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For questions during the exam:
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Exam in TTK4130 Modeling and Simulation

Thursday, June 9th 2016

09:00 – 13:00

Permitted aids (code A): All written and handwritten examination support materials are permitted.

Answers in English, Norwegian, or a mixture of the two accepted.

Grades available: As specified by regulations.

Problem 1 (28 %)

Consider the following simulation method:

$$
\begin{aligned}
\mathbf{k}_1 &= \mathbf{f}(\mathbf{y}_n, t_n) \\
\mathbf{k}_2 &= \mathbf{f}(\mathbf{y}_n + \frac{h}{4}\mathbf{k}_1 + \frac{h}{4}\mathbf{k}_2, t_n + \frac{h}{2}) \\
\mathbf{k}_3 &= \mathbf{f}(\mathbf{y}_n + h\mathbf{k}_2, t_n + h) \\
\mathbf{y}_{n+1} &= \mathbf{y}_n + \frac{h}{6}(\mathbf{k}_1 + 4\mathbf{k}_2 + \mathbf{k}_3)
\end{aligned}
$$

(2 %) (a) Is this method explicit or implicit? How many stages does it have?

Solution: Implicit Runge-Kutta, three stages.

(3 %) (b) Write up the Butcher array for this method.

Solution:

$$
\begin{array}{c|ccc}
0 & & & \\
1/2 & 1/4 & 1/4 & \\
1 & 0 & 1 & \\
\hline
 & 1/6 & 2/3 & 1/6
\end{array}
$$

(1 %) (c) How can you see from the Butcher array whether this is an implicit or explicit method?

Solution: Non-zero elements on the diagonal of the A-matrix implies that the method is implicit.

(4 %) (d) Comment on how much work it is to implement and solve this algorithm, compared to a general three-stage implicit Runge-Kutta method.

Solution: A general three-stage implicit method must solve a nonlinear equation system with $3d$ variables at each time-step, where d is the dimension of $\mathbf{y}$; whereas this method must solve a nonlinear system with d variables to find $\mathbf{k}_2$, while $\mathbf{k}_1$ and $\mathbf{k}_3$ can be explicitly calculated.

(8 %) (e) Find the stability function for this method. *Hint*:

$$
\begin{pmatrix} a & 0 & 0 \\ d & b & 0 \\ 0 & e & c \end{pmatrix}^{-1} = \begin{pmatrix} \frac{1}{a} & 0 & 0 \\ -\frac{d}{ab} & \frac{1}{b} & 0 \\ \frac{de}{abc} & -\frac{e}{bc} & \frac{1}{c} \end{pmatrix}
$$

Solution: The stability function is

$$
\begin{aligned}
R(s) &= 1 + s\begin{pmatrix} 1/6 & 2/3 & 1/6 \end{pmatrix}\left[\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} - s\begin{pmatrix} 0 & 0 & 0 \\ 1/4 & 1/4 & 0 \\ 0 & 1 & 0 \end{pmatrix}\right]^{-1}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\
&= 1 + s\begin{pmatrix} 1/6 & 2/3 & 1/6 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ -s/4 & 1 - s/4 & 0 \\ 0 & -s & 1 \end{pmatrix}^{-1}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \\
&= \frac{1 + \frac{3}{4}s + \frac{1}{4}s^2 + \frac{1}{24}s^3}{1 - \frac{1}{4}s}
\end{aligned}
$$

(4 %) (f) Is the method A-stable? L-stable? Justify the answer.

Solution: Not A-stable, and therefore not L-stable either, since $|R(h\lambda)| \to \infty$ when $\lambda \to -\infty$.

(6 %) (g) What is the order of the method? Justify the answer. Use the fact that for a Runge-Kutta method of order p, the stability function $R(s)$ approximates e^s with error $O(s^{p+1})$. (*Hint*: If you do long calculations/derivations, then you are probably attacking this the wrong (or at least not the most straightforward) way.)

