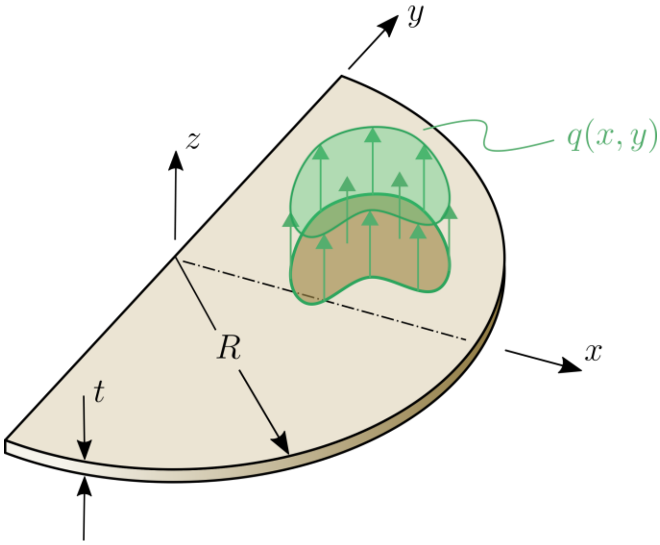
TKT4134 MECHANICS 4

EXERCISE 9

Exercise 9 deals with plates in bending from Chapter 7 of the Lecture notes. Problem 1 examines the boundary conditions and the differential equation for plates. We calculate plate moments using formulas given in the Lecture notes. Problem 2 deals with Navier's solution for simply supported rectangular plates from Chapter 7.7. In Problem 3, we revisit the completeness and compatibility criteria of the finite element method.

Problem 1



The figure shows a half-circle shaped plate with radius R and constant thickness t. The plate has plate stiffness D and Poisson's ratio ν. Assume that the plate deflection is described by the following function

$$w(x, y) = A(x^2 + y^2 - R^2)^2 x$$

The plate is loaded by a distributed load $q(x, y)$, which will be found in b).

a) Find the boundary conditions that $w(x, y)$ satisfies along the left boundary, $x = 0$, and the curved boundary, $r = R$.

Hint: The different plate supports are given in Figure 7-9 of the Lecture notes. The curved boundary is described using polar coordinates, by making the substitutions

$$x = r\sin\theta\,, \qquad y = r\cos\theta\,, \qquad r^2 = x^2 + y^2$$

Solution indication: Along curved boundary, $r = R$, the plate is clamped.

Solution: Along the boundary $x = 0$:

$$w(x = 0) = 0$$

$$\left.\begin{matrix} w,_{xx}(x = 0) = 0 \\ w,_{yy}(x = 0) = 0 \end{matrix}\right\} \Rightarrow M_x(x = 0) = 0$$

The edge $x = 0$ is simply supported as the displacement and bending moment are both zero.

The curved edge, $r = R$, is described using polar coordinates. We substitute $x = r\sin\theta$ and $y = r\cos\theta$:

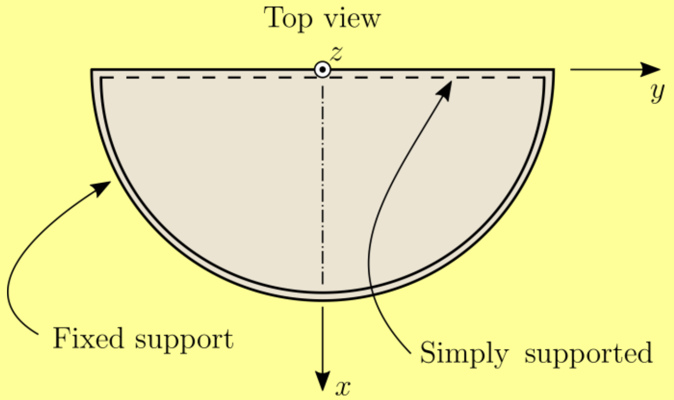
$$w(r,\theta) = A(r^2 - R^2)^2 r\sin\theta$$

Along the boundary $r = R$:

$$w(r = R) = 0$$

$$w_{,r} = 4A(r^2 - R^2)r^2\sin\theta + A(r^2 - R^2)^2\sin\theta \Longrightarrow w_{,r}\,(r = R) = 0$$

No displacement and no rotation along the curved edge yield a clamped (fixed) support. An illustration of the plate with its boundary conditions is given in the figure below.



b) You are given that $q(x = R, 0) = -q_0$. Insert the given function $w(x, y)$ into the differential equation for the plate and find the constant A based on this information. What is the mathematic expression for $q(x, y)$? Make a sketch of the function $q(x, y)$.

