POLITECNICO DI MILANO

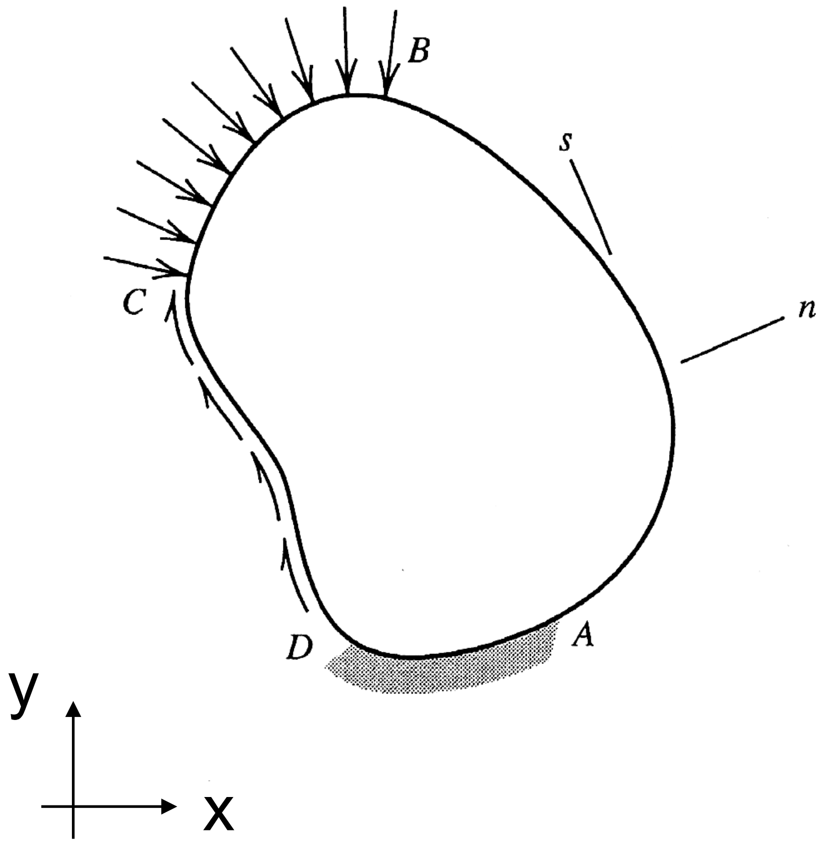
Finite Element Simulation For Mechanical Design

More on the formulation of FE method

2D elements

A. Bernasconi

2D elastic problem



We seek

u(x,y) e v(x,y)

Such that:

- Internal and external constraints are satisfied
- Equilibrium equations are satisfied

Solution needs taking into account:

Strain-displacement relationships

$$\varepsilon_x = \partial u / \partial x$$

$$\varepsilon_y = \partial v / \partial y$$

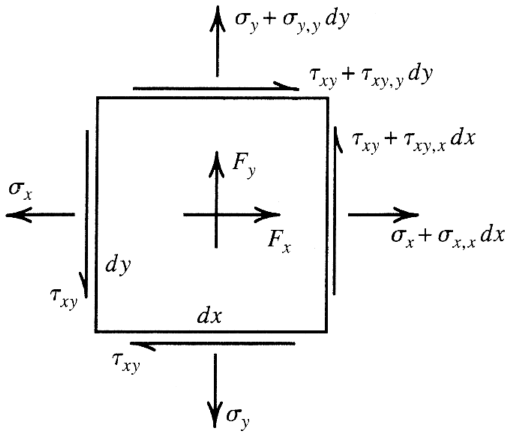
$$\gamma_{xy} = \partial u / \partial y + \partial v / \partial x$$

Compatibility

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \partial/\partial x & 0 \\ 0 & \partial/\partial y \\ \partial/\partial y & \partial/\partial x \end{bmatrix} \begin{Bmatrix} u \\ v \end{Bmatrix}$$

In matrix form



$$\frac{\partial \sigma_x}{\partial x} + \frac{\partial \tau_{xy}}{\partial y} + F_x = 0$$

$$\frac{\partial \sigma_y}{\partial y} + \frac{\partial \tau_{xy}}{\partial x} + F_y = 0$$

Equilibrium

And stress-strain relationship (constitutive model)

Plane stress

$$\begin{cases} \varepsilon_x = \sigma_x / E - \nu \, \sigma_y / E \\ \varepsilon_y = \sigma_y / E - \nu \, \sigma_x / E \\ \gamma_{xy} = \tau_{xy} / G \end{cases}$$

In matrix form

$$\{\varepsilon\} = [E]^{-1}\{\sigma\} \qquad \{\sigma\} = [E]\{\varepsilon\}$$

$$[E] = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix}$$

Note: it does not mean that stresses and strains form a vector; it is only a useful formalism

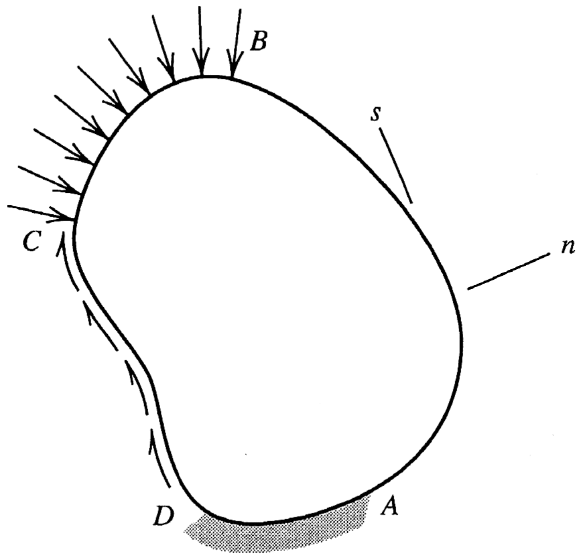
$$U = \frac{1}{2}\sigma_x\varepsilon_x + \frac{1}{2}\sigma_y\varepsilon_y + \frac{1}{2}\tau_{xy}\gamma_{xy} = \frac{1}{2}\{\varepsilon\}^T\{\sigma\} = \frac{1}{2}\{\varepsilon\}^T[E]\{\varepsilon\}$$

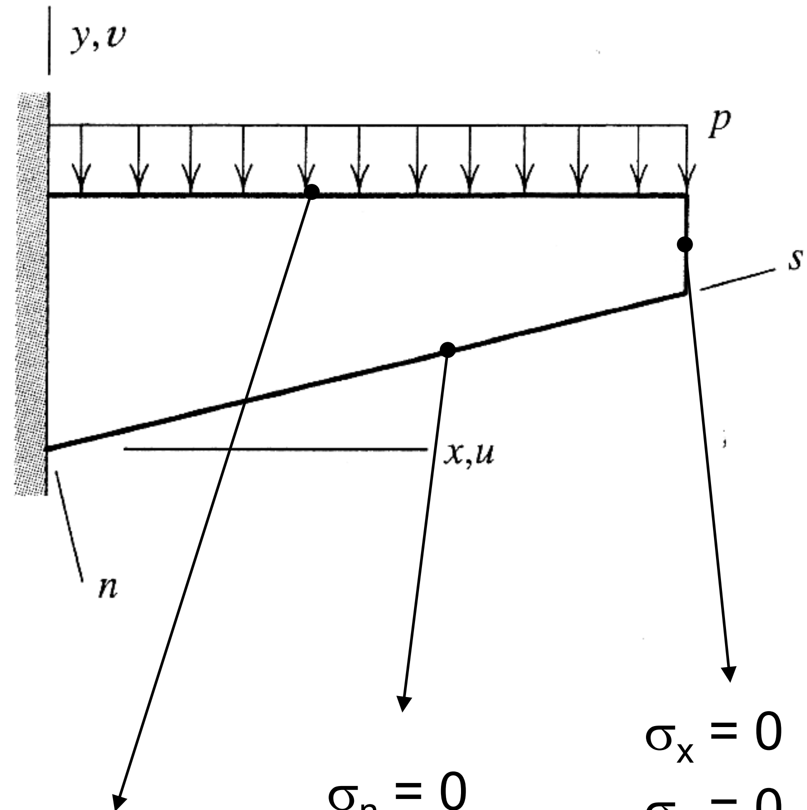
Plane strain

$$\{\varepsilon\} = [E]^{-1}\{\sigma\}$$

$$[E] = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} (1-\nu) & \nu & 0 \\ \nu & (1-\nu) & 0 \\ 0 & 0 & (1-2\nu)/2 \end{bmatrix}$$

Boundary conditions





$\sigma_x = 0$
$\sigma_y = -p$
$\tau_{xy} = 0$

$\sigma_n = 0$
$\tau_n = 0$

$\sigma_x = 0$
$\sigma_y = 0$
$\tau_{xy} = 0$

The elastic problem

If the displacement field, the stress field and the corresponding (via stress-strain relationships) strain field satisfy simultaneously:

1. equilibrium
2. compatibility
3. boundary conditions

The solution is exact and unique

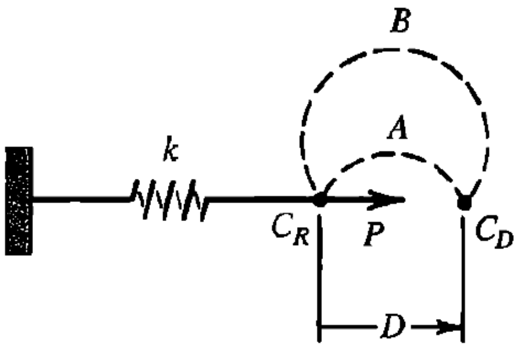
To obtain the solution, a **system of differential equations** needs being solved

The closed form solution is known only for certain classes of problems (axial loading, bending, shear, torsion of beams, bending of plates, thins disks, …)

With the finite element method, the system of differential equations is substituted by a system of algebraic equations.

