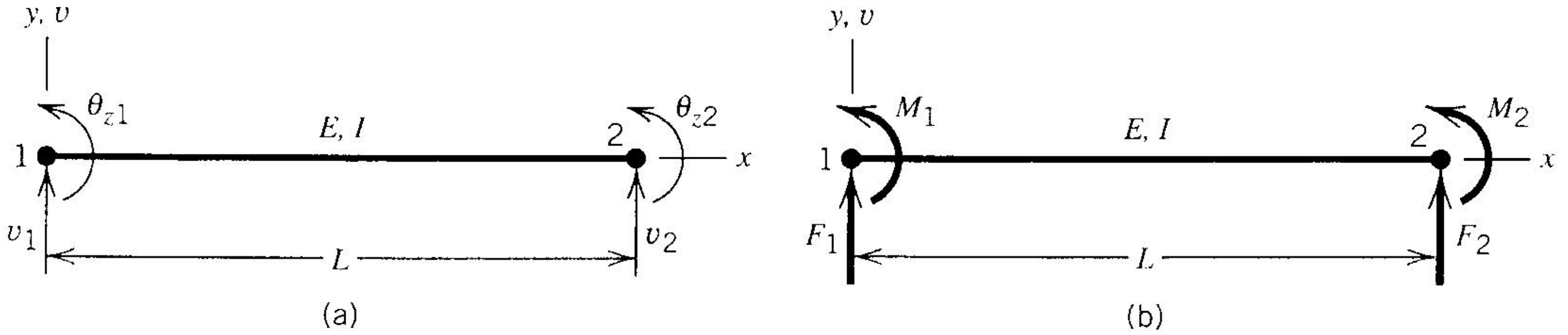
POLITECNICO DI MILANO

Finite Element Simulation For Mechanical Design

Beam elements

A. Bernasconi

Beam elements: Euler-Bernoulli theory



2D Element

Only M and V

4 d.o.f.: v_1, θ_1, v_2, θ_2

v = v(x)

For forces and moments applied to the ends, the deformed shape v(x) of a beam with constant EJ is cubic => cubic formulation

$$v''(x) = -\frac{M(x)}{EJ}; \quad M = Ax + B; \quad v(x) = ax^3 + bx^2 + cx + d$$

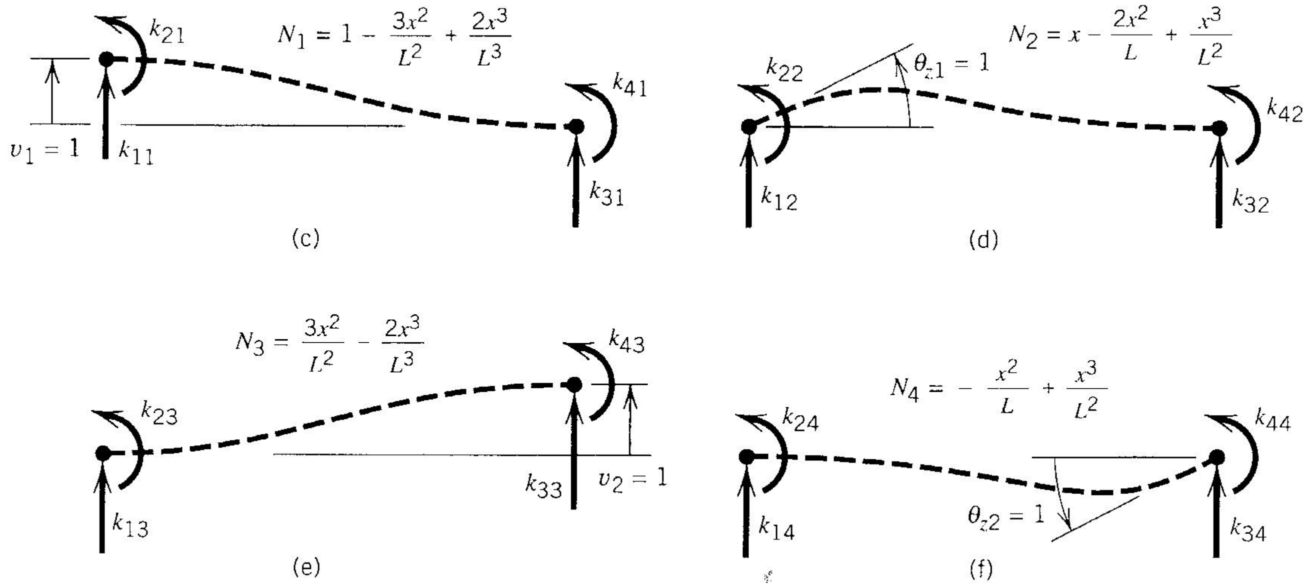
Deformed shape associated to each d.o.f.

