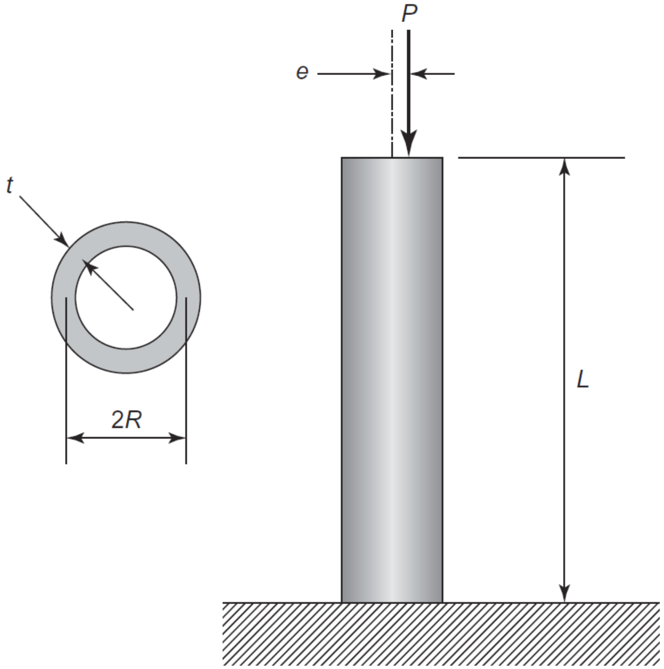
POLITECNICO DI MILANO

Finite Element Simulation For Mechanical Design

Stress stiffening and buckling

Prof. Andrea Bernasconi

Problem's data



P Load, 50 kN

L Length, 5 m

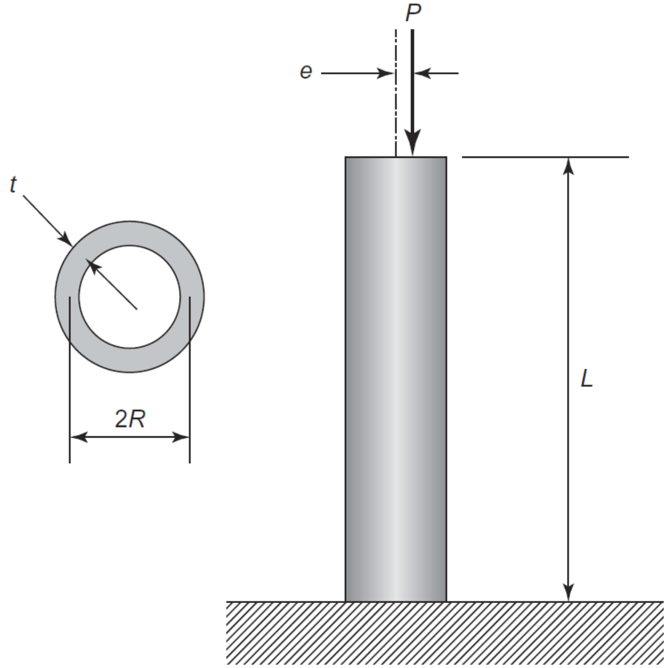
R Mean radius, 53.7 mm

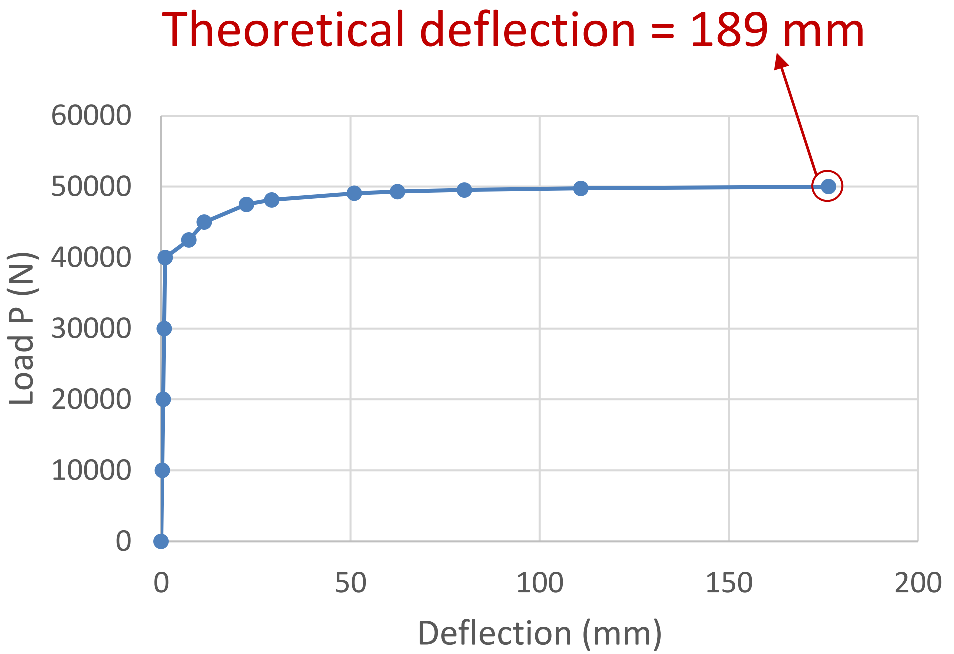
t, wall thickness, 5 mm

E Young's modulus, 210 GPa

e Eccentricity (2% of radius), 0.02R, = 1.074 mm

Introduction



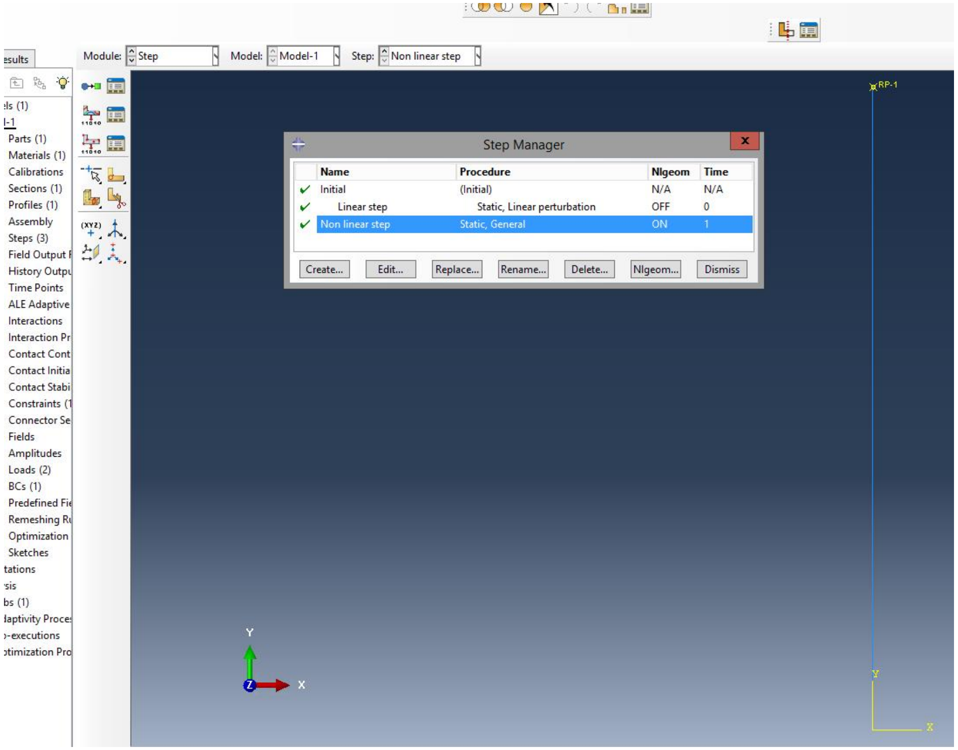


Beam length	L	5000	mm
Beam mean radius	R	53.7	mm
Wall thickness	t	5	mm
Eccentricity	e	0.02 R	
Axial load	P	50000	N

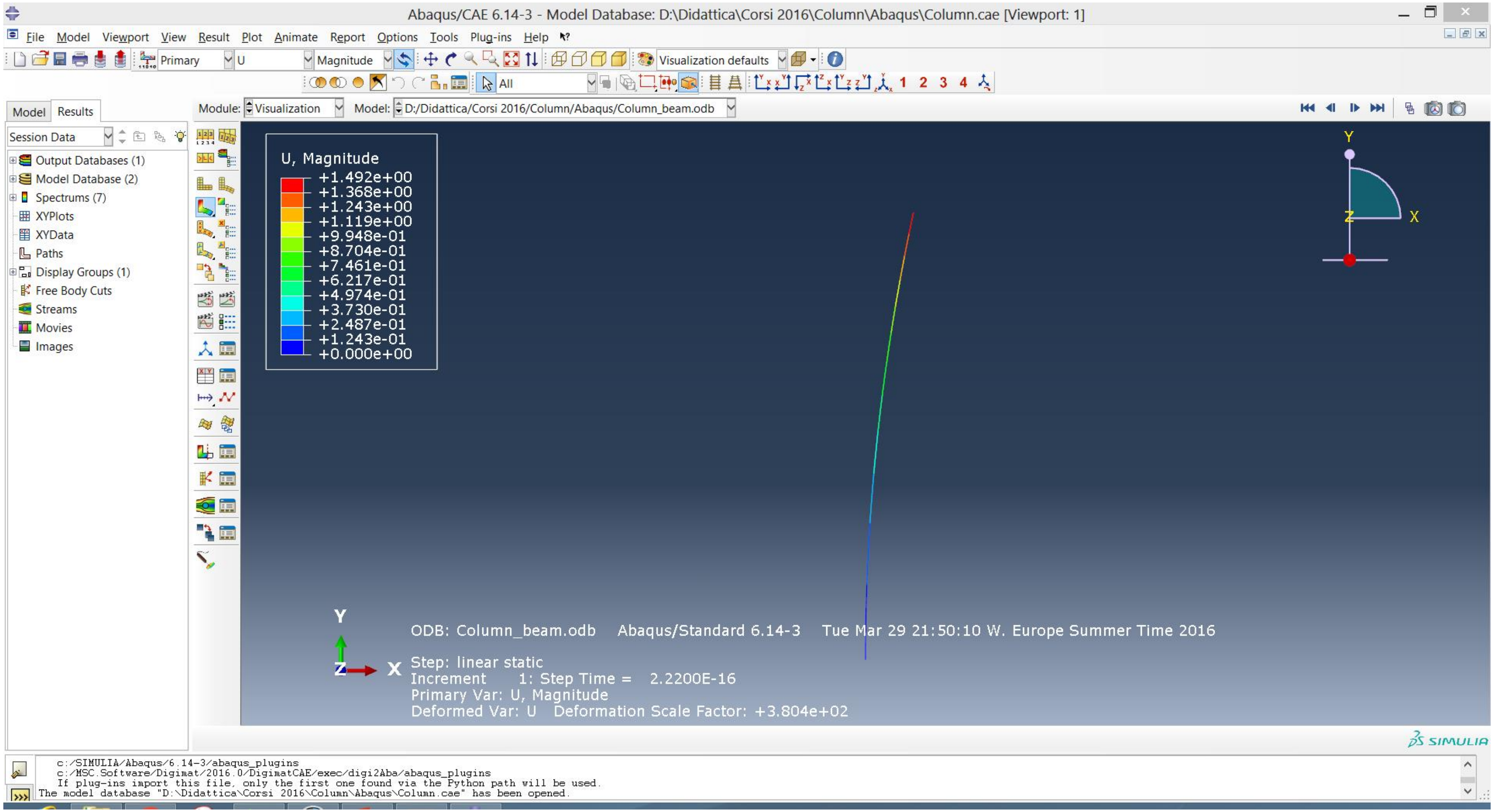
Let’s analyse a column with eccentric load using beam elements

