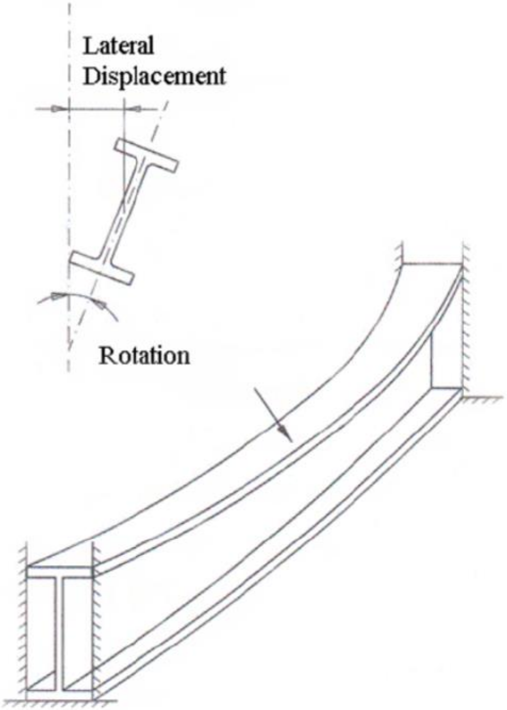
POLITECNICO DI MILANO

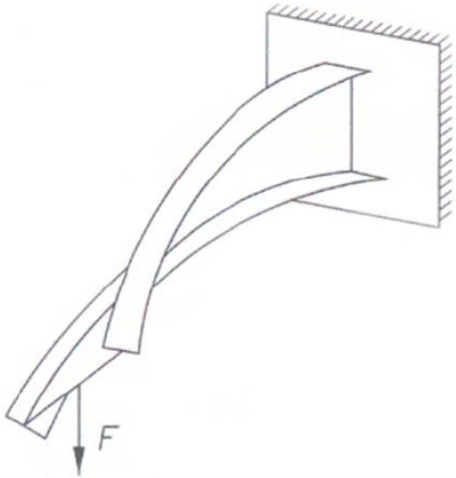
Finite Element Simulation For Mechanical Design

Stress stiffening and buckling

Prof. Andrea Bernasconi

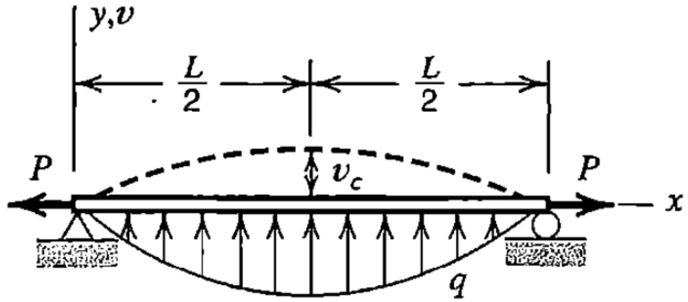
Lateral torsional buckling





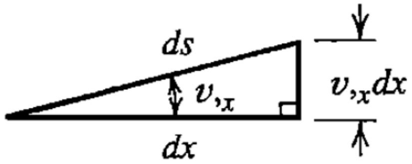
Lateral Torsional Buckling of a Beam Girder

Analysis of a laterally loaded beam



$$ds = \sqrt{1 + v_{,x}^2}\; dx$$

$$ds \approx (1 + \tfrac{1}{2} v_{,x}^2)\, dx$$



$$U_b = \frac{1}{2}\int_0^L EI_z v_{,xx}^2\, dx$$

$$\varepsilon_m = \frac{ds - dx}{dx} = \frac{ds}{dx} - 1 \qquad \text{hence} \qquad \varepsilon_m \approx \left(1 + \frac{1}{2} v_{,x}^2\right) - 1 = \frac{1}{2} v_{,x}^2$$

$$v_{,x} = \frac{\mathrm{d}v}{\mathrm{d}x}$$

Green strain tensor

$$\varepsilon_x = u_{,x} + \frac{1}{2}\left(u_{,x}^2 + v_{,x}^2 + w_{,x}^2\right)$$

$$\varepsilon_y = v_{,y} + \frac{1}{2}\left(u_{,y}^2 + v_{,y}^2 + w_{,y}^2\right)$$

$$\varepsilon_z = w_{,z} + \frac{1}{2}\left(u_{,z}^2 + v_{,z}^2 + w_{,z}^2\right)$$