It is cubic.

We can find it by imposing that the equation of the deformed shape

$$v = \beta_1 + \beta_2 x + \beta_3 x^2 + \beta_4 x^3$$

Respects the imposed boundary conditions (by substituting the coordinates of the nodes and finding the expressions for β_i as a function of imposed nodal displacements)



Shape functions

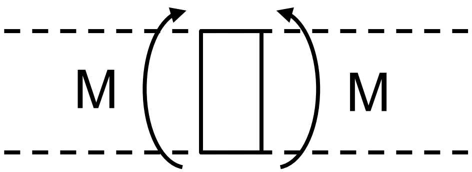
The four deformed shapes associated to the 4 d.o.f. are the shape function of the beam element, N_i

$$v(x) = N_1(x)v_1 + N_2(x)\theta_1 + N_3(x)v_2 + N_4(x)\theta_2$$

$$v(x) = \begin{bmatrix} N_1(x) & N_2(x) & N_3(x) & N_4(x) \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_{z1} \\ v_2 \\ \theta_{z2} \end{Bmatrix} = [N(x)]\{d\}$$

$$v = \begin{bmatrix} 1-\dfrac{3x^2}{L^2}+\dfrac{2x^3}{L^3} & x-\dfrac{2x^2}{L}+\dfrac{x^3}{L^2} & \dfrac{3x^2}{L^2}-\dfrac{2x^3}{L^3} & -\dfrac{x^2}{L}+\dfrac{x^3}{L^2} \end{bmatrix} \begin{Bmatrix} v_1 \\ \theta_{z1} \\ v_2 \\ \theta_{z2} \end{Bmatrix}$$

Stiffness matrix



$$\frac{dU}{dx} = \int_V \frac{1}{2}\sigma\varepsilon dV = \frac{1}{2}Mv''$$

$$v(x) = [N(x)]\{d\}$$

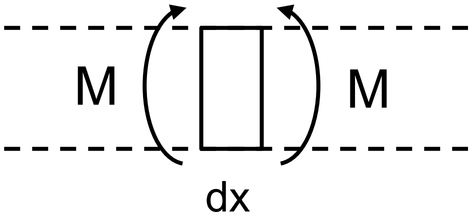
$$dU = \frac{1}{2}M\frac{d^2v}{dx^2}dx$$

$$v''(x) = [N''(x)]\{d\} = [B(x)]\{d\}$$

$$\left\{\frac{d^2v}{dx^2}\right\} = [B]\{d\}_{el}$$

$$M = [EJ]\left\{\frac{d^2v}{dx^2}\right\}$$

Stiffness matrix



$$dU = \frac{1}{2} M \frac{d^2 v}{dx^2} dx$$

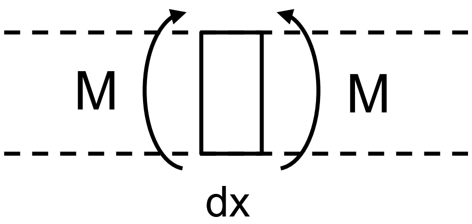
$$\left\{\frac{d^2 v}{dx^2}\right\} = [B]\{d\}_{el}$$

