POLITECNICO DI MILANO

Finite Element Simulation For Mechanical Design

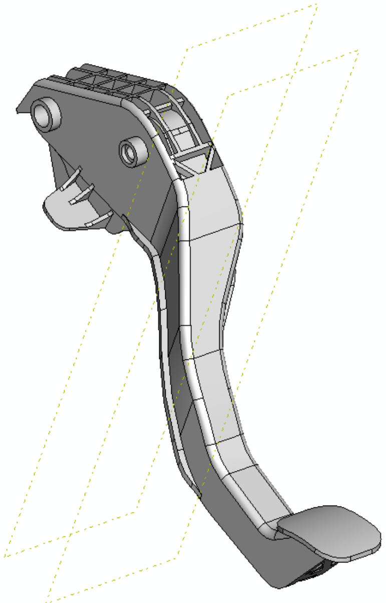
Isoparametric formulation

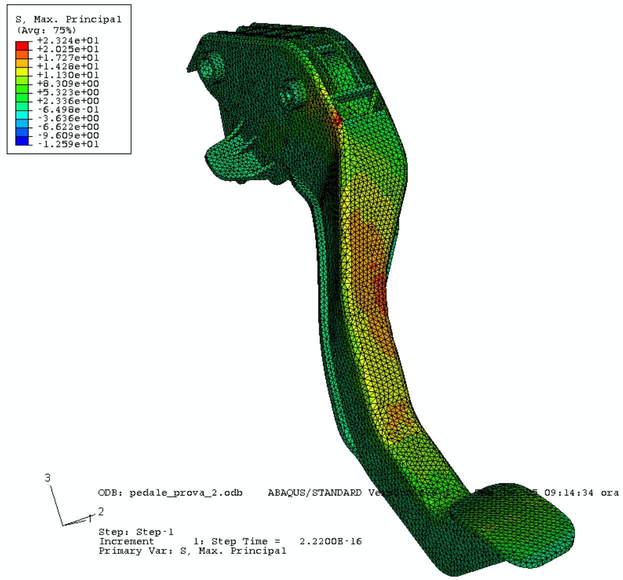
Full and reduced integration - hourglass

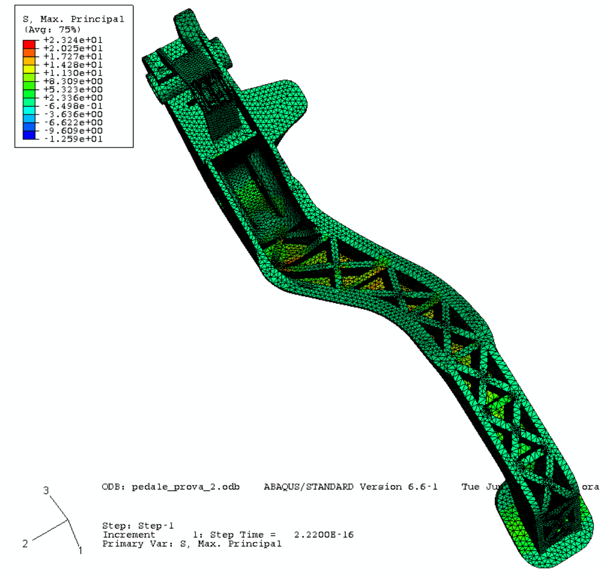
Stress evaluation

A. Bernasconi

Meshing parts of complex shape





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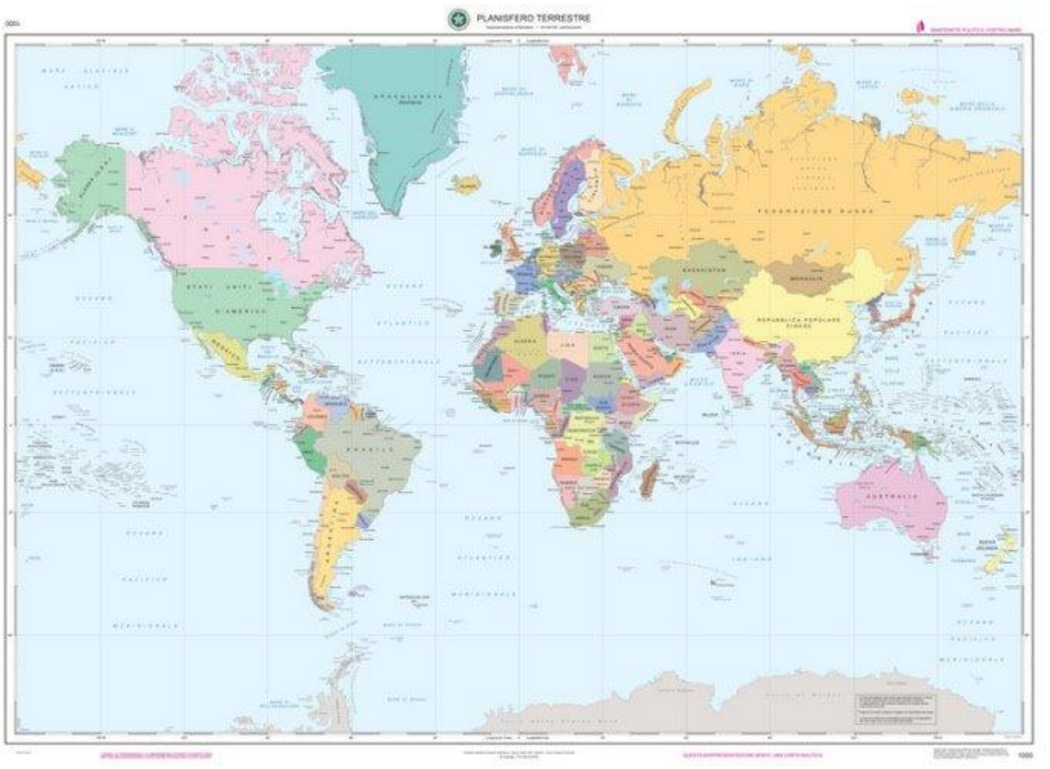
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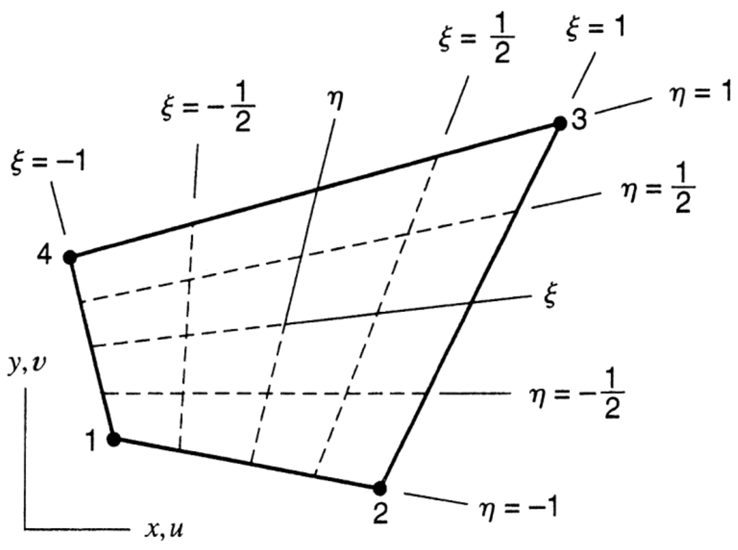
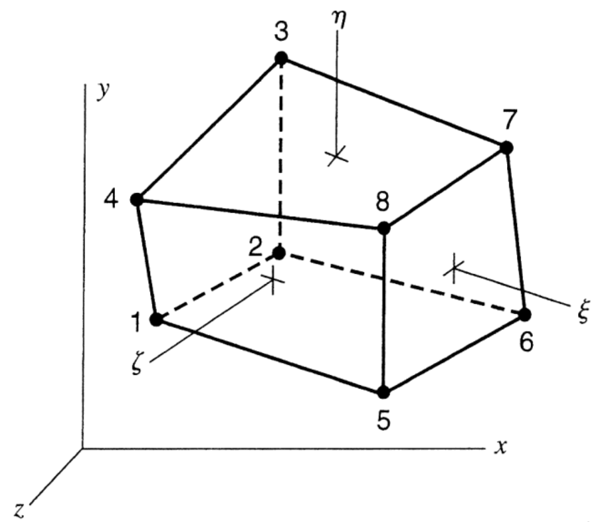
Isoparametric elements

