the homogenous transformation matrix representing the orientation and position of frame b relative to frame a?

(6 %) (b) What is $\mathbf{T}_a^b$?

Problem 4 (24 %)

Heat exchangers are basic unit processes which are found in almost every plant in the chemical process industries. As the name suggests, heat exchangers are used for energy (heat) exchange between a hot and a cold fluid stream (the hot stream heats the cold stream).

Heat exchangers are constructed in various ways, for example to maximize the energy transfer. Often they are considered as distributed systems since the temperatures will vary along the stream lines inside the heat exchanger. However, in the first part of this task we will develop a simple heat-exchanger model based on a very simple geometry (which can be an approximation for more complex geometries) and where we assume the temperatures at the hot and cold side (T_h and T_c) are spatially constant (that is, we assume the temperatures are averaged/lumped).

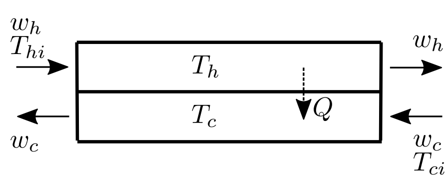

Figure 3: Simple heat exchanger

Consider Figure 3. The the hot and cold mass flow rates w_j, $j = \{h, c\}$ are assumed constant, such that the mass on each side are constant. The inlet temperatures are T_{hi} on the hot side, and T_{ci} on the cold side. The heat transfer from hot to cold side is

$$Q = UA\,(T_h - T_c)$$

where U is a heat transfer coefficient and A is the "effective contact area". The volumes of the hot and cold side are denoted V_h and V_c, respectively.

(10 %) (a) Derive a model for the temperatures T_h and T_c. Assume that the densities ρ_j and specific heats c_{pj}, $j = \{h, c\}$ are constant. Assume incompressible liquids and constant pressure, such that specific internal energy and specific enthalpy are equal.

(10 %) (b) The model found above, can be written

$$\begin{pmatrix} \dot{T}_h \\ \dot{T}_c \end{pmatrix} = \begin{pmatrix} -a_1 - k_1 & k_1 \\ k_2 & -a_2 - k_2 \end{pmatrix} \begin{pmatrix} T_h \\ T_c \end{pmatrix} + \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \begin{pmatrix} T_{hi} \\ T_{ci} \end{pmatrix}$$

where a_j and k_j, $j = \{h, c\}$ are appropriate positive constants (found in (a)). Let $\mathbf{x} = (x_1, x_2)^\mathsf{T} = (T_h, T_c)^\mathsf{T}$. Show that this model is passive from $\mathbf{u} = (u_1, u_2)^\mathsf{T} = (T_{hi}, T_{ci})^\mathsf{T}$ to $\mathbf{y} = (y_1, y_2)^\mathsf{T} = \left(\frac{a_1}{k_1} x_1, \frac{a_2}{k_2} x_2\right)^\mathsf{T}$, using the storage function

$$V(\mathbf{x}) = \frac{1}{2} \mathbf{x}^\mathsf{T} P \mathbf{x}, \quad P = \begin{pmatrix} \frac{1}{k_1} & 0 \\ 0 & \frac{1}{k_2} \end{pmatrix}.$$

A hint that may or may not be useful, is a special case of Gershgorin's Theorem:

$$\begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} \text{ is negative definite if } q_{11} < 0,\ q_{11} + q_{12} < 0 \text{ and } q_{22} < 0,\ q_{22} + q_{21} < 0.$$

Assume now that the temperatures of both streams are not averaged (not spatially constant), that is, that they vary along the streamlines. We often say in this case that the variables (temperatures) are *distributed*, and this is naturally modelled by partial differential equations. We will not do that here, but we will ask you to use your intuition and physical understanding to suggest how the temperature profiles will look.

(2 %) (c) Make a sketch of the temperature profiles in the hot and cold stream of a counter-flow heat exchanger (in principle like the one shown in Figure 3, where the hot and cold fluid enter on different ends). Draw them both in the same diagram, with position along x-axis and temperature along y-axis. Mark T_{ci} and T_{hi} in the sketch.

(2 %) (d) Do the same for a heat exchanger with parallel flows (where the hot and cold fluids enter at the same end, compared to Figure 3 where they enter on opposite ends). Why do you think counter-flow heat exchangers might be a better set-up than a heat exchanger with parallel flows?