$$M = [EJ]\left\{\frac{d^2 v}{dx^2}\right\}$$

$$dU = \frac{1}{2}\{M\}^T \left\{\frac{d^2 v}{dx^2}\right\} dx = \frac{1}{2}\left\{\frac{d^2 v}{dx^2}\right\}^T [EJ]\left\{\frac{d^2 v}{dx^2}\right\} dx =$$

$$= \frac{1}{2}\{d\}_{el}^T [B]^T [EJ][B]\{d\}_{el} dx$$

Stiffness matrix



$$dU = \frac{1}{2}\{d\}_{el}^T[B]^T[EJ][B]\{d\}_{el}dx$$

$$U = \int_L dU$$

$$U = \int_L \frac{1}{2}\{d\}_{el}^T[B]^T[EJ][B]\{d\}_{el}dx = \frac{1}{2}\{d\}_{el}^T[K]\{d\}_{el}$$

$$[K] = \int_L [B]^T[EJ][B]dx$$

$$v = \left[1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3} \quad x - \frac{2x^2}{L} + \frac{x^3}{L^2} \quad \frac{3x^2}{L^2} - \frac{2x^3}{L^3} \quad -\frac{x^2}{L} + \frac{x^3}{L^2} \right] \begin{Bmatrix} v_1 \\ \theta_{z1} \\ v_2 \\ \theta_{z2} \end{Bmatrix}$$

$$\frac{d^2 v(x)}{dx^2} = \left[\frac{d^2}{dx^2} \mathbf{N} \right] \{d\}_{el} = [B]\{d\}_{el}$$

$$[B] = \left[-\frac{6}{L^2} + \frac{12x}{L^3} \quad -\frac{4}{L} + \frac{6x}{L^2} \quad \frac{6}{L^2} - \frac{12x}{L^3} \quad -\frac{2}{L} + \frac{6x}{L^2} \right]$$

$$[k] = \int_L [B]^T [EJ][B] dx$$

expanding the matrix products and integrating, we obtain ...

$$[k] = \begin{bmatrix} \frac{12EJ}{L^3} & \frac{6EJ}{L^2} & -\frac{12EJ}{L^3} & \frac{6EJ}{L^2} \\ \frac{6EJ}{L^2} & \frac{4EJ}{L} & -\frac{6EJ}{L^2} & \frac{2EJ}{L} \\ -\frac{12EJ}{L^3} & -\frac{6EJ}{L^2} & \frac{12EJ}{L^3} & -\frac{6EJ}{L^2} \\ \frac{6EJ}{L^2} & \frac{2EJ}{L} & -\frac{6EJ}{L^2} & \frac{4EJ}{L} \end{bmatrix}$$

Limitations

The assumption of a cubic $v(x)$ is valid for a straight beam, having constant section and loaded at the ends.

A distributed load implies a $v(x)$ which is a fourth order polynomial

=> Solution for distributed loads is approximate, but improves as the mesh size decreases

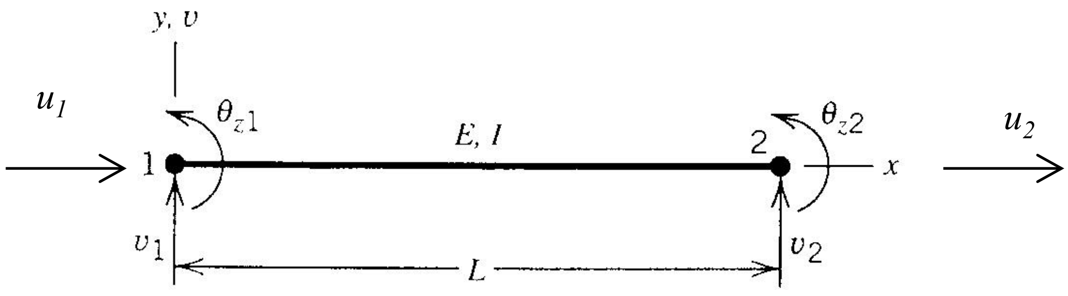
Stresses

Stresses are evaluated using the equation $\sigma_x = My/J$, where M is associated to the beam curvature through nodal displacements {d}

$M = EJd^2v/dx^2 = EJ\ [B]\ \{d\}$

Being the terms of [B] linear, M varies linearly with x through the element's length

2D beam + axial load = 6 d.o.f.