We’ll define two steps:

- Linear static, to compare deflection with result of the non linear step
- Non linear static analysis, accounting for large displacements

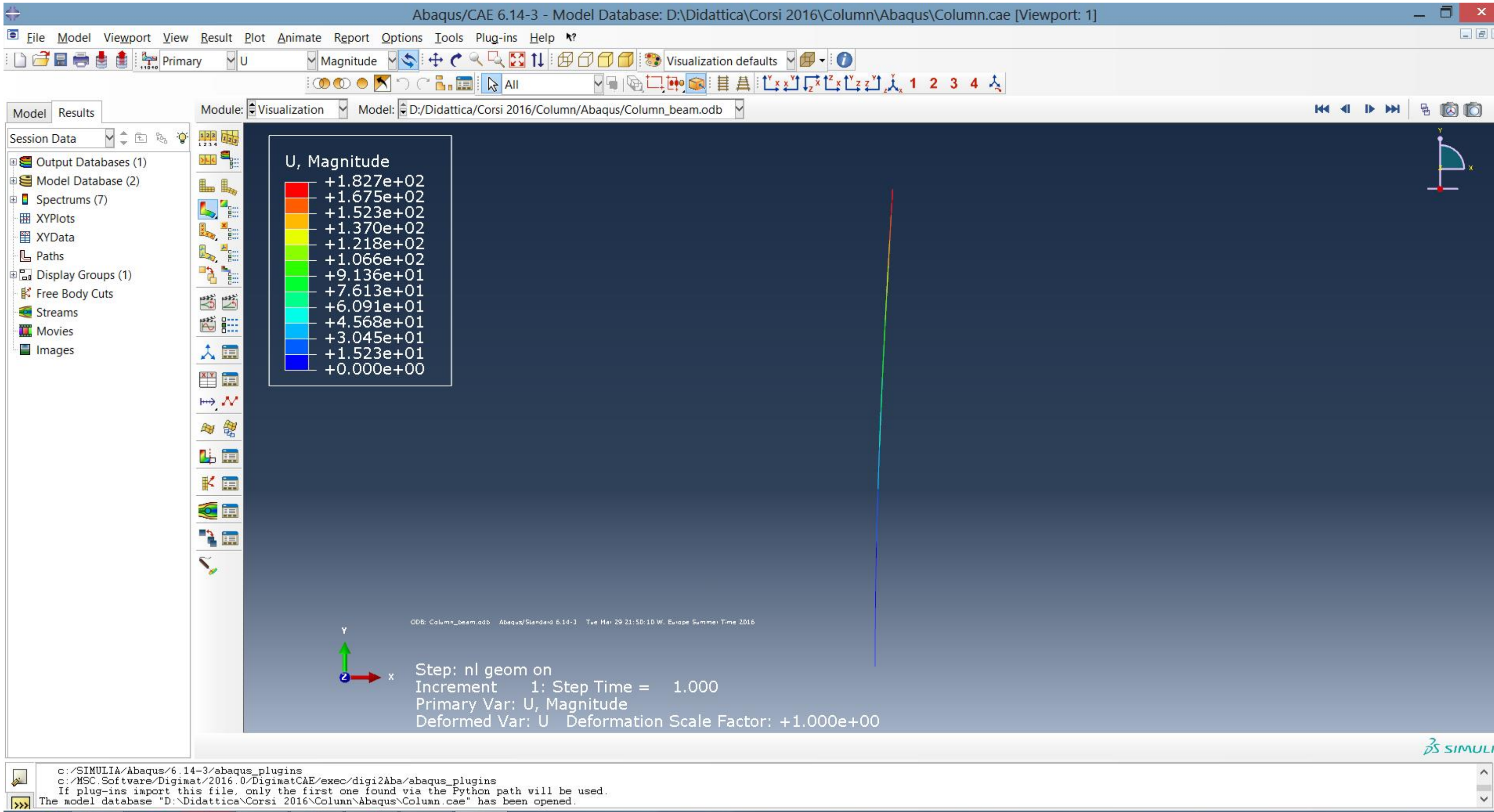


Results: linear static



Deflection is lower than that calculated analytically

Results: non-linear static

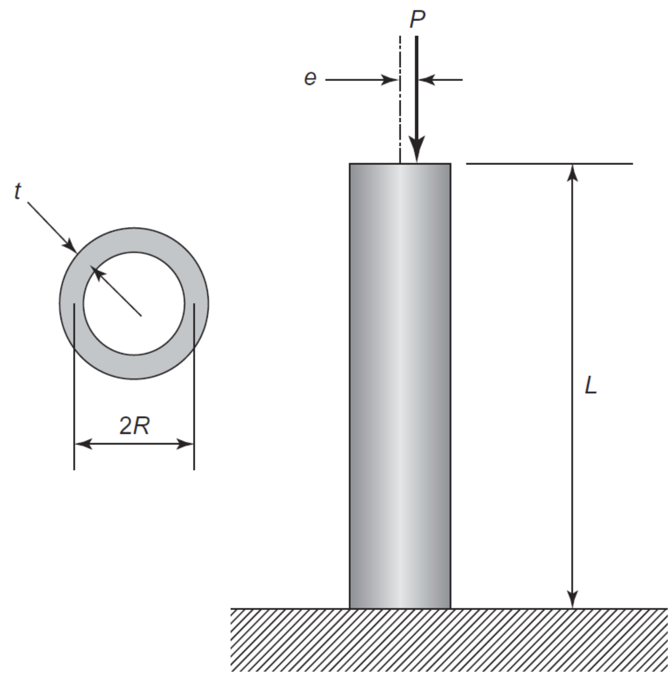


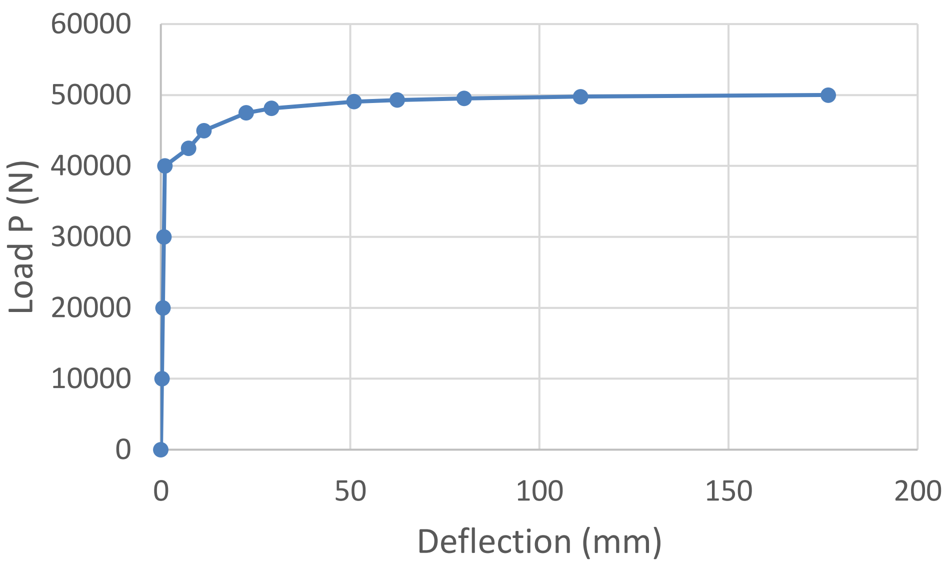
Deflection is now very close to the one obtained analytically

Introduction

The solution of the elastic problem of a slender column under compression by an eccentric load showed that:

- the deformed shape is hugely different if it is calculated assuming a linear elastic response (1.3 mm) or a non linear elastic one (182 mm, by FEA).
- the end tip deflection calculated by a non-linear analysis increases non linearly as the load approach some limit value (Euler critical load)





Beam length	L	5000	mm
Beam mean radius	R	53.7	mm
Wall thickness	t	5	mm
Eccentricity	e	0.02	R
Axial load	P	50000	N

Comments

Larger deflections than in the linear case

The linear elastic solution for deflection is 1.3 mm

The non-linear elastic solution was (FE solution with shell elements) is 182 mm.

For P = 40,000 N, the non-linear solution still almost coincides with the linear one.

