POLITECNICO DI MILANO

Finite Element Simulation For Mechanical Design

Bar elements

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FEM: Finite Element Method

- It is a numerical method
- Initially proposed for solving the problem of solving stresses in structures
- Applicable to different physical problems, where field quantities need being calculated
- It is proposed as a method for finding an **approximated numerical solution** to a **specific problem**
- It differs from **exact analytical solutions** to a **class of problems**, like the infinite plate with a circular hole solved using Airy’s stress function approach

Let’ start introducing bar (truss) elements

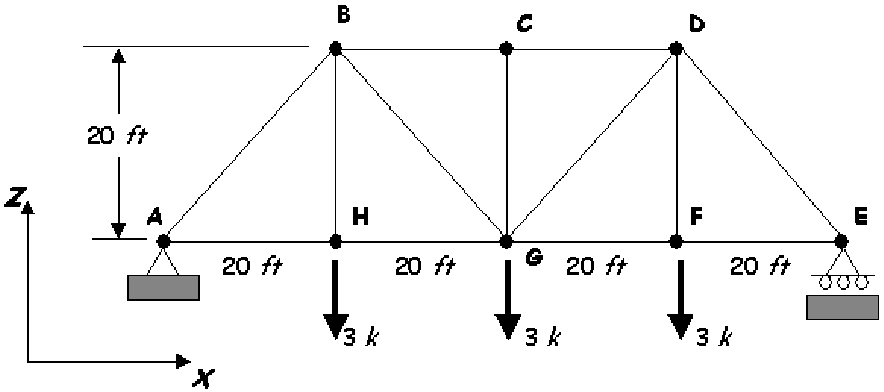
1 D elements

Two types:

Bar: only internal force is axial

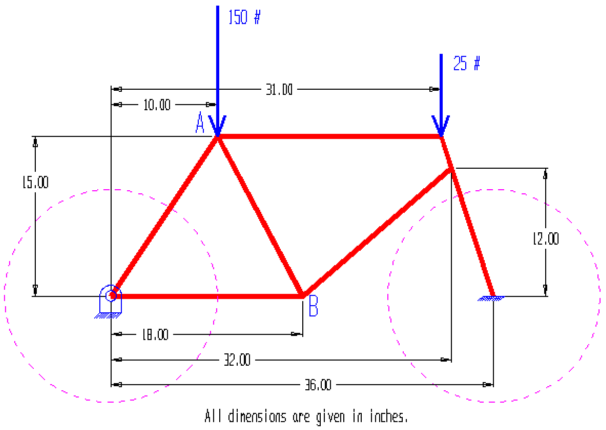
Example: truss structure

d.o.f. are displacements only



Beam: internal forces are axial, shear bending moments and torque

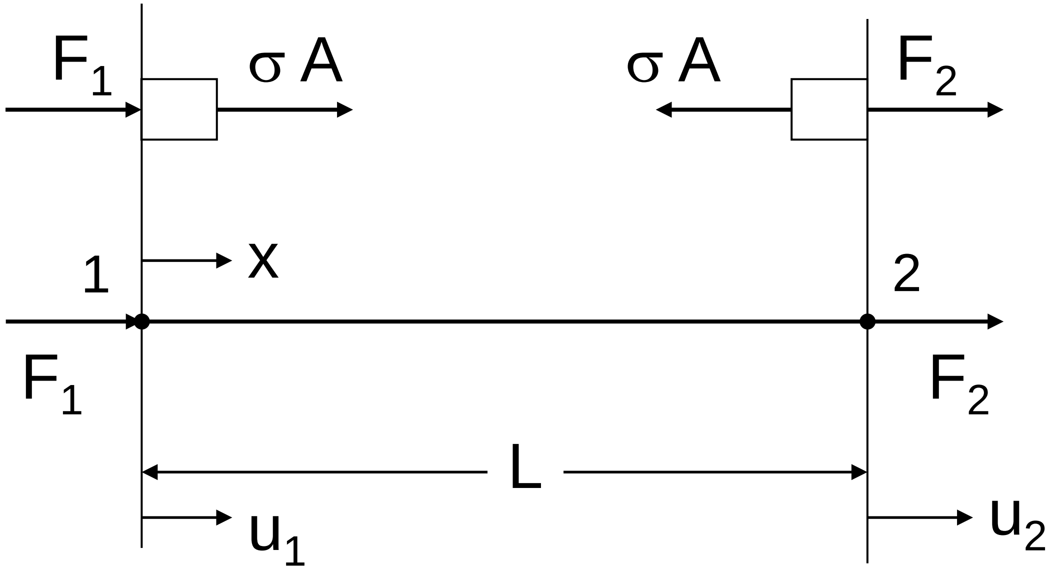
Example: bicycle frame



d.o.f. are displacements and rotations

Truss (bar) elements

Let's study a bar having length L, modulus E, section A, along its axis (coordinate x)



The structure is modelled based on nodes.