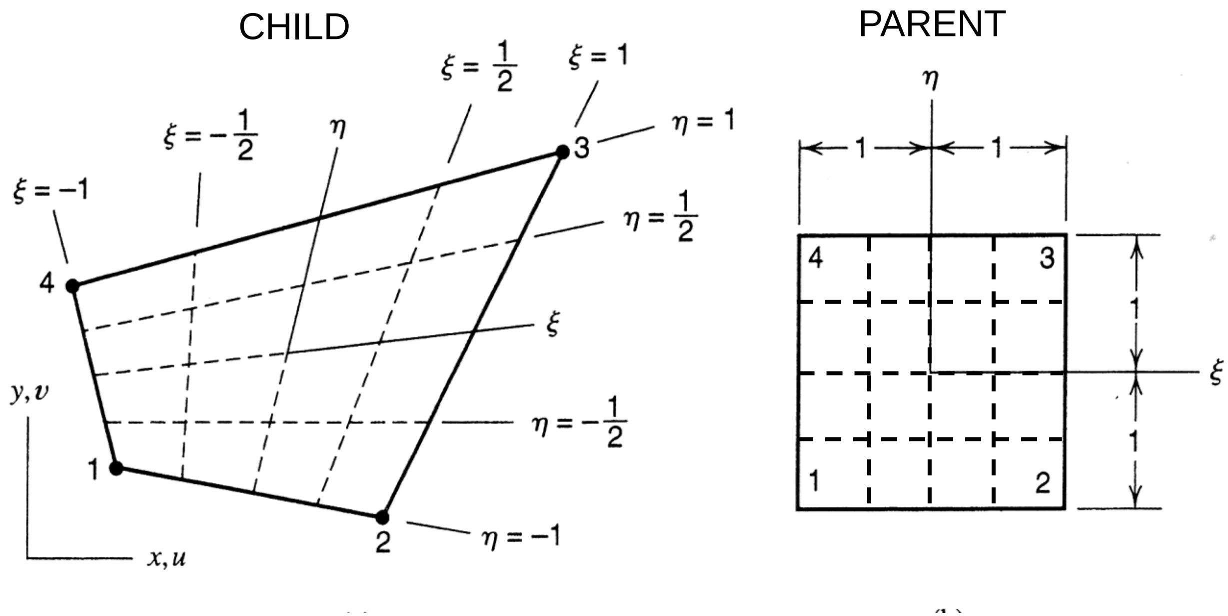


2D



3D

Isoparametric elements



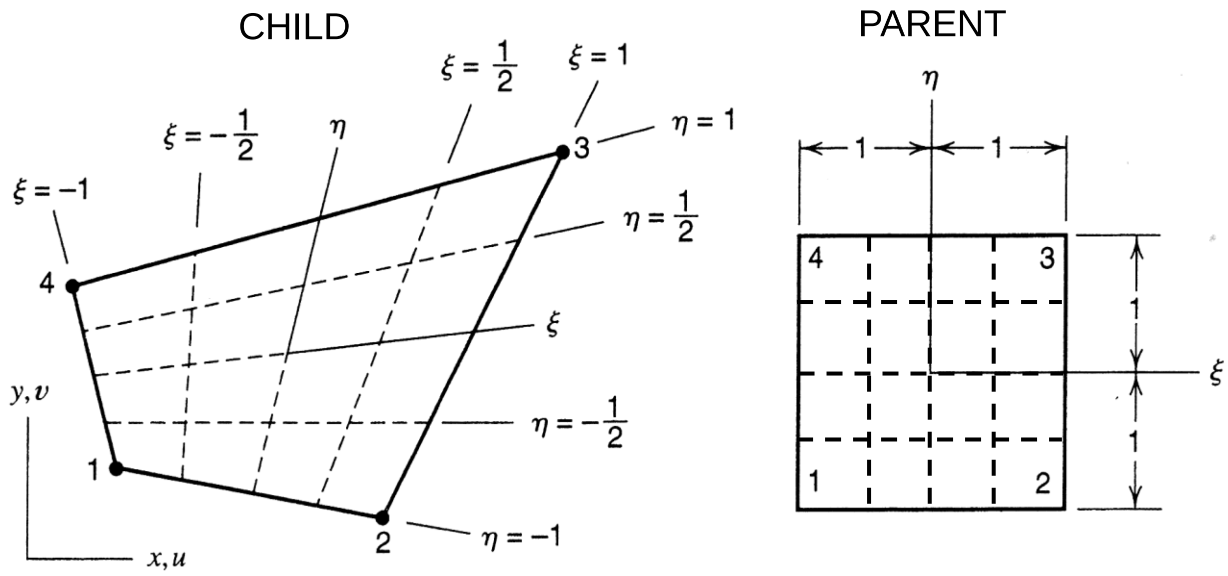
Mapping in the ξ, η space

Shape functions are used to map displacements and points in the *x*,*y* space

Shape functions are defined in the ξ, η space

$$u = \sum N_i(\xi,\eta) u_i \quad x = \sum N_i(\xi,\eta) x_i$$
$$v = \sum N_i(\xi,\eta) v_i \quad y = \sum N_i(\xi,\eta) y_i$$

Isoparametric elements

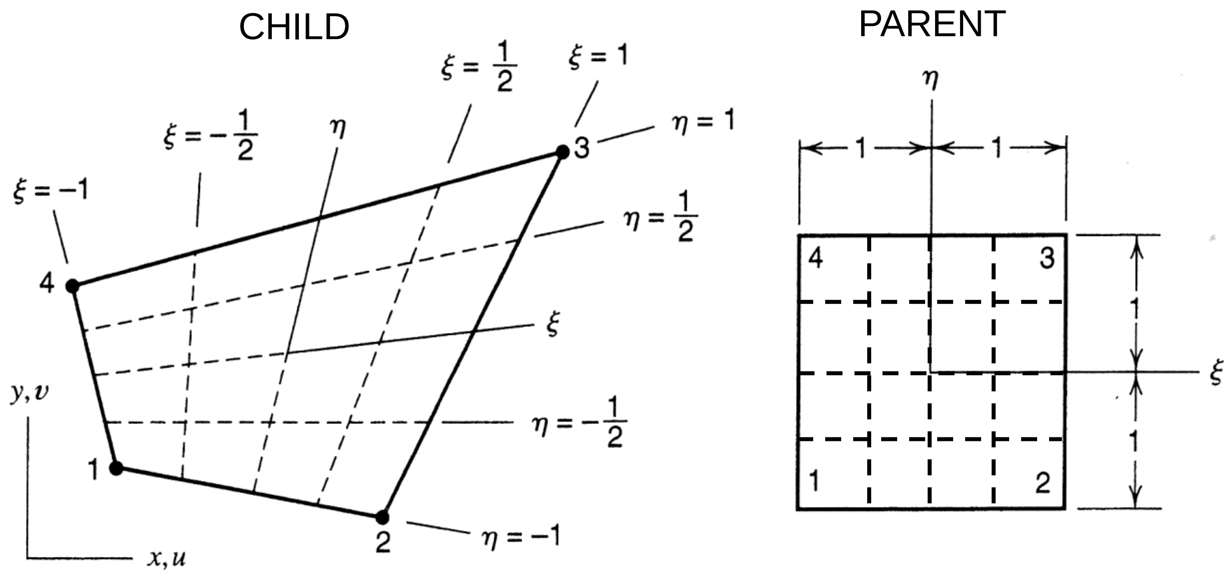


Same bi-linear functions as those of the undistorted rectangular element in ξ, η.

$$N_1 = \frac{(1-\xi)(1-\eta)}{4} \quad N_2 = \frac{(1+\xi)(1-\eta)}{4}$$

$$N_3 = \frac{(1+\xi)(1+\eta)}{4} \quad N_4 = \frac{(1-\xi)(1+\eta)}{4}$$

Isoparametric elements



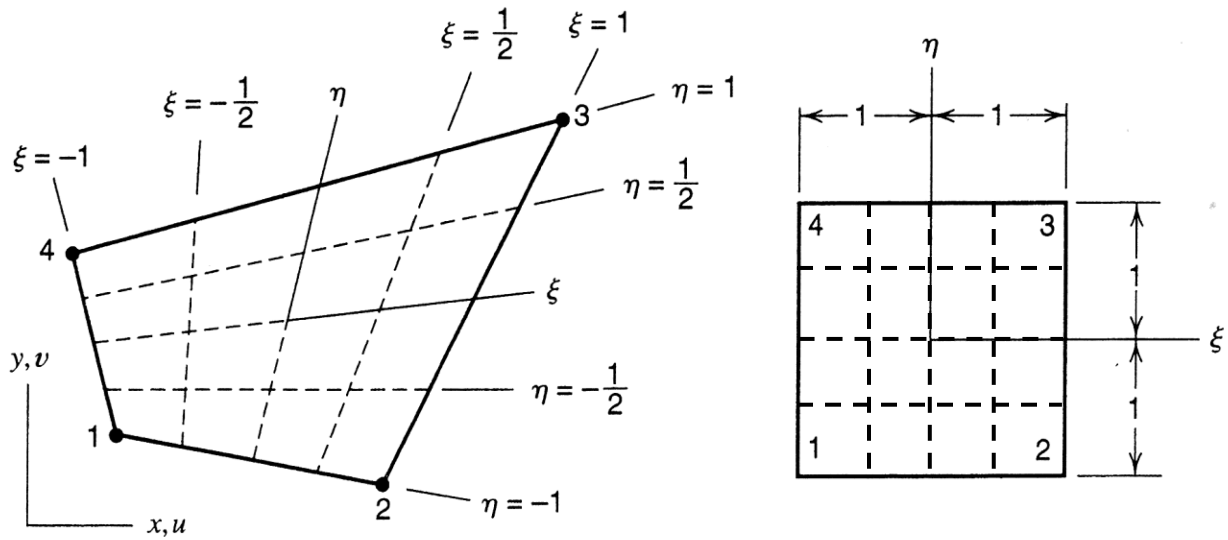
Compatibility is ensured (elements have shared edges in both reference systems)

Example: edge 1-2 (η=-1.)

$$x = N_1(\xi,-1)x_1 + N_2(\xi,-1)x_2$$
$$y = N_1(\xi,-1)y_1 + N_2(\xi,-1)y_2$$

By eliminating ξ, the equation of a straight line passing through points (x_1,y_1) e (x_2,y_2) can be obtained.

Isoparametric elements



The same considerations are valid for displacements, that vary linearly along lines for constant ξ, η, and quadratically along the other directions

$$u = N_1(\xi,-1)u_1 + N_2(\xi,-1)u_2$$
$$v = N_1(\xi,-1)v_1 + N_2(\xi,-1)v_2$$

Strain-dispacement

Strain-displacement relationships involve partial derivatives ∂ /∂x e ∂ /∂y (physical space).

$$\varepsilon_x = \frac{\partial u}{\partial x} \qquad \varepsilon_y = \frac{\partial v}{\partial y} \qquad \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

What is the relationship between (∂ /∂x, ∂ /∂y) and (∂ /∂ξ, ∂ /∂η) ?

