PROBLEM 1:

Let f be entire and assume that $|f(z)| \le |z|^{10}$ for all $z \in \mathbb{C}$.

a) Show that $f^{(m)}(0) = 0$ for all $m \ge 11$

b) Show that f is a polynomial of degree less than or equal to 10.

There are several ways of solving this problem.

One very smart:

Observe that $\frac{f(z)}{z^{10}} = g(z)$ is bounded, hence have a removable singularity at 0. Hence g is entire and bounded so Liouvilles theorem implies that g is constant so $\frac{f(z)}{z^{10}} = c \Rightarrow f(z) = cz^{10}$ and both a) and b) follows.

A more standard method is to use:

a) $$f^{(k)}(0) = \frac{k!}{2\pi i} \int_{|z|=R} \frac{f(\xi)}{\xi^{k+1}}\, d\xi$$

for all R

$$\Rightarrow |f^{(k)}(0)| \le \frac{k!}{2\pi} \int_{|z|=R} \frac{|f(\xi)|}{|\xi|^{k+1}}\, d|\xi|$$

$$\le \frac{k!}{2\pi} \int_{|z|=R} \frac{R^{10}}{R^{k+1}}\, d|\xi| = \frac{k!}{2\pi} \frac{R^{10}}{R^{k+1}} 2\pi R$$

$$= k!\, R^{10-k}$$

If $k \geq 11$ it follows that $R^{10-k} \to 0$ when $R \to \infty$.

Hence $f^{(k)}(0) = 0$ when $k \geq 11$.

b) Since f is entire it follows that

$$f(z) = \sum_{k=0}^{\infty} \frac{f^{(k)}(0)}{k!} z^k$$

for all $z \in \mathbb{C}$, but $f^{(k)}(0) = 0$ when $k \geq 11$, hence

$$f(z) = \sum_{k=0}^{10} \frac{f^{(k)}(0)}{k!} z^k$$

PROBLEM 2:

$h(z) = z^3 + 3z^2 + 17z + 50 = f(z) + g(z)$

where $f(z) = z^3$ and $g(z) = 3z^2 + 17z + 15$

When $|z| = 10$ we have $|f(z)| = 10^3 = 1000$

while $|g(z)| \leq 3|z|^2 + 17|z| + 50 =$

$300 + 170 + 50 = 520$

when $|z| = 10$

So $|f(z)| > |g(z)|$ on $\{z : |z| = 10\}$

Rouche's theorem $\Rightarrow$ $f+g$ and f have the same number of zero's in $\{z : |z| < 0\}$. The function f have a triple root at 0 so $f+g$ will have 3 roots in $\{z : |z| < 10\}$.

PROBLEM 3:

$$\gamma(t) = 4e^{it}, \quad 0 \le t \le 2\pi$$

$$I = \int_\gamma \frac{e^z - 1}{z} \left(\frac{1}{z^4 + 3 + 3i} \right) = \int_\gamma f(z)$$

$$= 2\pi i \sum_k \operatorname{Res}(f, z_k) \quad \text{where}$$

z_k are the singularities of f inside $\{z : |z| < 4\}$.

Potential singularities:

$$z = 0$$

$$z_k \text{ where } z_k^4 + 3 + 3i = 0$$

$$\frac{e^z - 1}{z} = \frac{1 + \sum_{j=1}^{\infty} \frac{1}{j!} z^j - 1}{z} = \sum_{j=1}^{\infty} \frac{1}{j!} z^{j-1}$$

so f have a removable singularity at 0.

