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For questions during the exam:
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Exam in TTK4130 Modeling and Simulation

Friday, May 27th 2011
09:00 – 13:00

Permitted aids (code A): All written and handwritten examination support materials are permitted.

Note: A Norwegian text is appended.

Answers in English, Norwegian, or a mixture of the two accepted.

Grades available: As specified by regulations.

Problem 1 (26 %)

The gyroscopic pendulum consists of a physical pendulum with a rotating symmetric disc at the end, spinning about an axis parallel to the axis of rotation of the pendulum. See Figure 1. The stiff rod has mass m_1, length ℓ_1 and moment of inertia I_1. The position of the rod's center of gravity is given by ℓ_{c1} (cf. figure). The disc has mass m_2 and moment of inertia I_2. The pendulum is attached to a fixed coordinate system (axis x and y).

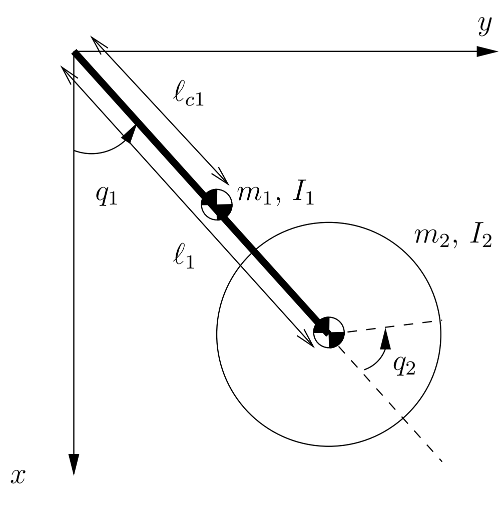

Figure 1: Gyroscopic pendulum

The rotating disc is actuated by a torque τ (which could be generated e.g. by a DC-motor). The gyroscopic pendulum is sometimes used as an experiment to illustrate nonlinear control theory.

We will develop the equations of motion for the gyroscopic pendulum.

(4 %) (a) Choose appropriate generalized coordinates for this system. The figure should give you some hints. What are the corresponding generalized forces?

> **Solution:** The angles q_1 and q_2 are appropriate generalized coordinates (parameterizations of degrees of freedom) (2p). There is no generalized force corresponding to q_1, while τ is the generalized force for q_2 (2p).

(6 %) (b) What is the angular velocity of the disc (that is, of a coordinate system fixed in the disc) in the earth-fixed coordinate system?

Solution: Let $\vec{k}$ be a vector that points out of the paper (z-axis). Then the angular velocity of the disc (of a coordinate system fixed in the disc, coordinate system 3) referred to the coordinate system that is fixed at the end of the rod (coordinate system 2) is

$$\vec{\omega}_{23} = \dot{q}_2 \vec{k}.$$

The angular velocity of this coordinate system referenced to the fixed coordinate system is

$$\vec{\omega}_{12} = \dot{q}_1 \vec{k}.$$

We know that the angular velocity of composite rotations is the sum of the angular velocities of the rotations (book p. 242), that is,

$$\vec{\omega}_{13} = (\dot{q}_1 + \dot{q}_2)\, \vec{k}.$$

(10 %) (c) Find the kinetic and potential energy for the system as functions of the generalized coordinates.

Solution: Position and velocity of rod's center of mass (y-axis pointing to the right, x-axis pointing downwards):

$$\mathbf{r}_{c1} = \begin{pmatrix} \ell_{c1} \cos q_1 \\ \ell_{c1} \sin q_1 \end{pmatrix}, \quad \mathbf{v}_{c1} = \begin{pmatrix} -\ell_{c1} \dot{q}_1 \sin q_1 \\ \ell_{c1} \dot{q}_1 \cos q_1 \end{pmatrix}$$

