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For questions during the exam:
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Exam in TTK4130 Modeling and Simulation

Wednesday, May 20 2015
09:00 – 13:00

Permitted aids (code A): All written and handwritten examination support materials are permitted.

Answers in English, Norwegian, or a mixture of the two accepted.

Grades available: As specified by regulations.

Problem 1 (12 %)

(6 %) (a) Find the stability function for the Runge-Kutta method given by the Butcher array below. The answer should contain calculations.

$$\begin{array}{c|ccc} 0 & & & \\ 1/3 & 1/3 & & \\ 2/3 & 0 & 2/3 & \\ \hline & 1/4 & 0 & 3/4 \end{array}$$

(4 %) (b) A dynamic system has an oscillatory mode with eigenvalues $\lambda = \pm 1j$ (that is, eigenvalues on the imaginary axis). Is the method in (a), with steplength $h = 1$, stable for this mode? Calculations required for full score.

(2 %) (c) Is the method A-stable? L-stable?

Problem 2 (28 %)

(8 %) (a) Consider the second-order system $\ddot{y} = -\omega_0^2 y$. For what combination of ω_0 and steplength h will the simulation be stable, if you use i) explicit Euler, ii) modified Euler, iii) Heun's method (ERK3, third order explicit Runge-Kutta) and iv) implicit Euler?

Figure 1 might be of use.

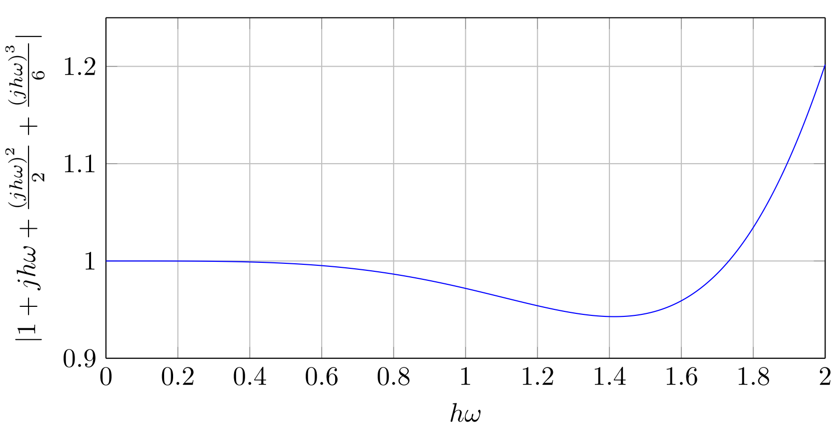

Figure 1: Helpful figure?

For some second-order systems (e.g. mechanical systems) a method called "semi-implicit Euler" is popular. This method is not mentioned in the book. For a second-order system on the form

$$\dot{y}_1 = y_2 \tag{1a}$$

$$\dot{y}_2 = f(y_1, y_2, t) \tag{1b}$$

this method can can be written

$$
\begin{aligned}
y_{1,n+1} &= y_{1,n} + h y_{2,n+1} \\
y_{2,n+1} &= y_{2,n} + h f(y_{1,n}, y_{2,n}, t_n)
\end{aligned}
$$

Like explicit and implicit Euler, one can show that this method is accurate of first order (you are not asked to show this).

(4 %) (b) Why do you think this method is called "semi-implicit"? How complex is it to implement this method (for systems like (1)), compared with explicit Euler and implicit Euler?

(10 %) (c) If we apply this method to the system in (a), for what combination of h and ω_0 will it be stable? Hint: You cannot apply the normal test-system $\dot{y} = \lambda y$ to this method. Instead, use the system in (a) as test system and analyze eigenvalues of the resulting discrete-time system. The following might become useful:

$$
\begin{pmatrix} 1 & -h \\ 0 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} 1 & h \\ 0 & 1 \end{pmatrix}
$$

$$
\text{for } g_{1,2}(s) = 1 - \frac{s^2}{2} \pm \frac{\sqrt{s^2\left(s^2-4\right)}}{2} \text{ we have } \begin{cases} |g_{1,2}(s)| = 1 & |s| \leq 2 \\ |g_1(s)| > 1 \text{ or } |g_2(s)| > 1 & |s| > 2 \end{cases}
$$

(6 %) (d) Based on the results in (c), why do you think this method is popular for these systems, compared to explicit Euler? (If you did not manage to solve (c), assume that there exist reasonable combinations of h and ω_0 that makes the method stable.)