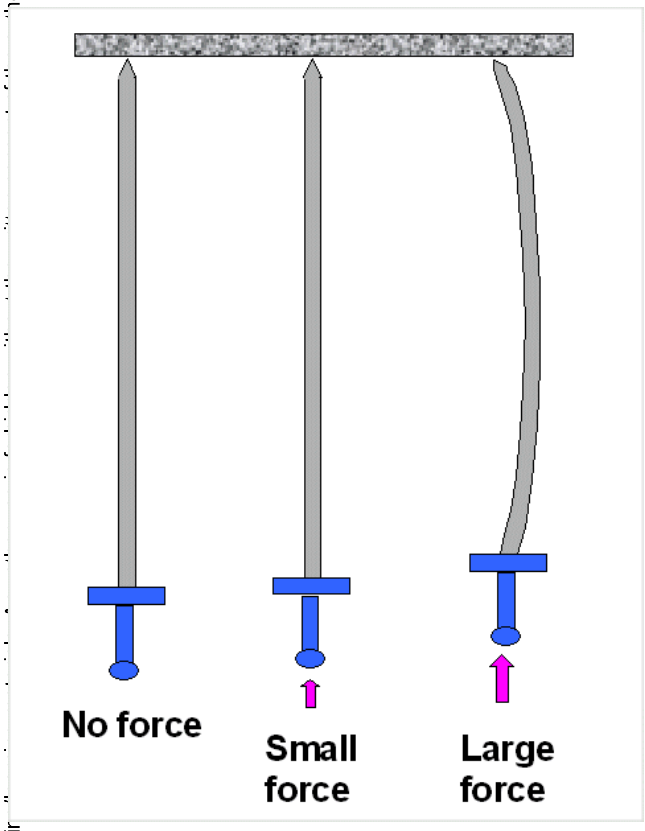
Even if the eccentricity is very small, the deflection becomes very large as P approaches Euler’s critical load between 40 kN and 50 kN.

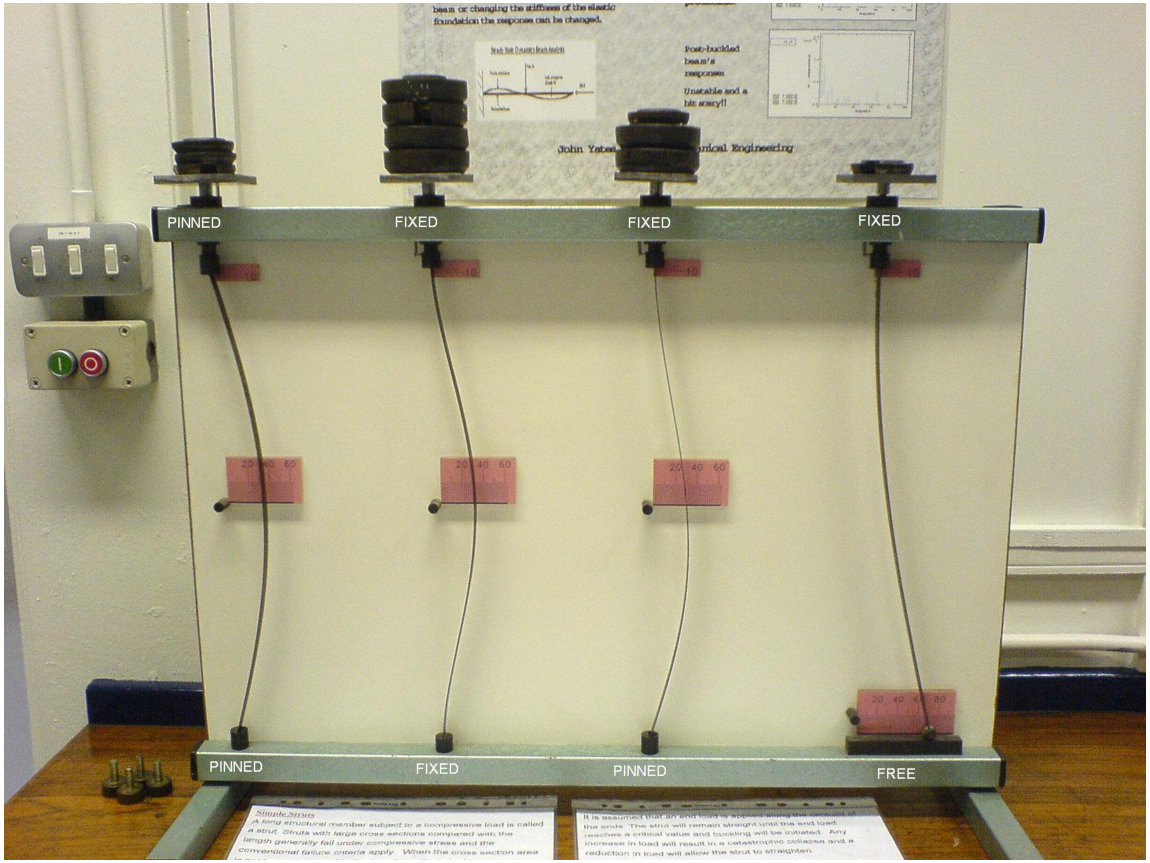
Larger deflections if defections are taken into account

Conversely, if we consider the ideal case of zero eccentricity, deflection is null.

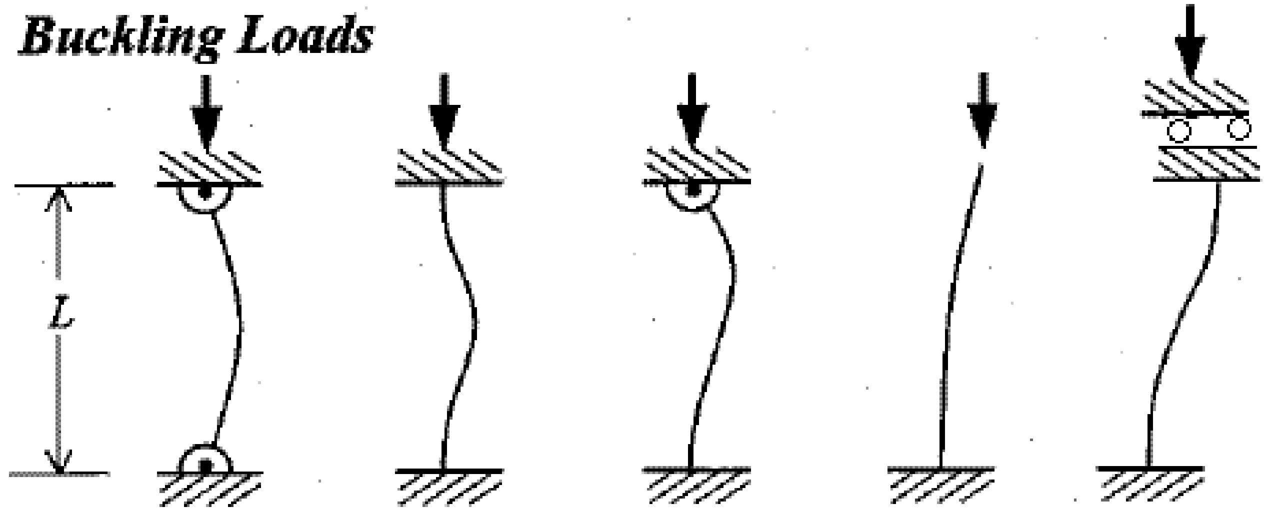
Then, if we consider a real world case and assume that uncertainty exists about the position of the load application point, we have to introduce an eccentricity: deflection (and stresses) will be small, provided that the applied load is sufficiently lower than the Euler’s critical load.

Buckling of slender beams





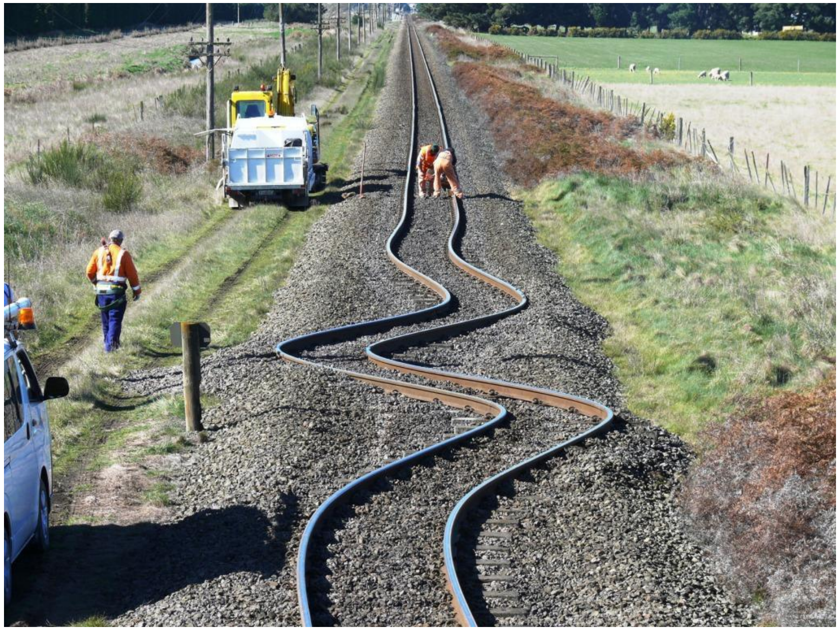
Euler formula for slender beams



Buckling Load	$\dfrac{\pi^2 EI}{L^2}$	$\dfrac{4\pi^2 EI}{L^2}$	$\dfrac{2.045\pi^2 EI}{L^2}$	$\dfrac{\pi^2 EI}{4L^2}$	$\dfrac{\pi^2 EI}{L^2}$
Effective Length	L	$0.5L$	$0.699L$	$2L$	L

Buckling of slender beams

https://construct.typepad.com/25seven/2009/01/engineering-buckling-limit-of-frame-tubing.html#more



https://blogs.agu.org/landslideblog/2010/11/02/the-canterbury-earthquake-images-of-the-distorted-railway-line/