Hint: The differential equation for plates reads (see Chapter 7.5 of the Lecture notes)

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D}$$

Solution indication:

$$A = -\frac{q_0}{192RD}$$

Solution: Partial derivatives of the displacement function

$$w = A(x^2+y^2-R^2)^2 x$$

$$w_{,x} = 4Ax^2(x^2+y^2-R^2) + A(x^2+y^2-R^2)^2 \qquad w_{,y} = 4Axy(x^2+y^2-R^2)$$

$$w_{,xx} = 4Ax(5x^2+3y^2-3R^2) \qquad w_{,yy} = 4Ax(x^2+3y^2-R^2)$$

$$w_{,xxx} = 12A(5x^2+y^2-R^2) \qquad w_{,yyy} = 24Axy$$

$$w_{,xxxx} = 120Ax \qquad w_{,xxyy} = 24Ax \qquad w_{,yyyy} = 24Ax$$

The differential equation for the plate reads

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D}$$

or in the notation used above

$$w_{,xxxx} + 2w_{,xxyy} + w_{,yyyy} = \frac{q}{D}$$

Thus

$$24Ax + 2 \cdot 24Ax + 120Ax = \frac{q}{D}$$

$$\Longrightarrow q(x,y) = 192ADx$$

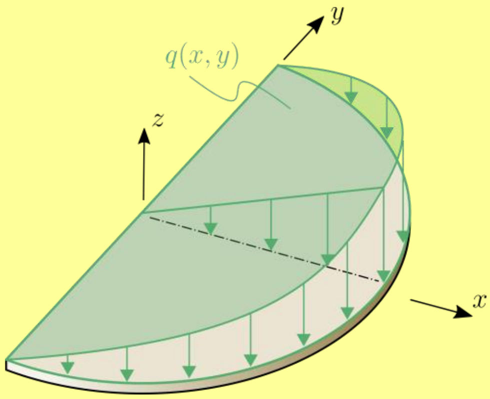
With $q(R,0) = -q_0$

$$\Longrightarrow \underline{A = -\frac{q_0}{192RD}}$$

The distributed load q reads

$$q(x,y) = -q_0 \frac{x}{R}$$

The load is sketched in the figure to the right. We have linear variation of q along the x-direction.



c) Calculate the plate moment M_x as a function of x along the line $y = 0$. Make a sketch of $M_x(x, y = 0)$.

Hint: The plate bending moment is given by (See Chapter 7.2 in the Lecture notes)

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right)$$

Solution indication:

$$M_x(x, y=0) = -\frac{q_0}{?\,R} x[(5+\nu)x^2 - (3+\nu)R^2]$$

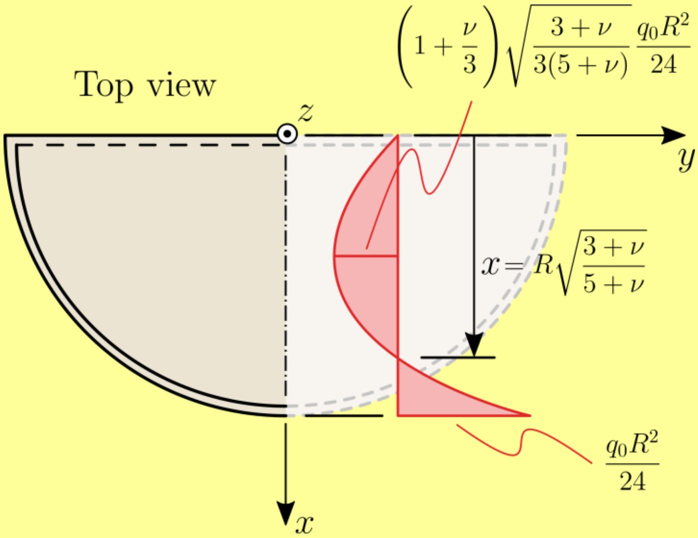
Solution: The plate bending moment

$$M_x(x,y) = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right) = -\frac{q_0}{48R}x[(5+\nu)x^2 + 3(1+\nu)y^2 - (3+\nu)R^2]$$