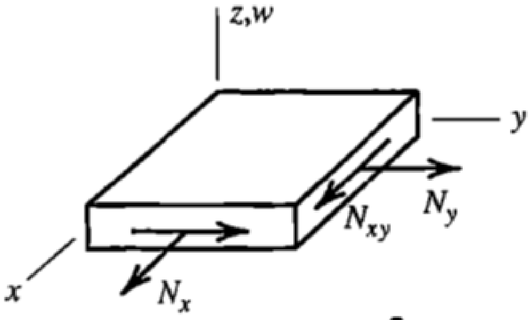
$$\gamma_{xy} = u_{,y} + v_{,x} + \left(u_{,x}\, u_{,y} + v_{,x}\, v_{,y} + w_{,x}\, w_{,y}\right)$$

$$\gamma_{yz} = v_{,z} + w_{,y} + \left(u_{,y}\, u_{,z} + v_{,y}\, v_{,z} + w_{,y}\, w_{,z}\right)$$

$$\gamma_{zx} = w_{,x} + u_{,z} + \left(u_{,z}\, u_{,x} + v_{,z}\, v_{,x} + w_{,z}\, w_{,x}\right)$$

To be used when large motions are involved, but strains remain small

Plate elements



$$\varepsilon_x = \frac{1}{2}w_{,x}^2 \qquad \varepsilon_y = \frac{1}{2}w_{,y}^2 \qquad \gamma_{xy} = w_{,x}w_{,y}$$

$$U_m = \int\left(\frac{1}{2}N_x w_{,x}^2 + \frac{1}{2}N_y w_{,y}^2 + N_{xy}w_{,x}\, w_{,y}\right) dA$$

$$U_m = \frac{1}{2}\iint \begin{Bmatrix} w_{,x} \\ w_{,y} \end{Bmatrix}^T \begin{bmatrix} N_x & N_{xy} \\ N_{xy} & N_y \end{bmatrix} \begin{Bmatrix} w_{,x} \\ w_{,y} \end{Bmatrix} dx\, dy = \frac{1}{2}\{\mathbf{d}\}^T[\mathbf{k}_\sigma]\{\mathbf{d}\}$$

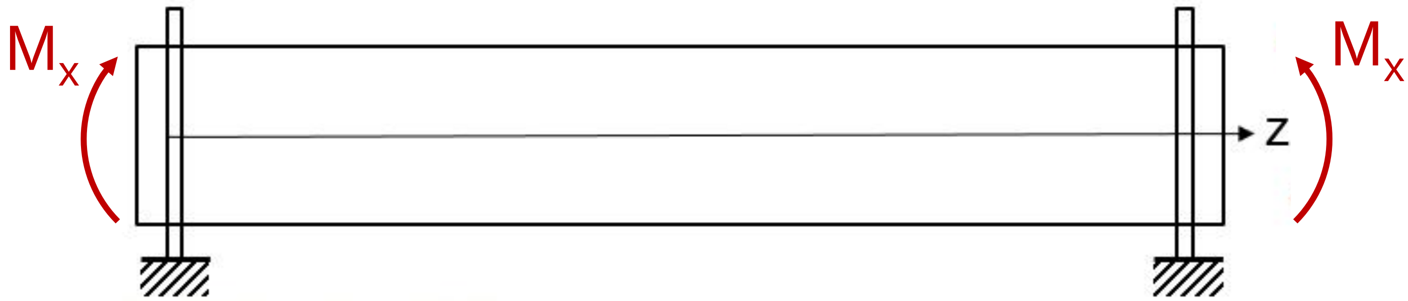
$$w = \lfloor \mathbf{N} \rfloor \underset{n\times 1}{\{\mathbf{d}\}} \qquad \text{yields} \qquad \begin{Bmatrix} w_{,x} \\ w_{,y} \end{Bmatrix} = \underset{2\times n}{[\mathbf{G}]}\{\mathbf{d}\}$$

