

For questions during the exam:  
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Exam in TTK4130 Modeling and Simulation  
 Monday, May 15 2017  
 09:00 – 13:00

Permitted aids (code A): All written and handwritten examination support materials are permitted.

Answers in English, Norwegian, or a mixture of the two accepted.

Grades available: As specified by regulations.

**Problem 1 (19 %)**

Consider the following variable-step Runge-Kutta method with the extended Butcher tableau:

0	
2/3	2/3
	1/4    3/4
	1      0

The following Taylor's expansion may be of use:

$$\mathbf{f}(\mathbf{y}_n + h\alpha\mathbf{k}, t_n + \alpha h) = \mathbf{f}(\mathbf{y}_n, t_n) + \alpha h \frac{d\mathbf{f}}{dt}(\mathbf{y}_n, t_n) + O(h^2). \quad (1)$$

- (2 %)    (a) Is the method explicit or implicit? Explain!
- (2 %)    (b) Find the order of the Runge-Kutta method with fewer stages (explanation required).
- (5 %)    (c) Find the order of the other Runge-Kutta method (calculations required).

The following pendulum equation is given

$$\ddot{\theta} = -\frac{g}{l}\theta. \quad (6)$$

- (6 %)    (d) For which steplength  $h$  is the higher order method stable for the pendulum with length  $l = 1m$  and  $g = 10m/s^2$ ?
- (4 %)    (e) This is the Butcher tableau of an *adaptive* method. Explain in a few sentences how the adaptivity of the method works and what advantages this includes.

**Problem 2 (14 %)**

A pendulum (Fig. 1) was modeled with the Newton-Euler approach in the inertia system.

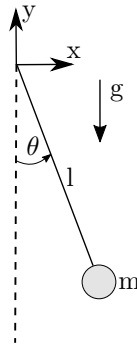


Figure 1: Pendulum.

The following DAE-system was derived

$$m\ddot{x} = -\frac{S}{l}x, \quad (10a)$$

$$m\ddot{y} = -\frac{S}{l}y - mg, \quad (10b)$$

$$0 = l^2 - x^2 - y^2, \quad (10c)$$

where  $S$  is the force in the rope.

- (2 %) (a) Determine differential and algebraic variables of the system.
- (6 %) (b) Determine the differential index of the system.
- (6 %) (c) Write down the Modelica code for the index 1 system based the system (10) and your results in the previous task. The name of the model should be **Pendulum**.  
*If you were not able to determine the index you can use only the system (10) (5%).*

**Problem 3 (30 %)**

A hollow cylinder (mass  $m_1$ , radius  $R$ ,  $\delta \ll R$ ) is pivoted on two mass-less bearings. In the hollow cylinder a plate (mass  $m_2$ , radius  $r$ ) rolls without sliding. The moments of inertia are denoted by  $I_i$ . The movement happens in a 2D plane (Fig. 2).

- (2 %) (a) Find a connection between  $\dot{\psi}$ ,  $\dot{\phi}$  and  $\dot{\theta}$ .  
*Hint: If you are not able to find a connection, continue the task with the following:  $(R + r)\dot{\phi} + r\dot{\psi} - R\dot{\theta} = 0$*
- (2 %) (b) Why are  $\psi$  and  $\theta$  a good choice as generalized coordinates?
- (4 %) (c) Find the kinetic energy of the system.
- (2 %) (d) Find the potential energy of the system.
- (8 %) (e) Find the equation of motion of the system.
- (6 %) (f) Determine the period of the oscillation in case of a small initial deflection in  $\theta$ .
- (6 %) (g) Derive the moment of inertia for the hollow cylinder.  
*Hint: The derived moment of inertia should have a similar form as the moment of inertia of a rod:  $I = \frac{1}{12}ml^2$  (this task can be solved independently from the rest).*

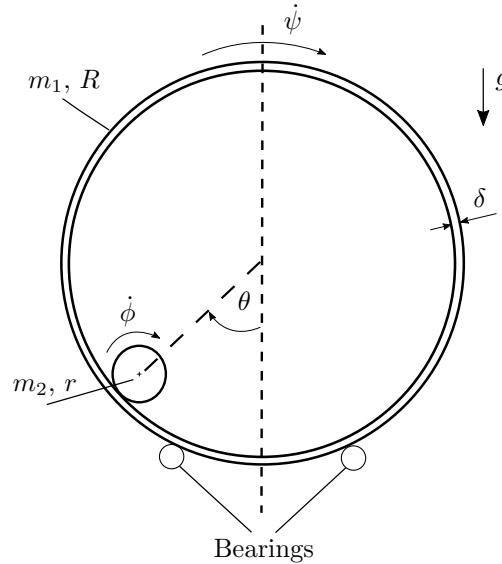


Figure 2: Hollow cylinder with rolling plate.