(a)

$$[k]=\begin{array}{c}\left[\begin{array}{cccccc} AE/L & 0 & 0 & -AE/L & 0 & 0 \\ 0 & 12EJ/L^3 & 6EJ/L^2 & 0 & -12EJ/L^3 & 6EJ/L^2 \\ 0 & 6EJ/L^2 & 4EJ/L & 0 & -6EJ/L^2 & 2EJ/L \\ -AE/L & 0 & 0 & AE/L & 0 & 0 \\ 0 & -12EJ/L^3 & -6EJ/L^2 & 0 & 12EJ/L^3 & -6EJ/L^2 \\ 0 & 6EJ/L^2 & 2EJ/L & 0 & -6EJ/L^2 & 4EJ/L \end{array}\right]\end{array}\begin{array}{c} u_1 \\ v_1 \\ \theta_{z1} \\ u_2 \\ v_2 \\ \theta_{z2} \end{array}$$

Shear flexible beam element

$$[k]=\left[\begin{array}{ccc|ccc} X & 0 & 0 & -X & 0 & 0 \\ 0 & Y_1 & Y_2 & 0 & -Y_1 & Y_2 \\ 0 & Y_2 & Y_3 & 0 & -Y_2 & Y_4 \\ \hline -X & 0 & 0 & X & 0 & 0 \\ 0 & -Y_1 & -Y_2 & 0 & Y_1 & -Y_2 \\ 0 & Y_2 & Y_4 & 0 & -Y_2 & Y_3 \end{array}\right] \begin{array}{c} u_1 \\ v_1 \\ \theta_1 \\ u_2 \\ v_2 \\ \theta_2 \end{array}$$

$$X=\frac{AE}{L} \qquad Y_1=\frac{12EJ_z}{(1+\Phi_y)L^3} \qquad Y_2=\frac{6EJ_z}{(1+\Phi_y)L^2}$$

$$Y_3=\frac{(4+\Phi_y)EJ_z}{(1+\Phi_y)L} \qquad Y_4=\frac{(2-\Phi_y)EJ_z}{(1+\Phi_y)L} \qquad \Phi_y=\frac{12EJ_z k_y}{AGL^2}$$