Forces and boundary conditions can be applied only at nodes.

Coordinate generalizzate

The displacement field across the element is expressed by

$$u(x) = \beta_1 + \beta_2 x$$

Where the coefficients β are defined as generalized coordinates and, for the bar element, corrispond to displacement amplitudes

Let's find β_1 e β_2 by imposing appropriate boundary conditions

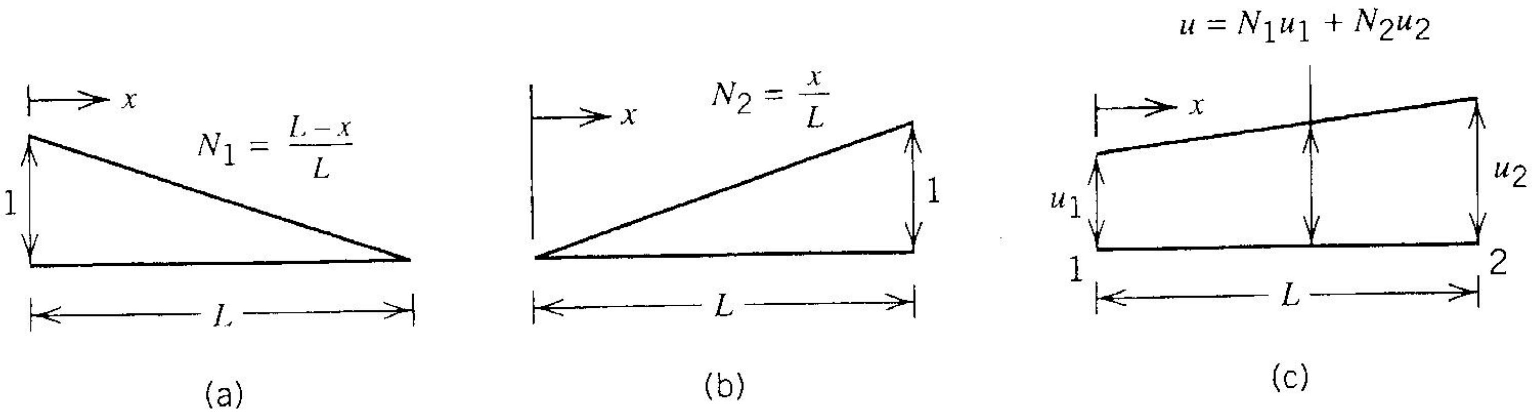
$$\begin{cases} x = 0 & u = u_1 \\ x = L & u = u_2 \end{cases}$$

$$u(x) = u_1 + \frac{u_2 - u_1}{L} x = rigid\ disp. + \varepsilon \cdot x$$

Rearranging the expression to highlight nodal displacements

$$u(x) = \left(\frac{L - x}{L}\right) u_1 + \frac{x}{L} u_2$$

Shape functions and strains



$$u(x) = \begin{bmatrix} \frac{L-x}{L} & \frac{x}{L} \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = [N]\{d\}$$

$\{d\}$, element's nodal displacements vector

$$\varepsilon_x = \frac{du(x)}{dx} = \frac{d}{dx}([N]\{d\}) = \frac{d[N]}{dx}\{d\} = [B]\{d\}$$

$$[B] = \frac{d}{dx}\begin{bmatrix} \frac{L-x}{L} & \frac{x}{L} \end{bmatrix} = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix}$$

thus:

$$\{\varepsilon\} = [B]\{d\} = -\frac{1}{L}u_1 + \frac{1}{L}u_2 = \frac{u_2 - u_1}{L}$$