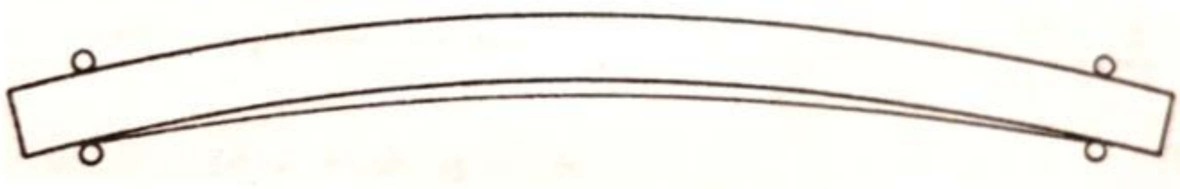
$$[\mathbf{k}_\sigma] = \iint [\mathbf{G}]^T \begin{bmatrix} N_x & N_{xy} \\ N_{xy} & N_y \end{bmatrix} [\mathbf{G}]\, dx\, dy$$

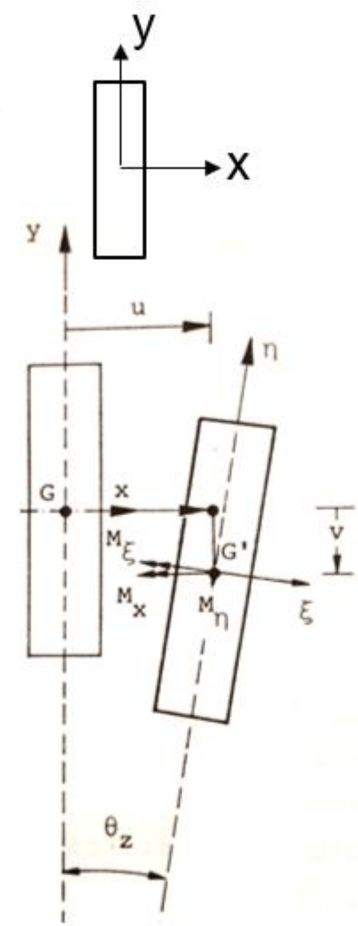
Lateral torsional buckling of beams

- Lateral-torsional buckling of double symmetric profiles
- Under small Moments M_x, only *v* deflection occurs
- If M_x increases, reaching a critical value, the beam can deflect laterally, the axis being curved spatially
- Thus there is an additional lateral displacement *v* and an associated cross-sectional rotation θ_z

Lateral torsional buckling of beams

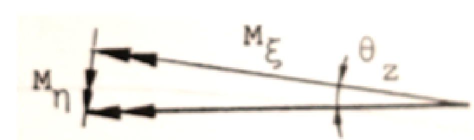




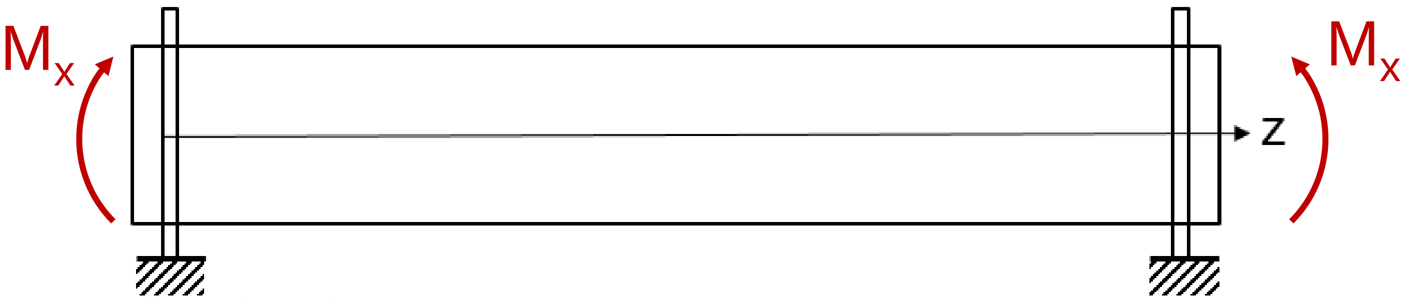


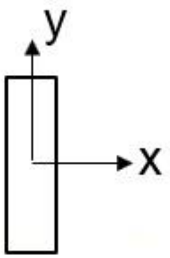
Due to twist, the moment M_x is decomposed in two additional flexural moments:

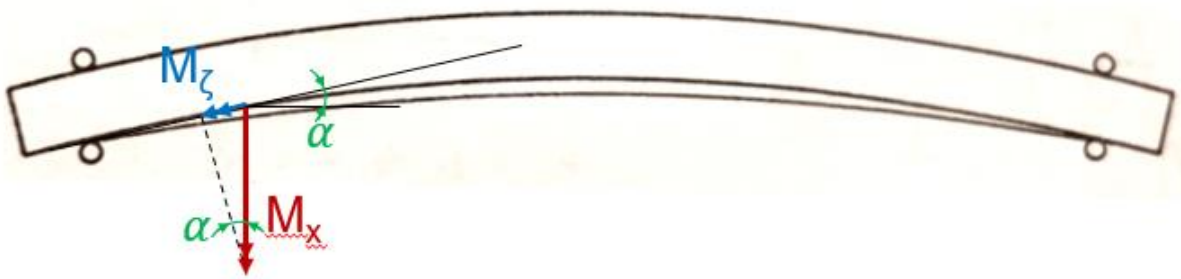
$$M_\eta = M_x \sin\theta_z \cong M_x\theta_x$$
$$M_\xi = M_x \cos\theta_z \cong \theta_z$$

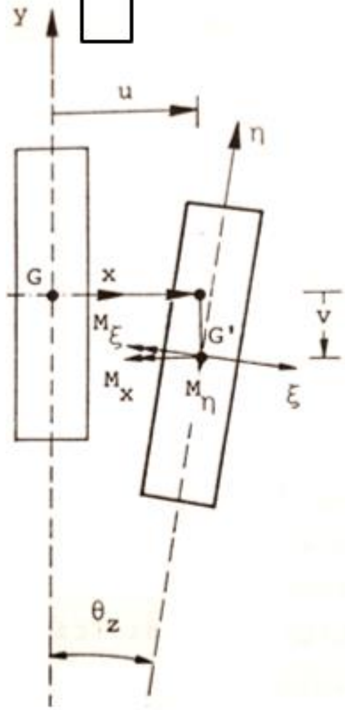


Lateral torsional buckling of beams









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$$M_\eta = M_x \sin\theta_z \cong M_x\theta_x$$
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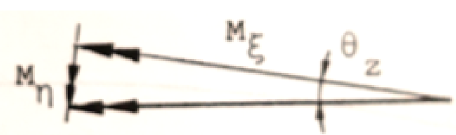
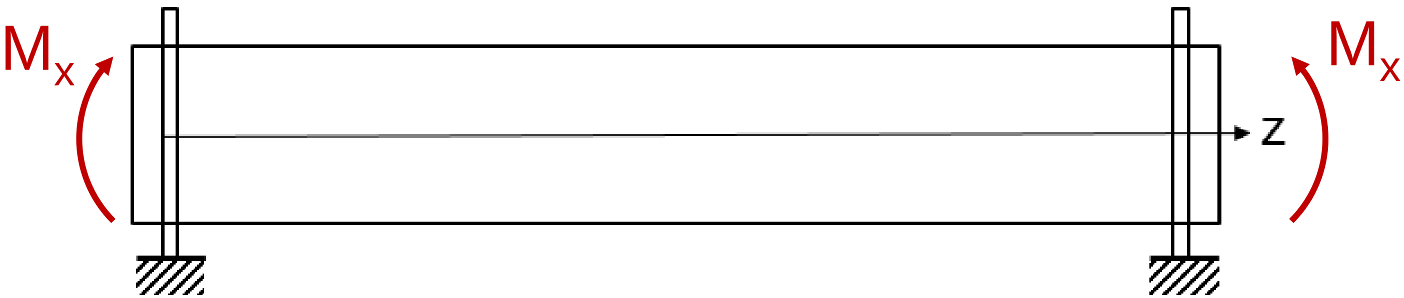
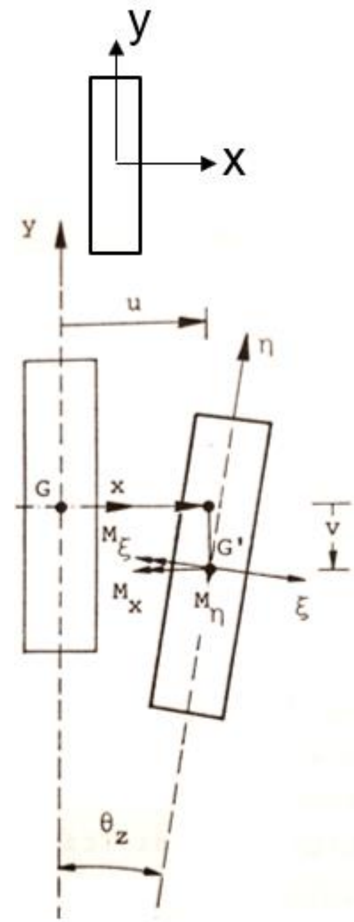
Due to the lateral displacement, the moment Mx also has a torsional component:

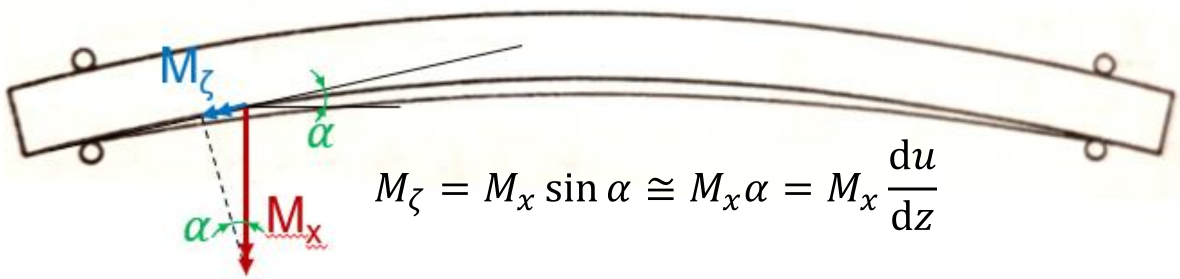
$$M_\zeta = M_x \sin\alpha \cong M_x\alpha = M_x\frac{\mathrm{d}u}{\mathrm{d}z}$$



Lateral torsional buckling of beams







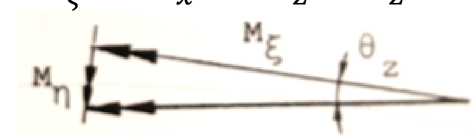
$$M_\zeta = M_x \sin\alpha \cong M_x\alpha = M_x \frac{\mathrm{d}u}{\mathrm{d}z}$$

We thus have both u(z) and v(z) due to flexural loads:

$$\frac{\mathrm{d}^2 u}{\mathrm{d}z^2} = \frac{M_\eta}{EJ_y} = \frac{M_x}{EJ_y}\theta_z \quad (1)$$

$$\frac{\mathrm{d}^2 v}{\mathrm{d}z^2} = -\frac{M_\xi}{EJ_x} = -\frac{M_x}{EJ_x}$$

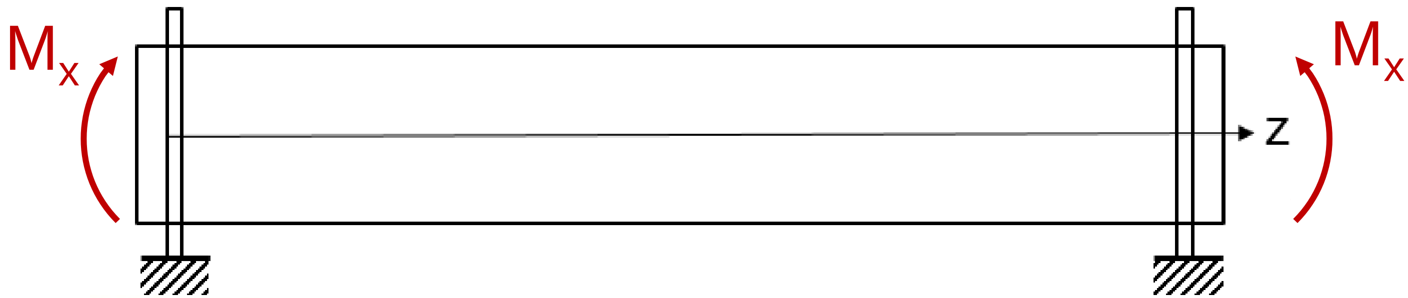
$$M_\eta = M_x \sin\theta_z \cong M_x\theta_x$$
$$M_\xi = M_x \cos\theta_z \cong \theta_z$$

