POLITECNICO DI MILANO

Finite Element Simulation For Mechanical Design

Plasticity in FE modelling

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Introduction

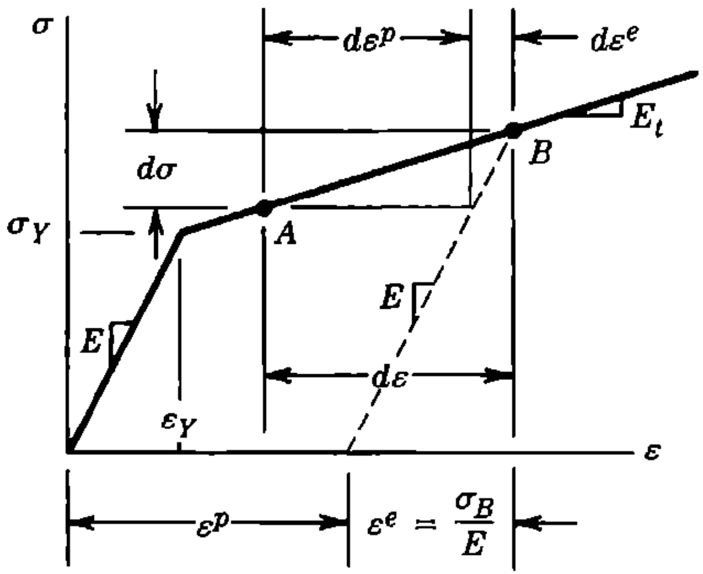
Plasticity: deformations that are not fully recovered once loads are removed

Here we treat time-independent plasticity:

- No creep
- No effect of the strain rate

We refer to isotropic materials and small strain formulation

Uniaxial stress (σ–ε relationship assumed bilinear for the sake of simplicity): loading



F, yield function

Yielding is defined by F = 0

For uniaxial loading

$$F = |\sigma| - \sigma_Y$$

For $\epsilon \geq \epsilon_Y$

$$d\epsilon = d\epsilon^e + d\epsilon^p$$

$$d\sigma = E\, d\varepsilon^e \qquad d\sigma = E(d\varepsilon - d\varepsilon^p) \qquad d\sigma = E_t\, d\varepsilon \qquad d\sigma = H_p\, d\varepsilon^p$$

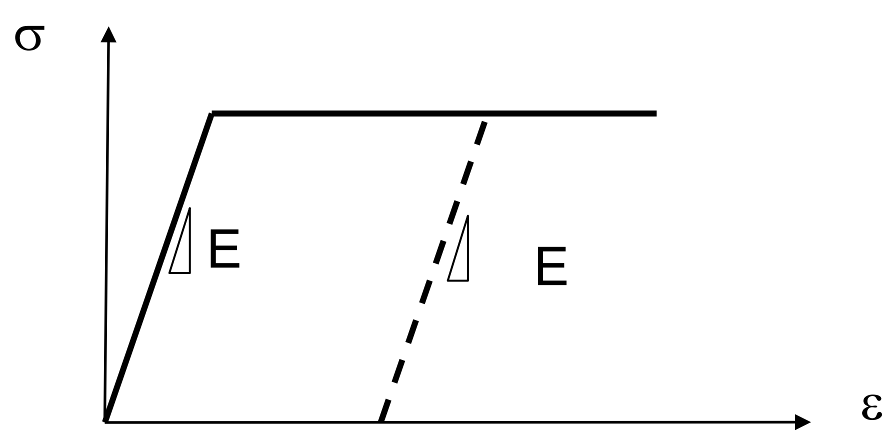
Strain hardening parameter

H_p, strain hardening parameter, or plastic modulus

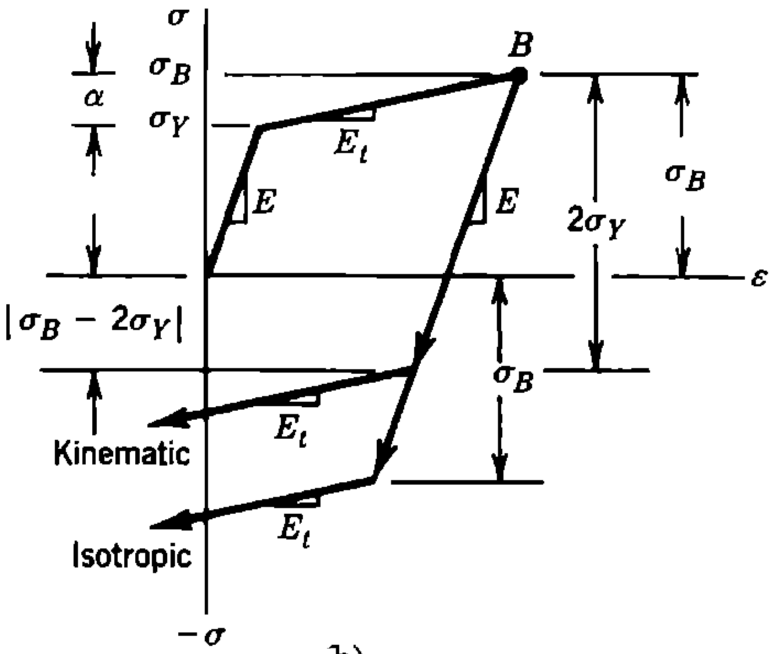
$$H_p = \frac{E_t}{1-(E_t/E)} \qquad \text{or} \qquad E_t = E\left(1-\frac{E}{E+H_p}\right)$$

If $H_p = 0$ (for which $E_t = 0$), the material is said **elastic-perfectly plastic**