Example: u

$$\begin{cases} u = \sum N_i(\xi,\eta)u_i \\ v = \sum N_i(\xi,\eta)v_i \end{cases}$$

$$\begin{cases} x = \sum N_i(\xi,\eta)x_i \\ y = \sum N_i(\xi,\eta)y_i \end{cases}$$

$$\Rightarrow$$

$$\frac{\partial u}{\partial \xi} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \xi} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \xi}$$

$$\frac{\partial u}{\partial \eta} = \frac{\partial u}{\partial x}\frac{\partial x}{\partial \eta} + \frac{\partial u}{\partial y}\frac{\partial y}{\partial \eta}$$

Thus $$\begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix} = \overset{2\times 2}{[J]} \begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix}$$

$$[J] = \begin{bmatrix} \partial x/\partial \xi & \partial y/\partial \xi \\ \partial x/\partial \eta & \partial y/\partial \eta \end{bmatrix}$$ **Jacobi's matrix**

$$\frac{\partial \overset{x}{\cancel{u}}}{\partial \xi} = \sum \frac{\partial N_i}{\partial \xi} \overset{x}{\cancel{u}}_i$$

Jacobi's matrix

$$[J]=\begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix}$$

The terms of the matrix are related to the shape functions

$$J_{11}=\sum \frac{\partial N_i}{\partial \xi} x_i$$

$$J_{12}=\sum \frac{\partial N_i}{\partial \xi} y_i$$

$$J_{22}=\sum \frac{\partial N_i}{\partial \eta} x_i$$

$$J_{21}=\sum \frac{\partial N_i}{\partial \eta} y_i$$

Jacobian

We need the derivatives in the physical space ∂ /∂x e ∂ /∂y

The relationship must be inverted

$$\begin{Bmatrix} \dfrac{\partial u}{\partial x} \\ \dfrac{\partial u}{\partial y} \end{Bmatrix} = [J]^{-1} \begin{Bmatrix} \dfrac{\partial u}{\partial \xi} \\ \dfrac{\partial u}{\partial \eta} \end{Bmatrix}$$

$$[J]^{-1} = \frac{1}{J} \begin{bmatrix} J_{22} & -J_{12} \\ -J_{21} & J_{11} \end{bmatrix} = [\Gamma]$$

$$J = \det[J] = J_{11}J_{22} - J_{21}J_{12}$$

Jacobian

J = A/4 for the undistorted rectangular element

N.B. expressions valid for counter clockwise numbering of nodes

Stiffness matrix

$$\begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 1 & 0 \end{bmatrix} \begin{Bmatrix} \partial u/\partial x \\ \partial u/\partial y \\ \partial v/\partial x \\ \partial v/\partial y \end{Bmatrix}; \qquad \begin{Bmatrix} \partial u/\partial x \\ \partial u/\partial y \\ \partial v/\partial x \\ \partial v/\partial y \end{Bmatrix} = \begin{bmatrix} [\Gamma] & \\ & [\Gamma] \end{bmatrix} \begin{Bmatrix} \partial u/\partial \xi \\ \partial u/\partial \eta \\ \partial v/\partial \xi \\ \partial v/\partial \eta \end{Bmatrix}$$

$$\begin{Bmatrix} \partial u/\partial \xi \\ \partial u/\partial \eta \\ \partial v/\partial \xi \\ \partial v/\partial \eta \end{Bmatrix} = \begin{bmatrix} \partial N_1/\partial \xi & 0 & \partial N_2/\partial \xi & 0 & \partial N_3/\partial \xi & 0 & \partial N_4/\partial \xi & 0 \\ \partial N_1/\partial \eta & 0 & \partial N_2/\partial \eta & 0 & \partial N_3/\partial \eta & 0 & \partial N_4/\partial \eta & 0 \\ & \partial N_1/\partial \xi & 0 & \partial N_2/\partial \xi & 0 & \partial N_3/\partial \xi & 0 & \partial N_4/\partial \xi \\ & \partial N_1/\partial \eta & 0 & \partial N_2/\partial \eta & 0 & \partial N_3/\partial \eta & 0 & \partial N_4/\partial \eta \end{bmatrix} \begin{Bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{Bmatrix}$$

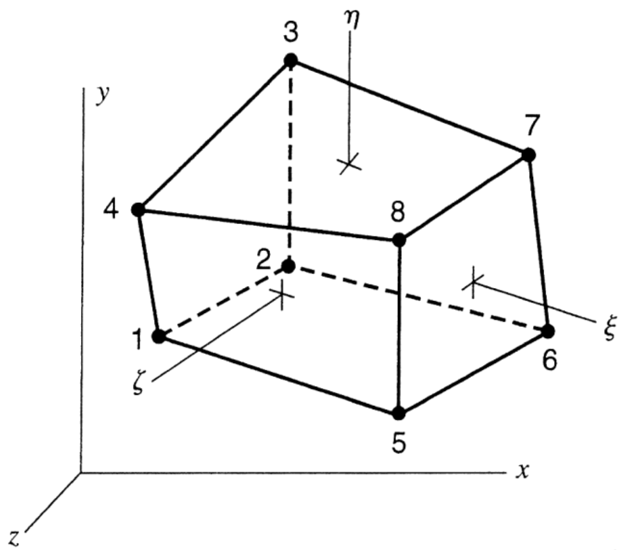
$$\{\boldsymbol{\varepsilon}\} = [B]\{d\} \quad = [J]^{-1}[\partial \mathbf{N}/\partial \boldsymbol{\Xi}]\{d\}$$

$$[k] = \int_V [B]^T[E][B]t\,dxdy = \int_{-1}^{+1}\int_{-1}^{+1}[B]^T[E][B]\,tJ\,d\xi d\eta$$

$$dx = d\left(\sum N_i(\xi,\eta)x_i\right) \qquad dy = d\left(\sum N_i(\xi,\eta)y_i\right) \qquad dx\,dy = J\,d\xi\,d\eta$$

The terms of B_{ij} are ratios of polynomials (except for the undistorted elements), that require numerical integration

Isoparametric 3D elements



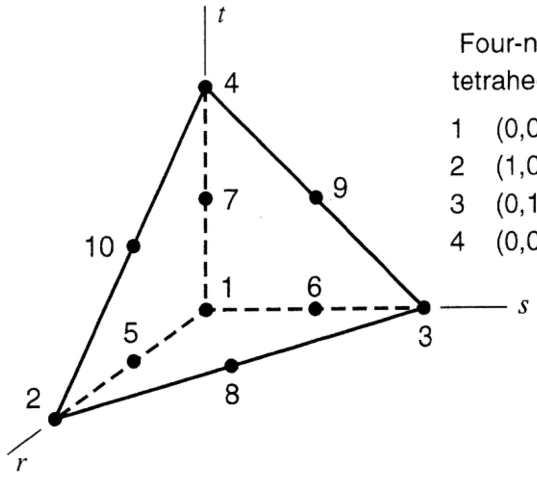
$$\begin{Bmatrix} x \\ y \\ z \end{Bmatrix} = \sum_i N_i \begin{Bmatrix} x_i \\ y_i \\ z_i \end{Bmatrix} \quad \text{and} \quad \begin{Bmatrix} u \\ v \\ w \end{Bmatrix} = \sum_i N_i \begin{Bmatrix} u_i \\ v_i \\ w_i \end{Bmatrix}$$

$$N_i = \frac{1}{8}(1 \pm \xi)(1 \pm \eta)(1 \pm \zeta)$$

$$\underset{24\times24}{[K]} = \int_{-1}^{1}\int_{-1}^{1}\int_{-1}^{1} \underset{24\times6}{[B]^T} \underset{6\times6}{[E]} \underset{6\times24}{[B]} J d\xi d\eta d\zeta$$

While for the undistorted element the elements of B are polynomials and therefore the integral can be evaluated analytically, this integral cannot be obtained in closed form, it has to be evaluated numerically

Tetrahedra



Four-node tetrahedron		Added nodes for 10-node tetrahedron	
1	(0,0,0)	5	$(\frac{1}{2},0,0)$
2	(1,0,0)	6	$(0,\frac{1}{2},0)$
3	(0,1,0)	7	$(0,0,\frac{1}{2})$
4	(0,0,1)	8	$(\frac{1}{2},\frac{1}{2},0)$
		9	$(0,\frac{1}{2},\frac{1}{2})$
		10	$(\frac{1}{2},0,\frac{1}{2})$

4 nodes

$$N_1 = 1-r-s-t \qquad N_2 = r \qquad N_3 = s \qquad N_4 = t$$

10 nodes

$$N_1 = (1-r-s-t)(1-2r-2s-2t)$$
$$N_2 = r(2r-1) \qquad N_5 = 4r(1-r-s-t) \qquad N_8 = 4rs$$
$$N_3 = s(2s-1) \qquad N_6 = 4s(1-r-s-t) \qquad N_9 = 4st$$
$$N_4 = t(2t-1) \qquad N_7 = 4t(1-r-s-t) \qquad N_{10} = 4tr$$

