TTK4250 Sensor Fusion
Assignment 1

Hand in: *Wednesday 4. September 16.00* in room D238, to teaching assistants in exercise class or on Blackboard.

Tasks are to be solved on paper if you are not told otherwise, and you are supposed to show how you got to a particular answer. It is, however, encouraged to use MATLAB, Maple, etc. to verify your answers. Rottmann's mathematical formula collection is allowed at both the exercises and the exam.

Task 1: *The CDF of any random variable is Uniform(0,1)*

Let X be a random variable with cumulative distribution function $P_X(x) = \Pr(X \leq x)$ (Here X denotes the random variable, while x is a free variable representing a particular realization). Show that the random variable $Y = P_X(X)$ is uniformly distributed over $[0, 1]$.

Hint: Start with the CDF of Y, denoted $P_Y(y)$, and its definition (see Definition 2.2.7). You should be able to rewrite it in terms of the definition of $P_X(x)$ and with x as a function of y. You might need the monotonicity of $P_X(x)$, and that $P_X(P_X^{-1}(y)) = y$ for a properly defined inverse function.

Task 2: *Some results regarding the Poisson distribution*

(a) Show that the generating function of a Poisson distributed random variable N with parameter λ is $e^{\lambda(t-1)}$.

Hint: $\sum_{i=0}^{\infty} \frac{x^i}{i!} = e^x$

(b) Show that the generating function of a Binomial distributed Random variable M, with probability parameter r and number parameter n, is $(1 - r + rt)^n$.

Hint: M can be seen as the sum of n Bernoulli variables that has probability r of being 1, and $1-r$ of being 0, or you might need the binomial theorem that states $\sum_{i=0}^{n} \frac{n!}{i!(n-i)!} a^i b^{(n-i)} = (a + b)^n$.

(c) Let $r = \frac{\lambda}{n}$ and take the limit as n approaches infinity in the generating function of the Binomial distribution and comment.

Hint: $\lim_{n\to\infty} \left(1 + \frac{x}{n}\right)^n = e^x$

(d) Use both PMF and GF to show that the distribution of $N = N_1 + N_2$ is Poisson distributed with parameter $\lambda = \lambda_1 + \lambda_2$, where N_1 and N_2 are independent Poisson random variables with parameters λ_1 and λ_2 respectively. Which way of doing it would you prefer?

Hint: the binomial theorem states that $\sum_{i=0}^{n} \frac{n!}{i!(n-i)!} a^i b^{(n-i)} = (a + b)^n$

Task 3: *Estimating the number of boats in a region*

Assume boats are arriving to a region with the interval between them i.i.d. according to an exponential distribution with rate λ. That is, $p\,(t_i - t_{i-1}) = \lambda e^{-\lambda(t_i - t_{i-1})}$, where t_i is the arrival time of the i'th boat.

It is installed a radar in order to detect how many boats there are in the region at time T. The radar scans the area, processes the data and reports how many boats it has counted, m. Since the radar has been tuned to minimize the probability of counting something that is not a boat, it only detects and counts a boat with probability $P_\mathrm{D} \in (0, 1)$ independently from other counts. Since each detection is Bernoulli with parameter P_D we have that $\Pr(n_D|n) = \mathrm{Binomial}(n_D; P_\mathrm{D}, n) = \binom{n}{n_D} P_\mathrm{D}^{n_D}(1 - P_\mathrm{D})^{n-n_D}$, we have in fact that $\Pr(n_D, n - n_D|n) = \Pr(n_D|n)$ since the binomial is a probabilistic unordered "split" into the two categories detected and undetected.

The total number measured is however $m = n_D + m_{fa}$, where m_{fa} is Poisson distributed with parameter Λ.

(a) The arrival process is assumed to start at time 0. It is, however, given that boat 1 has not yet arrived at $t_0 \geq 0$. Using Bayes rule, show that $p(t_1|t_1 \geq t_0) = \lambda e^{-\lambda(t_1 - t_0)}$ for $t_1 \geq t_0$ and zero otherwise.

This result ($p(x|x \geq a) = p(x - a)$) is known as memorylessness, and is one of the properties that makes the exponential distribution often used for continuous time arrival processes. The exponential distribution is in fact the only continuous distribution to have this property. Can you think of why this property is called memorylessness, and why it is usefull?

(b) What is the distribution of $t_n - t_0 = \sum_{i=1}^{n} t_i - t_{i-1}$ given $t_1 \geq t_0$?

Hint: Example 2.9

(c) What is the probability that boat $n + 1$ did not arrive before T given that boat n arrived at time $t_n \leq T$? That is, find $\Pr(t_{n+1} > T|t_n)$

Hint: $\Pr(t_{n+1} > T|t_n) = \Pr(t_{n+1} - t_n > T - t_n|t_n)$

(d) Use the last two results to show the probability for n boats to arrive between t_0 and T, given by $\Pr(n) = \Pr(t_n \leq T, t_{n+1} > T|t_1 \geq t_0)$, is Poisson.

Hint: you should be able to form $p(t_n, t_{n+1} > T|t_1 \geq t_0)$ and then integrate t_n over a suitable interval.

(e) Assume that you know that there where no boats at time 0. Show that the marginal distribution for n_D and $n_U = n - n_D$ can be written as the product of two independent Poisson distributions $p(n_D, n_U) = p(n_D)p(n_U)$.

Hint: it might be easier to find $p(n_D, n)$ and then do a transform with $n = n_U + n_D$ afterwards.

