Norges teknisk-naturvitenskapelige universitet
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LØSNINGSFORSLAG
EKSAMEN I TTK4130
MODELLERING OG SIMULERING
02. Juni 2008
Tid: 09:00-13:00

Hjelpemidler:

A: Alle kalkulatorer, trykte og håndskrevne hjelpemidler tillatt.

Sensur:

Sensuren vil bli avsluttet i henhold til gjeldende regelverk.

$\boxed{\text{Problem 1}}$

$\boxed{\text{a}}$ Stability function of the methods, A- and/or L- stability:

<u>Method 1:</u>

Applying the test system, $\dot{y} = \lambda y$ yields

$$\begin{aligned}
k_1 &= f\left(\mathbf{y}_n + \frac{h}{2}\mathbf{k}_1, t_n + \frac{h}{2}\right) \\
&= \lambda\left(x_n + \frac{h}{2}k_1\right) \\
&\rightarrow \; k_1\left(1 - \frac{h\lambda}{2}\right) = \lambda y_n \\
&\rightarrow \; k_1 = \frac{\lambda}{1 - \frac{h\lambda}{2}} y_n \\
\mathbf{y}_{n+1} &= \mathbf{y}_n + h\mathbf{k}_1 \\
&= y_n + \frac{h\lambda}{1 - \frac{h\lambda}{2}} y_n \\
&= \left[\frac{1 - \frac{h\lambda}{2} + h\lambda}{1 - \frac{h\lambda}{2}}\right] y_n \\
&= \left[\frac{1 + \frac{h\lambda}{2}}{1 - \frac{h\lambda}{2}}\right] y_n
\end{aligned}$$

Hence,

$$R\left(h\lambda\right) = \underline{\underline{\frac{1 + \frac{h\lambda}{2}}{1 - \frac{h\lambda}{2}}}}$$

Note that this is the implicit midpoint method (pg. 537), which is an IRK-method.

A-stable: Let $\lambda = \sigma + j\omega$, with $\sigma \leq 0$. It can be verified

$$\left|R\left(h\lambda\right)\right| = \frac{\sqrt{\left(1 - \left|\frac{h\sigma}{2}\right|\right)^2 + \left(\frac{h\omega}{2}\right)^2}}{\sqrt{\left(1 + \left|\frac{h\sigma}{2}\right|\right)^2 + \left(\frac{h\omega}{2}\right)^2}} \leq 0, \quad \forall \sigma \leq 0$$

Thus, the method is A-stable.

L-stable: The method is A-stable but

$$\lim_{\omega \rightarrow \infty} \left|R\left(j\omega h\right)\right| = 1 \neq 0$$

Thus, the method is <u>not</u> L-stable.

<u>Method 2:</u>

This is Heun's method (pg. 528). The stability function is given as

$$R(h\lambda) = 1 + h\lambda + \frac{1}{2}(h\lambda)^2 + \frac{1}{6}(h\lambda)^3$$

A-stable: From pg. 546 in the textbook, we know that none of ERK is A-stable. Thus, the method is not A-stable.

L-stable: The method is not A-stable. Thus, it is not L-stable.

<u>Method 3:</u>
This is Euler's method. The stability function is given as

$$R(h\lambda) = 1 + h\lambda$$

A-stable: negative. See above.

L-stable: negative. See above.

$\boxed{\text{b}}$ Eigenvalues

$$\lambda_{1,2} = \pm j$$

Stability: (marginal) stable.

Property of the solution $\left[\theta(t), \dot{\theta}(t)\right]^\top$: since $\lambda_{1,2} = \pm j$ it follows thus

$$\left|\left[\theta(t), \dot{\theta}(t)\right]^\top\right| = \left|\left[\theta(0), \dot{\theta}(0)\right]^\top\right| = 1, \forall t \geq 0$$

$\boxed{\text{c}}$ Due to the results above, one should choose method 1 (i.e. since $\lambda_{1,2} = \pm j$ and only method 1 ensures that $|R(h\lambda)| = 1, \forall h$).