Stress-strain: matrix form of the linear elastic constitutive model

$$\sigma = E\,\varepsilon \longrightarrow \{\sigma\} = [E]\{\varepsilon\}$$

Bar element

Any element

$$\sigma \cdot \varepsilon = \{\sigma\}^T\{\varepsilon\} = \{\varepsilon\}^T[E]\{\varepsilon\}$$

Expressions of an element's strain energy

$$U = \frac{1}{2}\int_V \sigma \cdot \varepsilon \, dV = \frac{1}{2}\int_V \{\sigma\}^T\{\varepsilon\} dV = \frac{1}{2}\int_V \{\varepsilon\}^T[E]\{\varepsilon\} dV$$

$$\{\varepsilon\} = [B]\{d\}$$

$$U = \frac{1}{2}\int_V \{d\}^T[B]^T[E][B]\{d\}\, dV = \frac{1}{2}\{d\}^T\int_V [B]^T[E][B]\, dV\, \{d\}$$

Stiffness matrix

$$[k]=\int_V [B]^T [E][B]dV$$

[**E**] depends on material's properties

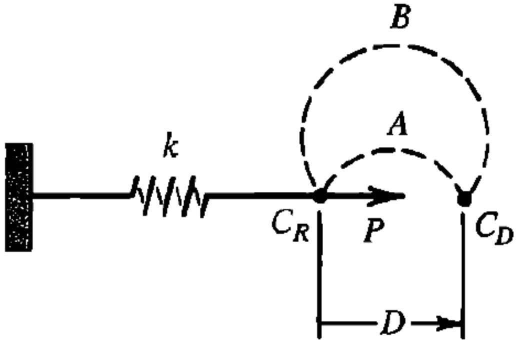
[**B**] , such that {**ε**}=[**B**] {**d**}

$$[B] = \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix}$$

$$[k] = \int_o^L \begin{Bmatrix} -\frac{1}{L} \\ \frac{1}{L} \end{Bmatrix} E \begin{bmatrix} -\frac{1}{L} & \frac{1}{L} \end{bmatrix} A dx = \begin{bmatrix} \frac{EA}{L} & -\frac{EA}{L} \\ -\frac{EA}{L} & \frac{EA}{L} \end{bmatrix}$$

Principle of stationary potential energy

Among all possible configurations (i.e. which respect internal compatibility and essential* boundary conditions) of a conservative system, those that satisfy the equilibrium equations make the potential energy stationary with respect to small admissible variations of the displacement



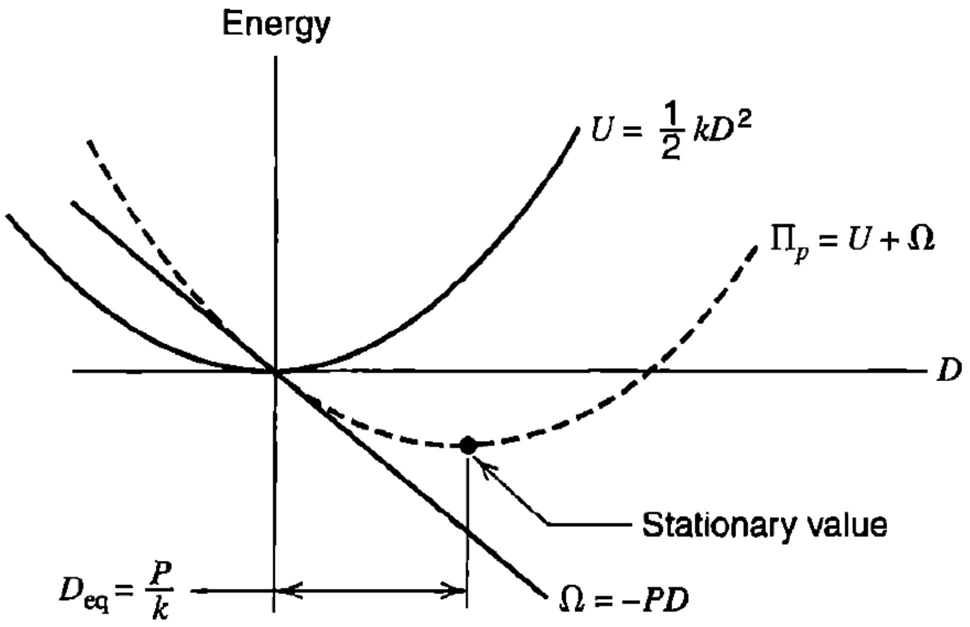
$$\Pi_p = U + \Omega \qquad\qquad U = \frac{1}{2}kD^2 \quad \text{and} \quad \Omega = -PD$$