Solution: We recognize the stability function as a Pade approximation, $R(s) = P_1^3(s)$ (see book p. 549). The error in a Pade approximation $P_m^k(s)$ to e^s is $O(s^{k+m+1})$ (see bottom p. 549). Using the fact in the problem, we see that the order $p = k + m = 3 + 1 = 4$.

Problem 2 (32 %)

The double inverted pendulum on a cart (DIPC) poses a challenging control problem. In a DIPC system, two rods are connected together on a moving cart as shown in Figure 1. The length of the first rod is denoted by l_1 and the length of the second rod by l_2. The mass of the cart is denoted by m_0, its length by l_0 and its width by b_0. The height of the cart is denoted by h_0. Both rods have a mass, which are denoted by m_1 and m_2. All masses are assumed to be concentrated into the centre of mass. The moments of inertia are denoted by I_i. Furthermore, the force τ is acting on the cart.

(4 %) (a) Choose generalized coordinates, and find the positions of the centers of mass for each of the three bodies (the cart and the two rods).

Solution: The position of the cart is given as

$$
r_0 = \begin{pmatrix} \theta_0 \\ h_0 - \frac{1}{2}b_0 \end{pmatrix}, \tag{1}
$$

the position of the centre of mass of the first rod is given by

$$
r_1 = \begin{pmatrix} \theta_0 + \frac{1}{2}l_1 \sin\theta_1 \\ h_0 + \frac{1}{2}l_1 \cos\theta_1 \end{pmatrix} \tag{2}
$$

and the position of the second rod is given by

$$
r_2 = \begin{pmatrix} \theta_0 + l_1 \sin\theta_1 + \frac{1}{2}l_2 \sin\theta_2 \\ h_0 + l_1 \cos\theta_1 + \frac{1}{2}l_2 \cos\theta_2 \end{pmatrix}. \tag{3}
$$