Position and velocity of disc's center of mass:

$$\mathbf{r}_{c2} = \begin{pmatrix} \ell_1 \cos q_1 \\ \ell_1 \sin q_1 \end{pmatrix}, \quad \mathbf{v}_{c2} = \begin{pmatrix} -\ell_1 \dot{q}_1 \sin q_1 \\ \ell_1 \dot{q}_1 \cos q_1 \end{pmatrix}$$

The kinetic energy for the rod:

$$\begin{aligned} T_1 &= \frac{1}{2} m_1 \mathbf{v}_{c1}^{\mathsf{T}} \mathbf{v}_{c1} + \frac{1}{2} I_1 \dot{q}_1^2 \\ &= \frac{1}{2} m_1 \ell_{c1}^2 \dot{q}_1^2 + \frac{1}{2} I_1 \dot{q}_1^2 \end{aligned}$$

The potential energy for the rod:

$$U_1 = -m_1 \ell_{c1} g \cos q_1.$$

The kinetic energy for the disc:

$$\begin{aligned} T_2 &= \frac{1}{2} m_2 \mathbf{v}_{c2}^{\mathsf{T}} \mathbf{v}_{c2} + \frac{1}{2} I_2 (\dot{q}_1 + \dot{q}_2)^2 \\ &= \frac{1}{2} m_2 \ell_1^2 \dot{q}_1^2 + \frac{1}{2} I_2 (\dot{q}_1 + \dot{q}_2)^2 \end{aligned}$$

The potential energy for the disc:

$$U_1 = -m_2 \ell_1 g \cos q_1.$$

The total kinetic energy is then $T = T_1 + T_2$, and potential energy $U = U_1 + U_2$.
Full score for correct energy (or correct except for angular velocity found in (b)). Miscalculations or wrong generalized coordinates give 6-8p, less for graver errors.

(6 %) (d) Derive the equations of motion for the system.

Solution: From the Lagrangian $L = T - U$, the equations of motion are

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_1}\right) - \frac{\partial L}{\partial q_1} = 0,$$
$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial L}{\partial \dot{q}_2}\right) - \frac{\partial L}{\partial q_2} = \tau.$$

This gives

$$\left(m_1\ell_{c1}^2 + m_2\ell_1^2 + I_1 + I_2\right)\ddot{q}_2 + I_2\ddot{q}_2 + m_1 g\ell_{c1}\sin q_1 + m_2 g\ell_1 \sin q_1 = 0,$$
$$I_2\ddot{q}_1 + I_2\ddot{q}_2 = \tau.$$

Full score for correct use of Lagrange equations of motion (even with wrong answer).

Problem 2 (28 %)

In this problem, we will consider four coordinate systems,

- coordinate system a with axes $\vec{a}_1$, $\vec{a}_2$, and $\vec{a}_3$,
- coordinate system b with axes $\vec{b}_1$, $\vec{b}_2$, and $\vec{b}_3$,
- coordinate system c with axes $\vec{c}_1$, $\vec{c}_2$, and $\vec{c}_3$,
- coordinate system d with axes $\vec{d}_1$, $\vec{d}_2$, and $\vec{d}_3$.

The rotation from a to b is described by a rotation α about $\vec{a}_3$, from b to c by a rotation β about $\vec{b}_2$, and from c to d by a rotation γ about $\vec{c}_1$. (Here, α, β, and γ are angles.)

(8 %) (a) Find the rotation matrix $\mathbf{R}_d^a$. The answer should contain the elements of this rotation matrix.

