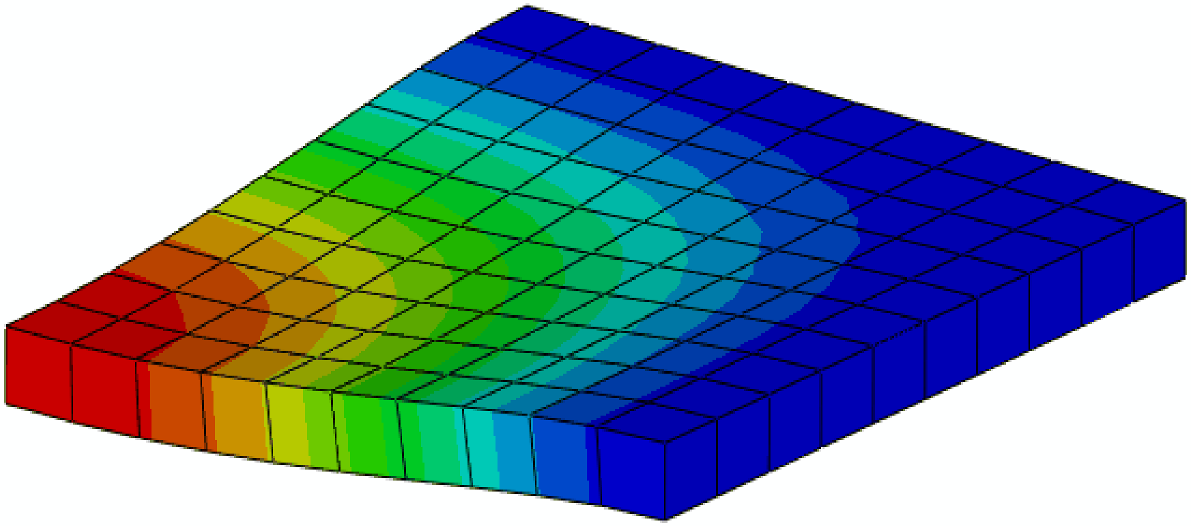
POLITECNICO DI MILANO

Finite Element Simulation For Mechanical Design

Plate and Shell elements

A. Bernasconi

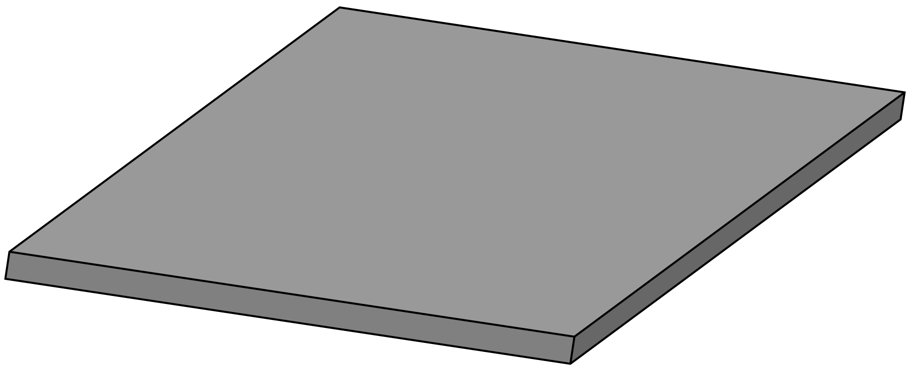
What is the right element for modelling thin structures?



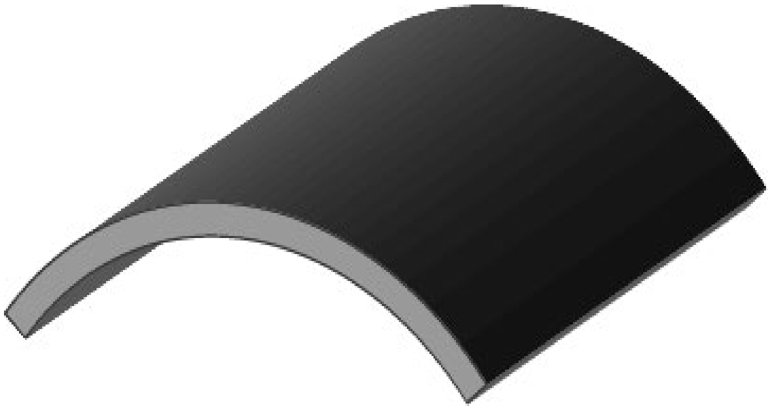
To capture stress gradients, a large number of solid 3D elements would be required

More efficient element exist: shell elements

Plates and shells



Plates



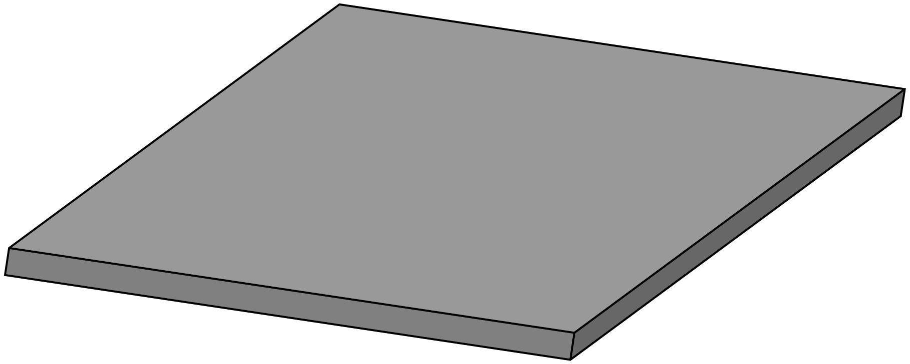
Shells

Thickness << other dimensions

Plates = planar structures

Shells = have initial curvature/s

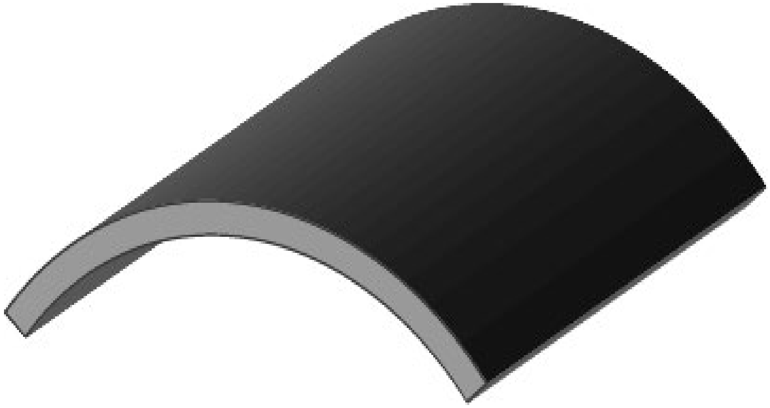
Plates

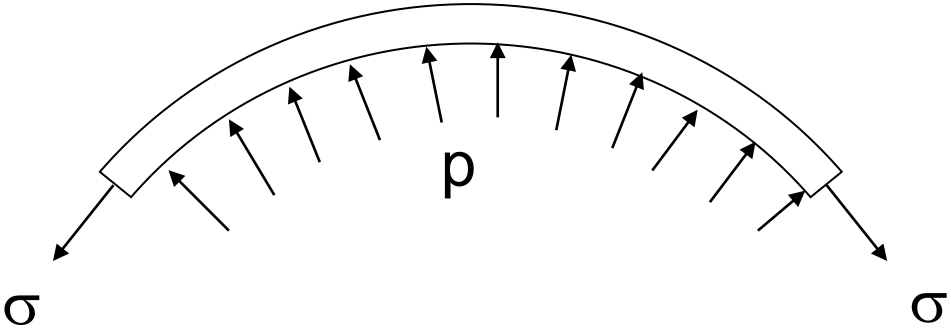


Plates react to transverse loads by deflecting (bending)

Analogous to straight beams in 2D, but with important differences

In shells, membrane stresses prevail

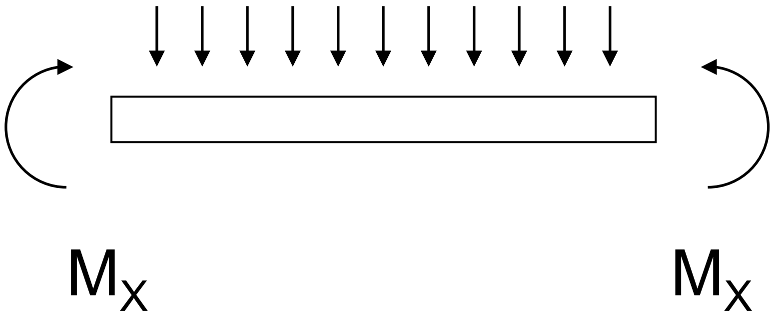


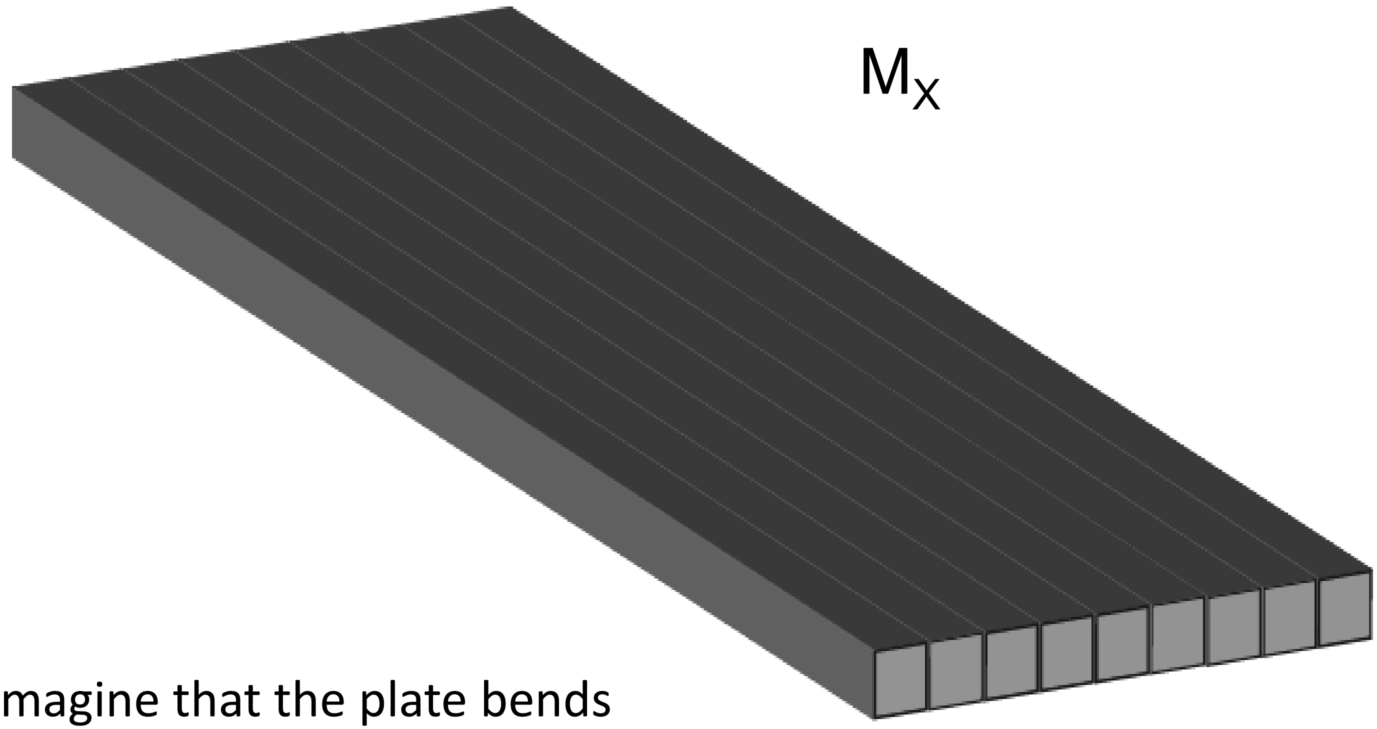


Example: thin-walled cylinder having wall thickness *s* and average diameter *D*, under internal pressure *p*:

$$\sigma = \frac{pD}{2s}$$

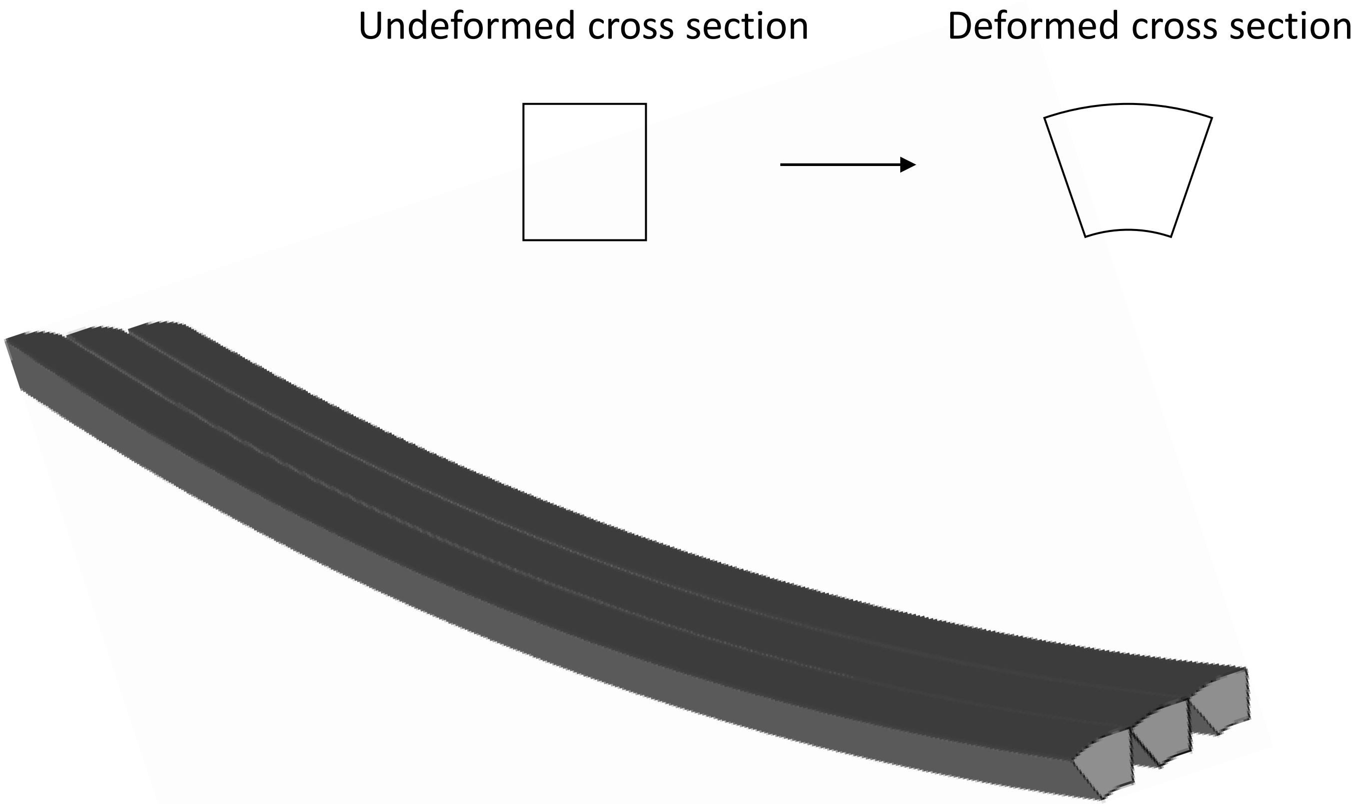
Can a plate be modelled like a set of independent beams?





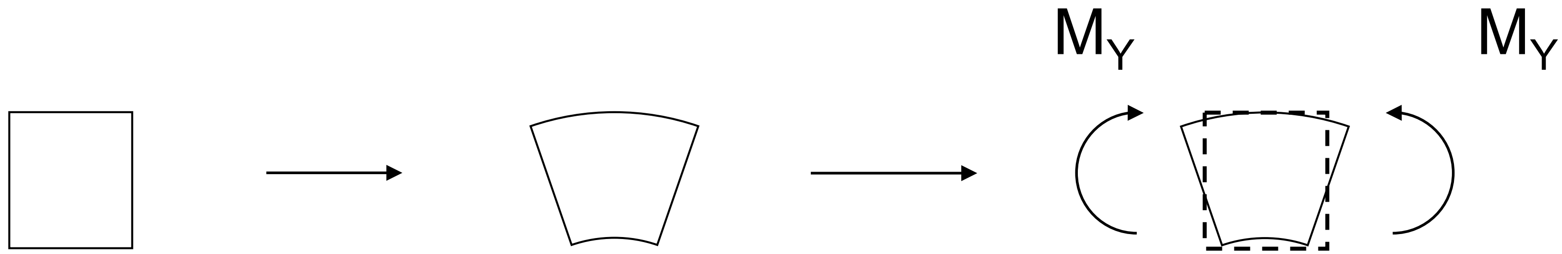
Let's imagine that the plate bends under the action of a vertical distributed load

Answer: NO



Due to lateral contraction, the plate would crack longitudinally

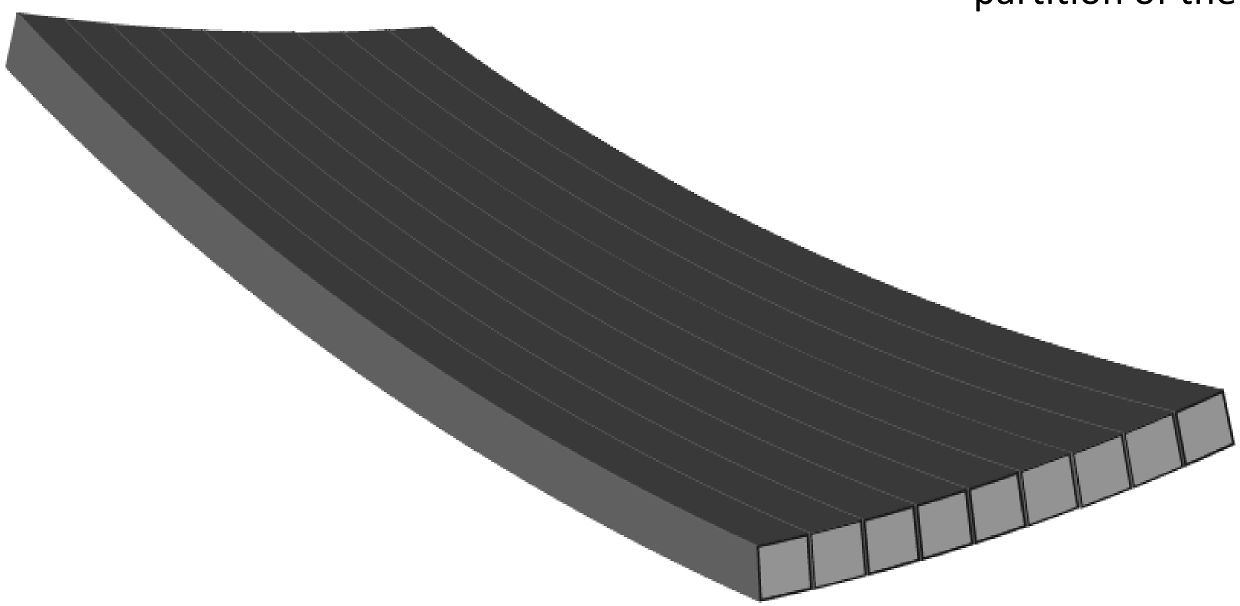
To restore compatibility of displacements, the presence of M_x necessarily implies M_y