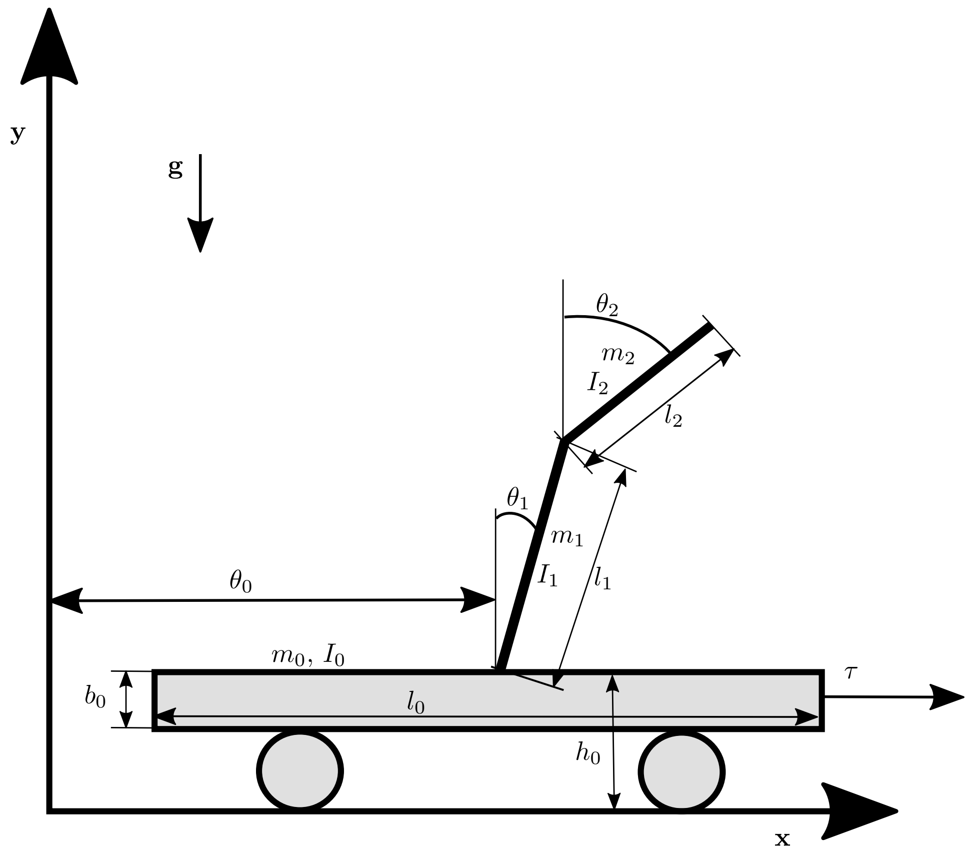

Figure 1: Double inverted pendulum on a cart

If the origin of the coordinate system is chosen differently the terms can look slightly different. However, if the coordinate system is chosen differently then in Fig. 1, it has to be mentioned.

(10 %) (b) Find the kinetic energy of the system. (*Hint:* the following identity may simplify the expressions: $\cos(x-y) = \cos x \cos y + \sin x \sin y$.)

Solution:

- The velocities can be calculated by derivation of the positions

$$v_0 = \begin{pmatrix} \dot{\theta}_0 \\ 0 \end{pmatrix}, \tag{4a}$$

$$v_1 = \begin{pmatrix} \dot{\theta}_0 + \frac{1}{2} l_1 \dot{\theta}_1 \cos\theta_1 \\ -\frac{1}{2} l_1 \dot{\theta}_1 \sin\theta_1 \end{pmatrix}, \tag{4b}$$

$$v_2 = \begin{pmatrix} \dot{\theta}_0 + l_1 \dot{\theta}_1 \cos\theta_1 + \frac{1}{2} l_2 \dot{\theta}_2 \cos\theta_2 \\ -l_1 \dot{\theta}_1 \sin\theta_1 - \frac{1}{2} l_2 \dot{\theta}_2 \sin\theta_2 \end{pmatrix}. \tag{4c}$$

- The kinetic energy is given by

$$T = \sum_i T_i, \tag{5}$$

with

$$T_i = \frac{1}{2} m_i v_i^T v_i + \frac{1}{2} I_i \omega_i^T \omega_i. \tag{6}$$

Consequently, the kinetic energy of each body part is given by

$$T_0 = \frac{1}{2} m_0 \dot{\theta}_0, \tag{7a}$$

$$T_1 = \frac{1}{2} m_1 \left(\dot{\theta}_0^2 + l_1 \dot{\theta}_0 \dot{\theta}_1 \cos\theta_1 + \frac{1}{4} l_1^2 \dot{\theta}_1^2 \right) + \frac{1}{2} I_1 \dot{\theta}_1^2, \tag{7b}$$

$$T_2 = \frac{1}{2} m_2 \left(\dot{\theta}_0^2 + l_1^2 \dot{\theta}_1^2 + \frac{1}{4} l_2^2 \dot{\theta}_2^2 + 2 l_1 \dot{\theta}_0 \dot{\theta}_1 \cos\theta_1 + l_2 \dot{\theta}_0 \dot{\theta}_2 \cos\theta_2 + l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \right) + \frac{1}{2} I_2 \dot{\theta}_2^2, \tag{7c}$$

with

$$\cos(\theta_1 - \theta_2) = \cos\theta_1 \cos\theta_2 + \sin\theta_1 \sin\theta_2. \tag{8}$$

(4 %) (c) Find the potential energy of the system.

Solution: The potential energy is given by

$$U = \sum_i U_i \tag{9}$$

$$U_0 = m_0 g (h_0 - \frac{1}{2} b_0), \tag{10a}$$

$$U_1 = m_1 g (h_0 + \frac{1}{2} l_1 \cos\theta_1), \tag{10b}$$

$$U_2 = m_2 g (h_0 + l_1 \cos\theta_1 + \frac{1}{2} l_2 \cos\theta_2). \tag{10c}$$

If the origin of the coordinate system is chosen differently the terms can look slightly different. However, if the coordinate system is chosen differently then in Fig. 1, it has to be mentioned.