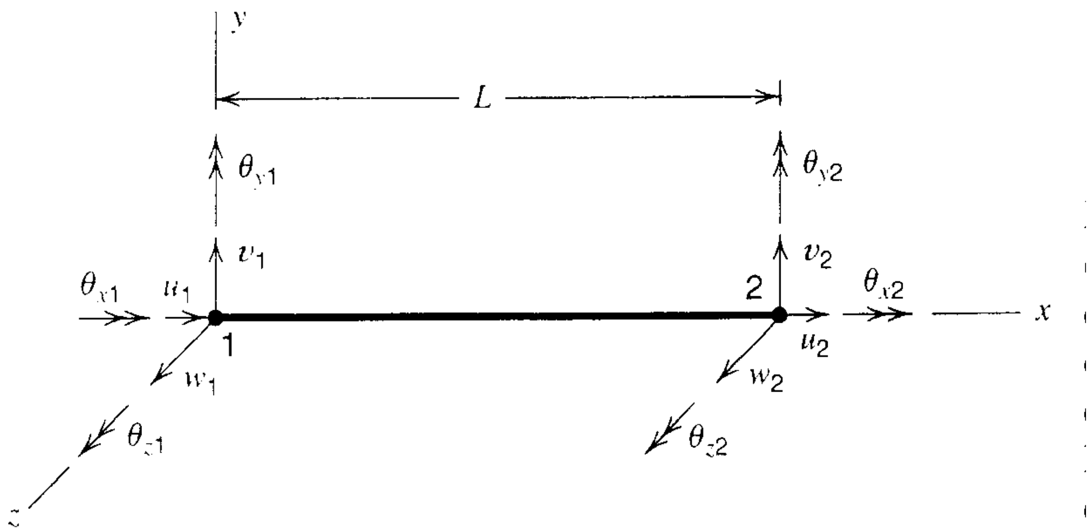
$$J_z=\rho_z^2 A$$

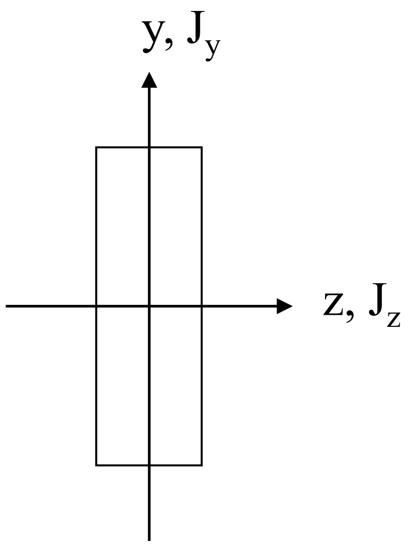
Φ_y accounts for shear flexibility (k_y = 1,2 for rectangular section)

Note: Φ_y tends to 0 for slender beams

$$\Phi_y \propto \frac{\rho_z^2}{L^2}$$

3D beam elements





More in general, the y and z axes refer to the principal axes of inertia

12 d.o.f. (3 translational and 3 rotational for each node)

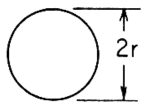
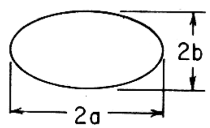
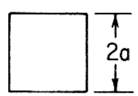
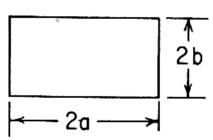
For the θ_z d.o.f. we have to introduce the torsional stiffness

$\theta_x = M_t / k_t$

where $k_t = G\, J_p / L$ is valid only for the circular section

For a generic section, $k_t = G\, K / L$ (con $K < J_p$)

Torsional stiffness

Form and dimensions of cross sections, other quantities involved, and case no.	Formula for K in $\theta = \frac{TL}{KG}$
1. Solid circular section 	$K = \frac{1}{2}\pi r^4$
2. Solid elliptical section 	$K = \frac{\pi a^3 b^3}{a^2 + b^2}$
3. Solid square section 	$K = 2.25a^4$
4. Solid rectangular section 	$K = ab^3\left[\frac{16}{3} - 3.36\frac{b}{a}\left(1 - \frac{b^4}{12a^4}\right)\right]$ for $a \geq b$