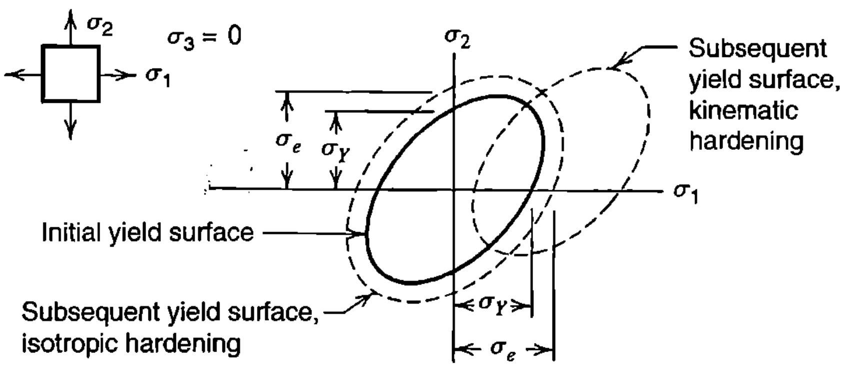
If the material has not yet yielded or is unloading $E_t = E$



Uniaxial plasticity: hardening and unloading



- Unloading takes place elastically
- Conditions remain elastic until the limit of the elastic range is reached
- The span of the elastic range is defined by the hardening rule

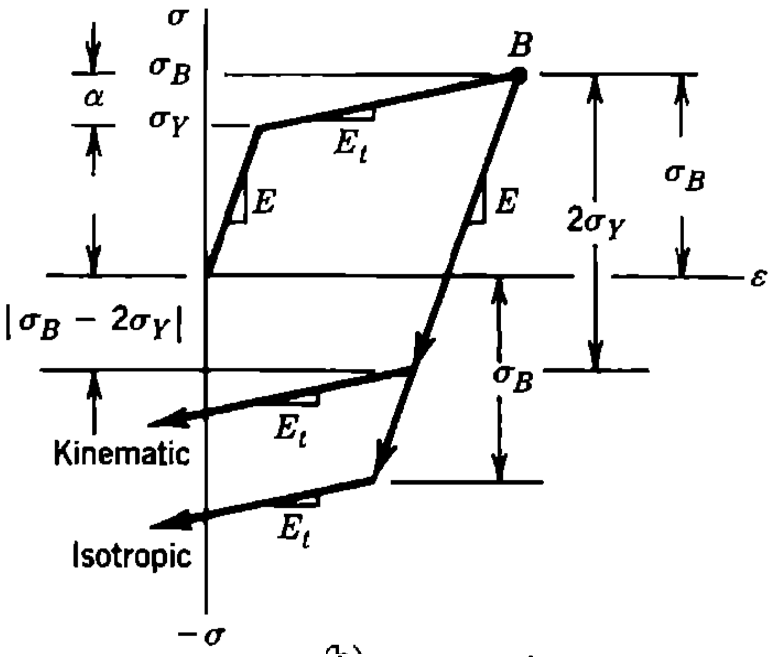


Hardening rules:

- ISOTROPIC (no Bauschinger effect)
- KYNEMATIC (no increase of the elastic range)

The two rules can be used in combination

Uniaxial plasticity: hardening and unloading



- Unloading takes place elastically
- Conditions remain elastic until the limit of the elastic range is reached
- The span of the elastic range is defined by the hardening rule

Hardening rules:

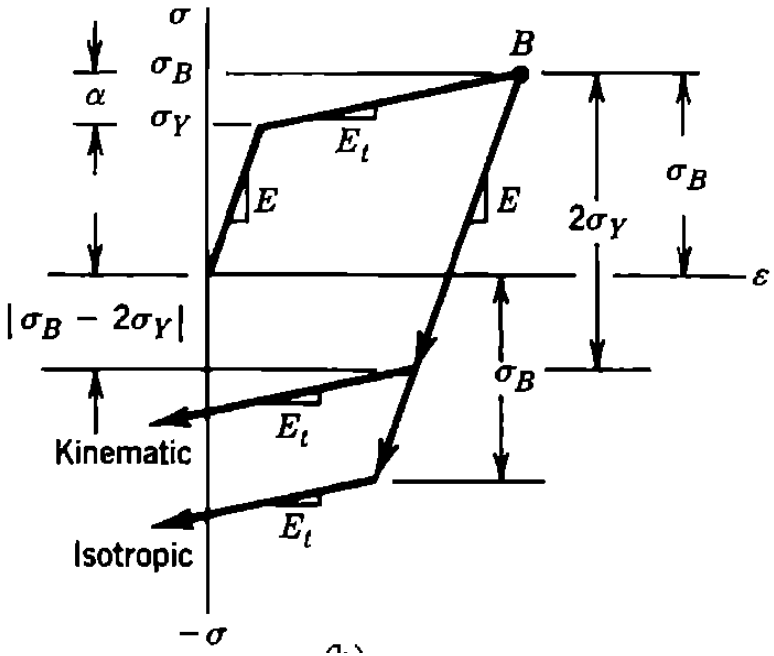
Largest magnitude of the stress reached previously, $\sigma_0 = \sigma_Y + \alpha$

- ISOTROPIC (no Bauschinger effect) $F = |\sigma| - \sigma_0$
- KYNEMATIC (no increase of the elastic range) $F = |\sigma - \alpha| - \sigma_Y$