$\boxed{\text{d}}$ Let $h = 0.1$ seconds. The global error at $t = 0.2$ sec. is given as

$$\begin{aligned} \mathbf{E}_2 &= \mathbf{y}_2 - \mathbf{y}(t = 0.2 \text{ sec}) \\ &= \mathbf{y}_2 - \begin{bmatrix} 0.9801 \\ -0.1987 \end{bmatrix} \end{aligned}$$

and

$$\begin{aligned} \mathbf{y}_0 &= [1, 0]^\top \\ \mathbf{y}_1 &= \mathbf{y}_0 + h\mathbf{f}(y_0, t_0) = [1, -0.1]^\top \\ \mathbf{y}_2 &= \mathbf{y}_1 + h\mathbf{f}(y_1, t_1) = [0.99, -0.2]^\top \end{aligned}$$

Hence,

$$\mathbf{E}_2 = [0.0099 - 0.0013]^{\top}$$

The stability condition is

$$|R(h\lambda)| \leq 1$$

Thus, the numerical solutions obtained with method 3 are not stable, i.e. $|\mathbf{y}_{n+1}| > |\mathbf{y}_n|, \forall n$ (the method is actually not stable for *any* $h \in \mathbb{R} \setminus \{0\}$).

$\boxed{\text{Problem 2}}$

$\boxed{\text{a}}$ Compute the rotation matrix that describes the rotation of 30° in roll and 30° in pitch, by multiplying two simple rotation matrices.

A.:

$$\begin{bmatrix} 0,5 & 0 & 0,86603 \\ 0 & 1 & 0 \\ -0,86603 & 0 & 0,5 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0,5 & -0,86603 \\ 0 & 0,86603 & 0,5 \end{bmatrix} = \\ = \begin{bmatrix} 0,5 & 0,75 & 0,433 \\ 0 & 0,5 & -0,86603 \\ -0,86603 & 0,433 & 0,25 \end{bmatrix} \tag{1}$$

$\boxed{\text{b}}$ Let $\mathbf{a}_1$ be a vector with $\|\mathbf{a}_1\| > 0$. Describe, justifying, all the rotation matrices $\mathbf{R}$ for which the following equality holds:

$$\|\mathbf{R}\mathbf{a}_1 - \mathbf{a}_1\| = 0 \tag{2}$$

(Hint: Parametrize $\mathbf{R}$ as $\mathbf{R}_{k,\theta}$).

A.: Parametrization of R

$$R = \cos(\theta) I + \sin(\theta) k^{\times} + (1 - \cos(\theta)) kk^T \tag{3}$$

Let us note solve the problem for a vector $a_2 = a_1 / \|\mathbf{a}_1\| = \lambda a_1$, $\lambda > 0$. We have $\|\mathbf{R}\mathbf{a}_1 - \mathbf{a}_1\| = \|\lambda \mathbf{R}\mathbf{a}_2 - \boldsymbol{\lambda}\mathbf{a}_2\| = |\lambda| \|\mathbf{R}\mathbf{a}_2 - \mathbf{a}_2\|$, so, by solving the equation $\|\mathbf{R}\mathbf{a}_2 - \mathbf{a}_2\| = 0$ we solve 2. In order for this to be verified, $Ra_2 = a_2$.

$$Ra_2 = \left[\cos(\boldsymbol{\theta}_2)\mathbf{I} + \sin(\boldsymbol{\theta}_2)\mathbf{k}^{\times} + (1 - \cos(\boldsymbol{\theta}_2))\mathbf{k}\mathbf{k}^T\right] . a_2 \tag{4}$$

In order for this expression to be equal to a_2, the term $\sin(\theta_2) k_2^{\times}$ must be identically zero, as $k_2^{\times} a_2$ gives a vector that is orthogonal to a_2. This only happens if $k \| a$ or $\theta_2 = k_1 \pi$, $k_1 \in \mathbb{Z}$.

Let us consider the case when $k \| a$. As both k and a are unit vectors, $k = \pm a$.

$$\begin{aligned}\mathbf{R}\mathbf{a}_2 &= \left[\cos(\boldsymbol{\theta}_2)\mathbf{I} + \sin(\boldsymbol{\theta}_2)\mathbf{k}^\times + (1-\cos(\boldsymbol{\theta}_2))\mathbf{k}\mathbf{k}^T\right].\mathbf{a}_2 = \\ &= \cos(\boldsymbol{\theta}_2)\mathbf{I}\mathbf{a_2} + (1-\cos(\boldsymbol{\theta}_2))\mathbf{k}\mathbf{k}^T\mathbf{a}_2 \end{aligned} \tag{5}$$