**Problem 4 (14 %)**

A simulation model for an autonomous fire truck should be found. For test purposes a small scale model of the fire truck was build (Fig. 4).



Figure 4: Fire truck - Small scale model.

The project has several groups working on different parts of the model. Your task is to find the equation of motion of the ladder on the fire truck to precisely control the position of the hose outlet. The only subtask we will consider here is to find the connection between inertial coordinate frame, which is aligned with the truck, and the body frame of the hose.

The movement can be described by three rotation and one translation. Two rotations are in the attachment of the ladder to the truck. One of these rotations rotates the ladder around  $\theta$  in the horizontal plane, the other rotates the ladder around  $\psi$  up or down. Afterward a translation  $l(t)$  has to be performed from the attachment to the joint of the hose. The hose joint can be rotated around  $\phi$  up or down.

- (6 %) (a) In the first step the coordinate systems (inertial and body frames) have to be defined. Draw an abstraction (body diagram) of the process that shows the inertial coordinate system with the axes

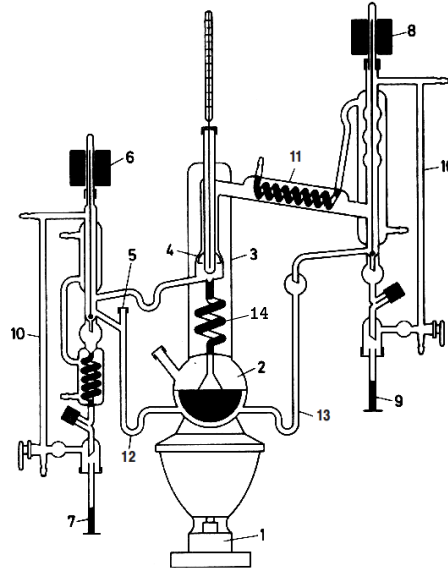
$(\{x_i, y_i, z_i\})$ . The origin is in the attachment of the ladder and one axis is aligned with the vehicle. Furthermore, draw the two body coordinate systems with the axes  $(\{x_a, y_a, z_a\})$  and  $(\{x_b, y_b, z_b\})$ . The first body coordinate system is the one after the first rotation with its origin in the attachment. The second has its origin in the joint of the hose at the end of the ladder. Include in your drawing also the angular velocities  $\dot{\theta}$ ,  $\dot{\psi}$  and  $\dot{\phi}$  and the translation  $l(t)$ .

- (6 %) (b) Define homogenous transformation matrices to calculate the position of the hose outlet in the inertial frame. Show how the overall transformation matrix can be found. You do not need to do the matrix multiplication for full score.
- (2 %) (c) Define the position of the attachment relative to the hose outlet. Matrix multiplications are not required for full score.

### Problem 5 (23 %)

In this task a measurement device used to determine the Vapor-liquid equilibrium (VLE) should be modeled (Fig 6). The VLE is a phase equilibrium in which the liquid and vapor are in thermodynamic equilibrium. For a pure substance the VLE is determined by temperature and pressure. For composites the composition of liquid and vapor phase has to be given in addition.

It is not necessary to understand the details of the device to solve the task. Nevertheless, the basic principle of the device is to distill a composite fluid, that is, to separate the fluid by selective evaporation and condensation. A vessel containing the fluid is heated. The vapor pulls a small amount of liquid with it into the Cottrel-pump. The temperature of the vapor-liquid mixture that exits the pump is in equilibrium and measured. Afterward the two phases (liquid and vapor) are separated and recycled to the vessel. The vapor must be completely condensed before it enters the vessel, which is done in a condenser. The pressure of the device can be controlled by a vacuum-system. Moreover, samples can be taken from the recirculation pipes to determine the concentration in vapor and liquid phase.



1 Vesselheater, 2 Vessel, 3 Vacuumjacket, 4 Temperature measurement point, 5 Inputnozzle, 6 Magnetvalve - liquid sample, 7 Receiver, 8 Magnetvalve - vapor sample, 9 Receiver, 10 Vacuum pipe, 11 Condenser, 12 Circulation pipe, 13 Circulation pipe, 14 Cottrel-pump

Figure 6: Measurement device

In our model we will assume that the binary (containing two components A and B) boiling point diagram should be determined. A boiling point diagram shows the VLE for a composite. For this purpose the device is modeled as a one-tray distillation column without feed nor product outlet. The structure of the model is shown in Fig. 7. The part  $K_1$  contains the vessel, the Cottrel-pump and the liquid reflux