From the first lesson:

Among all possible configurations (i.e. which respect internal compatibility and essential* boundary conditions) of a conservative system, those that satisfy the equilibrium equations make the potential energy stationary with respect to small admissible variations of the displacement



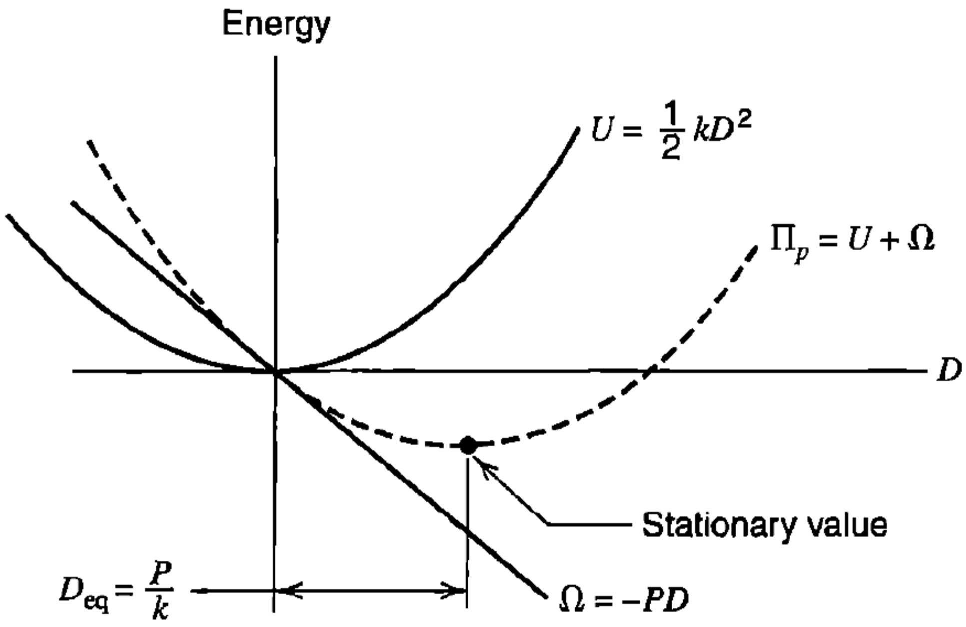
$$\Pi_p = U + \Omega \qquad\qquad U = \frac{1}{2}kD^2 \quad \text{and} \quad \Omega = -PD$$

$$\Pi_p = \frac{1}{2}kD^2 - PD$$

$$d\Pi_p = (kD_{eq} - P)dD = 0 \qquad \text{hence} \qquad D_{eq} = P/k$$

*, essential boundary conditions are expressed in terms of field variables (displacements in the case of structural analyses), non-essential to higher order derivatives of field variables

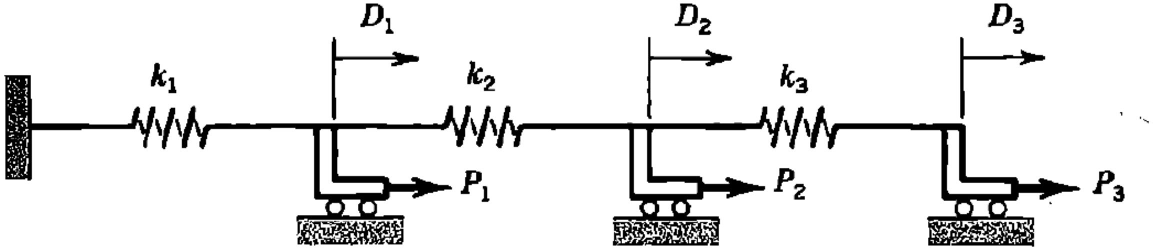
Functional Π_p



Potential energy is a **functional**, an integral expression that contains implicitly the differential equations governing the problem

System having *n* d.o.f.

$$d\Pi_p = 0 \qquad \text{where} \qquad d\Pi_p = \frac{\partial \Pi_p}{\partial D_1} dD_1 + \frac{\partial \Pi_p}{\partial D_2} dD_2 + \cdots + \frac{\partial \Pi_p}{\partial D_n} dD_n$$



$$\Pi_p = \frac{1}{2} k_1 D_1^2 + \frac{1}{2} k_2 (D_2 - D_1)^2 + \frac{1}{2} k_3 (D_3 - D_2)^2 - P_1 D_1 - P_2 D_2 - P_3 D_3$$

$$\begin{aligned} k_1 D_1 - k_2 (D_2 - D_1) - P_1 &= 0 \\ k_2 (D_2 - D_1) - k_3 (D_3 - D_2) - P_2 &= 0 \\ k_3 (D_3 - D_2) - P_3 &= 0 \end{aligned}$$

$$\begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix}$$

Total strain energy

For a system having *n* d.o.f.:

$$U = \frac{1}{2}\{\mathbf{D}\}^T[\mathbf{K}]\{\mathbf{D}\} \quad \text{and} \quad \Omega = -\{\mathbf{D}\}^T\{\mathbf{R}\}$$

For a continuum body:

$$U_0 = \int \{\boldsymbol{\sigma}\}^T\{d\boldsymbol{\varepsilon}\}$$

$$= \int \sigma_x \, d\varepsilon_x + \int \sigma_y \, d\varepsilon_y + \int \sigma_z \, d\varepsilon_z + \int \tau_{xy} \, d\gamma_{xy} + \int \tau_{yz} \, d\gamma_{yz} + \int \tau_{zx} \, d\gamma_{zx}$$

$$U_0 = \frac{1}{2}\{\boldsymbol{\sigma}\}^T\{\boldsymbol{\varepsilon}\} = \frac{1}{2}\{\boldsymbol{\varepsilon}\}^T[\mathbf{E}]\{\boldsymbol{\varepsilon}\}$$

Potential energy of an elastic body carrying conservative loads

$$\Pi = \int_V \left(\frac{1}{2}\{\boldsymbol{\varepsilon}\}^T[\mathbf{E}]\{\boldsymbol{\varepsilon}\} - \{\boldsymbol{\varepsilon}\}^T[\mathbf{E}]\{\boldsymbol{\varepsilon}_0\} + \{\boldsymbol{\varepsilon}\}^T\{\boldsymbol{\sigma}_0\} \right) dV - \int_V \{\mathbf{u}\}^T\{\mathbf{F}\}\, dV$$
$$- \int_S \{\mathbf{u}\}^T\{\boldsymbol{\Phi}\}\, dS - \{\mathbf{D}\}^T\{\mathbf{P}\}$$

F, volume forces

$\boldsymbol{\Phi}$, surface forces

P, concentrated loads

D, displacement or rotations associated to concentrated loads (either forces or moments)

The Rayleigh – Ritz method

Let's first introduce an approximate expression of the displacement field, by appropriate functions (admissible) and generalized coordinates or d.o.f.

$$u = \sum_{i=1}^{l} a_i f_i \qquad v = \sum_{i=l+1}^{m} a_i f_i \qquad w = \sum_{i=m+1}^{n} a_i f_i$$

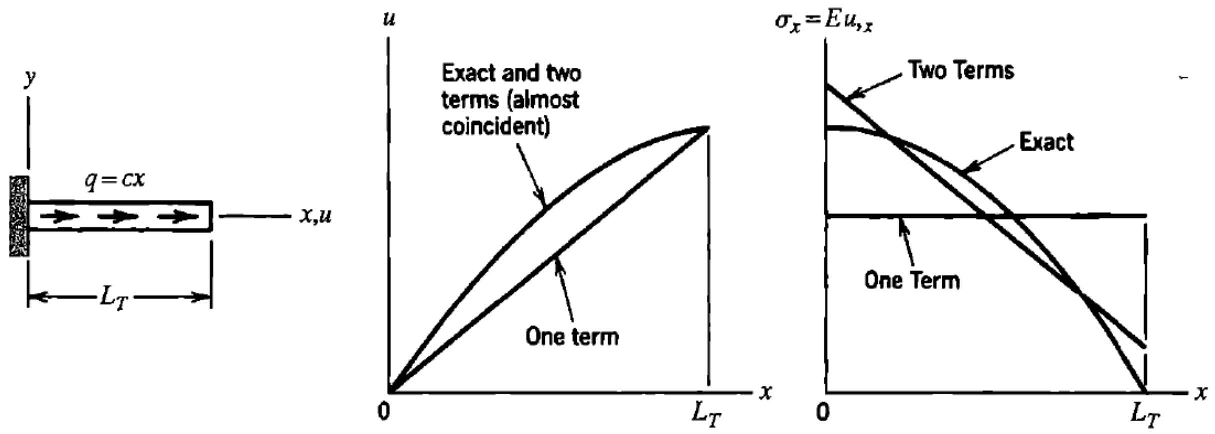
The condition of stationary potential energy is given by

$$\frac{\partial \Pi_p}{\partial a_i} = 0 \quad \text{for} \quad i = 1, 2, \ldots, n$$

This allows for transforming the initial differential problem into a system of algebraic equations of the generalized coordinates.

The solution is approximate and approximation improves as the number of terms $a_i f_i$ increases.