Problem 3 (40 %)

In this problem, we will study how to develop a model of the Quanser "3-DOF Helicopter", see Figure 2, using Lagrangian mechanics. This helicopter is popular as a lab-setup to illustrate advanced control concepts to engineering students. We will assume that the helicopter consists of three point masses, one at each rotor and one counterweight, and that the rods connecting these points are massless. The helicopter can move in three degrees of freedom about a base point; these degrees of freedom are usually parametrized with a travel angle, an elevation angle, and a pitch angle. These angles are illustrated in Figure 3, along with lengths. We disregard friction, and we ignore the angle of the counterweight we can see in Figure 2 (not present in Figure 3).

The two rotors (front rotor and back rotor) give a lift force of magnitude F_f and F_b, respectively. If we assume a coordinate system fixed in each rotor with z-axis pointing in the direction of these forces, these forces can be written

$$
\mathbf{f}_f^f = \begin{pmatrix} 0 \\ 0 \\ F_f \end{pmatrix} \quad \text{and} \quad \mathbf{f}_b^b = \begin{pmatrix} 0 \\ 0 \\ F_b \end{pmatrix},
$$

where superscript f and b denote the coordinate system fixed in the front and back rotor, respectively.

(4 %) (a) Why do we prefer Lagrangian mechanics over using the Newton-Euler equations of motion for this system? What should we choose as generalized coordinates?

(10 %) (b) The positions of the front motor, back motor and counterweight in the inertial system, as functions

Figure 2: Quanser 3-DOF Helicopter

of the generalized coordinates, are given by the following equations:

$$\text{Front motor :} \qquad \mathbf{r}_f^i = \begin{pmatrix} x_f \\ y_f \\ z_f \end{pmatrix} = \begin{pmatrix} L_h \cos\lambda\cos p - L_h \sin\lambda \sin\epsilon \sin p - L_a \sin\lambda\cos\epsilon \\ L_h \sin\lambda \cos p + L_h \cos\lambda \sin\epsilon \sin p + L_a \cos\lambda \cos\epsilon \\ -L_h \cos\epsilon \sin p + L_a \sin\epsilon \end{pmatrix}$$

$$\text{Back motor :} \qquad \mathbf{r}_b^i = \begin{pmatrix} x_b \\ y_b \\ z_b \end{pmatrix} = \begin{pmatrix} -L_h \cos\lambda\cos p + L_h \sin\lambda \sin\epsilon \sin p - L_a \sin\lambda\cos\epsilon \\ L_h \sin\lambda \cos p - L_h \cos\lambda \sin\epsilon \sin p + L_a \cos\lambda \cos\epsilon \\ L_h \cos\epsilon \sin p + L_a \sin\epsilon \end{pmatrix}$$

$$\text{Counterweight :} \qquad \mathbf{r}_c^i = \begin{pmatrix} x_c \\ y_c \\ z_c \end{pmatrix} = \begin{pmatrix} L_w \sin\lambda\cos\epsilon \\ -L_w \cos\lambda\cos\epsilon \\ -L_w \sin\epsilon \end{pmatrix}$$

Explain how the equations for positions for the front rotor can be derived by defining five (or fewer) homogenous transformation matrices (for example rotation, rotation, translation, rotation, translation). You do not have to do the matrix multiplications for full score (if you want to check your answer, you may want to anyway), but you should write down each of the homogenous transformation matrices.

(6 %) (c) Find an expression for the kinetic energy for the counterweight (mass m_c) as a function of the generalized coordinates (and their derivatives), and explain how you would find the total kinetic energy (you do not have to find the expression for the total kinetic energy, only explain the procedure). Assume the mass for front and back rotor is m_f and m_b, respectively.

(4 %) (d) Find the potential energy for the system, as a function of the generalized coordinates.

(8 %) (e) The generalized forces for travel λ and elevation ϵ are

$$\tau_\lambda = (F_b - F_f)\,(L_h \sin\epsilon - L_a \cos\epsilon \sin p)$$
$$\tau_\epsilon = (F_b + F_f) L_a \cos p$$

Set up equations on coordinate form, involving rotation matrices, to calculate these from $\mathbf{f}_f^f$ and $\mathbf{f}_b^b$. You do not have to (you should not) do the actual calculations/differentiations/matrix multiplications.

(4 %) (f) What is τ_p, the generalized force for p? (You can rely on "physical insight" for finding τ_p, you do not have to use the procedure in the above question.)