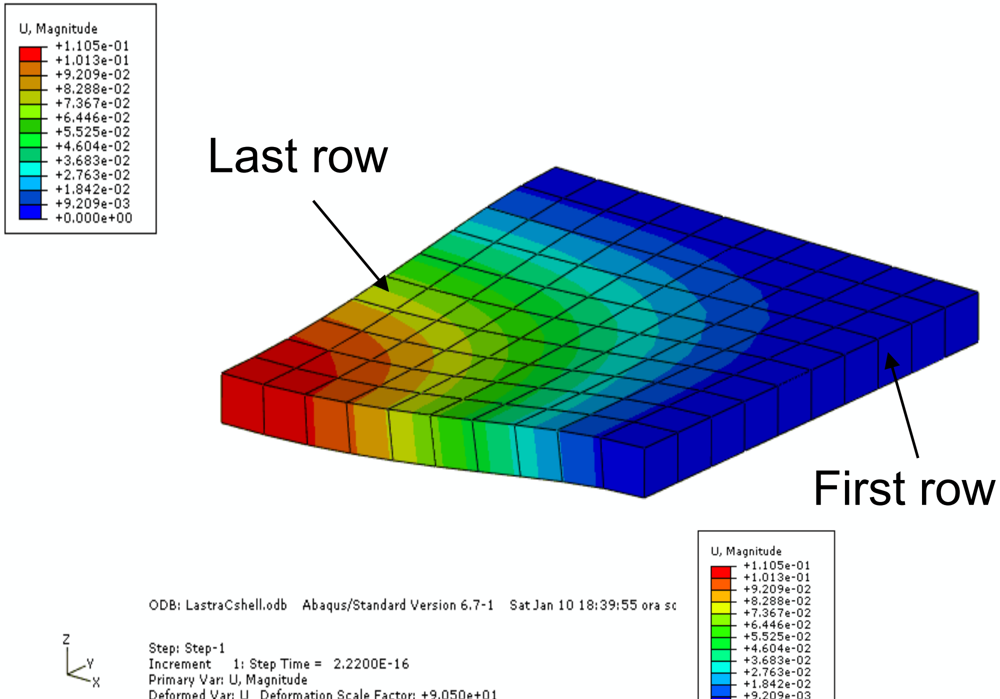
Undeformed cross section

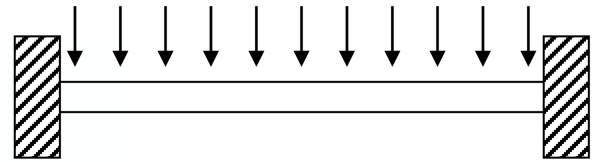
Deformed cross section of one beam

Deformed cross section of the corresponding partition of the plate



Moreover: a torsional moment exists M_{XY}

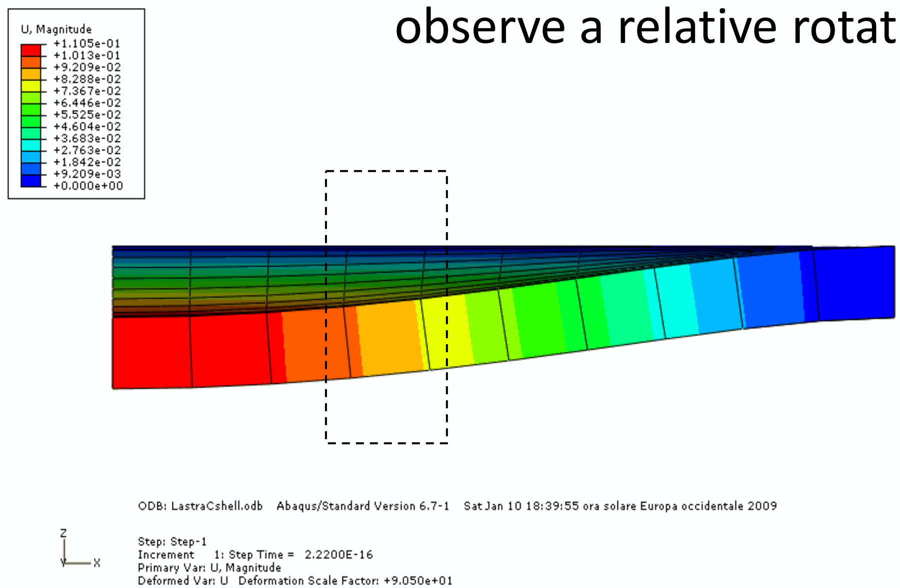




Square plate with clamped edges and uniform transverse load:

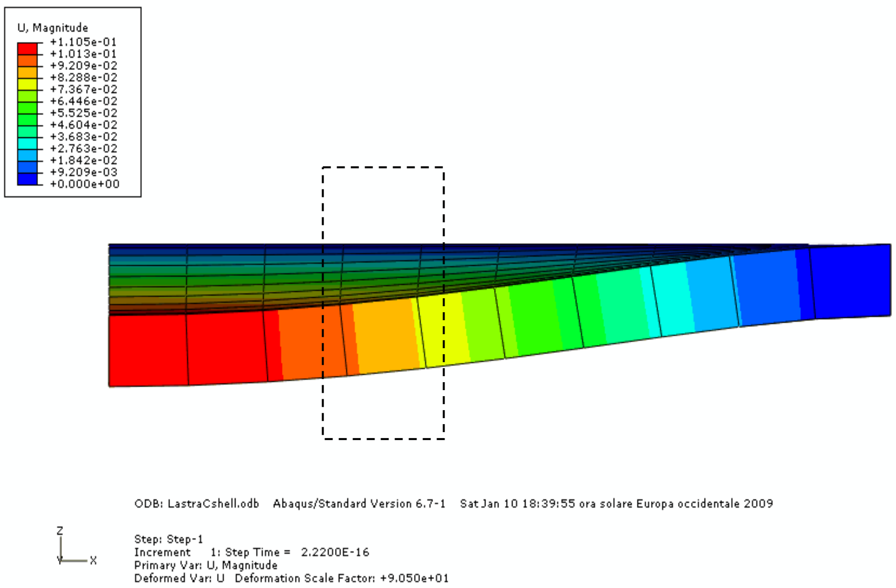
If we consider the first and the last row of elements, we can observe a relative rotation

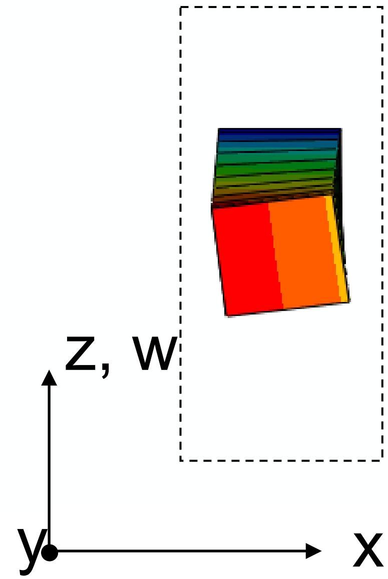
Note: ¼ model of a clamped plate

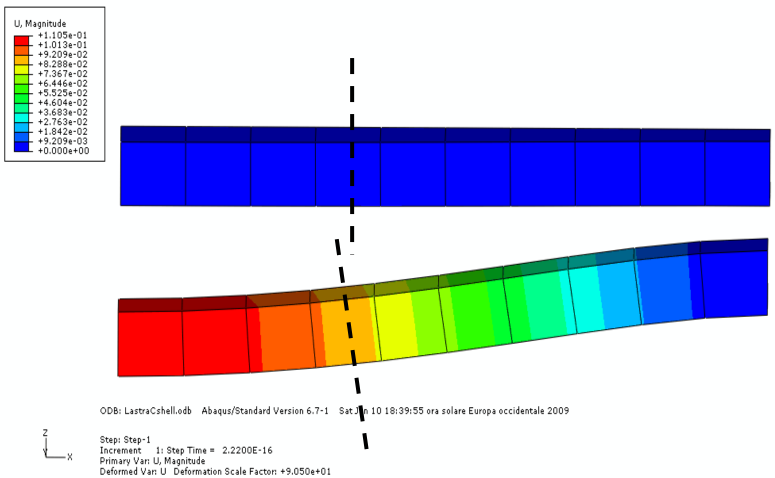


Let's focus on one row of elements

…torsion is clearly apparent

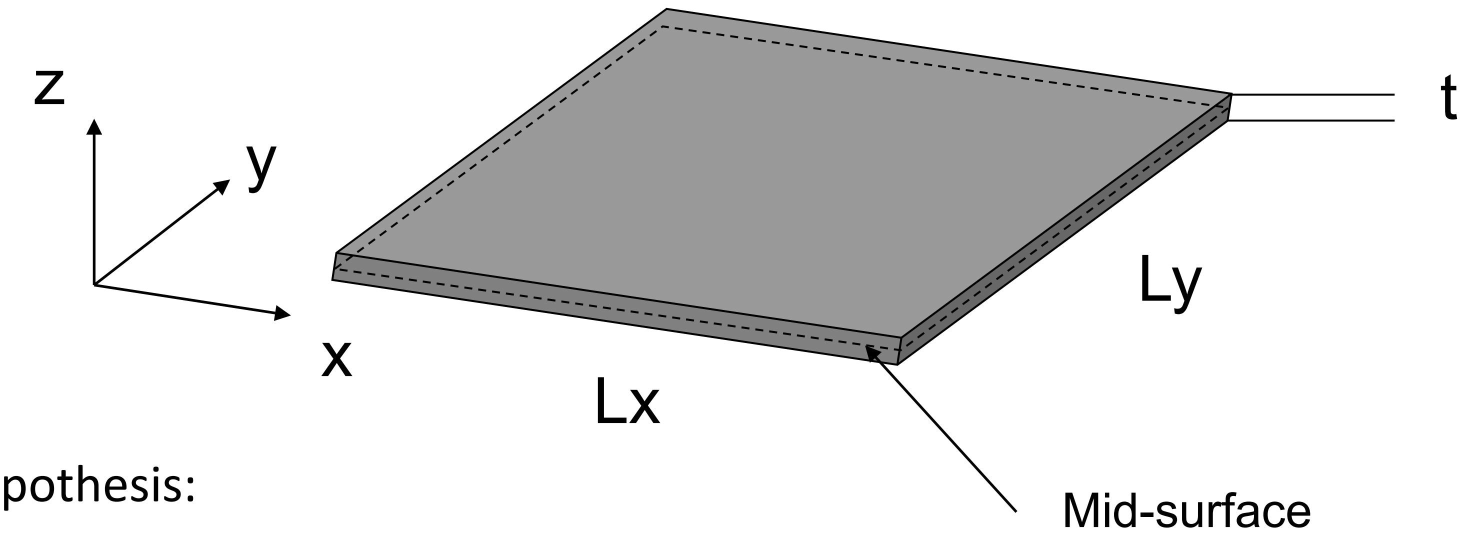






Torsion is related to the relative rotation (about the y axis) of the cross sections along the same coordinate x

Kirchhoff theory



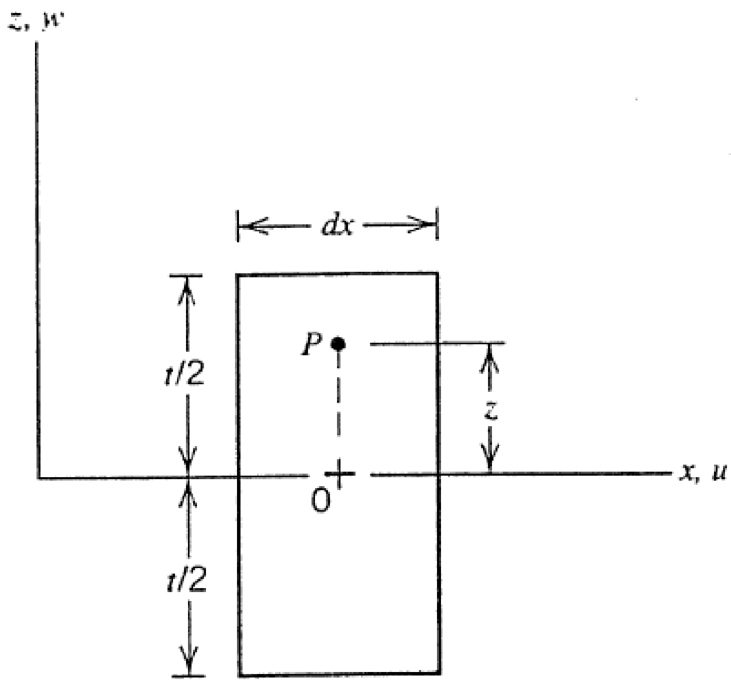
Hypothesis:

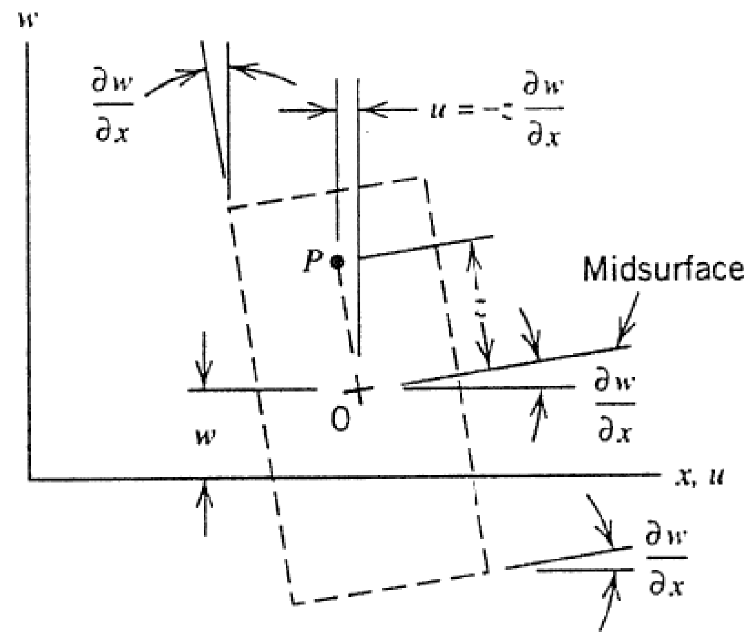
1. Lx, Ly >> t
2. Shear contribution to the deformed shape is negligible
3. A straight segment initially perpendicular to the mid-surface remains straight and perpendicular to it also in the deformed configuration

Kirchhoff theory: strain-displacement relationships

$z = 0$, midsurface

$$w = w(x, y)$$





$$u = -z\frac{\partial w}{\partial x} \qquad v = -z\frac{\partial w}{\partial y} \quad \Rightarrow$$

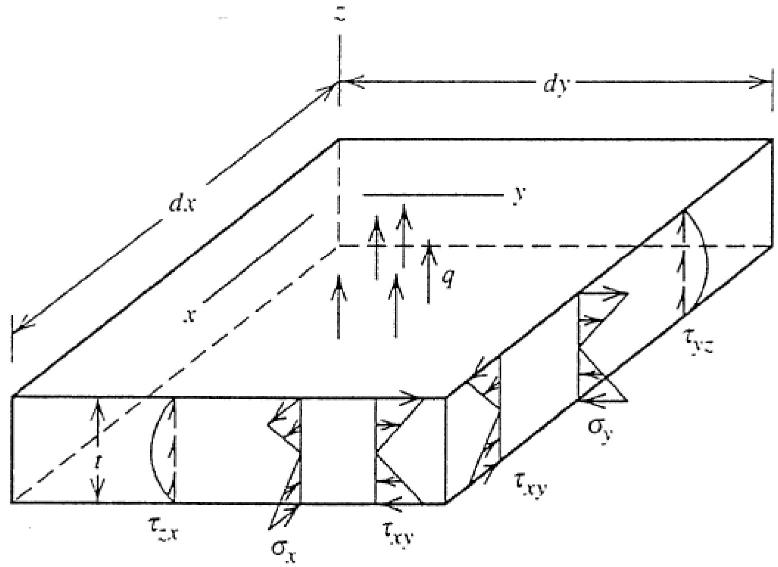
$$\varepsilon_x = \frac{\partial u}{\partial x} = -z\frac{\partial^2 w}{\partial x^2}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = -z\frac{\partial^2 w}{\partial y^2} \quad , \quad \left(\gamma_{xz} = \gamma_{yz} = 0, \quad assumed\right)$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -2z\frac{\partial^2 w}{\partial x \partial y}$$

$$\frac{\partial^2 w}{\partial x \partial y} = \frac{\partial^2 w}{\partial y \partial x}$$

By introducing a linear elastic constitutive model

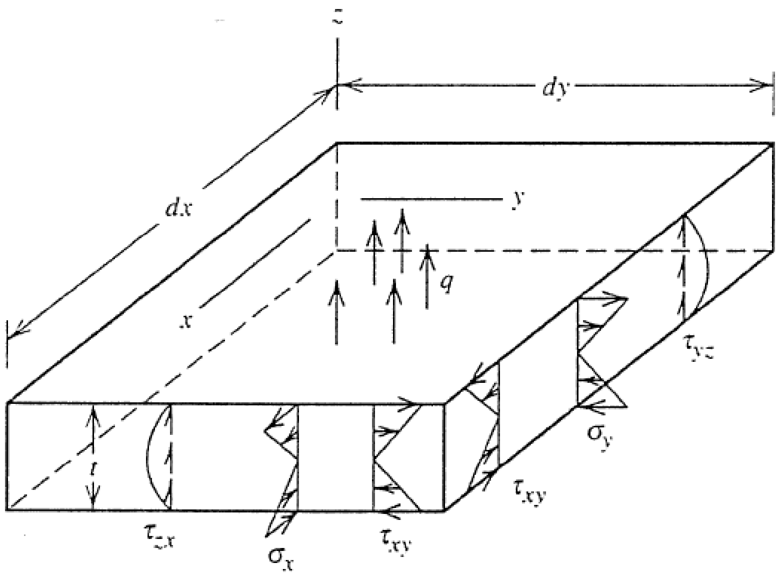


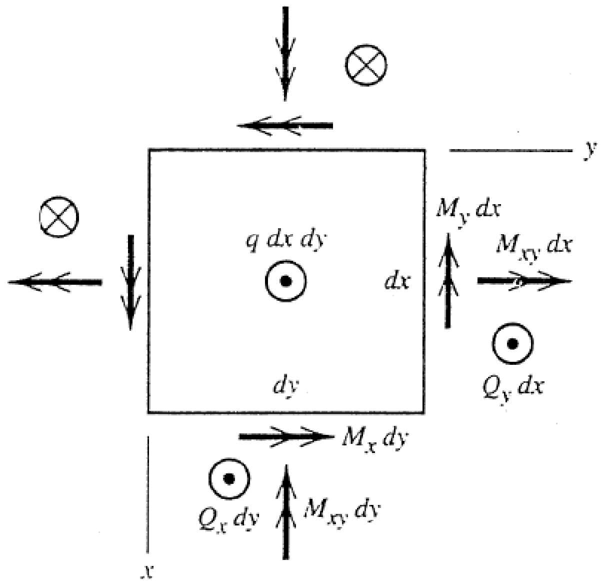
Small thickness => hp. $\sigma_z = 0$ through the thickness => plane stress

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix}$$

Note: τ_{zx} and $\tau_{yz} \neq 0$ (parabolic distribution) even if $\gamma_{zx} = \gamma_{yz} = 0$

Internal forces





By integrating stresses along z (Q and M, being referred to dx or to dy, have dimensions of Forces or Moments per unit length, respectively)

$$M_x = \int_{-t/2}^{+t/2} \sigma_x z dz \qquad Q_x = \int_{-t/2}^{+t/2} \tau_{xz} dz$$

$$M_y = \int_{-t/2}^{+t/2} \sigma_y z dz \qquad Q_y = \int_{-t/2}^{+t/2} \tau_{yz} dz$$

$$M_{xy} = \int_{-t/2}^{+t/2} \tau_{xy} z dz$$

maximum (absolute values) stresses :

$$\sigma_{\mathrm{x,max}} = \pm \frac{6M_x}{t^2}, \quad \sigma_{\mathrm{y,max}} = \pm \frac{6M_y}{t^2}, \quad \tau_{\mathrm{xy,max}} = \pm \frac{6M_{xy}}{t^2}$$

Relationships between moments and curvatures (twist)

$$\begin{Bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{Bmatrix} = \frac{E}{1-\nu^2} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & (1-\nu)/2 \end{bmatrix} \begin{Bmatrix} -z\dfrac{\partial^2 w}{\partial x^2} \\ -z\dfrac{\partial^2 w}{\partial y^2} \\ -2z\dfrac{\partial^2 w}{\partial x \partial y} \end{Bmatrix}$$

$$M_x = \int_{-t/2}^{+t/2} \sigma_x z dz = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu \frac{\partial^2 w}{\partial y^2}\right)$$

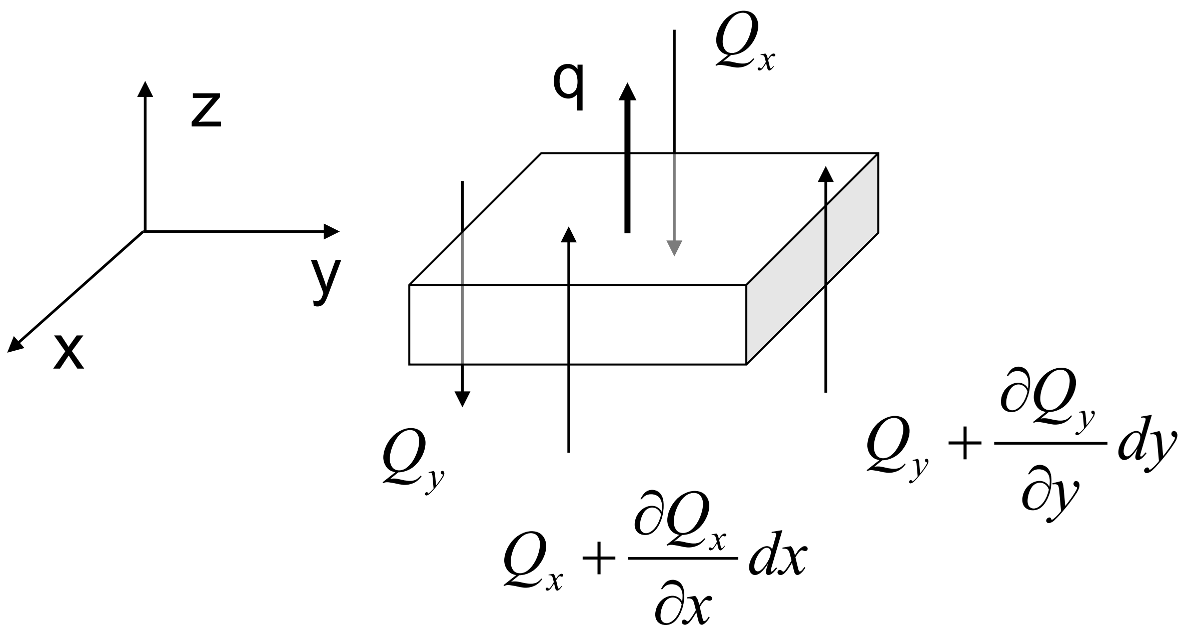
$$M_y = \int_{-t/2}^{+t/2} \sigma_y z dz = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu \frac{\partial^2 w}{\partial x^2}\right) \quad , \quad \text{where } D = \frac{Et^3}{12(1-\nu^2)}$$

$$M_{xy} = \int_{-t/2}^{+t/2} \tau_{xy} z dz = -(1-\nu) D \frac{\partial^2 w}{\partial x \partial y}$$