Numerical integration

To evaluate

$$I = \int_{x_1}^{x_2} g(x)dx \quad \Rightarrow \int_{-1}^{+1} f(\xi)d\xi$$

Change variable

$$x = \frac{1}{2}(1-\xi)x_1 + \frac{1}{2}(1+\xi)x_2$$

$$dx = \frac{x_2 - x_1}{2} d\xi$$

To obtain

$$f(\xi) = g(\xi)J; \qquad J = \frac{dx}{d\xi} = \frac{x_2 - x_1}{2}$$

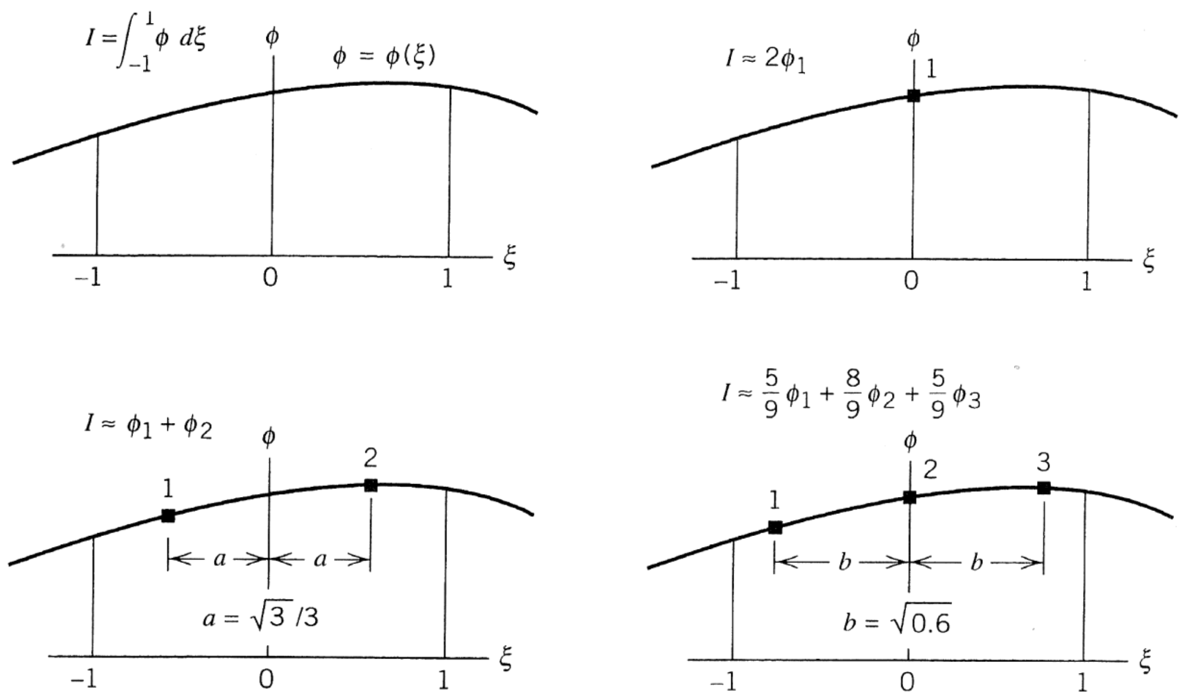
That is integrated between -1 and +1

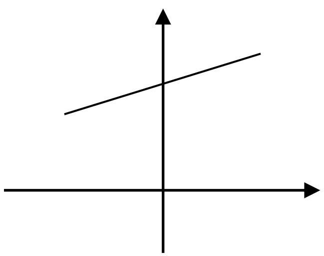
$$I = \int_{-1}^{+1} f(\xi)d\xi$$

Gauss method

Integral $I = \int_{-1}^{+1} f(\xi)\,d\xi$ is approximated by $I \approx \sum_{i=1}^{n} W_i f_i$

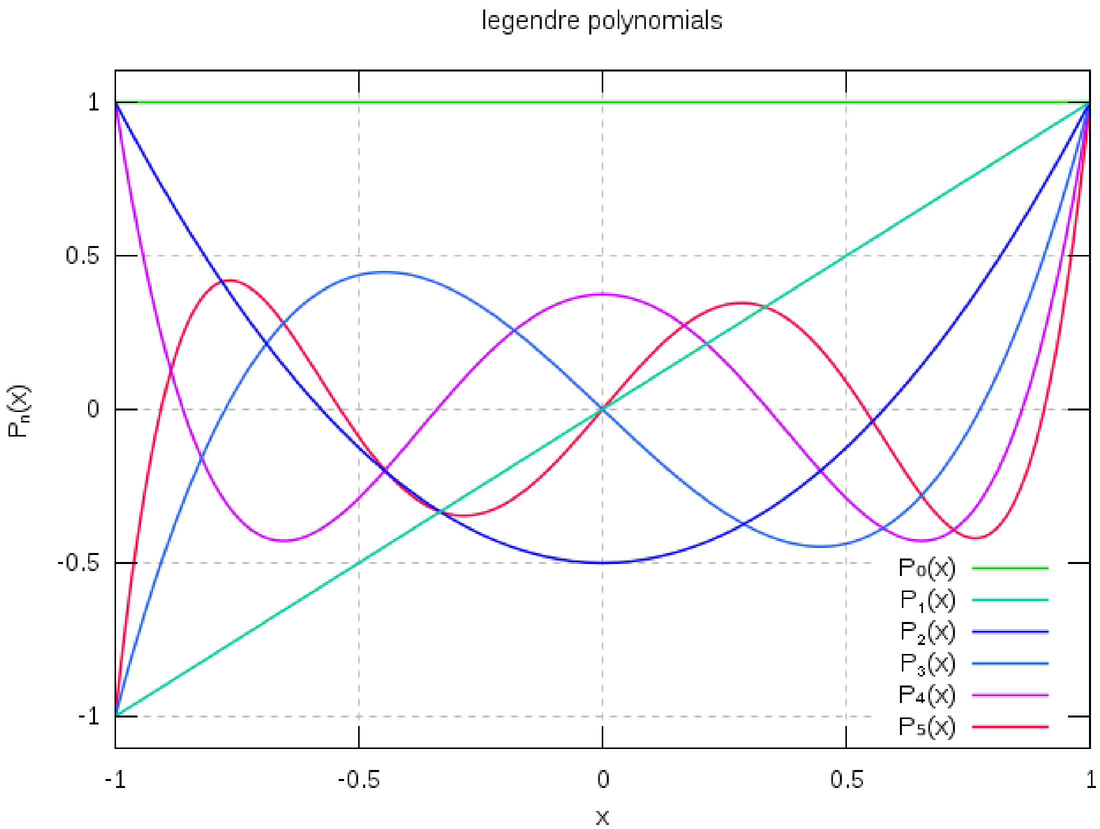
where W_i are appropriate weights for values f_i of the function, evaluated at **Gauss points**





Gauss = roots of the Legendre polynomials

$$P_n(x) = \frac{1}{2^n n!} \frac{d^n}{dx^n}\left[(x^2-1)^n\right].$$



$P_n(x)$

1

x

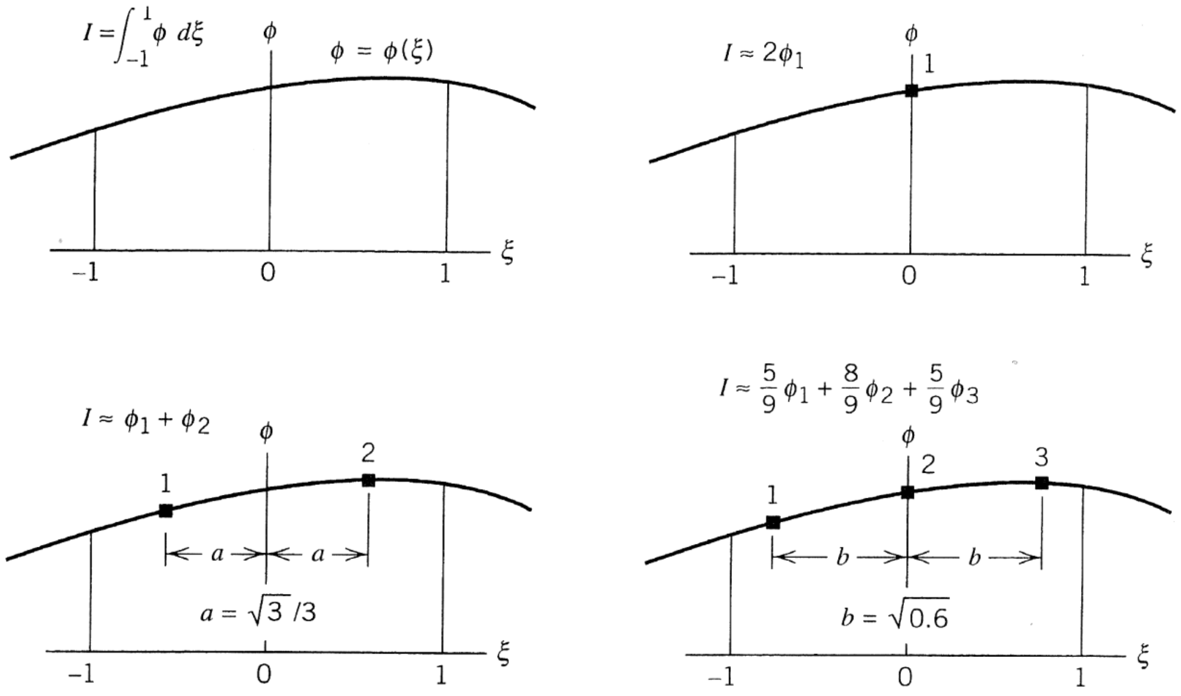
$\frac{1}{2}(3x^2-1)$

$\frac{1}{2}(5x^3-3x)$

Gauss method

$$I \approx \sum_{i=1}^{n} W_i f_i$$

The number *n* and the position of the Gauss points are chosen depending on the order 2*n*-1 of the polynomial that one wants to integrate exactly

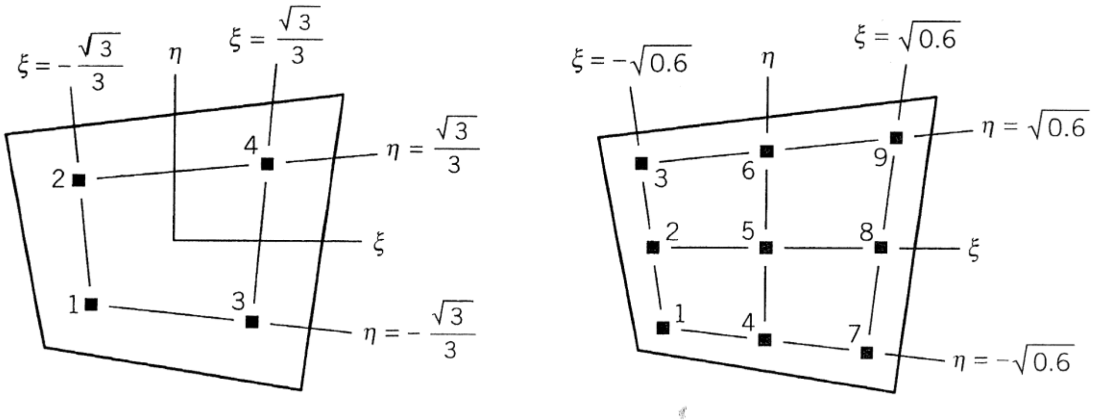


N.B. points are symmetrical

If f(ξ) is a polynomial, with a suitable *n* of Gauss points, the exact integral is obtained; if it is a ratio between polynomials, the result is approximated