Example



$$\Pi_p = \int_0^{L_T} \frac{1}{2} E u,_x^2 A\, dx - \int_0^{L_T} u\, cx\, dx$$

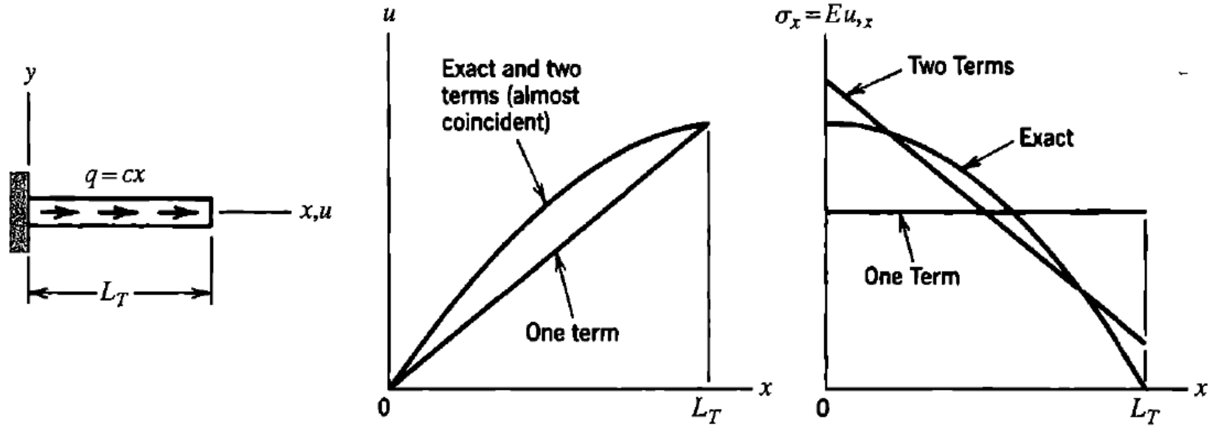
$$u = \sum_{i=1}^{n} a_i f_i \quad = a_1 x + a_2 x^2 + a_3 x^3 + \cdots + a_n x^n$$

By introducing the functional, an algebraic function of the generalized coordinates is obtained

$$\Pi_p = \frac{AEL_T}{2} a_1^2 - \frac{cL_T^3}{3} a_1$$

$$u = a_1 x. \qquad \frac{d\Pi_p}{da_1} = 0 \qquad a_1 = \frac{cL_T^2}{3AE}$$

$$u = \frac{cL_T^2}{3AE} x \qquad \sigma_x = Eu,_x = \frac{cL_T^2}{3A}$$



$$u = a_1 x + a_2 x^2$$

$$AEL_T \begin{bmatrix} 1 & L_T \\ L_T & 4L_T^2/3 \end{bmatrix} \begin{Bmatrix} a_1 \\ a_2 \end{Bmatrix} = \frac{cL_T^3}{12} \begin{Bmatrix} 4 \\ 3L_T \end{Bmatrix}$$

$$u = \frac{cL_T}{12AE}(7L_T x - 3x^2)$$

$$\sigma_x = Eu_{,x} = \frac{cL_T}{12A}(7L_T - 6x)$$

$$u = a_1 x + a_2 x^2 + a_3 x^3$$

Would yield the exact solution (but it is not always so)

$$u = \frac{c}{6AE}(3L_T^2 x - x^3)$$

From classical form to FE form

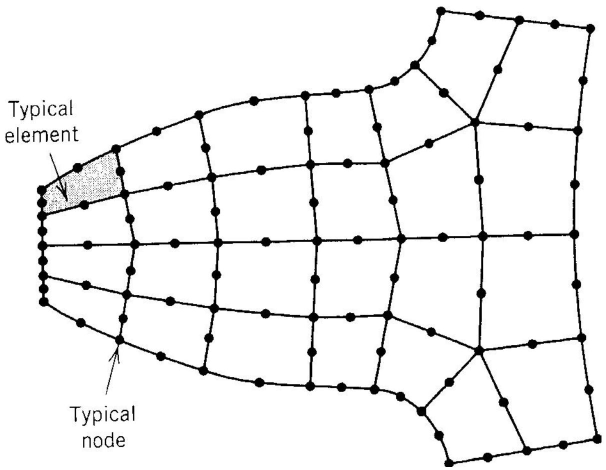
The classical form of the Rayleigh-Ritz method is based on the definition of a displacement field that approximates the solution **over the entire domain**

The d.o.f. (generalized coordinates) do not always have a recognizable physical meaning

The FE form is based on an approximation by **piecewise interpolation over subdomains**, i.e. over the finite elements

The d.o.f. have a physical meaning that can be easily recognized (nodal displacements) and define a configuration of the system that always respect compatibility and boundary conditions

First step: the structure is divided into subdomains (elements)

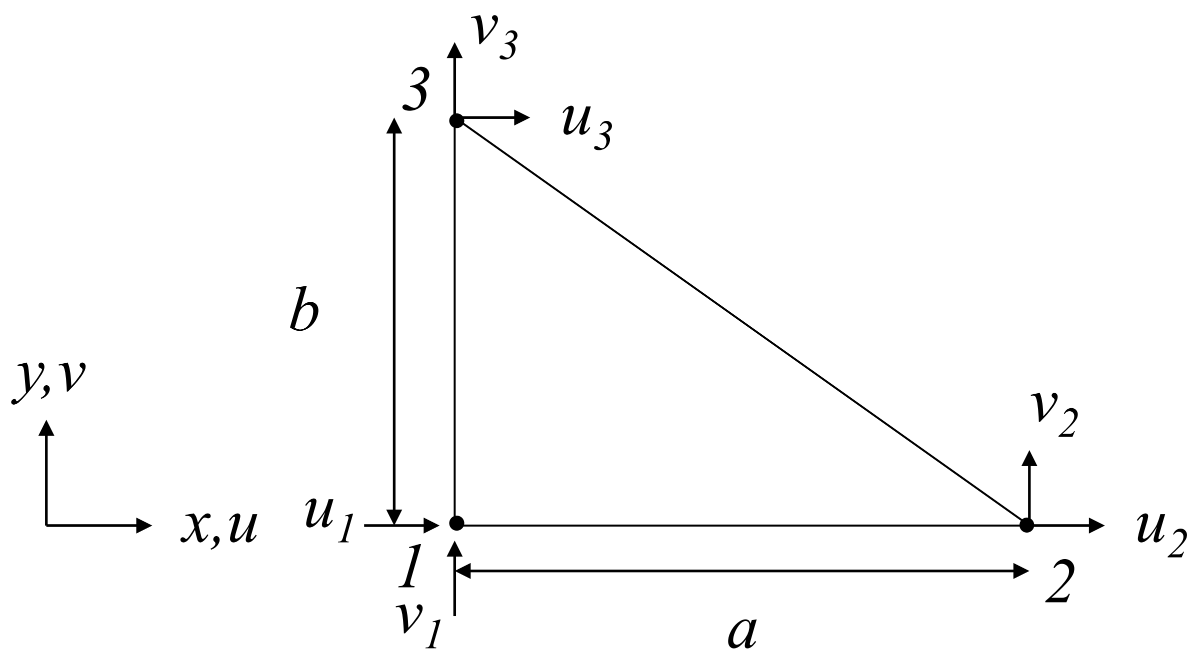


The entire geometry is subdivided into subdomains, called elements, forming a mesh.

The displacement field is described by simple functions, usually polynomials, over each subdomain.

The **nodes** (not necessarily the vertices), and not the edges, are the connectors between the elements

The elements and the displacement field



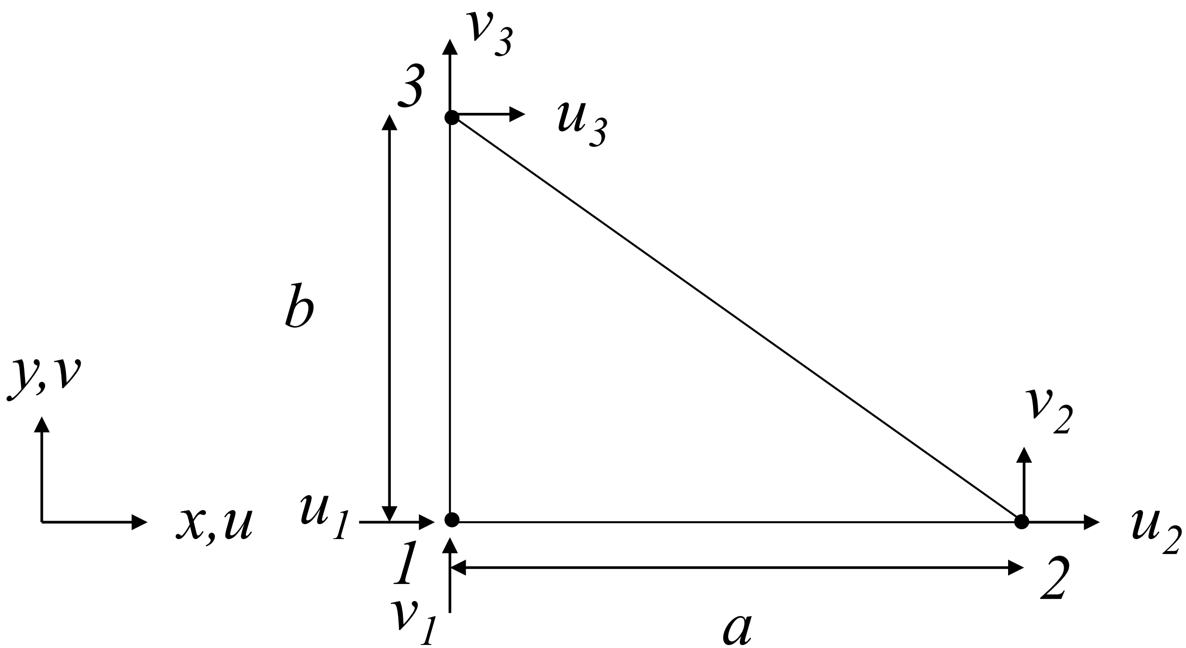
Polynomial interpolating the displacements

$$u(x,y) = \beta_1 + \beta_2 x + \beta_3 y$$
$$v(x,y) = \beta_4 + \beta_5 x + \beta_6 y$$

β_i, **generalized coordinates**, that can be expressed in terms of the nodal displacements

If $u(0,0)=u_1$, then $b_1=u_1$, and so on…

Expression of the displacement field inside the element as a function of nodal displacements



$$u(x,y) = \left(1 - \frac{x}{a} - \frac{y}{b}\right)u_1 + \frac{x}{a}u_2 + \frac{y}{b}u_3$$

$$v(x,y) = \left(1 - \frac{x}{a} - \frac{y}{b}\right)v_1 + \frac{x}{a}v_2 + \frac{y}{b}v_3$$

The β_i, like u_i e v_i, are 6 and are enough to describe the displacement field at any point inside the element => 6 d.o.f.