Kynematic shift

The two rules can be used in combination

Hardening rules



Prior to yielding, $\alpha = 0$ and $\sigma_0 = \sigma_Y$

Elastic conditions: when α and σ are such that $F < 0$

Yielding: when $F = 0$

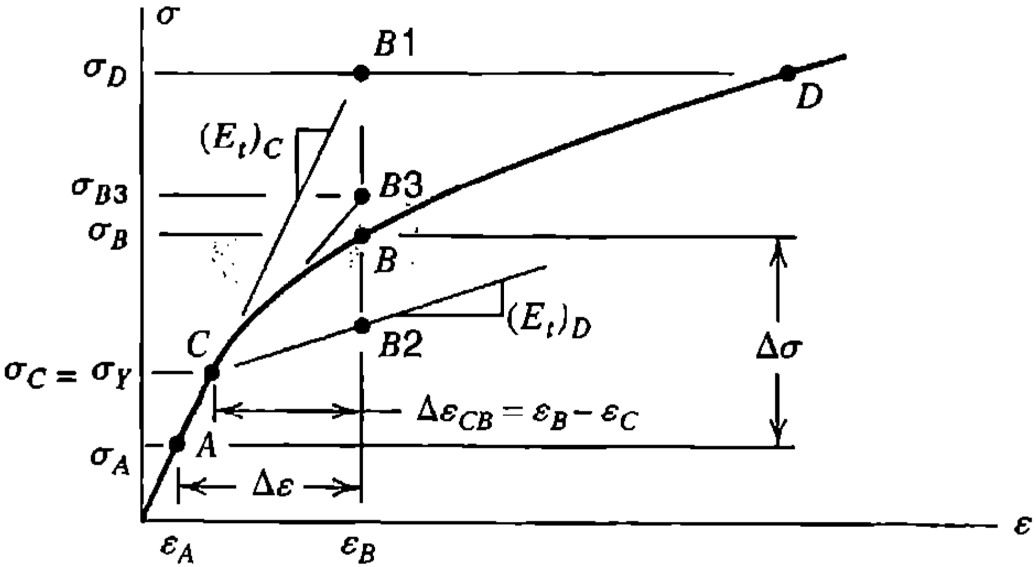
$F > 0$ is not physically possible

Continued or renewed plastic straining with strain hardening alters α

Let's start from point A, which lies in the linear elastic range.

We know σ_A, ε_A and ε_B

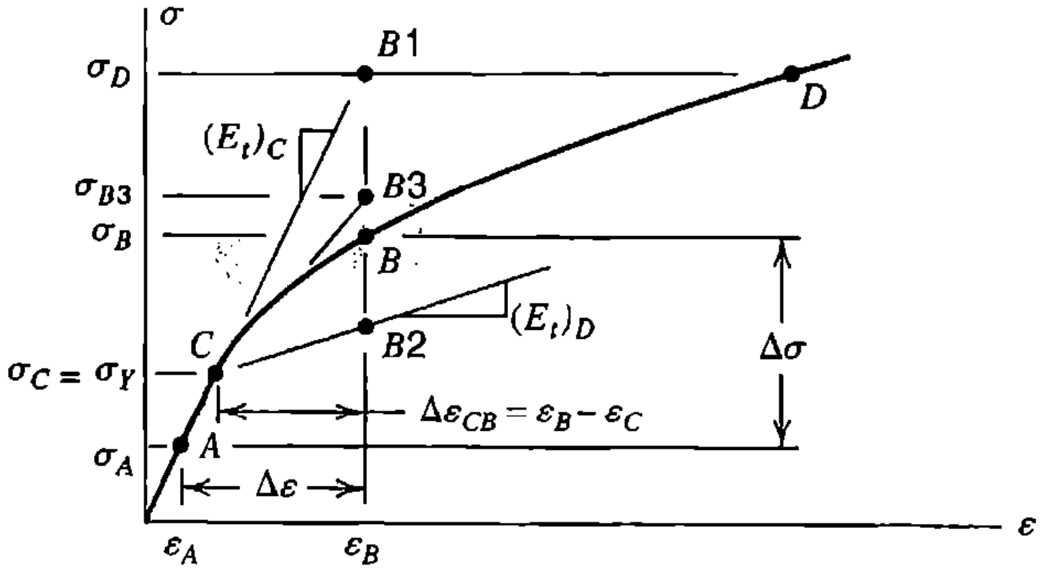
σ_B must be calculated



In order to resemble the calculations in a multiaxial stress state, we cannot "read the curve"

Calculations in uniaxial stress

Let’s start from point A, which lies in the linear elastic range. We seek for point C, at the limit of the elastic range



$$F_C = \left|\sigma_A + \beta\,\Delta\sigma_{\text{trial}}\right| - \sigma_Y = 0$$