For beams

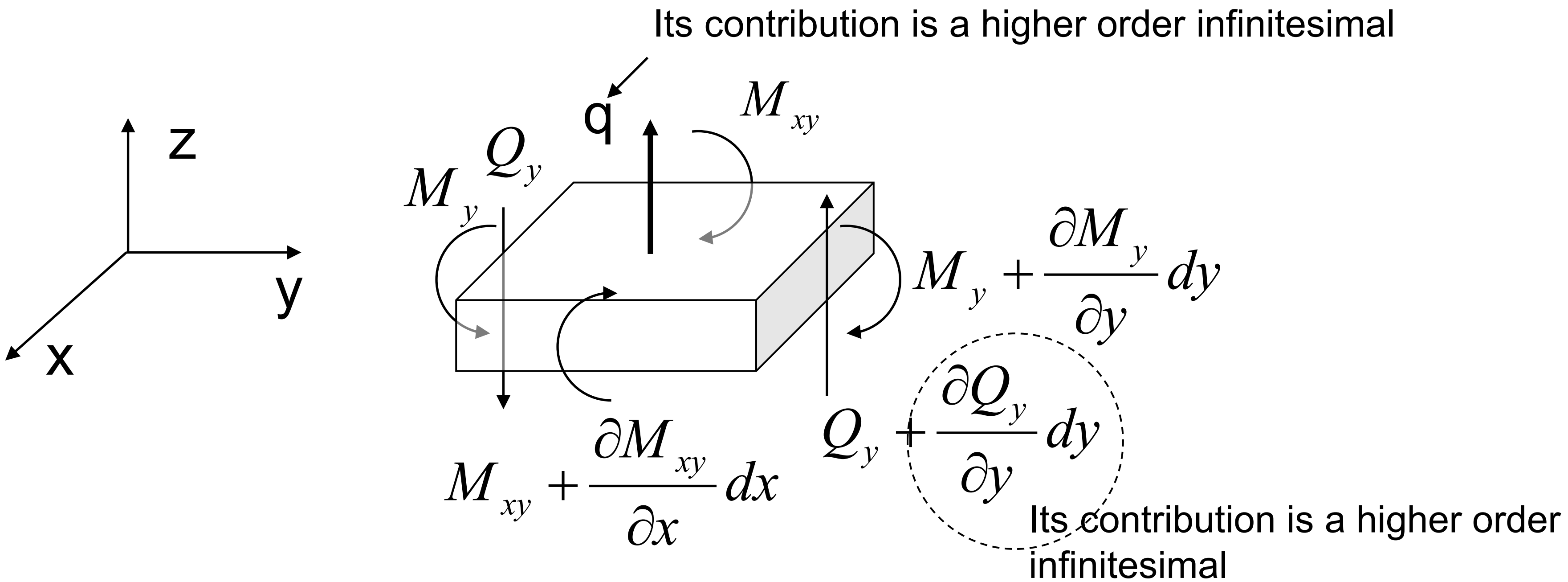
$$M_x = \int_{-t/2}^{+t/2} \sigma_x z b dz = -D' \frac{\partial^2 w}{\partial x^2}, \text{where } D' = \frac{Ebt^3}{12}$$

Equilibrium equations along z



$$q\,dx\,dy - Q_x dy + \left(Q_x + \frac{\partial Q_x}{\partial x} dx \right) dy - Q_y dx + \left(Q_y + \frac{\partial Q_y}{\partial y} dy \right) dx = q + \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = 0$$

Equilibrium equations: rotation about the x axis

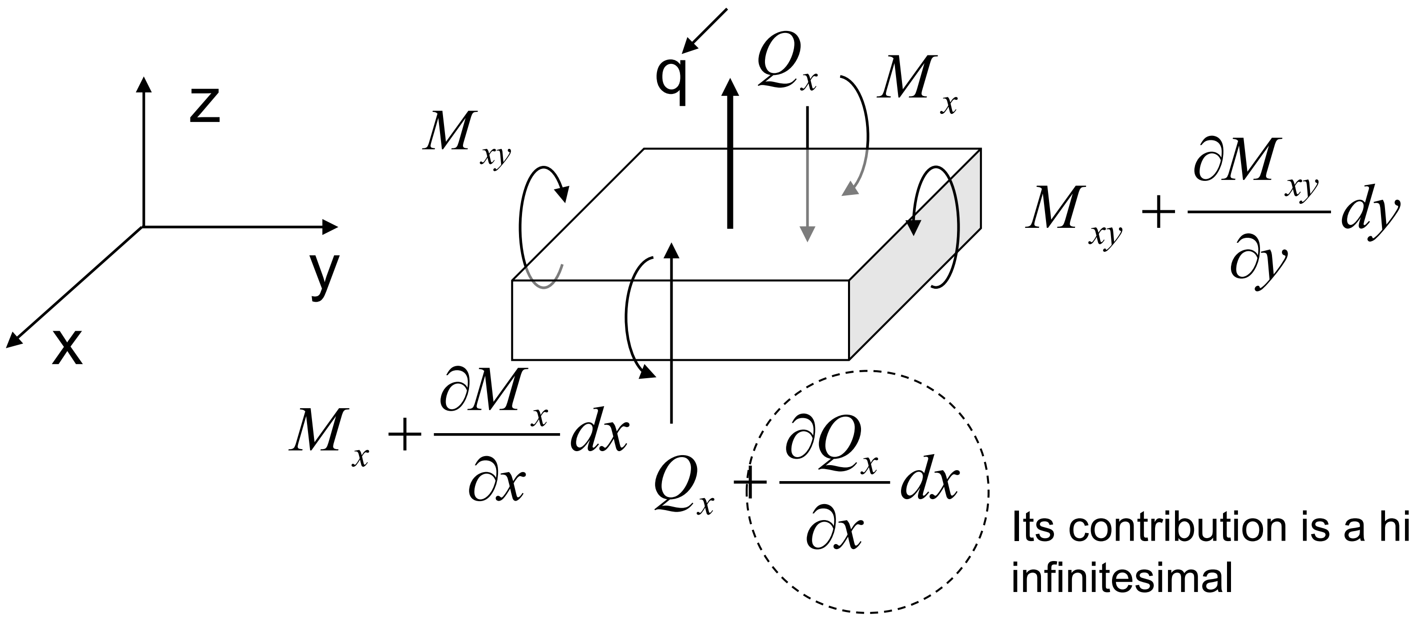


$$Q_y dx dy + M_y dx - \left(M_y + \frac{\partial M_y}{\partial y} dy \right) dx + M_{xy} dy - \left(M_{xy} + \frac{\partial M_{xy}}{\partial x} dx \right) dy = 0$$

$$Q_y - \frac{\partial M_y}{\partial y} - \frac{\partial M_{xy}}{\partial x} = 0$$

Equilibrium equations: rotation about the y axis

Its contribution is a higher order infinitesimal



Its contribution is a higher order infinitesimal

$$Q_x dxdy + M_x dy - \left(M_x + \frac{\partial M_x}{\partial x} dx \right) dy + M_{xy} dx - \left(M_{xy} + \frac{\partial M_{xy}}{\partial y} dy \right) dx = 0$$

$$Q_x - \frac{\partial M_x}{\partial x} - \frac{\partial M_{xy}}{\partial y} = 0$$

Substituting…

$$q+\frac{\partial Q_x}{\partial x}+\frac{\partial Q_y}{\partial y}=0 \qquad M_x=-D\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)$$

$$Q_x-\frac{\partial M_x}{\partial x}-\frac{\partial M_{xy}}{\partial y}=0 \quad e \quad M_y=-D\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right) \quad \Rightarrow$$

$$Q_y-\frac{\partial M_y}{\partial y}-\frac{\partial M_{xy}}{\partial x}=0 \qquad M_{xy}=-(1-\nu)D\frac{\partial^2 w}{\partial x\partial y}$$

$$\Rightarrow \quad q+\frac{\partial^2 M_x}{\partial x^2}+\frac{\partial^2 M_{xy}}{\partial y\partial x}+\frac{\partial^2 M_y}{\partial y^2}+\frac{\partial^2 M_{xy}}{\partial x\partial y}=0 \quad \Rightarrow$$

$$\Rightarrow \quad q-D\left(\frac{\partial^4 w}{\partial x^4}+\nu\frac{\partial^4 w}{\partial x^2\partial y^2}\right)-(1-\nu)D\frac{\partial^4 w}{\partial x^2\partial y^2}-D\left(\frac{\partial^4 w}{\partial y^4}+\nu\frac{\partial^4 w}{\partial y^2\partial x^2}\right)-(1-\nu)D\frac{\partial^4 w}{\partial x^2\partial y^2}=0 \quad \Rightarrow$$

$$\Rightarrow \quad D\left(\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}\right)=q$$

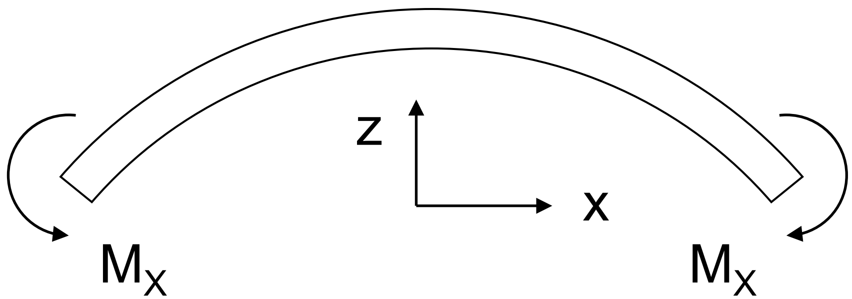
Elastic surface equation

It's analogous to the equation that describe bending of 2D beams $\mathrm{EJ}\ v^{\mathrm{IV}}=\mathrm{q}$

Elastic curve equation

Difference between beams and plates

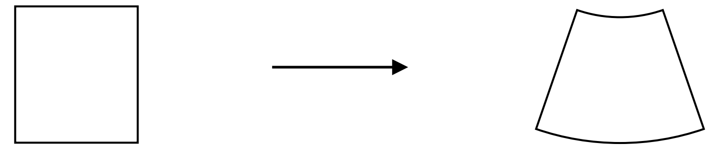
If a beam is bent:



$\partial^2 w/\ \partial x^2 \neq 0$

Undeformed cross section

Deformed cross section

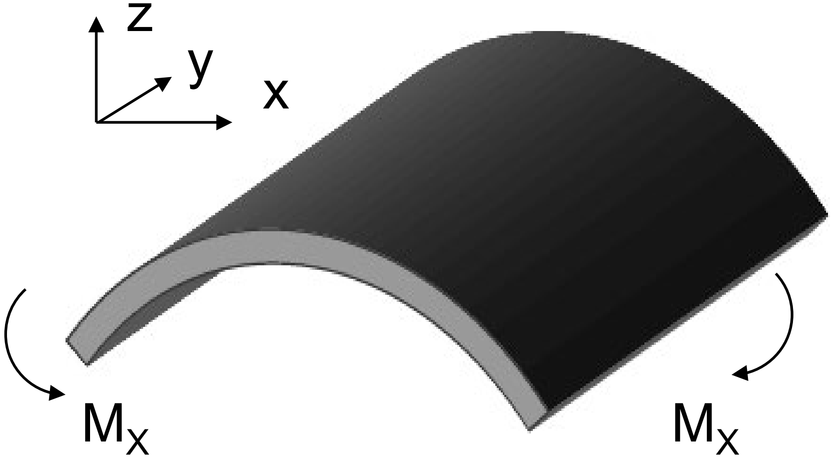


$M_X = -EJ\ \partial^2 w/\ \partial x^2$

$\partial^2 w/\ \partial y^2 \neq 0$

$M_y = 0$

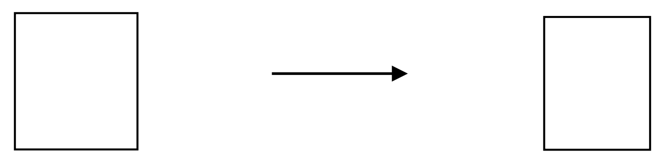
If a plate is bent over a cylindrical surface:



$\partial^2 w/\ \partial x^2 \neq 0$ $\quad$ $\partial^2 w/\ \partial y^2 = 0$ =>

Undeformed cross section

Deformed cross section



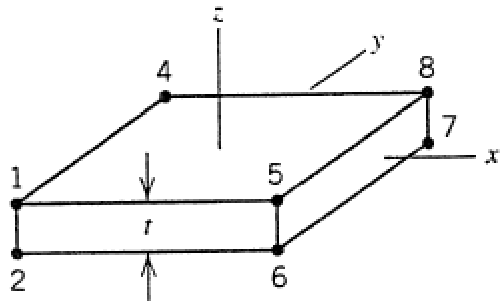
$$\begin{cases} M_x = -D\dfrac{\partial^2 w}{\partial x^2} \\ M_y = -\nu D\dfrac{\partial^2 w}{\partial x^2} \end{cases} \Rightarrow \quad M_y = \nu M_x$$

Anticlastic moment

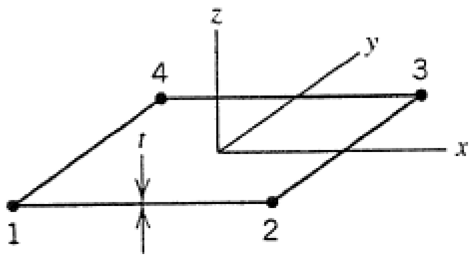
Note. For a unit width EJ =E $t^3/12$ for the beam, D = $Et^3/[12(1-\nu^2)]$ for the plate

Shell elements

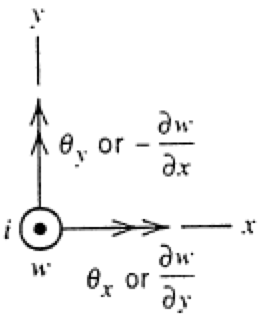
Analogous to beam elements in 2D problems



3D element

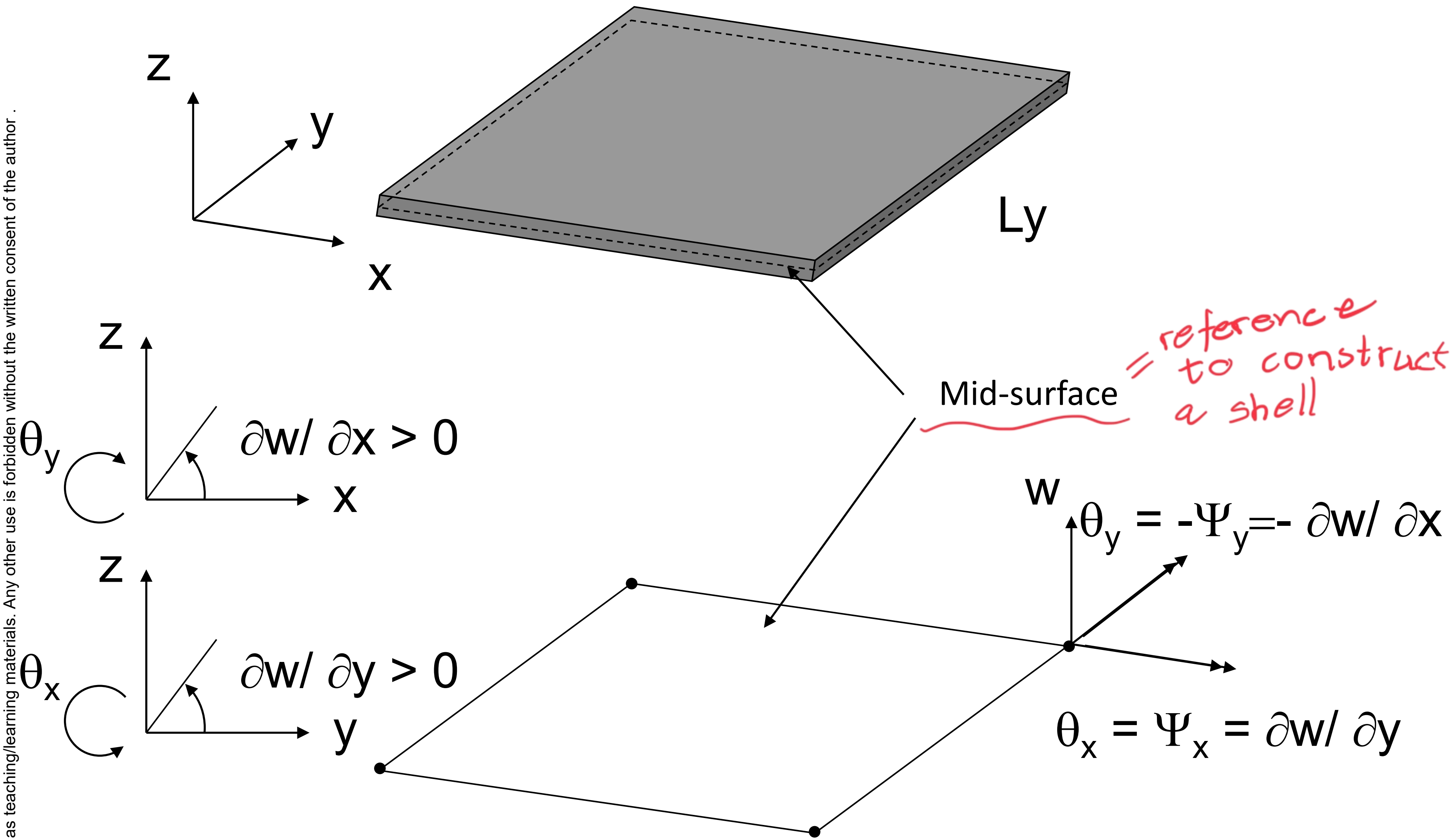


Shell element

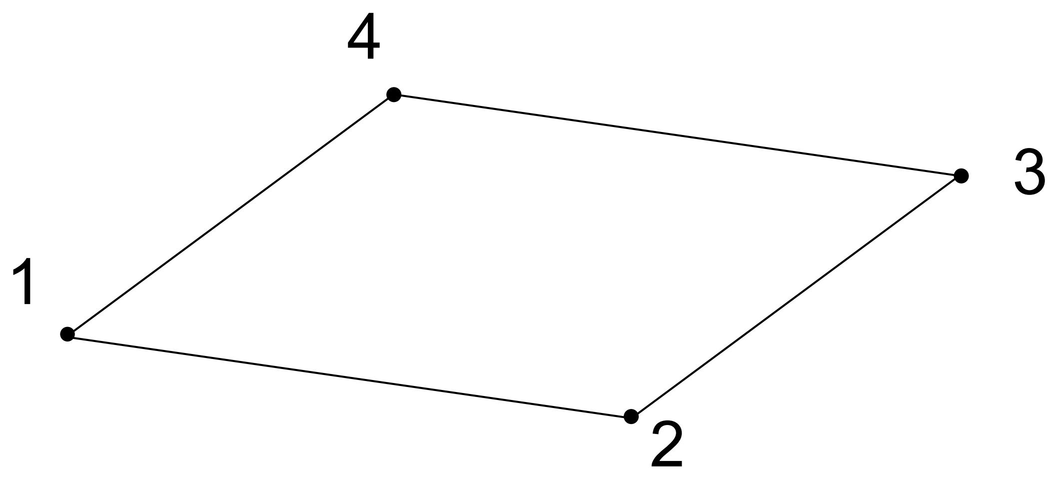


D.o.f. associated to one node of a shell elements

Notation for displacements in Kirchhoff shell elements



Shape functions of Kirchhoff elements



$$w = [N]\begin{Bmatrix} w_1 \\ \psi_{x1} \\ \psi_{y1} \\ w_2 \\ \vdots \\ \psi_{yn} \end{Bmatrix}; \quad \psi_{yi} = \frac{\partial w_i}{\partial x}, \quad \psi_{xi} = \frac{\partial w_i}{\partial y}$$

Stiffness matrix

$$[K] = \int_A [B]^T [D][B] dA$$

Integration along z has already been done implicitly

A, area of the shell element

where

$$[\mathrm{B}] = \begin{Bmatrix} \dfrac{\partial^2}{\partial x^2} \\ \dfrac{\partial^2}{\partial y^2} \\ \dfrac{\partial^2}{\partial x \partial y} \end{Bmatrix} [N]$$

e

$[D]$ è

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \end{Bmatrix} = -\begin{bmatrix} D & \nu D & 0 \\ \nu D & D & 0 \\ 0 & 0 & (1-\nu)\dfrac{D}{2} \end{bmatrix} \begin{Bmatrix} \dfrac{\partial^2 w}{\partial x^2} \\ \dfrac{\partial^2 w}{\partial y^2} \\ 2\dfrac{\partial^2 w}{\partial x \partial y} \end{Bmatrix}$$

[D]

Limitations of the Kirchhoff elements

We seek compatibility of displacements and their first derivatives between adjacent elements

4 node elements have 12 d.o.f.

