POLITECNICO DI MILANO

Finite Element Simulation For Mechanical Design

Geometric non-linearity

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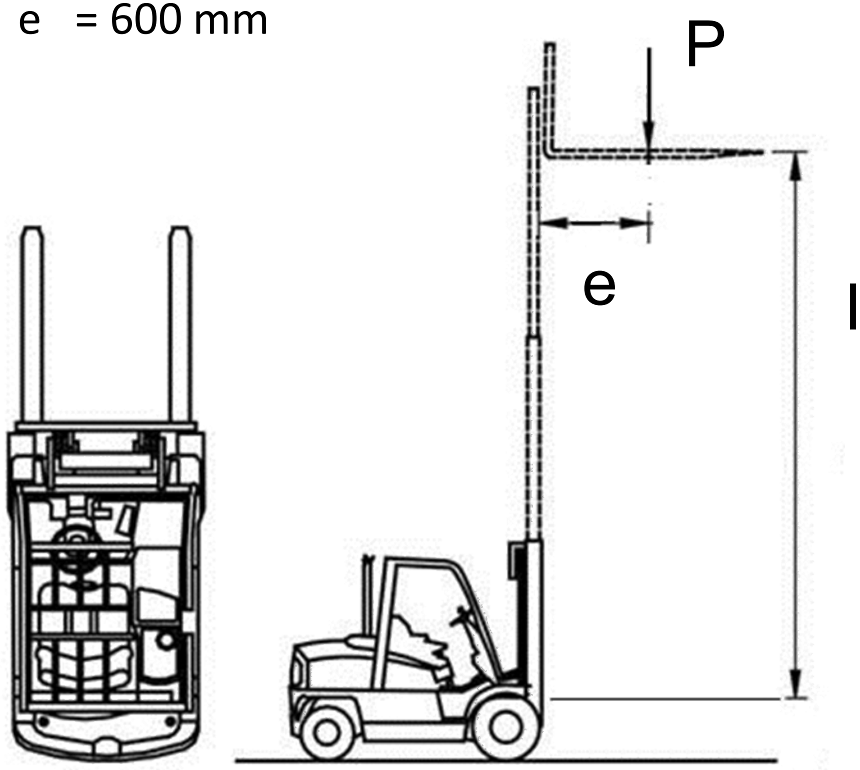
Is the assumption of linearity correct?