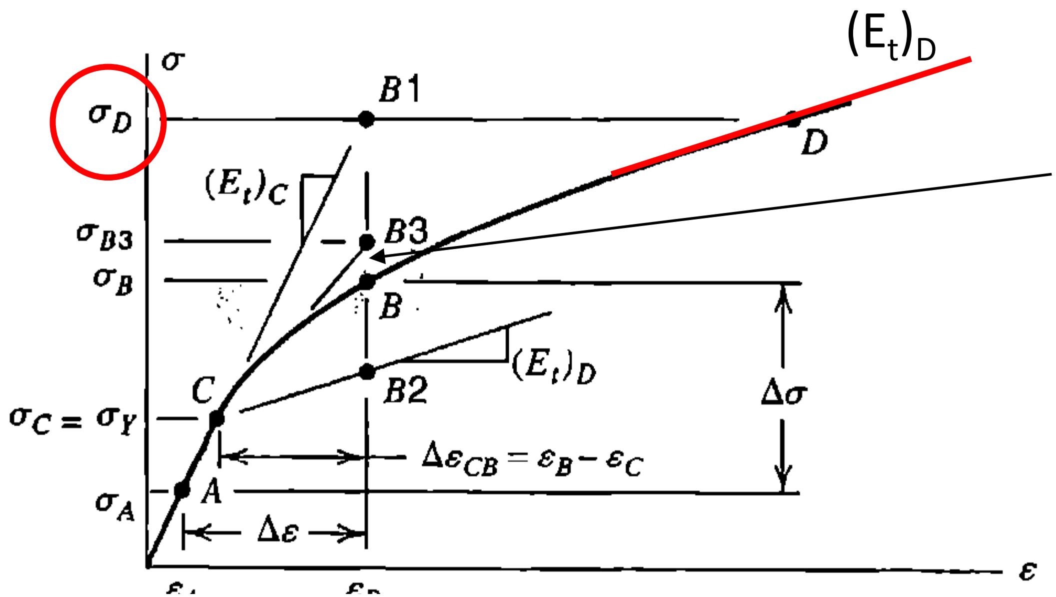
where $\Delta\sigma_{\text{trial}} = \sigma_D - \sigma_A$

$$\beta = \frac{\sigma_Y - \sigma_A}{\sigma_D - \sigma_A}$$

$$\Delta\sigma_{\text{trial}} = \sigma_D - \sigma_A = E\Delta\varepsilon$$

Post yield calculations in uniaxial stress by two-step integration

Starting from point C



The error can be reduced by reducing the step size

1[st] step $(\Delta\sigma)_1 = (E_t)_C\,(\varepsilon_B - \varepsilon_C) = \sigma_D - \sigma_C$ and $\sigma_{B1} \approx \sigma_C + (\Delta\sigma)_1 = \sigma_D$

2[nd] step $(\Delta\sigma)_2 = (E_t)_D(\varepsilon_B - \varepsilon_C)$ and $\sigma_{B2} \approx \sigma_C + (\Delta\sigma)_2$

$$\sigma_{B3} = \sigma_C + \frac{1}{2}\Big[(\Delta\sigma)_1 + (\Delta\sigma)_2\Big] \quad \text{or} \quad \sigma_{B3} = \sigma_C + \frac{1}{2}\Big[(E_t)_C + (E_t)_D\Big]\,\Delta\varepsilon_{CB}$$

$$y_{n+1} = y_n + \Big[(1-\gamma)(dy/dx)_n + \gamma(dy/dx)_{n+1}\Big]\,\Delta x$$

Generalized trapezoidal rule
We used γ=0.5, i.e. Runge-Kutta second order method

General formulation: incremental plasticity relations

In a 3D stress state, results are history-dependent

Therefore we have to establish relationships between increment of stresses and increment of strains.

$$\{d\boldsymbol{\varepsilon}\} = \{d\boldsymbol{\varepsilon}^e\} + \{d\boldsymbol{\varepsilon}^p\}$$

$$\{d\boldsymbol{\sigma}\} = [\mathbf{E}]\{d\boldsymbol{\varepsilon}^e\} \quad \text{or} \quad \{d\boldsymbol{\sigma}\} = [\mathbf{E}]\Big(\{d\boldsymbol{\varepsilon}\} - \{d\boldsymbol{\varepsilon}^p\}\Big)$$

$$\{d\boldsymbol{\sigma}\} = \lfloor d\sigma_x \;\; d\sigma_y \;\; d\sigma_z \;\; d\tau_{xy} \;\; d\tau_{yz} \;\; d\tau_{zx} \rfloor^T$$

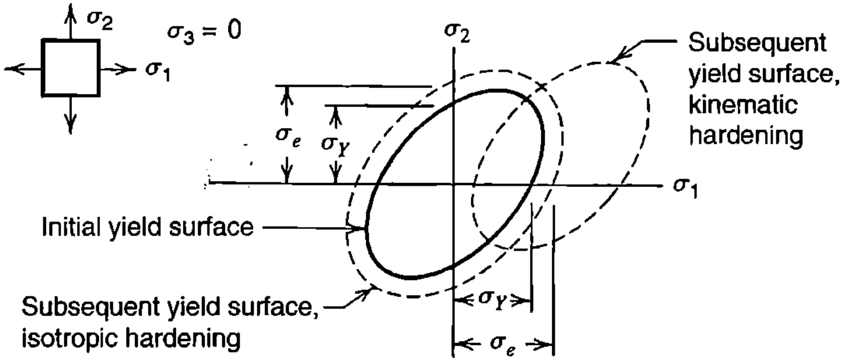
We need three ingredients:

- **Yield criterion** (relates the state of stress to the onset of yielding)
- **Flow rule** (relates the state of stress to increments of plastic strain)
- **Hardening rule** (describes how the yield criterion is modified by straining beyond initial yield)

Yield function

$$F = F\left(\{\sigma\},\{\alpha\},W_p \right)$$

α and W_p describe how the yield surface (equation F=0) changes in response to plastic strains



F = 0 at yield and during plastic flow

During plastic flow:

- α and/or W_p change
- stresses remain on the yield surface (dF = 0)

During unloading dF < 0

F > 0 is not physically possible

Flow rule: plastic potential

The flow rule is stated in terms of a function Q called **plastic potential**

$$\{d\boldsymbol{\varepsilon}^p\} = \left\{\frac{\partial Q}{\partial \boldsymbol{\sigma}}\right\} d\lambda$$