(14 %) (d) Find the equations of motion of the system.

Solution: The Lagrange equation of motion

$$\frac{d}{dt} \frac{\partial L}{\partial \dot{q}_i} - \frac{\partial L}{\partial q_i} = \tau_i \tag{11}$$

is used. L is defined as

$$L(\mathbf{q}, \dot{\mathbf{q}}, t) = T(\mathbf{q}, \dot{\mathbf{q}}, t) - U(\mathbf{q}), \tag{12}$$

where T is the kinetic energy and U the potential energy of the system.

- The resulting Lagrangian is

$$\begin{aligned} L =& \frac{1}{2} \Big[(m_0 + m_1 + m_2) \dot{\theta}_0^2 + \left(\frac{1}{2} m_1 l_1^2 + m_2 l_1^2 + I_1 \right) \dot{\theta}_1^2 + \\ & \left(\frac{1}{2} m_2 l_2^2 + I_2 \right) \dot{\theta}_2^2 + (m_1 l_1 + 2 m_2 l_1) \dot{\theta}_0 \dot{\theta}_1 \cos\theta_1 + \\ & m_2 l_2 \dot{\theta}_0 \dot{\theta}_2 \cos\theta_2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \cos(\theta_1 - \theta_2) \Big] - \\ & \left(\frac{1}{2} m_1 + m_2 \right) g l_1 \cos\theta_1 - \frac{1}{2} m_2 g l_2 \cos\theta_2 - (m_0 + m_1 + m_2) h_0 g + \frac{1}{2} m_0 g b_0. \end{aligned} \tag{13}$$

- The derivation of the Lagrangian with respect to $\dot{\mathbf{q}}$ is

$$\frac{\partial}{\partial \dot{\mathbf{q}}} L(\mathbf{q}, \dot{\mathbf{q}}, t) = \begin{pmatrix} (m_0 + m_1 + m_2)\dot{\theta}_0 + \frac{1}{2}\left((m_1 + 2m_2) l_1 \dot{\theta}_1 \cos\theta_1 + m_2 l_2 \dot{\theta}_2 \cos\theta_2\right) \\ (\frac{1}{2} m_1 l_1^2 + m_2 l_1^2 + I_1)\dot{\theta}_1 + \frac{1}{2}\left((m_1 + 2m_2) l_1 \dot{\theta}_0 \cos\theta_1 + m_2 l_1 l_2 \dot{\theta}_2 \cos(\theta_1 - \theta_2)\right) \\ (\frac{1}{2} m_2 l_2^2 + I_2)\dot{\theta}_2 + \frac{1}{2} m_2 l_2 \left(\dot{\theta}_0 \cos\theta_2 + l_1 \dot{\theta}_1 \cos(\theta_1 - \theta_2)\right) \end{pmatrix} \tag{14}$$

- The derivation of Eq. 14 with respect to time is

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\mathbf{q}}}\right) = \begin{pmatrix} (\sum_i m_i)\ddot{\theta}_0 + \frac{1}{2}\left(l_1(m_1 + 2m_2)(\ddot{\theta}_1 \cos\theta_1 - \dot{\theta}_1^2 sin\theta_1) + m_2 l_2(\ddot{\theta}_2 \cos\theta_2 - \dot{\theta}_2^2 \sin\theta_2)\right) \\ (\frac{1}{2} m_1 l_1^2 + m_2 l_1^2 + I_1)\ddot{\theta}_1 + \frac{1}{2}\Big(l_1(m_1 + 2m_2)(\ddot{\theta}_0 \cos\theta_1 - \dot{\theta}_0\dot{\theta}_1 \sin\theta_1) + ... \\ m_2 l_1 l_2 \big[\ddot{\theta}_2 \cos(\theta_1 - \theta_2) - \dot{\theta}_1\dot{\theta}_2 \sin(\theta_1 - \theta_2) + \dot{\theta}_2^2 \sin(\theta_1 - \theta_2)\big]\Big) \\ (\frac{1}{2} m_2 l_2^2 + I_2)\ddot{\theta}_2 + \frac{1}{2}\Big(m_2 l_2(\ddot{\theta}_0 \cos\theta_2 - \dot{\theta}_0\dot{\theta}_2 \sin\theta_2) + m_2 l_1 l_2 \big[\ddot{\theta}_1 \cos(\theta_1 - \theta_2) - ... \\ \dot{\theta}_1^2 \sin(\theta_1 - \theta_2) + \dot{\theta}_1\dot{\theta}_2 \sin(\theta_1 - \theta_2)\big]\Big) \end{pmatrix} \tag{15}$$

