Exam Solutions

Problem 1:

Notice that if $f(z) = 10z^2$,
then $|f(z)| = 10|z|^2 = 10$ on $\{z : |z| = 1\}$
while $|g(z)| = |z + \cos z| \leq$
$|z| + |\cos z| = |z| + \frac{1}{2}|e^{iz} + e^{-iz}|$
$\leq 1 + \frac{1}{2}(|e^{iz}| + |e^{-iz}|)$
$= 1 + \frac{1}{2}(e^{-y} + e^{y}) \leq 1 + e$

when $|z| = |x + iy| = 1$

So $|f(z)| > |g(z)|$ on $\{z : |z| = 1\}$

Now $f(z) = 10z^2$ do have a double 0 at $z = 0$. Hence by Rouches theorem $f(z) + g(z)$ have 2 0's in $\Delta = \{z : |z| < 1\}$.

Problem 2

Notice that $\displaystyle\int_0^\infty \frac{dx}{(2+x^2)(4+x^2)}$

$$= \int_0^1 \frac{dx}{(2+x^2)(4+x^2)} + \int_1^\infty \frac{dx}{(2+x^2)(4+x^2)}$$

Now $\dfrac{1}{(2+x^2)(4+x^2)} \leq \dfrac{1}{x^4}$ when $1 \leq x < \infty$

and $\displaystyle\int_1^\infty \frac{1}{x^4}\,dx < \infty$ so our integral converges

Further $\dfrac{1}{(2+x^2)(4+x^2)}$ is an even

function so

$$I = \int_0^\infty \frac{1}{(2+x^2)(4+x^2)}\,dx = \frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{(4+x^2)(2+x^2)}\,dx$$

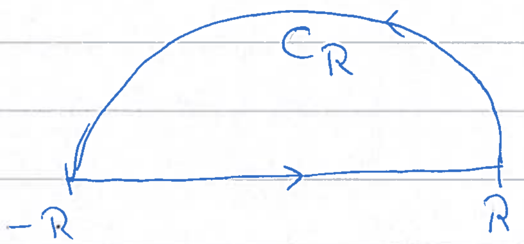
Let $R>0$ be large

and look at

$$\int_{-R}^{R} \frac{1}{(2+x^2)(4+x^2)}\,dx + \int_{C_R} \frac{1}{(2+z^2)(4+z^2)}\,dz$$

$$= \int_{\Gamma_R} \frac{1}{(2+z^2)(4+z^2)}\,dz$$

where $C_R = \{ Re^{i\theta} : 0 \le \theta \le \pi \}$



$$f(z) = \frac{1}{(2+z^2)(4+z^2)}$$ have

singularities at the points

$$2+z^2 = 0$$
$$4+z^2 = 0$$

3

$2 + z^2 = 0$ at $z^2 = -2$ or

$z_1 = i\sqrt{2}$ and $z_2 = -i\sqrt{2}$

while

$4 + z^2 = 0$ at $z^2 = -4$

or

$z_3 = i2$ and $z_4 = -iR$

Only

$z_1 = i\sqrt{2}$ and $z_3 = i2$

are surrounded by Γ_R

Hence $\displaystyle\int_{\Gamma_R} f(z)\,dz = 2\pi i \left[\operatorname*{Res}_{z=i\sqrt{2}} f(z) + \operatorname*{Res}_{z=i2} f(z) \right]$

when $R > 0$ is large.

$$f(z) = \frac{1}{(z-i\sqrt{2})(z+i\sqrt{2})(z-i2)(z+i2)}$$

$$\operatorname*{Res}_{z=i\sqrt{2}} f(z) = \frac{1}{(\sqrt{2}i + i\sqrt{2})(\sqrt{2}i - i2)(\sqrt{2}i + i2)}$$

$$= \frac{1}{2\sqrt{2}i} \frac{1}{(-2+4)} = \frac{1}{4\sqrt{2}i}$$

$$\operatorname*{Res}_{z=2i} f(z) = \frac{1}{(2+(2i)^2)(2i+2i)} = \frac{1}{-2}\,\frac{1}{4i}$$

$$= -\frac{1}{8i}$$

So

$$\int_{\Gamma_R} f(z)\,dz = 2\pi i\left(\frac{1}{4\sqrt{2}\,i} - \frac{1}{8i}\right) = 2\pi\left(\frac{1}{4\sqrt{2}} - \frac{1}{8}\right)$$

Now.