Shape functions

$$\begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & \dots \\ 0 & N_1 & 0 & N_2 & \dots \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ \dots \end{Bmatrix}$$

$$\begin{Bmatrix} u(x,y) \\ v(x,y) \end{Bmatrix} = [\mathbf{N}(x,y)]\{d\}$$

Then it can be expressed as:

Contains the partial derivatives of the shape functions

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} \partial/\partial x & 0 \\ 0 & \partial/\partial y \\ \partial/\partial y & \partial/\partial x \end{bmatrix} [\mathbf{N}]\{d\} = \partial[\mathbf{N}]\{d\} = [\mathbf{B}]\{d\}$$

Solution of the linear elastic problem

After having linked the displacement filed inside each element to the nodal displacements, we proceed to solving the problem in terms of nodal displacements.

After referring loads and constraints to nodal displacements (we'll se how later), a linear system is solved:

$$[K]\{D\} = \{P\}$$

$$\{D\} = [K]^{-1}\{P\}$$

where [K] is the stiffness matrix.

Let's see who we obtain this system.

The expression of Π_p for a set of N_{els} elements (in presence of initial stresses and strains) is

$$\Pi_p = \sum_{i=1}^{N_{els}} \int \left(\frac{1}{2}\{\boldsymbol{\varepsilon}\}^T [\mathbf{E}]\{\boldsymbol{\varepsilon}\} - \{\boldsymbol{\varepsilon}\}^T [\mathbf{E}]\{\boldsymbol{\varepsilon}_0\} + \{\boldsymbol{\varepsilon}\}^T \{\boldsymbol{\sigma}_0\} \right) dV$$

$$- \sum_{i=1}^{N_{els}} \int \{\mathbf{u}\}^T \{\mathbf{F}\} dV - \sum_{i=1}^{N_{els}} \int \{\mathbf{u}\}^T \{\mathbf{\Phi}\} dS - \{\mathbf{D}\}^T \{\mathbf{P}\}$$

Displacements are interpolated inside each element as a function of nodal displacements $\mathbf{d}_i$ of the element

$$\{\mathbf{u}\} = [\mathbf{N}]\{\mathbf{d}\}_i$$

FE form of the Rayleigh – Ritz method

$$\Pi_p = \frac{1}{2}\sum_{i=1}^{N_{\text{els}}}\{\mathbf{d}\}_i^T[\mathbf{k}]_i\{\mathbf{d}\}_i - \sum_{i=1}^{N_{\text{els}}}\{\mathbf{d}\}_i^T\{\mathbf{r}_e\}_i - \{\mathbf{D}\}^T\{\mathbf{P}\}$$

where:

$$[\mathbf{k}]_i = \int[\mathbf{B}]^T[\mathbf{E}][\mathbf{B}]\,dV$$

is the element’s stiffness matrix, and

Distributed loads reduced to nodes

$$\{\mathbf{r}_e\}_i = \int[\mathbf{B}]^T[\mathbf{E}]\{\boldsymbol{\varepsilon}_0\}\,dV - \int[\mathbf{B}]^T\{\boldsymbol{\sigma}_0\}\,dV + \int[\mathbf{N}]^T\{\mathbf{F}\}\,dV + \int[\mathbf{N}]^T\{\boldsymbol{\Phi}\}\,dS$$

is the vector of the element’s nodal loads

Stiffness matrix

The expression of the stiffness matrix

$$[K]=\int_V [B]^T [E][B]dV$$

[**E**] is the matrix of the material's properties

[**B**] is such that: {**ε**}=[**B**] {**d**}

The expression of [**k**] derives form the elastic strain energy

$$U=\frac{1}{2}\int_V \{\varepsilon\}^T \overbrace{[E]\{\varepsilon\}}^{\{\sigma\}}dV=$$

$$=\frac{1}{2}\{d\}^T\int_V [B]^T [E][B]dV\ \{d\}=$$

$$=\frac{1}{2}\{d\}^T[k]\{d\}$$

It is clear that the nature of [K] depends on shape functions

FE form of the Rayleigh – Ritz method

Now we have to express local d.o.f. in terms of global d.o.f.

$$\{\mathbf{d}\}_i = [\mathbf{L}]_i\{\mathbf{D}\} \qquad \text{e.g.} \begin{Bmatrix} d_1 \\ d_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix}$$

Using a matrix L that contains only zeros and ones (with many elements, L is very sparse).

The expression of the potential energy becomes

$$\Pi_p = \frac{1}{2}\{\mathbf{D}\}^T\left(\sum_{i=1}^{N_{\text{els}}}[\mathbf{L}]_i^T[\mathbf{k}]_i[\mathbf{L}]_i\right)\{\mathbf{D}\} - \{\mathbf{D}\}^T\sum_{i=1}^{N_{\text{els}}}[\mathbf{L}]_i^T\{\mathbf{r}_e\}_i - \{\mathbf{D}\}^T\{\mathbf{P}\}$$

And finally the condition of stationarity

$$\left(\sum_{i=1}^{N_{\text{els}}}[\mathbf{L}]_i^T[\mathbf{k}]_i[\mathbf{L}]_i\right)\{\mathbf{D}\} = \{\mathbf{P}\} + \sum_{i=1}^{N_{\text{els}}}[\mathbf{L}]_i^T\{\mathbf{r}_e\}_i \qquad \text{i.e. } [K]\{D\} = \{R\}$$

Compatibility

It expresses the absence of tearing and interpenetration.

Displacements must be expressed by continuous and single valued functions of the position

In terms of strains, compatibility is expressed as

$$\frac{\partial^2 \varepsilon_x}{\partial y^2} + \frac{\partial^2 \varepsilon_y}{\partial x^2} = \frac{\partial^2 \gamma_{xy}}{\partial x \partial y}$$

Not any strain field can satisfy this condition

Instead, any arbitrary displacement field, provided it is expressed by continuous and single valued functions, always satisfies it

$$\frac{\partial^2\, \partial u/\partial x}{\partial y^2} + \frac{\partial^2\, \partial v/\partial y}{\partial x^2} = \frac{\partial^2 \left(\partial u/\partial y + \partial v/\partial x\right)}{\partial x \partial y}$$

$$\frac{\partial^3 u}{\partial y^2 \partial x} + \frac{\partial^3 v}{\partial x^2 \partial y} = \frac{\partial^3 u}{\partial x \partial y^2} + \frac{\partial^3 v}{\partial x^2 \partial y} \quad \text{c.v.d.}$$

Finite elements are built to satisfy compatibility **inside the elements themselves**

The elastic problem and the FE method

We have seen that if the displacement field and the corresponding stress field satisfy at the same time:

1. equilibrium
2. boundary conditions
3. compatibility

The elastic solution is exact and unique

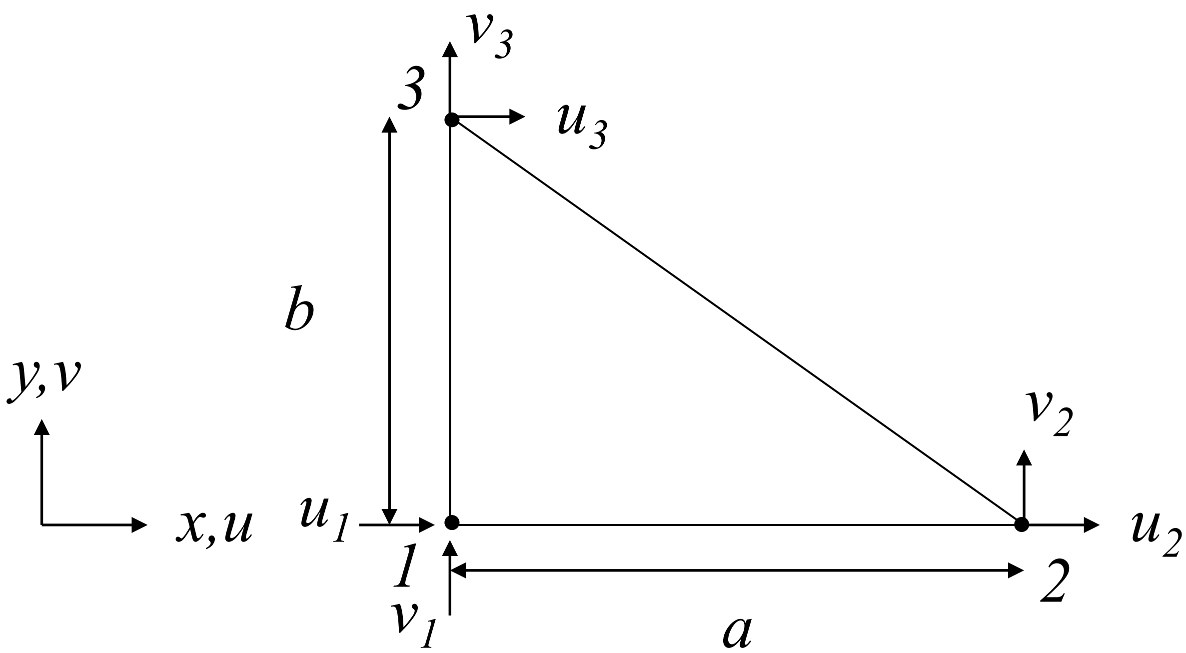
By the finite element method:

1. Compatibility is satisfied **inside** each element (the expression of the displacement filed is a polynomial function)
2. Equilibrium equations are not always satisfied, at least not in every point of the model (they are satisfied in an average/integral sense)
3. The same applies to the boundary conditions

With mesh refinement, conditions 2 and 3 are approached, provided that **completeness** is satisfied:

- Elements can describe a constant strain field
- Compatibility of displacements is guaranteed along the edges

Compatibility of displacements along edges



Let's consider edge 1-3 (it belongs to the straight line of equation x=0)

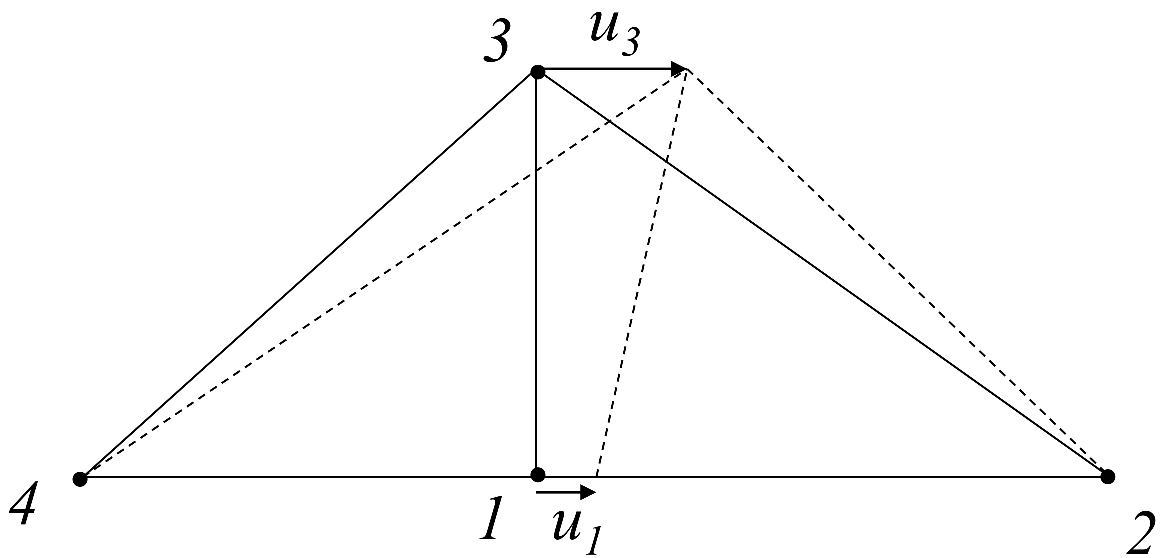
$$u(x,y)=\left(1-\frac{y}{b}\right)u_1+\frac{y}{b}u_3$$

$$v(x,y)=\left(1-\frac{y}{b}\right)v_1+\frac{y}{b}v_3$$

The relationship:

- is linear
- depends only on values at the extremes (u_1 e u_3)

Compatibility of displacements along edges

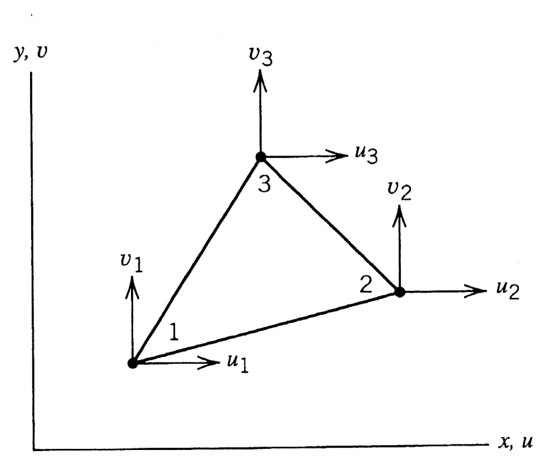


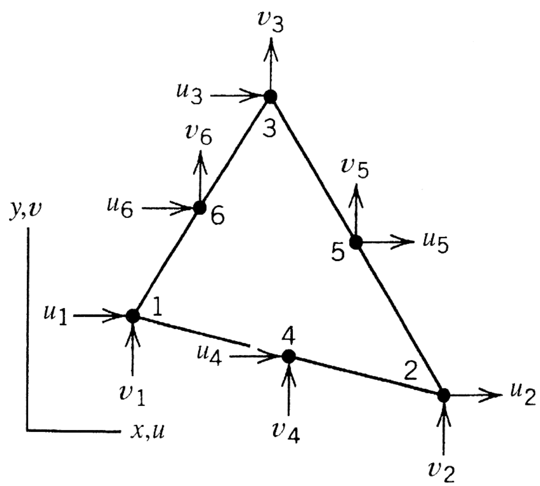
Similarly, it can be demonstrated that the same holds true for another element sharing edge 1-3 with the previous one

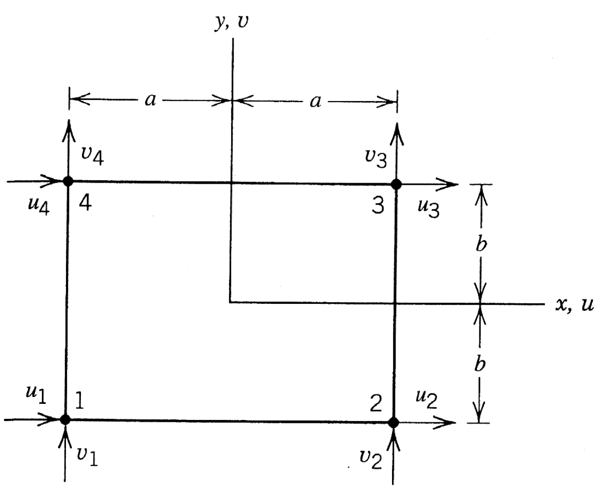
This prevents form tearing and interpenetration

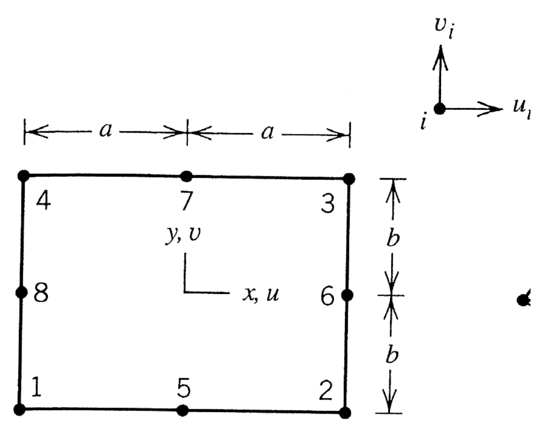
Combined with constant strain, it ensures that the solution converges as the mesh is refined.

Plane elements



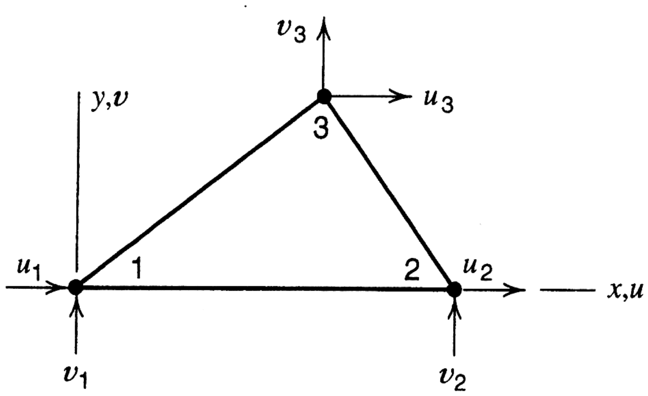






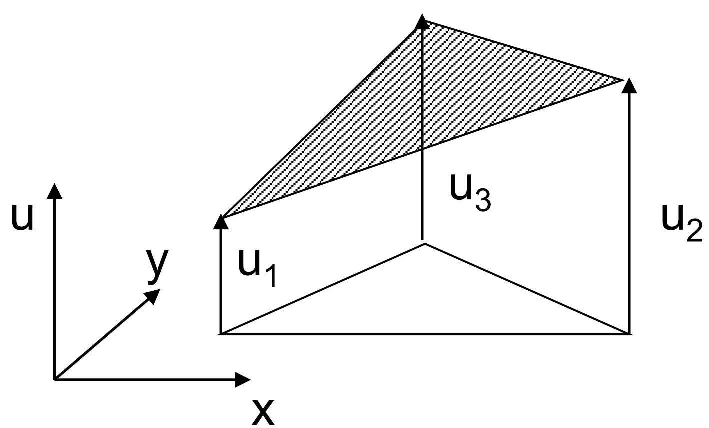
(+ special elements)