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$$z^4 + 3 + 3i = 0$$
$$\Updownarrow$$
$$z^4 = -3(1+i) = e^{i\pi}\, 3\sqrt{2}\, e^{i(\frac{\pi}{4} + 2k\pi)}$$

$$z^4 = (18)^{\frac{1}{2}} e^{i\frac{5\pi}{4}} e^{i2k\pi}$$

$$z_0 = (18)^{\frac{1}{8}} e^{i\frac{5\pi}{16}}$$

$$z_1 = (18)^{\frac{1}{8}} e^{i\frac{5\pi}{16}} i$$

$$z_2 = (18)^{\frac{1}{18}} e^{i\frac{5\pi}{16}} i^2$$

$$z_3 = (18)^{\frac{1}{18}} e^{i\frac{5\pi}{16}} i^3$$

z_0, z_1, z_2, z_3 are all simple poles for $f(z)$

Write

$$f(z) = \frac{\frac{e^z - 1}{z}}{z^4 + 3 + 3i} = \frac{g(z)}{h(z)}$$

$$\operatorname{Res}_{z_k} f(z) = \frac{g(z_k)}{h'(z_k)} = \frac{\frac{e^{z_k} - 1}{z_k}}{4 z_k^3} = \frac{e^{z_k} - 1}{4 z_k^4}$$

$\Downarrow$

$$\int_\gamma \frac{e^z - 1}{z} \left(\frac{1}{z^4 + 3 + 3i} \right) = 2\pi i \sum_{k=0}^{3} \frac{e^{z_k} - 1}{4 z_k^4}$$

PROBLEM 4:

$$\int_0^\infty \frac{x \sin x}{x^4 + 1}\, dx = \int_0^1 \frac{x \sin x}{x^4 + 1}\, dx + \int_1^\infty \frac{x \sin x}{x^4 + 1}\, dx$$

$$\left| \int_1^\infty \frac{x \sin x}{x^4 + 1}\, dx \right| \leq \int_1^\infty \frac{|x \sin x|}{x^4 + 1}\, dx \leq \int_1^\infty \frac{1}{x^3}\, dx$$

$\int_1^\infty \frac{1}{x^3}\, dx$ converges so our integral will converge.

$\frac{x \sin x}{x^4+1} dx$ is symmetric so

$$I = \int_0^{\infty} \frac{x \sin x}{x^4+1} dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{x \sin x}{x^4+1} dx$$

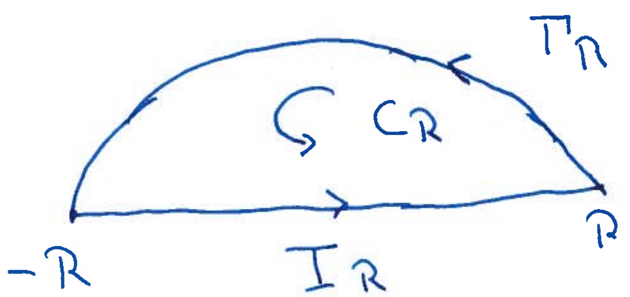
To compute

$\int_{-\infty}^{\infty} \frac{x \sin x}{x^4+1} dx$ we observe that

$$\int_{-\infty}^{\infty} \frac{x \sin x}{x^4+1} dx = \operatorname{Im} \int_{-\infty}^{\infty} \frac{x e^{ix}}{x^4+1} dx$$

$$= \operatorname{Im} \lim_{R \to \infty} \int_{-R}^{R} \frac{x e^{ix}}{x^4+1} dx$$

add $T_R = \{ R e^{i\theta} : 0 \le \theta \le \pi \}$ to $[-R, R]$



$C_R = I_R \cup T_R$

Let R be large

Now

$$\int_{C_R} \frac{z e^{iz}}{z^4+1}\,dz = 2\pi i \sum \operatorname{Res}\left(\frac{z e^{iz}}{z^4+1}, z_k\right)$$

where z_k are the singularities enclosed by C_R.