Local buckling: web and flange buckling in girders

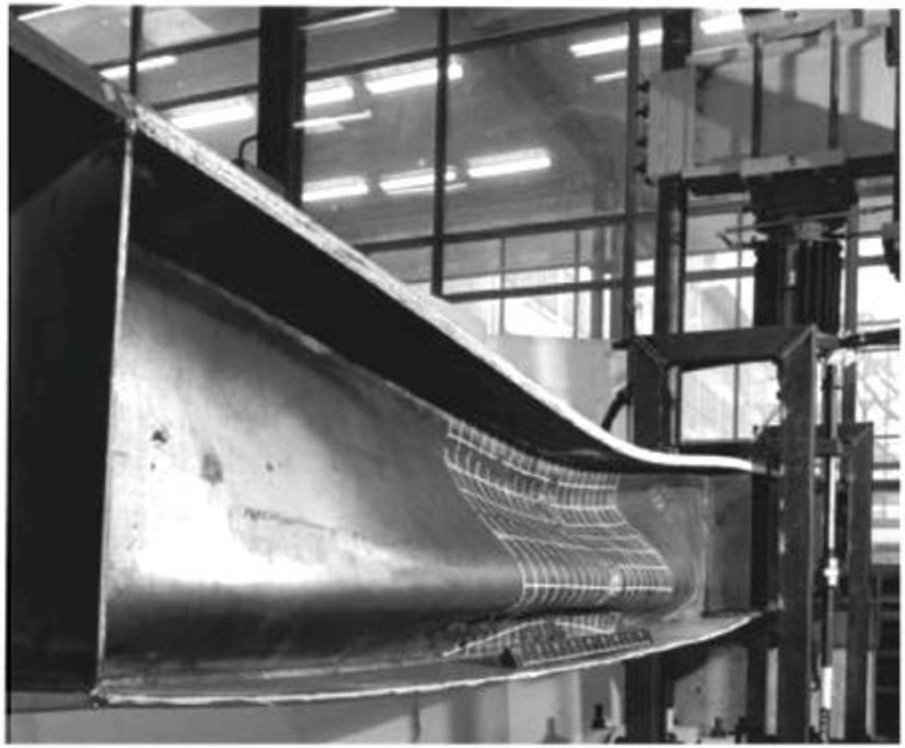
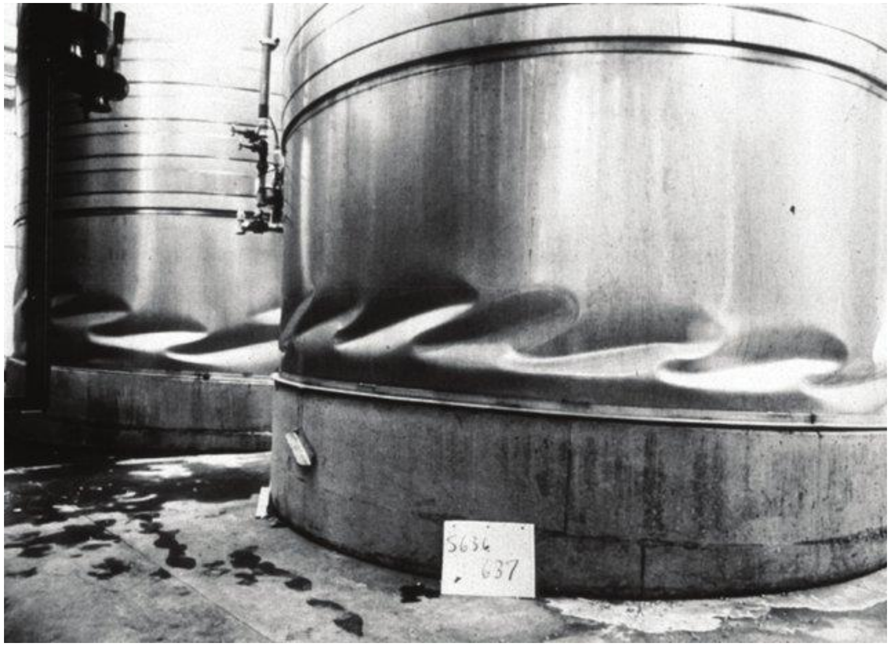


Figure 2. Vertical web buckling.

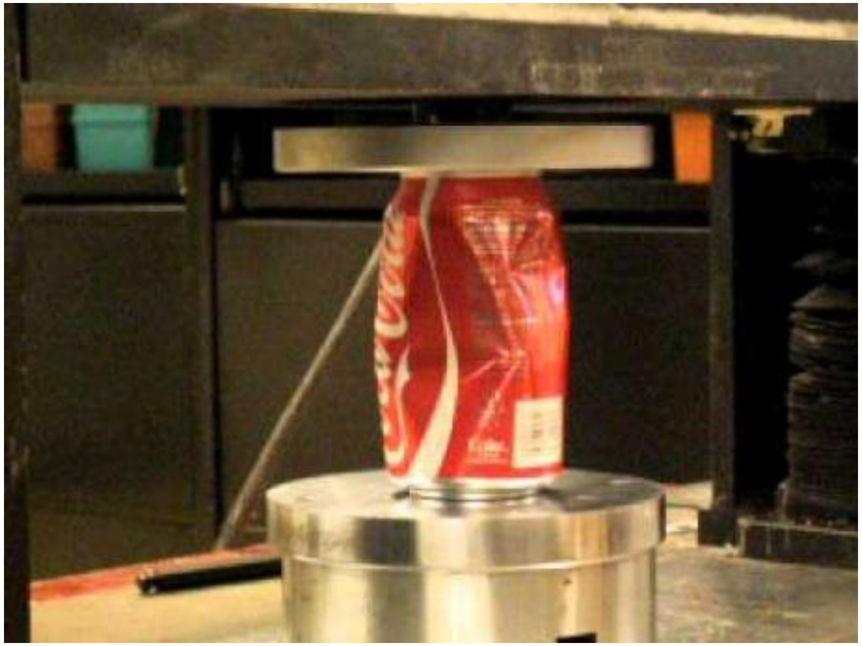
http://www.scielo.br/scielo.php?script=sci_arttext&pid=S0100-73862001000400003

http://www.bgstructuralengineering.com/BGSCM13/BGSCM006/BGSCM00603.htm

Local buckling in shells

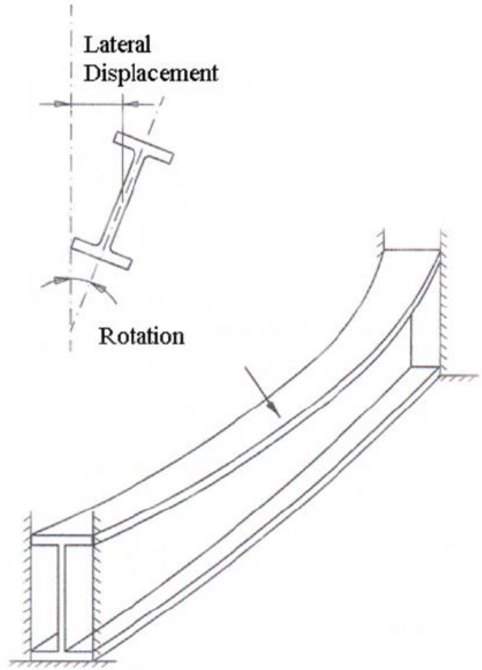


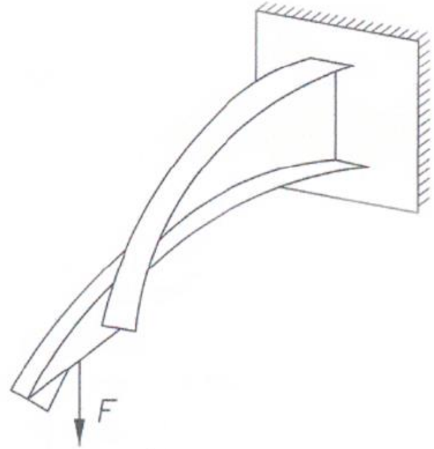
http://shellbuckling.com/buckledShells.php



https://youtu.be/3V-WGhqfRa0?t=14

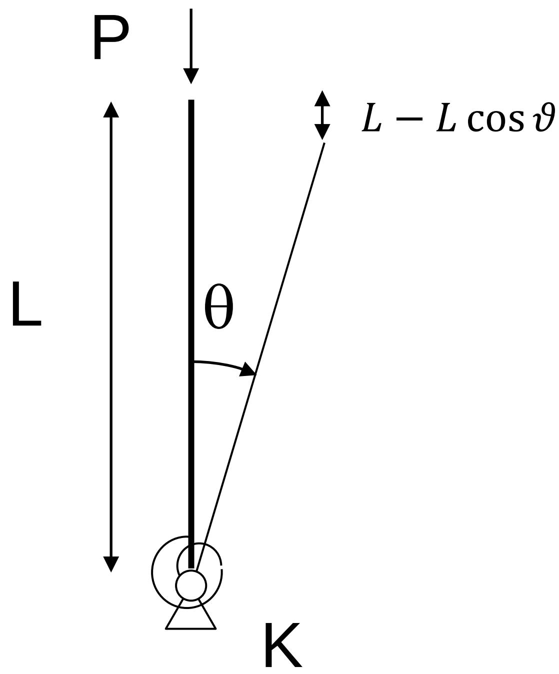
Lateral torsional buckling





Lateral Torsional Buckling of a Beam Girder

Analysis of the equilibrium of a 1 d.o.f system



Potential energy of the perturbed system

$$\Pi = \frac{1}{2}K\theta^2 - PL(1 - \cos\theta)$$

By expanding *cos(θ)* in Taylor's series, we obtain the linearized form

$$\Pi = \frac{1}{2}(K - PL)\theta^2$$

We observe that:

Π is minimum if K-PL > 0

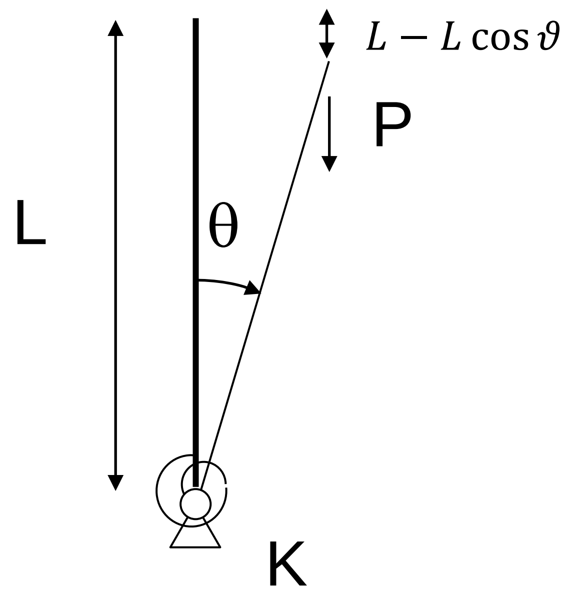
Π is not minimum if K – PL < 0

For small displacements, the only possible equilibrium configuration is $\theta = 0$

However, only when K – PL > 0 it is a stable configuration.

