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For questions during the exam:
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Exam in TTK4130 Modeling and Simulation

Thursday, May 30th 2014

09:00 – 13:00

Permitted aids (code A): All written and handwritten examination support materials are permitted.

Note: A Norwegian text is appended.

Answers in English, Norwegian, or a mixture of the two accepted.

Grades available: As specified by regulations.

Problem 1 (20 %)

(8 %) (a) Consider the class of explicit, two-stage Runge-Kutta methods. These have Butcher array

$$\begin{array}{c|cc} 0 & & \\ \alpha & \alpha & \\ \hline & b_1 & b_2 \end{array}.$$

Assume α a free parameter, $0 \leq \alpha \leq 1$, and determine $b_1(\alpha)$ and $b_2(\alpha)$ to satisfy the conditions for the method to be of second order. Use the following series expansion:

$$\mathbf{f}(\mathbf{y}_n + h\alpha\mathbf{k}_1, t_n + \alpha h) = \mathbf{f}(\mathbf{y}_n, t_n) + \alpha h \frac{\mathrm{d}\mathbf{f}}{\mathrm{d}t}(\mathbf{y}_n, t_n) + O(h^2)$$

where $\mathbf{k}_1 = \mathbf{f}(\mathbf{y}_n, t_n)$.

Solution: For this method, with

$$\begin{aligned} \mathbf{k}_1 &= \mathbf{f}(\mathbf{y}_n, t_n) \\ \mathbf{k}_2 &= \mathbf{f}(\mathbf{y}_n + h\alpha\mathbf{k}_1, t_n + \alpha h) \end{aligned}$$

we get

$$\mathbf{y}_{n+1} = \mathbf{y}_n + h\,(b_1 k_1 + b_2 k_2) = \mathbf{y}_n + (b_1 + b_2) h \mathbf{f}(\mathbf{y}_n, t_n) + b_2 \alpha h^2 \frac{\mathrm{d}\mathbf{f}}{\mathrm{d}t}(\mathbf{y}_n, t_n) + O(h^3).$$

Comparing with (14.23), we see that the method is of second order if (see p. 527 book)

$$\begin{aligned} b_1 + b_2 &= 1 \\ b_2 \alpha &= \frac{1}{2} \end{aligned}$$

giving

$$\begin{aligned} b_1(\alpha) &= 1 - \frac{1}{2\alpha} = \frac{2\alpha - 1}{2\alpha} \\ b_2(\alpha) &= \frac{1}{2\alpha}. \end{aligned}$$

For $\alpha = 1/2$, we get the modified Euler method, while for $\alpha = 1$ we get the improved Euler method.

(6 %) (b) Determine the stability function of these methods (with $b_1(\alpha)$ and $b_2(\alpha)$ chosen as in (a)). Can you use α to maximize the stability region?

> **Solution:** Note that this can be solved independently of (a).
> Short answer: An ERK where $\sigma = p \leq 4$ (in this case, $\sigma = p = 2$) has stability function
>
> $$R(h\lambda) = 1 + h\lambda + \frac{(h\lambda)^2}{2} + ... + \frac{(h\lambda)^p}{p!} \overset{p=2}{=} 1 + h\lambda + \frac{(h\lambda)^2}{2}$$
>
> See p. 533 in book.
> Longer answer: Calculate explicitly the stability function and see that it is not dependent on α.
> We cannot use α to influence the stability region as long as we require the method to be of order 2.

(6 %) (c) Given the following ordinary differential equation:

$$\dot{y}_1 = y_1(y_1 - 2) + e^{y_2}$$
$$\dot{y}_2 = 50y_2(y_2 - 2)$$

Determine the maximum stepsize h for which this Runge-Kutta method is (linearly) stable in a neighborhood of the origin.

> **Solution:** The Jacobian of the system is
>
> $$A = \begin{pmatrix} -2 & 1 \\ 0 & -100 \end{pmatrix}$$
>
> with eigenvalues $\lambda_1 = -2$ and $\lambda_2 = -100$. Since λ_2 is the fastest eigenvalue, this is the interesting one. Inserting it into $R(h\lambda)$, we get
>
> $$R(-100h) = 1 - 100h + 5000h^2.$$
>
> The absolute value $|R(h\lambda_2)|$ is smaller than 1 if $h < 1/50$.