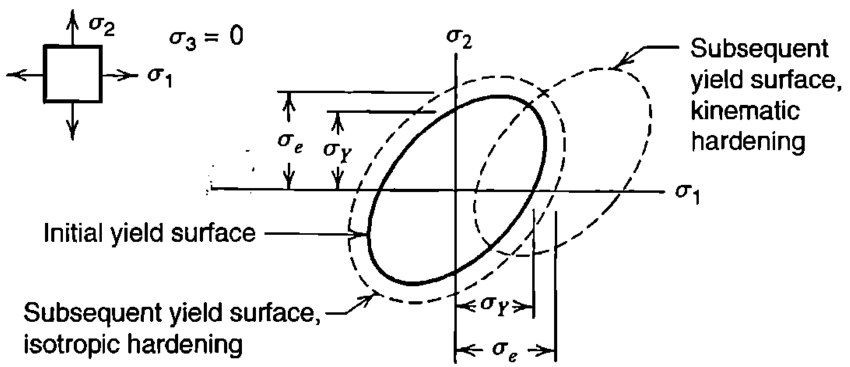
i.e. $d\varepsilon_x^p = (\partial Q/\partial \sigma_x)\, d\lambda$ and so on

$d\lambda$ is called a **plastic multiplier**

If Q = F, the flow rule is called **associated**

Flow rules for ductile metals are usually associated

Hardening can be modelled as isotropic or kinematic, either separately or in combination



Kinematic hardening:

$$\{\alpha\} = \int \lceil C \rfloor \{d\varepsilon^p\}$$

Translation of the yield surface

Isotropic hardening:

$$W_p = \int \{\sigma\}^T \{d\varepsilon^p\}$$

Plastic work per unit volume, describes growth of the yield surface

In general, H_p is not constant

$$\{d\alpha\} = \lceil C \rfloor \{d\varepsilon^p\} \quad \text{in which} \quad \lceil C \rfloor = \frac{2}{3} H_p \left\lceil 1 \quad 1 \quad 1 \quad \frac{1}{2} \quad \frac{1}{2} \quad \frac{1}{2} \right\rfloor$$

Plastic flow takes place at constant volume $\qquad d\varepsilon_x^p + d\varepsilon_y^p + d\varepsilon_z^p = 0$

hence $\qquad \lfloor 1 \quad 1 \quad 1 \quad 0 \quad 0 \quad 0 \rfloor \{\alpha\} = \alpha_x + \alpha_y + \alpha_z = 0$

During an increment of plastic straining, dF = 0, thus from $F = F\left(\{\boldsymbol{\sigma}\},\{\boldsymbol{\alpha}\},W_p\right)$

$$\left\{\frac{\partial F}{\partial \boldsymbol{\sigma}}\right\}^T \{d\boldsymbol{\sigma}\} + \left\{\frac{\partial F}{\partial \boldsymbol{\alpha}}\right\}^T \{d\boldsymbol{\alpha}\} + \frac{\partial F}{\partial W_p} dW_p = 0$$

$$\{d\boldsymbol{\sigma}\} = [\mathbf{E}]\left(\{d\boldsymbol{\varepsilon}\} - \{d\boldsymbol{\varepsilon}^p\}\right) \Longrightarrow \{d\boldsymbol{\sigma}\} = [\mathbf{E}]\left(\{d\boldsymbol{\varepsilon}\} - \left\{\frac{\partial Q}{\partial \boldsymbol{\sigma}}\right\} d\lambda\right),$$

$$\{d\boldsymbol{\varepsilon}^p\} = \left\{\frac{\partial Q}{\partial \boldsymbol{\sigma}}\right\} d\lambda \Longrightarrow dW_p = \{\boldsymbol{\sigma}\}^T \left\{\frac{\partial Q}{\partial \boldsymbol{\sigma}}\right\} d\lambda,$$

$$\left.\begin{array}{l} \{\boldsymbol{\alpha}\} = \int \lceil \mathbf{C} \rfloor \{d\boldsymbol{\varepsilon}^p\} \\ \{d\boldsymbol{\varepsilon}^p\} = \left\{\dfrac{\partial Q}{\partial \boldsymbol{\sigma}}\right\} d\lambda \end{array}\right\} \Longrightarrow \{d\boldsymbol{\alpha}\} = \lceil \mathbf{C} \rfloor \left\{\frac{\partial Q}{\partial \boldsymbol{\alpha}}\right\} d\lambda$$

Incremental stress-strain relations

Substituting and solving for the plastic multiplier

$$d\lambda = \lfloor \mathbf{P}_\lambda \rfloor \{ d\boldsymbol{\varepsilon} \}$$

$$\lfloor \mathbf{P}_\lambda \rfloor = \frac{\left\{ \frac{\partial F}{\partial \boldsymbol{\sigma}} \right\}^T [\mathbf{E}]}{\left\{ \frac{\partial F}{\partial \boldsymbol{\sigma}} \right\}^T [\mathbf{E}] \left\{ \frac{\partial Q}{\partial \boldsymbol{\sigma}} \right\} - \left\{ \frac{\partial F}{\partial \boldsymbol{\alpha}} \right\}^T \lceil \mathbf{C} \rfloor \left\{ \frac{\partial Q}{\partial \boldsymbol{\alpha}} \right\} - \frac{\partial F}{\partial W_p} \{ \boldsymbol{\sigma} \}^T \left\{ \frac{\partial Q}{\partial \boldsymbol{\sigma}} \right\}}$$

$$\{ d\boldsymbol{\sigma} \} = [\mathbf{E}] \left(\{ d\boldsymbol{\varepsilon} \} - \left\{ \frac{\partial Q}{\partial \boldsymbol{\sigma}} \right\} d\lambda \right), \quad \Longleftarrow \quad d\lambda = \lfloor \mathbf{P}_\lambda \rfloor \{ d\boldsymbol{\varepsilon} \}$$

$$\{ d\boldsymbol{\sigma} \} = [\mathbf{E}_{ep}] \{ d\boldsymbol{\varepsilon} \} \quad \text{where} \quad [\mathbf{E}_{ep}] = [\mathbf{E}] \left(\lceil \mathbf{I} \rfloor - \left\{ \frac{\partial Q}{\partial \boldsymbol{\sigma}} \right\} \lfloor \mathbf{P}_\lambda \rfloor \right)$$

