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September 21, 2019

Problem 1 I&S 4.5

a) With $R(s)$ known

We have

$$y = \frac{Z(s)}{R(s)}u \implies R(s)y = Z(s)u \tag{1}$$

Defining

$$\alpha_n = \begin{bmatrix} s^n & s^{n-1} & \dots & s & 1 \end{bmatrix}^\top, \tag{2}$$

we then obtain an adaptive scheme where our unknowns parameters are separated into a single vector θ^*,

$$R(s)y = \begin{bmatrix} b_{n-1} & b_{n-2} & \dots & b_1 & b_0 \end{bmatrix} \alpha_{n-1}u = {\theta^*}^\top \alpha_{n-1}u. \tag{3}$$

However, we notice how the right hand side in this expression is not realizable. To ensure it is realizable, we filter it by deviding by a Hurwitz polinomial on both sides. We define

$$\Lambda(s) = \begin{bmatrix} \lambda_n & \lambda_{n-1} & \dots & \lambda_1 & \lambda_0 \end{bmatrix} \alpha_n = \lambda\alpha_n \tag{4}$$

and obtain

$$\frac{R(s)}{\Lambda(s)}y = \theta^{*\top}\frac{\alpha_{n-1}u}{\Lambda(s)}. \tag{5}$$

Letting

$$\lambda = \begin{bmatrix} 1 & \alpha_{n-1} & \dots & a_1 & a_0 \end{bmatrix}, \tag{6}$$

we get

$$z = {\theta^*}^\top \phi \tag{7}$$

where $z = y$ and $\phi = \frac{\alpha_{n-1}u}{\Lambda(s)} = \frac{\alpha_{n-1}}{R(s)}$ are known signals and θ^* is our unknown parameter vector. I realize now that could have been deduced directly from (1).

b) With Z(s) known

Starting from (1) to

$$R(s)y = Z(s)u \tag{8}$$

and rewriting the lhs yields

$$R(s)y = (s^n + \begin{bmatrix} a_{n-1} & a_{n-2} & \dots & a_1 a_0 \end{bmatrix} \alpha_{n-1})y = (s^n + \theta^{*\top}\alpha_{n-1})y = Z(s)u. \tag{9}$$

Then we write

$$\theta^{*\top}\alpha_{n-1}y = Z(s)u - s^n y \tag{10}$$

We filter both sides to make the rhs realizable. Defining $\Lambda(s)$ as a Hurwitz polynomial of degree n, i.e.

$$\Lambda(s) = \lambda\alpha_n \tag{11}$$

we obtain

$$z = \theta^{*\top}\phi \tag{12}$$

with known signals $z = \frac{Z(s)u - s^n y}{\Lambda(s)}$, $\phi = \frac{\alpha_{n-1}y}{\Lambda(s)}$ and unknown parameter vector θ^*.

Problem 2

We have

$$\nabla J(\theta) = -\phi\frac{2(z - \theta^\top\phi}{2m^2} = 0 \tag{13}$$

which implies

$$\begin{aligned}
& \phi\theta^\top\phi = \phi z \\
\implies\quad & \phi^\top\phi\theta^\top\phi = \phi^\top\phi z \\
\implies\quad & \phi^\top\phi\phi^\top\theta = \phi^\top\phi z \\
\implies\quad & \phi^\top\theta\phi^\top\phi = \phi^\top\phi z \\
\implies\quad & \phi^\top\theta = \frac{\phi^\top\phi z}{\phi^\top\phi} \\
\implies\quad & \theta(t) = \frac{\phi z}{\phi^\top\phi}.
\end{aligned} \tag{14}$$

Problem 3

We will show that $\omega_0 = F\omega$, where $F \in \mathcal{R}^{m\times n}$ with $m \leq n$ is a constant matrix and $\omega \in \mathcal{L}_\infty$ is PE, is PE iff. F has rank m. From the definition of PE from I&S (4.3.39), we have

$$\begin{aligned}
& \alpha_1 I \geq \frac{1}{T_0}\int_t^{t+T_0} \omega_0(\tau)\omega_0^\top(\tau)d\tau \geq \alpha_0 I, \quad \forall t \geq 0 \\
\iff\ & \alpha_1 I \geq \frac{1}{T_0}\int_t^{t+T_0} F\omega(\tau)(F\omega)^\top(\tau)d\tau \geq \alpha_0 I, \quad \forall t \geq 0 \\
\iff\ & \alpha_1 I \geq \frac{1}{T_0}\int_t^{t+T_0} F\omega(\tau)\omega^\top(\tau)F^\top d\tau \geq \alpha_0 I, \quad \forall t \geq 0
\end{aligned} \tag{15}$$

Because f is a constant matrix, we can put it outside the integral, obtaining

$$\alpha_1 I \geq F\left(\frac{1}{T_0}\int_t^{t+T_0} \omega(\tau)\omega^\top(\tau)d\tau\right)F^\top \geq \alpha_0 I, \quad \forall t \geq 0. \tag{16}$$

Furthermore, since ω is PE,

$$\beta_1 I \geq \frac{1}{T_0}\int_t^{t+T_0} \omega(\tau)\omega^\top(\tau)d\tau \geq \beta_0 I. \tag{17}$$

We call

$$G = \frac{1}{T_0}\int_t^{t+T_0} \omega(\tau)\omega^\top(\tau)d\tau, \tag{18}$$

and rewrite to

$$\beta_1 \geq G \geq \beta_0. \tag{19}$$

For any matrix $M \geq 0 \iff QMQ^\top \geq 0$, we obtain

$$\beta_1 FF^\top \geq FGF^\top \geq \beta_0 FF^\top. \tag{20}$$

Now, if F is full rank, we have

$$\beta_1' I \geq FF^\top \geq \beta_0' I \tag{21}$$

for some constants $\beta_1' \geq 0$ and $\beta_0' \geq 0$, meaning that

$$\alpha_1 I \geq FGF^\top \geq \alpha_0 I \tag{22}$$

and ω_0 is PE. The result in (21) is found in the proof of Lemma 5.6.2 in I&S. ■