Problem 2 (20 %)

In this problem, and the next, we will develop a model for all degrees of freedom for a quadrotor, modeling the quadrotor as a rigid body. See Figure 1 for definition of coordinate systems, and a "free body diagram" with forces and moments acting on the quadrotor.

(8 %) (a) To specify the orientation of the quadrotor, the Z-X-Y Euler angles are sometimes used. These are specified by first a rotation α about the (inertial) z-axis, then β about the intermediate (rotated) x-axis, and finally γ about the body y-axis. Write up an expression for the rotation matrix $\mathbf{R}_b^i = \mathbf{R}_b^i(\boldsymbol{\phi})$ as a function of the Euler angles $\boldsymbol{\phi} = (\alpha, \beta, \gamma)^\mathsf{T}$.

> **Solution:** The rotation matrix is found by writing up the simple rotations in the order they

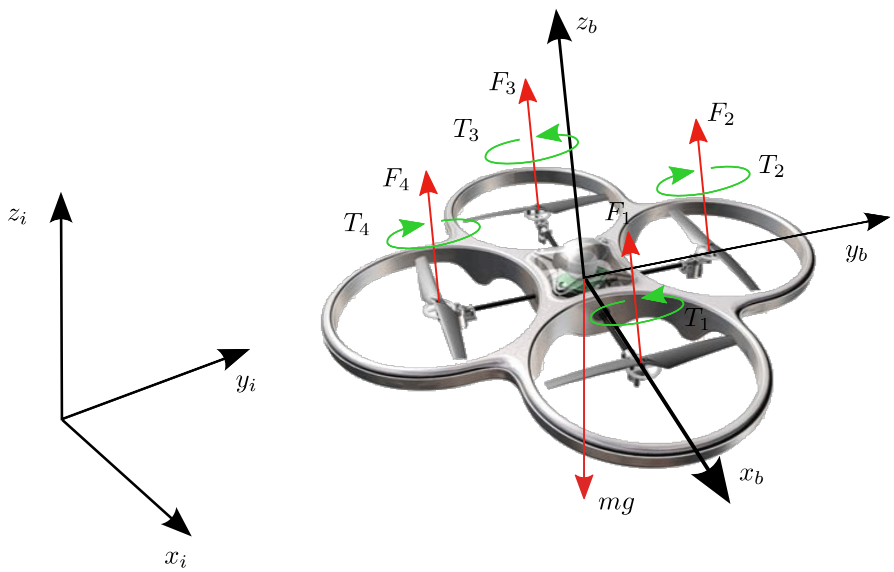

Figure 1: Coordinate systems and forces/moments.

appear:

$$
\begin{aligned}
\mathbf{R}_b^i &= \mathbf{R}_{z,\alpha}\mathbf{R}_{x,\beta}\mathbf{R}_{y,\gamma} \\
&= \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\beta & -\sin\beta \\ 0 & \sin\beta & \cos\beta \end{pmatrix} \begin{pmatrix} \cos\gamma & 0 & \sin\gamma \\ 0 & 1 & 0 \\ -\sin\gamma & 0 & \cos\gamma \end{pmatrix} \\
&= \begin{pmatrix} \cos\alpha\cos\gamma - \sin\alpha\sin\beta\sin\gamma & -\cos\beta\sin\alpha & \cos\alpha\sin\gamma + \cos\gamma\sin\alpha\sin\beta \\ \cos\gamma\sin\alpha + \cos\alpha\sin\beta\sin\gamma & \cos\alpha\cos\beta & \sin\alpha\sin\gamma - \cos\alpha\cos\gamma\sin\beta \\ -\cos\beta\sin\gamma & \sin\beta & \cos\beta\cos\gamma \end{pmatrix}
\end{aligned}
$$

These Euler angles are used for instance in Vijay Kumar's lab at University of Pennsylvania, and Raffaello D'Andrea's lab at ETH. Other groups use other conventions.

(12 %) (b) Find the kinematic differential equations for this choice of Euler angles. Assume that the angular velocity is given in body-frame. (It is not necessary to perform a matrix inversion for full score.)

Solution: The answer depends on whether the angular velocity is given in the inertial or body system. The latter is more natural, and is assumed in this problem.