If $k = +a_2$

$$\begin{aligned}\mathbf{R}\mathbf{a}_2 &= \cos(\boldsymbol{\theta}_2)\mathbf{I}\mathbf{a_2} + (1-\cos(\boldsymbol{\theta}_2))\mathbf{k}(\mathbf{k}^T\mathbf{a}_2) = \\ &= \cos(\boldsymbol{\theta}_2)\mathbf{I}\mathbf{a_2} + (1-\cos(\boldsymbol{\theta}_2))\mathbf{a}_2(\mathbf{1}) = \mathbf{a}_2 \end{aligned} \tag{6}$$

which solves 2 for any θ_2.
If $k = -a_2$

$$\begin{aligned}\mathbf{R}\mathbf{a}_2 &= \cos(\boldsymbol{\theta}_2)\mathbf{I}\mathbf{a_2} + (1-\cos(\boldsymbol{\theta}_2))\mathbf{k}(\mathbf{k}^T\mathbf{a}_2) = \\ &= \cos(\boldsymbol{\theta}_2)\mathbf{I}\mathbf{a_2} + (1-\cos(\boldsymbol{\theta}_2))(-\mathbf{a}_2)(-\mathbf{1}) = \mathbf{a}_2 \end{aligned} \tag{7}$$

which also solves the problem for any θ_2.
In the case when $\theta_2 = k_1\pi$, we have

$$\begin{aligned}\mathbf{R}\mathbf{a}_2 &= \cos(\boldsymbol{\theta}_2)\mathbf{I}\mathbf{a_2} + (1-\cos(\boldsymbol{\theta}_2))\mathbf{k}\mathbf{k}^T\mathbf{a}_2 = \\ &= \left\{ \begin{array}{c} -\mathbf{I}\mathbf{a_2} + 2\mathbf{k}\mathbf{k}^T\mathbf{a}_2 = (2\mathbf{k}\mathbf{k}^T - \mathbf{I})\mathbf{a}_2,\ k_1 \textit{ odd} \\ \mathbf{I}\mathbf{a_2} = \mathbf{a}_2,\ k_1 \textit{ even} \end{array} \right. \end{aligned} \tag{8}$$

In the case of k_1 odd, the only solution comes when $k = \pm a_2$, which is identical to the solutions presented above. When k_1 is even, 2 is verified. But $R_{\mathbf{k},2k_2\pi} = R_{\mathbf{k},0} = I$.

Therefore, for 2 to be verified, the axis of rotation k must be equal to $\pm a_2$, which means, $k = \pm\lambda a_1$ and $R = R_{\mathbf{k},\theta}$, for any θ; the other solution (trivial) is that there is no rotation at all, $\theta = 0$, for any $\mathbf{k}$, and $R = I$.

$\boxed{\text{c}}$ Let $\mathbf{k}, \theta$ parametrize the rotation matrix $\mathbf{R}_d^a$. Show that

$$\mathbf{R}_d^a - (\mathbf{R}_d^a)^{-1} = 2\mathbf{k}^\times \sin(\theta) \tag{9}$$

A.: Note that

$$\mathbf{R}_d^a = \cos(\theta)\mathbf{I} + \sin(\theta)\mathbf{k}^\times + (1-\cos(\theta))\mathbf{k}\mathbf{k}^T \tag{10}$$

and

$$\begin{aligned}(\mathbf{R}_d^a)^{-1} &= (\mathbf{R}_d^a)^T = \cos(-\theta)\mathbf{I} + \sin(-\theta)\mathbf{k}^\times + (1-\cos(-\theta))\mathbf{k}\mathbf{k}^T = \\ &= \cos(\theta)\mathbf{I} - \sin(\theta)\mathbf{k}^\times + (1-\cos(\theta))\mathbf{k}\mathbf{k}^T \end{aligned} \tag{11}$$

Subtracting 11 from 10 we get the result.

$\boxed{\text{Problem 3}}$

$\boxed{\text{a}}$ Let $\mathbf{a}^b = [\ 0 \quad 0 \quad 9.8\ ]^T$. Let $\mathbf{R}(\boldsymbol{\eta},\boldsymbol{\epsilon}) = \mathbf{R}_a^b$ be a rotation matrix, with $(\boldsymbol{\eta},\boldsymbol{\epsilon}) = (\frac{\sqrt{2}}{2}, [\ 0 \quad \frac{\sqrt{2}}{2} \quad 0\ ]^T)$. Compute $\mathbf{a}^a$ using quaternion products.
A.:

$$\mathbf{a}^a = \mathbf{R}_b^a = (\mathbf{R}_a^b)^T = \mathbf{R}(\boldsymbol{\eta}, -\boldsymbol{\epsilon}). \tag{12}$$