Integrals in 2D

$$I = \int_{-1}^{+1}\int_{-1}^{+1} f(\xi,\eta)\, d\xi\, d\eta$$



$$I \approx \sum_{i=1}^{n}\sum_{j=1}^{m} W_i W_j f(\xi_i,\eta_j)$$

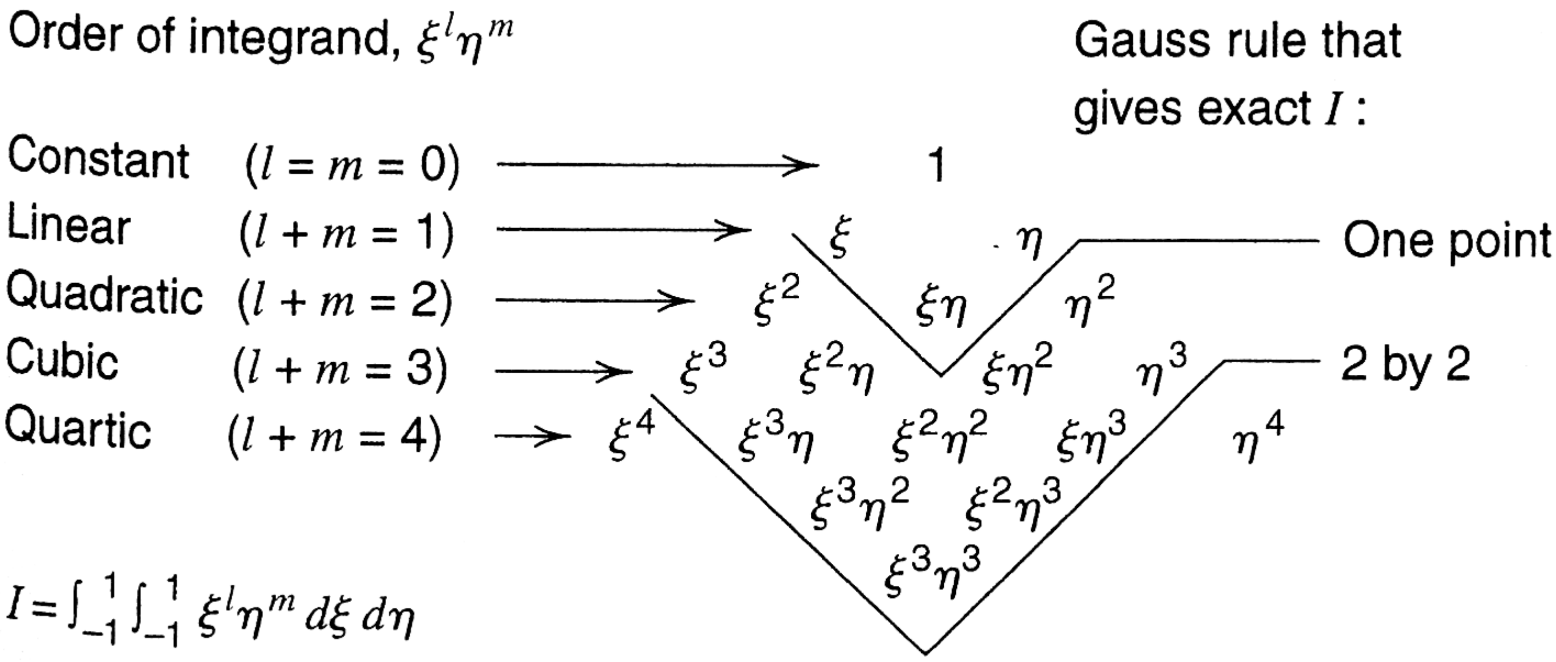
Usually n = m

For n = m = 1, 1 Gauss point in $\xi = 0$ and $\eta = 0$, $I = 4 f_1$

For n = m = 2, $I = f_1 + f_2 + f_3 + f_4$

Choice of the number of Gauss points

The number of points that allows for exact integration of a polynomial is given by the Pascal triangle



Full or reduced integration? (e.g. CP4 vs CP4R)

The stiffness matrix is obtained by integrating

•Polynomials, for undistorted elements (J_{ij} constants)

•Ratios of polynomials, when the element is distorted (J_{ij} are polynomials)

We distinguish between:

•**Full integration**, when the number of Gauss points is enough to integrate exactly the terms of the stiffness matrix of a undistorted element

•**Reduced integration**, when the points are less that those required to integrate exactly the terms of the undistorted element

Full integration may lead to overestimate the element stiffness

Reduced integration may activate zero energy modes (hourglass)

Full and reduced integration of [K]

- Full integration: exact integration of k_{ij} for undistorted elements
- In undistorted elements, the Jacobian J is constant => integrands are polynomials, therefore closed form integration is possible
- For distorted elements, J is a function of iso-parametric coordinates, that transforms the integrands into ratios of polynomials => exact integration cannot be achieved
- Full integration refers to undistorted elements
- By the Gauss method, it is always possible to adopt an integration rule with enough integration points to achieve full integration
- Any other rule that adopts a lower number of Gauss points is defined as reduced

Full and reduced integration of solid 2D and 3D elements: summary

	Gauss integration rule	
Element type	Full	Reduced
Planar 4 nodes	2 x 2	1
Planar 8 nodes	3 x 3	2 x 2
Planar 9 nodes	3 x 3	2 x 2
Solid 8 nodes	2 x 2 x 2	1
Solid 20 nodes	3 x 3 x 3	2 x 2 x 2

Reduced integration

It is justified by the following reasons:

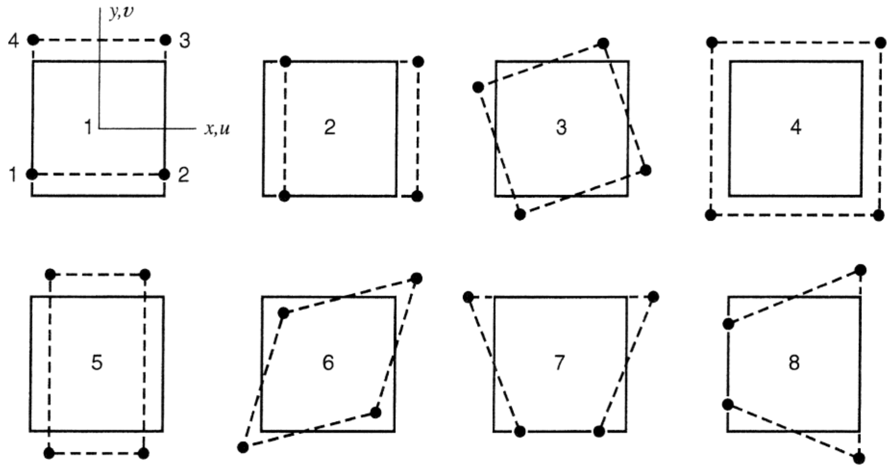
1. better evaluation of the stiffness of the real structure:
   - Any FEM model is by nature MORE RIGID than the real structure (the displacement field that can be described by the shape functions is limited to a polynomials of a given order, which in turn is determined by the number of nodes);
   - Some elements, e.g. solid linear ones, if fully integrated suffer from shear locking, therefore if subjected to bending reduced integration is required
   - Increasing the number of Gauss points:
     - The model may become too stiff
     - The calculation time may increase

More accuracy in the evaluation of the integrals ≠> higher accuracy of the FEM solution

2. reduction of the calculation times, particularly for non-linear problems and dynamic steps

However, zero energy modes may appear, that are particularly disturbing in the case of dynamic analyses

Effects of reduced integration



$$\frac{1}{2} d^T [k] d = U$$

Possible deformation modes of a Q4 element

Gauss points, for the evaluation of the elements of [K], behave like “strain sensors”

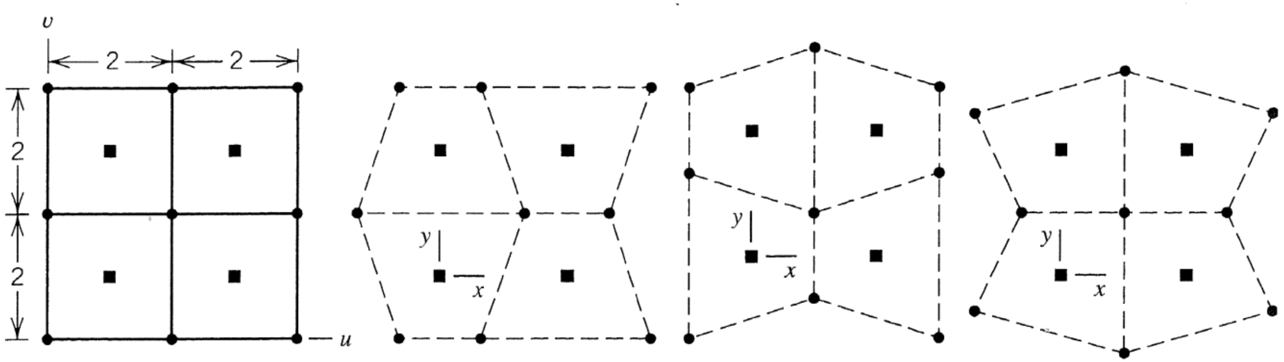
To calculate an integral by the Gauss method, values of the function at given points is used.

If at these locations a particular displacement field results into zero strain at those points (the Gauss points), the terms of [k] of the element will be such that the element will not oppose any elastic reaction to that displacement field.

An example, referring to a Q4 element, are cases 7 and 8 in the figure, for which at point $\xi=0$, $\eta=0$ (Gauss point for reduced integration) one obtains

$\varepsilon_x = \varepsilon_y = \gamma_{xy} = 0$.

