Problem 1 Let $u(x, y) = x + e^x \cos y$.

a) Show that u is harmonic.

b) Find a function $v(x, y)$ such that $f(z) = f(x + iy) = u(x, y) + iv(x, y)$ is analytic.

Problem 2 Show that $f(z) = z^5 + 4z - 6$ has five zeros in the annulus $A = \{z \in \mathbb{C} : 1 < |z| < 2\}$.

Problem 3 Use the residue theorem to compute

$$\int_0^\infty \frac{\cos x}{(x^2 + 1)^2} dx.$$

Show all estimates.

Problem 4

a) Prove that if z lies on the circle $|z - r| = r > 0$ then

$$\operatorname{Re}\left(\frac{1}{z}\right) = \frac{1}{2r}$$

for $z \neq 0$.

b) Find the image of $\Omega = \{z \in \mathbb{C} : \ |z| < 1, \operatorname{Im}(z) > 0\}$ under the mapping

$$f(z) = \frac{1 - z}{1 + z} = \frac{2}{z + 1} - 1.$$

c) Find a conformal from $G = \{z \in \mathbb{C} : \operatorname{Re}(z) < 0, 0 < \operatorname{Im}(z) < 1\}$ to the upper half plane $\mathcal{H}^+ = \{z \in \mathbb{C} : \operatorname{Im}(z) > 0\}$.

Problem 5 Let $f(z)$ be an entire function such that $|f(z)| \leqslant K|e^z|$ for some constant $K > 0$. Prove that $f(z) = ce^z$ for some constant c.

Problem 6 Let $f(z) = 1/(z^n - 1)$ with n a positive integer.

a) For a given $r > 0$, find all minima of $|f(z)|$ on the closed disk $\overline{D_r(0)} = \{z \in \mathbb{C} : |z| \leqslant r\}$.

b) Consider the line segment

$$\gamma_\theta = \{re^{i\theta} \in \mathbb{C} : r \in [\rho_1, \rho_2]\}$$

where $0 < \rho_1 < \rho_2 < 1$ and $\theta \in [0, 2\pi)$. Prove that

$$\left| \int_{\gamma_\theta} f(z) dz \right| \leqslant \frac{\rho_2 - \rho_1}{1 - \rho_2^n}$$

for all $\theta \in [0, 2\pi)$.

c) Show that there exists a choice of θ for which

$$\left| \int_{\gamma_\theta} f(z) dz \right| \leqslant \frac{\rho_2 - \rho_1}{\rho_1^n + 1}.$$