POLITECNICO DI MILANO

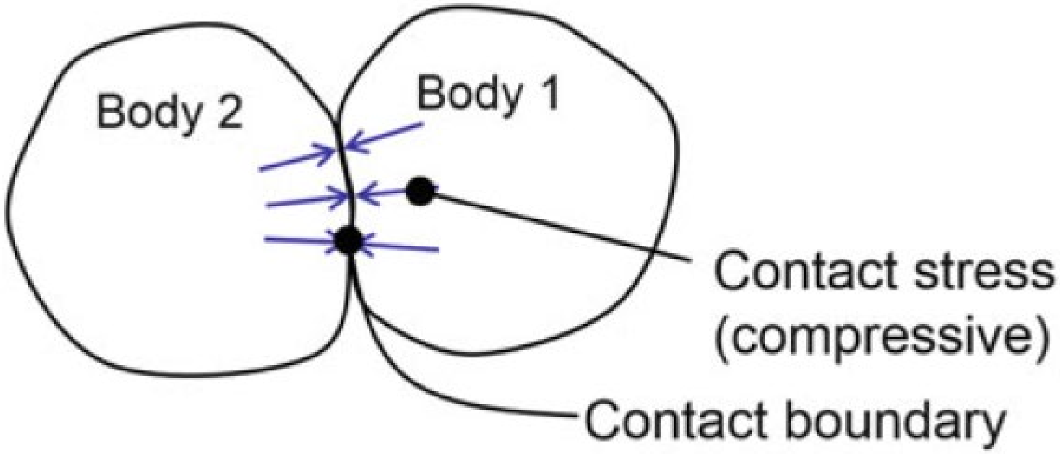
Finite Element Simulation For Mechanical Design

Introduction to contact analyses

Prof. Andrea Bernasconi

Introducion to the contact problem

It is a non-linear problem

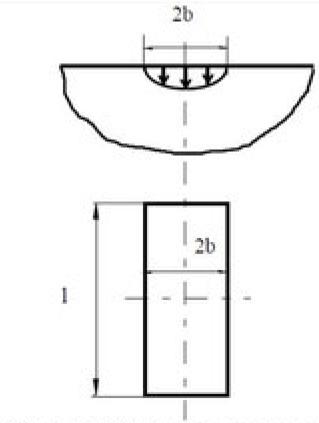
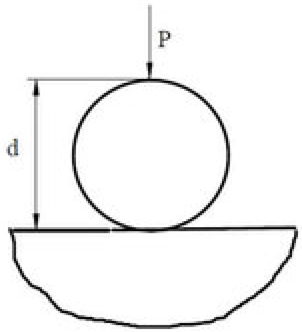
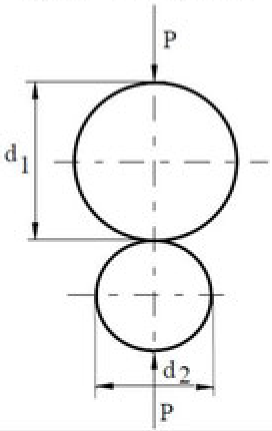


Mating surfaces experience normal and tangential tractions

The contact area grows gradually as the force pressing the bodies one against the other increases

Stresses develop below the contact

Analytical solutions exist for selected problems (Hertz)

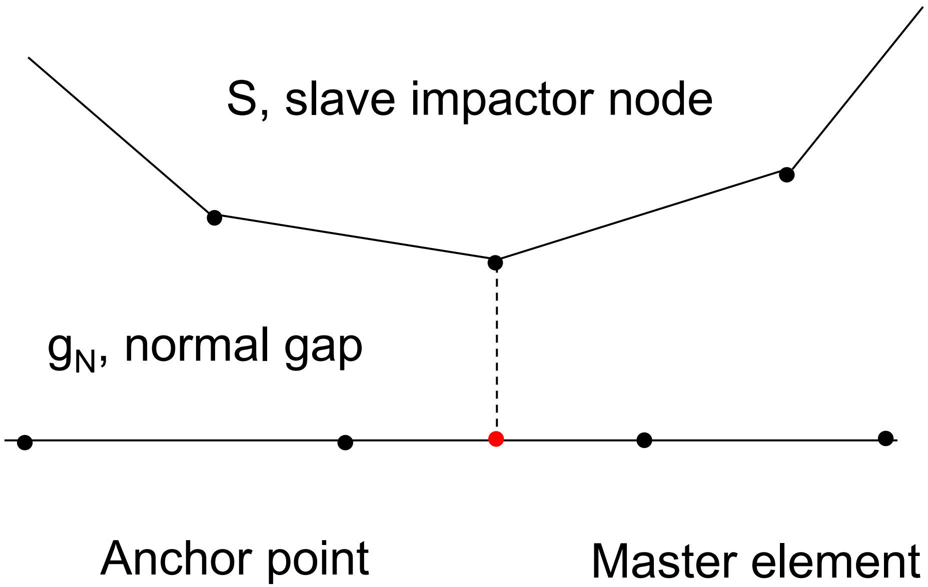
 Rectangular contact area $2b\cdot l$	Cylinder on surface 	Cylinder on cylinder 
Contact width	$b=\sqrt{\frac{2}{\pi}}\sqrt{\frac{P}{L}\,d\,C}$	$b=\sqrt{\frac{2}{\pi}}\sqrt{\frac{P}{L}\frac{C}{\frac{1}{d_1}+\frac{1}{d_2}}}$
Maximum pressure	$p_{max}=\frac{4}{\pi}\cdot p_{media}=\frac{4}{\pi}\frac{P}{2bL}$	
position of the maximum Guest stress	$\zeta_0=\frac{z}{b}=0.489$	
Maximum Guest stress	$\sigma_{max,Guest}=0.801\cdot p_{mean}=0.801\frac{P}{2bL}$	

FE formulation of the contact problem

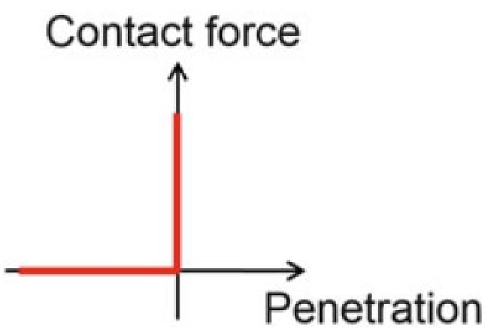
To enforce contact, a condition of no penetration between surfaces must be imposed

Master and slave surfaces must be defined

Control points must be identified, to define the normal gap



Hard frictionless contact



Equilibrium: $\delta\Pi = 0$ + additional constraints of the type $g_N \geq 0$

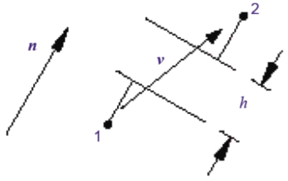
Summary

- **Contact Formulations**
  - Contact Discretization
  - Contact Enforcement Methods
  - Relative Sliding between Bodies
  - Output of Contact Results
  - Finite Element types for Contact Analysis
- **Benchmark (3D Cube on Cube)**
  - Taylor’s contact patch test
- **Case study (2D Disc on Block)**
  - Statement of the problem
  - Analytical solutions
  - Pre/Post-processing

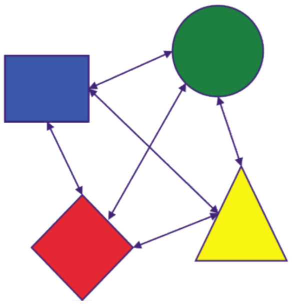
Contact Modeling

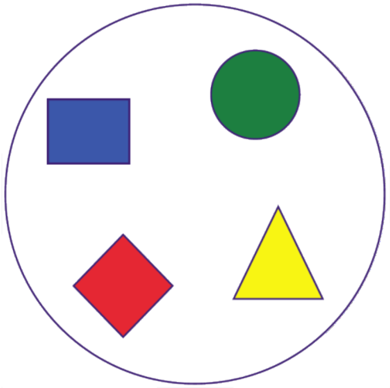
Types of Contact Modeling

- *Contact Elements*
  - User-defined elements for each contact constraint
- *Contact pairs*
  - Many pairings for assemblies
- *General Contact*
  - Model all interactions between free surfaces



$$h = d + \boldsymbol{n}\cdot(\boldsymbol{u}^2 - \boldsymbol{u}^1) \geq 0$$

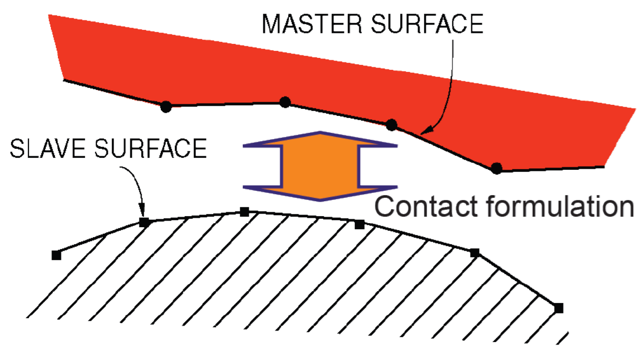




Contact Formulations

Basic features of a contact formulation:

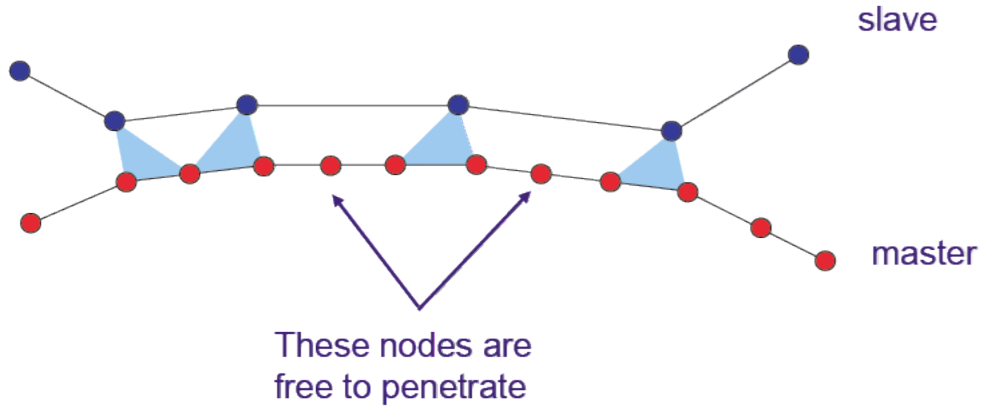
- **Contact discretization**
  - *Where is the constraint applied?*
    - Node-to-surface
    - Surface-to-surface
- **Contact enforcement**
  - *How is the constraint enforced?*
    - Direct (Lagrange Multipliers)
    - Penalty Method
    - Augmented Lagrange
- **Contact tracking (relative sliding)**
  - *How does the constraint evolve?*
    - Finite sliding
    - Small sliding

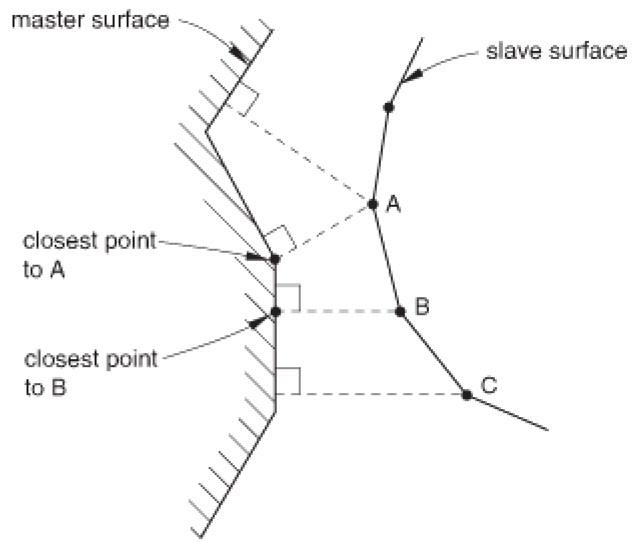


Contact discretization techniques

Contact discretization techniques:

- **Node-to-Surface**
  - Nodes on one surface (the *slave surface*) contact the segments on the other surface (the master surface)
  - Contact enforced at discrete points (the *slave nodes*)





The location of the contact constraints depend on the contact discretization technique used

Guidelines for choosing master/slave

When a contact pair contains two surfaces, the two surfaces are not allowed to include any of the same nodes and you must choose which surface will be the slave and which will be the master.

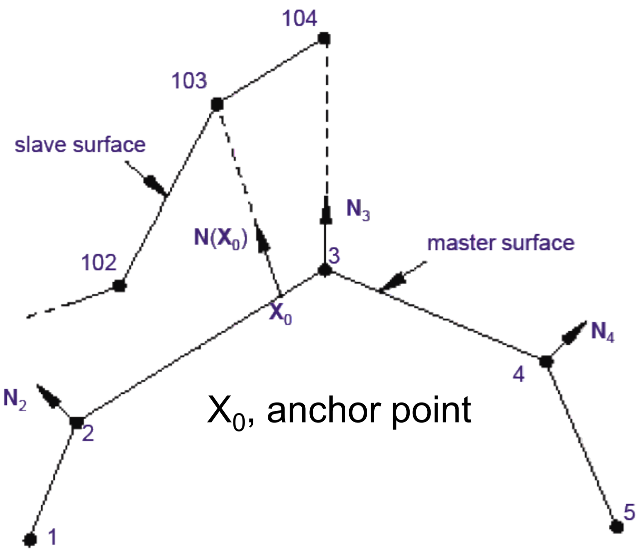
For simple contact pairs consisting of two deformable surfaces, the following basic guidelines can be used:

- The larger of the two surfaces should act as the master surface (not to lose the contact).
- If the surfaces are of comparable size, the surface on the stiffer body should act as the master surface (to limit penetration).
- If the surfaces are of comparable size and stiffness, the surface with the coarser mesh should act as the master surface (otherwise some nodes of the master may penetrate into the slave).

Contact discretization techniques

Contact discretization techniques:

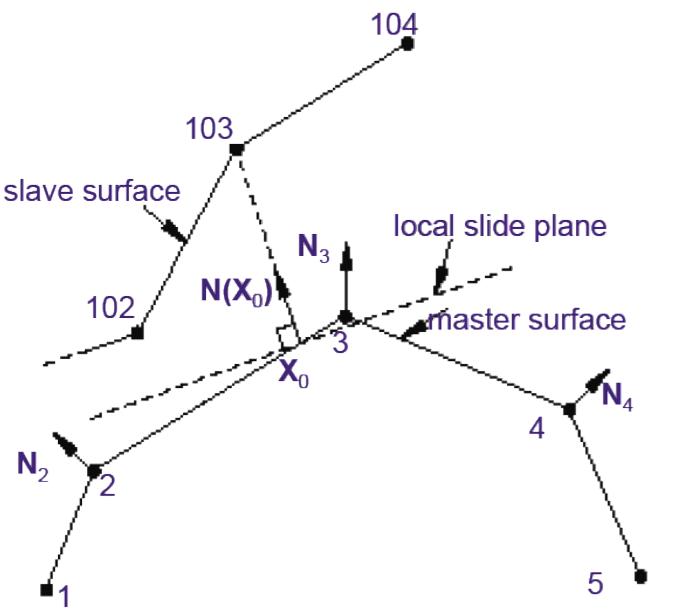
- **Node-to-Surface**
  - Normal and tangent unit vectors are defined using a surface element on the master surface
  - Contact enforced at discrete points (the *slave nodes*)



X_0, anchor point

N(x) = smoothly varying "normal" vector on the master surface.

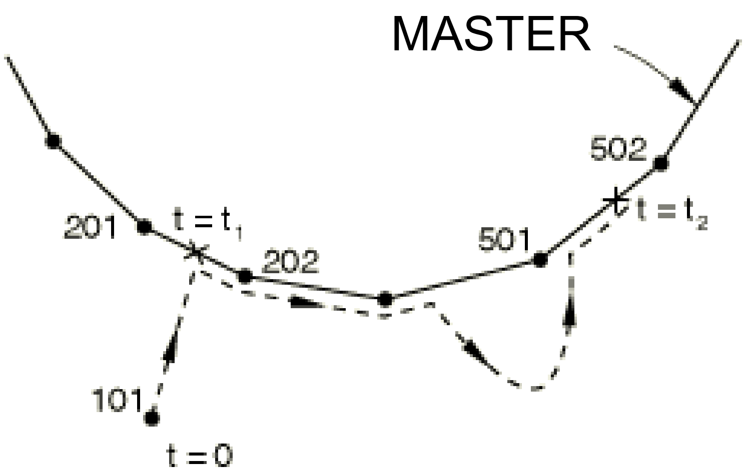
Master surface normal vectors



Anchor point and slide plane for small-sliding, node-to-surface contact

Finite-sliding

The algorithm tracks the position of node 101 relative to the master surface as the bodies deform. Figure shows the possible evolution of the contact between node 101 and its master surface.