Compatibility of zero energy modes for some element types



a) b) c)

$$a)\quad u = Cxy \qquad v = 0$$
$$b)\quad u = 0 \qquad v = -Cxy$$
$$c)\quad u = Cy(1-x) \qquad v = Cx(y-1)$$

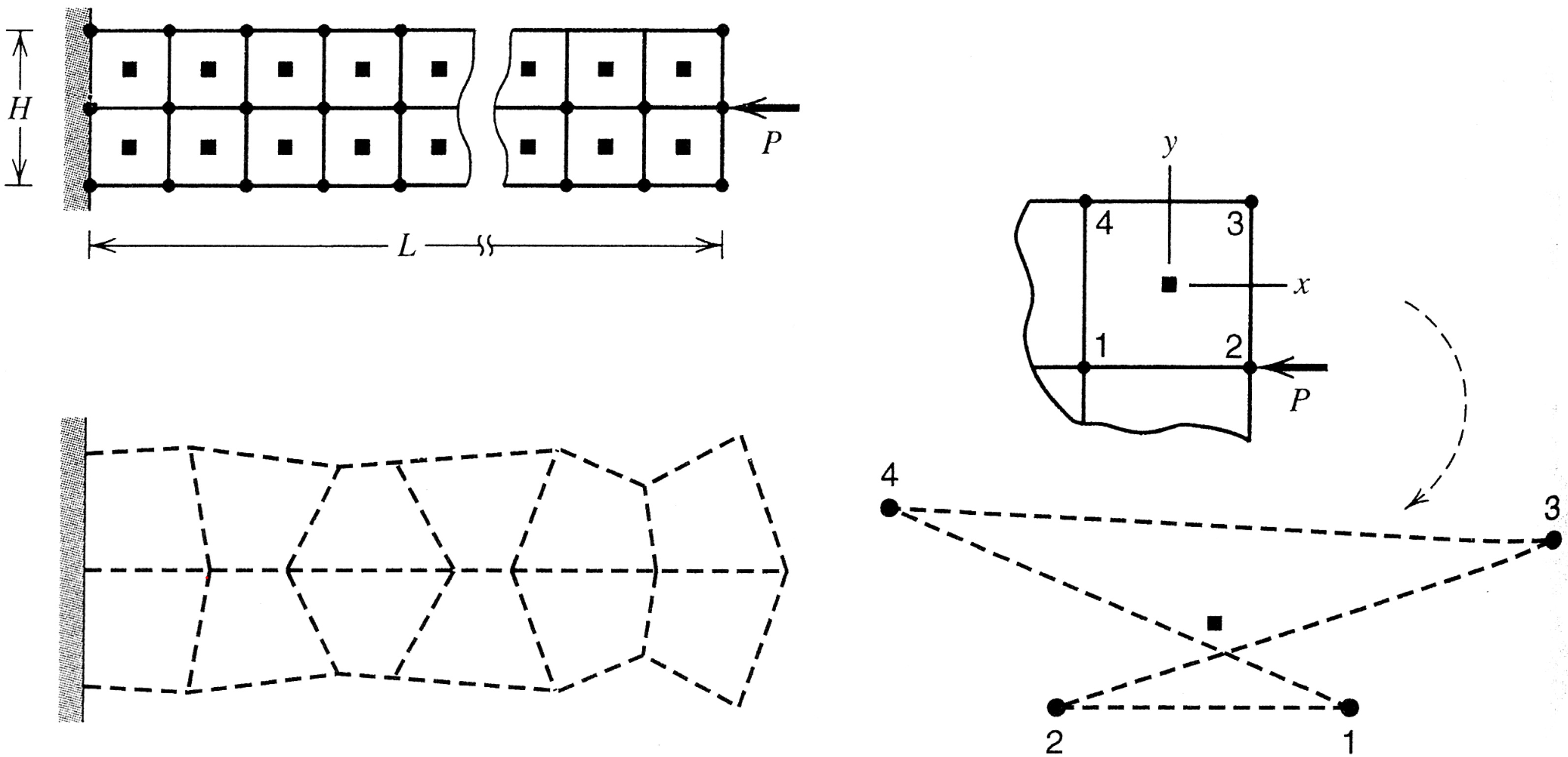
For all these modes $\varepsilon_x = \varepsilon_y = \gamma_{xy} = 0$ at x = y =0, and they are compatible across neighbouring elements, exposing the model to the risk of instability.

It may happen even if the boundary conditions prevent rigid motions

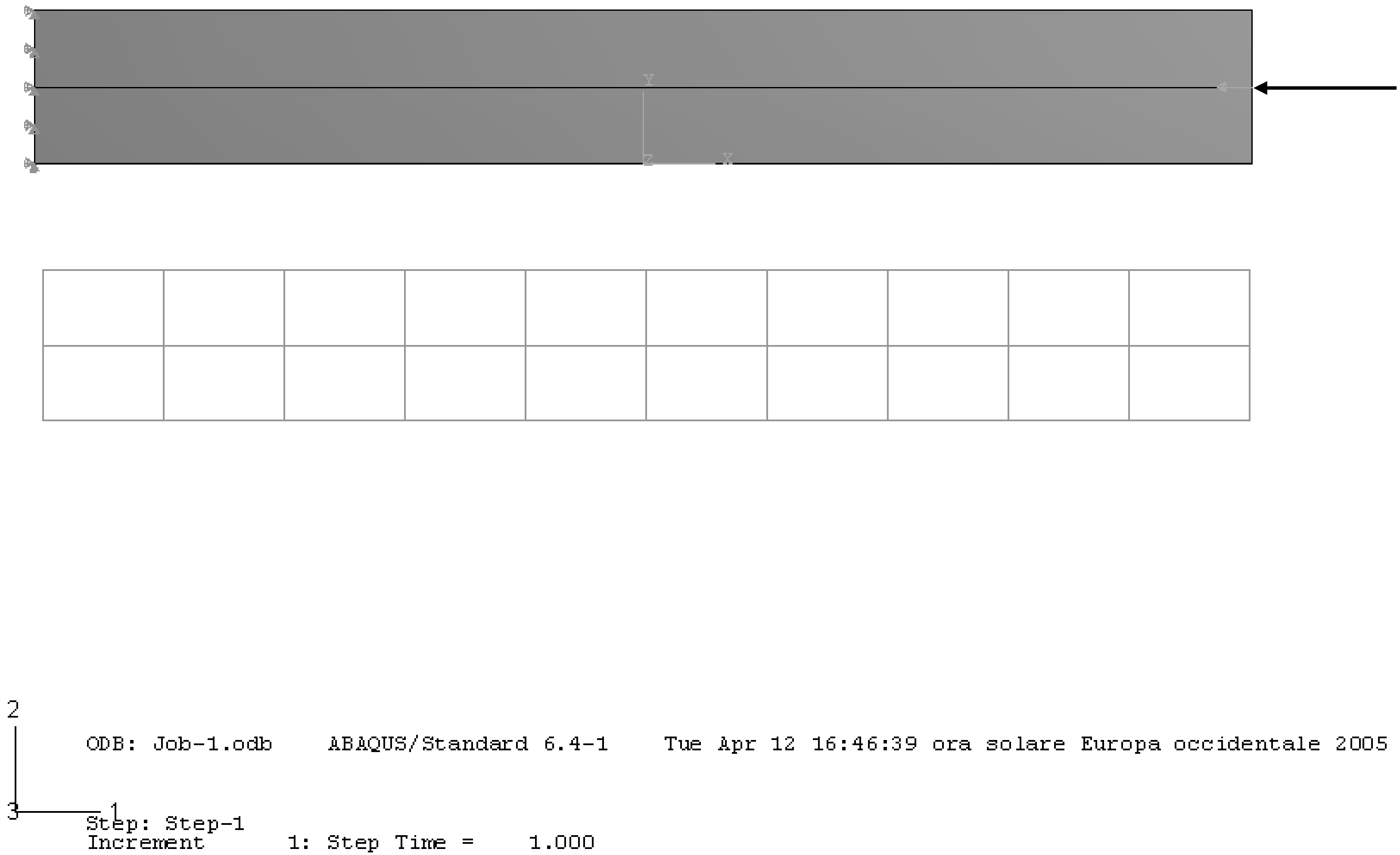
The same happens with non rectangular iso-parametric elements.

These modes are superimposed onto real ones, making it difficult to identify them