$$\left|\int_{C_R} f(z)\,dz\right| \leq \int_{C_R} |f(z)|\,|dz|$$

$$= \int_{C_R} \frac{1}{|2+z^2|\,|4+z^2|}\,|dz|$$

$$\leq \int_{C_R} \frac{1}{(|z|^2-2)(|z|^2-4)}\,|dz| = \frac{\pi R}{(R^2-2)(R^2-4)}$$

So this integral goes to 0 when $R \to \infty$

Hence.

$$\int_{-\infty}^{\infty} \frac{1}{(2+x^2)(4+x^2)}\,dx = 2\pi\left(\frac{1}{4\sqrt{2}} - \frac{1}{8}\right)$$

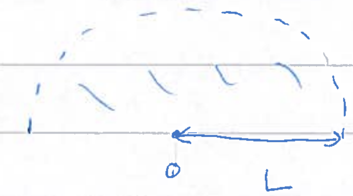
or.

$$\frac{1}{2}\int_{-\infty}^{\infty} \frac{1}{(2+x^2)(4+x^2)}\,dx = \int_{0}^{\infty} \frac{1}{(2+x^2)(4+x^2)}\,dx = \pi\left(\frac{\sqrt{2}-1}{8}\right)$$

PROBLEM 3:

f is analytic in $\{z : |z|<1, \operatorname{Im} z>0\}$
and continuous on

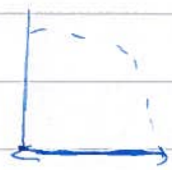
$\{z : |z|<1, \operatorname{Im} z > 0\} \cup (0,1)$



Further $f(x) = 1$ on L in

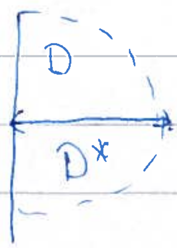
particular f is real valued on L.

Study f on.



$D = \{|z| : |z|<1$ and $0 < \arg z < \frac{\pi}{2}\}$
and
$D \cup L$.

Let $D^* = \{z : \bar{z} \subset D\}$.



Schwarz reflection principle says that

$$\tilde{f}(z) = \begin{cases} f(z), & z \in D \cup L \\ \overline{f(\bar{z})} & \end{cases}$$

then $\tilde{f}$ is analytic in

$D \cup L \cup D^*$

Further $\tilde{f}_L = 1$ so

$\tilde{f}(z) - 1 = 0$ on all of L.

Hence the zero set of $\tilde{f}(z) - 1$ do not consist of only isolated points so $\tilde{f}(z) = \phi$ for

all $z \in D \cup L \cup D^*$.

This means that $f(z) = \phi$ when $z \in D$ so $f(z) = 1$ in

$\{z : |z| < 1 \text{ and } \operatorname{Im} z > 0\} = \Omega$.

Problem 4 :

$$h(z) = \frac{\sin z - z}{z^3} + \sin\left(\frac{1}{z-i}\right) + \frac{e^{(z-1)}-1}{(z-1)}$$

a) h have (potential) singularities at $z = 0$, $z = i$ and $z = 1$

b) $\underline{\underline{z = 0}}$:

The part of h that is singular at this point is

$$\frac{\sin z - z}{z^3}$$

$$\sin z = \cancel{z} - \frac{1}{6} z^3 + \frac{1}{5!} z^5 - \cdots$$

so

so $\sin z - z = -\frac{1}{6} z^3 + \frac{1}{5!} z^5 - \cdots$

and

$$\frac{\sin z - z}{z^3} = -\frac{1}{6} + \frac{1}{5!} z^2 - \cdots$$

Hence $\frac{\sin z - z}{z^3}$ is bounded near 0 and therefor the singularity is <u>removable</u>.

$\underline{\underline{z = i}}$

The part of h that is singular at $\underline{\underline{i}}$ is $\sin\left(\frac{1}{z-i}\right)$

$$\sin\left(\frac{1}{z-i}\right) = \sum_{j=0}^{\infty} (-1)^j \left(\frac{1}{z-i}\right)^{2j+1} \frac{1}{(2j+1)!}$$

Hence $z = i$ can not be a pole or removable so $z = i$ is an $\underline{\underline{\text{essential}}}$ singularity.