To study the system for K – PL < 0, we have to assume large displacements

Analysis of the equilibrium of a 1 d.o.f system



Potential energy of the perturbed system

$$\Pi = \frac{1}{2}K\theta^2 - PL(1-\cos\theta)$$

By imposing $\delta\Pi = 0$, we obtain

$$p\sin\theta - \theta = 0$$

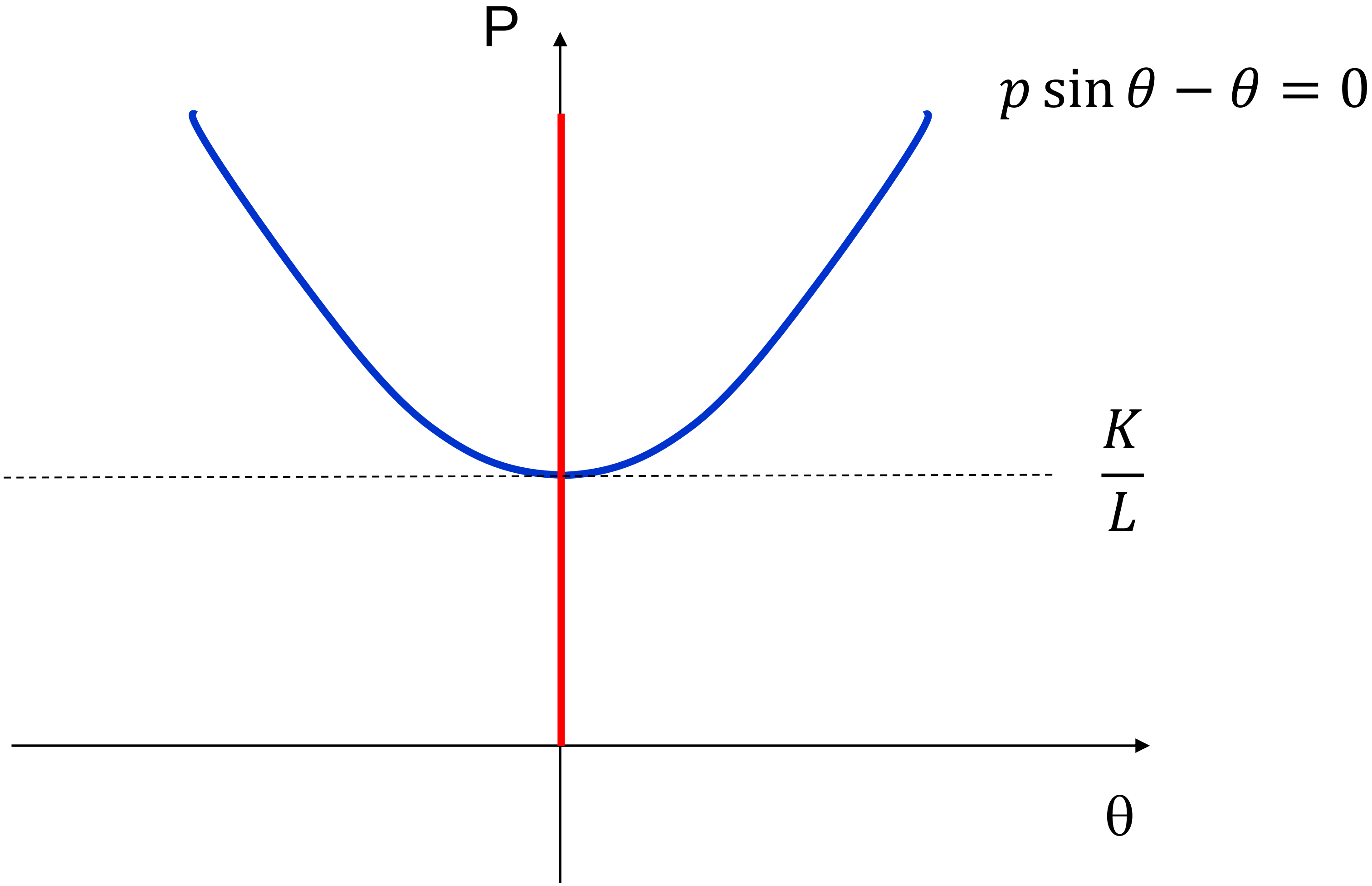
where

$$p = \frac{PL}{K}$$

For P< K/L, the only possible equilibrium configuration is θ = 0

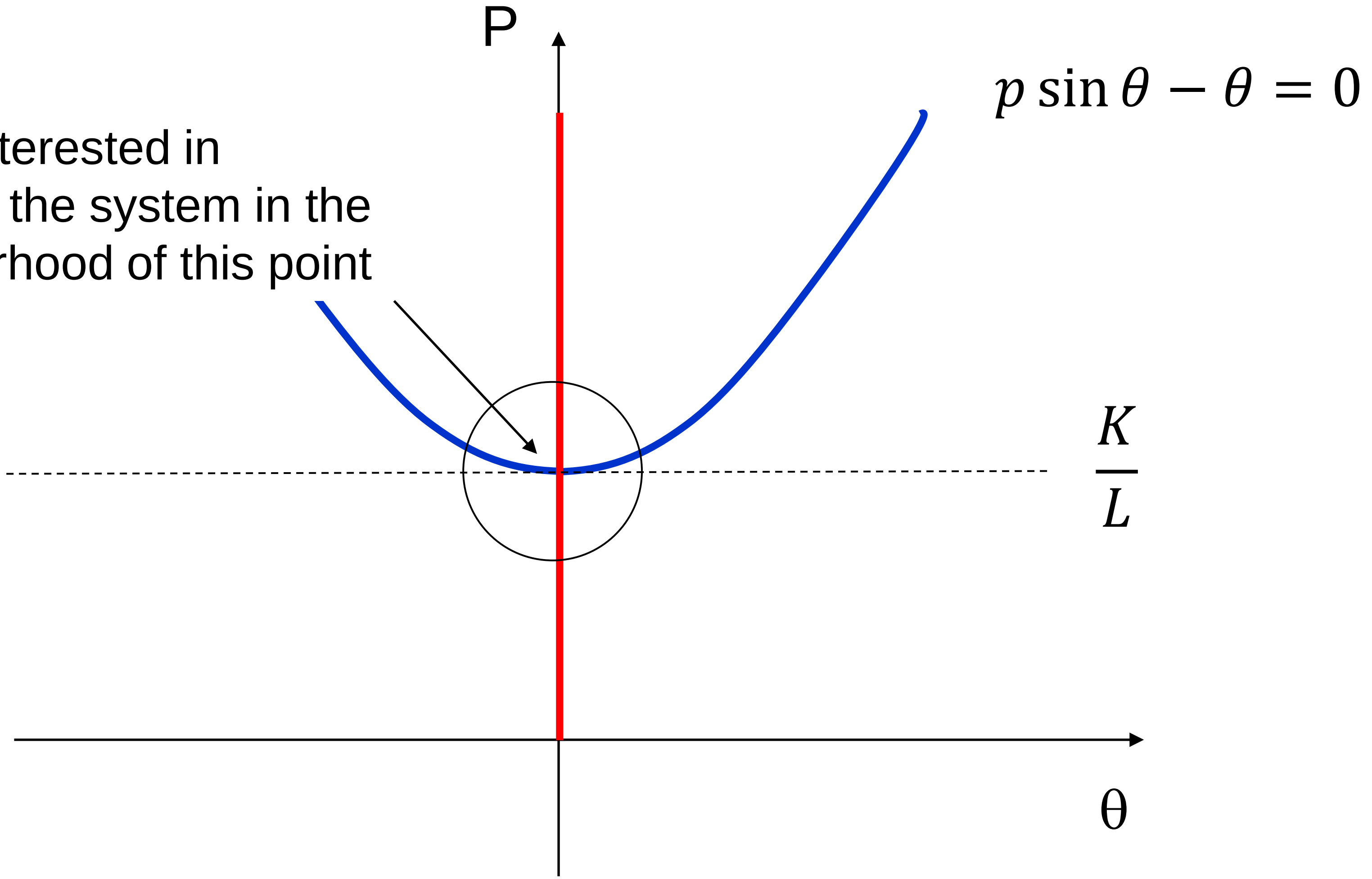
For P> K/L, besides θ = 0, another equilibrium configuration exists, given by the solution of the equation $p\sin\theta - \theta = 0$

Bifurcation of equilibrium

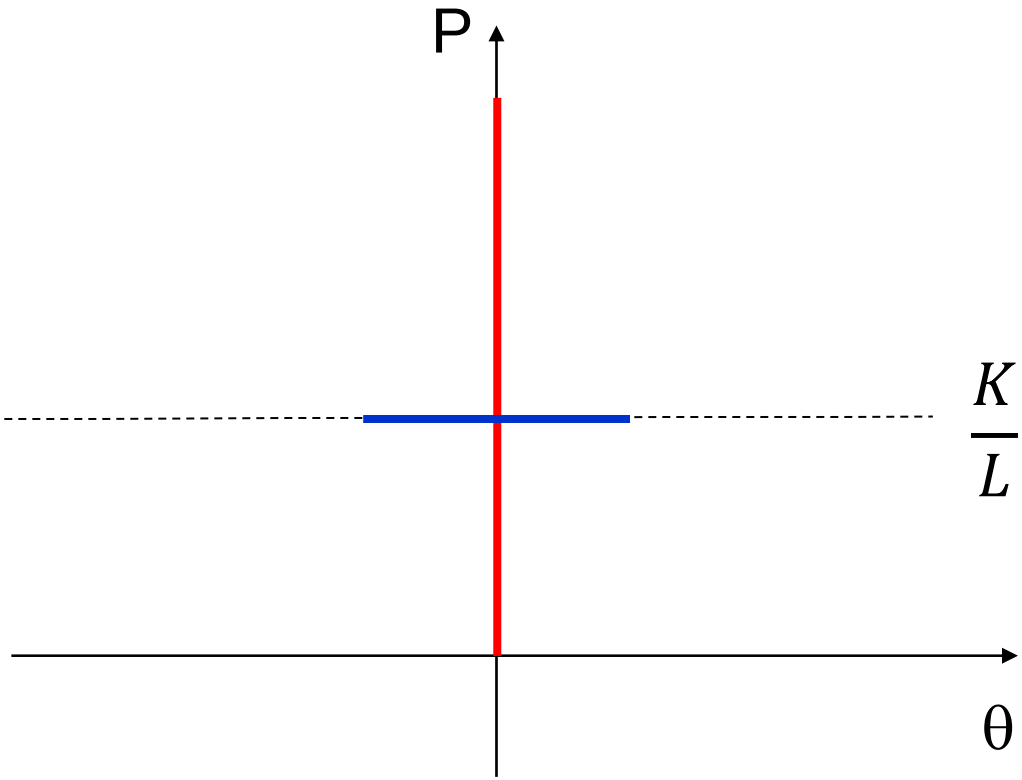


Bifurcation of equilibrium

We are interested in analysing the system in the neighbourhood of this point



Bifurcation of equilibrium



By partially removing large displacement, i.e.