Potential singularities:

$$z^4+1=0$$
$$\Updownarrow$$
$$z^4=-1=e^{i(\pi+2\pi k)}$$

$$z_0 = e^{i\frac{\pi}{4}}$$
$$z_1 = e^{i\frac{3\pi}{4}}$$
$$z_2 = e^{i\frac{5\pi}{4}}$$
$$z_3 = e^{i\frac{7\pi}{4}}$$

Only z_0 and z_1 are enclosed by C_R so

$$\int_{C_R} \frac{z e^{iz}}{z^4+1}\,dz = 2\pi i\left[\operatorname{Res}\left(\frac{z e^{iz}}{z^4+1}, e^{i\frac{\pi}{4}}\right) + \operatorname{Res}\left(\frac{z e^{iz}}{z^4+1}, e^{i\frac{3\pi}{4}}\right)\right]$$

$$\operatorname{Res}\left(\frac{z e^{iz}}{z^4+1}, e^{i\frac{\pi}{4}}\right) = \frac{e^{i\frac{\pi}{4}} e^{ie^{i\frac{\pi}{4}}}}{4\left(e^{i\frac{\pi}{4}}\right)^3}$$

$$= \frac{e^{i\left(\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)}}{4\left(e^{i\frac{\pi}{4}}\right)^2}$$

$$= \frac{1}{4i}\left(e^{i\frac{1}{\sqrt{2}}} e^{-\frac{1}{\sqrt{2}}}\right)$$

$$= \frac{e^{-\frac{1}{\sqrt{2}}}}{4i}\left(\cos\frac{1}{\sqrt{2}} + i\sin\frac{1}{\sqrt{2}}\right)$$

$$\operatorname{Res}\left(\frac{z e^{iz}}{z^4+1}, e^{i\frac{3\pi}{4}}\right) = \frac{e^{i\frac{3\pi}{4}} e^{ie^{i\frac{3\pi}{4}}}}{4\left(e^{i\frac{3\pi}{4}}\right)^3}$$

$$= \frac{e^{i\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right)}}{4\left(e^{i\frac{3\pi}{4}}\right)^2}$$

$$= \frac{e^{-\frac{1}{\sqrt{2}}} e^{-i\frac{1}{\sqrt{2}}}}{4(-i)}$$

$$= -\frac{1}{4i} e^{-\frac{1}{\sqrt{2}}}\left(\cos\frac{1}{\sqrt{2}} - i\sin\frac{1}{\sqrt{2}}\right)$$

So

$$\int_{C_R} \frac{z e^{iz}}{z^4+1}\, dz = 2\pi i \frac{1}{4i} e^{-\frac{1}{\sqrt{2}}}\left(\cos\frac{1}{\sqrt{2}} + i\sin\frac{1}{\sqrt{2}} - \cos\frac{1}{\sqrt{2}} + i\sin\frac{1}{\sqrt{2}}\right)$$

$$= i\pi e^{-\frac{1}{\sqrt{2}}} \sin\frac{1}{\sqrt{2}}$$

$$\left| \int_{\Gamma_R} \frac{z e^{iz}}{z^4+1} dz \right| \le \int_{\Gamma_R} \frac{|z||e^{iz}|}{|z|^4-1} d|z|$$

$$|e^{iz}| = |e^{i(x+iy)}| = |e^{-y+ix}| = e^{-y} \le 1$$

when $y \ge 0$.

Hence

$$\left| \int_{\Gamma_R} \frac{z e^{iz}}{z^4+1} dz \right| \le \int_{\Gamma_R} \frac{R}{R^4-1} d|z|$$

$$= \frac{\pi R^2}{R^4-1} \xrightarrow[R\to\infty]{} 0$$

So

$$\lim_{R\to\infty} \int_{C_R} \frac{z e^{iz}}{z^4+1} dz = \int_{-\infty}^{\infty} \frac{x e^{ix}}{x^4+1} dx$$