$\underline{\underline{z = 1}}$

The singular part of h here is

$$\frac{e^{(z-1)} - 1}{(z-1)} = \frac{1 + (z-1) + \frac{1}{2}(z-1)^2 + \cdots \quad - 1}{z-1}$$

$$= 1 + \frac{1}{2}(z-1) + \frac{1}{3!}(z-1)^2 + \cdot$$

Again $z = 1$ is a $\underline{\underline{\text{removable}}}$ singularity.

c)

$z = i$ is an essential singularity so the image

$$h(\Delta(i, \tfrac{1}{100})) = \{w ; w = h(z), z \in \Delta(i, \tfrac{1}{100})\}$$

is dense in $\mathbb{C}$

From this we see that

I. NOT ALL $z \in \Delta(i, \frac{1}{100})$ will satisfy.

$$|h(z) - i| > 10^{-30}$$

II We can find points.

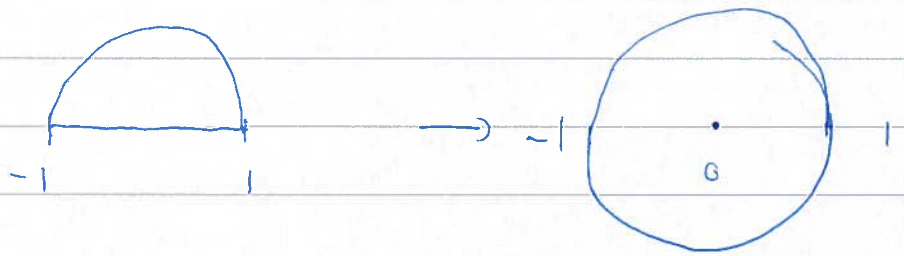
$z \in \Delta(i, \frac{1}{100})$ which are very far away from i so we can find some $z_1 \in \Delta(i, \frac{1}{100})$ such that

$$|h(z_1) - i| > 10^{-30}$$

Problem 5:

a) Find a conformal map.

$\varphi : \Omega = \{z : |z|<1, \operatorname{Im} z > 0\} \to \Delta$



Find T such that

$-1 \to 0$ and $1 \to \infty$

then $T(\Omega)$ will be a sector with angle $\frac{\pi}{2}$

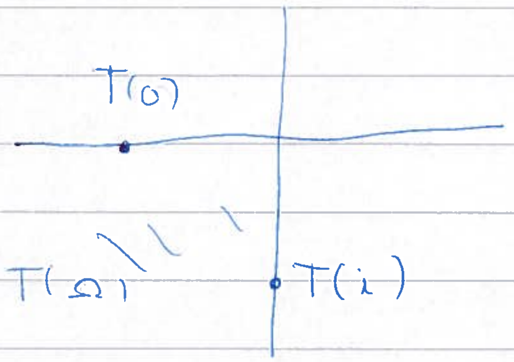
Try $T(z) = \frac{z+1}{z-1}$

Observe that.

$T(0) = -1$

and $T(i) = \frac{i+1}{i-1} = -\frac{(i+1)^2}{2} = -\frac{2i}{2} = -i$

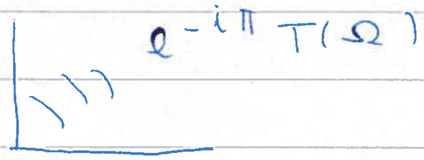
Hence $T(\Omega)$ is the third quadrant



Rotate such that $T(\Omega)$ becomes the first quadrant

$$e^{-i\pi} T(\cancel{\Omega} z) = e^{-i\pi} \frac{z+1}{z-1}$$

$$= \frac{z+1}{1-z}$$



Now send the first quadrant to the upper half plane

$$\{z : \operatorname{Im} z > 0\} = H^+$$

$w \to w^2$ so

$$\left(e^{i\pi} T(z)\right)^2 = \left(\frac{z+1}{z-1}\right)^2$$

sends Ω to H^+

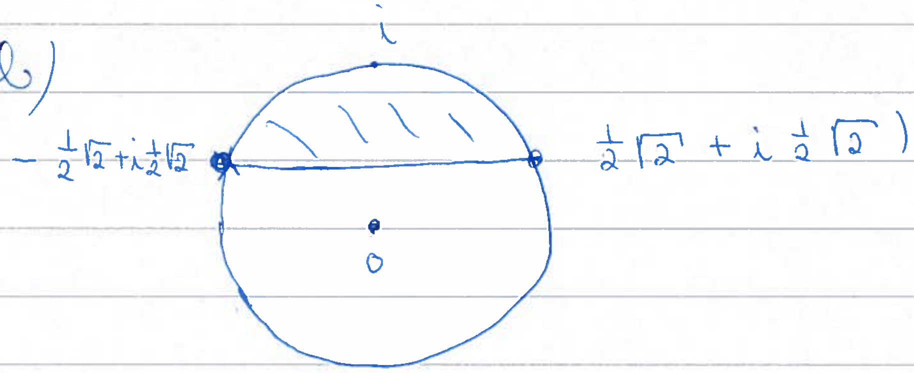
Now $\xi \to \dfrac{\xi - i}{\xi + i}$ sends H^+ to the unit disc.