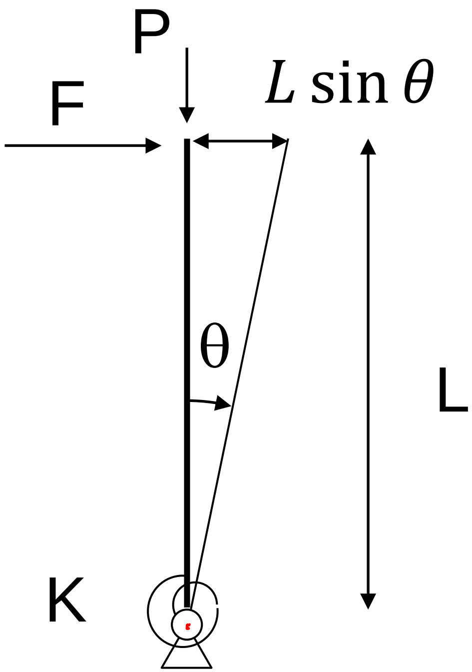
- assuming small displacements
- writing equilibrium equation in the deformed configuration

We obtain $(1-p)\theta = 0$, which admits $\theta \neq 0$ when $P = \frac{K}{L}$

We only know that a deformed configuration is possible.

No information about the amount of displacement can be obtained (for this, large displacements need being introduced into the equilibrium equations)

Effect of an axial load upon the deflection due to a lateral load



Potential energy of the perturbed system

$$\Pi = \frac{1}{2}K\theta^2 - PL(1-\cos\theta) - FL\sin\theta$$

By imposing $\delta\Pi = 0$, we obtain

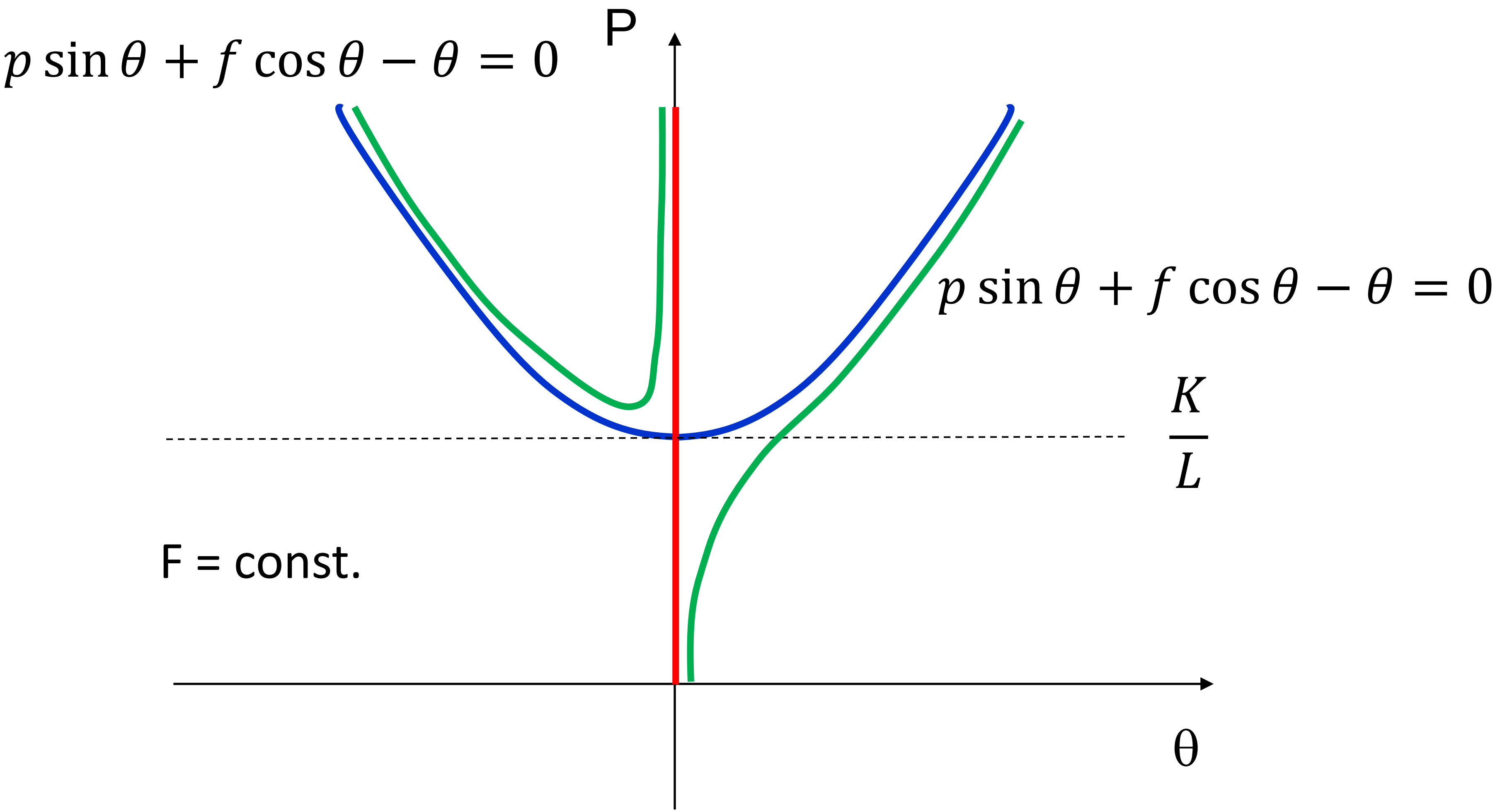
$$p\sin\theta + f\cos\theta - \theta = 0$$

where

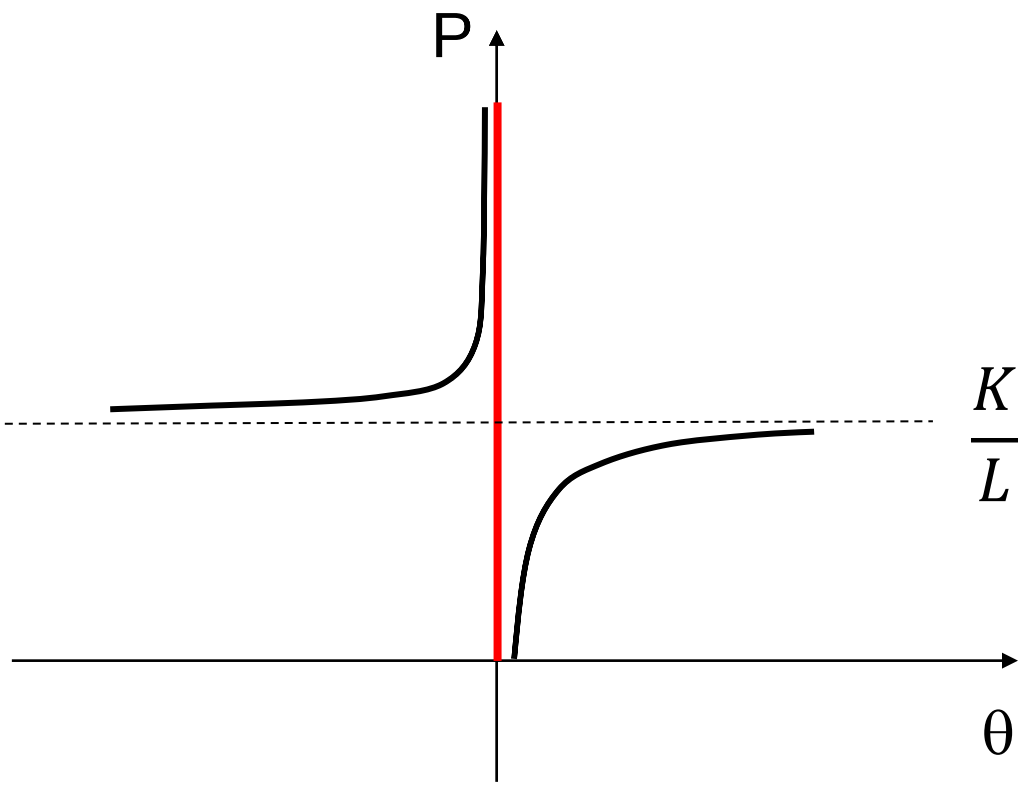
$$p = \frac{PL}{K}, f = \frac{FL}{K}$$

For a given applied load F, the solution depends also upon the value of P

Effect of an axial load upon the deflection due to a lateral load



By assuming small displacements



$$(1-p)\theta = f$$

$$\left(1-\frac{PL}{K}\right)\theta = \frac{FL}{K}$$

$$(K-PL)\theta = FL$$

The force P modifies the stiffness of the system, with respect to deflection caused by the lateral force F