Triangular elements



Linear displacements

$$u = \beta_1 + \beta_2 x + \beta_3 y$$
$$v = \beta_4 + \beta_5 x + \beta_6 y$$

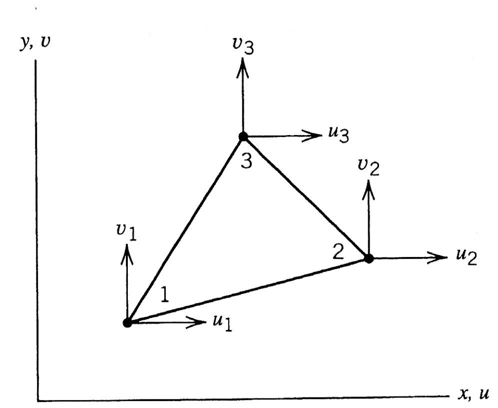


The same holds for v(x,y)

Triangular elements: [B] matrix

Constant strain (and stresses)

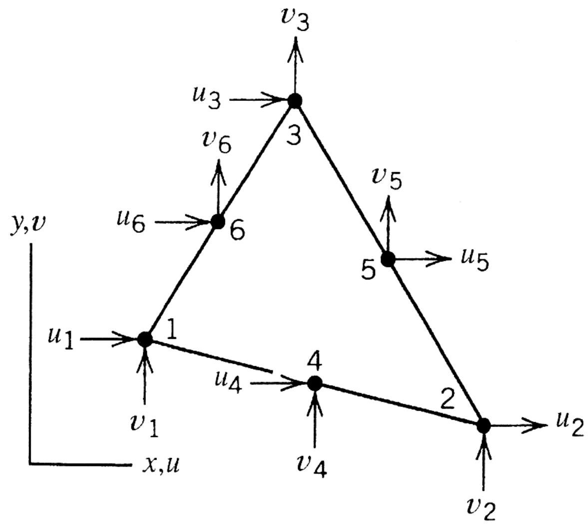
$$\varepsilon_x = \beta_2$$
$$\varepsilon_y = \beta_6$$
$$\gamma_{xy} = \beta_3 + \beta_5$$

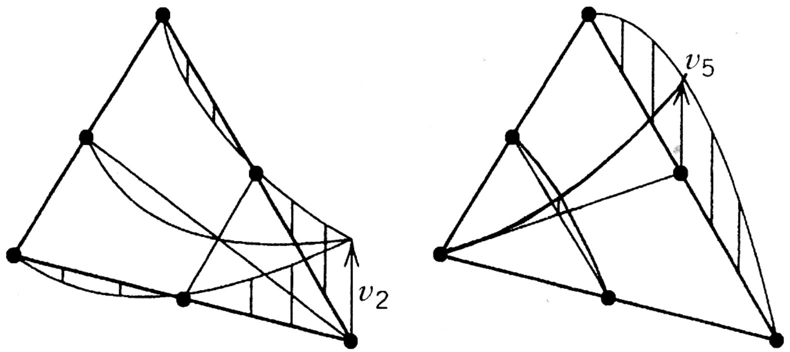


$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{1}{2A} \begin{bmatrix} y_2 - y_3 & 0 & y_3 - y_1 & 0 & y_1 - y_2 & 0 \\ 0 & x_3 - x_2 & 0 & x_1 - x_3 & 0 & x_2 - x_1 \\ x_3 - x_2 & y_2 - y_3 & x_1 - x_3 & y_3 - y_1 & x_2 - x_1 & y_1 - y_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{Bmatrix}$$

$$A = (x_2 - x_1)(y_3 - y_1) - (x_3 - x_1)(y_2 - y_1)$$

6 nodes triangular element





Quadratic shape functions

$$u = \beta_1 + \beta_2 x + \beta_3 y + \beta_4 x^2 + \beta_5 xy + \beta_6 y^2$$
$$v = \beta_7 + \beta_8 x + \beta_9 y + \beta_{10} x^2 + \beta_{11} xy + \beta_{12} y^2$$

Quadratic displacements

$$\varepsilon_x = \beta_2 + 2\beta_4 x + \beta_5 y$$

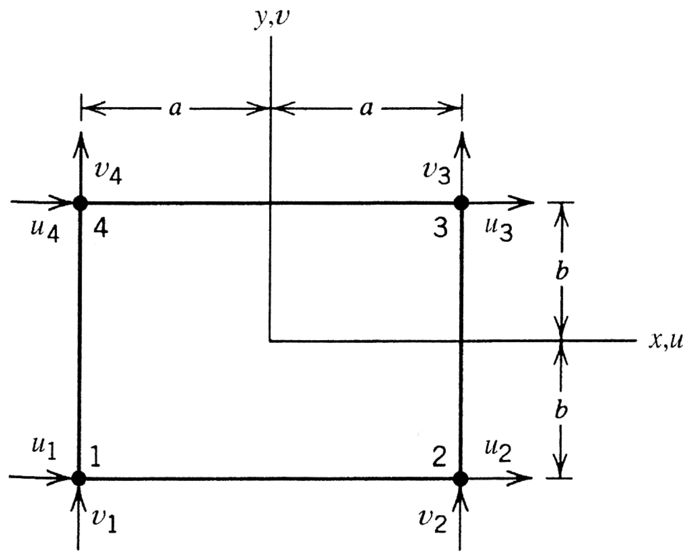
Describes correctly beam bending

$$\varepsilon_y = \beta_9 + \beta_{11} x + 2\beta_{12} y$$
$$\gamma_{xy} = (\beta_3 + \beta_8) + (\beta_5 + 2\beta_{10})x + (2\beta_6 + \beta_{11})y$$

Linear strains

(bending OK)

Rectangular bi-linear element with 4 nodes



$$u = \beta_1 + \beta_2 x + \beta_3 y + \beta_4 xy$$
$$v = \beta_5 + \beta_6 x + \beta_7 y + \beta_8 xy$$

$$\varepsilon_x = \beta_2 + \beta_4 y$$
$$\varepsilon_y = \beta_7 + \beta_8 x$$
$$\gamma_{xy} = (\beta_3 + \beta_6) + \beta_4 x + \beta_8 y$$

The constant strain case is included.

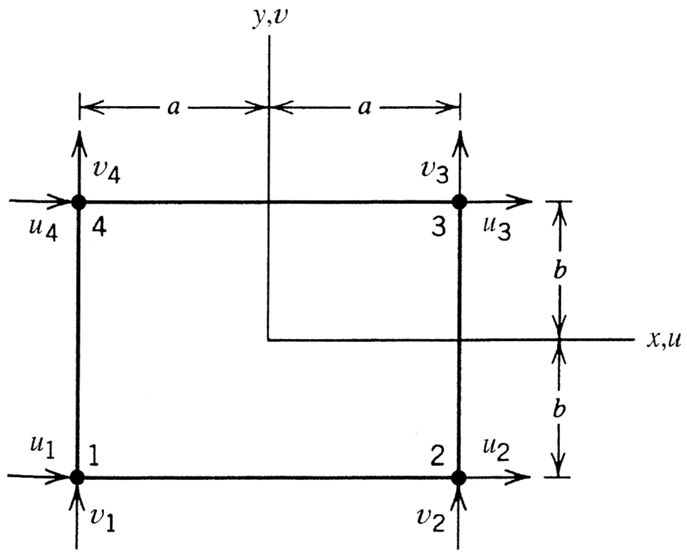
Along the edges (x=+/- a e y = +/-b) i the fields u(x,y) e v(x,y) become linear and depend only upon values at vertices.

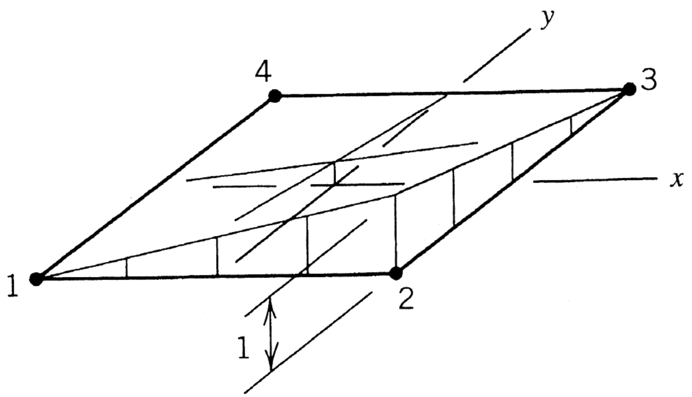
In this case, the condition of **completeness** is satisfied**:**

- Ability of describing a constant strain field
- Compatibility of displacements between elements

It ensures convergence as the mesh is refined

Shape functions





Shape functions are **bi-linear**

$$N_1 = \frac{(a-x)(b-y)}{4ab} \qquad N_2 = \frac{(a+x)(b-y)}{4ab}$$

$$N_3 = \frac{(a+x)(b+y)}{4ab} \qquad N_4 = \frac{(a-x)(b+y)}{4ab}$$

Shape functions and matrix [k]

$$\begin{Bmatrix} u \\ v \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 & 0 \\ 0 & N_1 & 0 & N_2 & 0 & N_3 & 0 & N_4 \end{bmatrix} \{d\}$$

$$\{d\} = \{u_1 \quad v_1 \quad u_2 \quad v_2 \quad u_3 \quad v_3 \quad u_4 \quad v_4\}^T$$