E_{ep}

$$[\mathbf{E}_{ep}] = [\mathbf{E}]\left(\lceil \mathbf{I} \rfloor - \left\{\frac{\partial Q}{\partial \boldsymbol{\sigma}}\right\} \lfloor \mathbf{P}_\lambda \rfloor\right)$$

$[E_{ep}]$ can be regarded as a generalized form of tangent modulus E_t

It is symmetric for Q = F

$[E_{ep}] = [E]$:

- for unloading from a plastic state (F = 0 and dF < 0) or
- when yielding has yet to appear (F < 0)

Use of $[E_{ep}]$ in element formulation provides the tangent stiffness

$$[\mathbf{k}_t] = \int_V [\mathbf{B}]^T [\mathbf{E}_{ep}][\mathbf{B}]\, dV$$

By assembling the equations for a FE structure, one obtains the relationship between displacement increments and load increments when there is plastic straining

$$[\mathbf{K}_t]\{d\mathbf{D}\} = \{d\mathbf{R}\}$$

Associative plasticity (metals): von Mises theory

It postulates that yielding takes place when the effective stress reaches a limiting value

$$\sigma_e = \frac{1}{\sqrt{2}}\left[(\sigma_x-\sigma_y)^2+(\sigma_y-\sigma_z)^2+(\sigma_z-\sigma_x)^2+6\left(\tau_{xy}^2+\tau_{yz}^2+\tau_{xy}^2\right)\right]^{1/2}$$

In this context, deviatoric stresses play an important role

$$\{\mathbf{s}_\sigma\} = \begin{Bmatrix} s_x \\ s_y \\ s_z \end{Bmatrix} = \begin{Bmatrix} \sigma_x-\sigma_m \\ \sigma_y-\sigma_m \\ \sigma_z-\sigma_m \end{Bmatrix} = \frac{1}{3}\begin{Bmatrix} 2\sigma_x-\sigma_y-\sigma_z \\ 2\sigma_y-\sigma_z-\sigma_x \\ 2\sigma_z-\sigma_x-\sigma_y \end{Bmatrix} \qquad \{\mathbf{s}_\tau\} = \begin{Bmatrix} s_{xy} \\ s_{yz} \\ s_{zx} \end{Bmatrix} = \begin{Bmatrix} \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix}$$

$$s_x+s_y+s_z = 0 \qquad \sigma_m = \frac{\sigma_x+\sigma_y+\sigma_z}{3}$$

$$\sigma_e = \sqrt{\frac{3}{2}}\left[s_x^2+s_y^2+s_z^2+2\left(s_{xy}^2+s_{yz}^2+s_{zx}^2\right)\right]^{1/2}$$

Being Q = F,

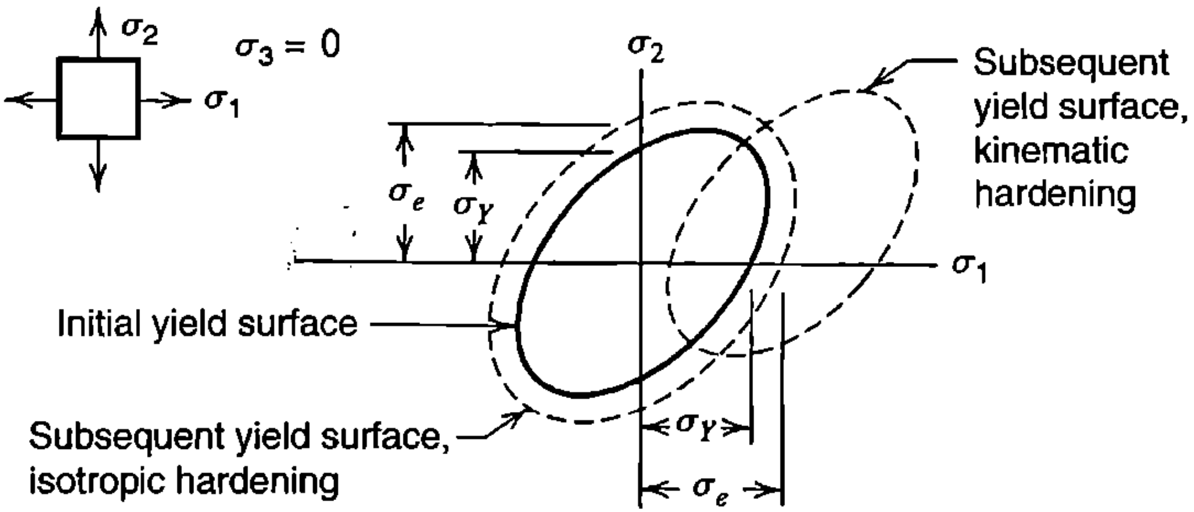
$$d\boldsymbol{\varepsilon}_p = \frac{\partial F}{\partial \boldsymbol{\sigma}} d\lambda$$

It implies that $d\boldsymbol{\varepsilon}_p$ is normal to the surface of equation F = 0 in the stress space

