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Problem 1

a)

We write the plant as

$$(s^3 + a_2 s^2 + a_1 s + a_0)y = (b_2 s^2 + b_1 s + b_0)u \tag{1}$$
$$s^3 y + [a_2, a_1, a_0]\alpha_2 y = [b_2, b_1, b_0]\alpha_2 u. \tag{2}$$

We then get

$$s^3 y = [b_2, b_1, b_0, a_2, a_1, a_0][\alpha_2 u, -\alpha_2 y]^\top \tag{3}$$

where we let $\theta^{*\top} = [b_2, b_1, b_0, a_2, a_1, a_0]$ be our vector of unknown parameters. This form however is not realizable, as we have a triple derivative of y. We combat this by defining a Hurwitz polynomial

$$\Lambda(s) = s^3 + \lambda_2 s^2 + \lambda_1 s + \lambda_0 \tag{4}$$

and filter both sides with it's reciprocal, obtaining

$$z = \frac{s^3 y}{\Lambda(s)} = \theta^{*\top}[\frac{\alpha_2}{\Lambda(s)u}, -\frac{\alpha_2}{\Lambda(s)y}]^\top. \tag{5}$$

That is, we have defined $\phi = [\frac{\alpha_2}{\Lambda(s)u}, -\frac{\alpha_2}{\Lambda(s)y}]^\top$, hence having the usual form $z = \theta^{*\top}\phi$.

b)

If we instead know $[a_2, a_1, a_0]$ and wish to parametrize the plant in terms of $\theta^{*\top} = [b_2, b_1, b_0]$ we rewrite (1) to

$$(s^3 + [a_2, a_1, a_0]\alpha_2)y = [b_2, b_1, b_0]\alpha_2 u. \tag{6}$$

Using the same filter as before, we obtain the realizable adaptive law

$$z = \frac{s^3 + [a_2, a_1, a_0]\alpha_2}{\Lambda} y = [b_2, b_1, b_0]\frac{\alpha_2}{\Lambda} u = \theta^{*\top}\phi. \tag{7}$$

If we were to let $\lambda_i = a_i$ in Λ, the fraction would in fact cancel and we would have $y = \theta^{*\top}\phi$.

c)

On the flip side, if we know $[b_2, b_1, b_0]$, we find

$$[b_2, b_1, b_0]\,\alpha_2 u = (s^3 + [a_2, a_1, a_0]\alpha_2)y \tag{8}$$

which in turn yields

$$\frac{s^3 y - [b_2, b_1, b_0]\alpha_2}{\Lambda} u = -[a_2, a_1, a_0]\frac{\alpha_2}{\Lambda} y. \tag{9}$$

Given that $[b_2, b_1, b_0] = [0, 0, 1]$, we then obtain $s^3 y - [b_2, b_1, b_0]\alpha_2 = s^3 y - u$, and so $z = \frac{s^3 y - u}{\Lambda}$, $\theta^{*\top} = [b_2, b_1, b_0]$ and $\phi = -\frac{\alpha_2}{\Lambda} y$.

Problem 2

a)

We define states

$$\begin{aligned} x_1 &= x, \quad \text{and} \\ x_2 &= \dot{x}. \end{aligned} \tag{10}$$

Rewriting the system on the form

$$\begin{aligned} \dot{x}_1 &= x_2, \\ \dot{x}_2 &= \frac{k}{M} x_1 - \frac{f}{M} x_2 + \frac{1}{M} u \end{aligned} \tag{11}$$

then lets us write the system on the general state space representation

$$\dot{x} = Ax + Bu \tag{12}$$

with

$$A = \begin{bmatrix} 0 & 1 \\ -\frac{k}{M} & -\frac{f}{M} \end{bmatrix}, \quad \text{and} \quad B = \begin{bmatrix} 0 \\ \frac{1}{M} \end{bmatrix}. \tag{13}$$

b)

Using s as a differential operator we obtain

$$M s^2 x + f s x + k x = u \tag{14}$$

which yields the transfer function from x to u

$$\frac{x}{u}(s) = \frac{1}{M s^2 + f s + k} = \frac{\frac{1}{M}}{s^2 + \frac{f}{M} s + \frac{k}{M}}. \tag{15}$$

c)

We now want to find an adaptiv law with $\theta^{*\top} = [M, f, k]$. We write

$$u = [M, f, k]\alpha_2 x, \tag{16}$$

but the right hand side in this is not proper and hence non-realizable. We thus define a Hurwitz polynomial

$$\Lambda = s^2 + \lambda_1 s + \lambda_0 \tag{17}$$

and devide both sides with it obtaining the adaptive scheme

$$z = \frac{u}{\Lambda} = [M, f, k]\frac{\alpha_2}{\Lambda}x. \tag{18}$$

That is, $\phi = \frac{\alpha_2}{\Lambda}x$.