If sign of P changes, the apparent stiffness increases: **stress stiffening**

Stress stiffening

The term PL in the equilibrium equation can be seen as an additional stiffening term

$$(K + K_\sigma)\theta = FL$$

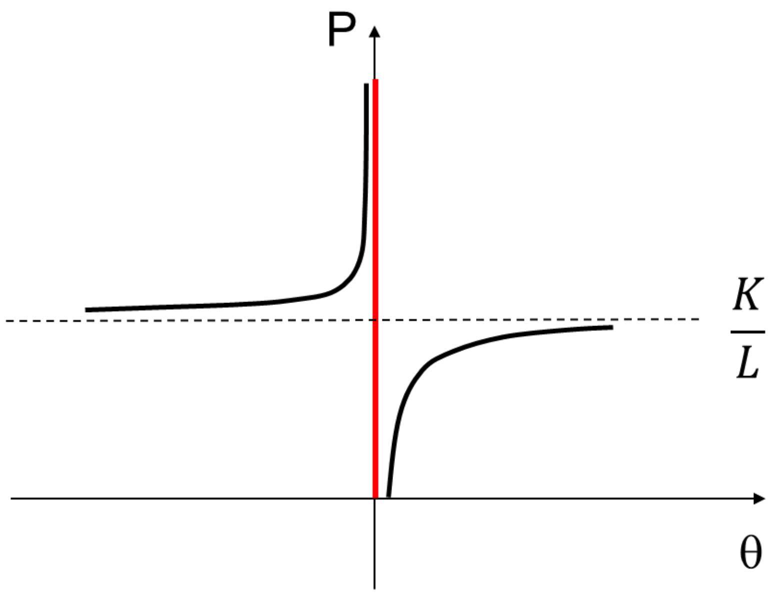
K_σ is a function of the load P

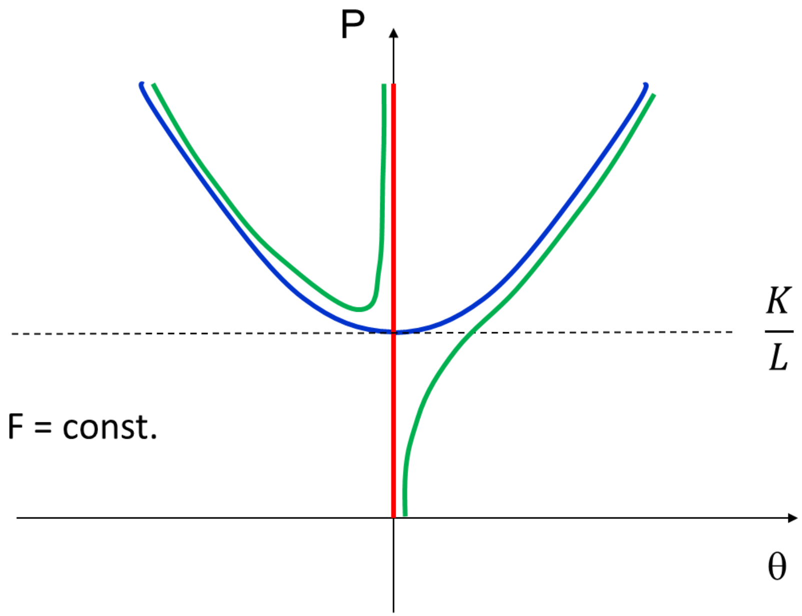
The apparent stiffness:

- diminishes if P is compressive
- increases if P is tensile

If $(K + K_\sigma)$ = 0, the system behaves as if its stiffness were null. This happens when P = KL (bifurcation load)

Comparison of solutions





Small displacements solution indicate that, as P approaches K/L, deflection becomes unbounded

This can happen even if F = 0

Large displacements solution indicate that, even if F is small, as P approaches K/L, deflection is bounded but can be very large

Avoid large displacements

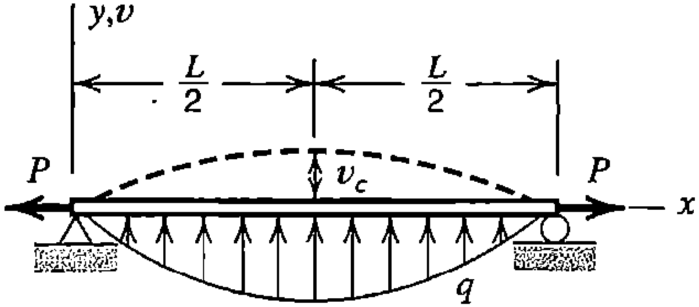
Generally, we want to avoid large displacements.

The small displacement analysis, although not providing us with exact values of displacements, allows for **identifying the bifurcation point**.

By keeping the compression load sufficiently lower than the critical load, we can ensure that:

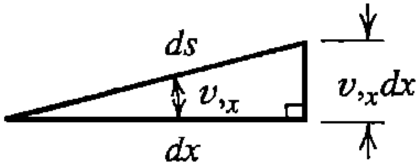
- as the lateral loads increase, deflection does not grow faster that the applied load
- when the system is perturbed, the perturbed configuration is not too different form the unperturbed one

Analysis of a laterally loaded beam



$$ds = \sqrt{1 + v_{,x}^2}\ dx$$

$$ds \approx \left(1 + \frac{1}{2}v_{,x}^2\right) dx$$

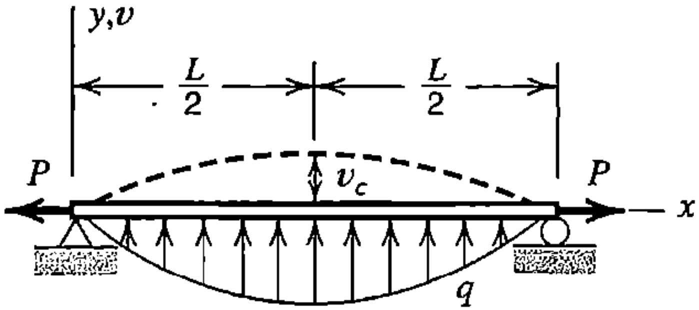


$$U_b = \frac{1}{2}\int_0^L EI_z v_{,xx}^{\,2}\ dx$$

$$\varepsilon_m = \frac{ds - dx}{dx} = \frac{ds}{dx} - 1 \qquad \text{hence} \qquad \varepsilon_m \approx \left(1 + \frac{1}{2}v_{,x}^{\,2}\right) - 1 = \frac{1}{2}v_{,x}^{\,2}$$

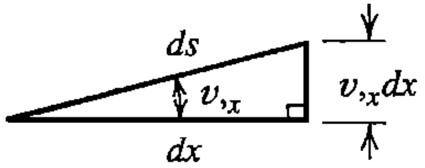
$$v_{,x} = \frac{\mathrm{d}v}{\mathrm{d}x}$$

Analysis of a laterally loaded beam



$$ds = \sqrt{1 + v_{,x}^2}\, dx$$

$$ds \approx \left(1 + \frac{1}{2} v_{,x}^2\right) dx$$



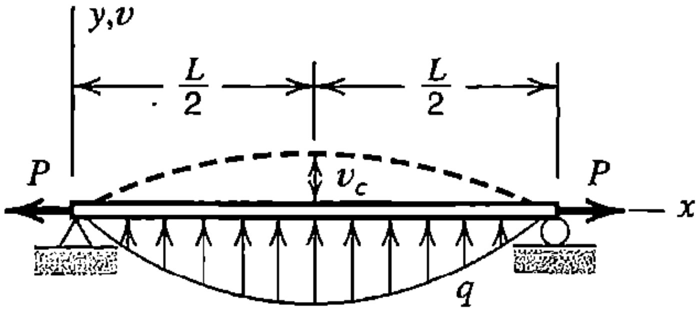
$$U_m = \int_0^L P \varepsilon_m \, dx \quad \text{or} \quad U_m = \frac{1}{2}\int_0^L P v_{,x}^2 \, dx$$

$$v = v_c \sin(\pi x / L)$$

$$U_b = \frac{\pi^4 E I_z}{4L^3} v_c^2 \qquad U_m = \frac{\pi^2 P}{4L} v_c^2$$

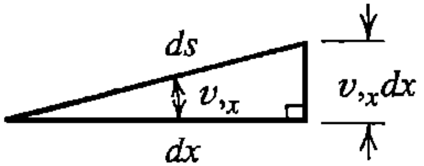
Note: P pre-exists, thus d$U = P\varepsilon_m dx$, not 1/2…

Analysis of a laterally loaded beam



$$ds = \sqrt{1 + v_{,x}^2}\, dx$$

$$ds \approx \left(1 + \tfrac{1}{2} v_{,x}^2\right) dx$$



$$q = q_c \sin(\pi x / L)$$

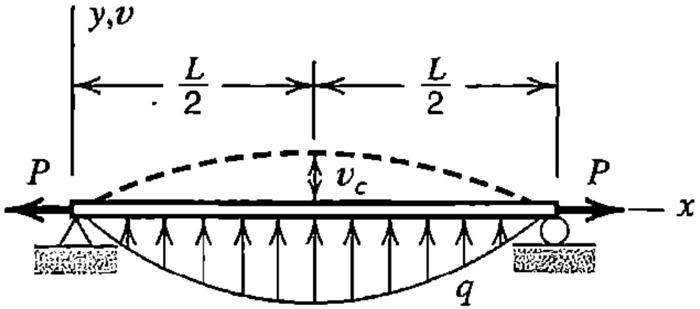
$$\Pi_p = U_b + U_m + \Omega \qquad \text{where} \qquad \Omega = -\int_0^L vq\, dx = -\frac{q_c L}{2} v_c$$

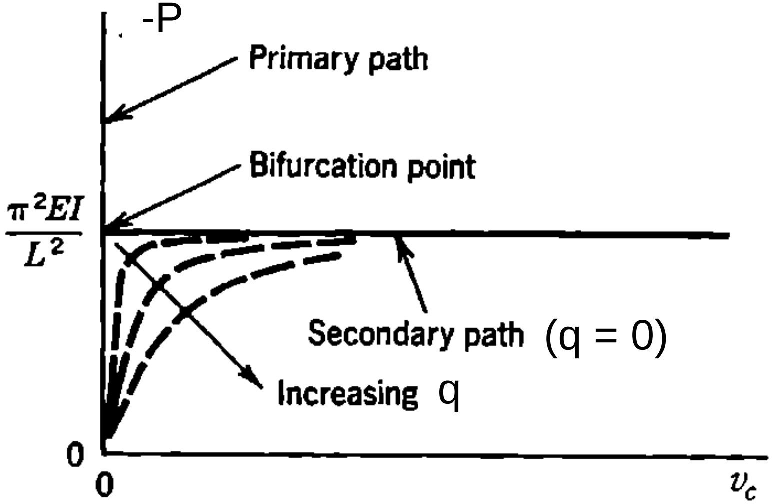
$$d\Pi_p / dv = 0$$

$$v_c = \frac{q_c L}{2(k + k_\sigma)} \qquad \text{where} \qquad k = \frac{\pi^4 E I_z}{2L^3} \quad \text{and} \quad k_\sigma = \frac{\pi^2 P}{2L}$$

stress stiffening

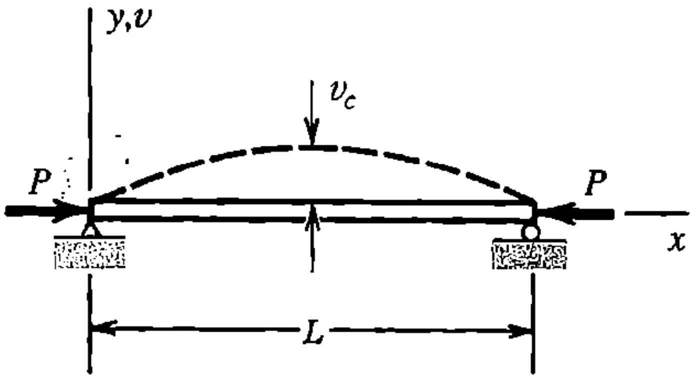
Effect of a compressive load on the lateral deflection