$$\begin{aligned}
\begin{pmatrix} 0 \\ \mathbf{a}^a \end{pmatrix} &= \begin{pmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ -\frac{\sqrt{2}}{2} \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ \mathbf{a}^b \end{pmatrix} \otimes \begin{pmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ -\frac{\sqrt{2}}{2} \\ 0 \end{pmatrix} = \\
&= \begin{pmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ -\frac{\sqrt{2}}{2} \\ 0 \end{pmatrix} \otimes \begin{pmatrix} 0 \\ 0 \\ 0 \\ 9.8 \end{pmatrix} \otimes \begin{pmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ -\frac{\sqrt{2}}{2} \\ 0 \end{pmatrix} = \\
&= \frac{\sqrt{2}}{2} \begin{pmatrix} 0 \\ -9,8 \\ 0 \\ 9,8 \end{pmatrix} \otimes \begin{pmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ -\frac{\sqrt{2}}{2} \\ 0 \end{pmatrix} = \\
&= \frac{\sqrt{2}}{2}\frac{\sqrt{2}}{2} \begin{pmatrix} 0 \\ -19,6 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ -9,8 \\ 0 \\ 0 \end{pmatrix}
\end{aligned} \tag{13}$$

$\mathbf{a}^a = [\ -9,8 \quad 0 \quad 0\ ]^T$.

$\boxed{\text{b}}$ Let the rotation matrix $\mathbf{R}_a^c$ be given as a simple rotation of 60° in roll. Compute the unit quaternion associated to the rotation matrices $\mathbf{R}_a^c$ and $\mathbf{R}_b^c$.
A.: q_1 associated with $\mathbf{R}_a^c$

$$\begin{aligned}
\eta_1 &= \cos(\frac{\pi/3}{2}) = \frac{\sqrt{3}}{2} \\
\epsilon &= \sin(\frac{\pi/3}{2}) \left[\ 1 \quad 0 \quad 0\ \right]^T = \left[\ 1/2 \quad 0 \quad 0\ \right]^T
\end{aligned}$$

q_2 associated with $\mathbf{R}_b^c$

$$\mathbf{R}_b^c = \mathbf{R}_a^c \mathbf{R}_b^a$$

$$\begin{bmatrix} \eta_2 \\ \epsilon_2 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ 1/2 \\ 0 \\ 0 \end{bmatrix} \otimes \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 0 \\ -\frac{\sqrt{2}}{2} \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{6}}{4} \\ \frac{\sqrt{2}}{4} \\ -\frac{\sqrt{6}}{4} \\ -\frac{\sqrt{2}}{4} \end{bmatrix}$$

Problem 4

Given the system

$$\mathbf{M}(\mathbf{q})\ddot{\mathbf{q}} + [\mathbf{C}(\mathbf{q},\dot{\mathbf{q}}) + \mathbf{D}(\mathbf{q},\dot{\mathbf{q}})]\dot{\mathbf{q}} + \mathbf{K}\mathbf{q} = \boldsymbol{\tau} \tag{14}$$

Let

$$\begin{aligned} \mathbf{u} &= \boldsymbol{\tau} \\ \mathbf{y} &= \dot{\mathbf{q}} \end{aligned}$$

Define the storage function

$$V(\mathbf{q},\dot{\mathbf{q}}) = \frac{1}{2}\dot{\mathbf{q}}^\top \mathbf{M}(\mathbf{q})\dot{\mathbf{q}} + \frac{1}{2}\mathbf{q}^\top \mathbf{K}\mathbf{q}$$

- Since $\mathbf{K} > 0$ and $\mathbf{M}(\mathbf{q}) = \mathbf{M}(\mathbf{q})^\top > 0$, $\forall \mathbf{q} \neq 0$. Thus, $V(\mathbf{q},\dot{\mathbf{q}}) > 0, \forall (\mathbf{q},\dot{\mathbf{q}}) \neq 0$
- Taking the time derivative of V along the solution trajectories of (14) yields

$$\begin{aligned} \dot{V} &= \dot{\mathbf{q}}^\top \boldsymbol{\tau} - \dot{\mathbf{q}}^\top \mathbf{D}\dot{\mathbf{q}} + \frac{1}{2}\dot{\mathbf{q}}^\top \left(\dot{\mathbf{M}} - 2\mathbf{C}\right)\dot{\mathbf{q}} \\ &= \dot{\mathbf{q}}^\top \boldsymbol{\tau} - \dot{\mathbf{q}}^\top \mathbf{D}\dot{\mathbf{q}} \\ &= \mathbf{y}^\top \mathbf{u} - g(\mathbf{q},\dot{\mathbf{q}}) \end{aligned}$$

  where the skew symmetric property of $\left(\dot{\mathbf{M}} - 2\mathbf{C}\right)$ has been applied, and