The total angular velocity is the sum of the angular velocities of each rotation (6.269), but we need to transform the angular velocities to a common coordinate system when summing. In this case this common system is the body system:

$$\begin{aligned}\boldsymbol{\omega}_{ib}^{b} &= \mathbf{R}_{y,-\gamma}\mathbf{R}_{x,-\beta}\begin{pmatrix}0\\0\\\dot{\alpha}\end{pmatrix} + \mathbf{R}_{y,-\gamma}\begin{pmatrix}\dot{\beta}\\0\\0\end{pmatrix} + \begin{pmatrix}0\\\dot{\gamma}\\0\end{pmatrix}\\
&= \begin{pmatrix}-\sin\gamma\cos\beta\dot{\alpha} + \cos\gamma\dot{\beta}\\ \sin\beta\dot{\alpha} + \dot{\gamma}\\ \cos\gamma\cos\beta\dot{\alpha} + \sin\gamma\dot{\beta}\end{pmatrix}\\
&= \begin{pmatrix}-\sin\gamma\cos\beta & \cos\gamma & 0\\ \sin\beta & 0 & 1\\ \cos\gamma\cos\beta & \sin\gamma & 0\end{pmatrix}\begin{pmatrix}\dot{\alpha}\\\dot{\beta}\\\dot{\gamma}\end{pmatrix}\\
&= \mathbf{E}_b(\phi)\dot{\boldsymbol{\phi}}\end{aligned}$$

where $\dot{\boldsymbol{\phi}} = (\dot{\alpha}, \dot{\beta}, \dot{\gamma})^{\mathsf{T}}$ and

$$\mathbf{E}_b(\boldsymbol{\phi}) = \begin{pmatrix}-\sin\gamma\cos\beta & \cos\gamma & 0\\ \sin\beta & 0 & 1\\ \cos\gamma\cos\beta & \sin\gamma & 0\end{pmatrix}.$$

The kinematic differential equations are then

$$\dot{\boldsymbol{\phi}} = \mathbf{E}_b^{-1}(\boldsymbol{\phi})\boldsymbol{\omega}_{ib}^{b}.$$

Compare (6.316) for the choice of Euler angles used in the book (roll-pitch-yaw Euler angles). Not asked for: The inverse of $\mathbf{E}_b(\boldsymbol{\phi})$ is

$$\mathbf{E}_b^{-1}(\boldsymbol{\phi}) = \frac{1}{\cos\beta}\begin{pmatrix}-\sin\gamma & 0 & \cos\gamma\\ \cos\gamma\cos\beta & 0 & \sin\gamma\cos\beta\\ \sin\beta\sin\gamma & \cos\beta & -\cos\gamma\sin\beta\end{pmatrix}$$

and we see that we have a singularity for $\beta = \pi/2 + k\pi$, $k = 0, \pm 1, \pm 2, \ldots$.

Problem 3 (22 %)

In this problem, we will continue to develop the complete dynamic model of the quadrotor by modeling the kinetics. The forces and moments acting on the quadrotor are illustrated in Figure 1. The body system has origin in the center of mass, and the quadrotor has mass m and an inertia matrix $\mathbf{M}_{b/c}^{b}$. Note that the moments T_i due to rotation of the rotors give moments acting about the z_b-axis, and that the rotor forces F_i will give cause to moments about the x_b and y_b axis, with "arm" (distance from center of mass to rotor) L for all rotors. Note also that T_i has a "sign" defined in the figure, due to the default direction of rotation of the rotors.

(4 %) (a) Why is it natural to use the Newton-Euler equations of motions as starting point, rather than the Lagrange equations of motion?

Solution: We will model the quadcopter in all ("six") degrees of freedom, therefore there are no "forces of constraints" to eliminate.

(6 %) (b) Write up expressions for the force and torque vectors acting on the center of mass, $\mathbf{F}_{bc}^{b}$ and $\mathbf{T}_{bc}^{b}$, decomposed in the body system, as function of the forces and torques defined in Figure 1.

Solution:

$$\mathbf{F}^b_{bc} = \mathbf{R}^b_i(\boldsymbol{\phi}) \begin{pmatrix} 0 \\ 0 \\ -mg \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \sum_{i=1}^4 F_i \end{pmatrix}, \quad \mathbf{T}^b_{bc} = \begin{pmatrix} L(F_2 - F_4) \\ L(F_3 - F_1) \\ T_1 - T_2 + T_3 - T_4 \end{pmatrix}$$

(The very observant will have noticed that the sign of T_i in the figure is wrong when you consider the propeller configuration used on this particular quadcopter.)