3D – Cube on cube

Cube 10x10 mm

Pressure 10 MPa

Prevent rigid motions!

cube-on-cube.cae

cube-on-cube.jnl

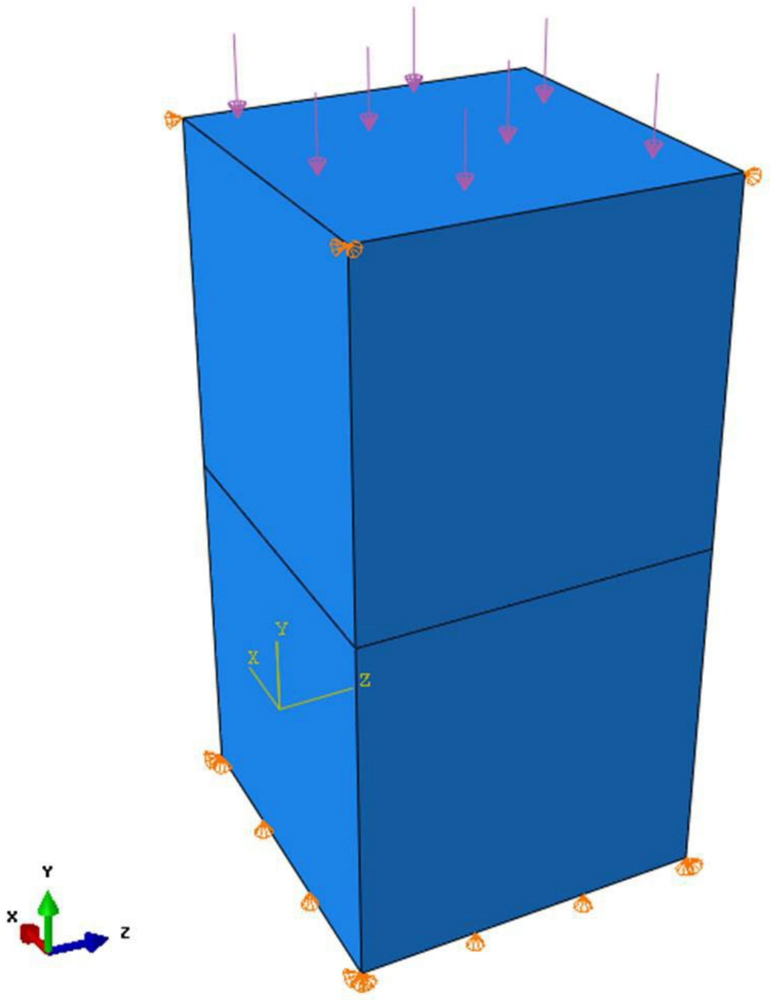
C3D8R_LM_N-to-S_SS

C3D8R_LM_S-to-S_SS

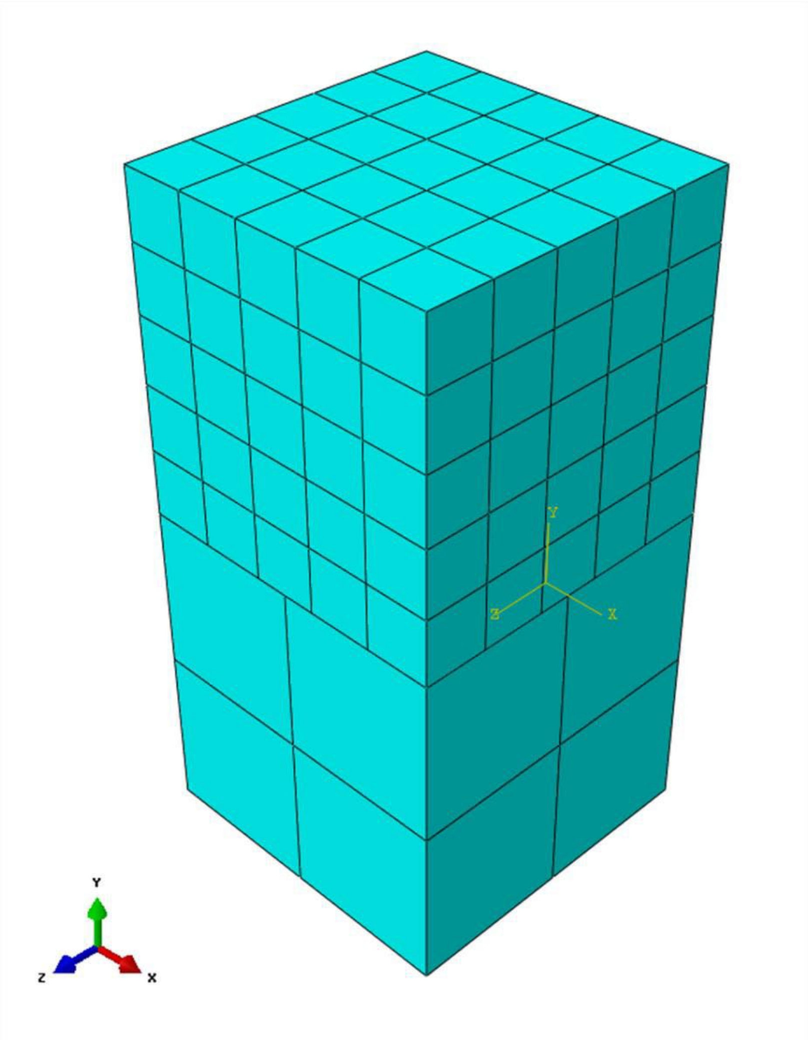
C3D8R_PM_N-to-S_SS

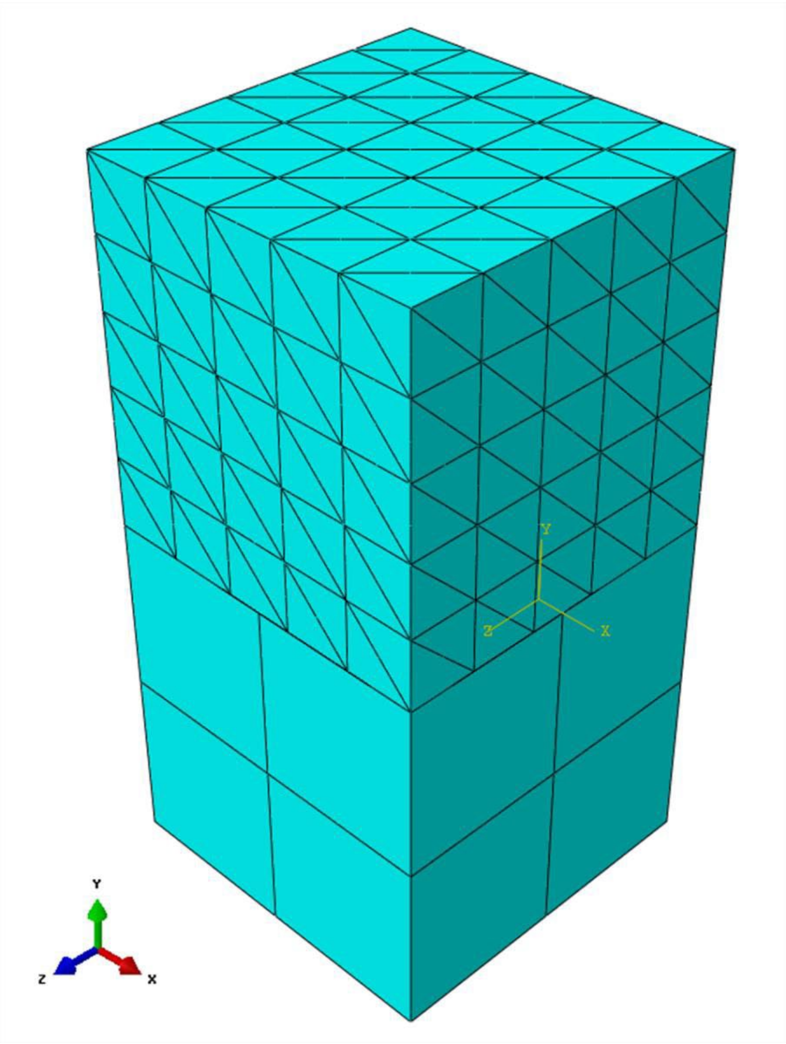
