NTNU – Trondheim
Norwegian University of
Science and Technology

For questions during the exam:
Lars Imsland, tel. 47 23 19 49.

Exam in TTK4130 Modeling and Simulation

Thursday, May 30th 2014
09:00 – 13:00

Permitted aids (code A): All written and handwritten examination support materials are permitted.

Note: A Norwegian text is appended.

Answers in English, Norwegian, or a mixture of the two accepted.

Grades available: As specified by regulations.

Problem 1 (20 %)

(8 %) (a) Consider the class of explicit, two-stage Runge-Kutta methods. These have Butcher array

$$\begin{array}{c|cc} 0 & & \\ \alpha & \alpha & \\ \hline & b_1 & b_2 \end{array}.$$

Assume α a free parameter, $0 \leq \alpha \leq 1$, and determine $b_1(\alpha)$ and $b_2(\alpha)$ to satisfy the conditions for the method to be of second order. Use the following series expansion:

$$\mathbf{f}(\mathbf{y}_n + h\alpha\mathbf{k}_1, t_n + \alpha h) = \mathbf{f}(\mathbf{y}_n, t_n) + \alpha h \frac{\mathrm{d}\mathbf{f}}{\mathrm{d}t}(\mathbf{y}_n, t_n) + O(h^2)$$

where $\mathbf{k}_1 = \mathbf{f}(\mathbf{y}_n, t_n)$.

(6 %) (b) Determine the stability function of these methods (with $b_1(\alpha)$ and $b_2(\alpha)$ chosen as in (a)). Can you use α to maximize the stability region?

(6 %) (c) Given the following ordinary differential equation:

$$\begin{aligned} \dot{y}_1 &= y_1(y_1 - 2) + e^{y_2} \\ \dot{y}_2 &= 50 y_2 (y_2 - 2) \end{aligned}$$

Determine the maximum stepsize h for which this Runge-Kutta method is (linearly) stable in a neighborhood of the origin.

Problem 2 (20 %)

In this problem, and the next, we will develop a model for all degrees of freedom for a quadrotor, modeling the quadrotor as a rigid body. See Figure 1 for definition of coordinate systems, and a “free body diagram” with forces and moments acting on the quadrotor.

(8 %) (a) To specify the orientation of the quadrotor, the Z-X-Y Euler angles are sometimes used. These are specified by first a rotation α about the (inertial) z-axis, then β about the intermediate (rotated) x-axis, and finally γ about the body y-axis. Write up an expression for the rotation matrix $\mathbf{R}_b^i = \mathbf{R}_b^i(\boldsymbol{\phi})$ as a function of the Euler angles $\boldsymbol{\phi} = (\alpha, \beta, \gamma)^{\mathsf{T}}$.

(12 %) (b) Find the kinematic differential equations for this choice of Euler angles. Assume that the angular velocity is given in body-frame. (It is not necessary to perform a matrix inversion for full score.)

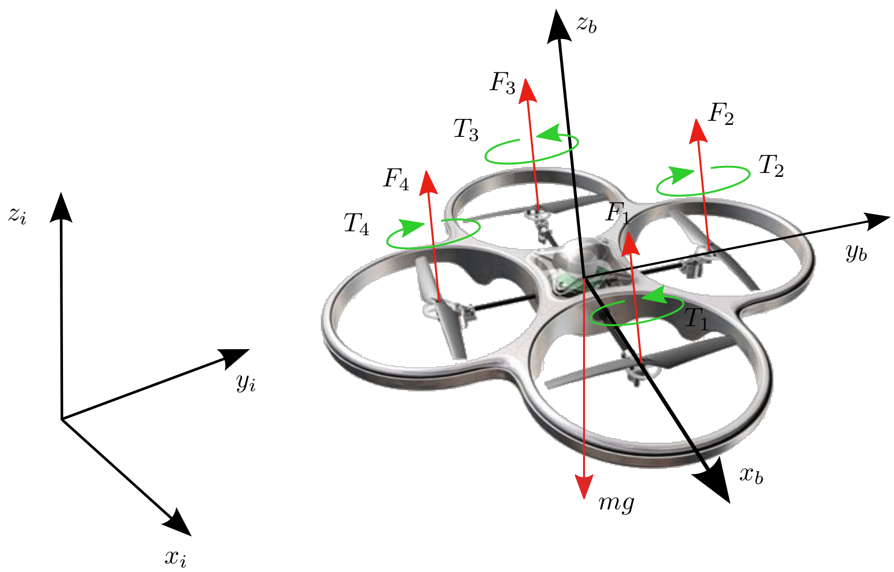

Figure 1: Coordinate systems and forces/moments.