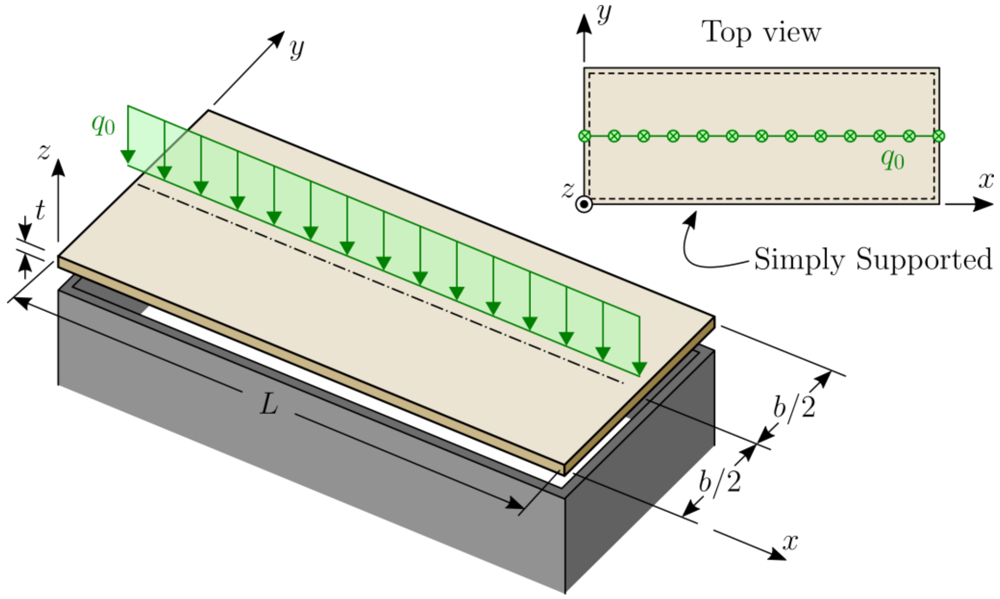
Insert $y = 0$

$$M_x(x, y=0) = -\frac{q_0}{48R}x[(5+\nu)x^2 - (3+\nu)R^2]$$

Note that $M_x(x=0) = 0$ as this is a simply supported edge, while $M_x(x=R) \neq 0$ due to the clamped support. Maximum moment occurs at $x = R$ as the Poisson's ratio is confined by $-1 < \nu < 0.5$.



Problem 2



The figure shows a rectangular simply supported steel plate subjected to a line load with constant intensity $q_0 = -20$ N/mm (downwards). The geometry of the plate and the position of external load are shown in the figure. You are given that

$L = 4000$ mm
$t = 10$ mm
$b = 1000$ mm

Steel has a Young's modulus $E = 210\,000$ N/mm^2 and Poisson's ratio $\nu = 0.3$.

a) We shall use Navier's solution to solve this problem. Show that the coefficients q_{mn} in the double Fourier sine series of the load are given by

$$q_{mn} = \frac{4q_0}{m\pi b} \sin\left(\frac{n\pi}{2}\right) [1 - \cos(m\pi)]$$

Hint: The coefficients q_{mn} in the double Fourier sine series of the load are given in Chapter 7.7 of the Lecture notes as

$$q_{mn} = \frac{4}{bL} \int_0^L \int_0^b q(x,y) \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi y}{b}\right) \mathrm{d}y\, \mathrm{d}x$$