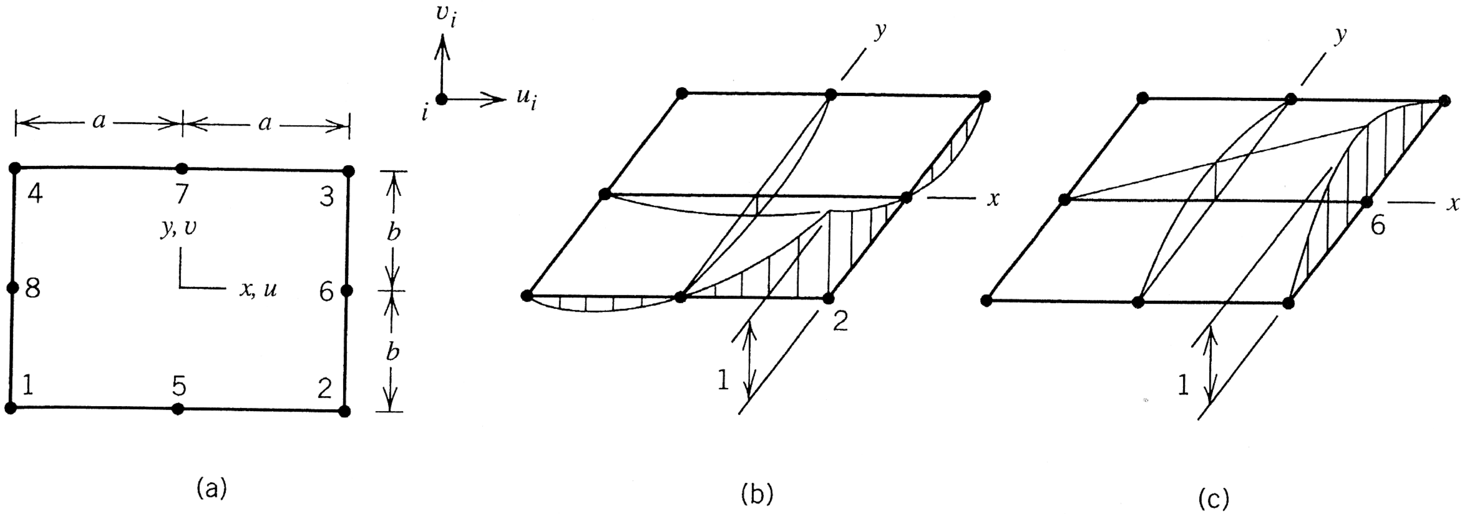
$$[B] = \partial[N]$$

$$\partial = \begin{bmatrix} \partial/\partial x & 0 \\ 0 & \partial/\partial y \\ \partial/\partial y & \partial/\partial x \end{bmatrix}$$

$$\underset{8\times 8}{[K]} = \int_{-b}^{b}\int_{-a}^{a} \underset{8\times 3}{[B]^T} \underset{3\times 3}{[E]} \underset{3\times 8}{[B]} t\,dx\,dy$$

Matrix [B] contains linear functions => [k] can be integrated in closed form

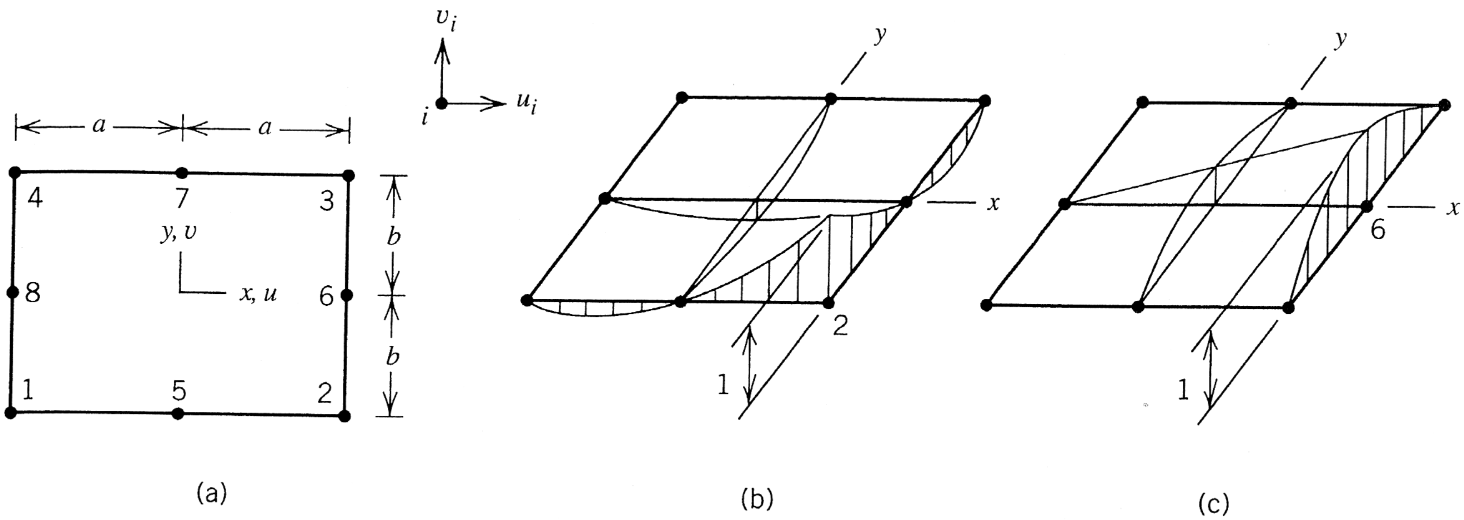
8 node elements



$$u = \beta_1 + \beta_2 x + \beta_3 y + \beta_4 x^2 + \beta_5 xy + \beta_6 y^2 + \beta_7 x^2 y + \beta_8 xy^2$$
$$v = \beta_9 + \beta_{10} x + \beta_{11} y + \beta_{12} x^2 + \beta_{13} xy + \beta_{14} y^2 + \beta_{15} x^2 y + \beta_{16} xy^2$$

$$\varepsilon_x = \beta_2 + 2\beta_4 x + \beta_5 y + 2\beta_7 xy + \beta_8 y^2$$
$$\varepsilon_y = \beta_{11} + \beta_{13} x + 2\beta_{14} y + \beta_{15} x^2 + 2\beta_{16} xy$$
$$\gamma_{xy} = (\beta_3 + \beta_{10}) + (\beta_5 + 2\beta_{12})x + (2\beta_6 + \beta_{13})y + \beta_7 x^2 + 2(\beta_8 + \beta_{15})xy + \beta_{16} y^2$$

8 node elements



$$u = \beta_1 + \beta_2 x + \beta_3 y + \beta_4 x^2 + \beta_5 xy + \beta_6 y^2 + \beta_7 x^2 y + \beta_8 xy^2$$
$$v = \beta_9 + \beta_{10} x + \beta_{11} y + \beta_{12} x^2 + \beta_{13} xy + \beta_{14} y^2 + \beta_{15} x^2 y + \beta_{16} xy^2$$

$$\varepsilon_x = \beta_2 + 2\beta_4 x + \beta_5 y + 2\beta_7 xy + \beta_8 y^2$$
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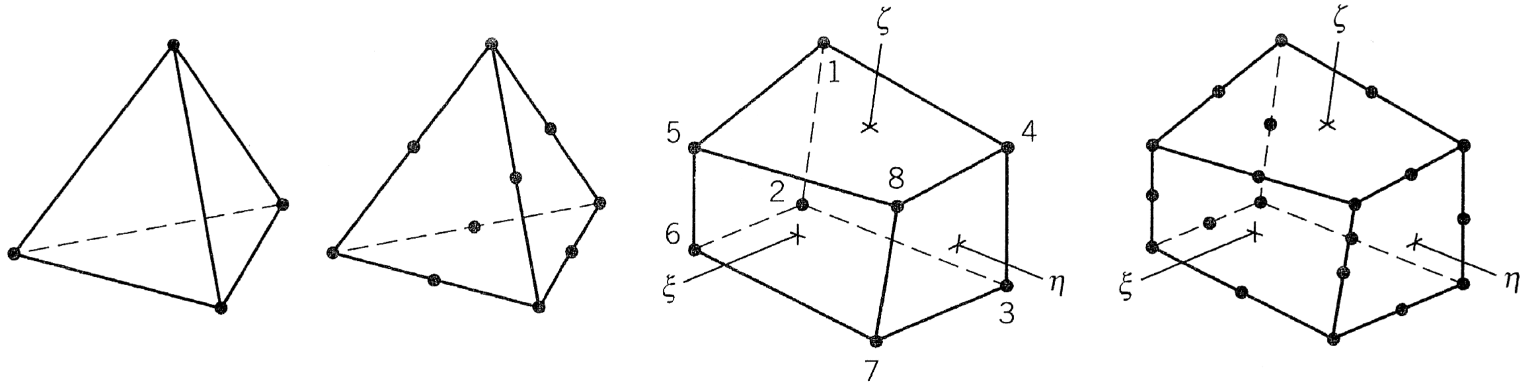
$$\gamma_{xy} = (\beta_3 + \beta_{10}) + (\beta_5 + 2\beta_{12})x + (2\beta_6 + \beta_{13})y + \beta_7 x^2 + 2(\beta_8 + \beta_{15})xy + \beta_{16} y^2$$

Solid elements



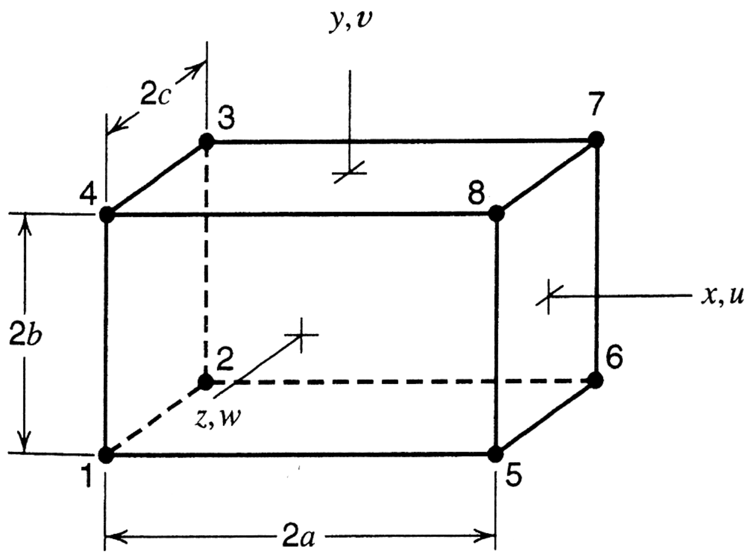
- 4 nodes tetrahedra
- 10 nodes tetrahedra
- 8 nodes hexahedra
- 20 nodes hexahedra

Each node has 3 d.o.f.

