Department of Engineering Cybernetics

TTK4215 Adaptive Control

Solution 12

The following system is given

$$\dot{x}_1 = x_2 + \theta_1^{*T} g_1(x_1) \tag{1}$$
$$\dot{x}_2 = x_3 \tag{2}$$
$$\dot{x}_3 = u + \theta_2^{*T} g_2(x_1, x_2, x_3) \tag{3}$$

where g_1, g_2 are known nonlinear differentiable functions and u is the control input.
a) Looking only at equation (1), it is clear that if

$$x_2 = -c_1 x_1 - \theta_1^{*T} g_1(x_1)$$

would drive x_1 to zero. Therefore, we select the virtual control

$$q_1(x_1) = -c_1 x_1 - \theta_1^{*T} g_1(x_1),$$

and define

$$z_2 = x_2 - q_1(x_1).$$

We then obtain the dynamics

$$\begin{aligned}
\dot{x}_1 &= x_2 + \theta_1^{*T} g_1(x_1) \\
&= x_2 + \theta_1^{*T} g_1(x_1) + q_1(x_1) - q_1(x_1) \\
&= x_2 + \theta_1^{*T} g_1(x_1) - c_1 x_1 - \theta_1^{*T} g_1(x_1) - q_1(x_1) \\
&= x_2 - c_1 x_1 - q_1(x_1) \\
&= -c_1 x_1 + z_2 \\
\dot{z}_2 &= \dot{x}_2 - \frac{\partial q_1}{\partial x_1}\dot{x}_1 \\
&= x_3 - \frac{\partial q_1}{\partial x_1}(-c_1 x_1 + z_2)
\end{aligned}$$

Considering

$$V_1 = \frac{x_1^2}{2} + \frac{z_2^2}{2}$$

we have

$$\begin{aligned}
\dot{V}_1 &= x_1 \dot{x}_1 + z_2 \dot{z}_2 \\
&= x_1(-c_1 x_1 + z_2) \\
&\quad + z_2\left(x_3 - \frac{\partial q_1}{\partial x_1}(-c_1 x_1 + z_2)\right) \\
&= -c_1 x_1^2 \\
&\quad + z_2\left(x_1 + x_3 - \frac{\partial q_1}{\partial x_1}(-c_1 x_1 + z_2)\right)
\end{aligned}$$

At this point, it is clear that if x_3 is considered the control input, it can be selected such that

$$x_1 + x_3 - \frac{\partial q_1}{\partial x_1}\left(-c_1 x_1 + z_2\right) = -c_2 z_2$$

to stabilize the (x_1, z_2) dynamics. Select therefore the virtual control

$$q_2\left(x_1, z_2\right) = -c_2 z_2 - x_1 + \frac{\partial q_1}{\partial x_1}\left(-c_1 x_1 + z_2\right)$$

and define

$$z_3 = x_3 - q_2\left(x_1, z_2\right).$$

We then obtain the dynamics

$$\begin{aligned}
\dot{x}_1 &= -c_1 x_1 + z_2 \\
\dot{z}_2 &= x_3 - \frac{\partial q_1}{\partial x_1}\left(-c_1 x_1 + z_2\right) \\
&= x_3 - \frac{\partial q_1}{\partial x_1}\left(-c_1 x_1 + z_2\right) + q_2\left(x_1, z_2\right) - q_2\left(x_1, z_2\right) \\
&= x_3 - \frac{\partial q_1}{\partial x_1}\left(-c_1 x_1 + z_2\right) - c_2 z_2 - x_1 + \frac{\partial q_1}{\partial x_1}\left(-c_1 x_1 + z_2\right) - q_2\left(x_1, z_2\right) \\
&= -c_2 z_2 + z_3 - x_1 \\
\dot{z}_3 &= \dot{x}_3 - \frac{\partial q_2}{\partial x_1}\dot{x}_1 - \frac{\partial q_2}{\partial z_2}\dot{z}_2 \\
&= u + \theta_2^{*T} g_2\left(x_1, x_2, x_3\right) - \frac{\partial q_2}{\partial x_1}\left(-c_1 x_1 + z_2\right) - \frac{\partial q_2}{\partial z_2}\left(-c_2 z_2 + z_3 - x_1\right)
\end{aligned}$$