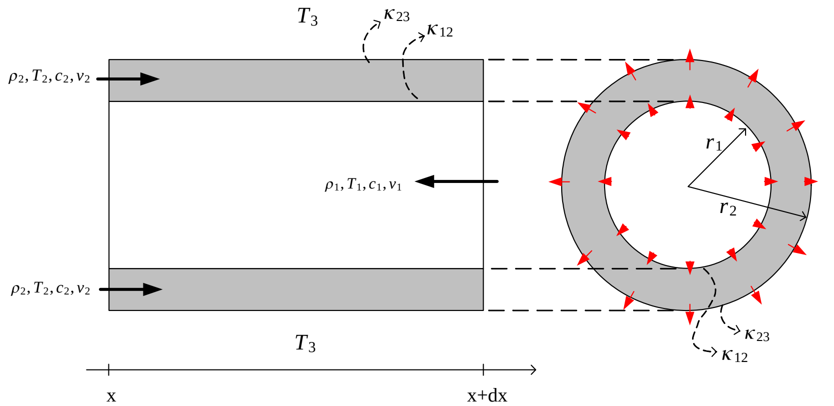
$$g(\mathbf{q},\dot{\mathbf{q}}) = \dot{\mathbf{q}}^\top \mathbf{D}(\mathbf{q},\dot{\mathbf{q}})\dot{\mathbf{q}} \geq 0, \quad \forall \mathbf{q},\dot{\mathbf{q}} \in \mathbb{R}^n$$

According to the results on pg. 63, and eq. (2.149) it follows that the system (14) is passive with $\mathbf{u} = \boldsymbol{\tau}$ and $\mathbf{y} = \dot{\mathbf{q}}$.

Problem 5

Consider an infinitesimal volume of the system (Figure).

Energy-balance:



Consider the inner-volume. We have

$$\begin{aligned} \frac{d}{dt}\int_{V_c}\rho\left(u+\frac{1}{2}\vec{v}^2+\phi\right)dV &= -\int_{\partial V_c}\rho\left(u+\frac{p}{\rho}+\frac{1}{2}\vec{v}^2+\phi\right)\vec{v}\cdot\vec{n}\,dA \\ &\quad -\int_{\partial V_c}\vec{j}_Q\cdot\vec{n}\,dA \end{aligned}$$

Applying the assumptions

- neglect the potential - and kinetic energy
- constant pressure

gives

$$\frac{d}{dt}\int_{V_c}\rho_1 u_1 dV = -\int_{\partial V_c}\rho_1 u_1\vec{v}\cdot\vec{n}\,dA - \int_{\partial V_c}\vec{j}_Q\cdot\vec{n}\,dA$$

and using

$$\begin{aligned} u_1 &= c_1 T_1 \\ \int_{\partial V_c}\vec{j}_Q\cdot\vec{n}\,dA &= \kappa_{12}\left(T_1-T_2\right)2\pi r_1 dx \end{aligned}$$

Hence,

$$\begin{aligned} \frac{d}{dt}\left(\rho_1 c_1 T_1\pi r_1^2 dx\right) &= \rho_1 c_1 T_1 v_1\pi r_1^2\big|_x - \rho_1 c_1 T_1 v_1 \pi r_1^2\big|_{x+dx} \\ &\quad -\kappa_{12}\left(T_1-T_2\right)2\pi r_1 dx \\ c_1\pi r_1^2 dx\frac{d}{dt}\left(\rho_1 T_1\right) &= c_1 v_1\pi r_1^2\left(\rho_1 T_1\big|_x - \rho_1 T_1\big|_{x+dx}\right) - 2\pi r_1 dx\kappa_{12}\left(T_1-T_2\right) \\ \frac{\partial}{\partial t}\left(\rho_1 T_1\right) &= -v_1\frac{\partial}{\partial x}\left(\rho_1 T_1\right) - \frac{2\kappa_{12}}{c_1 r_1}\left(T_1-T_2\right) \end{aligned}$$

Since $\rho_1 =$ constant, it follows thus

$$c_1\rho_1\left[\frac{\partial T_1}{\partial t}+v_1\frac{\partial T_1}{\partial x}\right]=-\frac{2\kappa_{12}}{r_1}\left(T_1-T_2\right)$$