It can be demonstrated that the plastic multiplier is the same as the effective plastic strain corresponding to σ_e , which is expressed by

$$d\lambda = d\varepsilon_e^p = \sqrt{\frac{2}{3}}\left[\left(d\varepsilon_x^p\right)^2 + \left(d\varepsilon_y^p\right)^2 + \left(d\varepsilon_z^p\right)^2 + \frac{1}{2}\left\langle\left(d\gamma_{xy}^p\right)^2 + \left(d\gamma_{yz}^p\right)^2 + \left(d\gamma_{zx}^p\right)^2\right\rangle\right]^{1/2}$$

Hardening rules



$$F = \sigma_e - \sigma_0$$

Isotropic hardening can be described by a strain hardening expressions (instead of work hardening) , where σ_0 is the largest value of σ_e reached in previous plastic straining

Mixed hardening rule

$$F = \left[\frac{3}{2} \left\langle (s_x - \eta\alpha_x)^2 + (s_y - \eta\alpha_y)^2 + (s_z - \eta\alpha_z)^2 \right\rangle + 3\left\langle (s_{xy} - \eta\alpha_{xy})^2 + (s_{yz} - \eta\alpha_{yz})^2 + (s_{zx} - \eta\alpha_{zx})^2 \right\rangle \right]^{1/2} - \eta\sigma_Y - (1-\eta)\sigma_0$$

$$\frac{\partial F}{\partial \boldsymbol{\alpha}} = -\eta \frac{\partial F}{\partial \boldsymbol{\sigma}}$$

η = 0, purely isotropic

η = 1 purely kynematic

In $\left\{\frac{\partial F}{\partial \boldsymbol{\sigma}}\right\}^T \{d\boldsymbol{\sigma}\} + \left\{\frac{\partial F}{\partial \boldsymbol{\alpha}}\right\}^T \{d\boldsymbol{\alpha}\} + \frac{\partial F}{\partial W_p} dW_p = 0$ $\quad \frac{\partial F}{\partial W_p} dW_p$ is replaced by

$$\frac{\partial F}{\partial \sigma_0} d\sigma_0 \quad \text{where} \quad \frac{\partial F}{\partial \sigma_0} = -(1-\eta) \quad \text{and} \quad d\sigma_0 = H_p d\lambda$$

$$\frac{\partial F}{\partial \boldsymbol{\alpha}} = -\eta \frac{\partial F}{\partial \boldsymbol{\sigma}}$$

Replacing in

$$\lfloor \mathbf{P}_\lambda \rfloor = \frac{\left\{\frac{\partial F}{\partial \boldsymbol{\sigma}}\right\}^T [\mathbf{E}]}{\left\{\frac{\partial F}{\partial \boldsymbol{\sigma}}\right\}^T [\mathbf{E}]\left\{\frac{\partial Q}{\partial \boldsymbol{\sigma}}\right\} - \left\{\frac{\partial F}{\partial \boldsymbol{\alpha}}\right\}^T \lceil \mathbf{C} \rfloor \left\{\frac{\partial Q}{\partial \boldsymbol{\alpha}}\right\} - \boxed{\frac{\partial F}{\partial W_p}\{\boldsymbol{\sigma}\}^T\left\{\frac{\partial Q}{\partial \boldsymbol{\sigma}}\right\}}}$$

$$\frac{\partial Q}{\partial \boldsymbol{\sigma}} = \frac{\partial F}{\partial \boldsymbol{\sigma}} \qquad \frac{\partial F}{\partial \boldsymbol{\alpha}} = -\eta \frac{\partial F}{\partial \boldsymbol{\sigma}} \qquad \frac{\partial F}{\partial \sigma_0} d\sigma_0$$

$$\frac{\partial Q}{\partial \boldsymbol{\alpha}} = \frac{\partial F}{\partial \boldsymbol{\alpha}} \qquad d\sigma_0 = H_p d\lambda$$

$$\frac{\partial F}{\partial \sigma_0} = -(1-\eta)$$

We obtain

$$\lfloor \mathbf{P}_\lambda \rfloor = \frac{\left\{\frac{\partial F}{\partial \boldsymbol{\sigma}}\right\}^T [\mathbf{E}]}{\left\{\frac{\partial F}{\partial \boldsymbol{\sigma}}\right\}^T \left(\boxed{[\mathbf{E}] + \eta \lceil \mathbf{C} \rfloor}\right)\left\{\frac{\partial F}{\partial \boldsymbol{\sigma}}\right\} + \boxed{(1-\eta) H_p}}$$

$$\left\{\frac{\partial F}{\partial \boldsymbol{\sigma}}\right\} = \frac{1}{2[\cdots]^{1/2}} \left\{\frac{\partial}{\partial \boldsymbol{\sigma}}[\cdots]\right\}$$

$$\Big[\frac{3}{2}\left\langle (s_x - \eta\alpha_x)^2 + (s_y - \eta\alpha_y)^2 + (s_z - \eta\alpha_z)^2 \right\rangle + 3\Big\langle (s_{xy} - \eta\alpha_{xy})^2 + (s_{yz} - \eta\alpha_{yz})^2$$

$$+ (s_{zx} - \eta\alpha_{zx})^2 \Big\rangle \Big]^{1/2}$$

$$\lfloor \mathbf{P}_\lambda \rfloor = \frac{\left\{\dfrac{\partial F}{\partial \boldsymbol{\sigma}}\right\}^T [\mathbf{E}]}{\left\{\dfrac{\partial F}{\partial \boldsymbol{\sigma}}\right\}^T \left([\mathbf{E}] + \eta \lceil \mathbf{C} \rfloor\right)\left\{\dfrac{\partial F}{\partial \boldsymbol{\sigma}}\right\} + (1-\eta) H_p}$$

$$\left\{\frac{\partial F}{\partial \boldsymbol{\sigma}}\right\} = \frac{1}{2[\cdots]^{1/2}} \left\{\frac{\partial}{\partial \boldsymbol{\sigma}}[\cdots]\right\} = \frac{1}{2\left[\eta \sigma_Y + (1-\eta)\sigma_0\right]} \left\{\frac{\partial}{\partial \boldsymbol{\sigma}}[\cdots]\right\}$$