Solution: From

$$\mathbf{R}_b^a = \mathbf{R}_{z,\alpha} = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 1 \end{pmatrix},$$
$$\mathbf{R}_c^b = \mathbf{R}_{y,\beta} = \begin{pmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{pmatrix},$$
$$\mathbf{R}_d^c = \mathbf{R}_{x,\gamma} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\gamma & -\sin\gamma \\ 0 & \sin\gamma & \cos\gamma \end{pmatrix},$$

we get

$$\mathbf{R}_d^a = \mathbf{R}_b^a\mathbf{R}_c^b\mathbf{R}_d^c = \mathbf{R}_{z,\alpha}\mathbf{R}_{y,\beta}\mathbf{R}_{x,\gamma}$$
$$= \begin{pmatrix} \cos\alpha\cos\beta & -\sin\alpha\cos\gamma + \cos\alpha\sin\beta\sin\gamma & \sin\alpha\sin\gamma + \cos\alpha\sin\beta\cos\gamma \\ \sin\alpha\cos\beta & \cos\alpha\cos\gamma + \sin\alpha\sin\beta\sin\gamma & -\cos\alpha\sin\gamma + \sin\alpha\sin\beta\cos\gamma \\ -\sin\beta & \cos\beta\sin\gamma & \cos\beta\cos\gamma \end{pmatrix}.$$

6p for correct simple rotations, full score for full rotation matrix.

(8 %) (b) The angle/axis parameters $\vec{k}$ and θ correspond to $\mathbf{R}_d^a$. Show that

$$\mathbf{R}_d^a - (\mathbf{R}_d^a)^\mathsf{T} = 2\mathbf{k}^\times \sin\theta.$$

Solution: From

$$\mathbf{R}_d^a = \mathbf{R}_{\mathbf{k},\theta} = \cos\theta\mathbf{I} + \mathbf{k}^\times \sin\theta + \mathbf{k}\mathbf{k}^\mathsf{T}(1-\cos\theta)$$

we see that the first and last term are symmetrical, and hence

$$\begin{aligned}\mathbf{R}_d^a - \left(\mathbf{R}_d^a\right)^\mathsf{T} &= \mathbf{k}^\times \sin\theta - \left(\mathbf{k}^\times\right)^\mathsf{T} \sin\theta \\ &= 2\mathbf{k}^\times \sin\theta\end{aligned}$$

since $\left(\mathbf{k}^\times\right)^\mathsf{T} = -\mathbf{k}^\times$.

(4 %) (c) In which of the coordinate systems a, b, c, and d is the coordinate vector $\mathbf{k}$ specified?

Solution: $\mathbf{k}$ is given in (is the same in) a and d. (2p for each.)

(8 %) (d) Let $\alpha = \frac{\pi}{2}$, $\beta = 0$, and $\gamma = -\frac{\pi}{2}$. Make a sketch of coordinate system a and d, and find the parameters k and θ that correspond to $\mathbf{R}_d^a$ for these values.

Solution: We find that

$$\mathbf{R}_d^a = \begin{pmatrix} 0 & 0 & -1 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}.$$

For sketching, it may be helpful to note that

$$\mathbf{R}_d^a = \left(\mathbf{d}_1^a \mathbf{d}_2^a \mathbf{d}_3^a\right),$$

where $\mathbf{d}_i^a$ is the coordinate vector of $\vec{d}_i$ in a. See Figure 2.

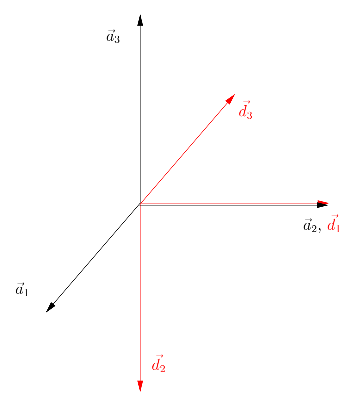

Figure 2: Coordinate systems

To find $\mathbf{k}$ and θ, we use the result from (b) (we could also have used e.g. Shepperd's method):

$$\mathbf{R}_d^a - \left(\mathbf{R}_d^a\right)^\mathsf{T} = 2\mathbf{k}^\times \sin\theta$$

$$\begin{pmatrix} 0 & -1 & -1 \\ 1 & 0 & 1 \\ 1 & -1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & -2k_3\sin\theta & 2k_2\sin\theta \\ 2k_3\sin\theta & 0 & -2k_1\sin\theta \\ -2k_2\sin\theta & 2k_1\sin\theta & 0 \end{pmatrix}$$

Toghether with the requirement that $\mathbf{k}$ is a unit vector, this gives four equations for four unknowns.