T=Torque=M_t

Stiffness matrix of a 3D beam element

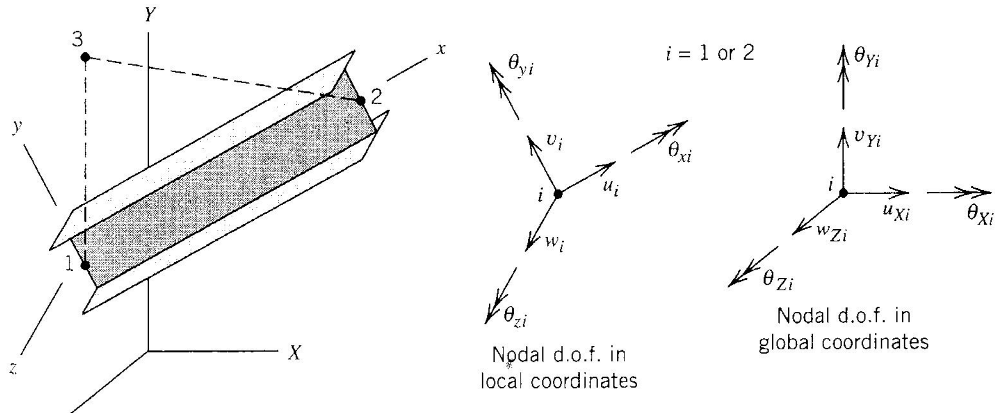
$$
[k]=\left[\begin{array}{cccccc|cccccc}
X & 0 & 0 & 0 & 0 & 0 & -X & 0 & 0 & 0 & 0 & 0 \\
 & Y_1 & 0 & 0 & 0 & Y_2 & 0 & -Y_1 & 0 & 0 & 0 & Y_2 \\
 & & Z_1 & 0 & -Z_2 & 0 & 0 & 0 & -Z_1 & 0 & -Z_2 & 0 \\
 & & & S & 0 & 0 & 0 & 0 & 0 & -S & 0 & 0 \\
 & & & & Z_3 & 0 & 0 & 0 & Z_2 & 0 & Z_4 & 0 \\
 & & & & & Y_3 & 0 & -Y_2 & 0 & 0 & 0 & Y_4 \\
\hline
 & & & & & & X & 0 & 0 & 0 & 0 & 0 \\
 & & & & & & & Y_1 & 0 & 0 & 0 & -Y_2 \\
 & & & & & & & & Z_1 & 0 & Z_2 & 0 \\
 & & & & & & & & & S & 0 & 0 \\
 & & & & & & & & & & Z_3 & 0 \\
 & & & & & & & & & & & Y_3
\end{array}\right]
\begin{array}{c}
u_1 \\ v_1 \\ w_1 \\ \theta_{x1} \\ \theta_{y1} \\ \theta_{z1} \\ u_2 \\ v_2 \\ w_2 \\ \theta_{x2} \\ \theta_{y2} \\ \theta_{z2}
\end{array}
$$

Z_i like Y_i, but with J_y instead of J_z

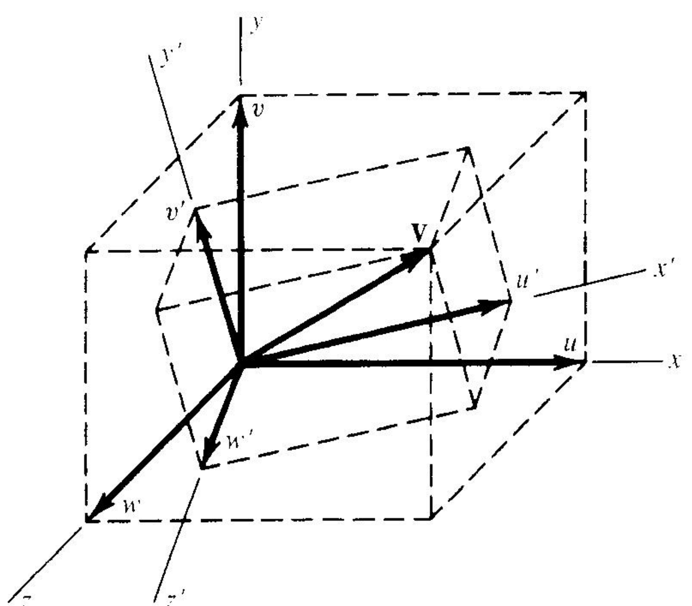
$S = GK / L$ (torsional stiffness)

Transformation into the global system

$$[K]=[T]^T[K'][T]$$



$$[T]=\begin{bmatrix}[\Lambda] & 0 & 0 & 0\\ 0 & [\Lambda] & 0 & 0\\ 0 & 0 & [\Lambda] & 0\\ 0 & 0 & 0 & [\Lambda]\end{bmatrix}$$

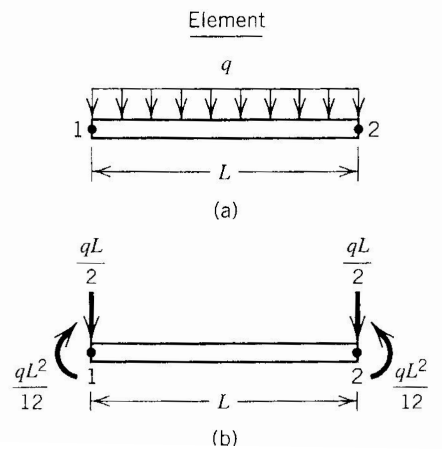


Direction cosines between axes:

	x	y	z
x'	l_1	m_1	n_1
y'	l_2	m_2	n_2
z'	l_3	m_3	n_3

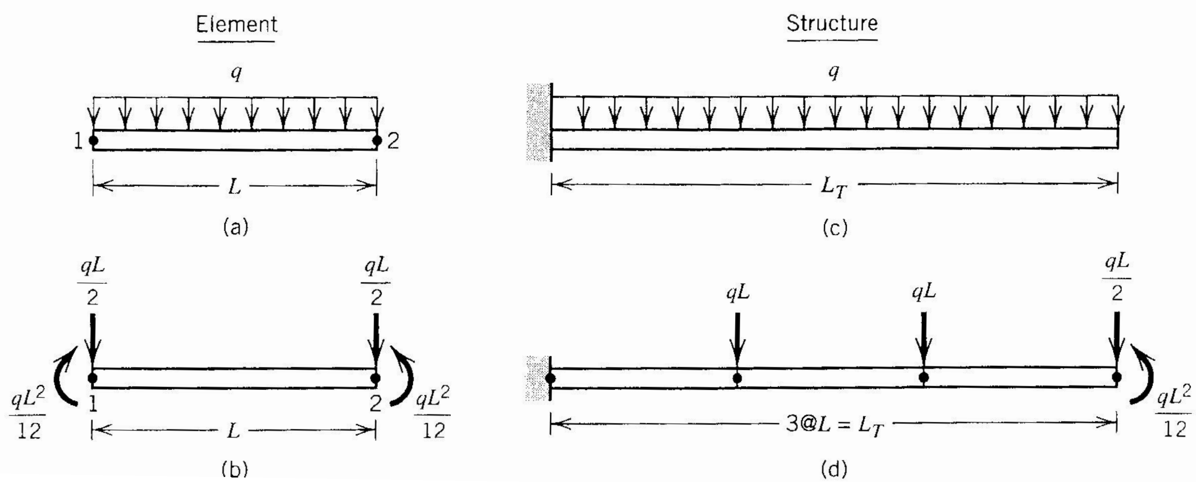
$$[\Lambda]=\begin{bmatrix}l_1 & m_1 & n_1\\ l_2 & m_2 & n_2\\ l_3 & m_3 & n_3\end{bmatrix}$$

Equivalent nodal loads



A distributed load is equivalent to nodal loads and moments

In the case of several elements, moments may cancel out



Equivalent nodal loads

F_1, F_2, M_1 e M_2 must do the same work as q, thus

$$-F_1 v_1 - M_1 \theta_{z1} - F_2 v_2 + M_2 \theta_{z2} = -\int_0^L q v(x) dx$$

with

$$v(x) = \beta_1 + \beta_2 x + \beta_3 x^2 + \beta_4 x^3$$

To obtain F_1, F_2, M_1 e M_2 we have to introduce the shape functions

Equivalent nodal loads

$$v(x) = N_1(x)v_1 + N_2(x)\theta_{z1} + N_3(x)v_2 + N_4(x)\theta_{z2}$$

$$F_1 v_1 + M_1\theta_{z1} + F_2 v_2 - M_2\theta_{z2} =$$

$$= \int_0^L q\left(N_1(x)v_1 + N_2(x)\theta_{z1} + N_3(x)v_2 + N_4(x)\theta_{z2}\right)dx$$

$$F_1 = \int_0^L qN_1(x)dx = \int_0^L q\left(1 - \frac{3x^2}{L^2} + \frac{2x^3}{L^3}\right)dx = \frac{qL}{2}$$