Similarly, we have

$$\begin{aligned}\frac{d}{dt}\left(\rho_2 c_2 T_2\pi\left[r_2^2-r_1^2\right]dx\right) &= \rho_2 c_2 T_2 v_2\pi\left(r_2^2-r_1^2\right)\big|_x \\ &\quad -\rho_2 c_2 T_2 v_2\pi\left(r_2^2-r_1^2\right)\big|_{x+dx} \\ &\quad +\kappa_{12}\left(T_1-T_2\right)2\pi r_1 dx-\kappa_{23}\left(T_2-T_3\right)2\pi r_2 dx\end{aligned}$$

Dividing $c_2\pi\left[r_2^2-r_1^2\right]dx$ yields

$$\begin{aligned}c_2\frac{\partial}{\partial t}\left(\rho_2 T_2\right) &= -c_2 v_2\frac{\partial}{\partial x}\left(\rho_2 T_2\right) \\ &\quad +\frac{2r_1\kappa_{12}}{\left[r_2^2-r_1^2\right]}\left(T_1-T_2\right) \\ &\quad -\frac{2r_2\kappa_{23}}{\left[r_2^2-r_1^2\right]}\left(T_2-T_3\right)\end{aligned}$$

Hence

$$c_2\rho_2\left[\frac{dT_2}{dt}+v_2\frac{\partial T_2}{\partial x}\right]=\frac{2r_1\kappa_{12}}{\left[r_2^2-r_1^2\right]}\left(T_1-T_2\right)-\frac{2r_2\kappa_{23}}{\left[r_2^2-r_1^2\right]}\left(T_2-T_3\right)$$

Problem 6

a The pressure at the bottom of the tank:

$$p=p_{\mathrm{atm}}+\rho h g$$

b The mass balance yields

$$\begin{aligned}\frac{d}{dt}\left(\rho A h\right) &= \rho v A\big|_{\mathrm{inn}}-\rho v A\big|_{\mathrm{ut}} \\ &= \rho q\big|_{\mathrm{inn}}-\rho q\big|_{\mathrm{ut}}\end{aligned}$$

Since the fluid is incompressible, we have

$$\frac{dh}{dt}=\frac{1}{A}q_{\mathrm{inn}}-\frac{1}{A}q_{\mathrm{ut}}$$

Moreover,

$$\begin{aligned} q_{\text{ut}} &= c_v u \sqrt{\frac{2}{\rho}\Delta p} \\ &= c_v u \sqrt{\frac{2}{\rho}\left(\rho h g\right)} \\ &= c_v u \sqrt{2g}\sqrt{h} \end{aligned}$$

Thus,

$$\frac{dh}{dt} = \frac{1}{A} q_{\text{inn}} - \frac{1}{A} c_v u \sqrt{2g}\sqrt{h} = \alpha q_{\text{inn}} - \beta\sqrt{h} \tag{15}$$

i.e.

$$\begin{aligned} \alpha &= \frac{1}{A} \\ \beta &= \frac{c_v u}{A}\sqrt{2g} \end{aligned}$$

$\boxed{\text{c}}$ The linearized model

$$\frac{d}{dt}\Delta h = \alpha \Delta q_{\text{inn}} - \frac{\beta}{2\sqrt{h^*}}\Delta h, \quad h^* \neq 0 \tag{16}$$

where

$$\begin{aligned} \Delta h &= h - h^* \\ \Delta q_{\text{inn}} &= q_{\text{inn}} - q_{\text{inn}}^* \end{aligned}$$

$\boxed{\text{d}}$ The eigenvalues of (16),

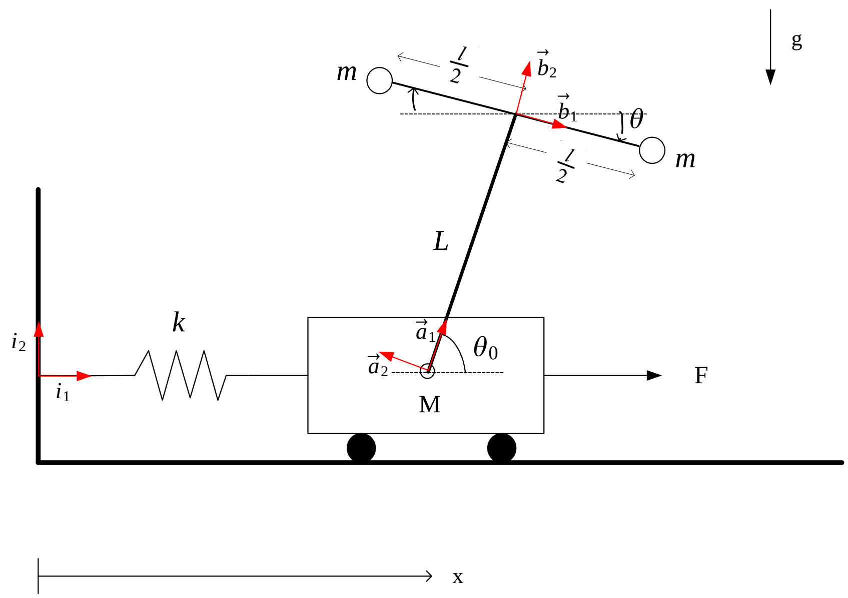
$$\lambda = -\frac{\beta}{2\sqrt{h^*}} < 0$$