$$\Pi_p = \frac{1}{2}kD^2 - PD$$

$$d\Pi_p = (kD_{eq} - P)dD = 0 \quad \text{hence} \quad D_{eq} = P/k$$

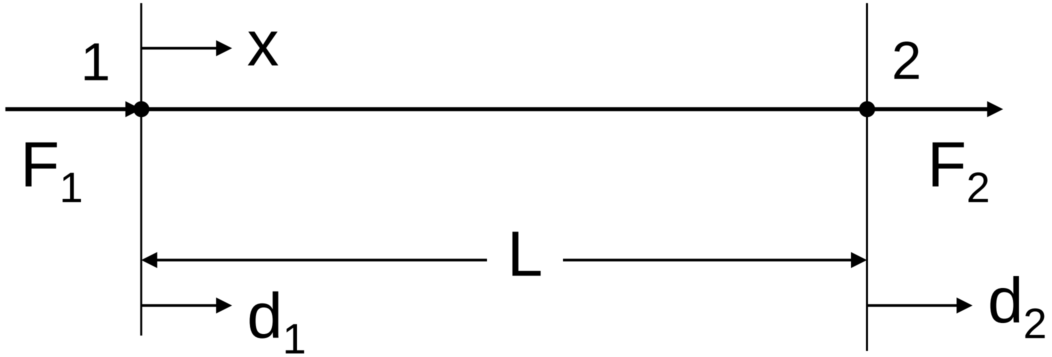
*, essential boundary conditions are expressed in terms of field variables (displacements in the case of structural analyses), non-essential to higher order derivatives of field variables

Functional Π_p



Potential energy is a **functional**, an integral expression that contains implicitly the differential equations governing the problem

Equilibrium: minimum potential energy



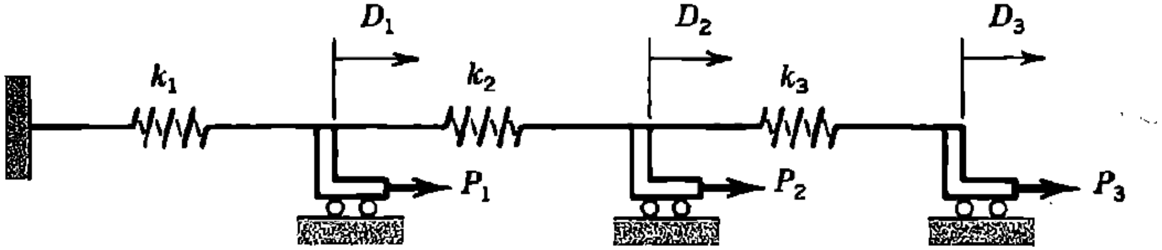
$$\Pi_p = U + \Omega$$

$$\Pi_p = \frac{1}{2}\{d\}^T\,[K]\{d\} - \{d\}^T\,\{F\}$$

$$\frac{\partial \Pi_p}{\partial d} = 0 \qquad [K]\{d\} - \{F\} = \{0\}$$

System having *n* d.o.f.

$$d\Pi_p = 0 \qquad \text{where} \qquad d\Pi_p = \frac{\partial \Pi_p}{\partial D_1} dD_1 + \frac{\partial \Pi_p}{\partial D_2} dD_2 + \cdots + \frac{\partial \Pi_p}{\partial D_n} dD_n$$



$$\Pi_p = \frac{1}{2} k_1 D_1^2 + \frac{1}{2} k_2 (D_2 - D_1)^2 + \frac{1}{2} k_3 (D_3 - D_2)^2 - P_1 D_1 - P_2 D_2 - P_3 D_3$$

$$\begin{aligned} k_1 D_1 - k_2 (D_2 - D_1) - P_1 &= 0 \\ k_2 (D_2 - D_1) - k_3 (D_3 - D_2) - P_2 &= 0 \\ k_3 (D_3 - D_2) - P_3 &= 0 \end{aligned} \qquad \begin{bmatrix} k_1 + k_2 & -k_2 & 0 \\ -k_2 & k_2 + k_3 & -k_3 \\ 0 & -k_3 & k_3 \end{bmatrix} \begin{Bmatrix} D_1 \\ D_2 \\ D_3 \end{Bmatrix} = \begin{Bmatrix} P_1 \\ P_2 \\ P_3 \end{Bmatrix}$$