Considering

$$V_2 = V_1 + \frac{z_3^2}{2},$$

we have

$$\begin{aligned}
\dot{V}_2 &= x_1 \dot{x}_1 + z_2 \dot{z}_2 + z_3 \dot{z}_3 \\
&= x_1\left(-c_1 x_1 + z_2\right) \\
&\quad + z_2\left(-c_2 z_2 + z_3 - x_1\right) \\
&\quad + z_3\left(u + \theta_2^{*T} g_2\left(x_1, x_2, x_3\right) - \frac{\partial q_2}{\partial x_1}\left(-c_1 x_1 + z_2\right) - \frac{\partial q_2}{\partial z_2}\left(-c_2 z_2 + z_3 - x_1\right)\right) \\
&= -c_1 x_1^2 + x_1 z_2 \\
&\quad - c_2 z_2^2 - x_1 z_2 + z_3 z_2 \\
&\quad + z_3\left(u + \theta_2^{*T} g_2\left(x_1, x_2, x_3\right) - \frac{\partial q_2}{\partial x_1}\left(-c_1 x_1 + z_2\right) - \frac{\partial q_2}{\partial z_2}\left(-c_2 z_2 + z_3 - x_1\right)\right) \\
&= -c_1 x_1^2 - c_2 z_2^2 \\
&\quad + z_3\left(z_2 + u + \theta_2^{*T} g_2\left(x_1, x_2, x_3\right) - \frac{\partial q_2}{\partial x_1}\left(-c_1 x_1 + z_2\right) - \frac{\partial q_2}{\partial z_2}\left(-c_2 z_2 + z_3 - x_1\right)\right)
\end{aligned}$$

Now, select

$$u = -c_3 z_3 - z_2 - \theta_2^{*T} g_2\left(x_1, x_2, x_3\right) + \frac{\partial q_2}{\partial x_1}\left(-c_1 x_1 + z_2\right) + \frac{\partial q_2}{\partial z_2}\left(-c_2 z_2 + z_3 - x_1\right)$$

to get

$$\dot{V}_2 = -c_1 x_1^2 - c_2 z_2^2 - c_3 z_3^2.$$

This proves that the equilibrium $(\bar{x}_1, \bar{z}_2, \bar{z}_3) = (0, 0, 0)$ is globally exponentially stable, and it follows that $(x_1, z_2, z_3) \to 0$. From the definitions of the virtual controls it follows that $q_2(x_1, z_2) \to 0$ and since $g(0) = 0$ that $q_1(x_1) \to 0$, and therefore that $x_2, x_3 \to 0$.
b) Since θ_1^* and θ_2^* we follow the procudure called *Backstepping design with tuning functions.*
Step 1: Introduce $z_1 = x_1$ and $z_2 = x_2 - \alpha_1$, and rewrite the first equation as

$$\dot{z}_1 = z_2 + \alpha_1 + \theta_1^{*^T} g_1(x_1)$$

and view α_1 as a virtual control input. Considering

$$V_1 = \frac{1}{2} z_1^2 + \frac{1}{2} \tilde{\theta}_1^T \Gamma_1^{-1} \tilde{\theta}_1,$$

we have

$$\begin{aligned}
\dot{V}_1 &= z_1 \left(z_2 + \alpha_1 + \theta_1^{*^T} g_1(x_1) \right) + \tilde{\theta}_1^T \Gamma_1^{-1} \dot{\theta}_1 \\
&= z_1 \left(z_2 + \alpha_1 + \theta_1^{*^T} g_1(x_1) + \theta_1^T g_1(x_1) - \theta_1^T g_1(x_1) \right) + \tilde{\theta}_1^T \Gamma_1^{-1} \dot{\theta}_1 \\
&= z_1 \left(z_2 + \alpha_1 + \theta_1^T g_1(x_1) \right) + \tilde{\theta}_1^T \Gamma_1^{-1} \dot{\theta}_1 - \tilde{\theta}_1^T g_1(x_1) z_1 \\
&= z_1 \left(z_2 + \alpha_1 + \theta_1^T g_1(x_1) \right) + \tilde{\theta}_1^T \Gamma_1^{-1} \left(\dot{\theta}_1 - \Gamma_1 z_1 g_1(x_1) \right).
\end{aligned}$$