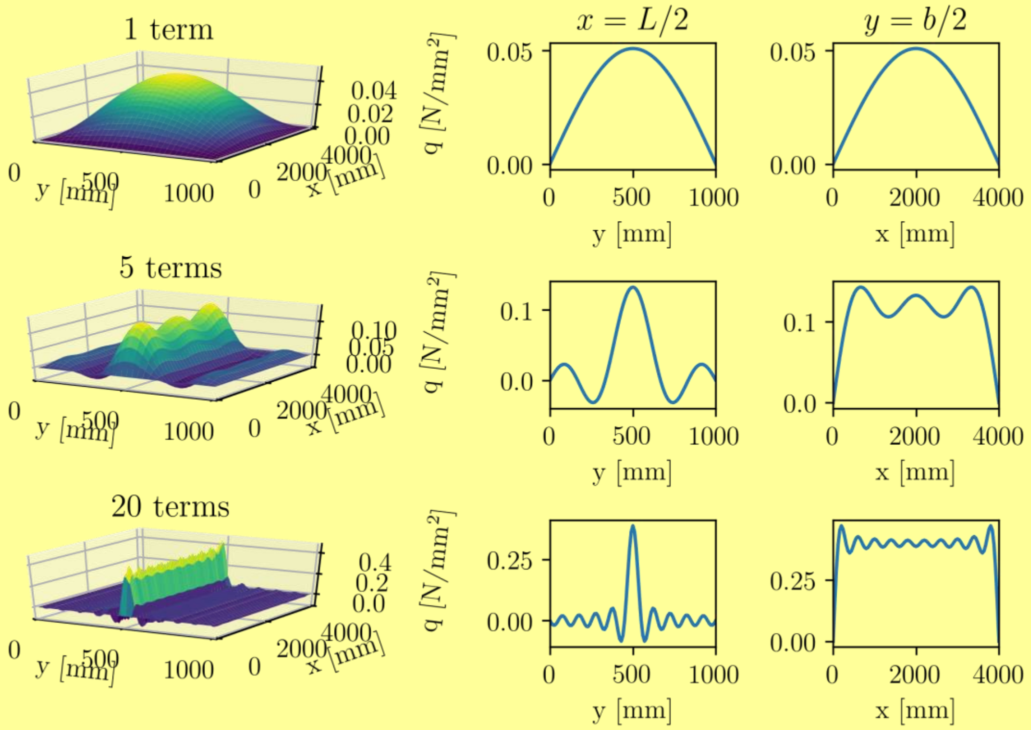
Solution: As $q\left(x, y \neq \frac{b}{2}\right) = 0$, we can simplify the expression as

$$q_{mn} = \frac{4q_0}{bL} \sin\left(\frac{n\pi}{2}\right) \int_0^L \sin\left(\frac{m\pi x}{L}\right) \mathrm{d}x = \frac{4q_0}{bL} \sin\left(\frac{n\pi}{2}\right) \left[-\frac{L}{m\pi} \cos\left(\frac{m\pi x}{L}\right)\right]_0^L$$
$$= \frac{4q_0}{m\pi b} \sin\left(\frac{n\pi}{2}\right) [1 - \cos(m\pi)]$$

Bonus: In Navier's solution, we approximate the load as a double Fourier sine series according to

$$q(x,y) = \sum_{m=1}^{M}\sum_{n=1}^{N} q_{nm} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi y}{b}\right)$$

The plots under shows $q(x,y)$ [N/mm^2] using 1, 5 and 20 terms. As the number of terms increases, the approximated load approaches the correct load. Note how the intensity of $q(x,y)$ increases as the load becomes more concentrated at the centre line.



b) Find the deflection $w\left(\frac{L}{2},\frac{b}{2}\right)$ of the midpoint of the plate. Use Navier's solution with a single term (i.e., $M = N = 1$).

Hint: The displacement field is defined as

$$w(x,y) = \sum_{m=1}^{M}\sum_{n=1}^{N} w_{mn} \sin\left(\frac{m\pi x}{L}\right) \sin\left(\frac{n\pi y}{b}\right)$$

with displacement coefficients

$$w_{mn} = \frac{q_{mn}}{D\left(\frac{m^2\pi^2}{L^2} + \frac{n^2\pi^2}{b^2}\right)^2}, \qquad D = \frac{Et^3}{12(1-\nu^2)}$$

Solution indication:

$$w = -24.08 \text{ mm}$$

Solution: The first term in Navier's solution reads

$$q_{11} = \frac{4q_0}{1 \cdot \pi b} \sin\left(\frac{1 \cdot \pi}{2}\right) [1 - \cos(1 \cdot \pi)] = \frac{8q_0}{\pi b}$$

$$w_{11} = \frac{q_{11}}{D\left(\frac{1^2\pi^2}{L^2} + \frac{1^2\pi^2}{b^2}\right)^2} = \frac{8q_0 b^3 L^4}{\pi^5 D (L^2 + b^2)^2}$$