By comparing elements, we see that we must have $k_1 = k_2 = -k_3$. The demand that $\mathbf{k}$ is a unit vector, then gives that

$$\mathbf{k} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$$

(or with opposite sign). With this choice of $\mathbf{k}$, we next find that $\theta = -\pi/3$ ($-60°$) or $\theta = -2\pi/3$ ($-120°$). Of these, only the latter is correct, as can be verified by calculating the rotation matrix from angle axis parameters using e.g. (6.140) (but both answers give full score). 4p for correct scetch (2-3p for smaller errors), 4p for correct $\mathbf{k}$ (2-3p for sensible calculations).

Side-remark: This problem was badly designed in that the "obvious" answer for θ is wrong. On the other hand, this illustrates why the more general Shepperd's method is preferable to the procedure outlined in this problem.

Problem 3 (26 %)

Given the following Butcher array:

$$\begin{array}{c|cc} 0 & 0 & 0 \\ 1 & 1-\alpha & \alpha \\ \hline & 1-\alpha & \alpha \end{array}$$

where $\alpha \in [0, 1]$ is a parameter.

(6 %) (a) Write up the equations for this method, for a system $\dot{\mathbf{y}} = \mathbf{f}(\mathbf{y}, t)$. Is the method explicit or implicit? Why?

Solution:

$$\begin{aligned} \mathbf{k}_1 &= \mathbf{f}(\mathbf{y}_n, t_n) \\ \mathbf{k}_2 &= \mathbf{f}(\mathbf{y}_n + h\left[(1-\alpha)\mathbf{k}_1 + \alpha\mathbf{k}_2\right], t_n + h) \\ \mathbf{y}_{n+1} &= \mathbf{y}_n + h\left[(1-\alpha)\mathbf{k}_1 + \alpha\mathbf{k}_2\right] \end{aligned}$$

The method is implicit for all $\alpha > 0$ since the equation for $\mathbf{k}_2$ then cannot be solved directly (is implicit). The method is explicit for $\alpha = 0$. (3p+2p+1p)

(6 %) (b) Derive the stability function for this method as a function of $s = h\lambda$ (correct answer without calculations give 50% score).

Solution: For calculations, we can use formula 14.142, or insert $\theta = 1 - \alpha$ in (14.139) to get

$$R(s) = \frac{1 + s(1-\alpha)}{1 - s\alpha}.$$

Using correct formulas but ending up with wrong answer also gives 50% score.

(8 %) (c) For which α is the method A-stable? Substantiate your answer.

Solution: The method is A-stable if

$$|R(s)| = \left|\frac{1+s(1-\alpha)}{1-s\alpha}\right| \leq 1, \qquad \text{for all } \mathrm{Re}\,[s] \leq 0$$

Introduce $s = \sigma + j\omega$, then $\mathrm{Re}\,[s] \leq 0$ translates to $\sigma \leq 0$.

$$\begin{aligned}
\left|\frac{1+s(1-\alpha)}{1-s\alpha}\right| &\leq 1 \\
|1+s(1-\alpha)| &\leq |1-s\alpha| \\
|1+(\sigma+j\omega)(1-\alpha)| &\leq |1-(\sigma+j\omega)\alpha| \\
(1+\sigma(1-\alpha))^2 + \omega^2(1-\alpha)^2 &\leq (1-\sigma\alpha)^2 + \omega^2\alpha^2 \\
2\sigma + \sigma^2(1-2\alpha) + \omega^2(1-2\alpha) &\leq 0
\end{aligned}$$

For this to hold for $\sigma = 0$, we see from the last term on the left hand side that we must have $1 - 2\alpha \leq 0$. Next we see that if this condition holds, then the left hand side is negative for all $\sigma \leq 0$.