$$= i\pi e^{-\frac{1}{\sqrt{2}}} \sin\frac{1}{\sqrt{2}}$$

Finally

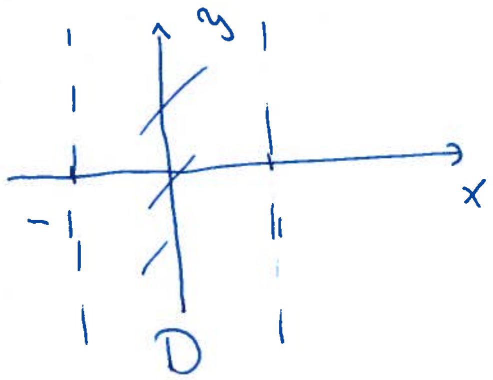
$$\int_0^{\infty} \frac{x \sin x}{x^4+1} dx = \frac{1}{2} \int_{-\infty}^{\infty} \frac{x \sin x}{x^4+1} dx = \frac{1}{2} \operatorname{Im} \int_{-\infty}^{\infty} \frac{x e^{ix}}{x^4+1} dx$$

$$= \frac{\pi}{2} e^{-\frac{1}{\sqrt{2}}} \sin\frac{1}{\sqrt{2}}$$

PROBLEMS:

Find a conformal map from

$D = \{ z \in \mathbb{C} : -1 < \operatorname{Re} z < 1 \}$

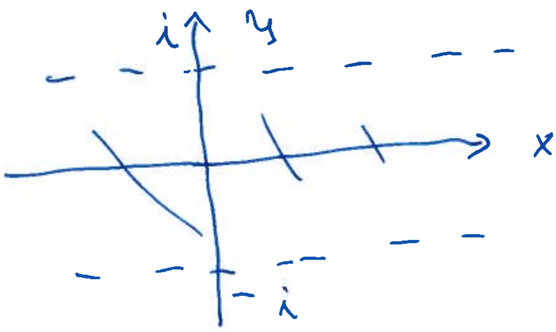


Step 1 : Rotate D by

$z \to e^{i\frac{\pi}{2}} z = iz$

The image is

$\{ z : -1 < \operatorname{Im} z < 1 \}$



Step 2 : Add i to iz

$iz + i$

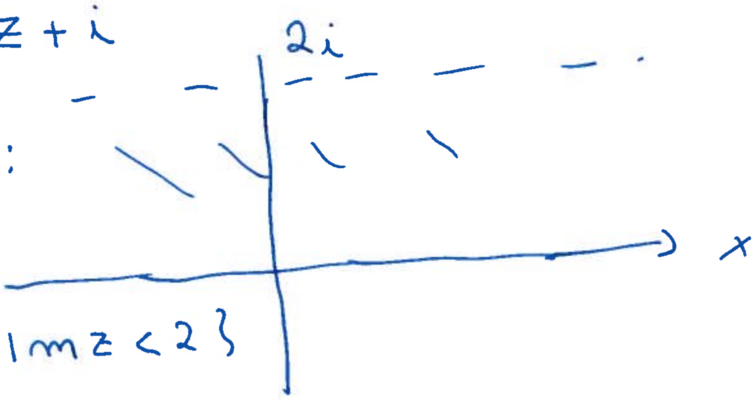


Image:

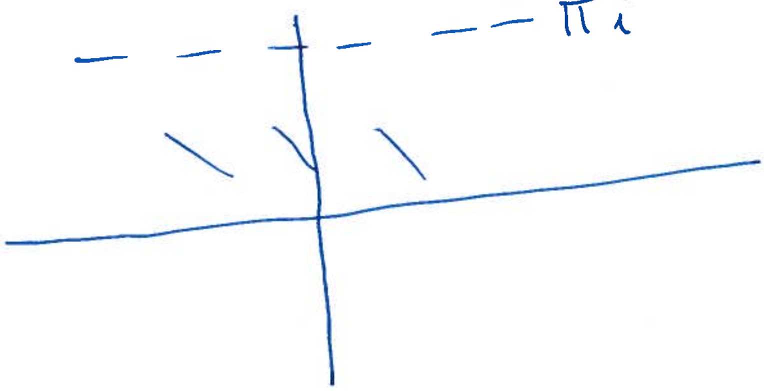
$\{ z : 0 < \operatorname{Im} z < 2 \}$

Step 3:

Multiplying by $\frac{\pi}{2}$, we get

$$i(z+1)\frac{\pi}{2}$$

Image



Step 4: Let $w \to e^{w}$, $w = i(z+1)\frac{\pi}{2}$

and we get $e^{i(z+1)\frac{\pi}{2}} = i e^{i\frac{\pi z}{2}}$

Image

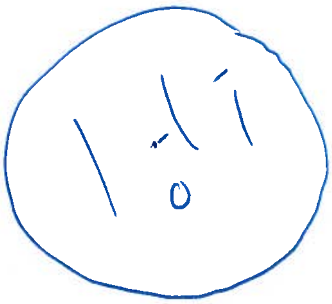
$\{z : \operatorname{Im} z > 0\}$

Step 5: $w \to \dfrac{w-i}{w+i}$

we get $\dfrac{i e^{i\frac{\pi}{2}z} - i}{i e^{i\frac{\pi}{2}z} + i} = \dfrac{e^{i\frac{\pi}{2}z} - 1}{e^{i\frac{\pi}{2}z} + 1}$

The image is

$\{z : |z| < 1\}$.



PROBLEMS:

f analytic in $\Delta = \{z : |z| < 1\}$,

$|f(z)| \le 1$, $f(0) = 0$ and $f'(0) = 0$.

We want to show that

$|f(z)| \le |z|^2$ and if

$|f(z_0)| = |z_0|^2$ for some $z_0 \neq 0$

it follows that $f(z) = e^{i\theta} z^2$

There are several ways to solve this problem

I choose to use Schwarz lemma on the function $g(z) = \frac{f(z)}{z}$

<u>Observe:</u>

$f(0)=0$ so g is analytic in Δ

also

$f'(0)=0$ so $g(0) = 0$

$|f(z)| \le 1$ so $\lim_{|z|\to 1} |g(z)| = \lim_{|z|\to 1} \left|\frac{f(z)}{z}\right| \le 1$

so the maximum principle implies that $|g(z)| \le 1$ for all $z \in \Delta$

Schwarz lemma $\Rightarrow$

$|g(z)| = \left|\frac{f(z)}{z}\right| \le |z|$

for all $z \in \Delta$

Hence $|f(z)| \le |z|^2$

Also if $\exists\, z_0 \ne 0$ such that $|g(z_0)| = |z_0|$ or $|f(z_0)| = |z_0|^2$ it follows that $g(z) = e^{i\theta} z$ or

$f(z) = e^{i\theta} z^2$ for all z.

PROBLEM 7:

a) $u(x,y) = x^3 - 2xy$ so

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 6x$ which is not zero unless $x=0$ so u is NOT harmonic in $\mathbb{C}$, hence there is NO v such that $u+iv$ is analytic in $\mathbb{C}$

b) $D = \mathbb{C} \setminus \gamma$, $\gamma(t) = (t, t^2)$, $0 \le t \le \infty$

D is SIMPLY connected and NOT all of $\mathbb{C}$. Riemann mapping theorem implies that there exist a conformal $\varphi : D \to \{z : |z| < 1\}$

The fact that φ is conformal implies that φ is analytic and NONCONSTANT.

c) If f is analytic in Δ, then f is also continuous. If $f(\frac{1}{n}) = 0$ for all $n = 2, 3, \ldots$ it follows that $f(0) = 0$ but 0 is not an isolated point in $\{\frac{1}{n} : n = 2, \ldots\} \cup \{0\}$. Nonconstant analytic function have isolated zeros. Hence f is constant.

d) Observe that $u(e^{it}) = |\cos t| = |\mathrm{Re}\, e^{it}|$ so u is continuous on the unit circle.

If $z = re^{i\theta}$ we let

$$\hat{u}(z) = \frac{1}{2\pi}\int_0^{2\pi} u(e^{it})\left(\frac{1-r^2}{1-2r\cos(\theta-t)+r^2}\right)dt$$