- The derivation of the Lagrangian with respect to $\mathbf{q}$ is

$$\frac{\partial}{\partial \mathbf{q}} L(\mathbf{q}, \dot{\mathbf{q}}, t) = \frac{1}{2} \begin{pmatrix} 0 \\ -(m_1 + 2m_2) l_1 \dot{\theta}_0 \dot{\theta}_1 \sin\theta_1 - m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + ... \\ (m_1 + 2m_2) g l_1 \sin\theta_1 \\ -m_2 l_2 \dot{\theta}_0 \dot{\theta}_2 \sin\theta_2 + m_2 l_1 l_2 \dot{\theta}_1 \dot{\theta}_2 \sin(\theta_1 - \theta_2) + m_2 g l_2 \sin\theta_2 \end{pmatrix}. \tag{16}$$

- The results (15) and (16) are put in Eq. 11, which gives

$$\tau = \left(\sum m_i\right)\ddot{\theta}_0 + \frac{1}{2}\Big(l_1(m_1 + 2m_2 l_1)(\ddot{\theta}_1 \cos\theta_1 - \dot{\theta}_1^2 sin\theta_1) + \\ m_2 l_2(\ddot{\theta}_2 \cos\theta_2 - \dot{\theta}_2^2 \sin\theta_2)\Big), \tag{17a}$$

$$0 = \left(\frac{1}{2} m_1 l_1^2 + m_2 l_1^2 + I_1\right)\ddot{\theta}_1 + \frac{1}{2}\Bigg(l_1(m_1 + 2m_2)\ddot{\theta}_0 \cos\theta_1 + \\ m_2 l_1 l_2 \big[\ddot{\theta}_2 \cos(\theta_1 - \theta_2) + \dot{\theta}_2^2 \sin(\theta_1 - \theta_2)\big] - (m_1 + 2m_2) g l_1 \sin\theta_1\Bigg), \tag{17b}$$

$$0 = \left(\frac{1}{2} m_2 l_2^2 + I_2\right)\ddot{\theta}_2 + \\ \frac{1}{2}\left(m_2 l_2 \ddot{\theta}_0 \cos\theta_2 + m_2 l_1 l_2 \big[\ddot{\theta}_1 \cos(\theta_1 - \theta_2) - \dot{\theta}_1^2 \sin(\theta_1 - \theta_2)\big] - m_2 g l_2 \sin\theta_2\right), \tag{17c}$$

which are the equations of motion for the DIPC system.

Problem 3 (16 %)

Figure 2 illustrates two coordinate frames in three dimensions. Note that all of the unit vectors shown and the dashed line segment (of 5 cm) are in the same plane, and unit vectors pointing into or out of the paper plane is not shown.

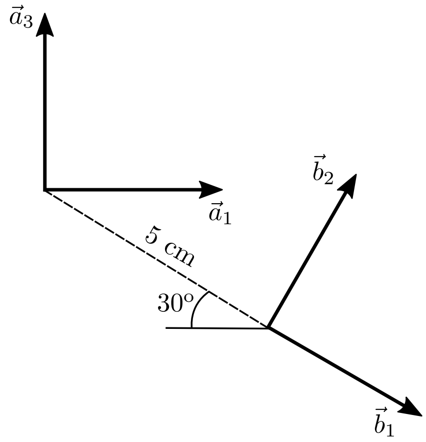

Figure 2: Two coordinate frames, rotated and translated.