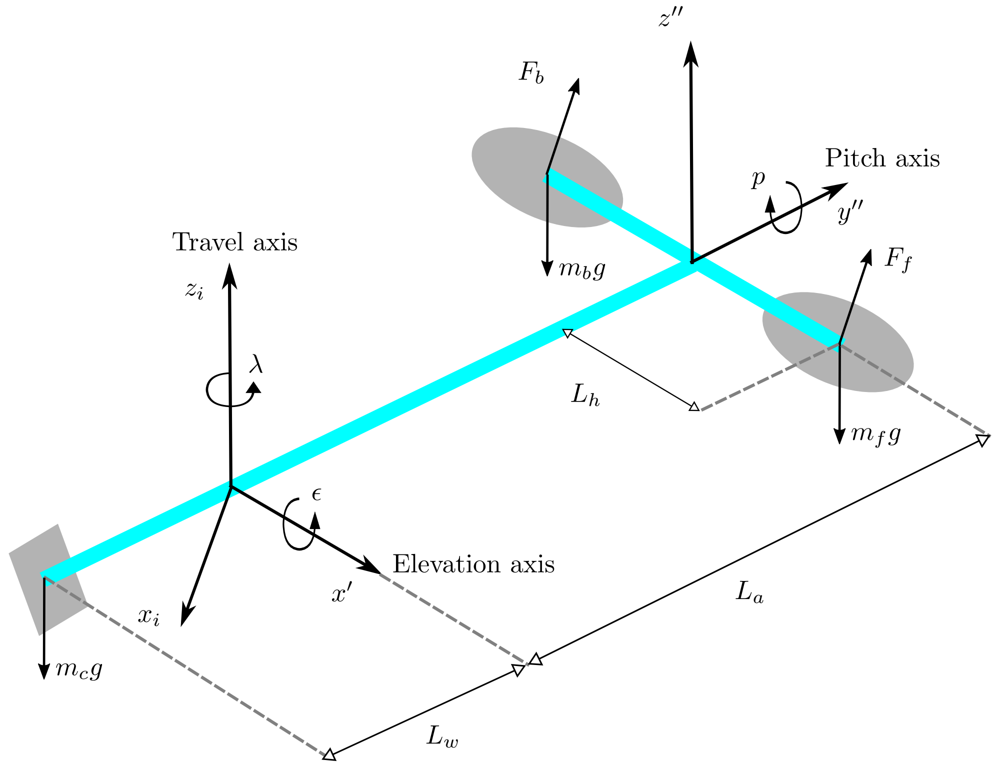

Figure 3: "Free body diagram" of 3-DOF helicopter. Note that angles are defined with positive direction according to right-hand rule about axis shown. The lift forces from the front and back rotors are denoted F_f and F_b, respectively.

(4 %) (g) Explain how you can find the equations of motion based on the results above. You should not find them, but explain the procedure.

Problem 4 (20 %)

In this problem, we consider a stirred tank that cools an inlet stream, see Figure 4. The tank is cooled by a "jacket" that contains a fluid of (presumably) lower temperature than the tank. The inlet stream to the tank has density ρ, temperature T_1, and massflowrate w_1. The outflow from the tank is

$$w_2 = Cu\sqrt{h},$$

where C is a constant and u is the valve opening. The liquid level is h. You can assume that the outflow is controlled such that the level does not exceed the height of the jacket.

The inlet and outlet massflowrates for the jacket is matched such that the jacket is always filled with fluid ($w_3 = w_4$). The cooling fluid has density ρ_c, and the inlet stream to the jacket has temperature T_3. Since the tank is stirred, we assume homogenous conditions, that is, the temperature T is the same everywhere in the tank. Similarly, we assume that the temperature T_c is the same everywhere in the jacket.

The cross-sectional area of the tank is A. The volume of the jacket is V_c.

The heat transfer from the tank to the jacket is

$$Q = Gh(T - T_c),$$

where h is the height of the liquid in the tank, and G a (constant) heat transfer coefficient. We assume that the jacket (and tank) is well insulated from the surroundings, meaning there are no other heat losses.

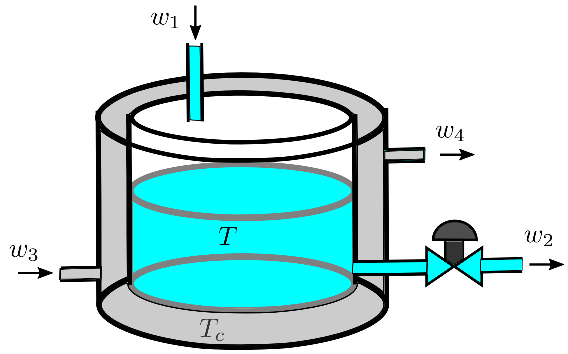

Figure 4: Tank with cooling jacket.

We assume both fluids incompressible, meaning that specific internal energy and enthalpy both can be assumed equal and proportional to temperature, with constant of proportionality being c_p and c_{pc} for the two fluids, respectively.

(20 %) (a) Set up differential equations for the temperatures T in the tank and T_c in the jacket, and the level h in the tank.