C3D10_PM_N-to-S_SS

C3D10_PM_S-to-S_SS



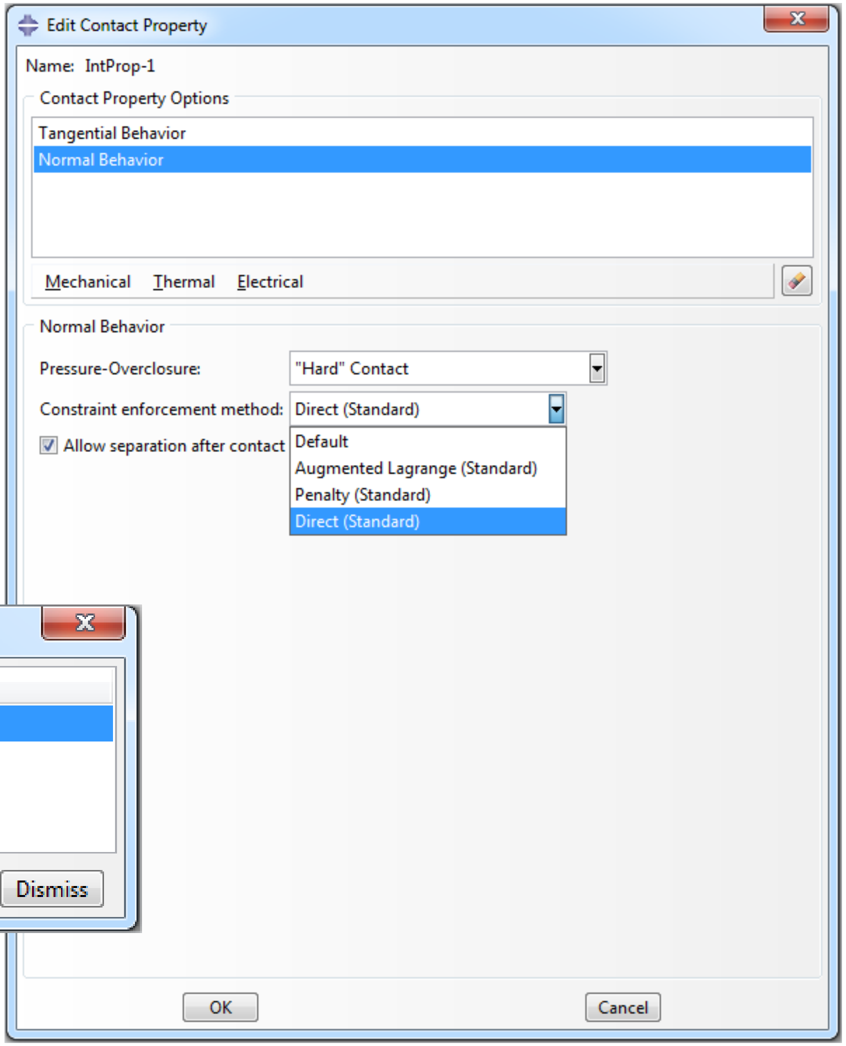
3D – Cube on cube

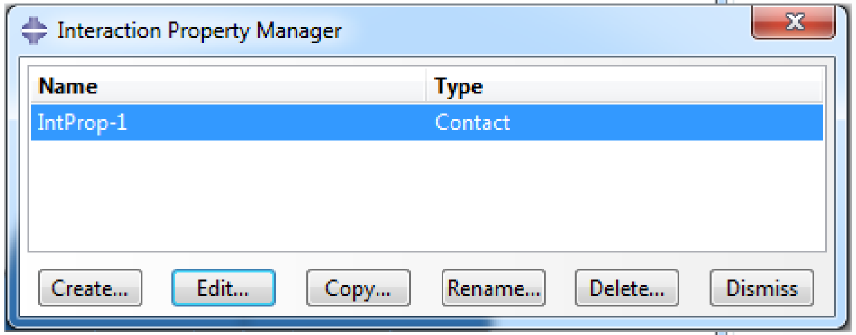




Contact property

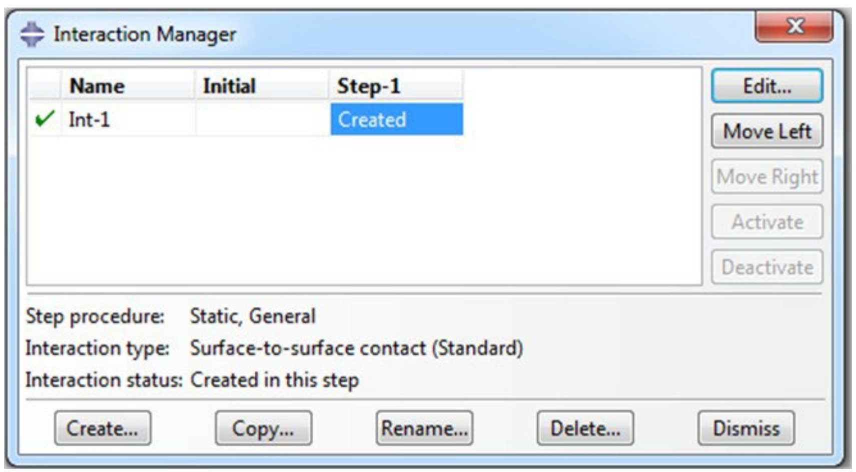
- Pressure-Overclousure
- Constraint enforcement method