If x_2 where the actual control input, we would have $z_2 \equiv 0$ and $x_2 \equiv \alpha_1$, and we would select $\dot{\theta} = \tau_1$ with

$$\tau_1(x_1) = \Gamma_1 z_1 g_1(x_1)$$

and

$$\alpha_1(x_1, \theta) = -c_1 z_1 - \theta_1^T g_1(x_1)$$

to achieve

$$\dot{V}_1 = -c_1 z_1^2.$$

Since this is not the case, we do not use $\dot{\theta}_1 = \tau_1$ as an update law, but retain τ_1 as our first *tuning function* and α_1 as our first *stabilizing function.* We thus postpone the decision about $\dot{\theta}_1$ and have the following

$$\begin{aligned}
\dot{V}_1 &= z_1 \left(z_2 + \alpha_1 + \theta_1^T g_1(x_1) \right) + \tilde{\theta}_1^T \Gamma_1^{-1} \left(\dot{\theta}_1 - \Gamma_1 z_1 g_1(x_1) \right) \\
&= -c_1 z_1^2 + z_1 z_2 + \tilde{\theta}_1^T \Gamma_1^{-1} \left(\dot{\theta}_1 - \tau_1(x_1) \right). \qquad (4)
\end{aligned}$$

The closed loop (inserting α_1) dynamics for z_1 is

$$\dot{z}_1 = -c_1 z_1 + z_2 - \tilde{\theta}_1^T g_1(x_1). \qquad (5)$$

Step 2: Introducing $z_3 = x_3 - \alpha_2$, we rewrite equation (2) as

$$\begin{aligned}
\dot{z}_2 &= \dot{x}_2 - \frac{\partial \alpha_1}{\partial x_1} \dot{x}_1 - \frac{\partial \alpha_1}{\partial \theta} \dot{\theta} \\
&= x_3 - \frac{\partial \alpha_1}{\partial x_1} \left(x_2 + \theta_1^{*^T} g_1(x_1) \right) - \frac{\partial \alpha_1}{\partial \theta} \dot{\theta} \\
&= z_3 + \alpha_2 - \frac{\partial \alpha_1}{\partial x_1} \left(x_2 + \theta_1^{*^T} g_1(x_1) \right) - \frac{\partial \alpha_1}{\partial \theta} \dot{\theta}
\end{aligned}$$

and use α_2 as a virtual control to stabilize the (z_1, z_2)-system using $V_2 = V_1 + z_2^2/2$. We have

$$\begin{aligned}
\dot{V}_2 &= -c_1 z_1^2 + z_1 z_2 + z_2 \left(z_3 + \alpha_2 - \frac{\partial \alpha_1}{\partial x_1} \left(x_2 + \theta_1^{*^T} g_1(x_1) \right) - \frac{\partial \alpha_1}{\partial \theta} \dot{\theta} \right) \\
&\quad + \tilde{\theta}_1^T \Gamma_1^{-1} \left(\dot{\theta}_1 - \tau_1(x_1) \right) \\
&= -c_1 z_1^2 + z_2 \left(z_1 + z_3 + \alpha_2 - \frac{\partial \alpha_1}{\partial x_1} x_2 - \frac{\partial \alpha_1}{\partial \theta} \dot{\theta} - \frac{\partial \alpha_1}{\partial x_1} \theta_1^{*^T} g_1(x_1) - \frac{\partial \alpha_1}{\partial x_1} \theta_1^T g_1(x_1) + \frac{\partial \alpha_1}{\partial x_1} \theta_1^T g_1(x_1) \right) \\
&\quad + \tilde{\theta}_1^T \Gamma_1^{-1} \left(\dot{\theta}_1 - \tau_1(x_1) \right) \\
&= -c_1 z_1^2 + z_2 \left(z_1 + z_3 + \alpha_2 - \frac{\partial \alpha_1}{\partial x_1} x_2 - \frac{\partial \alpha_1}{\partial \theta} \dot{\theta} - \frac{\partial \alpha_1}{\partial x_1} \theta_1^T g_1(x_1) \right) \\
&\quad + \tilde{\theta}_1^T \Gamma_1^{-1} \left(\dot{\theta}_1 - \tau_1(x_1) + \Gamma_1 z_2 \frac{\partial \alpha_1}{\partial x_1} g_1(x_1) \right).
\end{aligned}$$