With $M = N = 1$

$$w(x, y) = w_{11} \sin\left(\frac{\pi x}{L}\right) \sin\left(\frac{\pi y}{b}\right) = \frac{8q_0 b^3 L^4}{\pi^5 D (L^2 + b^2)^2} \sin\left(\frac{\pi x}{L}\right) \sin\left(\frac{\pi y}{b}\right)$$

Calculating the deflection in the centre $(x, y) = \left(\frac{L}{2}, \frac{b}{2}\right)$

$$w\left(\frac{L}{2}, \frac{b}{2}\right) = \frac{8q_0 b^3 L^4}{\pi^5 D (L^2 + b^2)^2} \sin\left(\frac{\pi}{2}\right) \sin\left(\frac{\pi}{2}\right) = \frac{8q_0 b^3 L^4}{\pi^5 D (L^2 + b^2)^2}$$

Inserting the numerical values

$$w\left(\frac{L}{2}, \frac{b}{2}\right) = \frac{8 \cdot \left(-20 \frac{\mathrm{N}}{\mathrm{mm}}\right) \cdot 1000^3 \cdot 4000^4\ \mathrm{mm}^7}{\pi^5 \cdot 19.23 \cdot 10^6\ \mathrm{Nmm} \cdot (4000^2 + 1000^2)^2\ \mathrm{mm}^4} = \underline{-24.08\ \mathrm{mm}}$$

where

$$D = \frac{Et^3}{12(1 - \nu^2)} = \frac{210000 \frac{\mathrm{N}}{\mathrm{mm}^2} \cdot 1000\ \mathrm{mm}^3}{12(1 - 0.3^2)} = 19.23 \cdot 10^6\ \mathrm{Nmm}$$

c) Calculate the maximal normal stresses σ_x and σ_y, as well as the maximal shear stress τ_{xy}. Use a single term in Navier's solution.

Hint: The stresses in plates are given in Chapter 7.2 of the Lecture notes, e.g.,

$$\sigma_x = -\frac{zE}{1 - \nu^2} \left(w_{,xx} + \nu w_{,yy}\right)$$

Where will the maximal stresses occur? Are there any similarities with a simply supported beam?

Solution indication:

$$\sigma_{y,\max} = 279.4\ \mathrm{N/mm^2}$$

Solution: Partial derivatives

$$w(x,y) = A\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{\pi y}{b}\right) \quad \text{with} \quad A = \frac{8q_0 b^3 L^4}{\pi^5 D(L^2+b^2)^2} = -24.08 \text{ mm}$$

$$w_{,x} = \frac{\pi}{L}A\cos\left(\frac{\pi x}{L}\right)\sin\left(\frac{\pi y}{b}\right) \qquad w_{,y} = \frac{\pi}{b}A\sin\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi y}{b}\right)$$

$$w_{,xx} = -\left(\frac{\pi}{L}\right)^2 A\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{\pi y}{b}\right) \qquad w_{,yy} = -\left(\frac{\pi}{b}\right)^2 A\sin\left(\frac{\pi x}{L}\right)\sin\left(\frac{\pi y}{b}\right)$$

$$w_{,xy} = w_{,yx} = \frac{\pi}{L}\frac{\pi}{b}A\cos\left(\frac{\pi x}{L}\right)\cos\left(\frac{\pi y}{b}\right)$$

Maximal normal stresses (in tension) occur where $\sin\left(\frac{\pi x}{L}\right) = \sin\left(\frac{\pi y}{b}\right) = 1$, i.e., in the centre $(x,y) = \left(\frac{L}{2},\frac{b}{2}\right)$ at the bottom of the plate (i.e., $z = -\frac{t}{2}$)

$$\sigma_{x,\max} = \sigma_x\left(\frac{L}{2},\frac{b}{2},-\frac{t}{2}\right) = -A\cdot\frac{t}{2}\frac{E}{1-\nu^2}\left(\left(\frac{\pi}{L}\right)^2 + \nu\left(\frac{\pi}{b}\right)^2\right) = \underline{99.4 \text{ N/mm}^2}$$

$$\sigma_{y,\max} = \sigma_y\left(\frac{L}{2},\frac{b}{2},-\frac{t}{2}\right) = -A\cdot\frac{t}{2}\frac{E}{1-\nu^2}\left(\left(\frac{\pi}{b}\right)^2 + \nu\left(\frac{\pi}{L}\right)^2\right) = \underline{279.4 \text{ N/mm}^2}$$