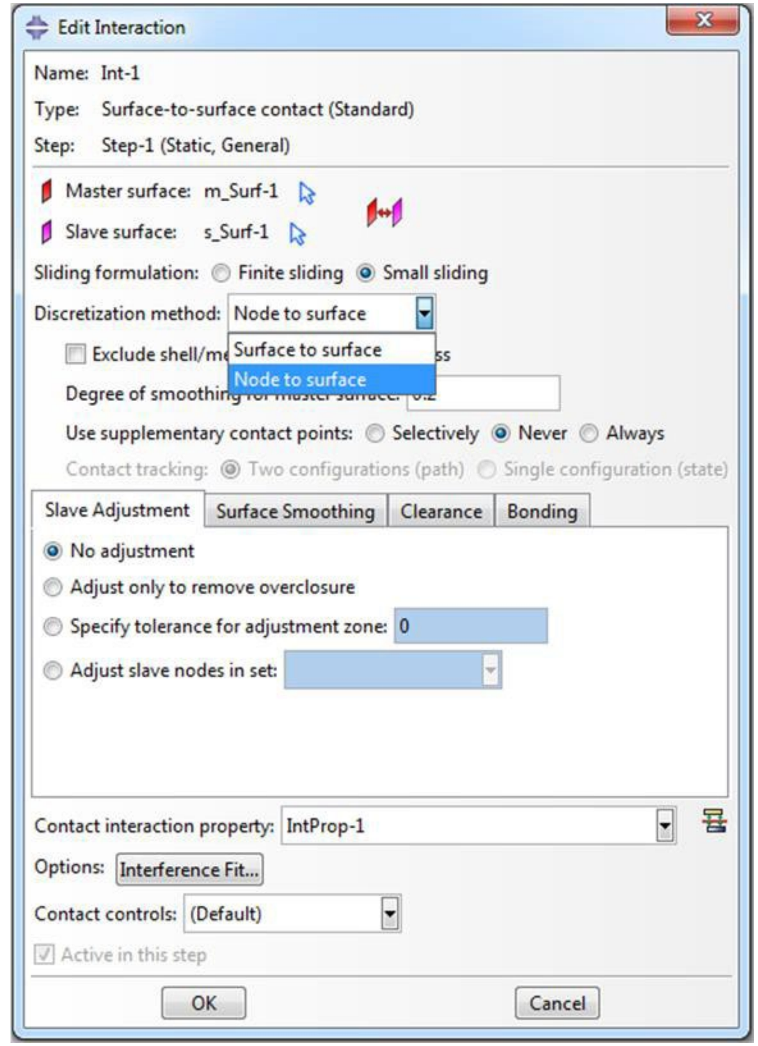




Interaction

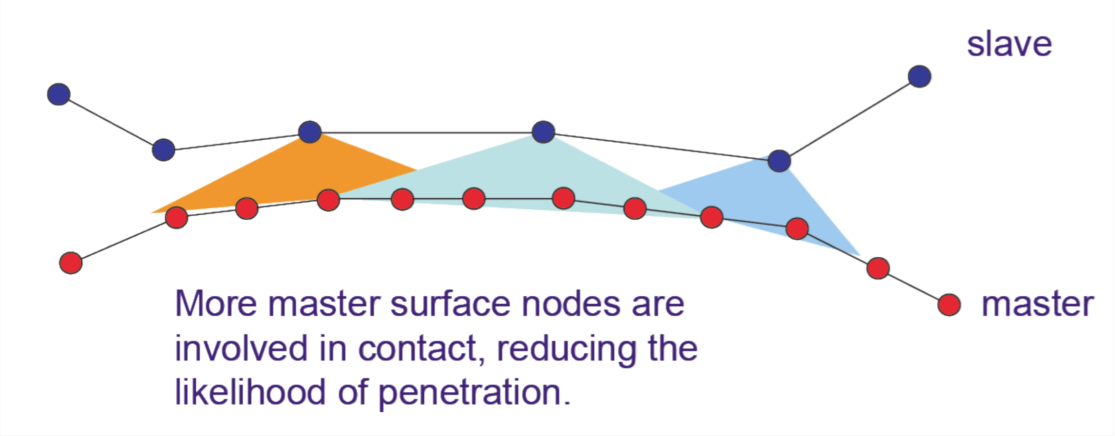
- Contact pair definition
- Contact discretization techniques
- Contact tracking

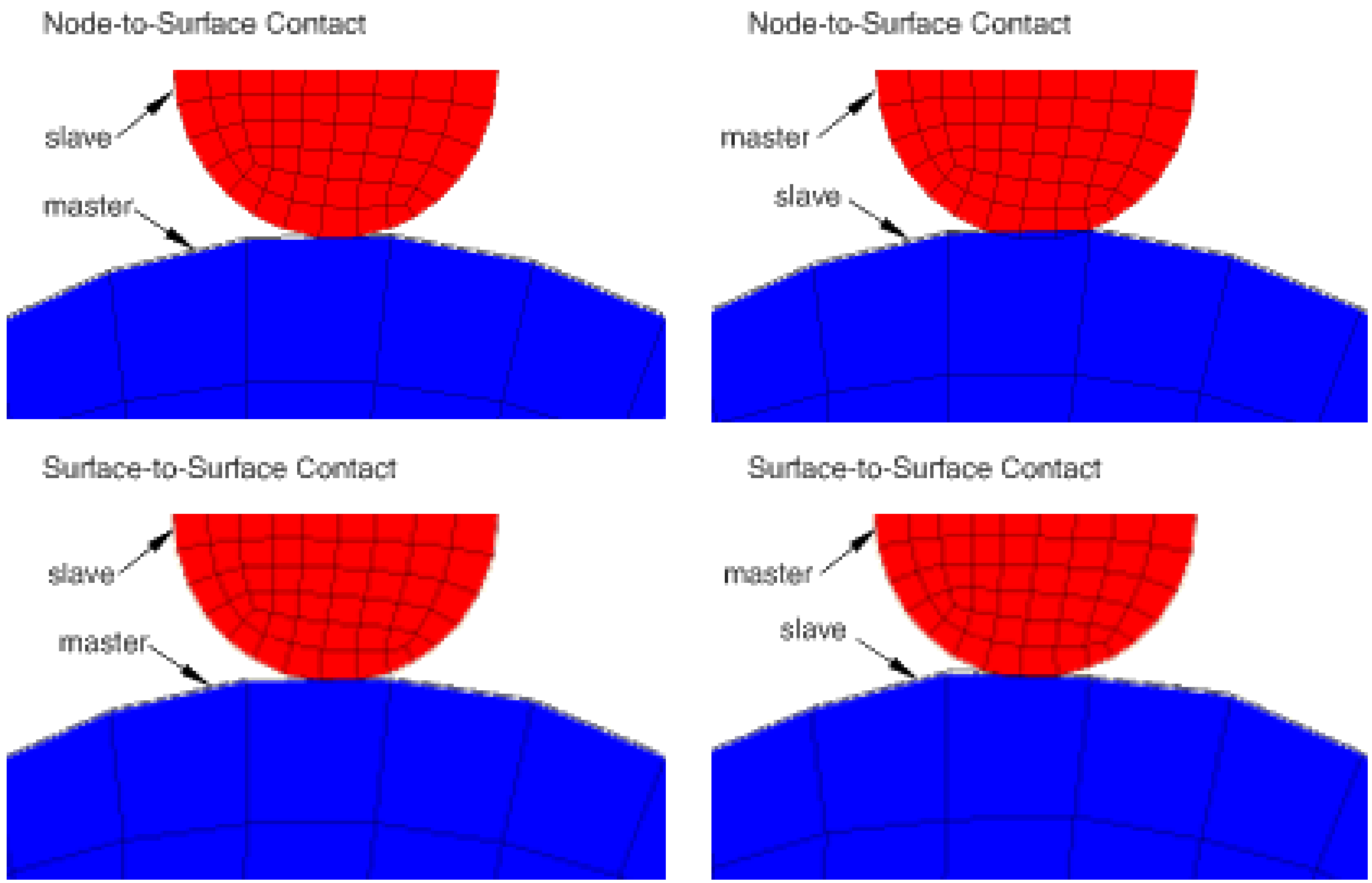




Contact discretization techniques:

- **Surface-to-Surface**
  - Contact enforced in an average sense over a region surrounding each slave node
  - Slave surface much more than just a collection of nodes




Node-to-Surface Contact
slave
master
Node-to-Surface Contact
master
slave
Surface-to-Surface Contact
slave
master
Surface-to-Surface Contact
master
slave

Contact discretization techniques

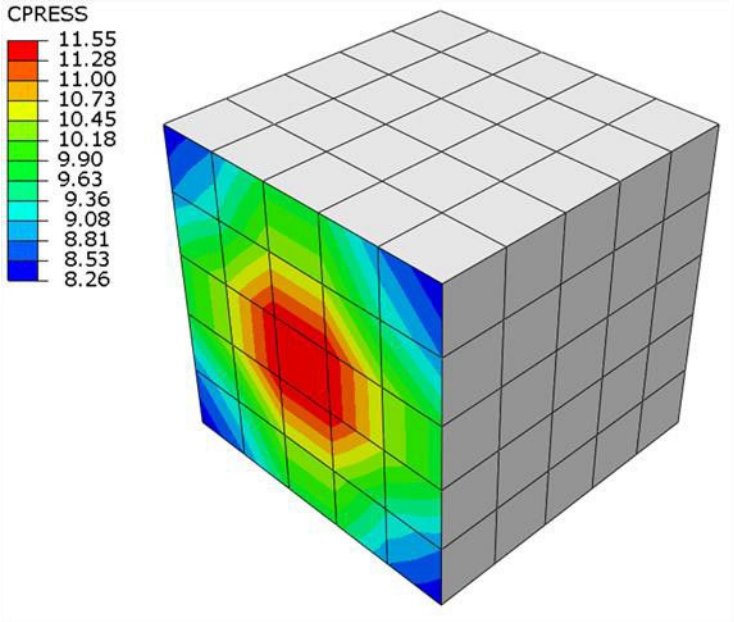
Node-to-Surface

Element type:

C3D8R

Contact Enforcement:

Lagrange Multiplier



Surface-to-Surface

Element type:

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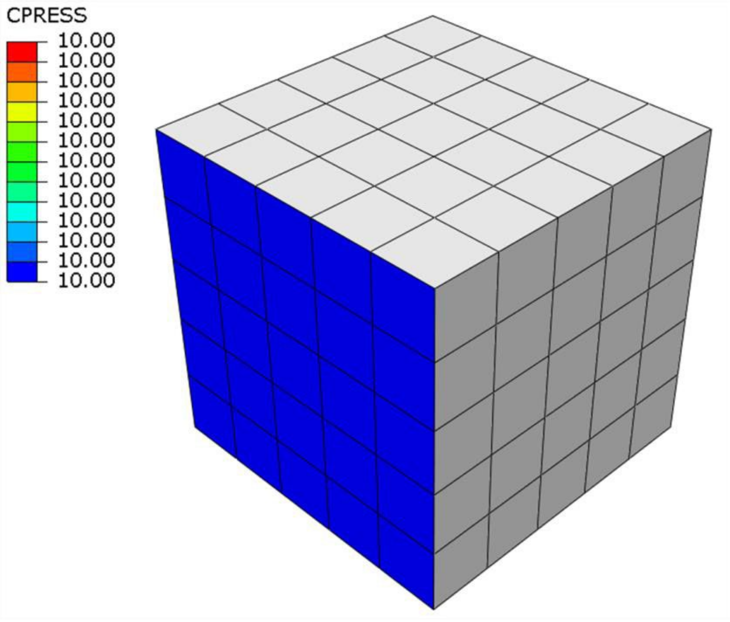
Contact Enforcement:

Lagrange Multiplier



Contact enforcement methods

Contact enforcement methods:

- **Direct enforcement method**
  - Strict enforcement of pressure-penetration relationship using the Lagrange multiplier method
- **Penalty method**
  - Approximate enforcement using penalty stiffness
- **Augmented Lagrange Method**
  - Approximate enforcement using penalty method with augmentation iterations

Standard Optimization Model

Find an n-vector x $(x_1, x_2, \ldots, x_n)$ of variables to

Minimize a cost function:

$$f(x) = f(x_1, x_2, \ldots, x_n)$$

subject to the *p* equality constraints:

$$h_j(x) = h_j(x_1, x_2, \ldots, x_n) = 0; \qquad j=1 \text{ to } p$$

and the m inequality constraints:

$$g_i(x) = g_i(x_1, x_2, \ldots, x_n) \le 0; \qquad i=1 \text{ to } m$$

Note that the simple bounds on design variables, such as $x_i \le 0$, or $x_{iL} \le x_i \le x_{iU}$, where x_{iL} and x_{iU} are the smallest and largest allowed values for x_i, are assumed to be included in the inequalities

Lagrange Multiplier Theorem

Consider the optimization problem defined as:

Minimize f(**x**)

subject to equality constraints

$h_i(\mathbf{x}) = 0$; i=1 to p

Let **x*** be a local minimum for the problem.

Then there exist unique Lagrange multipliers v_j^*, j=1 to p, such that

$$\frac{\partial f(\mathbf{x}^*)}{\partial x_i} + \sum_{j=1}^{p} v_j^* \frac{\partial h_j(\mathbf{x}^*)}{\partial x_i} = 0; \quad i = 1 \text{ to } n$$

$$h_j(\mathbf{x}^*) = 0; \quad j = 1 \text{ to } p$$

Lagrange Multiplier Theorem

It is convenient to write these conditions in terms of the Lagrange function,

$$L(\mathbf{x}, \mathbf{v}) = f(\mathbf{x}) + \sum_{j=1}^{p} v_j h_j(\mathbf{x})$$
$$= f(\mathbf{x}) + \mathbf{v}^{\mathrm{T}} \mathbf{h}(\mathbf{x})$$

Then, equations

$$\frac{\partial f(\mathbf{x}^*)}{\partial x_i} + \sum_{j=1}^{p} v_j^* \frac{\partial h_j(\mathbf{x}^*)}{\partial x_i} = 0; \quad i = 1 \text{ to } n$$

$$h_j(\mathbf{x}^*) = 0; \quad j = 1 \text{ to } p$$

become

$$\nabla L(\mathbf{x}^*, \mathbf{v}^*) = \mathbf{0},$$

$$\frac{\partial L(\mathbf{x}^*, \mathbf{v}^*)}{\partial x_i} = 0; \qquad i = 1 \text{ to } n$$

$$\frac{\partial L(\mathbf{x}^*, \mathbf{v}^*)}{\partial v_j} = 0 \Rightarrow h_j(\mathbf{x}^*) = 0; \quad j = 1 \text{ to } p$$

Differentiating by v_j, we recover the equality constraints

i.e. the Lagrange function is **stationary** with respect to both **x** and **v**.

Comments on the Lagrange Multiplier Theorem

- The Lagrange function may be **treated as an unconstrained function** in the variables **x** and **v** to determine the stationary points.
- Any point that does not satisfy the conditions of the theorem cannot be a local minimum point.
- A point satisfying the conditions need not be a minimum point either. It is simply a **candidate minimum point**, which can actually be an inflection or maximum point.
- The *n* variables **x** and the *p* multipliers **v** are the unknowns, and the necessary conditions provide enough equations to solve for them.
- the Lagrange multipliers v_i are **free in sign**; that is, they can be positive, negative, or zero (this is in contrast to the Lagrange multipliers for the inequality constraints, which are required to be non-negative, see KKT theorem later).

Necessary conditions for a general constrained problem

The Lagrange multipliers theorem can be extended to include inequality constraints

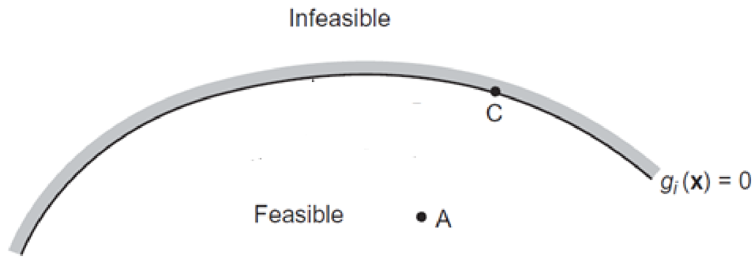
KKT - Karush-Kuhn-Tucker necessary conditions

$g_i(\mathbf{x}) \le 0$ is equivalent to the equality constraint $g_i(\mathbf{x})+s_i=0$, where $s_i \ge 0$ is a **slack variable**.

s_i are treated as unknowns along with the original variables.

When $s_i = 0$ (point C):

- the corresponding inequality constraint is satisfied at equality
- the constraint is active (tight) constraint
- there is no “slack” in the constraint.



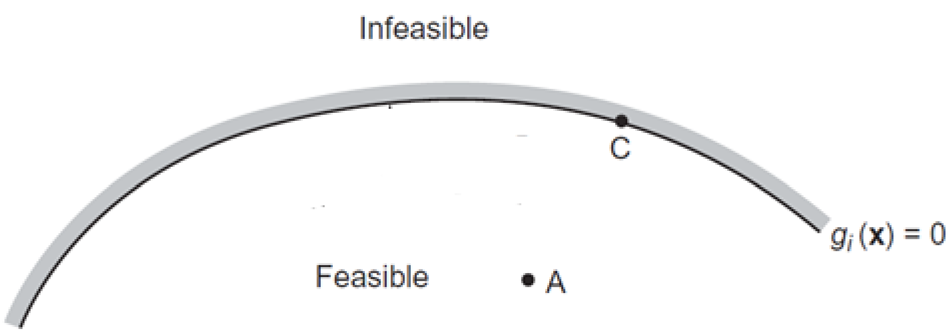
Necessary conditions for a general constrained problem

For any $s_i > 0$ (point A):

- the corresponding constraint is a strict inequality.
- the constraint is **inactive** and has slack given by s_i.

The status of an inequality constraint is determined as a part of the solution to the problem.

The condition slack ≥ 0 can be satisfied implicitly introducing $g_i(\mathbf{x})+s_i^2=0$



Active and inactive constraints: values of u_j

It can be demonstrated that an additional necessary condition for the Lagrange multipliers of "≤ type" constraints exists:

$$u_j \geq 0; j = 1 \text{ to } m$$

where u_j^* is the Lagrange multiplier for the jth inequality constraint.

Thus, the Lagrange multiplier for each "≤" inequality constraint must be non-negative:

- If the constraint is inactive at the optimum, its associated Lagrange multiplier is zero.
- If it is active (g_i=0), then the associated multiplier must be non-negative.

Let **x*** be a regular point of the feasible set that is a local minimum for f(**x**), subject to $h_i(\mathbf{x})=0$; i=1 to p; $g_j(\mathbf{x}) \leq 0$; j=1 to m.

Then there exist Lagrange multipliers **v*** (a p-vector) and **u*** (an m-vector) such that the Lagrange function is stationary with respect to x_j, v_i, u_j, and s_j at the point x*.