Thus, the linearized model (16) is asymptotically stable. Moreover, according to the results in ch. 1.2.7 on pg. 9, the nonlinear system (15) is also asymptotic stable around (q_{inn}^*, h^*). Actually, from physically point of view, the system can not be unstable, i.e. $0 \leq h \leq H_{\max}$, where $H_{\max}$ is the height of the tank.

$\boxed{\text{Oppg. 7}}$

Define the coordinate systems (Figure),

$$\begin{aligned} i &: \left\{ \vec{i}_1, \vec{i}_2, \vec{i}_3 \right\} \\ a &: \{ \vec{a}_1, \vec{a}_2, \vec{a}_3 \} \\ b &: \left\{ \vec{b}_1, \vec{b}_2, \vec{b}_3 \right\} \end{aligned}$$



where

$$\begin{aligned}
\vec{a}_1 &= \cos\theta_0\vec{i}_1 + \sin\theta_0\vec{i}_2 \\
\vec{a}_2 &= -\sin\theta_0\vec{i}_1 + \cos\theta_0\vec{i}_2 \\
\vec{a}_3 &= \vec{i}_3
\end{aligned}$$

and

$$\begin{aligned}
\vec{b}_1 &= \cos\theta\vec{i}_1 - \sin\theta\vec{i}_2 \\
\vec{b}_2 &= \sin\theta\vec{i}_1 + \cos\theta\vec{i}_2 \\
\vec{b}_3 &= \vec{i}_3
\end{aligned}$$

The position and velocity of

- Wagon:

$$\begin{aligned}
\vec{r}_v &= x\vec{i}_1 \\
\vec{v}_v &= \frac{{}^i d}{dt}\vec{r}_v = \dot{x}\vec{i}_1
\end{aligned}$$

- mass 1:

$$\begin{aligned}
\vec{r}_1 &= x\overrightarrow{i}_1 + L\vec{a}_1 + \frac{l}{2}\overrightarrow{b}_1 \\
\vec{v}_1 &= \frac{{}^i d}{dt}\vec{r}_1 \\
&= \frac{{}^i d}{dt}\left(x\overrightarrow{i}_1 + L\vec{a}_1 + \frac{l}{2}\overrightarrow{b}_1\right) \\
&= \dot{x}\vec{i}_1 + \frac{{}^b d}{dt}\left(\frac{l}{2}\overrightarrow{b}_1\right) + \vec{\omega}_{ib}\times\left(\frac{l}{2}\overrightarrow{b}_1\right) \\
&= \dot{x}\vec{i}_1 + \left(-\dot{\theta}\overrightarrow{b}_3\right)\times\left(\frac{l}{2}\overrightarrow{b}_1\right) \\
&= \dot{x}\vec{i}_1 - \frac{l}{2}\dot{\theta}\overrightarrow{b}_2
\end{aligned}$$

- mass 2:

$$\begin{aligned}
\vec{r}_2 &= x\overrightarrow{i}_1 + L\vec{a}_1 - \frac{l}{2}\overrightarrow{b}_1 \\
\vec{v}_2 &= \frac{{}^i d}{dt}\vec{r}_2 \\
&= \dot{x}\vec{i}_1 + \frac{l}{2}\dot{\theta}\overrightarrow{b}_2
\end{aligned}$$

a) The potential energy of the system

Let

$$\vec{g} = g\ \left(-\vec{i}_2\right)$$

where $g = 9.81\ m/s^2$ is the gravitational constant.

