1)

a) We have

$$u_x = 1 + e^x \cos y,$$

$$u_{xx} = e^x \cos y,$$

$$u_y = -e^x \sin y,$$

$$u_{yy} = -e^x \cos y.$$

Hence

$$\nabla u = u_{xx} + u_{yy} = 0$$

& u is harmonic.

b) In general, a function $f(z) = f(x+iy) = u(x,y) + iv(x,y)$

is analytic $\Longleftrightarrow$ u, v have continuous first partial derivatives

& satisfy the equations

$$u_x = v_y$$

$$u_y = -v_x$$

$\therefore$ We require

$$\left.\begin{aligned} v_x &= -u_y = e^x \sin y \\ \& \quad v_y &= u_x = 1 + e^x \cos y \end{aligned}\right\} \Rightarrow \left\{\begin{aligned} v &= e^x \sin y + g(y) \\ v &= y + e^x \sin y + h(x) \end{aligned}\right.$$

for some g, h. From this we infer

$$v = y + e^x \sin y + C$$

and so

$$\begin{aligned} f(z) = f(x+iy) &= u + iv \\ &= x + e^x \cos y + i(y + e^x \sin y + C) \\ &= x + iy + e^x(\cos y + i \sin y) + c \\ &= z + e^z + c \end{aligned}$$

which is analytic.

2)

On the region $|z|<2$ we compare $f(z) = z^5 + 4z - 6$

with $g(z) = z^5$ & apply Rouché's thm.

If $|z|=2$ then

$$|g(z)| = |z|^5 = 32$$

&

$$|f(z) - g(z)| = |4z - 6| \leq 4|z| + 6 = 14 < 32.$$

Therefore, by Rouche's thm $f(z)$ has the same number of zeros in the region $|z|<2$ as $g(z)$, which is five.

If $|z|\leq 1$ then

$$
\begin{aligned}
|f(z)| &= |z^5+4z-6| \\
&= |6-4z-z^5| \\
&\geq 6-4|z|-|z|^5 \qquad (\text{since } |\alpha-\beta|\geq|\alpha|-|\beta|) \\
&\geq 6-4-1 \qquad (\text{since } |z|\leq 1) \\
&= 1 \\
&> 0.
\end{aligned}
$$

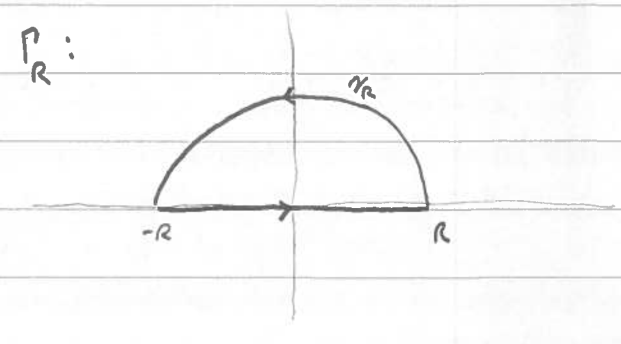
Hence, $f(z)$ has no zeros in $|z|\leq 1$ and so all five zeros are in $A=\{z\in\mathbb{C} : 1<|z|<2\}$.

3).

We have

$$\int_0^\infty \frac{\cos x}{(x^2+1)^2}\,dx = \frac{1}{2}\int_{-\infty}^{\infty}\frac{\cos x}{(x^2+1)^2}\,dx = \frac{1}{2}\operatorname{Re}\left(\int_{-\infty}^{\infty}\frac{e^{ix}}{(x^2+1)^2}\,dx\right).$$

We consider this last integral as the limit of the horizontal section of the contour



$$\Gamma_R = [-R,R]\cup\gamma_R,$$

where

$$\gamma_R = \{Re^{i\theta}\in\mathbb{C} : \theta\in[0,\pi]\}$$

Consider the integral

$$\int_{\Gamma_R} \frac{e^{iz}}{(z^2+1)^2}\, dz .$$

The integrand has double poles at $z = \pm i$. Only $z=i$ is contained in Γ_R. Therefore, by the residue thm

$$\int_{\Gamma_R} \frac{e^{iz}}{(z^2+1)^2}\, dz = 2\pi i \cdot \operatorname{res}\left(\frac{e^{iz}}{(z^2+1)^2}, z=i \right)$$

$$= 2\pi i \, \frac{d}{dz}\left((z-i)^2 \frac{e^{iz}}{(z^2+1)^2} \right)\Bigg|_{z=i}$$