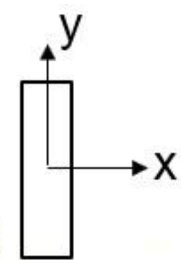


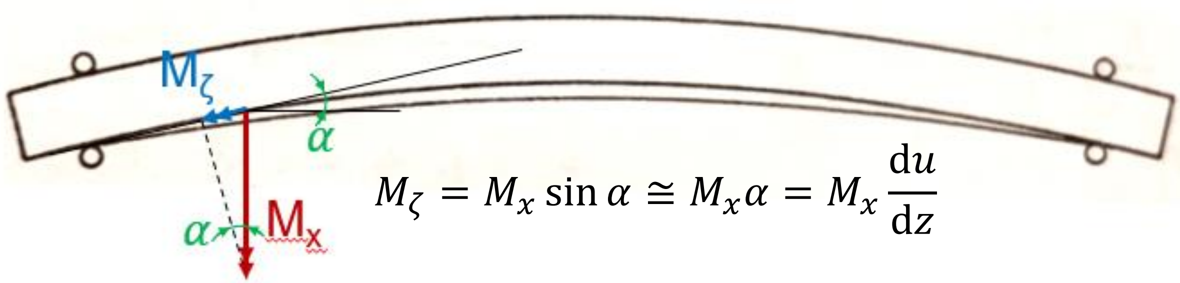
Regarding torsion and twist:

$$\frac{d\theta}{\mathrm{d}z} = -\frac{M_\zeta}{GJ_{xy}} = -\frac{M_x}{GJ_{xy}}\frac{\mathrm{d}u}{\mathrm{d}z} \quad (2)$$

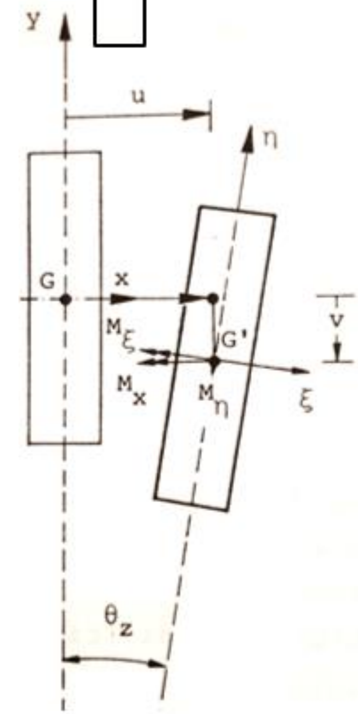
Lateral torsional buckling of beams







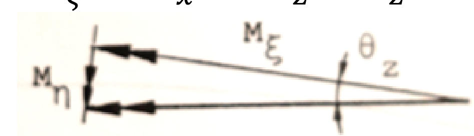
$$M_\zeta = M_x \sin\alpha \cong M_x\alpha = M_x \frac{\mathrm{d}u}{\mathrm{d}z}$$



Differentiating (2) and substituting (1):

$$\frac{d^2\theta}{\mathrm{d}z^2} + \frac{M_x^2}{EJ_xGJ_{xy}}\theta_z = 0$$

$$M_\eta = M_x \sin\theta_z \cong M_x\theta_x$$
$$M_\xi = M_x \cos\theta_z \cong \theta_z$$