Hence if

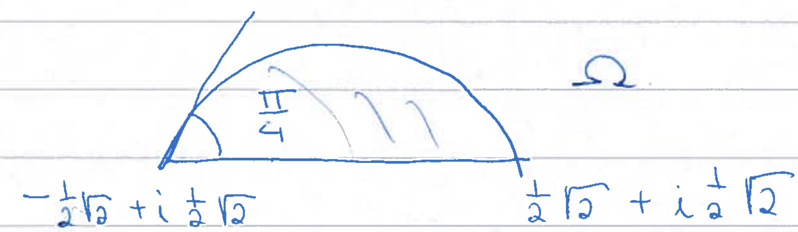
$$\frac{\left(\frac{z+1}{z-1}\right)^2 - i}{\left(\frac{z+1}{z-1}\right)^2 + i} = \varphi$$

then $\varphi(\Omega) = \Delta = \{z : |z| < 1\}$.

b)



$$\Omega = \left\{ z \; ; \; |z| < 1 \text{ and } \operatorname{Im} z > \frac{\sqrt{2}}{2} \right\}.$$

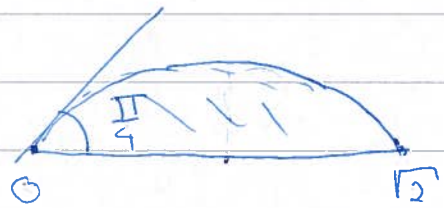


First let us send $-\frac{1}{2}\sqrt{2} + i\frac{1}{2}\sqrt{2}$ to 0

$$z - \left(-\tfrac{1}{2}\sqrt{2} + i\tfrac{1}{2}\sqrt{2}\right)$$
$$= z + \tfrac{1}{2}\sqrt{2} - i\tfrac{1}{2}\sqrt{2}$$

Then $\Delta \cap \{ \operatorname{Im} z = \frac{1}{2}\sqrt{2} \}$ is
~~going~~ going to

$$\{ z = x + iy : y = 0 \text{ and } 0 < x < \sqrt{2} \}$$



and the part of $\{ z : |z| = 1 \}$ above $\operatorname{Im} z = \frac{1}{2}\sqrt{2}$ to a piece of a circle which makes the angle $\frac{\pi}{4}$ with the real axis.

Now we want to send $\sqrt{2} \to \infty$, then we will get a sector with angle $\frac{\pi}{4}$

$$T(w) = \frac{1}{\sqrt{2} - w}$$

Then $(0, \sqrt{2})$ goes to

$$\{ \xi : \operatorname{Im} \xi = 0 , \sqrt{2} < \xi < \infty \}$$

Fix this $T_1(w) = \dfrac{w}{\sqrt{2} - w}$

Then $T_1(0 < \xi < \sqrt{2}) = \{\eta : 0 < \eta < \infty\}$.

and $T_1(\Omega + \frac{1}{2}\sqrt{2} - \frac{1}{2}\sqrt{2}\,i)$ will

be a sector of angle $\frac{\pi}{4}$ with one leg the positive real axis.

The question is if it is above or below $\operatorname{Im} \xi = 0$.

Test on the point $z = i$ on the boundary of Ω.

$$i \to i + \tfrac{1}{2}\sqrt{2} - i\tfrac{1}{2}\sqrt{2}$$

$$= (1 - \tfrac{1}{2}\sqrt{2})\,i + \tfrac{1}{2}\sqrt{2}$$

$$T_1\left((1 - \tfrac{1}{2}\sqrt{2})\,i + \tfrac{1}{2}\sqrt{2}\right) =$$

$$\frac{1 - \frac{1}{2}\sqrt{2}\,i + \frac{1}{2}\sqrt{2}}{(\frac{1}{2}\sqrt{2}) - (1 - \frac{1}{2}\sqrt{2})\,i} = \frac{(\sqrt{2} - 1) + i\sqrt{2}(1 - \frac{1}{2}\sqrt{2})}{2 - \sqrt{2}}$$

so Im is positive

So the image of

$$T_1(\Omega + \tfrac{1}{2}\sqrt{2} - \tfrac{1}{2}\sqrt{2}\, i)$$

is $\{\eta : 0 < \arg \eta < \frac{\pi}{4}\}$.

