Formelark

TKT4134 Mekanikk 4

Tredimensjonal elastisitetsteori

Cauchys lov:

$$t_x = \sigma_x n_x + \tau_{xy} n_y + \tau_{xz} n_z$$

$$t_y = \tau_{xy} n_x + \sigma_y n_y + \tau_{yz} n_z$$

$$t_z = \tau_{zx} n_x + \tau_{yz} n_y + \sigma_z n_z$$

Likevektsligningene:

$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + \frac{\partial \tau_{zx}}{\partial z} + b_x = 0$$

$$\frac{\partial \tau_{xy}}{\partial x} + \frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{yz}}{\partial z} + b_y = 0$$

$$\frac{\partial \tau_{zx}}{\partial x} + \frac{\partial \tau_{yz}}{\partial y} + \frac{\partial \sigma_z}{\partial z} + b_z = 0$$

Kinematisk sammenheng (tøyninger):

$$\varepsilon_x = \frac{\partial u}{\partial x}, \quad \varepsilon_y = \frac{\partial v}{\partial y}, \quad \varepsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}, \quad \gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}, \quad \gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

Hookes lov for isotropt, lineært elastisk materiale:

$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} 1-\nu & \nu & \nu & 0 & 0 & 0 \\ \nu & 1-\nu & \nu & 0 & 0 & 0 \\ \nu & \nu & 1-\nu & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2}(1-2\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{1}{2}(1-2\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & \frac{1}{2}(1-2\nu) \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix}$$

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix} = \frac{1}{E} \begin{bmatrix} 1 & -\nu & -\nu & 0 & 0 & 0 \\ -\nu & 1 & -\nu & 0 & 0 & 0 \\ -\nu & -\nu & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(1+\nu) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(1+\nu) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(1+\nu) \end{bmatrix} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix}$$

Tøyningsenergitetthet:

$$U_0 = \frac{1}{2}(\sigma_x \varepsilon_x + \sigma_y \varepsilon_y + \sigma_z \varepsilon_z + \tau_{xy}\gamma_{xy} + \tau_{yz}\gamma_{yz} + \tau_{zx}\gamma_{zx})$$

Matriseformulering av KLM-kravene

Likevekt:

$$\boldsymbol{\Delta}^T \boldsymbol{\sigma} + \boldsymbol{b} = \boldsymbol{0}$$

Kinematikk:

$$\boldsymbol{\varepsilon} = \boldsymbol{\Delta u}$$

Materiallov:

$$\boldsymbol{\sigma} = \boldsymbol{C\varepsilon}$$

Forskyvnings-, spennings- og tøyningsvektorene:

$$\boldsymbol{u} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}, \quad \boldsymbol{\sigma} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{bmatrix}, \quad \boldsymbol{\varepsilon} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{bmatrix}$$

Differensialoperatormatrisen:

$$\boldsymbol{\Delta} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix}$$

Energiprinsipper

Prinsippet om virtuelt arbeid:

$$\int_V \delta\boldsymbol{\varepsilon}^T \boldsymbol{\sigma} dV = \int_V \delta\boldsymbol{u}^T \boldsymbol{b} dV + \int_S \delta\boldsymbol{u}^T \boldsymbol{t} dS$$

Potensiell energi:

$$\Pi(\boldsymbol{u}) = \int_V \frac{1}{2} \boldsymbol{\varepsilon}^T \boldsymbol{C\varepsilon}\, dV - \int_V \boldsymbol{u}^T \boldsymbol{b} dV - \int_{S_t} \boldsymbol{u}^T \bar{\boldsymbol{t}} dS$$

Prinsippet om stasjonær potensiell energi:

$$\delta\Pi(\boldsymbol{u}) = 0$$

Rayleigh-Ritz-metoden

Interpolasjon:

$$\boldsymbol{u} = \boldsymbol{N}\boldsymbol{q}$$

Potensiell energi:

$$\Pi(\boldsymbol{q}) = \frac{1}{2}\boldsymbol{q}^T\boldsymbol{K}\boldsymbol{q} - \boldsymbol{q}^T\boldsymbol{R}$$

Stivhetsmatrise og lastvektor:

$$\boldsymbol{K} = \int_V \boldsymbol{B}^T\boldsymbol{C}\boldsymbol{B}\,dV, \quad \boldsymbol{R} = \int_V \boldsymbol{N}^T\boldsymbol{b}dV + \int_{S_t} \boldsymbol{N}^T\bar{\boldsymbol{t}}dS$$