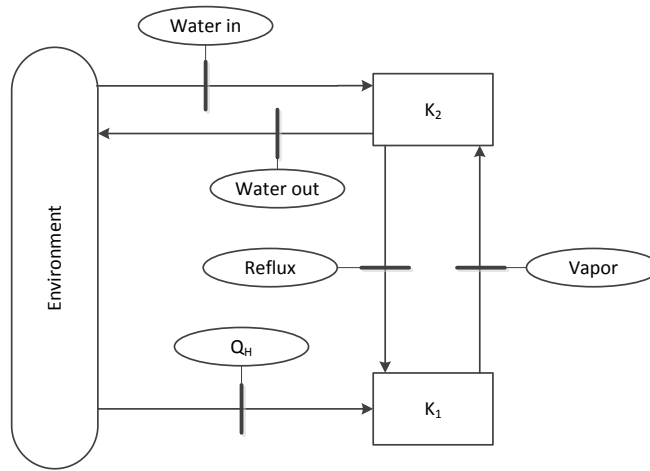


Figure 7: Model structure of measurement device.

from the thermometer. The part  $K_2$  represents the condenser.

In the modeling process the following assumptions are done:

- The part  $K_1$  is heated by heat flux  $\dot{Q}_H$ . The part is ideally-mixed. The volume is  $V$ . The composite can be described by the pressure  $p$ , the temperature  $T$ , and the mole-fraction in the liquid phase  $x_A$ ,  $x_B$  and in the vapor phase  $y_A$  and  $y_B$ .
- The part  $K_1$  is quasi-homogeneous, which means that the AB mixture is in equilibrium.
- The condenser ( $K_2$ ) works as a steady-state total condenser. The outlet feed  $\dot{R}$  is a saturated liquid (meaning it is exactly at boiling temperature). The mole-fractions are given by  $x_{R,A}$  and  $x_{R,B}$ . The specific enthalpy is given by  $h_R$ .  $K_2$  enters a known mole-flow  $\dot{W}_{in}$  of cooling water with a specific mole-intensive enthalpy  $h_{W,in}$ . The condenser pressure is  $p_K$ .
- The vacuum-system is not modeled and the pressure  $p_K$  is a known input-variable.
- Heat losses can be neglected.
- The vapor flow into the condenser can be calculated by  $\dot{V} = f(p, p_K)$ . Its specific enthalpy is given by  $h_V$ .
- Potential and kinetic energies are neglected in both parts.

(6%)

- (a) Derive the balance equations for the entire amount of substance  $n$ , the amount of substance of component A  $n_A$  and the internal energy  $U$  for part  $K_1$ .

To complete the model of the part  $K_1$  "closure-relation" have to be derived. The following substance equations can be used:

$$\text{spec. mol. vapor-enthalpy of AB-mixture: } h_V = h_V(T, p, y_A, y_B), \quad (38a)$$

$$\text{spec. mol. liquid-enthalpy of AB-mixture: } h_L = h_L(T, p, x_A, x_B), \quad (38b)$$

$$\text{molar vapor-density of AB-mixture: } \rho_{m,V} = \rho_{m,V}(T, p, y_A, y_B), \quad (38c)$$

$$\text{molar liquid-density of AB-mixture: } \rho_{m,L} = \rho_{m,L}(T, p, x_A, x_B), \quad (38d)$$

$$\text{Henry's law constant of A of AB-mixture: } K_A = K_A(T, p, y_A, y_B, x_A, x_B), \quad (38e)$$

$$\text{Henry's law constant of B of AB-mixture: } K_B = K_B(T, p, y_A, y_B, x_A, x_B). \quad (38f)$$

Henry's law describes the VLE data for mixtures, where the  $K$  values are defined by

$$K_i = \frac{y_i}{x_i}. \quad (39)$$

- (6 %) (b) Complete the model of  $K_1$  with the necessary closure relations (constitutive equations and constraints). Assume, that all input molecular and energy flows as well as the vapor flow are known.  
*Hint:* With 12 additional equations you can complete the model (However, depending on which equations you choose it might be necessary to have more than 12 equations).
- (4 %) (c) Check the degrees of freedom of the equation system! How many variables have to be determined (specified as inputs or parameters) to have a well-defined equations system? Give the variables of the input vector! Assume that the volume  $V$  is the only model parameter.
- (3 %) (d) Complete the Modelica model of the connector "port" that passes the entire mole flux  $J$ , the molar fraction  $y_A$  and the specific molar enthalpy  $h_j$ .

**connector port**

**end port;**

- (4 %) (e) Derive the balance equations for the total amount of substance, the amount of component A, the amount of cooling water as well as the energy of the condenser  $K_2$ .