Maximal shear stress τ_{xy} occurs at the boundaries where $\cos\left(\frac{\pi x}{L}\right) = \cos\left(\frac{\pi y}{b}\right) = 1$. Thus

$$\tau_{xy,\max} = \tau_{xy}\left(0,0,\frac{t}{2}\right) = -\frac{E}{1-\nu^2}(1-\nu)\frac{\pi^2 t}{2bL}A = \underline{48.0 \text{ N/mm}^2}$$

d) Use Python (or your preferred numerical tool) to solve b) and c) with more than one term. How many terms are needed in the Navier's solution to get a converged result (or at least have a sufficient accuracy)?

Hint: The notebook "Exercise 9 – Problem 2d" is uploaded to Jupyter Hub, which also has some hints on how to solve the problem.

Solution indication: Displacement is −21.3362 mm when 40 terms are used.

Solution: Python solution

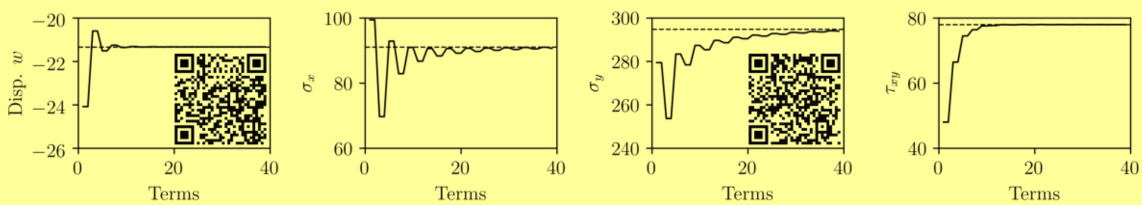
```python
# import sin, cos and pi from math library
from math import sin, cos, pi

# Define number of terms
terms = 40

# Define parameters
L  = 4000    # [mm]
t  = 10      # [mm]
b  = 1000    # [mm]
E  = 210000 # [N/mm2]
nu = 0.3
q0 = -20     # [N/mm]
```

```python
# Define coordinates
x = L/2.
y = b/2.
z = -t/2.

# Calculate plate stiffness D
D = E*t**3/(12.*(1-nu**2))

# Initialize displacement and 2. derivatives
w = 0
wxx = 0
wyy = 0
wxy = 0

# Go through all terms and calculate displacment and 2. derivatives
for m in range(1, terms+1):
    for n in range(1, terms+1):
        # Find load contribution qmn
        qmn = 4.*q0/(m*pi*b)*sin(n*pi/2.)*(1-cos(m*pi))
        # Find displacement coefficients wmn
        wmn = qmn/(D*(m**2*pi**2/L**2 + n**2*pi**2/b**2)**2)

        # Find displacment in given coordinate
        w   = w + wmn*sin(m*pi*x/L)*sin(n*pi*y/b)
        # Find derivatives in given coordinate
        wxx = wxx - wmn*(m*pi/L)**2*sin(m*pi*x/L)*sin(n*pi*y/b)
        wyy = wyy - wmn*(n*pi/b)**2*sin(m*pi*x/L)*sin(n*pi*y/b)
        wxy = wxy + wmn*(m*pi/L)*(n*pi/b)*cos(m*pi*x/L)*cos(n*pi*y/b)

# Find stresses
sigmax = -z*E/(1-nu**2)*(wxx+nu*wyy)
sigmay = -z*E/(1-nu**2)*(wyy+nu*wxx)
tauxy  = -z*E*(1-nu)/(1-nu**2)*wxy

# Print result
print('Displacement: ' + str(round(w,2)) + ' mm')
print('Sigmax: '       + str(round(sigmax,2)) + ' MPa')
print('Sigmay: '       + str(round(sigmay,2)) + ' MPa')
print('Tauxy: '        + str(round(tauxy,2)) + ' MPa')
```

Output using 40 terms:

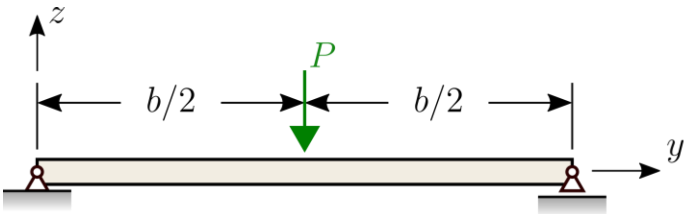
```
Displacement: -21.34 mm
Sigmax: 90.63 MPa
Sigmay: 293.91 MPa
Tauxy: 77.94 MPa (for x=y=0 and z=t/2)
```

The number of terms needed can be checked by a trial-and-error method. We have here instead plotted the response for an increasing number of terms. The horizontal line is the correct solution (obtained by using 100 terms).