Assembly of the stiffness matrices

$$[K]_1 = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix} \qquad [K]_2 = \begin{bmatrix} k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix}$$

refers to $\begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$ refers to $\begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix}$

The same matrices need being made compatible with the global disp. vector

$$\begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} \qquad [K]_1 = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 & 0 \\ 0 & 0 & 0 \end{bmatrix} \quad [K]_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix}$$

The global stiffness matrix

$$[K]_{TOT} = [K]_1 + [K]_2 = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 + k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix}$$

It is the sum of the [k] of each element, after making each [k] compatible with {D}

$$[K]_1 = \begin{bmatrix} k_1 & -k_1 \\ -k_1 & k_1 \end{bmatrix}$$ is compatible with $$\{d\}_1 = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix}$$

$$[K]_1 = \begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$ is compatible with $$\{D\} = \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

Expanded matrix

Expansion can be achieved using a matrix L that contains only zeros and ones:

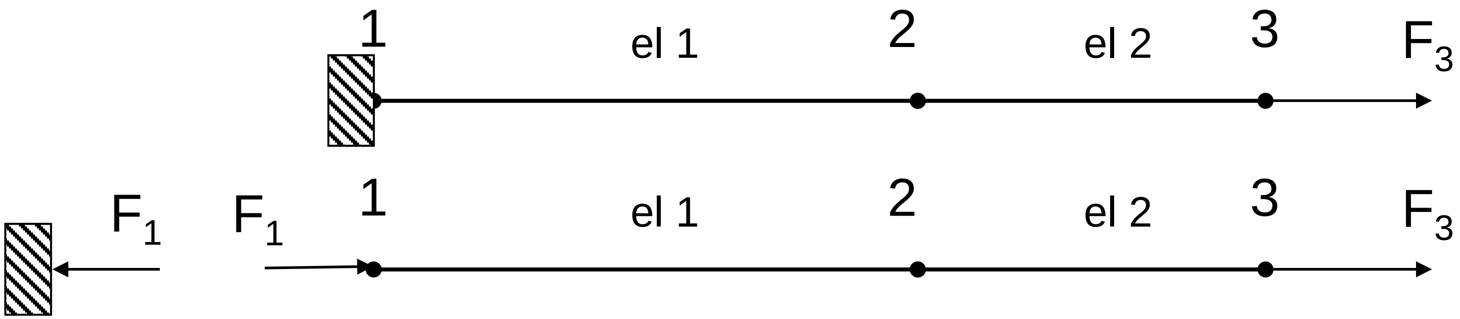
$$\{d\}_i = [L]_i\{D\}$$

$$\{d\}_1 = \begin{Bmatrix} u_1 \\ u_2 \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix}$$

$$[L]_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Note: creation of L_i matrices is a conceptual help, it is not the way matrices are stored and assembled in FE software

Boundary conditions



The system to be solved now includes the known displacement and the unknown reaction force and is

$$\begin{bmatrix} k_1 & -k_1 & 0 \\ -k_1 & k_1+k_2 & -k_2 \\ 0 & -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_1 \\ u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} F_1 \\ 0 \\ F_3 \end{Bmatrix}$$

The boundary condition is expressed by equation $u_1 = 0$

Thus, the system becomes

$$\begin{bmatrix} k_1+k_2 & -k_2 \\ -k_2 & k_2 \end{bmatrix} \begin{Bmatrix} u_2 \\ u_3 \end{Bmatrix} = \begin{Bmatrix} 0 \\ F_3 \end{Bmatrix}$$

$$k_1 u_1 - k_1 u_2 = F_1$$

Constraints

Usually, we have:

- $\{d\}_{\text{unknown}}$, unknown displacements
- $\{d\}_{\text{known}}$, known displacements (constraints or imposed displacements)

- $\{F\}_{\text{known}}$, known forces (applied forces)
- $\{F\}_{\text{unknown}}$, unknown loads (reactions)