$$= 2\pi i \, \frac{d}{dz}\left(\frac{e^{iz}}{(z+i)^2} \right)\Bigg|_{z=i}$$

$$= 2\pi i \left(\frac{ie^{iz}}{(z+i)^2} - 2\frac{e^{iz}}{(z+i)^3} \right)\Bigg|_{z=i}$$

$$= 2\pi i \left(\frac{ie^{-1}}{-4} - 2 \cdot \frac{e^{-1}}{8i^3} \right)$$

$$= 2\pi i \left(-\frac{ie^{-1}}{4} + \frac{e^{-1}}{4i} \right)$$

$$= \frac{\pi e^{-1}}{2} + \frac{\pi e^{-1}}{2}$$

$$= \frac{\pi}{e} .$$

Now, on γ_R we have

$$|e^{iz}| = |e^{ix-y}| = e^{-y} \leq 1$$

& $$\left|\frac{1}{(z^2+1)^2}\right| = \frac{1}{|z^2+1|^2} \leq \frac{1}{(|z|^2-1)^2} \qquad \text{(since } |\alpha+\beta| \geq |\alpha|-|\beta|\text{)}$$

$$= \frac{1}{(R^2-1)^2}$$

Hence,

$$\left|\int_{\gamma_R} \frac{e^{iz}}{(z^2+1)^2}\,dz\right| \leq (\text{length of } \gamma_R) \times \max_{z\in\gamma_R}\left|\frac{e^{iz}}{(z^2+1)^2}\right|$$

$$\leq \pi R \cdot \frac{1}{(R^2-1)^2}$$

$$\ll \frac{1}{R^3}$$

$$\to 0$$

as $R \to \infty$.

Putting everything together gives

$$\int_0^\infty \frac{\cos x}{(x^2+1)^2}\,dx = \frac{1}{2}\mathrm{Re}\left(\int_{-\infty}^\infty \frac{e^{ix}}{(x^2+1)^2}\,dx\right) = \frac{1}{2}\mathrm{Re}\left(\lim_{R\to\infty}\left[\left(\int_{\Gamma_R} - \int_{\gamma_R}\right)\frac{e^{iz}}{(z^2+1)^2}\,dz\right]\right)$$

$$= \frac{1}{2}\mathrm{Re}\left(\lim_{R\to\infty}\int_{\Gamma_R}\frac{e^{iz}}{(z^2+1)^2}\,dz\right) - 0$$

$$= \frac{1}{2}\mathrm{Re}\left(\frac{\pi}{e}\right)$$

$$= \frac{\pi}{2e}.$$

4

a) If $z = x+iy$ lies on $|z-r| = r$ then

$$(x-r)^2 + y^2 = r^2$$

$$\Rightarrow \quad x^2+y^2 = 2xr.$$

Therefore,

$$\mathrm{Re}\left(\frac{1}{z}\right) = \mathrm{Re}\left(\frac{\bar{z}}{|z|^2}\right) = \mathrm{Re}\left(\frac{x-iy}{x^2+y^2}\right)$$

$$= \frac{x}{x^2+y^2}$$

$$= \frac{x}{2xr}$$

$$= \frac{1}{2r}.$$

b) We see that $f(z)$ is the composition

$$z \mapsto z+1 \mapsto \frac{1}{z+1} \mapsto \frac{2}{z+1} \mapsto \frac{2}{z+1} - 1.$$

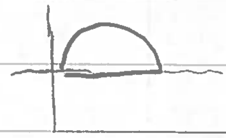
So, $\Omega = \{ |z|<1, \ \mathrm{Im}\, z > 0 \}$ is first mapped to

$$\Omega' = \{ |z-1|<1, \ \mathrm{Im}\, z>0 \}$$

via $z \mapsto z+1$.

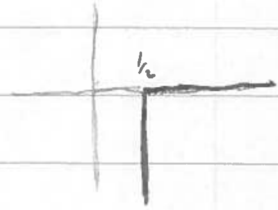
To see how inversion acts on Ω' we consider its boundary:

$$\{ |z-1|=1 , \ \operatorname{Im}(z) \geq 0 \} \cup [0,2] .$$

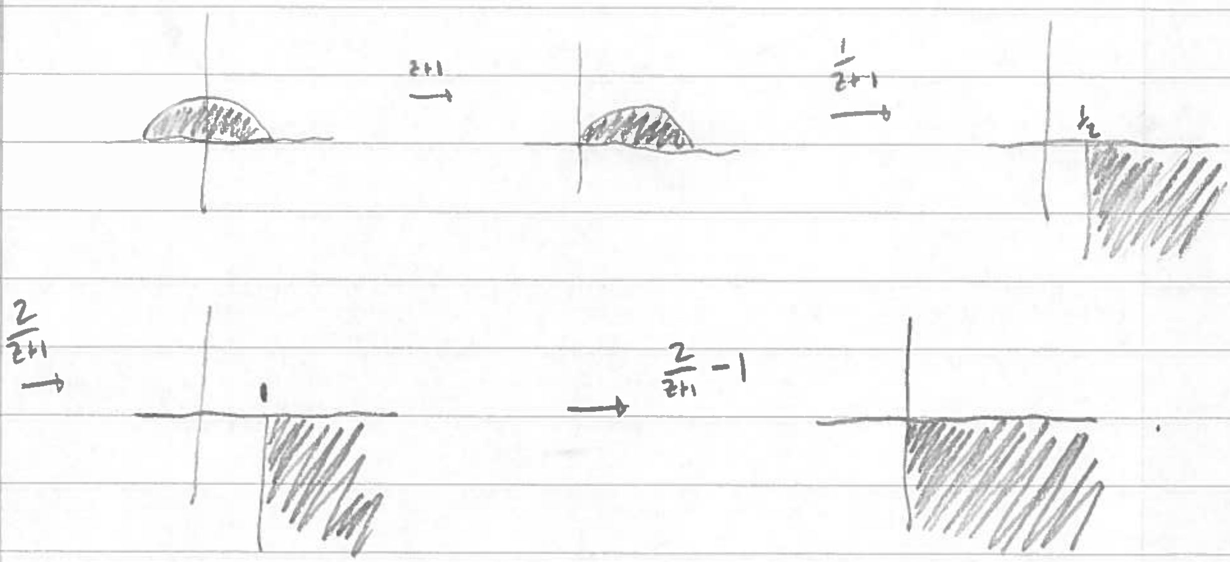


By part (a) this gets mapped to

$$\{ \operatorname{Re}(z) = \tfrac{1}{2} , \ \operatorname{Im}(z) \leq 0 \} \cup [\tfrac{1}{2}, \infty) .$$



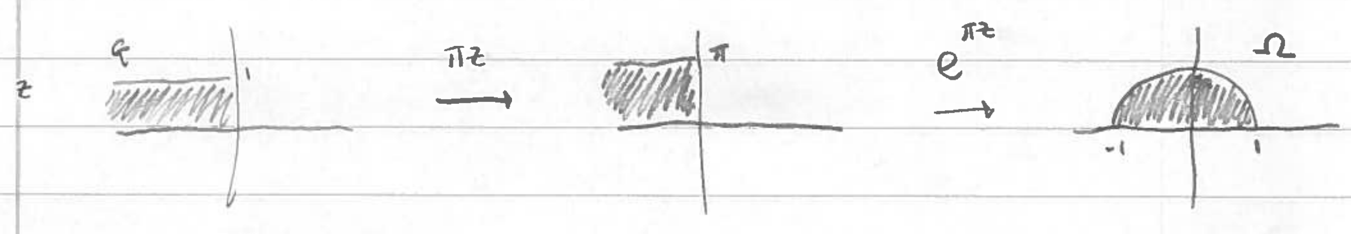
Thus, we have



The image of Ω under g is the fourth quadrant.

c) To map $G = \{ \operatorname{Re}(z) < 0,\ 0 < \operatorname{Im}(z) < 1 \}$ to $H^+ = \{ \operatorname{Im}(z) > 0 \}$

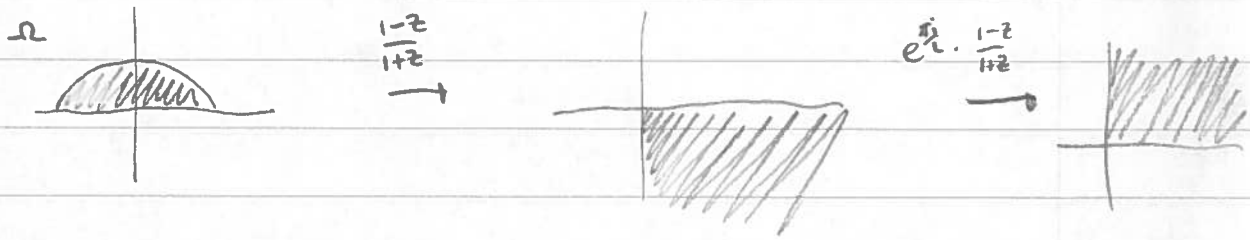
we first map G to Ω via the composition:

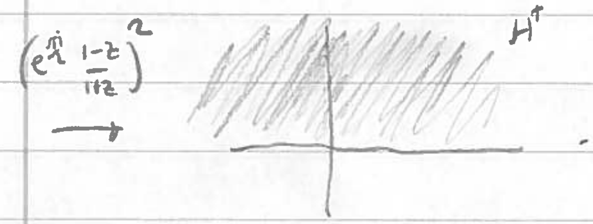


(Note $e^{\pi z} = e^{\pi x} \cdot e^{\pi i y} = re^{i\theta}$ with $r = e^{\pi x}$ & $\theta = \pi y$

& since $x < 0$, $0 < y < 1$ we have $r \in (0,1)$, $\theta \in (0, \pi)$)

We now map Ω to H^+ using part b) :





All in all, the map is given by

$$g(z) := -\left(\frac{1 - e^{\pi z}}{1 + e^{\pi z}} \right)^2 .$$

5) Since $f(z)$ is entire & $e^z \neq 0$ the function $g(z) = \frac{f(z)}{e^z}$

is entire. By assumption,

$$|g(z)| = \frac{|f(z)|}{|e^z|} \leq K$$

i.e. $g(z)$ is bounded. By Liouville's thm $g(z)$ is constant &

hence $\frac{f(z)}{e^z} = c \Rightarrow f(z) = ce^z$

6.)

a) The minima of $f(z) = \frac{1}{z^n - 1}$ occur at the maxima of

$g(z) = \frac{1}{f(z)} = z^n - 1$. By the maximum Modulus principle

this must occur on the boundary of $\overline{D_r(0)}$.

Thus, we can let $z = re^{i\theta}$, $\theta \in [0, 2\pi]$

and consider

$$|g(z)|^2 = |z^n - 1|^2 = |r^n e^{in\theta} - 1|^2 = (r^n e^{in\theta} - 1)(r^n e^{-in\theta} - 1)$$

$$= 1 + r^{2n} - 2r^n \cos n\theta .$$

The maxima of this function occur when $\cos n\theta = -1$

i.e. when $\theta = \frac{(2k-1)\pi}{n}$, $k = 1, 2, \ldots, n$.

Therefore, the minima of $|f(z)|$ on $\overline{D_r(0)}$ occur at the points $z = re^{i(2k-1)\pi/n}$, $k = 1, 2, \ldots, n$, where it attains the value

$$|f(z)| = \left| \frac{1}{-r^n - 1} \right| = \frac{1}{r^n + 1}.$$

b) By the estimation lemma,

$$\left| \int_{\gamma_\theta} f(z)\, dz \right| \leq (\text{length of } \gamma_\theta) \times \max_{z \in \gamma_\theta} |f(z)|$$

$$= (\rho_2 - \rho_1) \cdot \max_{r \in [\rho_1, \rho_2]} \left| \frac{1}{r^n e^{in\theta} - 1} \right|$$

$$\leq (\rho_2 - \rho_1) \max_{r \in [\rho_1, \rho_2]} \frac{1}{1 - |r^n e^{in\theta}|}, \quad (|\alpha - \beta| \geq |\beta| - |\alpha|)$$

$$= (\rho_2 - \rho_1) \cdot \max_{r \in [\rho_1, \rho_2]} \frac{1}{1 - r^n}$$

$$= \frac{\rho_2 - \rho_1}{1 - \rho_2^n}$$

c). Again, by the estimation lemma

$$\left| \int_{\gamma_\theta} f(z)\, dz \right| \leq (\rho_2 - \rho_1) \max_{z \in \gamma_\theta} |f(z)| .$$

We seek a choice of θ to minimize $\max_{z \in \gamma_\theta} |f(z)|$.

By part a) we know these minima occur on the lines with

$\theta = \frac{(2k-1)\pi}{n}$, $k = 1, 2, \ldots, n$.

With this choice of θ we have

$$\max_{z \in \gamma_\theta} |f(z)| = \max_{r \in [\rho_1, \rho_2]} \left| \frac{1}{r^n e^{in\theta} - 1} \right|$$

$$= \max_{r \in [\rho_1, \rho_2]} \left| \frac{1}{-r^n - 1} \right|$$

$$= \max_{r \in [\rho_1, \rho_2]} \frac{1}{r^n + 1}$$

$$= \frac{1}{\rho_1^n + 1} .$$