Elementmetoden

Interpolasjon:

$$\boldsymbol{u} = \boldsymbol{N}_e\boldsymbol{v}_e$$

Konnektivitet:

$$\boldsymbol{v}_e = \boldsymbol{a}_e\boldsymbol{r}$$

Potensiell energi:

$$\Pi(\boldsymbol{r}) = \frac{1}{2}\boldsymbol{r}^T\boldsymbol{K}\boldsymbol{r} - \boldsymbol{r}^T\boldsymbol{R}$$

Stivhetsmatrise og lastvektor for system og element:

$$\boldsymbol{K} = \sum_{e=1}^{n_{el}} \boldsymbol{a}_e^T\boldsymbol{k}_e\boldsymbol{a}_e\,, \quad \boldsymbol{R} = \boldsymbol{R}^k + \sum_{e=1}^{n_{el}} \boldsymbol{a}_e^T\boldsymbol{S}_e$$

$$\boldsymbol{k}_e = \int_{V_e} \boldsymbol{B}_e^T\boldsymbol{C}\boldsymbol{B}_e\,dV, \quad \boldsymbol{S}_e = \int_{V_e} \boldsymbol{N}_e^T\boldsymbol{b}dV + \int_{S_{te}} \boldsymbol{N}_e^T\bar{\boldsymbol{t}}dS$$

Systemligningene:

$$\boldsymbol{K}\boldsymbol{r} = \boldsymbol{R}$$

Kinematiske randkrav**:**

$$\begin{bmatrix} \boldsymbol{K}_{ff} & \boldsymbol{K}_{fs} \\ \boldsymbol{K}_{sf} & \boldsymbol{K}_{ss} \end{bmatrix} \begin{bmatrix} \boldsymbol{r}_f \\ \boldsymbol{r}_s \end{bmatrix} = \begin{bmatrix} \boldsymbol{R}_f \\ \boldsymbol{R}_s \end{bmatrix}$$

Skiveteori

Likevektsligningene:

$$\frac{\partial \sigma_x}{\partial x}+\frac{\partial \tau_{xy}}{\partial y}+b_x=0,\;\; \frac{\partial \tau_{xy}}{\partial x}+\frac{\partial \sigma_y}{\partial y}+b_y=0$$

Kinematiske relasjoner:

$$\varepsilon_x=\frac{\partial u}{\partial x},\;\; \varepsilon_y=\frac{\partial v}{\partial y},\;\; \gamma_{xy}=\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}$$

Materiallov for plan spenning:

$$\sigma_x=\frac{E}{1-\nu^2}\left(\varepsilon_x+\nu\varepsilon_y\right),\;\; \sigma_y=\frac{E}{1-\nu^2}\left(\varepsilon_y+\nu\varepsilon_x\right),\;\; \tau_{xy}=\frac{E}{2(1+\nu)}\gamma_{xy}$$

$$\varepsilon_x=\frac{1}{E}\left(\sigma_x-\nu\sigma_y\right),\quad \varepsilon_y=\frac{1}{E}\left(\sigma_y-\nu\sigma_x\right),\quad \gamma_{xy}=\frac{2(1+\nu)}{E}\tau_{xy}$$

Effektive elastiske konstanter for plan tøyning:

$$\bar{E}=\frac{E}{1-\nu^2},\;\; \bar{\nu}=\frac{\nu}{1-\nu}$$

Kinematiske og mekaniske randkrav:

$$u=\bar{u},\;\; v=\bar{v},\;\; (x,y)\in S_u$$

$$t_x=\bar{t}_x,\;\; t_y=\bar{t}_y,\;\; (x,y)\in S_t$$

Airys spenningsfunksjon for $b_x=b_y=0$:

$$\sigma_x=\frac{\partial^2\Phi}{\partial y^2},\;\; \sigma_y=\frac{\partial^2\Phi}{\partial x^2},\;\; \tau_{xy}=-\frac{\partial^2\Phi}{\partial x\partial y}$$

Kompatibilitetsligningen:

$$\frac{\partial^2\varepsilon_x}{\partial y^2}+\frac{\partial^2\varepsilon_y}{\partial x^2}=\frac{\partial^2\gamma_{xy}}{\partial x\partial y}$$

