

Problem 1 How many roots does $10z^2 + z + \cos z = 0$ have in the unit disc

$$\Delta = \{z \in \mathbb{C} : |z| < 1\}?$$

Problem 2 Find

$$\int_0^\infty \frac{dx}{(2+x^2)(4+x^2)}.$$

Show all estimates.

Problem 3 Let $\Omega = \{z : |z| < 1 \text{ and } \operatorname{Im}(z) > 0\}$. Assume that f is analytic in Ω and continuous in $\Omega \cup (0, 1) = D \cup L$. Use reflection to show that if $f(x) = 1$ for all $x \in L$, then $f(z) = 1$ for all $z \in \Omega$.

Problem 4 Let

$$h(z) = \frac{\sin z - z}{z^3} + \sin\left(\frac{1}{z-i}\right) + \frac{e^{(z-1)} - 1}{(z-1)}.$$

a) Find the isolated singularities of h .

b) What kind of singularities are they?

c) Is it possible that $|h(z) - i| > 10^{-30}$ in $\Delta(i, \frac{1}{100}) = \{z : |z - i| < \frac{1}{100}\}$. Explain why.

Problem 5

a) Find a conformal map from $\Omega = \{z : |z| < 1 \text{ and } \operatorname{Im}(z) > 0\}$ to

$$\Delta = \{z \in \mathbb{C} : |z| < 1\}?$$

b) Use the same idea to find a conformal map from

$$\Omega = \{z : |z| < 1 \text{ and } \operatorname{Im}(z) > \frac{\sqrt{2}}{2}\}$$

to the unit disc Δ . (Hint: Observe that the angle between the unit circle and the line $\operatorname{Im}(z) = \frac{\sqrt{2}}{2}$ at the point $z_0 = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}$ is $\frac{\pi}{4}$.)

Problem 6

Let $u(x, y) = x^3 - 3xy^2$.

a) Show that u is harmonic.

b) Find v such that $f = u + iv$ is analytic.

X c) What is the value of the Poisson integral

$$\frac{1}{2\pi} \int_0^{2\pi} \frac{(1 - r^2)(\cos^3 \theta - 3 \cos \theta \sin^2 \theta)}{1 - 2r \cos(\theta - \phi) + r^2} d\theta$$

when $r = 1/2$ and $\phi = \pi/4$.