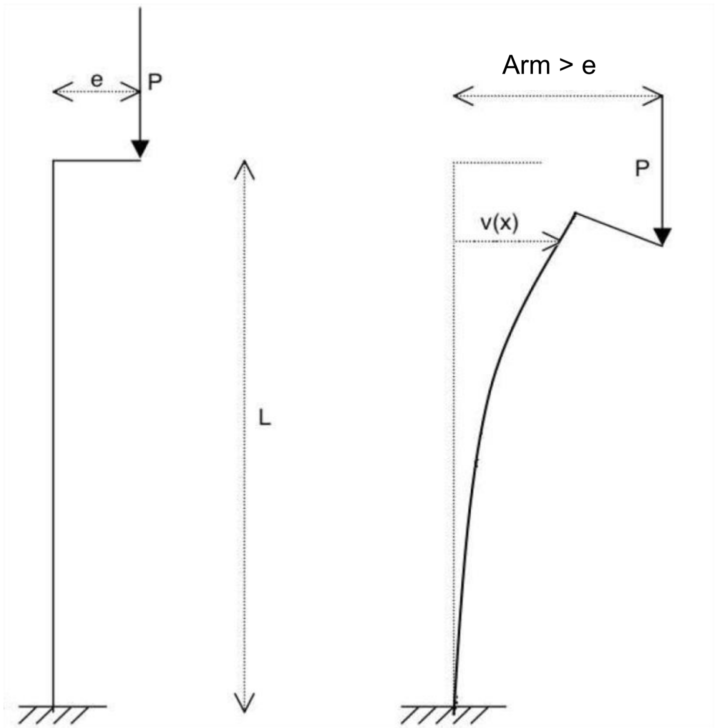
m = 2000 kg

P = mg/2 = 9810 N

l = 10 000 mm

e = 600 mm





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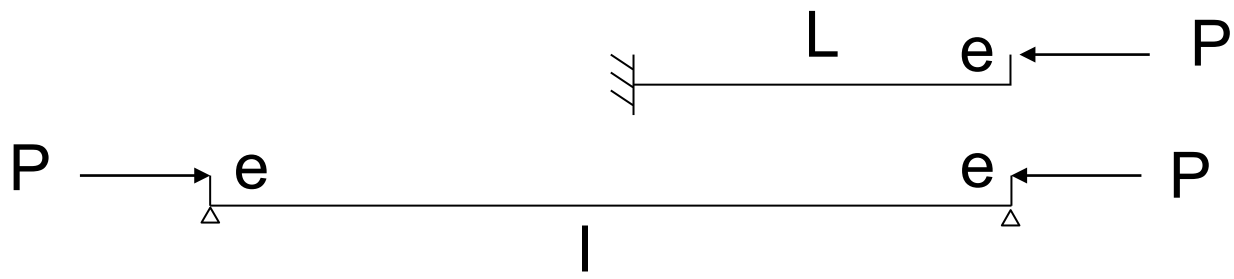
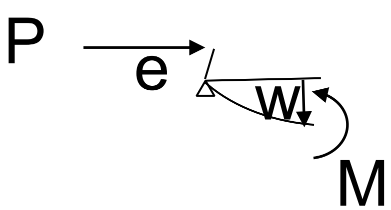
Notes

A cantilever beam with an axial load behaves like one half of a simply supported beam loaded axially at its ends



Let’s write equilibrium equations in the deformed configuration



$$M = P(e + w)$$

$$k = \frac{\| w'' \|}{(1 + w'^2)^{\frac{3}{2}}} \approx w''$$

Acceptable under the assumption of large displacements, but small rotations

$$w'' = -\frac{P}{EJ}(e + w)$$

$$w'' + \frac{P}{EJ} w = -\frac{P}{EJ} e$$

$$w = A \sin(mx) + \mathrm{B} \cos(mx) - e; \; m^2 = \frac{P}{EJ}$$

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Notes

Boundary conditions

$$w(0) = 0 \Rightarrow B - e = 0 \Rightarrow B = e$$

$$w(l) = 0 \Rightarrow A\sin(ml) + e\cos(ml) - e = 0 \Rightarrow A = e\tan\left(\frac{ml}{2}\right)$$

Deflection at the free end of the column w(L) is equal to the deflection in the middle of the equivalent simply supported beam w(l/2)

$$\delta = w\left(\frac{l}{2}\right) = e\tan\left(\frac{ml}{2}\right)\sin\left(\frac{ml}{2}\right) + e\cos\left(\frac{ml}{2}\right) - e = e/\cos\left(\frac{ml}{2}\right) - e$$

$$\delta = e\left[\sec\left(L\sqrt{\frac{P}{EJ}}\right) - 1\right]$$

If of interest, maximum absolute value of compressive stress is

$$\sigma = \frac{P}{A} + \frac{P}{J}(\delta + e)c = \frac{Pe}{J}\sec\left(L\sqrt{P/_{EJ}}\right)c$$

c, distance from the neutral axis to the extreme fiber

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Introduction to non-linearity

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Problems presented so fare are linear because the constitutive equations are linear, the boundary conditions are independent from the equilibrium configuration and the equilibrium configuration is close to the un-deformed one.

However, in many problems we introduce non-linearity in order to take into account:

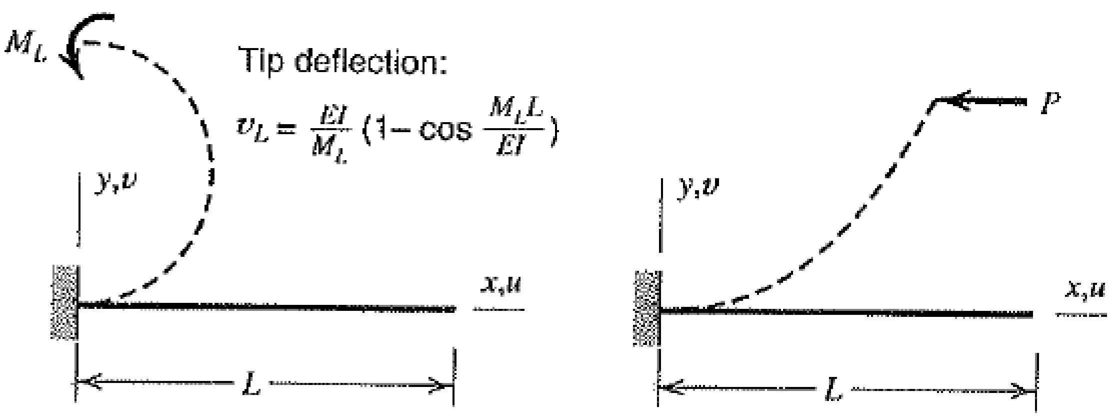
- Large displacements
- Non-linear constitutive equations (e.g. non-linear elasticity, plasticity, creep)
- Dependence of boundary conditions upon applied forces (e.g. contacts, gaps between adjacent parts closing or opening)

In all cases, an iterative process is required to obtain {D}

Let’s see an introduction to solution techniques for time-independent problems.

Non linearity for large displacements

- Elements introduced so far can be used to solve problems where strain and rotations are <<1
- How do we solve problems where the equilibrium configuration is very different from the undeformed one?
- In this case, stiffness is a function of the deformed configuration => nonlinear problem



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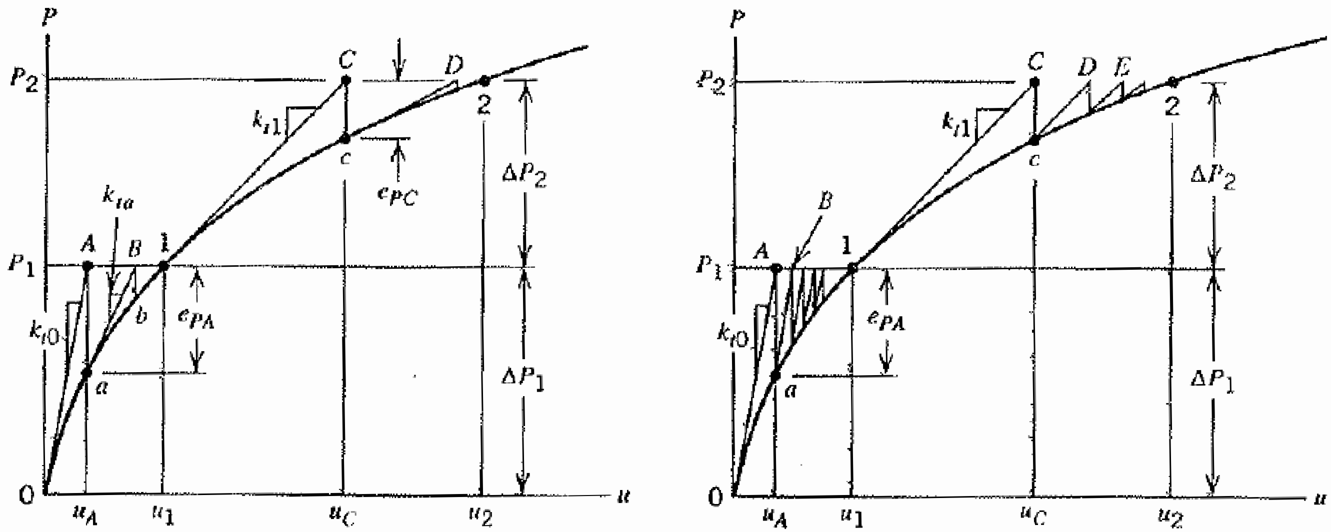
Solution methods for non linear problems