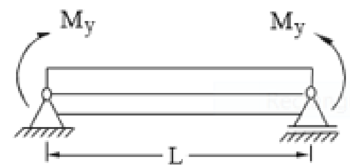
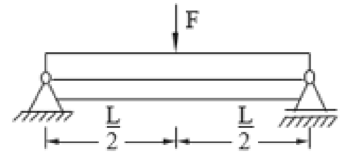
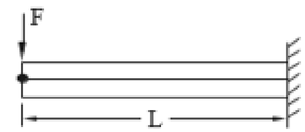
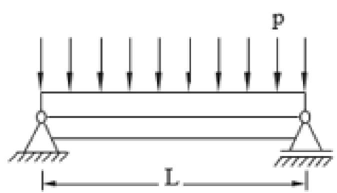
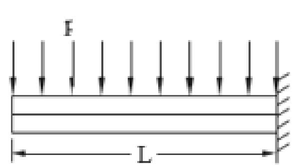


Lateral buckling solutions

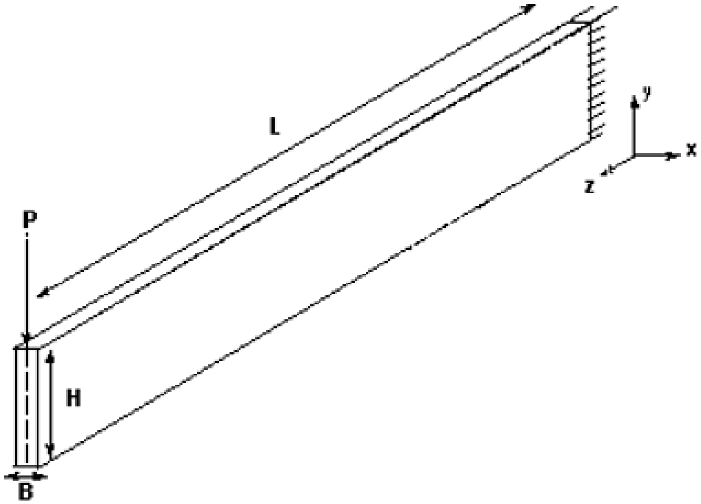
Solutions $\theta \neq 0$ for eq.

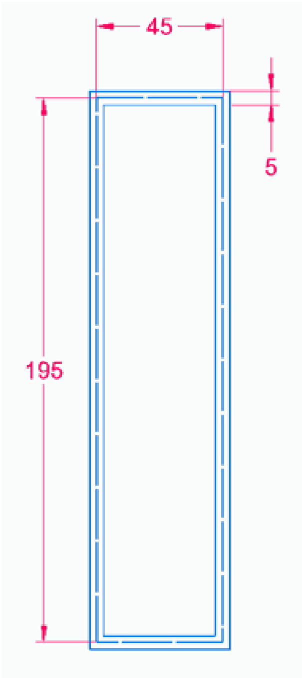
$$\frac{d^2\theta}{\mathrm{d}z^2} + \frac{M_x^2}{EJ_x GJ_{xy}}\theta_z = 0$$

can be found for different loading and boundary conditions, see table ->

Loading	M_y	Critical loading
1. 	—	$M_{crit} \approx \frac{\pi}{L}\sqrt{E \cdot J_z \cdot G \cdot J_t}$ Assumption: $J_z << J_y$
2. 	$\frac{F}{2} \cdot \frac{L}{2}$	$F_{crit} \approx \frac{16{,}93}{L^2}\sqrt{E \cdot J_z \cdot G \cdot J_t}$
3. 	$F \cdot L$	$F_{crit} \approx \frac{4{,}2}{L^2}\sqrt{E \cdot J_z \cdot G \cdot J_t}$
4. 	$\frac{1}{8} p \cdot L^2$	$p_{crit} \approx \frac{28{,}3}{L^3}\sqrt{E \cdot J_z \cdot G \cdot J_t}$
5. 	$\frac{p \cdot L^2}{2}$	$p_{crit} \approx \frac{,85}{\,3}\sqrt{E \cdot J_z \cdot G \cdot J_t}$

Exercise





L = 2000 m

Create a shell element model

Evaluate the critical load