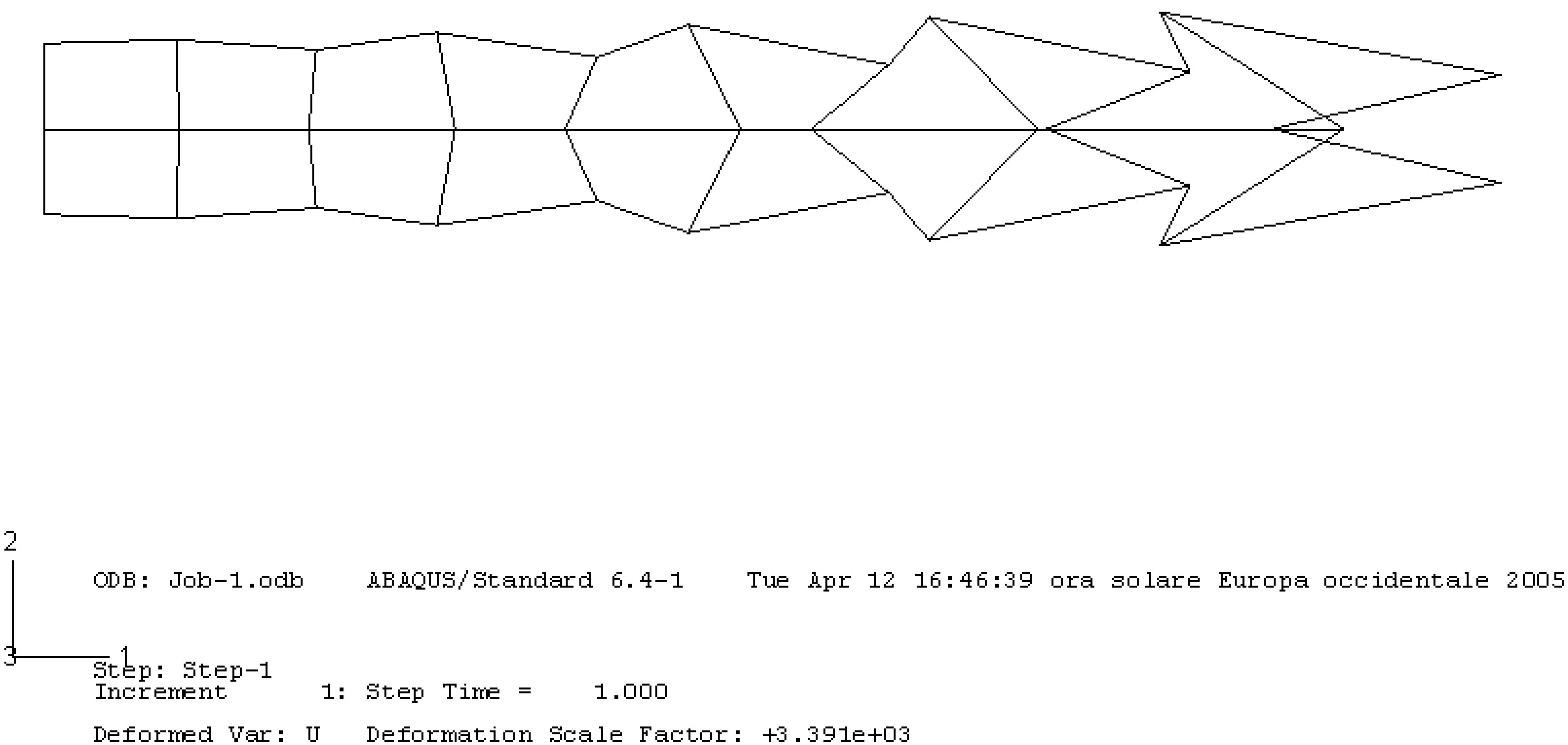
Compatibility of modes: hourglass



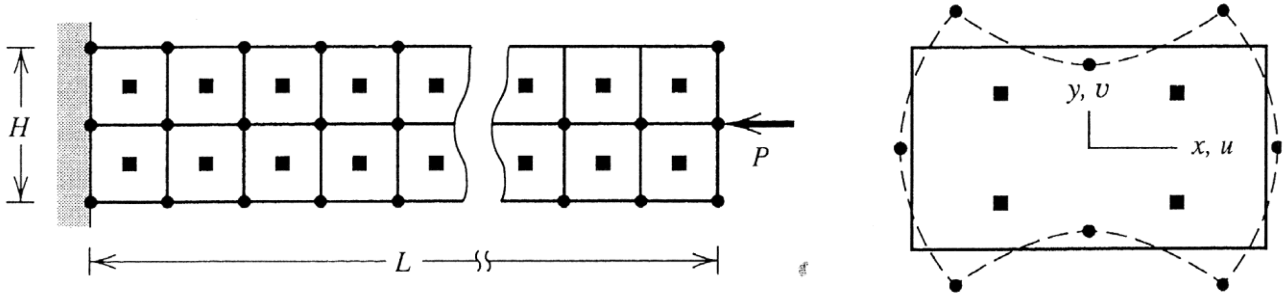
Compatibility of modes: hourglass



Mode Compatibily: *hourglass*

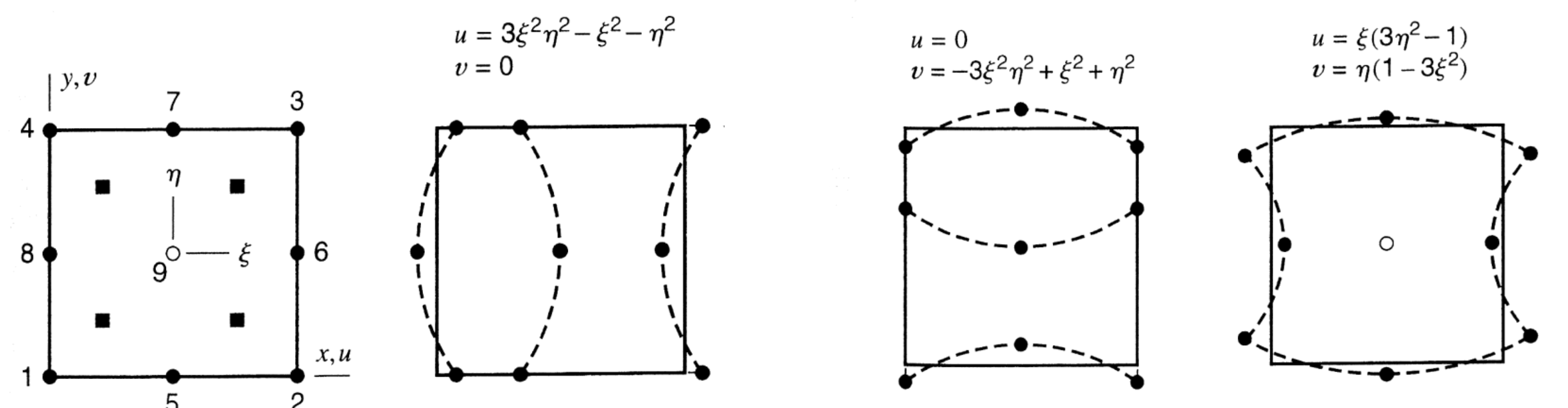


Compatibility of modes for higher order elements



Also Q8 elements integrated with a 2 x 2 law present similar hourglass modes, but they are nor of concern because they cannot propagate (two neighbouring elements cannot deform the same way)

The Q9 element with 2 X 2 integration has transmissible modes



Eight- or nine-node elements | Nine-node element only | Nine-node element only | Eight- and nine-node elements

Summary of the number of zero energy modes for solid 2D and 3D elements (either transmissible or not)

	Gauss integration rule		Hourglass modes
Element type	Full	Reduced	Number
Planar 4 nodes	2 x 2	1	2
Planar 8 nodes	3 x 3	2 x 2	1
Planar 9 nodes	3 x 3	2 x 2	3
Solid 8 nodes	2 x 2 x 2	1	12
Solid 20 nodes	3 x 3 x 3	2 x 2 x 2	6

Hourglass control

- In non-linear analyses and in dynamic analyses it is often necessary to update the stiffness matrix at each step.
- Therefore, the computational effort required by a full integration rule may become non negligible
- To be able to apply a reduced integration rule, to take advantage of the lower computational effort, algorithm of hourglass control have been developed
- These algorithms stabilize the stiffness matrix, to suppress or reduce the zero energy modes, without altering the stiffness matrix (fictitious terms are added to the stiffness matrix, to make energy non null under hourglass modes) with a lower computational effort than that required by full integration.

Accuracy of interpolation of f and f’

Stress evaluation

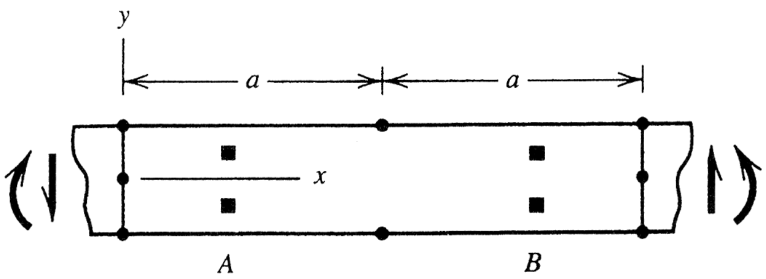
Once nodal displacements are known, stresses could be evaluated at any point of an element by the formula

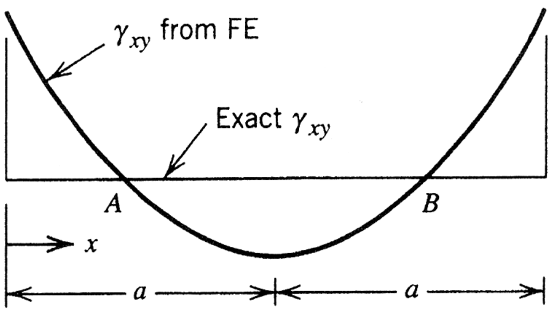
$$\{\sigma(x,y)\} = [E]\{\varepsilon\} \qquad \{\sigma(x,y)\} = [E][B(x,y)]\{d\}$$

where [B] is a function of the coordinates of the point being considered

Not all points display correct stress values, if evaluated by this method

E.g. in the case of bending of a Q8 element

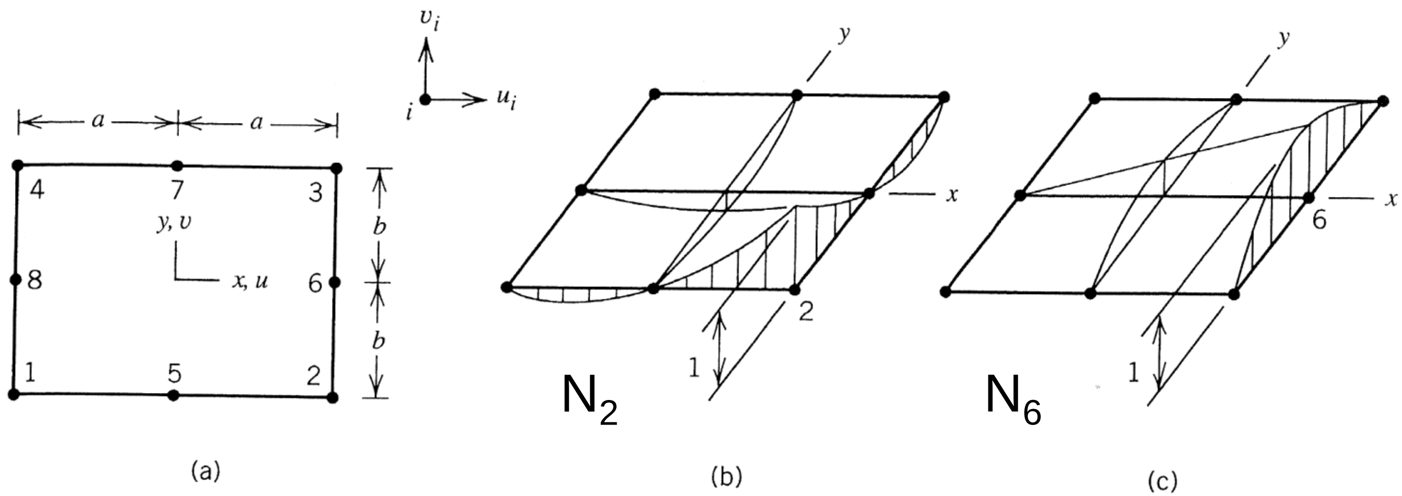




One observes that the correct value fo γ_{xy} is captured at Gauss points only, but not at nodes, where one usually wants them to be evaluated.