(12 %) (c) What are the equations of motion of the quadrotor, on vector form? The components of vectors equations in the answer should amount to 12 first-order differential equations, including the answer from Problem 2(b).

Solution: There are (at least) two different correct answers here, depending on whether the velocity is expressed in body-fixed or inertial coordinates (and whether angular velocity is expressed in body-fixed or inertial coordinates, see Problem 2(b)).

First, the force balance. With velocity in inertial coordinates (which perhaps is simplest and most natural in this case), we can write down

$$\begin{aligned} m\dot{\mathbf{v}}^i_c &= \begin{pmatrix} 0 \\ 0 \\ -mg \end{pmatrix} + \mathbf{R}^i_b(\boldsymbol{\phi}) \begin{pmatrix} 0 \\ 0 \\ \sum_{i=1}^4 F_i \end{pmatrix} \\ \dot{\mathbf{r}}^i_c &= \mathbf{v}^i_c \end{aligned}$$

where the latter equation is a kinematic differential equation.

Alternatively, with velocity in body-fixed coordinates, we get (since $\dot{\mathbf{v}}^i_c = \mathbf{R}^i_b(\boldsymbol{\phi}) \left(\dot{\mathbf{v}}^b_c + (\boldsymbol{\omega}^b_{ib})^\times \mathbf{v}^b_c \right)$)

$$\begin{aligned} m\dot{\mathbf{v}}^b_c &= -m(\boldsymbol{\omega}^b_{ib})^\times \mathbf{v}^b_c + \mathbf{R}^b_i(\boldsymbol{\phi}) \begin{pmatrix} 0 \\ 0 \\ -mg \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ \sum_{i=1}^4 F_i \end{pmatrix} \\ \dot{\mathbf{r}}^i_c &= \mathbf{R}^i_b(\boldsymbol{\phi}) \mathbf{v}^b_c. \end{aligned}$$

Second, the torque balance (Euler's equation) is

$$\mathbf{M}^b_{b/c} \dot{\boldsymbol{\omega}}^b_{ib} = \mathbf{T}^b_{bc} - \boldsymbol{\omega}^b_{ib} \times \left(\mathbf{M}^b_{b/c} \boldsymbol{\omega}^b_{ib} \right).$$

Together with

$$\dot{\boldsymbol{\phi}} = \mathbf{E}_b^{-1}(\boldsymbol{\phi}) \boldsymbol{\omega}^b_{ib}$$

from Problem 2(b), this specifies 12 ODEs.

(Problems 2 and 3 are based on the article: Daniel Mellinger, Nathan Michael and Vijay Kumar, *Trajectory generation and control for precise aggressive maneuvers with quadrotors*, The International Journal of Robotics Research 31(5):664–674, 2012.)

Problem 4 (20 %)

A rod rotates in a vertical plane with constant angular speed ω about a horizontal axis, such that the angle of the rod with respect to the horizontal plane is $\theta = \omega t$. A bead of mass m slides without friction on the rod, under the influence of gravity g. See Figure 2.

(14 %) (a) Find the equation of motion for the bead. *Hint*: Use the distance r from the center of rotation to

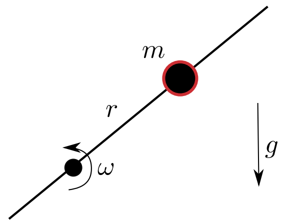

Figure 2: Bead on a rotating rod.

the bead as the generalized coordinate for the bead.

Solution: The kinetic and potential energy is

$$T = \frac{1}{2}m\left(\dot{r}^2 + r^2\omega^2\right), \quad U = mgr\cos\omega t$$

such that the Lagrangian is

$$\mathcal{L} = T - U = \frac{1}{2}m\left(\dot{r}^2 + r^2\omega^2\right) - mgr\cos\omega t.$$

Note that depending on how you assume the the rod is rotated at time $t = 0$ (the problem does not specify this), the potential energy could just as well be $mgr\sin\omega t$. Here I have assumed that the rod is vertical at time $t = 0$.

The equation of motion is (there are no generalized forces acting on the bead, in the direction of the generalized coordinate)

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial\mathcal{L}}{\partial\dot{r}} - \frac{\partial\mathcal{L}}{\partial r} = 0.$$

This gives

$$m\ddot{r} - mr\omega^2 + mg\cos\omega t = m\left(\ddot{r} - r\omega^2 + g\cos\omega t\right) = 0,$$

that is,

$$\ddot{r} - r\omega^2 + g\cos\omega t = 0.$$

(6 %) (b) Write pseudocode for integrating this equation of motion one timestep using Euler's method (explicit Euler) with step-length h. The pseudocode should include the solution to (a) (or, if you did not do (a), a qualified guess of the solution of (a)).