(10 %) (a) What is $\mathbf{T}_b^a$, the homogenous transformation matrix representing the orientation and position of frame b relative to frame a?

Solution: One way to find this is to multiply together two simple rotations (for instance 90 degrees about x-axis, -30 degrees about (rotated) z-axis) and one translation (5 cm along (rotated) x-axis). Another way is to use that the columns in the rotation matrix $\mathbf{R}_b^a$ are the unit coordinate vectors of frame b expressed in frame a, and find these by looking at the figure. The vector $\mathbf{r}_b^a$ is found as the location of the origin of frame b in frame a.

In any case, you should end up with

$$\mathbf{T}_b^a = \begin{pmatrix} \mathbf{R}_b^a & \mathbf{r}_b^a \\ 0\ 0\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 & 1/2 & 0 & 5\sqrt{3}/2\text{cm} \\ 0 & 0 & -1 & 0 \\ -1/2 & \sqrt{3}/2 & 0 & -5/2\text{cm} \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

(6 %) (b) What is $\mathbf{T}_a^b$?

Solution: The simplest is perhaps to use $\mathbf{R}_a^b = (\mathbf{R}_b^a)^\mathsf{T}$ and $\mathbf{r}_a^b = -\mathbf{R}_a^b \mathbf{r}_b^a$ to obtain

$$\mathbf{T}_a^b = \begin{pmatrix} \mathbf{R}_a^b & \mathbf{r}_a^b \\ 0\ 0\ 0 & 1 \end{pmatrix} = \begin{pmatrix} \sqrt{3}/2 & 0 & -1/2 & -5\text{cm} \\ 1/2 & 0 & \sqrt{3}/2 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

or use that the columns of $\mathbf{R}_a^b$ are the unit coordinate vectors of a expressed in b, and that $\mathbf{r}_a^b$ is the location of origin of frame a in frame b.

Problem 4 (24 %)

Heat exchangers are basic unit processes which are found in almost every plant in the chemical process industries. As the name suggests, heat exchangers are used for energy (heat) exchange between a hot and a cold fluid stream (the hot stream heats the cold stream).

Heat exchangers are constructed in various ways, for example to maximize the energy transfer. Often they are considered as distributed systems since the temperatures will vary along the stream lines inside the heat exchanger. However, in the first part of this task we will develop a simple heat-exchanger model based on a very simple geometry (which can be an approximation for more complex geometries) and where we assume the temperatures at the hot and cold side (T_h and T_c) are spatially constant (that is, we assume the temperatures are averaged/lumped).

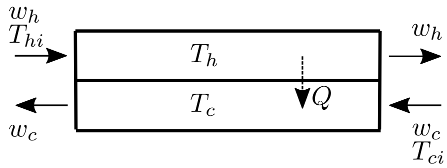

Figure 3: Simple heat exchanger

Consider Figure 3. The the hot and cold mass flow rates w_j, $j = \{h, c\}$ are assumed constant, such that the mass on each side are constant. The inlet temperatures are T_{hi} on the hot side, and T_{ci} on the cold side. The heat transfer from hot to cold side is

$$Q = UA\,(T_h - T_c)$$

where U is a heat transfer coefficient and A is the "effective contact area". The volumes of the hot and cold side are denoted V_h and V_c, respectively.

(10 %) (a) Derive a model for the temperatures T_h and T_c. Assume that the densities ρ_j and specific heats c_{pj}, $j = \{h, c\}$ are constant. Assume incompressible liquids and constant pressure, such that specific internal energy and specific enthalpy are equal.