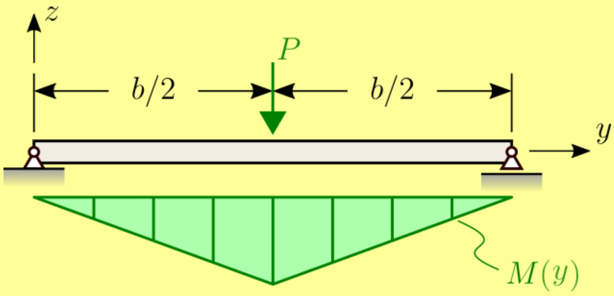
The displacement converges fast, and a proper solution is obtained with about 5 terms (error of 0.84% compared to the correct solution). The normal stresses σ_x and σ_y converge somewhat slower because they are obtained by differentiating the displacement field twice. However, the solution is still good with 5 terms (error of 3.87% and 1.92% compared to the correct solution, respectively). Lastly, notice how all even- number terms do not influence the solution. This is because both the plate and load are symmetric.

e) As the width b of the plate is rather small compared to the length L, it might be sufficient to handle the plate as a simply supported beam subjected to a point load $P = q_0 L = 80$ kN within reasonable accuracy. Use elementary beam theory and calculate the maximal deflection and bending stress. Comment on the result.



Solution: Second areal moment

$$I = \frac{1}{12} \cdot 4000 \cdot 10^3 = 333\,333.33 \text{ mm}^4$$



Maximum deflection at $y = \frac{b}{2}$

$$w_{\max} = w\left(\frac{L}{2}\right) = -\frac{Pb^3}{48EI} = -23.81 \text{ mm}$$

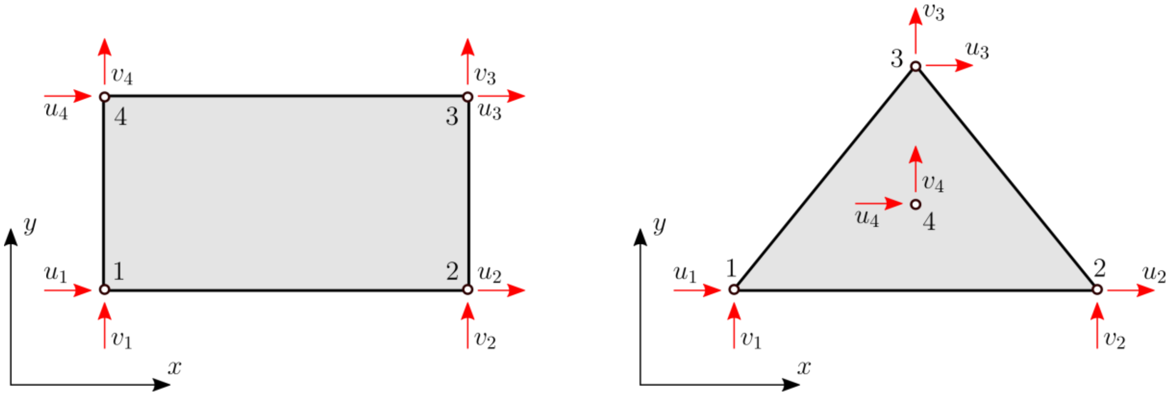
Maximum stress at $y = \frac{b}{2}$ in the outer material fibre

$$\sigma_{y,\max} = \frac{M_{\max}}{I}\frac{t}{2} = \frac{Pb}{4}\frac{t}{2I} = 300\,\frac{\text{N}}{\text{mm}^2}$$

Comments:

- The stiffness in x-direction is neglected. The displacement becomes too large.
- In Example 7-1 of the Lecture notes, we have defined the modified Young's modulus $\bar{E} = \frac{E}{1-\nu^2}$ to account for an increase in stiffness due to the assumption of a plane strain state which applies for one-way slabs (norsk: enveisplater). With this adjustment, the deflection at $x = \frac{b}{2}$ becomes $w_{\max} = -21.67$ mm, i.e., an error of only 1.5% compared to the correct solution in d).
- In summary, the beam theory is a good approximation when $b \ll L$.