Problem 3 (24 %)

In this problem, we will continue to develop the complete dynamic model of the quadrotor by modeling the kinetics. The forces and moments acting on the quadrotor are illustrated in Figure 1. The body system has origin in the center of mass, and the quadrotor has mass m and an inertia matrix $\mathbf{M}^b_{b/c}$. Note that the moments T_i due to rotation of the rotors give moments acting about the z_b-axis, and that the rotor forces F_i will give cause to moments about the x_b and y_b axis, with "arm" (distance from center of mass to rotor) L for all rotors. Note also that T_i has a "sign" defined in the figure, due to the default direction of rotation of the rotors.

(4 %) (a) Why is it natural to use the Newton-Euler equations of motions as starting point, rather than the Lagrange equations of motion?

(6 %) (b) Write up expressions for the force and torque vectors acting on the center of mass, $\mathbf{F}^b_{bc}$ and $\mathbf{T}^b_{bc}$, decomposed in the body system, as function of the forces and torques defined in Figure 1.

(12 %) (c) What are the equations of motion of the quadrotor, on vector form? The components of vectors equations in the answer should amount to 12 first-order differential equations, including the answer from Problem 2(b).

(Problems 2 and 3 are based on the article: Daniel Mellinger, Nathan Michael and Vijay Kumar, *Trajectory generation and control for precise aggressive maneuvers with quadrotors*, The International Journal of Robotics Research 31(5):664–674, 2012.)

Problem 4 (18 %)

A rod rotates in a vertical plane with constant angular speed ω about a horizontal axis, such that the angle of the rod with respect to the horizontal plane is $\theta = \omega t$. A bead of mass m slides without friction on the rod, under the influence of gravity g. See Figure 2.

(14 %) (a) Find the equation of motion for the bead. *Hint*: Use the distance r from the center of rotation to the bead as the generalized coordinate for the bead.

(6 %) (b) Write pseudocode for integrating this equation of motion one timestep using Euler's method (explicit Euler) with step-length h. The pseudocode should include the solution to (a) (or, if you did not do (a)), a qualified guess of the solution of (a)).

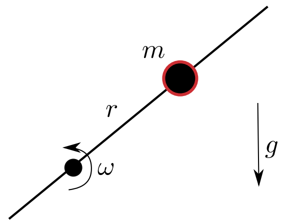

Figure 2: Bead on a rotating rod.

Problem 5 (18 %)

Consider the closed tank in Figure 3, of total height L. An incompressible liquid of density ρ flows into the tank through the valve on the left, and leaves the tank through the valve on the right. The flow through the valves are given by the valve equation (assuming turbulent flow through the valve). The liquid in the tank has a liquid level h, and the total liquid volume is $V = Ah$ (A is the tank cross-sectional area). The volume V_g above the liquid varies as h varies, and is filled with a gas, for which the ideal gas law applies:

$$p_g V_g = m_g \bar{R} T.$$

We assume that m_g (the mass of gas in the volume), $\bar{R}$ (the specific gas constant) and T (the gas temperature) are known and constant.

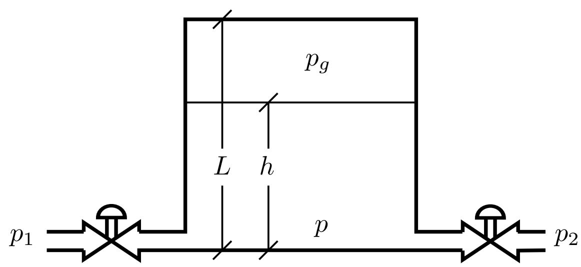

Figure 3: Closed tank with in- and outflow.

(12 %) (a) Set up a differential equation for the liquid level h.

(6 %) (b) Assume you want to simulate this system with an explicit Runge-Kutta method with automatic adjustment of step-length. When will this give trouble?