If x_3 were the actual control, we would let $z_3 \equiv 0$ and eliminate the uncertainty $\tilde{\theta}$ from $\dot{V}_2$ by selecting $\dot{\theta} = \tau_2$ with

$$\tau_2(x_1, x_2, \theta) = \tau_1(x_1) - \Gamma_1 z_2 \frac{\partial \alpha_1}{\partial x_1} g_1(x_1),$$

and make $\dot{V}_2 = -c_1 z_1^2 - c_2 z_2^2$ by letting

$$\alpha_2(x_1, x_2, \theta_1) = -c_2 z_2 - z_1 + \frac{\partial \alpha_1}{\partial x_1} x_2 + \frac{\partial \alpha_1}{\partial \theta_1} \tau_2 + \frac{\partial \alpha_1}{\partial x_1} \theta_1^T g_1(x_1).$$

Notice that in the virtual control law, $\dot{\theta}_1$ is replaced by the tuning function τ_2. Since x_3 is not our control, we do not use $\dot{\theta}_1 = \tau_2$ as our update law, but retain τ_2 as our second tuning function and α_2 as our second stabilizing function. We then have

$$\begin{aligned}
\dot{V}_2 &= -c_1 z_1^2 + z_2 \left(z_3 - c_2 z_2 + \frac{\partial \alpha_1}{\partial \theta_1} \tau_2 - \frac{\partial \alpha_1}{\partial \theta_1} \dot{\theta}_1 \right) \\
&\quad + \tilde{\theta}_1^T \Gamma_1^{-1} \left(\dot{\theta}_1 - \tau_2(x_1, x_2, \theta_1) \right) \\
&= -c_1 z_1^2 - c_2 z_2^2 + z_2 z_3 + \left(\tilde{\theta}_1^T \Gamma_1^{-1} - z_2 \frac{\partial \alpha_1}{\partial \theta_1} \right) \left(\dot{\theta}_1 - \tau_2(x_1, x_2, \theta_1) \right).
\end{aligned}$$

The closed loop z_2-dynamics is

$$\begin{aligned}
\dot{z}_2 &= -z_1 - c_2 z_2 + z_3 + \frac{\partial \alpha_1}{\partial x_1} x_2 + \frac{\partial \alpha_1}{\partial \theta_1} \tau_2 + \frac{\partial \alpha_1}{\partial x_1} \theta_1^T g_1(x_1) - \frac{\partial \alpha_1}{\partial x_1} \left(x_2 + \theta_1^{*^T} g_1(x_1) \right) - \frac{\partial \alpha_1}{\partial \theta_1} \dot{\theta}_1 \\
&= -z_1 - c_2 z_2 + z_3 + \tilde{\theta}_1^T g_1(x_1) \frac{\partial \alpha_1}{\partial x_1} + \frac{\partial \alpha_1}{\partial \theta_1} \left(\tau_2 - \dot{\theta}_1 \right).
\end{aligned}$$

Step 3: With $z_3 = x_3 - \alpha_2$ we rewrite equation (3) as

$$\begin{aligned}
\dot{z}_3 &= \dot{x}_3 - \frac{\partial \alpha_2}{\partial x_1}\dot{x}_1 - \frac{\partial \alpha_2}{\partial x_2}\dot{x}_2 - \frac{\partial \alpha_2}{\partial \theta_1}\dot{\theta}_1 \\
&= u + \theta_2^{*^T} g_2(x_1, x_2, x_3) - \frac{\partial \alpha_2}{\partial x_1}\left(x_2 + \theta_1^{*^T} g_1(x_1)\right) - \frac{\partial \alpha_2}{\partial x_2}x_3 - \frac{\partial \alpha_2}{\partial \theta_1}\dot{\theta}_1 \\
&= u + \theta_2^T g_2(x_1, x_2, x_3) - \tilde{\theta}_2^T g_2(x_1, x_2, x_3) - \frac{\partial \alpha_2}{\partial x_1}\left(x_2 + \theta_1^{*^T} g_1(x_1)\right) - \frac{\partial \alpha_2}{\partial x_2}x_3 - \frac{\partial \alpha_2}{\partial \theta_1}\dot{\theta}_1 \\
&= u + \theta_2^T g_2(x_1, x_2, x_3) - \frac{\partial \alpha_2}{\partial x_2}x_3 - \frac{\partial \alpha_2}{\partial x_1}\left(x_2 + \theta_1^{*^T} g_1(x_1)\right) \\
&\quad -\tilde{\theta}_2^T g_2(x_1, x_2, x_3) - \frac{\partial \alpha_2}{\partial \theta_1}\dot{\theta}_1.
\end{aligned}$$