Solution: The mass balance is trivial ($\dot{m}_h = w_h - w_h = 0$ and same for cold side). The energy balance for the hot and cold sides are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho_h V_h c_{ph} T_h\right) = w_h h(T_{hi}) - w_h h(T_h) - Q = w_h c_{ph}(T_{hi} - T_h) - UA\,(T_h - T_c)$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\rho_c V_c c_{pc} T_c\right) = w_c h(T_{ci}) - w_c h(T_c) + Q = w_c c_{pc}(T_{ci} - T_c) + UA\,(T_h - T_c)$$

which means

$$\frac{\mathrm{d}}{\mathrm{d}t} T_h = \frac{w_h}{\rho_h V_h}(T_{hi} - T_h) - \frac{UA}{\rho_h V_h c_{ph}}\,(T_h - T_c)$$
$$\frac{\mathrm{d}}{\mathrm{d}t} T_c = \frac{w_c}{\rho_c V_c}(T_{ci} - T_c) + \frac{UA}{\rho_c V_c c_{pc}}\,(T_h - T_c)$$

(10 %) (b) The model found above, can be written

$$\begin{pmatrix} \dot{T}_h \\ \dot{T}_c \end{pmatrix} = \begin{pmatrix} -a_1 - k_1 & k_1 \\ k_2 & -a_2 - k_2 \end{pmatrix} \begin{pmatrix} T_h \\ T_c \end{pmatrix} + \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \begin{pmatrix} T_{hi} \\ T_{ci} \end{pmatrix}$$

where a_j and k_j, $j = \{h, c\}$ are appropriate positive constants (found in (a)). Let $\mathbf{x} = (x_1, x_2)^\mathsf{T} = (T_h, T_c)^\mathsf{T}$. Show that this model is passive from $\mathbf{u} = (u_1, u_2)^\mathsf{T} = (T_{hi}, T_{ci})^\mathsf{T}$ to $\mathbf{y} = (y_1, y_2)^\mathsf{T} = \left(\frac{a_1}{k_1} x_1, \frac{a_2}{k_2} x_2\right)^\mathsf{T}$, using the storage function

$$V(\mathbf{x}) = \frac{1}{2}\mathbf{x}^\mathsf{T} P \mathbf{x}, \quad P = \begin{pmatrix} \frac{1}{k_1} & 0 \\ 0 & \frac{1}{k_2} \end{pmatrix}.$$

A hint that may or may not be useful, is a special case of Gershgorin's Theorem:

$$\begin{pmatrix} q_{11} & q_{12} \\ q_{21} & q_{22} \end{pmatrix} \text{ is negative definite if } q_{11} < 0,\ q_{11} + q_{12} < 0 \text{ and } q_{22} < 0,\ q_{22} + q_{21} < 0.$$

Solution: The derivative of the storage function is

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}t}V(\mathbf{x}) &= \mathbf{x}^\mathsf{T}\begin{pmatrix}\frac{1}{k_1} & 0\\ 0 & \frac{1}{k_2}\end{pmatrix}\left(\begin{pmatrix}-a_1-k_1 & k_1\\ k_2 & -a_2-k_2\end{pmatrix}\mathbf{x}+\begin{pmatrix}a_1 & 0\\ 0 & a_2\end{pmatrix}\mathbf{u}\right)\\
&= \mathbf{x}^\mathsf{T}\begin{pmatrix}-\frac{a_1}{k_1}-1 & 1\\ 1 & -\frac{a_2}{k_2}-1\end{pmatrix}\mathbf{x}+\mathbf{x}^\mathsf{T}\begin{pmatrix}\frac{a_1}{k_1} & 0\\ 0 & \frac{a_2}{k_2}\end{pmatrix}\mathbf{u}\\
&= -\frac{a_1}{k_1}x_1^2-\frac{a_2}{k_2}x_2^2-x_1^2+2x_1x_2-x_2^2+\frac{a_1}{k_1}x_1u_1+\frac{a_2}{k_2}x_2u_1\\
&= -\frac{a_1}{k_1}x_1^2-\frac{a_2}{k_2}x_2^2-(x_1-x_2)^2+\frac{a_1}{k_1}x_1u_1+\frac{a_2}{k_2}x_2u_1\\
&= \frac{a_1}{k_1}x_1u_1+\frac{a_2}{k_2}x_2u_1-g(\mathbf{x})\\
&= \mathbf{y}^\mathsf{T}\mathbf{u}-g(\mathbf{x})
\end{aligned}$$

where the dissipation function $g(\mathbf{x})=\frac{a_1}{k_1}x_1^2+\frac{a_2}{k_2}x_2^2+(x_1-x_2)^2$ is clearly positive. By (2.149) in the book, the system is passive from $\mathbf{u}$ to $\mathbf{y}$.