- 6 nodes wedges
- 15 nodes wedges

(not shown in the picture)

8 node hexahedra



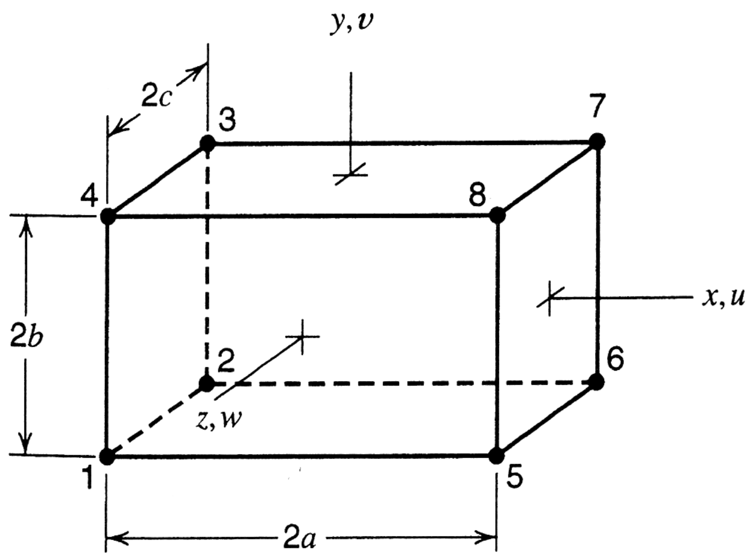
$$u = a_1 + a_2 x + a_3 y + a_4 z + a_5 xy + a_6 yz + a_7 zx + a_8 xyz$$

$$v = a_9 + a_{10} x + a_{11} y + a_{12} z + a_{13} xy + a_{14} yz + a_{15} zx + a_{16} xyz$$

$$w = a_{17} + a_{18} x + a_{19} y + a_{20} z + a_{21} xy + a_{22} yz + a_{23} zx + a_{24} xyz$$

24 d.o.f.

8 node hexahedra



$$\{u\} = [N]\{d\}_{el}$$

$$\begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \begin{bmatrix} N_1 & 0 & 0 & \cdots & N_8 & 0 & 0 \\ 0 & N_1 & 0 & \cdots & 0 & N_8 & 0 \\ 0 & 0 & N_1 & \cdots & 0 & 0 & N_8 \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ w_1 \\ \vdots \\ u_8 \\ v_8 \\ w_8 \end{Bmatrix}$$

$$N_i = \frac{(a \pm x)(b \pm y)(c \pm z)}{8abc}$$

Stiffness matrix of solid elements

Considering $\{\varepsilon\}=[B]\{d\}_{el}=[\partial][N]\{d\}_{el}$ where $\{\varepsilon\}=[\partial]\{u\}$

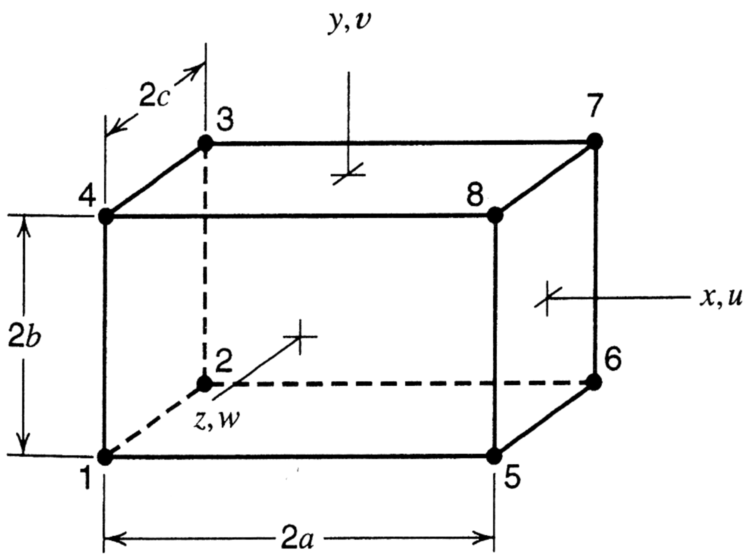
$$\begin{Bmatrix}\varepsilon_x\\ \varepsilon_y\\ \varepsilon_z\\ \gamma_{xy}\\ \gamma_{yz}\\ \gamma_{zx}\end{Bmatrix}=\begin{bmatrix}\frac{\partial}{\partial x} & 0 & 0\\ 0 & \frac{\partial}{\partial y} & 0\\ 0 & 0 & \frac{\partial}{\partial z}\\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0\\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y}\\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x}\end{bmatrix}\begin{Bmatrix}u\\ v\\ w\end{Bmatrix}=\underbrace{\begin{bmatrix}\frac{\partial}{\partial x} & 0 & 0\\ 0 & \frac{\partial}{\partial y} & 0\\ 0 & 0 & \frac{\partial}{\partial z}\\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0\\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y}\\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x}\end{bmatrix}\begin{bmatrix}N_1 & 0 & 0 & \cdots & N_8 & 0 & 0\\ 0 & N_1 & 0 & \cdots & 0 & N_8 & 0\\ 0 & 0 & N_1 & \cdots & 0 & 0 & N_8\end{bmatrix}}_{[B]}\begin{Bmatrix}u_1\\ v_1\\ w_1\\ \vdots\\ u_8\\ v_8\\ w_8\end{Bmatrix}$$

The stiffness matrix is obtained as $\underset{24\times24}{[K]}=\int_{-c}^{c}\int_{-b}^{b}\int_{-a}^{a}\underset{24\times6}{[B]^T}\,\underset{6\times6}{[E]}\,\underset{6\times24}{[B]}\,dx\,dy\,dz$

Limitations of solid elements

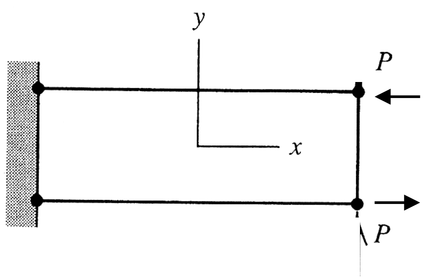
8 node solid elements can reproduce stress distributions which vary linearly through the thickness (e.g. along y for a bending moment along z)

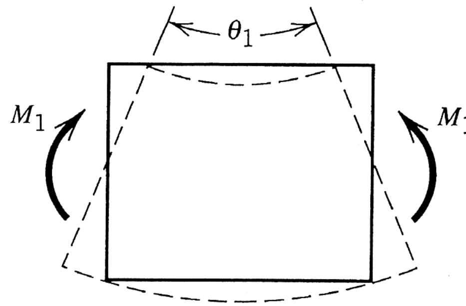
However, stresses are constant along the x direction



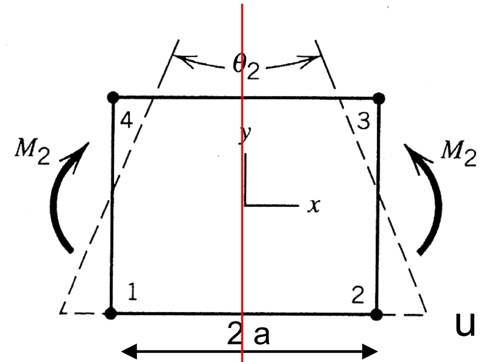
Moreover, if fully integrated, they are prone to shear locking

Shear locking





Real beams deform like this



$u = \theta_2\, b/2$

The deformed shape of the element is different

$$\varepsilon_x = -\frac{\vartheta_1 y}{2a} \qquad \varepsilon_y = +\frac{\nu \vartheta_1 y}{2a} \qquad \gamma_{xy} = 0$$

$$\varepsilon_x = -\frac{\vartheta_2 y}{2a} \qquad \varepsilon_y = 0 \qquad \gamma_{xy} = -\frac{\vartheta_2 x}{2a}$$

90° angles deform => parasitic shear γ appears

$$M_1 = M_2 \Rightarrow \frac{\vartheta_2}{\vartheta_1} = \frac{1-\nu^2}{1+\dfrac{1-\nu}{2}\left(\dfrac{a}{b}\right)^2}$$

$$\frac{\vartheta_2}{\vartheta_1} \to 0 \quad \text{per} \quad \frac{a}{b} \to \infty$$

Parasitic shear absorbs strain energy, thus bending angle is lower

Effect of shear locking

σ_x const. along x

τ_{xy} greater for elements with greater bending deformation