An early proposal for *w(x,y)* was:

$$w = \begin{bmatrix} 1 & x & y & x^2 & xy & y^2 & x^3 & x^2 y & xy^2 & y^3 & x^3 y & xy^3 \end{bmatrix} \{\mathbf{a}\}$$

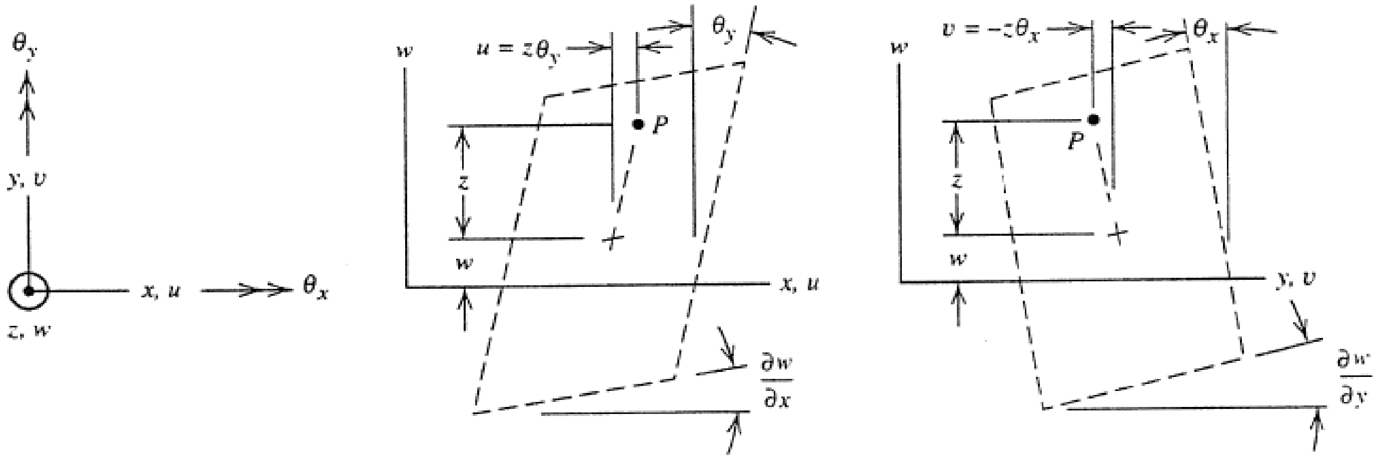
The element is incompatible in normal slope (along a shared side x = cost., the first derivative of w with respect to x is generally different for adjacent elements)

Triangular Kirchhoff elements present similar difficulties.

Instead of seeking complex formulation of pure Kirchhoff elements allowing for compatibility of all derivatives of displacement w, the Discrete Kirchhoff (DK) formulation was preferred and resulted more efficient.

They are Mindlin elements, i.e. shear deformable, where zero transverse strain is enforced at selected locations.

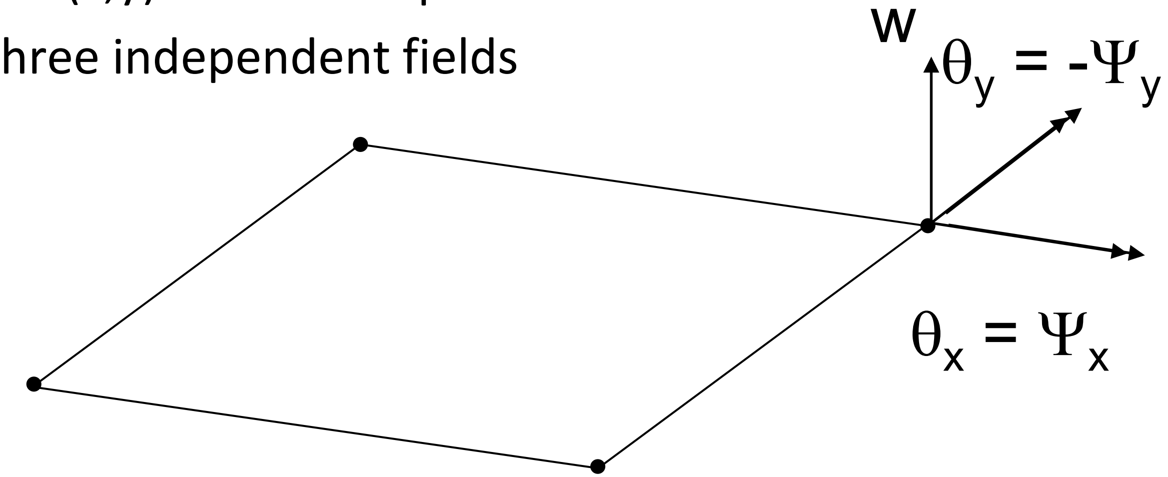
Mindlin elements



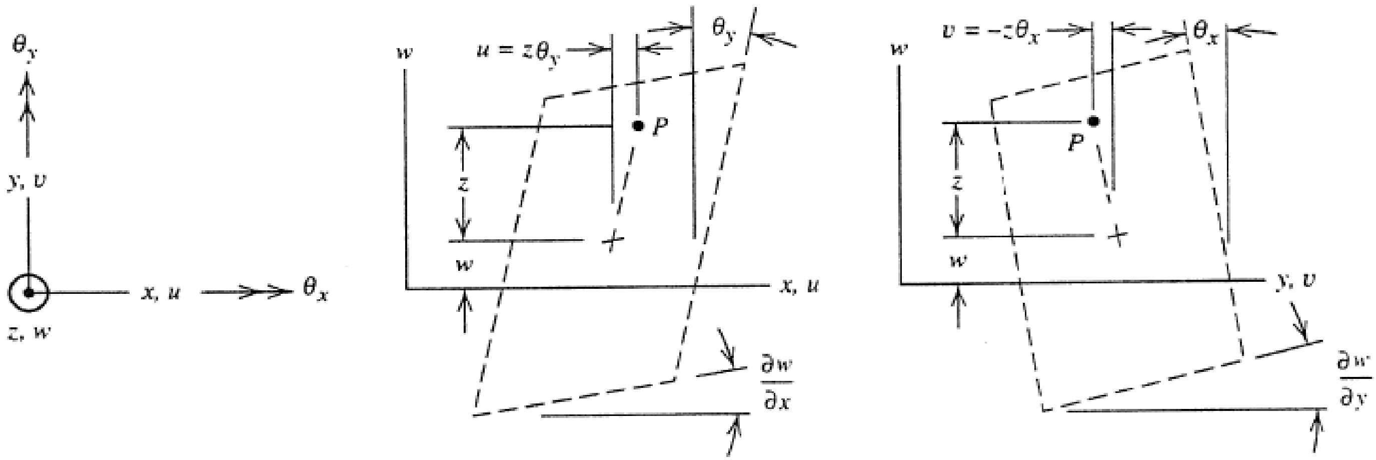
They account for shear deformation

Rotations and derivatives of w(x,y) are not coupled

w(x,y), θ_y(x,y) e θ_x(x,y) are three independent fields



Mindlin theory



$$\varepsilon_x = \frac{\partial u}{\partial x} = -z\frac{\partial \psi_y}{\partial x}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = -z\frac{\partial \psi_x}{\partial y}$$

$$u = -z\psi_y \qquad v = -z\psi_x$$

$$\Rightarrow \gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -z\left(\frac{\partial \psi_y}{\partial y} + \frac{\partial \psi_x}{\partial x}\right)$$

$$\gamma_{yz} = \frac{\partial w}{\partial y} + \frac{\partial v}{\partial z} = \frac{\partial w}{\partial y} - \psi_x$$

$$\gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = \frac{\partial w}{\partial x} - \psi_y$$

Can be non-zero

Relationship between internal forces and displacements and rotations for Mindlin elements

$$\begin{Bmatrix} M_x \\ M_y \\ M_{xy} \\ Q_x \\ Q_y \end{Bmatrix} = -\begin{bmatrix} D & \nu D & 0 & 0 & 0 \\ \nu D & D & 0 & 0 & 0 \\ 0 & 0 & (1-\nu)\frac{D}{2} & 0 & 0 \\ 0 & 0 & 0 & kGt & 0 \\ 0 & 0 & 0 & 0 & kGt \end{bmatrix} \begin{Bmatrix} \frac{\partial \psi_y}{\partial x} \\ \frac{\partial \psi_x}{\partial y} \\ \frac{\partial \psi_y}{\partial y} + \frac{\partial \psi_x}{\partial x} \\ \psi_y - \frac{\partial w}{\partial x} \\ \psi_x - \frac{\partial w}{\partial y} \end{Bmatrix}$$

$$k = \frac{5}{6}$$

Rotations and displacement *w* are coupled through the transverse shear forces

Stiffness matrix of the Mindlin elements

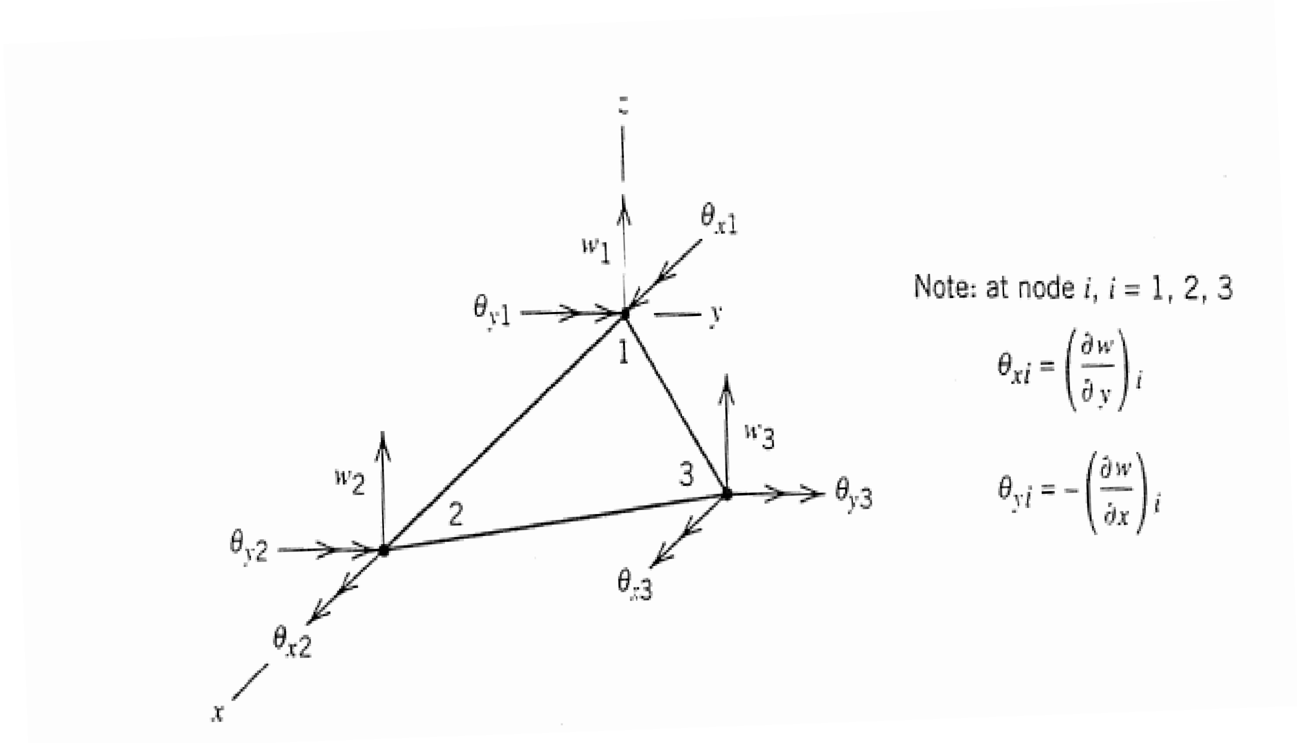
$$\begin{Bmatrix} w \\ \psi_x \\ \psi_y \end{Bmatrix} = \sum_{i=1}^{n} \begin{bmatrix} N_i & 0 & 0 \\ 0 & N_i & 0 \\ 0 & 0 & N_i \end{bmatrix} \begin{Bmatrix} w_i \\ \psi_{xi} \\ \psi_{yi} \end{Bmatrix}$$