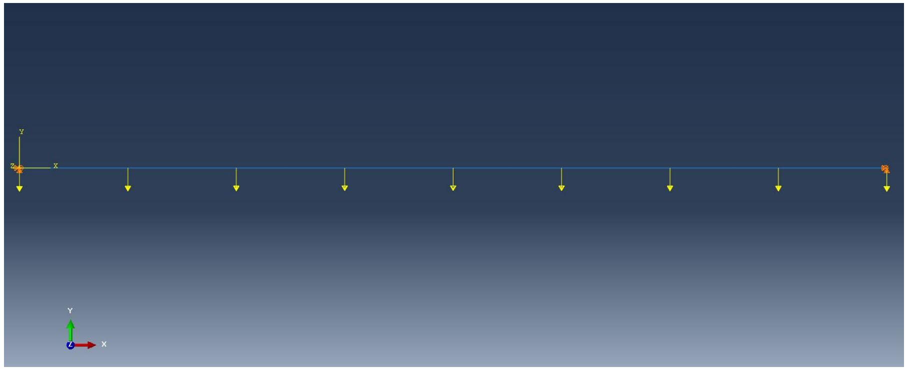
$$F_2 = \int_0^L qN_3(x)dx = \int_0^L q\left(\frac{3x^2}{L^2} - \frac{2x^3}{L^3}\right)dx = \frac{qL}{2}$$

$$M_1 = \int_0^L qN_2(x)dx = \int_0^L q\left(x - \frac{2x^2}{L} - \frac{x^3}{L^2}\right)dx = \frac{qL^2}{12}$$

$$M_2 = -\int_0^L qN_4(x)dx = \int_0^L q\left(-\frac{x^3}{L^2} + \frac{2x^2}{L}\right)dx = \frac{qL^2}{12}$$

Examples with analytical checks

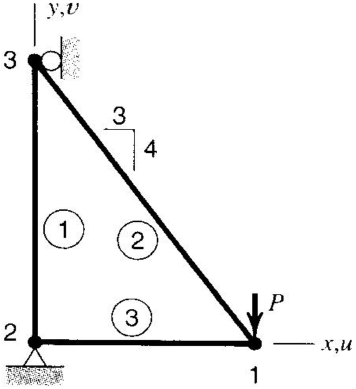
1. Pinned-pinned beam with:
   1. Concentrated load
   2. Distributed load



To understand how displacements and stresses are differently affected by the mesh size

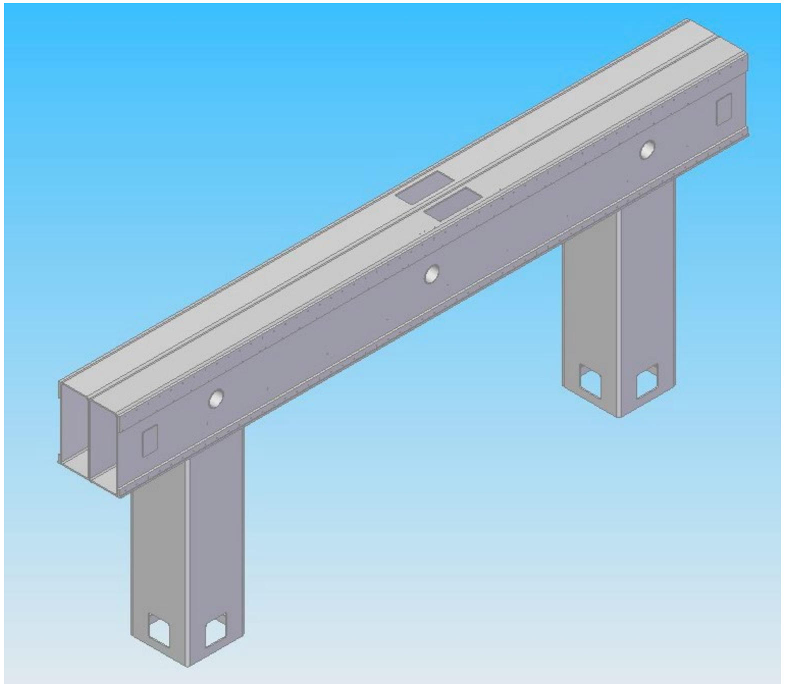
2. Triangular frame previously used for the bar element exercises, now with beam elements

To understand how the element type affect the modelling strategy: use of connectors



Examples on “are our modelling assumptions correct?”

Structure of a machine tool



Examples on “are our modelling assumptions correct?”

Mast of a fork lift