Consider now the Lyapunov function candidate $V_3 = V_2 + z_3^2/2 + \frac{1}{2}\tilde{\theta}_2^T \Gamma_2^{-1}\tilde{\theta}_2$. We get

$$\begin{aligned}
\dot{V}_3 &= -c_1 z_1^2 - c_2 z_2^2 + z_2 z_3 + \left(\tilde{\theta}_1^T \Gamma_1^{-1} - z_2 \frac{\partial \alpha_1}{\partial \theta_1}\right)\left(\dot{\theta}_1 - \tau_2(x_1, x_2, \theta_1)\right) + z_3 \dot{z}_3 + \tilde{\theta}_2^T \Gamma_2^{-1}\dot{\theta}_2 \\
&= -c_1 z_1^2 - c_2 z_2^2 + z_2 z_3 - z_2 \frac{\partial \alpha_1}{\partial \theta_1}\left(\dot{\theta}_1 - \tau_2(x_1, x_2, \theta_1)\right) \\
&\quad + z_3\left(u + \theta_2^T g_2(x_1, x_2, x_3) - \frac{\partial \alpha_2}{\partial x_2}x_3 - \frac{\partial \alpha_2}{\partial x_1}\left(x_2 + \theta_1^T g_1(x_1)\right) - \frac{\partial \alpha_2}{\partial \theta_1}\dot{\theta}_1\right) \\
&\quad + \tilde{\theta}_1^T \Gamma_1^{-1}\left(\dot{\theta}_1 - \tau_2(x_1, x_2, \theta_1) + \Gamma_1 g_1(x_1) z_3 \frac{\partial \alpha_2}{\partial x_1}\right) \\
&\quad + \tilde{\theta}_2^T \Gamma_2^{-1}\left(\dot{\theta}_2 - \Gamma_2 g_2(x_1, x_2, x_3) z_3\right)
\end{aligned}$$

To eliminate the uncertain terms, we select $\dot{\theta}_1 = \tau_3$ with

$$\tau_3(x1, x2, x3, \theta_1) = \tau_2(x_1, x_2, \theta_1) - \Gamma_1 g_1(x_1) z_3 \frac{\partial \alpha_2}{\partial x_1},$$

and

$$\dot{\theta}_2 = \Gamma_2 g_2(x_1, x_2, x_3) z_3.$$

We then have

$$\begin{aligned}
\dot{V}_3 &= -c_1 z_1^2 - c_2 z_2^2 + z_2 z_3 - z_2 \frac{\partial \alpha_1}{\partial \theta_1}\left(\dot{\theta}_1 - \tau_2(x_1, x_2, \theta_1)\right) \\
&\quad + z_3\left(u + \theta_2^T g_2(x_1, x_2, x_3) - \frac{\partial \alpha_2}{\partial x_2}x_3 - \frac{\partial \alpha_2}{\partial x_1}\left(x_2 + \theta_1^T g_1(x_1)\right) - \frac{\partial \alpha_2}{\partial \theta_1}\dot{\theta}_1\right)
\end{aligned}$$

Noticing that

$$\begin{aligned}
\dot{\theta}_1 - \tau_2(x_1, x_2, \theta_1) &= \tau_3(x1, x2, x3, \theta_1) - \tau_2(x_1, x_2, \theta_1) \\
&= -\Gamma_1 g_1(x_1) z_3 \frac{\partial \alpha_2}{\partial x_1},
\end{aligned}$$

we have

$$\begin{aligned}
\dot{V}_3 &= -c_1 z_1^2 - c_2 z_2^2 + z_2 z_3 + z_2 z_3 \frac{\partial \alpha_1}{\partial \theta_1}\Gamma_1 g_1(x_1) \frac{\partial \alpha_2}{\partial x_1} \\
&\quad + z_3\left(u + \theta_2^T g_2(x_1, x_2, x_3) - \frac{\partial \alpha_2}{\partial x_2}x_3 - \frac{\partial \alpha_2}{\partial x_1}\left(x_2 + \theta_1^T g_1(x_1)\right) - \frac{\partial \alpha_2}{\partial \theta_1}\dot{\theta}_1\right)
\end{aligned}$$

and we get

$$\begin{aligned}
\dot{V}_3 &= -c_1 z_1^2 - c_2 z_2^2 \\
&\quad + z_3 \left(u + z_2 + z_2 \frac{\partial \alpha_1}{\partial \theta_1} \Gamma_1 g_1(x_1) \frac{\partial \alpha_2}{\partial x_1} + \theta_2^T g_2(x_1, x_2, x_3) - \frac{\partial \alpha_2}{\partial x_2} x_3 - \frac{\partial \alpha_2}{\partial x_1} \left(x_2 + \theta_1^T g_1(x_1) \right) - \frac{\partial \alpha_2}{\partial \theta_1} \tau_3
\end{aligned}$$