Therefore, the procedure for the evaluation of stresses consists of evaluating them at Gauss points and then extrapolate them at the points of interest using the shape functions.

Why is this happening?



The shape functions, to satisfy the condition

$$N_i = 1 \text{ at node } i; \quad N_i = 0 \text{ at any other node } i \neq j$$

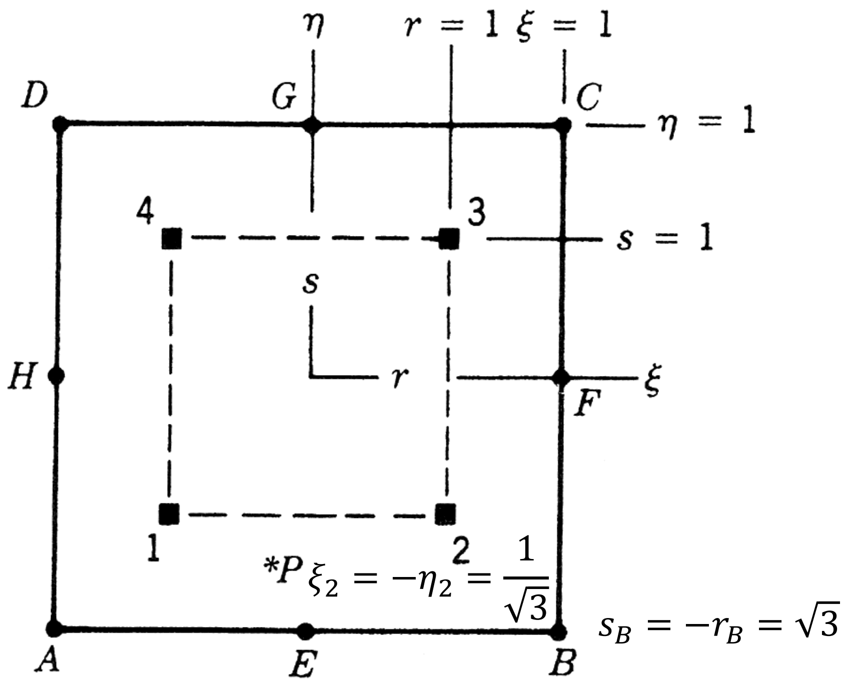
must have a shape characterized by **strong gradients** in proximity of the vertices.

Consequently, their derivatives (and strains are the derivatives of the displacements) have larger values close to nodes than in the interior of the elements.

This, in general, makes the evaluation of the strains by differentiation of the displacements erroneous and consequently also stress values, that are related to strains by the constitutive equations.

Inside the element, particularly close to the Gauss points, the variation of displacements is smoother and therefore the evaluation of strains is more accurate.

Example of extrapolation of stresses from Gauss points



1. Coordinate transformation

$$r = \sqrt{3}\xi$$
$$s = \sqrt{3}\eta$$

2. Extrapolation

$$\sigma_P = \sum N_i \sigma_i \quad i = 1,2,3,4$$

Using the shape functions

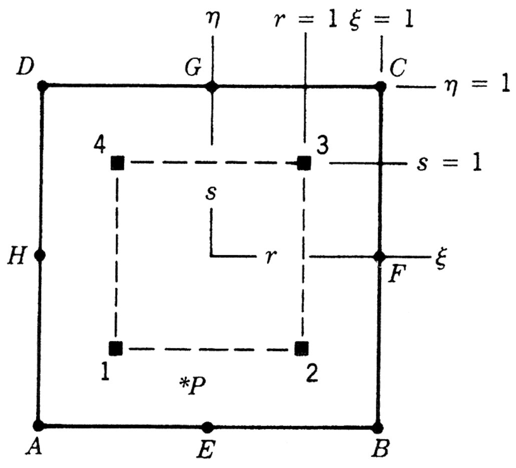
$$N_1 = \frac{(1-r)(1-s)}{4} \quad N_2 = \frac{(1+r)(1-s)}{4}$$

$$N_3 = \frac{(1+r)(1+s)}{4} \quad N_4 = \frac{(1-r)(1+s)}{4}$$

Example: at node A

$$\sigma_{xA} = 1.866\,\sigma_{x1} - 0.5\sigma_{x2} + 0.134\,\sigma_{x3} - 0.5\sigma_{x4}$$

Extrapolation of stresses from Gauss points



In general, the expression of stresses is bi-linear

$$\sigma_x = C_1 + C_2\xi + C_3\eta + C_4\xi\eta$$

This means that for a Q8 element it is one order lower than the expression of displacements (u,v)

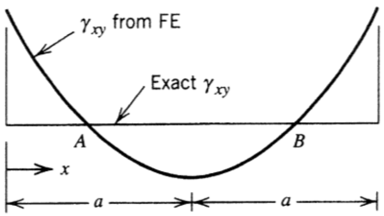
Nevertheless, the accuracy of the representation of the stress field is better than the one that could be obtained by differentiation of the displacement field.

N.B. #1: the Gauss point used for stress evaluation may differ from those used for the integration of [K]

N.B. #2: averaging stresses at nodes does not usually compensate the inaccuracy related to the differentiation of the shape functions.

Example:

(



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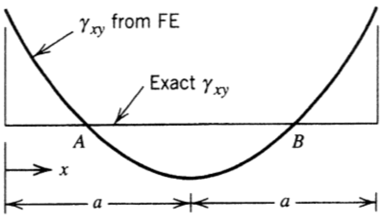

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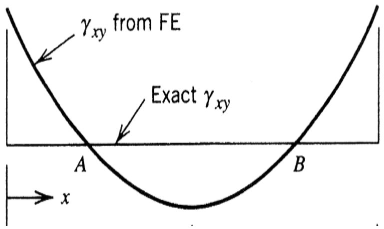

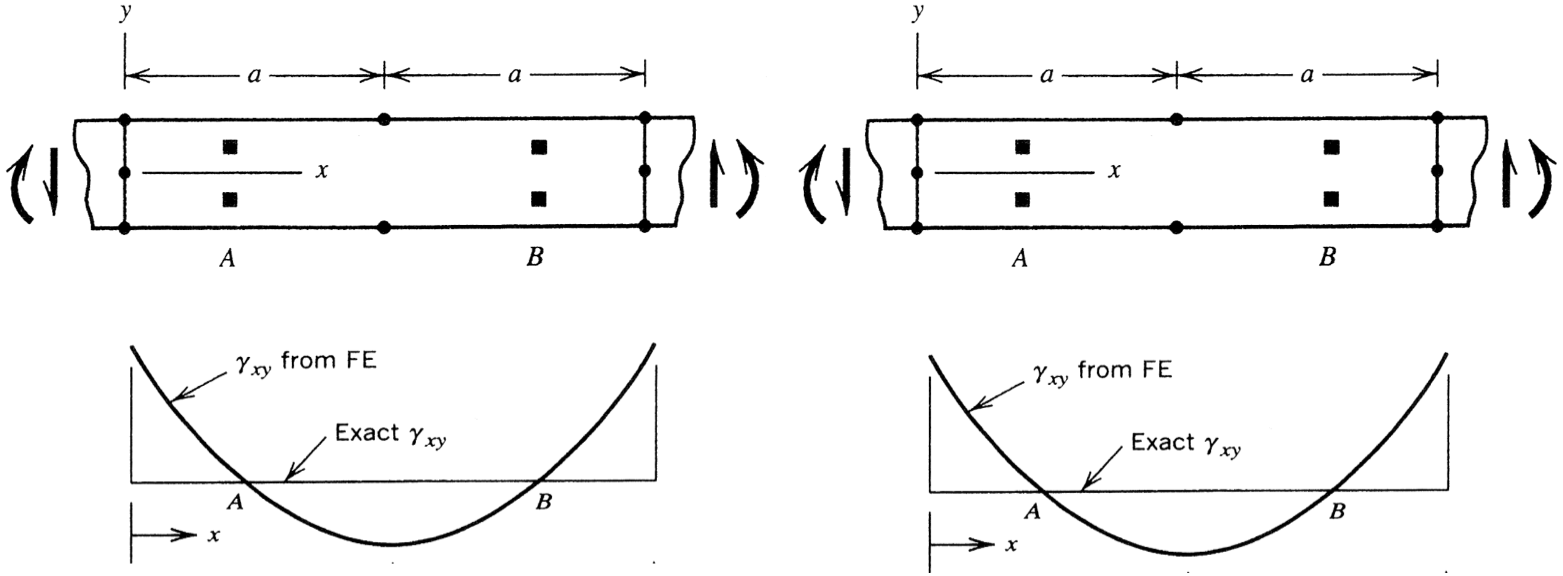
y
a
a
x
A
B
γxy from FE
Exact γxy
A
B
x
y
a
a
x
A
B
γxy from FE
Exact γxy
A
B
x

How are results displayed in FE viewers?

Displacements:

- Displacements are evaluated directly at nodes
- Values are unique, no need of any other transformation
- Inside the element, values are displayed by **linear interpolation** from nodal values

Stresses:

- Once extrapolated from Gauss points to nodes, nodal values are **averaged at nodes** (only if the difference is less than a certain threshold that can be set by the user)
- This allows for a smoother representation, but it may give an **erroneous impression of an accurate solution**, e.g. in the case of a too coarse mesh.
- To assess the accuracy of the results better, it is advisable:
  - To display stress without averaging values at nodes.
  - Adjust the threshold to highlight the differences.