In other words, the method is A-stable for $\frac{1}{2} \leq \alpha \leq 1$.

(6 %) (d) For which α is the method L-stable? Substantiate your answer.

Solution: The method is L-stable if it is A-stable and $|R(j\omega)| \to 0$ as $\omega \to \infty$. We see that the denominator and nominator of $R(j\omega)$ have the same degree in ω for all $0 < \alpha < 1$, which means it cannot go to 0 as ω to ∞ for these values of α. For $\alpha = 0$ (Explicit Euler), the method is not A-stable, but for $\alpha = 1$ (Implicit Euler) the method is A-stable and have higher degree in the nominator than in the denominator, meaning that $|R(j\omega)| \to 0$ as $\omega \to \infty$.

That is, the method is L-stable only if $\alpha = 1$.

(A detailed calculation is of course also accepted.)

Problem 4 (20 %)

When ice forms on water, the "rate of freezing" is mainly a function of air temperature (but also other factors, such as water temperature, water salinity, wind, etc.). Assume that this rate of freezing (per unit volume) is known, and denote it $S_h = S_h(\mathbf{x}, t)$.

Let $h = h(\mathbf{x}, t)$ denote the thickness of ice, and assume that ice density is constant.

(4 %) (a) Explain *briefly* what the equation

$$\frac{\mathrm{D}}{\mathrm{D}t}\iiint_{V_m} h \mathrm{d}V = \iiint_{V_m} S_h \mathrm{d}V$$

express.

Solution: The integral of the ice thickness h in a material volume,

$$\iiint_{V_m} h \mathrm{d}V$$

is a measure of the amount of ice within that volume. The integral on the right hand side is a measure of rate of change of ice (thickness) within the volume.

Since we are looking at a material volume, transport in and out of the volume is not a concern.

Full score for answers understanding the "gists" of volumes and rates.

(6 %) (b) Set up a partial differential equation (on divergence form) for the dynamics of ice thickness. Assume that the velocity $\vec{v}$ of ice is known,

$$\mathbf{v} = \begin{pmatrix} u \\ v \\ w \end{pmatrix},$$

where the velocity in z-direction is zero, $w = 0$.
Hint: Reynolds theorem on divergence form is

$$\frac{\mathrm{D}}{\mathrm{D}t} \iiint_V \phi \mathrm{d}V = \iiint_V \frac{\partial \phi}{\partial t} + \vec{\nabla} \cdot [\phi \vec{v}] \,\mathrm{d}V$$

Solution: Using the hint, we get that

$$\frac{\mathrm{D}}{\mathrm{D}t} \iiint_V h \mathrm{d}V = \iiint_V \frac{\partial h}{\partial t} + \vec{\nabla} \cdot [h \vec{v}] \,\mathrm{d}V = \iiint_V S_h \mathrm{d}V$$

As the volume is arbitrary, the integrands must be equal, which gives us

$$\frac{\partial h}{\partial t} + \vec{\nabla} \cdot [h \vec{v}] = S_h$$

which can be written (with $w = 0$)

$$\frac{\partial h}{\partial t} + \frac{\partial hu}{\partial x} + \frac{\partial hv}{\partial y} = S_h.$$

(This is the continuity equation for ice.)

(4 %) (c) If the velocity was not known, where should we start if we were to set up a model for the velocity? The answer should be a single sentence, and not contain mathematics.

Solution: We should start by setting up the momentum (impulse) balance.

(6 %) (d) Outline (briefly, in words or simple mathematics) a way to simulate the model found in (b).

Solution: We could grid the area we are interested in, and use finite differences for the spatial derivatives. This gives a (large) ODE that could be solved (simulated) with standard ODE methods.
(The answer should contain grid, finite differences (or alternative methods) (4p) and explain how the equations are simulated in time (2p).)