Now $\eta \to \eta^4$ sends this set to

$$H^+ = \{\zeta : \operatorname{Im} \zeta > 0\}.$$

Finally $\zeta \to \dfrac{\zeta - i}{\zeta + i}$ sends

H^+ to $\{z : |z| < 1\}$

To wrap thing up.

$$z \to w = z + \tfrac{1}{2}\sqrt{2} - \tfrac{1}{2}\sqrt{2}\, i$$

$$w \to \frac{w}{\sqrt{2} - w} = \eta$$

$$\eta \to \eta^4 = \zeta$$

$\zeta \to \dfrac{\zeta - i}{\zeta + i}$ gives a

map $\varphi = \left[\dfrac{\left(\dfrac{z + \frac{1}{2}\sqrt{2} - \frac{1}{2}\sqrt{2}\, i}{\frac{1}{2}\sqrt{2} - z + \frac{1}{2}\sqrt{2}\, i}\right)^4 - i}{\left(\dfrac{z + \frac{1}{2}\sqrt{2} - \frac{1}{2}\sqrt{2}\, i}{\frac{1}{2}\sqrt{2} - z + \frac{1}{2}\sqrt{2}\, i}\right)^4 + i}\right]$

sends Ω to Δ.

Problem 6:

Let $u = x^3 - 3xy^2$

a) $$\frac{\partial u}{\partial x} = 3x^2 - 3y^2$$

$$\frac{\partial^2 u}{\partial x^2} = 6x$$

$$\frac{\partial u}{\partial y} = -6xy$$

$$\frac{\partial^2 u}{\partial y^2} = -6x$$

Hence $$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x - 6x = 0$$

so u is harmonic

b) If $f = u + iv$ is analytic, then the the Cauchy-Riemann equations gives.

$$v_y = u_x = 3x^2 - 3y^2$$

and

$$v_x = -u_y = 6xy.$$

By taking antiderivatives we obtain:

$$v = 3x^2y - y^3 + c_1(x)$$

and

$$v = 3x^2y + c_2(y)$$

If we let $c_1(x) = 0$ and $c_2(y) = -y^3$

we see that

$$v(x,y) = 3x^2y - y^3$$

will work.

c) Observe that if

$$z = x + iy = e^{i\theta} = \cos\theta + i\sin\theta$$

then $u(x,y) = \cos^3\theta - 3\cos\theta\sin^2\theta$

so

$$\frac{1}{2\pi}\int_0^{2\pi} \frac{(1-r^2)(\cos^3\theta - 3\cos\theta\sin^2\theta)}{1 - 2r\cos(\theta - \varphi) + r^2)}\, d\theta$$

$$= \frac{1}{2\pi}\int_0^{2\pi} P_r(\theta - \varphi)\, u(e^{i\theta})\, d\theta$$

u is harmonic in $\Delta = \{ |z| < 1 \}$
and continuous on $\bar{\Delta} = \{ |z| \le 1 \}$

Hence

$$\frac{1}{2\pi}\int_0^{2\pi} P_r(\theta - \phi)\, u(e^{i\theta})\, d\theta$$

$$= u(re^{i\phi})$$

If $r = \frac{1}{2}$ and $\phi = \frac{\pi}{4}$

Then

$$re^{i\phi} = \frac{1}{2}\cos\frac{\pi}{4} + i\,\frac{1}{2}\sin\frac{\pi}{4}$$

so $u(re^{i\phi}) = \left(\frac{1}{2}\cos\frac{\pi}{4}\right)^3 - 3\left(\frac{1}{2}\cos\frac{\pi}{4}\right)\left(\frac{1}{2}\sin\frac{\pi}{4}\right)^2$

$$= \frac{1}{8}\left[\frac{1}{2}\sqrt{2}\right]^3 - 3\,\frac{1}{8}\left[\frac{1}{2}\sqrt{2}\right]\left[\frac{1}{2}\sqrt{2}\right]^2$$

$$= -\frac{\sqrt{2}}{16}.$$