Differensialligningen:

$$\frac{\partial^4\Phi}{\partial x^4}+2\frac{\partial^4\Phi}{\partial x^2\partial y^2}+\frac{\partial^4\Phi}{\partial y^4}=0$$

Timoshenko bjelketeori

Kinematikk:

$$u = z\theta_y(x), \;\; v = 0, \;\; w = w(x)$$

Moment og skjærkraft:

$$M = \int_A z\sigma_x \, dA, \;\; V = \int_A \tau_{xz} \, dA$$

Likevektsligninger:

$$V - \frac{dM}{dx} = 0, \quad \frac{dV}{dx} + q = 0$$

Materiallov:

$$\sigma_x = E\varepsilon_x, \;\; \tau_{xz} = \kappa G \gamma_{xz}, \quad G = \frac{E}{2(1+\nu)}$$

Arealtreghetsmoment:

$$I = \int_A z^2 \, dA$$

Differensialligningene**:**

$$\frac{d}{dx}\left(EI\frac{d\theta_y}{dx}\right) - \kappa GA\left(\theta_y + \frac{dw}{dx}\right) = 0, \quad \frac{d}{dx}\left[\kappa GA\left(\theta_y + \frac{dw}{dx}\right)\right] + q = 0$$

Euler-Bernoulli bjelketeori

Kinematikk:

$$u = -z\frac{dw(x)}{dx}, \;\; v = 0, \;\; w = w(x)$$

Moment og skjærkraft:

$$M = \int_A z\sigma_x \, dA, \;\; V = \int_A \tau_{xz} \, dA$$

Likevektsligninger:

$$V - \frac{dM}{dx} = 0, \quad \frac{dV}{dx} + q = 0$$

Materiallov:

$$\sigma_x = E\varepsilon_x$$

Arealtreghetsmoment:

$$I = \int_A z^2 \, dA$$

Differensialligningen**:**

$$EI\frac{d^4 w(x)}{dx^4} = q(x)$$

Plateteori

Kinematikk:

$$u = -z\frac{\partial w}{\partial x}, \quad v = -z\frac{\partial w}{\partial y}, \quad w = w(x,y)$$

Momenter per lengdeenhet:

$$M_x = \int_{-h/2}^{h/2} \sigma_x z dz\,, \quad M_y = \int_{-h/2}^{h/2} \sigma_y z dz\,, \quad M_{xy} = M_{yx} = \int_{-h/2}^{h/2} \tau_{xy} z dz$$

Skjærkrefter per lengdeenhet:

$$V_x = \int_{-h/2}^{h/2} \tau_{zx} dz\,, \quad V_y = \int_{-h/2}^{h/2} \tau_{yz} dz$$

Likevektsligningene:

$$\frac{\partial V_x}{\partial x} + \frac{\partial V_y}{\partial y} + q = 0, \quad \frac{\partial M_{xy}}{\partial x} + \frac{\partial M_y}{\partial y} = V_y, \quad \frac{\partial M_x}{\partial x} + \frac{\partial M_{xy}}{\partial y} = V_x$$

Moment-krumningsrelasjonene:

$$M_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right), \quad M_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right), \quad M_{xy} = -(1-\nu)D\frac{\partial^2 w}{\partial x\,\partial y}$$

Skjærkraft-krumningsrelasjoner:

$$V_x = -D\frac{\partial}{\partial x}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right), \quad V_y = -D\frac{\partial}{\partial y}\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)$$

Differensialligningen:

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2\,\partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{q}{D}$$

Randskjærkrefter:

$$V_x^* = V_x + \frac{\partial M_{xy}}{\partial y}, \quad V_y^* = V_y + \frac{\partial M_{xy}}{\partial x}$$

Hjørnekraft for rektangulær plate:

$$R = 2M_{xy}(x_c, y_c)$$

Tøyningsenergi:

$$U = \int_A \frac{D}{2}\left[\left(\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)^2 - 2(1-\nu)\left(\frac{\partial^2 w}{\partial x^2}\frac{\partial^2 w}{\partial y^2} - \left(\frac{\partial^2 w}{\partial x \partial y}\right)^2\right)\right] dA$$

Platestivhet:

$$D = \frac{Eh^3}{12(1-\nu^2)}$$