Problem 3



The figure above shows two plane elements: one triangular and one rectangular. Both elements have four nodes i, and the two degrees of freedom u_i and v_i in each node.

Assume that the displacements within the two elements are interpolated by generalized degrees of freedom

$$\mathbf{q}_{\mathrm{e}} = \begin{bmatrix} \mathbf{q}_x \\ \mathbf{q}_y \end{bmatrix}$$

and the following interpolation functions

$$u = \begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} 1 & x & y & xy & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & x & y & xy \end{bmatrix} \begin{bmatrix} \mathbf{q}_x \\ \mathbf{q}_y \end{bmatrix} = \begin{bmatrix} \mathbf{N}_{\mathrm{q}} & \mathbf{0} \\ \mathbf{0} & \mathbf{N}_{\mathrm{q}} \end{bmatrix} \begin{bmatrix} \mathbf{q}_x \\ \mathbf{q}_y \end{bmatrix} = \mathbf{N}_{\mathrm{e}} \mathbf{q}_{\mathrm{e}}$$

Evaluate if the two elements satisfy the *completeness* and *compatibility criteria*.

Hint: The *completeness* and *compatibility criteria* are discussed in Chapter 3.4 of the Lecture notes. To check the *compatibility criterion,* you should evaluate the displacement along the element edges. Are you able to describe the edge displacements with the available degrees of freedom?

Note that in the case of a triangle, at least one of the sides is slanted. In general, an element edge can therefore be expressed by $y = Ax + B$.

Solution: The *completeness criterion* states that the element must be able to describe arbitrary rigid-body motion and arbitrary constant strains.

With *compatibility*, we understand that the displacement field within an element must be continuous and sufficiently differentiable (smooth). Also, the displacement field should be compatible, i.e., vary continuously across inter-element boundaries so that no gaps or overlaps occur, and the strains remain finite.

Rectangular element

- Completeness criterion: The element can describe all rigid body motions and constant strain modes. The reason for this is that the three terms 1, x and y are included in $\mathbf{N}_0$. <u>The completeness criterion is satisfied</u>.
- Compatibility criterion: We wish that the displacement field is continuous along the inter-element boundaries. This is achieved if the displacement along a

boundary can be uniquely defined by the degrees of freedom that are shared between two neighboring elements.
We consider the displacement u along edge 1-2, where $y = \text{constant} = C$. The displacement reads

$$u(x, y = C) = 1 \cdot q_{x1} + x \cdot q_{x2} + C \cdot q_{x3} + x \cdot Cq_{x4}$$
$$= \overbrace{(q_{x1} + Cq_{x3})}^{A} + \overbrace{(q_{x2} + Cq_{x4})}^{B}\, x$$

As u varies linearly along the boundary, we can uniquely define A and B by the two degrees of freedom that are shared with the neighbouring element. The same is true for the other edges. The same reasoning is valid for displacement v.
$\Longrightarrow$ The element satisfies the compatibility criterion.

Triangular element

- Completeness criterion: Following the same explanation as for the rectangular element, we can conclude that the completeness criterion is satisfied.
- Compatibility criterion: In a triangular element, at least one of the edges is slanted, and can be expressed as $y = Ax + B$. Inserting this relation into the expression for the displacement u, we get

$$u(x, y = Ax + B) = q_{x1} + xq_{x2} + (Ax + B)q_{x3} + x(Ax + B)q_{x4}$$
$$= \underbrace{(q_{x1} + Bq_{x3})}_{C} + \underbrace{(q_{x2} + Aq_{x3} + Bq_{x4})}_{D}\, x + \underbrace{Aq_{x4}}_{E}\, x^2$$

Now u is a second-order polynomial and we cannot uniquely define the three unknown constants C, D and E with the two degrees of freedom common with the neighbouring element.
$\Longrightarrow$ The element does not satisfy the compatibility criterion.
If we had excluded the internal node 4, we would not have a problem with compatibility as the x^2 term would not appear in the expression for u.