- Newton- Raphson
- Modified Newton-Raphson
- Quasi-Newton

We assume that, as it would be for the case of [K]{D}={R}, k can be calculated for any u, but it is not possible to explicitly solve for *u* when *P* is prescribed

$$P = k(u)u$$



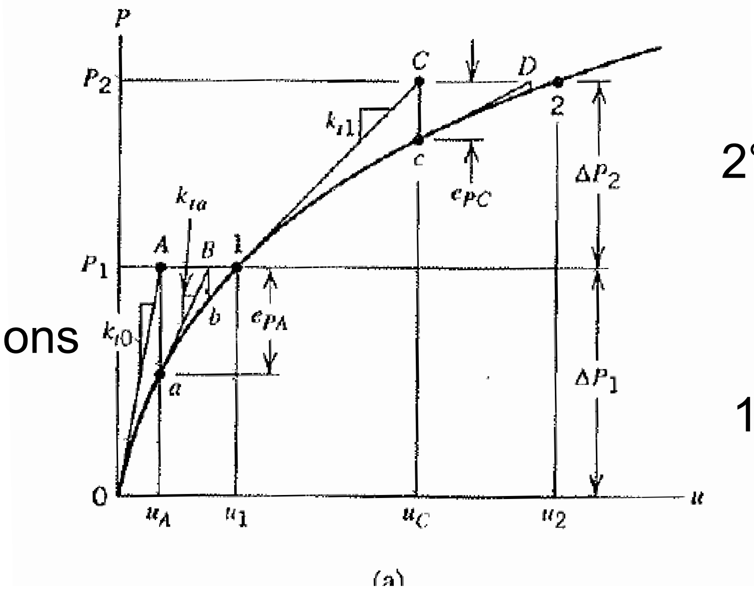
(a) (b)

Figure 17.2-2. Iterations to convergence at each of load levels P_1 and P_2. (a) Newton-Raphson iterations. (b) Modified Newton-Raphson iterations.

Newton's methods (1D example)

The load P is split into load increments

equilibrium iterations



2° increment

1° increment

We proceed iteratively, until convergence is achieved within tolerance

$$k_{t0}\,\Delta u = \Delta P_1 \qquad \Delta u = k_{t0}^{-1}\Delta P_1 \qquad u_A = 0 + \Delta u$$

Load imbalance is $e_{PA} = P_1 - k u_A$ where $k = k(u)$ is evaluated using displacement u_A

$$k_{ta}\,\Delta u = e_{PA} \qquad \Delta u = k_{ta}^{-1} e_{PA} \qquad u_B = u_A + \Delta u$$

$e_{PB} = P_1 - k u_B$ where $k = k(u)$ is evaluated using displacement u_B

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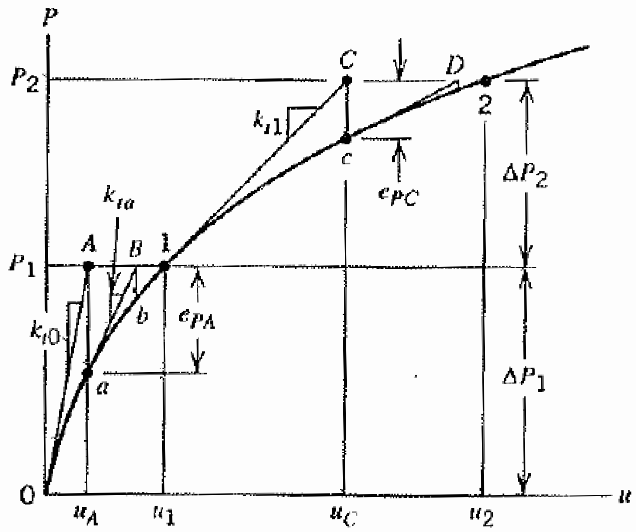
Then the second step starts