The calculations above are somewhat easier if using the hint. Then we see that

$$\begin{pmatrix}-\frac{a_1}{k_1}-1 & 1\\ 1 & -\frac{a_2}{k_2}-1\end{pmatrix}$$

is negative definite and we can use

$$g(\mathbf{x})=\mathbf{x}^\mathsf{T}\begin{pmatrix}-\frac{a_1}{k_1}-1 & 1\\ 1 & -\frac{a_2}{k_2}-1\end{pmatrix}\mathbf{x}$$

already in the second line.

Assume now that the temperatures of both streams are not averaged (not spatially constant), that is, that they vary along the streamlines. We often say in this case that the variables (temperatures) are *distributed*, and this is naturally modelled by partial differential equations. We will not do that here, but we will ask you to use your intuition and physical understanding to suggest how the temperature profiles will look.

(2 %) (c) Make a sketch of the temperature profiles in the hot and cold stream of a counter-flow heat exchanger (in principle like the one shown in Figure 3, where the hot and cold fluid enter on different ends). Draw them both in the same diagram, with position along x-axis and temperature along y-axis. Mark T_{ci} and T_{hi} in the sketch.

Solution: See Figure 4.

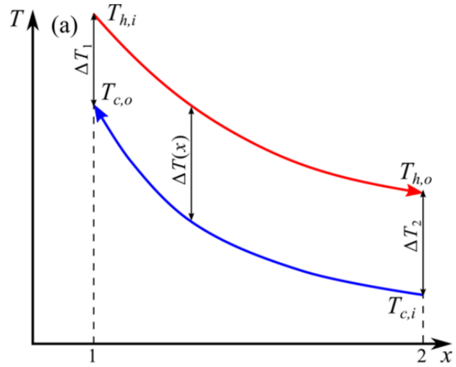

Figure 4: Counter-flow heat exchanger (Figure stolen from http://www.che.ufl.edu/unit-ops-lab/experiments/HE/HE-theory.pdf)

(2 %) (d) Do the same for a heat exchanger with parallel flows (where the hot and cold fluids enter at the same end, compared to Figure 3 where they enter on opposite ends). Why do you think counter-flow heat exchangers might be a better set-up than a heat exchanger with parallel flows?

Solution: See Figure 5.

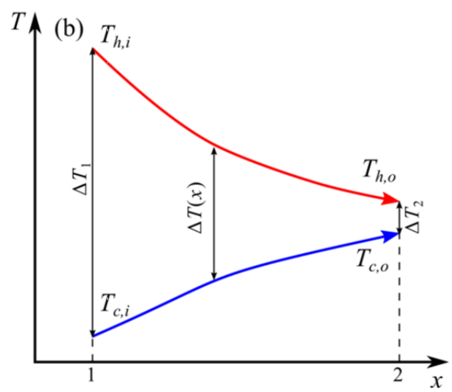

Figure 5: Parallel-flow heat exchanger (Figure stolen from http://www.che.ufl.edu/unit-ops-lab/experiments/HE/HE-theory.pdf)

A countercurrent design will give a similar temperature difference along the flow lines, meaning the heat transfer (proportional to temperature difference) will be relative evenly distributed. A parallel flow design will have a large heat transfer in one end, with much less transfer in the other end. This has some drawbacks, such as potential material challenges (due to large temperature differences), and parts of the heat exchanger being "inefficient" (low temperature differences).