- Spring.

$$U_k = \frac{1}{2}k\left(x - x_0\right)^2$$

- Wagon:

$$U_M = -M\ \vec{r}_v\cdot\vec{g} = 0$$

- Mass 1:

$$\begin{aligned}
U_1 &= -m\vec{r}_1\cdot\vec{g} = mg\left[x\overrightarrow{i}_1 + L\vec{a}_1 + \frac{l}{2}\overrightarrow{b}_1\right]\cdot\overrightarrow{i}_2 \\
&= mg\left[L\sin\theta_0 - \frac{l}{2}\sin\theta\right]
\end{aligned}$$

- Mass 2:

$$\begin{aligned} U_2 &= -m\vec{r}_1 \cdot \vec{g} \\ &= mg\left[x\overrightarrow{i}_1 + L\vec{a}_1 - \frac{l}{2}\overrightarrow{b}_1\right] \cdot \overrightarrow{i}_2 \\ &= mg\left[L\sin\theta_0 + \frac{l}{2}\sin\theta\right] \end{aligned}$$

Hence,

$$\begin{aligned} U &= U_k + U_M + U_1 + U_2 \\ &= \frac{1}{2}k\left(x - x_0\right)^2 + 2mgL\sin\theta_0 \end{aligned}$$

$\boxed{\text{b}}$ <u>The kinetic energy of the system</u>

- Wagon:

$$\begin{aligned} T_M &= \frac{1}{2}M\vec{v}_v \cdot \vec{v}_v \\ &= \frac{1}{2}M\dot{x}^2 \end{aligned}$$

- Mass 1:

$$\begin{aligned} T_1 &= \frac{1}{2}m\vec{v}_1 \cdot \vec{v}_1 \\ &= \frac{1}{2}m\left[\dot{x}^2 - l\dot{\theta}\dot{x}\ \vec{i}_1 \cdot \overrightarrow{b}_2 + \left(\frac{l}{2}\dot{\theta}\right)^2\right] \\ &= \frac{1}{2}m\left[\dot{x}^2 - l\dot{\theta}\dot{x}\ \sin\theta + \left(\frac{l}{2}\dot{\theta}\right)^2\right] \end{aligned}$$

- Mass 2:

$$\begin{aligned} T_2 &= \frac{1}{2}m\vec{v}_2 \cdot \vec{v}_2 \\ &= \frac{1}{2}m\left[\dot{x}^2 + l\dot{\theta}\dot{x}\ \sin\theta + \left(\frac{l}{2}\dot{\theta}\right)^2\right] \end{aligned}$$

Hence,

$$
\begin{aligned}
T &= T_M + T_1 + T_2 \\
&= \frac{1}{2}M\dot{x}^2 + m\left[\dot{x}^2 + \left(\frac{l}{2}\dot{\theta}\right)^2\right]
\end{aligned}
$$

☐ Define the Lagrange function:

$$
\begin{aligned}
L(\mathbf{q}, \dot{\mathbf{q}}) &= T(\mathbf{q}, \dot{\mathbf{q}}) - U(\mathbf{q}) \\
&= \frac{1}{2}M\dot{x}^2 + m\left[\dot{x}^2 + \left(\frac{l}{2}\dot{\theta}\right)^2\right] - \frac{1}{2}k(x - x_0)^2 - 2mgL\sin\theta_0
\end{aligned}
$$

where $\mathbf{q} = [x, \theta]^\top$.

Equations of motion:

$$
\begin{aligned}
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} &= F \\
\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} &= 0
\end{aligned}
$$

where

- $$
\begin{aligned}
\frac{\partial L}{\partial \dot{x}} &= \frac{\partial T}{\partial \dot{x}} \\
&= \frac{\partial}{\partial \dot{x}}\left\{\frac{1}{2}M\dot{x}^2 + m\left[\dot{x}^2 + \left(\frac{l}{2}\dot{\theta}\right)^2\right] - \frac{1}{2}k(x - x_0)^2 - 2mgL\sin\theta_0\right\} \\
&= (M + 2m)\dot{x}
\end{aligned}
$$

- $$\frac{\partial L}{\partial \dot{\theta}} = \frac{\partial T}{\partial \dot{\theta}} = \frac{ml^2}{2}\dot{\theta}$$

- $$\frac{\partial L}{\partial x} = -\frac{\partial U}{\partial x} = -k(x - x_0)$$

- $$\frac{\partial L}{\partial \theta} = 0$$

Hence, we get the equations of motion

Eq. 1

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{x}}\right) - \frac{\partial L}{\partial x} &= F \\ (M + 2m)\,\ddot{x} + k\,(x - x_0) &= F \end{aligned}$$

Eq. 2

$$\begin{aligned} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\theta}}\right) - \frac{\partial L}{\partial \theta} &= 0 \\ \frac{ml^2}{2}\ddot{\theta} &= 0 \end{aligned}$$