The system [K]{d} = {f} needs being rewritten, grouping the unknowns

$$\begin{bmatrix} [K_{11}] & [K_{12}] \\ [K_{21}] & [K_{22}] \end{bmatrix} \begin{Bmatrix} \{d\}_{\text{unknown}} \\ \{d\}_{\text{known}} \end{Bmatrix} = \begin{Bmatrix} \{F\}_{\text{known}} \\ \{F\}_{\text{unknown}} \end{Bmatrix}$$

constraints

reactions

Constraints

$$\begin{bmatrix} [K_{11}] & [K_{12}] \\ [K_{21}] & [K_{22}] \end{bmatrix} \begin{Bmatrix} \{d\}_{\text{unknown}} \\ \{d\}_{\text{known}} \end{Bmatrix} = \begin{Bmatrix} \{F\}_{\text{known}} \\ \{F\}_{\text{unknown}} \end{Bmatrix}$$

First unknown displacements are evaluated

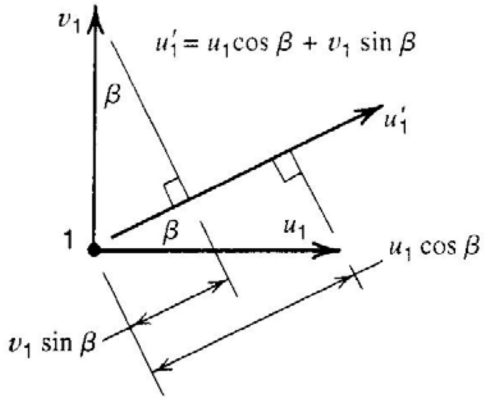
$$[K_{11}]\{d\}_{\text{unknown}} + [K_{12}]\{d\}_{\text{known}} = \{F\}_{\text{known}}$$

$$\{d\}_{\text{unknown}} = [K_{11}]^{-1}\left(\{F\}_{\text{known}} - [K_{12}]\{d\}_{\text{known}}\right)$$

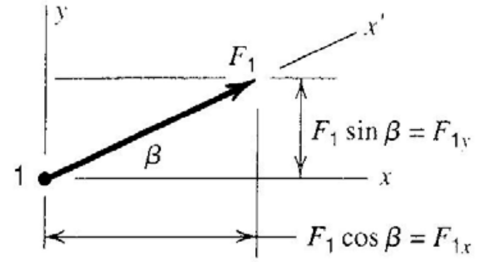
Then the reaction forces

$$\{F\}_{\text{unknown}} = [K_{21}]\{d\}_{\text{unknown}} + [K_{22}]\{d\}_{\text{known}}$$

Displacements and forces in the global reference



(a)



(b)

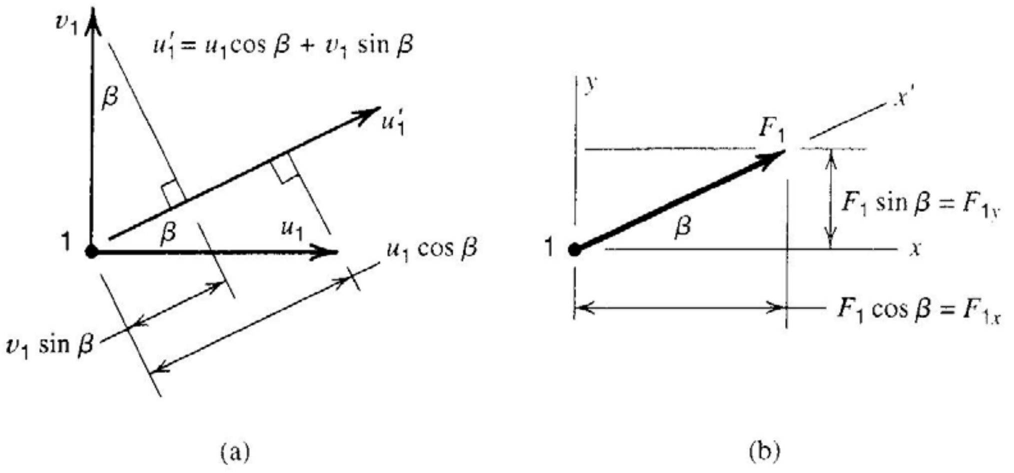
$$\begin{Bmatrix} u_1' \\ u_2' \end{Bmatrix} = \begin{bmatrix} \cos\beta & \sin\beta & 0 & 0 \\ 0 & 0 & \cos\beta & \sin\beta \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \end{Bmatrix}$$

$$[T] = \begin{bmatrix} \cos\beta & \sin\beta & 0 & 0 \\ 0 & 0 & \cos\beta & \sin\beta \end{bmatrix}; \qquad N.B.: \quad [T][T]^T = [I]$$

$$\begin{Bmatrix} F_{1x} \\ F_{1y} \\ F_{2x} \\ F_{2y} \end{Bmatrix} = [T]^T \begin{Bmatrix} F_1 \\ F_2 \end{Bmatrix}$$