1. Lagrangian Function for the Problem Written in the Standard Form:

$$
\begin{aligned}
L(\mathbf{x}, \mathbf{v}, \mathbf{u}, \mathbf{s}) &= f(\mathbf{x}) + \sum_{i=1}^{p} v_i h_i(\mathbf{x}) \\
&+ \sum_{j=1}^{m} u_j(g_j(\mathbf{x}) + s_j^2) \\
&= f(\mathbf{x}) + \mathbf{v}^{\mathrm{T}}\mathbf{h}(\mathbf{x}) + \mathbf{u}^{\mathrm{T}}(\mathbf{g}(\mathbf{x}) + \mathbf{s}^2)
\end{aligned}
$$

Karush-Kuhn-Tucker Optimality Conditions

2. Gradient Conditions

$$\frac{\partial L}{\partial x_k} = \frac{\partial f}{\partial x_k} + \sum_{i=1}^{p} v_i^* \frac{\partial h_i}{\partial x_k} + \sum_{j=1}^{m} u_j^* \frac{\partial g_j}{\partial x_k} = 0; \quad k = 1 \text{ to } n$$

$$\frac{\partial L}{\partial v_i} = 0 \Rightarrow h_i(\mathbf{x}^*) = 0; \quad i = 1 \text{ to } p$$

$$\frac{\partial L}{\partial u_j} = 0 \Rightarrow (g_j(\mathbf{x}^*) + s_j^2) = 0; \quad j = 1 \text{ to } m$$

3. Feasibility Check for Inequalities

$$s_j^2 \geq 0; \text{ or equivalently } g_j \leq 0; \quad j = 1 \text{ to } m$$

Karush-Kuhn-Tucker Optimality Conditions

4. Switching Conditions

$$\frac{\partial L}{\partial s_j} = 0 \Rightarrow 2u_j^* s_j = 0; \quad j = 1 \text{ to } m$$

5. Non-negativity of Lagrange Multipliers for Inequalities:

$$u_j^* \geq 0; \quad j = 1 \text{ to } m$$

6. Regularity Check: Gradients of the active constraints must be linearly independent. In such a case, the Lagrange multipliers for the constraints are unique.

Similarity with contact problems

Lagrange multiplier = reaction force

Slack = gap between possible contact points

Lagrange Multipliers method (frictionless contact)

$$\mathcal{L} = \Pi + \sum \lambda_i g_{Ni}$$

Applying KKT conditions,

$$\delta\mathcal{L} = \delta\Pi + \sum \lambda_i \delta g_{Ni} + \sum \delta\lambda_i g_{Ni}$$

must vanish for any feasible $\delta\boldsymbol{D}$ (first order conditions):

$$\frac{\partial\Pi}{\partial\boldsymbol{D}}\delta\boldsymbol{D} + \sum \lambda_i \delta g_{Ni}(\delta\boldsymbol{D}) = 0$$

Non negative gap

$$g_{Ni} \geq 0$$

Multiplier = contact force

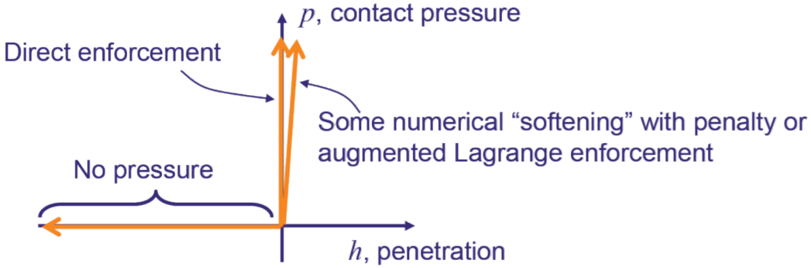
$$\lambda_i \geq 0$$

Switching condition

$$g_{Ni}\lambda_i = 0$$

The solution is found iteratively. The multipliers add to the set of variables

Penalty and Augmented Lagrangian methods



Penalty method

$$\Phi = \Pi + \sum_a \frac{1}{2} v^T \boldsymbol{C} v$$

v, constraint violations

a, active contact elements

A penalty term is added to the function to be made stationary

The penalty term is always greater than or equal zero

It is different from zero if the constraint is violated

C represents the "stiffness" of the boundary (penalty = contact strain energy)

Augmented Lagrangian method

$$\Phi = \Pi + \sum \lambda_i g_{Ni} + \sum_a \frac{1}{2} v^T \boldsymbol{C} v$$

Contact enforcement methods

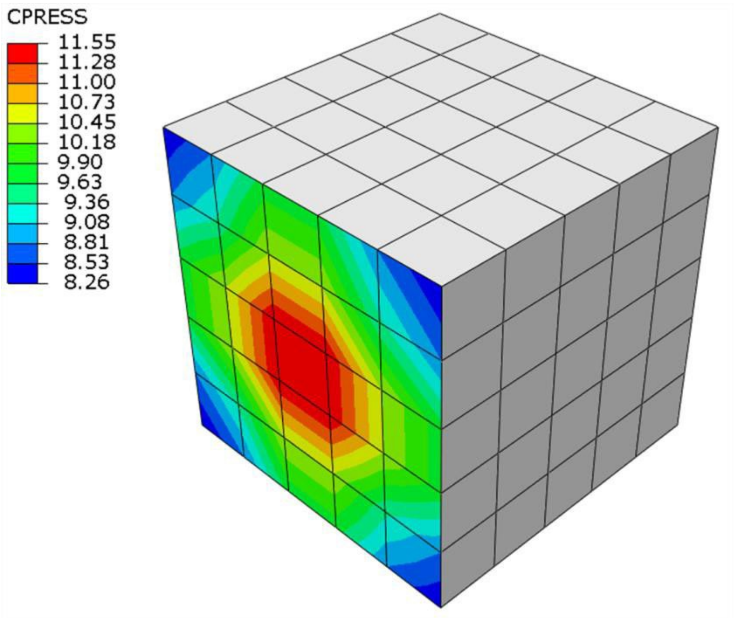
Lagrange Multiplier Method

Element type:

C3D8R

Contact Discretization:

Node-to-Surface



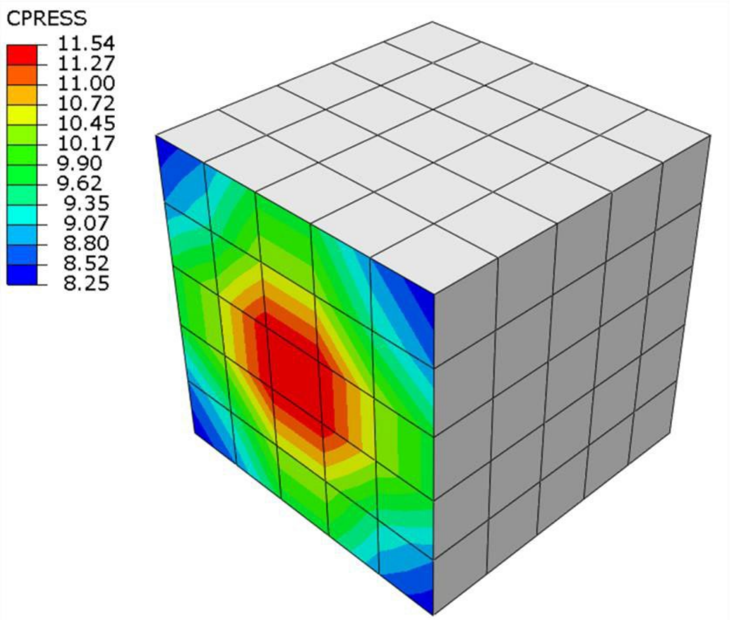
Penalty Method

Element type:

C3D8R

Contact Discretization:

Node-to-Surface



Contact enforcement methods

Lagrange Multiplier Method

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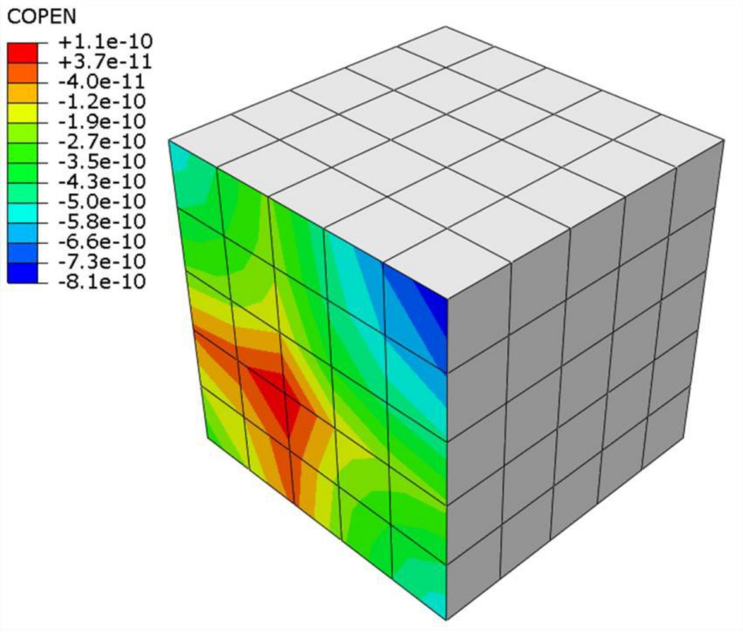