Problem 3

a)

From I&S page 50-51 we have the realization

$$\dot{\phi}_1 = \Lambda_c \phi + lu, \tag{19}$$

where

$$\Lambda_c = \begin{bmatrix} -\lambda_{n-1} & -\lambda_{n-2} & \dots & -\lambda_0 \\ 1 & 0 & \dots & 0 \\ \vdots & \ddots & & \vdots \\ 0 & \dots & 1 & 0 \end{bmatrix}, \quad \text{and} \quad l = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}. \tag{20}$$

Problem 4

a)

We have that

$$||u||_1 = \int_0^\infty |u|dt = \sum_{n=1}^{\infty} n\frac{1}{n^3} = \sum_{n=1}^{\infty} \frac{1}{n^2} = \frac{\pi^2}{6}, \tag{21}$$

where the last result is obtained from recognizing the sum as the one in the Basel problem. That implies $u \in \mathcal{L}_1$. However,

$$||u||_2 = \int_0^\infty dt = \sum_{n=1}^{\infty} n\frac{1}{n} = \frac{n=1}{\infty} \to \infty \implies u \notin \mathcal{L}_2 \tag{22}$$

and

$$||u||_\infty = \sup_{t\geq 0}(u) = \lim_{k\to\infty} k \implies u \notin \mathcal{L}_\infty. \quad \blacksquare \tag{23}$$

b)

We recognize $G(s)$ as the laplace transform of $g(t) = e^{-t}$ and $y = g * u$. Since g decays exponentially fast, it is clearly in $\mathcal{L}_1$. Thus, since $u \in \mathcal{L}_\infty$ corollary 3.3.1 in I&S (ii) gives that

$$y \in \mathcal{L}_1 \bigcap \mathcal{L}_\infty, \quad \text{and} \quad \lim_{t\to\infty} t = 0. \quad \blacksquare \tag{24}$$

Problem 5

a)

Consider theorem 3.5.1 in I&S. Clearly, (i) holds, since all poles are clearly the left half plane. In this case the relative order is $n^* = 1$. We have

$$\Re\left[G(jw)\right] = \Re\left[\frac{j\omega + 5}{(j\omega)^2 + 5j\omega + 4}\right] \tag{25}$$

Then, we find

$$\lim_{|\omega|\to\infty} \omega^2 \Re\left[\frac{j\omega + 5}{-\omega^2 + 5j\omega + 4}\right] = -1 \tag{26}$$

and as such, G is not SPR. It is neither PR either since the residue at -4 is

$$\operatorname{Res}(G, -4) = \lim_{s\to-4} (s+4)G(s) = \lim_{s\to-4} \frac{s+5}{s+1} = -\frac{1}{3} \not> 0. \tag{27}$$

b)

This tf is not PR since it has a non-simple pole (of order 2) in $s = -2$.

c)

This time all poles are again in the left half plane, but we have a zero in the rhp. We have

$$\begin{aligned}
\Re\left[G(jw)\right] &= \Re\left[\frac{j\omega - 2}{(j\omega)^2 + 8j\omega + 15}\right] \\
&= \Re\left[\frac{j\omega - 2}{-\omega^2 + 8j\omega + 15}\right] \\
&= \Re\left[\frac{(j\omega - 2)(-\omega^2 + 15 - 8j\omega)}{(-\omega^2 + 8j\omega + 15)(-\omega^2 + 15 - 8j\omega)}\right] \\
&= \frac{8\omega^2 - 2\omega^2 - 30}{(-\omega^2 + 15)^2 + 64\omega^2} \\
&= \frac{6\omega^2 - 30}{\omega^4 + 34\omega^2 + 15^2}.
\end{aligned} \tag{28}$$

Because

$$\text{Res}(G,-3) = \lim_{s\to -3}(s+3)G(s) = \lim_{s\to -3}\frac{s-2}{s+5} = -\frac{5}{2} \not> 0 \tag{29}$$

this is not PR.

d)

This is, by corollary 3.5.1, not PR since $|n^*| = 2 > 1$.