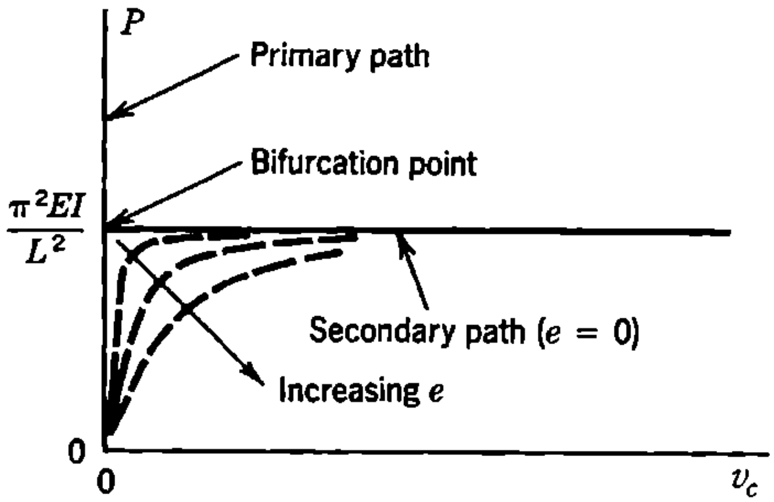




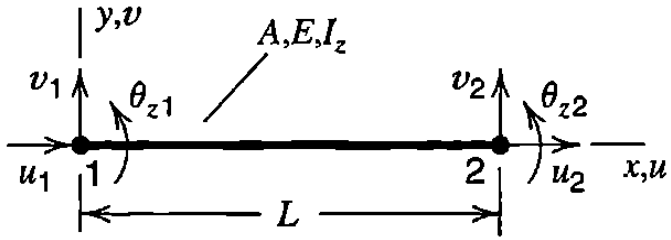
Effect of a compressive load applied with eccentricity

If the load P is kept constant but eccentricity *e* is increased:





Beam elements: stress stiffening



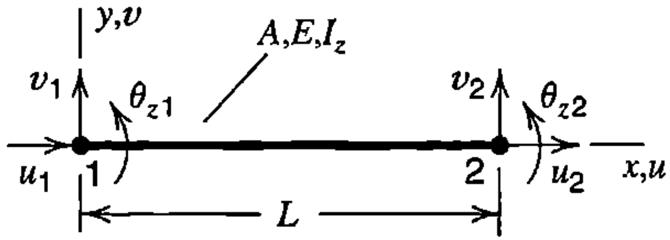
$$\dot{U}_m = \frac{1}{2}\int_0^L P v_{,x}^2 \, dx = \frac{1}{2}\int_0^L v_{,x}^T P v_{,x} \, dx = \frac{1}{2}\{\mathbf{d}\}^T[\mathbf{k}_\sigma]\{\mathbf{d}\}$$

$$v = \lfloor \mathbf{N} \rfloor \{\mathbf{d}\} \quad \text{and} \quad v_{,x} = \lfloor \mathbf{G} \rfloor \{\mathbf{d}\} \qquad \text{where} \qquad \lfloor \mathbf{G} \rfloor = \frac{d}{dx}\lfloor \mathbf{N} \rfloor$$

$$[\mathbf{k}_\sigma] = \int_0^L \lfloor \mathbf{G} \rfloor^T \lfloor \mathbf{G} \rfloor P \, dx$$

K_σ is a function of P

Beam elements: stress stiffening



$$[\mathbf{k}_\sigma] = \frac{P}{30L}\begin{bmatrix} 36 & 3L & -36 & 3L \\ 3L & 4L^2 & -3L & -L^2 \\ -36 & -3L & 36 & -3L \\ 3L & -L^2 & -3L & 4L^2 \end{bmatrix} \quad \text{for} \quad \{\mathbf{d}\} = \begin{Bmatrix} v_1 \\ \theta_{z1} \\ v_2 \\ \theta_{z2} \end{Bmatrix}$$

Linear bifurcation buckling analysis steps

1. The structure is loaded by an arbitrary level of external loads $\{R\}_{ref}$
2. The stress stiffness matrix $[K_\sigma]_{ref}$ is evaluated for stresses associated with loads $\{R\}_{ref}$
3. For other loads obtained by multiplying $\{R\}_{ref}$ by a scalar multiplier λ (load multiplier), i.e. $\{R\}=\lambda\{R\}_{ref}$, $[K_\sigma] = \lambda[K_\sigma]_{ref}$
4. Buckling displacement takes place relative to displacements $\{D\}_{ref}$ of the reference configuration

$$\left([K] + [K_\sigma]_{ref}\right)\{D\}_{ref} = \{R\}_{ref}$$

$$\left([K] + \lambda[K_\sigma]_{ref}\right)\{D\} = \lambda\{R\}_{ref}$$

5. At the bifurcation point, $\{\delta \boldsymbol{D}\}$ a perturbed configuration of equilibrium is possible with no change of the external loads:

$$\left([K] + \lambda[K_\sigma]_{ref}\right)\{D + \delta D\} = \lambda\{R\}_{ref}$$

Linear bifurcation buckling analysis steps

6. Subtracting

$$\left([K] + \lambda \ \ [K_\sigma]_{ref}\right)\{D\} = \lambda \ \ \{R\}_{ref}$$

from

$$\left([K] + \lambda \ \ [K_\sigma]_{ref}\right)\{D + \delta D\} = \lambda \ \ \{R\}_{ref}$$

yields

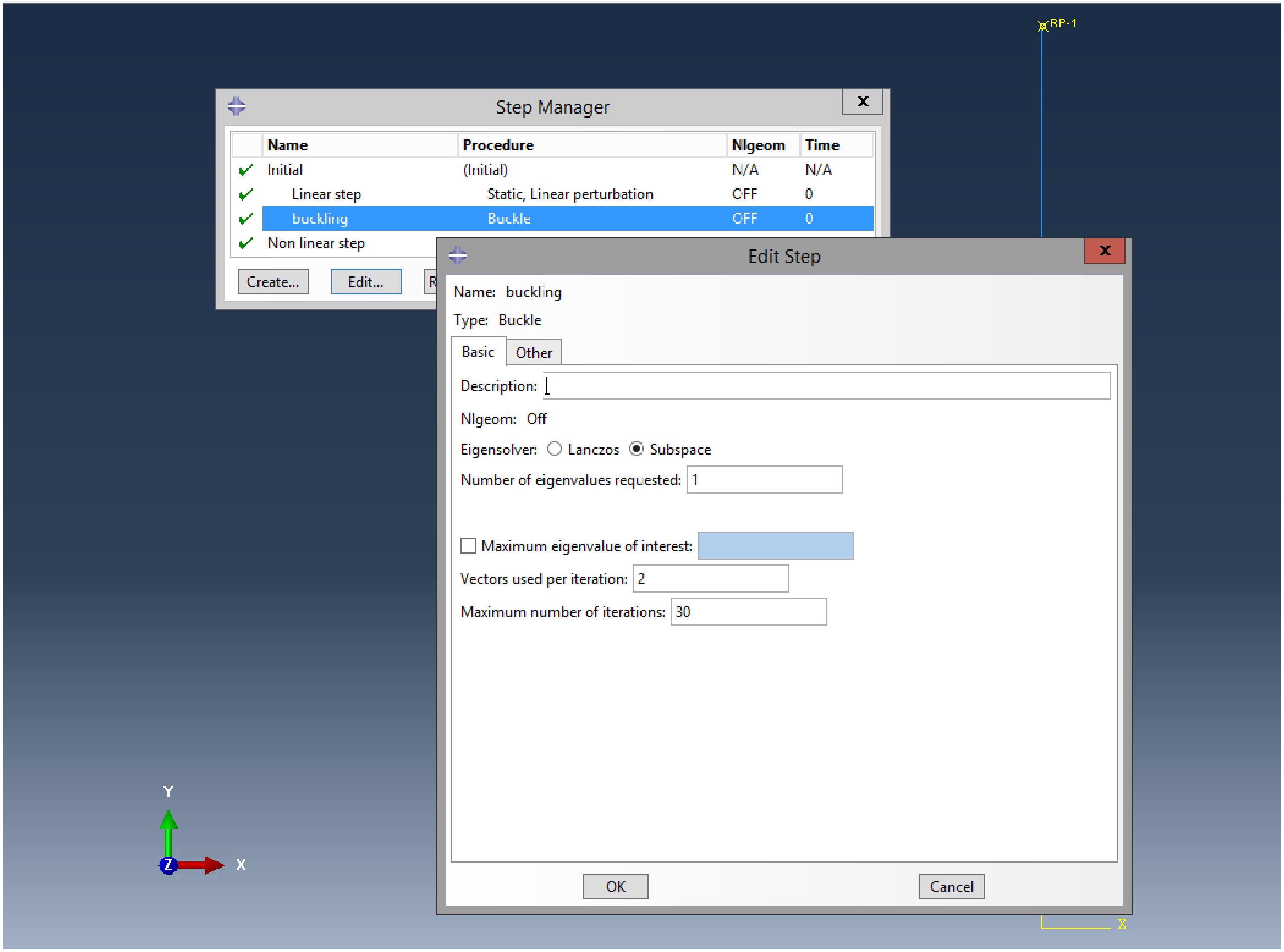
$$\left([K] + \lambda_{cr}[K_\sigma]_{ref}\right)\{\delta D\} = \{0\}$$

7. It is an eigenvalue problem: $\left|[K] + \lambda_{cr}[K_\sigma]_{ref}\right| = 0$; the smallest root defines the critical load

$$\{\mathbf{R}\}_{cr} = \lambda_{cr}\{\mathbf{R}\}_{ref}$$

8. The eigenvector $\{\delta D\}$ associated to λ_{cr} defines the shape of critical deformed configuration (being defined with an arbitrary multiplicative constant)

Definition of a buckling step in Abaqus



Buckling step: results