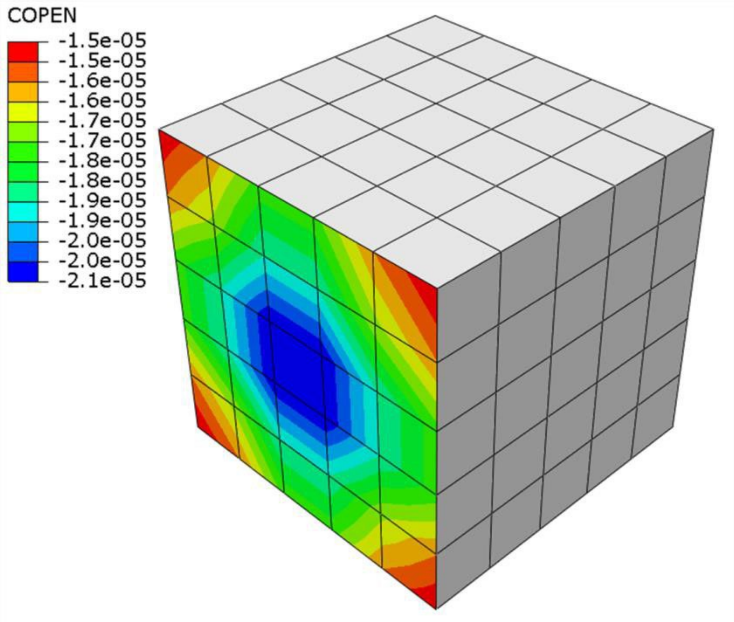
Penalty Method

Element type:

C3D8R

Contact Discretization:

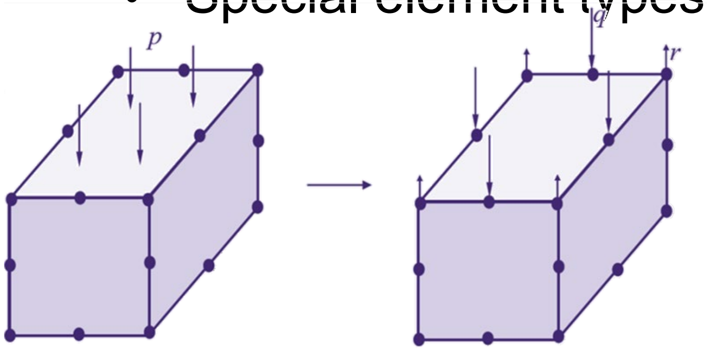
Node-to-Surface

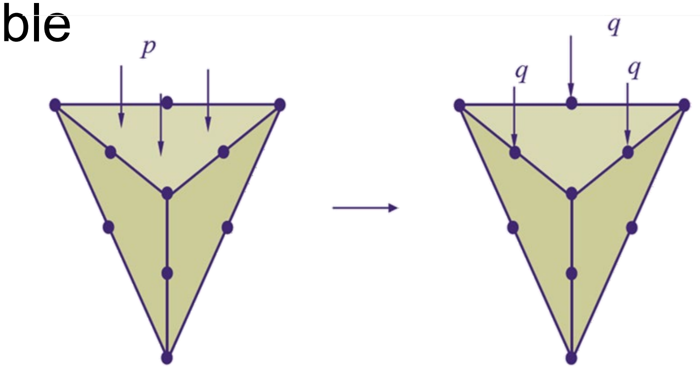


Finite Element Types

Finite Element type for contact analysis:

- **First-order**
  - General applicability for both N-to-S and S-to-S contact discretization
- **Second-order**
  - Work well with the surface-to-surface contact formulation
  - In some cases, do not work well with the node-to-surface formulation
  - Special element types available





Finite Element Types

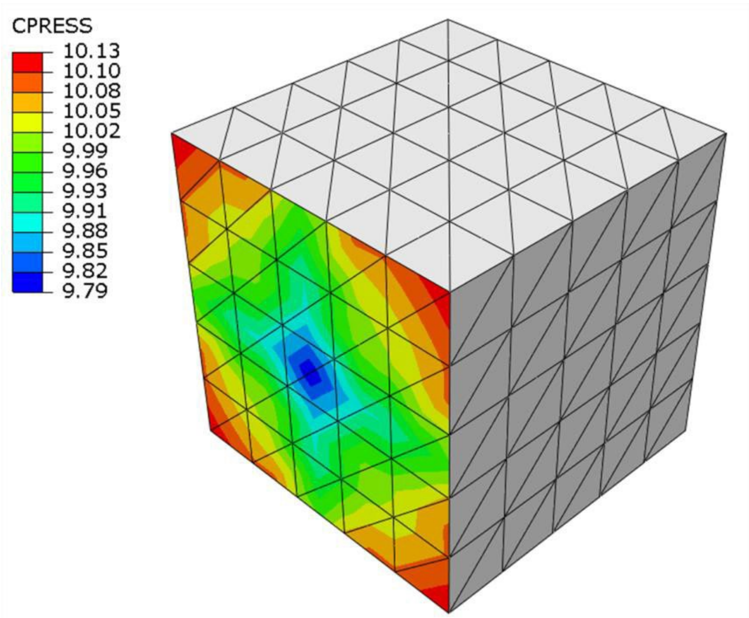
C3D10

Contact Enforcement:

Penalty method

Contact Discretization:

Node-to-Surface



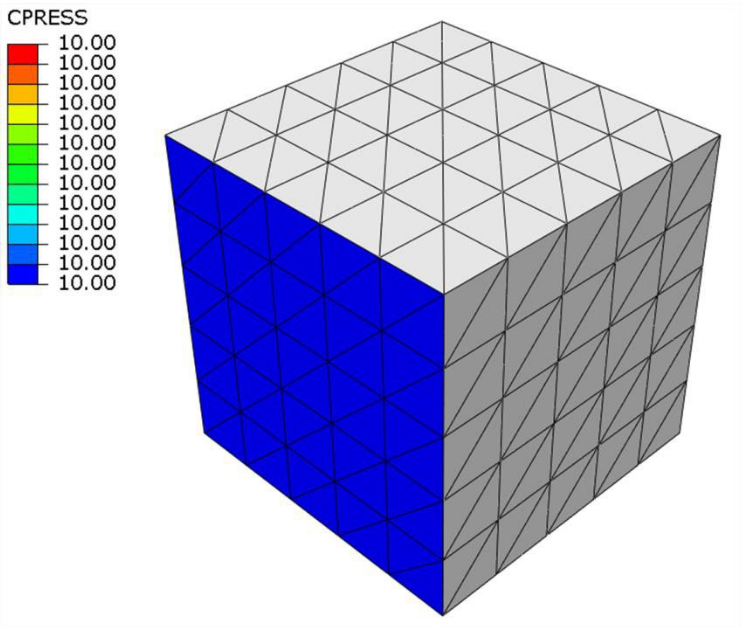
C3D10

Contact Enforcement:

Penalty method

Contact Discretization:

Surface-to-Surface



Contact tracking methods

Relative sliding between bodies:

- **Small-sliding formulation**
  - Approximation intended to reduce solution cost
  - Limited applicability
  - Planar representation of master surface per slave node based on the initial configuration
- **Finite-sliding formulation**
  - General applicability
  - Point of interaction on master surface is updated using the true representation of the master surface

Output of contact analysis

Nodal contact output to the ODB file:

- **Contact stresses**
  - Contact pressure (CPRESS)
  - Frictional shear stresses (CSHEAR1 and CSHEAR2)
- **Contact displacement**
  - Contact openings (COPEN)
  - Accumulated relative tangential motions (CSLIP1 and CSLIP2)

- Above output available as both field and history data

Output of contant analysis

- **Additional field output**
  - Contact error indicators (CSTRESSERI)
  - Contact nodal force vectors (CFORCE)
  - Nodal areas associated with active contact constraints (CNAREA)
  - Contact status (CSTATUS)
    - Enables contour plots of sticking/slipping/open status