During yielding F = 0

$$F = \left[\frac{3}{2}\left\langle (s_x - \eta\alpha_x)^2 + (s_y - \eta\alpha_y)^2 + (s_z - \eta\alpha_z)^2 \right\rangle + 3\left\langle (s_{xy} - \eta\alpha_{xy})^2 + (s_{yz} - \eta\alpha_{yz})^2 + (s_{zx} - \eta\alpha_{zx})^2 \right\rangle \right]^{1/2} - \eta\sigma_Y - (1-\eta)\sigma_0$$

von Mises theory

$$\lfloor \mathbf{P}_\lambda \rfloor = \frac{\left\{\dfrac{\partial F}{\partial \boldsymbol{\sigma}}\right\}^T [\mathbf{E}]}{\left\{\dfrac{\partial F}{\partial \boldsymbol{\sigma}}\right\}^T \left([\mathbf{E}] + \eta \lceil \mathbf{C} \rfloor\right)\left\{\dfrac{\partial F}{\partial \boldsymbol{\sigma}}\right\} + (1-\eta) H_p}$$

$$\left\{\frac{\partial F}{\partial \boldsymbol{\sigma}}\right\} = \frac{1}{2[\cdots]^{1/2}}\left\{\frac{\partial}{\partial \boldsymbol{\sigma}}[\cdots]\right\} = \frac{1}{2\left[\eta \sigma_Y + (1-\eta)\sigma_0\right]}\left\{\frac{\partial}{\partial \boldsymbol{\sigma}}[\cdots]\right\}$$

F = 0

$$\frac{\partial}{\partial \sigma_x}[\cdots] = 3\left[(s_x - \eta\alpha_x)\frac{\partial s_x}{\partial \sigma_x} + (s_y - \eta\alpha_y)\frac{\partial s_y}{\partial \sigma_x} + (s_z - \eta\alpha_z)\frac{\partial s_z}{\partial \sigma_x}\right]$$

$$\frac{\partial}{\partial \sigma_x}[\cdots] = 3\left[(s_x - \eta\alpha_x)\frac{2}{3} - (s_y - \eta\alpha_y)\frac{1}{3} - (s_z - \eta\alpha_z)\frac{1}{3}\right]$$

Differentiation with respect to σ_y and σ_z is performed similarly

von Mises theory

Remembering that 2 s_x - s_y - s_z = 3 s_x, same for α_x, and all other deviatoric and shift components

$$\left\{\frac{\partial F}{\partial \boldsymbol{\sigma}}\right\} = \frac{3}{\eta\sigma_Y + (1-\eta)\sigma_0}\left(\frac{1}{2}\begin{Bmatrix}\mathbf{s}_\sigma - \eta\boldsymbol{\alpha}_\sigma \\ \mathbf{0}\end{Bmatrix} + \begin{Bmatrix}\mathbf{0} \\ \mathbf{s}_\tau - \eta\boldsymbol{\alpha}_\tau\end{Bmatrix}\right)$$

$$\text{where} \qquad \{\boldsymbol{\alpha}\} = \begin{Bmatrix}\boldsymbol{\alpha}_\sigma \\ \boldsymbol{\alpha}_\tau\end{Bmatrix}$$

With F = Q and isotropic hardening (η=0), we have the **Prandtl-Reuss relations**

$$\frac{d\varepsilon_x^p}{s_x} = \frac{d\varepsilon_y^p}{s_y} = \frac{d\varepsilon_z^p}{s_z} = \frac{d\varepsilon_{xy}^p}{s_{xy}} = \frac{d\varepsilon_{yz}^p}{s_{yz}} = \frac{d\varepsilon_{zx}^p}{s_{zx}} = \frac{3}{2\sigma_0}d\lambda$$

Each plastic strain increment is proportional to its corresponding deviatoric stress (principal axes of strain increments coincide with those of stresses)

Computational procedures for time independent plastic analysis: given R, find D

Required material's data: E, σ_Y and values of H_p for any given value of σ_e

First step: find state A, where the structure is at the verge of yield.

Then, we apply a load increment $\{\Delta R\}$ and seek for state B

First, we generate $[K_t]_A$, and calculate

$$[\mathbf{K}_t]_A\{\Delta\mathbf{D}\}_{AB} = \{\Delta\mathbf{R}\}$$

After this first iteration, a load imbalance between the applied loads R and the nodal forces generated by the stresses in the elements, will be generated.

In facts, the problem is non-linear, therefore more iterations are needed to find $\{\Delta D\}_{AB}$.

Next step

- $\{\sigma\}$
- $\{\alpha\}$
- W_p (or σ_0)

are updated.

- A new element tangent stiffness is evaluated
- A new structure tangent stiffness $[\boldsymbol{K}_t]_B$ is evaluated
- A new load increment is applied

$$[\boldsymbol{K}_t]_B \, \{\Delta \boldsymbol{D}\}_{BC} = \{\Delta \boldsymbol{R}\}$$