Solution: Write first the system on standard (first-order) form ($y_1 = r$, $y_2 = \dot{r}$):

$$\begin{aligned}\dot{y}_1 &= y_2\\ \dot{y}_2 &= \omega^2 y_1 + g\cos\omega t.\end{aligned}$$

Then we get

1. $y_1(t_{k+1}) = y_1(t_k) + hy_2(t_k)$
2. $y_2(t_{k+1}) = y_2(t_k) + h\left(\omega^2 y_1(t_k) + g\cos\omega t_k\right)$
3. $t_{k+1} = t_k + h$

The two first lines are enough in an answer, as long as it is clear that the time-dependence is taken care of.

First-order system: 2p, correct Euler: 2p, time dependence: 1p

Problem 5 (18 %)
Consider the closed tank in Figure 3, of total height L. An incompressible liquid of density ρ flows into the tank through the valve on the left, and leaves the tank through the valve on the right. The flow through the valves are given by the valve equation (assuming turbulent flow through the valve). The liquid in the tank has a liquid level h, and the total liquid volume is $V = Ah$ (A is the tank cross-sectional area). The volume V_g above the liquid varies as h varies, and is filled with a gas, for which the ideal gas law applies:

$$p_g V_g = m_g \bar{R} T.$$

We assume that m_g (the mass of gas in the volume), $\bar{R}$ (the specific gas constant) and T (the gas temperature) are known and constant.

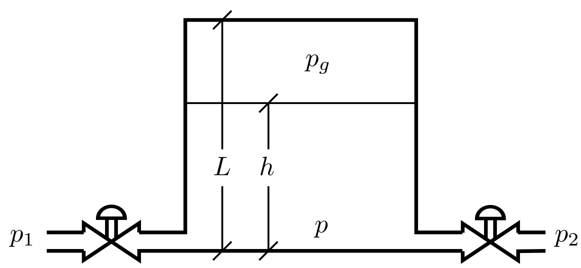

Figure 3: Closed tank with in- and outflow.

(12 %) (a) Set up a differential equation for the liquid level h.

Solution: The pressure in the bottom of the tank is

$$p = p_g + \rho g h = \frac{m\bar{R}T}{A(L-h)} + \rho g h.$$

The flows into and out of the tank are (see eq. (4.1) in the book)

$$q_1 = C_{d1} A_1 \sqrt{\frac{2}{\rho}(p_1 - p)}, \quad q_2 = C_{d2} A_2 \sqrt{\frac{2}{\rho}(p - p_2)}.$$

The mass balance for the liquid gives

$$\rho A \dot{h} = \rho q_1 - \rho q_2,$$

after inserting the above we get

$$\begin{aligned}
\dot{h} &= \frac{C_{d1}A_1}{A}\sqrt{\frac{2}{\rho}(p_1 - p)} - \frac{C_{d2}A_2}{A}\sqrt{\frac{2}{\rho}(p - p_2)} \\
&= \frac{C_{d1}A_1}{A}\sqrt{\frac{2}{\rho}\left(p_1 - \frac{m\bar{R}T}{A(L-h)} - \rho g h\right)} - \frac{C_{d2}A_2}{A}\sqrt{\frac{2}{\rho}\left(\frac{m\bar{R}T}{A(L-h)} + \rho g h - p_2\right)}
\end{aligned}$$

(6 %) (b) Assume you want to simulate this system with an explicit Runge-Kutta method with automatic adjustment of step-length. When will this give trouble?

Solution: The possible sources of troubles (for this system) are when the eigenvalue of the system approaches infinity. We can linearize to find an expression for the eigenvalue to find when this happens, but in this case we can also deduce directly that the problems occur when

$$\begin{aligned}\frac{m\bar{R}T}{A\,(L-h)} + \rho g h &\rightarrow p_1,\\ \frac{m\bar{R}T}{A\,(L-h)} + \rho g h &\rightarrow p_2,\\ h &\rightarrow L\end{aligned}$$

(One might argue that before the latter happens, the solver is likely to stop at the first problem, unless p_1 is very large.)