We now select

$$u = -c_3 z_3 - z_2 - z_2 \frac{\partial \alpha_1}{\partial \theta_1} \Gamma_1 g_1(x_1) \frac{\partial \alpha_2}{\partial x_1} - \theta_2^T g_2(x_1, x_2, x_3) + \frac{\partial \alpha_2}{\partial x_2} x_3 + \frac{\partial \alpha_2}{\partial x_1} \left(x_2 + \theta_1^T g_1(x_1) \right) + \frac{\partial \alpha_2}{\partial \theta_1} \tau_3$$

and we get

$$\dot{V}_3 = -c_1 z_1^2 - c_2 z_2^2 - c_3 z_3^2 \tag{6}$$

proving that $z_1, z_2, z_3, \theta_1, \theta_2 \in \mathcal{L}_\infty$ and $z_1, z_2, z_3 \in \mathcal{L}_2$. The closed-loop dynamics of z_3 is

$$\begin{aligned}
\dot{z}_3 &= u + \theta_2^T g_2(x_1, x_2, x_3) - \frac{\partial \alpha_2}{\partial x_2} x_3 - \frac{\partial \alpha_2}{\partial x_1} \left(x_2 + \theta_1^{*T} g_1(x_1) \right) \\
&\quad - \tilde{\theta}_2^T g_2(x_1, x_2, x_3) - \frac{\partial \alpha_2}{\partial \theta_1} \dot{\theta}_1 \\
&= -c_3 z_3 - z_2 - z_2 \frac{\partial \alpha_1}{\partial \theta_1} \Gamma_1 g_1(x_1) \frac{\partial \alpha_2}{\partial x_1} + \frac{\partial \alpha_2}{\partial x_1} \tilde{\theta}_1^T g_1(x_1) - \tilde{\theta}_2^T g_2(x_1, x_2, x_3).
\end{aligned}$$

We have

$$\begin{aligned}
\tau_1(x_1) &= \Gamma_1 z_1 g_1(x_1), \\
\alpha_1(x_1, \theta) &= -c_1 z_1 - \theta_1^T g_1(x_1)
\end{aligned}$$

so $\tau_1, \alpha_1 \in \mathcal{L}_\infty$. We have

$$\begin{aligned}
\tau_2(x_1, x_2, \theta) &= \tau_1(x_1) - \Gamma_1 z_2 \frac{\partial \alpha_1}{\partial x_1} g_1(x_1), \\
\alpha_2(x_1, x_2, \theta_1) &= -c_2 z_2 - z_1 + \frac{\partial \alpha_1}{\partial x_1} x_2 + \frac{\partial \alpha_1}{\partial \theta_1} \tau_2 + \frac{\partial \alpha_1}{\partial x_1} \theta_1^T g_1(x_1)
\end{aligned}$$

so $\tau_2, \alpha_2 \in \mathcal{L}_\infty$, since g_1 is continuously differentiable and $x_1 \in \mathcal{L}_\infty$. We have

$$\tau_3(x_1, x_2, x_3, \theta_1) = \tau_2(x_1, x_2, \theta_1) - \Gamma_1 g_1(x_1) z_3 \frac{\partial \alpha_2}{\partial x_1}$$

so $\tau_3 \in \mathcal{L}_\infty$ since g_1 is twice continuously differentiable and $x_1 \in \mathcal{L}_\infty$.It follows that $\dot{z}_1, \dot{z}_2, \dot{z}_3 \in \mathcal{L}_\infty$, and so $z_1, z_2, z_3 \to 0$ by Barbalat's lemma. From the definitions of $\tau_1, \alpha_1, \tau_2, \alpha_2$ we have $\tau_1, \alpha_1 \to 0$ which implies $x_2 \to 0$. This, in turn implies $\tau_2, \alpha_2 \to 0$ and therefore that $x_3 \to 0$.