$$\begin{Bmatrix} \dfrac{\partial \psi_x}{\partial x} \\ \dfrac{\partial \psi_y}{\partial y} \\ \dfrac{\partial \psi_x}{\partial y} + \dfrac{\partial \psi_y}{\partial x} \\ \psi_x - \dfrac{\partial w}{\partial x} \\ \psi_y - \dfrac{\partial w}{\partial y} \end{Bmatrix} = [\partial] \begin{Bmatrix} w \\ \psi_x \\ \psi_y \end{Bmatrix} = \begin{bmatrix} 0 & \partial/\partial x & 0 \\ 0 & 0 & \partial/\partial y \\ 0 & \partial/\partial y & \partial/\partial x \\ -\partial/\partial x & 1 & 0 \\ -\partial/\partial y & 0 & 1 \end{bmatrix} \begin{Bmatrix} w \\ \psi_x \\ \psi_y \end{Bmatrix} = [\partial] \sum_{i=1}^{n} \begin{bmatrix} N_i & 0 & 0 \\ 0 & N_i & 0 \\ 0 & 0 & N_i \end{bmatrix} \begin{Bmatrix} w_i \\ \psi_{xi} \\ \psi_{yi} \end{Bmatrix}$$

$$= [\mathbf{B}]\{\boldsymbol{d}\}$$

$$[k] = \int_A [\mathbf{B}]^T [D][\mathbf{B}] dA$$

DK elements

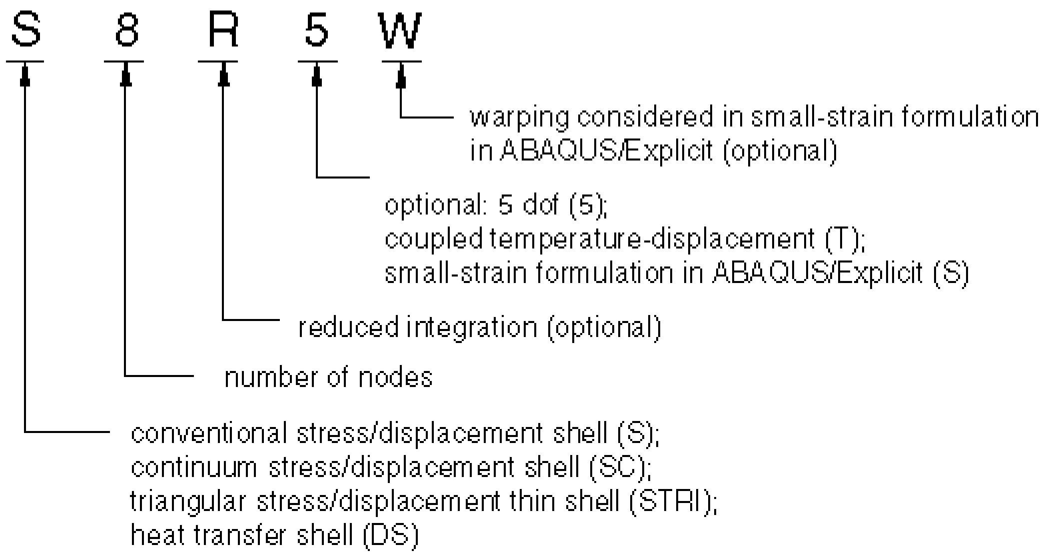


As in Mindlin elements, three independent fields are assumed (w and rotations)

Instead of imposing out of plane γ=0 over the whole element (pure Kirchhoff elements), or relating *w* to rotations through shear deformation (Mindlin), zero shear strain (γ=0) is imposed only at selected locations

Shell elements in Abaqus

Three-dimensional shell elements in Abaqus are named as follows:



Examples:

- S4R is a 4-node, quadrilateral, stress/displacement shell element with reduced integration and a large-strain formulation
- S8R5 is an 8-node, quadrilateral, second-order interpolation, stress/displacement shell element with reduced integration and 5 d.o.f. for each node.

Thin conventional shell elements

In Abaqus/Standard thin shells are needed in cases where transverse shear flexibility is negligible and the Kirchhoff constraint must be satisfied accurately (i.e., the shell normal remains orthogonal to the shell reference surface). For homogeneous shells this occurs when the thickness is less than about 1/15 of a characteristic length on the surface of the shell, such as the distance between supports or the wave length of a significant eigenmode. However, the thickness may be larger than 1/15 of the element length.

Abaqus/Standard has two types of thin shell elements: those that solve thin shell theory (the Kirchhoff constraint is satisfied analytically) and those that converge to thin shell theory as the thickness decreases (the Kirchhoff constraint is satisfied numerically).

The element that solves thin shell theory is STRI3.

The elements that impose the Kirchhoff constraint numerically are S4R5, STRI65, S8R5, S9R5. These elements should not be used for applications in which transverse shear deformation is important.

General-purpose conventional shell elements

These elements allow transverse shear deformation. They use thick shell theory as the shell thickness increases and become discrete Kirchhoff thin shell elements as the thickness decreases; the transverse shear deformation becomes very small as the shell thickness decreases.

Element types S3/S3R, S3RS, S4, S4R, S4RS.

Thick conventional shell elements

In Abaqus/Standard thick shells are needed in cases where transverse shear flexibility is important and second-order interpolation is desired. When a shell is made of the same material throughout its thickness, this occurs when the thickness is more than about 1/15 of a characteristic length on the surface of the shell, such as the distance between supports for a static case or the wavelength of a significant natural mode in dynamic analysis.

Abaqus/Standard provides element types S8R

Shell Element Outputs

- **Section Forces**
  - SF1, SF2, SF3 (in-plane shear), SF4 (out of plane shear), SF5 (out of plane shear)
- **Section Moments**
  - SM1, SM2, SM3 (torsion)
- **Stress/Strain**
  - S11, S22, S33, S12
  - TSHR31, TSHR32
- **…**

Shell element output variables (stress/strain components, as well as the section forces/strains) are expressed by default in the local coordinate system that can be a default reference system or a user defined one

Local directions on surface in space

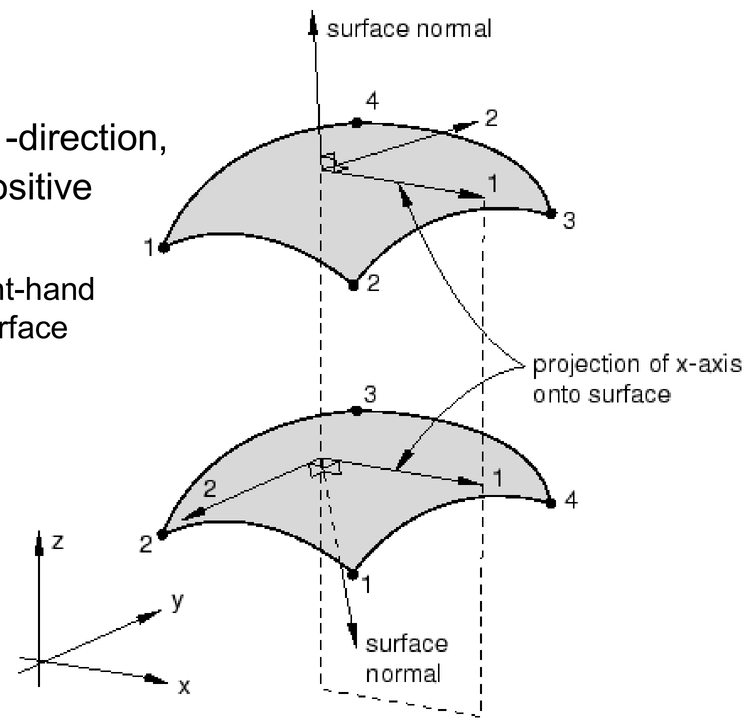
The convention used in Abaqus for such directions is as follows.

- The default local 1-direction is the projection of the global x-axis onto the surface.

  (If the global x-axis is within 0.1° of being normal to the surface, the local 1-direction is the projection of the global z-axis onto the surface.)

- The local 2-direction is then at right angles to the local 1-direction, so that the local 1-direction, local 2-direction, and the positive normal to the surface form a right-handed set.

  (The positive normal direction is defined in an element by the right-hand rotation rule going around the nodes of the element. The local surface directions can be redefined.)



Shell Element Outputs in Abaqus

Section Forces

- **SF1** Direct membrane force per unit width in local 1-direction.
- **SF2** Direct membrane force per unit width in local 2-direction.
- **SF3** Shear membrane force per unit width in local 1–2 plane.
- **SF4** Transverse shear force per unit width in local 1-direction
  (available only for S3/S3R, S3RS, S4, S4R, S4RS, S4RSW, S8R, and S8RT).
- **SF5** Transverse shear force per unit width in local 2-direction
  (available only for S3/S3R, S3RS, S4, S4R, S4RS, S4RSW, S8R, and S8RT).
- **SF6** Normal force
  (reported only for finite-strain shell elements and zero because of the plane stress constitutive assumption).
- The section force per unit length in the normal basis directions in a given shell section of thickness h can be defined on this basis as

$$(SF1, SF2, SF3, SF4, SF5) = \int_{-h/2-z_0}^{h/2-z_0} (\sigma_{11}, \sigma_{22}, \sigma_{12}, \sigma_{13}, \sigma_{23})\, dz,$$

where z_0 is the offset of the reference surface from the midsurface.

Shell Element Outputs in Abaqus

Section Moments

- **SM1** Bending moment force per unit width about local 2-axis.
- **SM2** Bending moment force per unit width about local 1-axis.
- **SM3** Twisting moment force per unit width in local 1–2 plane.

- The section moment per unit length in the normal basis directions in a given shell section of thickness h can be defined on this basis as

$$(SM1, SM2, SM3) = \int_{-h/2-z_0}^{h/2-z_0} (\sigma_{11}, \sigma_{22}, \sigma_{12}) z \, dz,$$

where z_0 is the offset of the reference surface from the midsurface.