(f) Use Bayes rule to find the distribution of the number of detected boats after you have received a measurement m, where $m = n_D + m_{fa}$. That is, find $p(n_D|m)$.

Hint: To find $p(m)$ you probably need to invoke an independence assumption and the result of task 2 (d).

(g) Find the MMSE and MAP estimate of n_D. You can use $\text{Binomial}(n_D; r, m)$ and insert for r afterwards if you prefer.

Hint for MMSE: You might need $n\text{Binomial}(n; r, m) = mr\text{Binomial}(n-1; r, m-1)$ and the binomial theorem.

Hint for MAP: Look at the sign of $p(n + 1) - p(n)$, and note how many peaks the distribution has. What can you say about this difference, specifically the sign, in relation to MAP and this/these peaks?

Task 4: *Transformation of Gaussian random variables*

Let $x \in \mathbb{R}^n$ be $\mathcal{N}(\mu, \Sigma)$. Find the distribution and see if you recognize it:

Hint: they are all given in the book.

(a) $z = \Sigma^{-\frac{1}{2}}(x - \mu)$, where $\Sigma^{\frac{1}{2}}(\Sigma^{\frac{1}{2}})^T = \Sigma$

Hint: If you are using theorem 2.4.1, you might need $\det(A^{\frac{1}{2}}) = \det(A)^{\frac{1}{2}}$, $(A^{-1})^T = (A^T)^{-1}$, and $\det(A^T) = \det(A)$ whenever A has full rank.

(b) Use transformation of random variables to find $y_i = z_i^2$, where z_i is the i'th variable in the vector z.

(c) $y = (x-\mu)^T\Sigma^{-1}(x-\mu) = z^T z = \sum z_i^2 = \sum y_i$.

Hint: The MGF of y_i is given in the book through example 2.7 and 2.9.

Task 5: *Sensor fusion*

In this task we want to find out if a boat is above the line $y = x + 2$. In order to do this we will fuse measurements from two sensors with our prior belief: A drone-mounted camera, and a maritime surveillance radar. You have some prior knowledge of the state of the boat. You get 1 measurement from each sensor that are processed so that you know them to be (approximately) Gaussian [2] conditioned on the position.

To be more specific, let us denote the state by x and our prior knowledge a Gaussian of the form $\mathcal{N}(x; \bar{x}, P)$. The measurement from the camera is given by $z^c = H^c x + v^c$ and the measurement from the radar by $z^r = H^r x + v^r$, where v^j, $j \in \{c, r\}$ denotes the measurement noise and is distributed according to $\mathcal{N}(0, R^c)$ and $\mathcal{N}(0, R^r)$, respectively.

The boat is assumed to move around according to the model $x^+ = Fx + w$, where w is $\mathcal{N}(0, Q)$. Note the similarity to the measurement models.

Only insert the numbers when asked to. The needed values are given by

$$\bar{x} = \begin{bmatrix} 0 & 0 \end{bmatrix}^T, \qquad P = 25 I_2, \qquad H^c = H^r = I_2,$$
$$R^c = \begin{bmatrix} 79 & 36 \\ 36 & 36 \end{bmatrix}, \qquad R^r = \begin{bmatrix} 28 & 4 \\ 4 & 22 \end{bmatrix}, \qquad z_c = \begin{bmatrix} 2 & 14 \end{bmatrix}^T, \qquad z_r = \begin{bmatrix} -4 & 6 \end{bmatrix}^T$$

(a) What is $p(z^c|x)$.

(b) Show that the joint $p(x, z^c)$ can be written as a Gaussian distribution.

Hint: conditional probability and the proof of theorem 3.3.1.

(c) Find the marginal $p(z^c)$ and the conditional $p(x|z^c)$ using the above and either theorems from the book or calculations.

(d) Can what was found above be reused to find the marginal $p(x^+)$ and/or $p(x|z^r)$? If so, state them.

(e) What is the MMSE and MAP estimate of x given z^c? You do not need to do calculations to find the answer, but briefly state what you would do if you had to.

(f) Using MATLAB. Use what you have found to condition x on z for each sensor to find the conditional mean and covariance, and plot and comment. Ie., insert the values to find the parameters of $p(x|z^c)$ and $p(x|z^r)$.

Hint: to plot a "g-sigma" covariance ellipse in MATLAB, you can use `[x, y] = [x_center, y_center]' + chol(P)' * g * [cos([0:0.01:2*pi]); sin([0:0.01:2*pi])]; plot(x, y)`

(g) Using matlab. Perform the update of the other sensor (for both) to get $p(x|z^r, z^c)$ and investigate. Are the distributions the same? does it matter which order we condition?

(h) You now want to know the probability that the boat is above the line, $\Pr(x_2 - x_1 > 2)$. Find this probability using the appropriate linear transform and the CDF

Hint: write `help normcdf` in MATLAB.

[2] In reality, a camera measures a bearing from a point while a radar measures in polar coordinates. However, with some knowledge of which plane/distance something is operating in, we can extract an approximate 3d cartesian measurement and approximate it as Gaussian (more on that later in the course). At a certain distance a Gaussian in polar coordinates with small enough covariance can safely be approximated by a Gaussian in cartesian coordinates.

Task 6: *Information matrix update*

$z = Hx + w$, where x is $\mathcal{N}(\bar{x}, P)$ and w is $\mathcal{N}(0, R)$. Show that the formula for the inverse of the covariance matrix of x conditional on z, here denoted $\hat{P}^{-1}$, is given by: $\hat{P}^{-1} = P^{-1} + H^T R^{-1} H$

Hint: Previous task, section 3.2-3.4 and P.1-P.2