The stiffness matrix in the global reference system

$$\{u'\}=[T]\{u\}$$



$$\{F\}=[T]^T\{F'\}$$

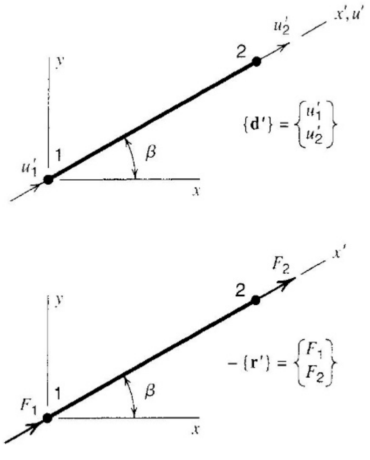
$$[K']\{u'\}=\{F'\} \quad \Rightarrow \quad [K'][T]\{u\}=\{F'\}$$

$$[T]^T[K'][T]\{u\}=[T]^T\{F'\}=\{F\}$$

$$[K]=[T]^T[K'][T]$$

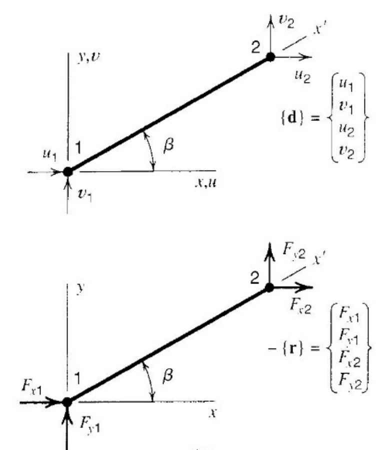
The stiffness matrix in the global reference system

$$[K]=[T]^T[K'][T]$$



(a)

[K’] 2 x 2



(b)

[K] 4 x 4

$$[T]=\begin{bmatrix} \cos\beta & \sin\beta & 0 & 0 \\ 0 & 0 & \cos\beta & \sin\beta \end{bmatrix}; \qquad N.B.: \quad [T][T]^T=[I]$$

$$[K]=\frac{AE}{L}\begin{bmatrix} \cos^2\beta & \cos\beta\sin\beta & -\cos^2\beta & -\cos\beta\sin\beta \\ \cos\beta\sin\beta & \sin^2\beta & -\cos\beta\sin\beta & -\sin^2\beta \\ -\cos^2\beta & -\cos\beta\sin\beta & \cos^2\beta & \cos\beta\sin\beta \\ -\cos\beta\sin\beta & -\sin^2\beta & \cos\beta\sin\beta & \sin^2\beta \end{bmatrix}$$

L_1 = 2000 mm

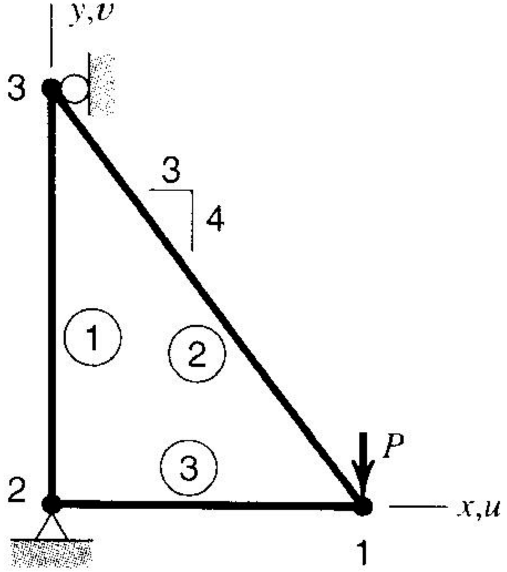
L_2 = 2500 mm

L_3 = 1500 mm

P = 10 kN

A = 100 mm^2

E = 206 GPa



Solve for the displacements of this truss