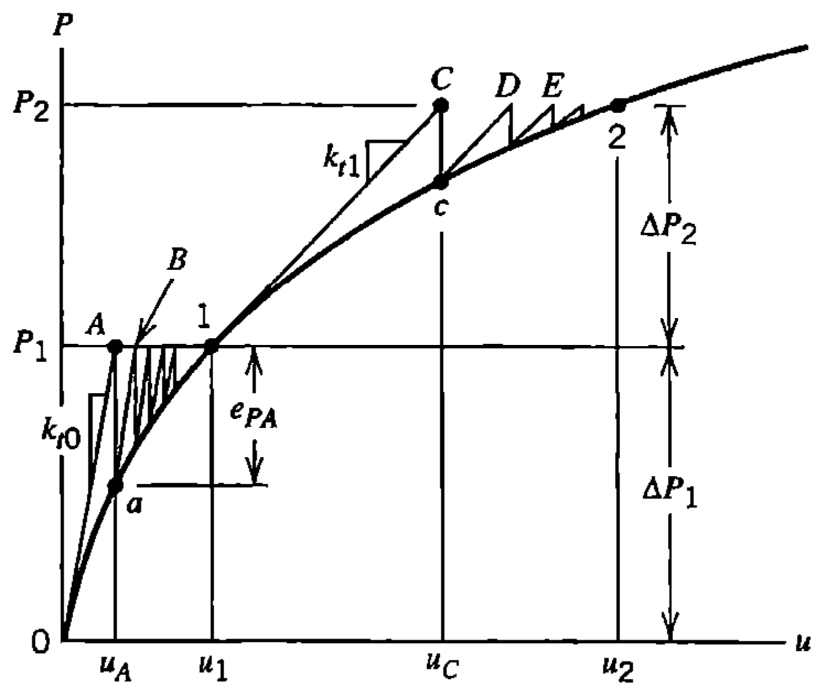


$$k_{t1}\Delta u = \Delta P_2 \qquad \Delta u = k_{t1}^{-1}\Delta P_2 \qquad u_C = u_1 + \Delta u$$

$$e_{PC} = P_2 - k u_C \qquad \text{where } k = k(u) \text{ is evaluated using displacement } u_C$$

Convergence is ensured by small increments

Modified Newton-Raphson



The same K_t is used for all iterations within the same increment step

Convergence is slower, but each iteration is less computationally expensive

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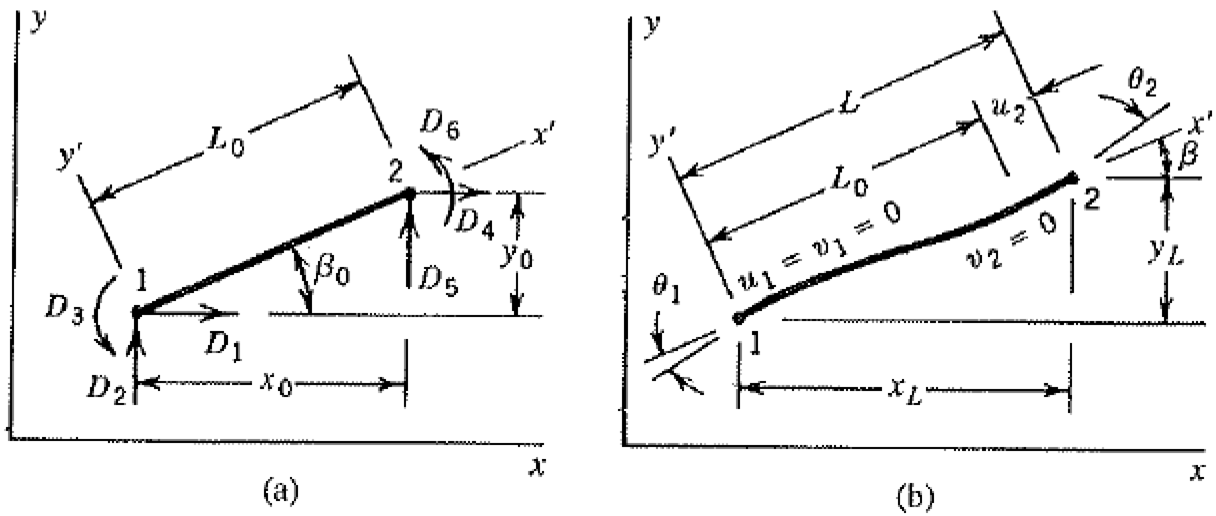
Co-rotational formulation of beam elements

Equilibrium equations need being written in the deformed configuration

Then, results need being transformed into the global reference frame.

Local deformations are tracked in the local reference.

Large displacements, small strains are assumed.



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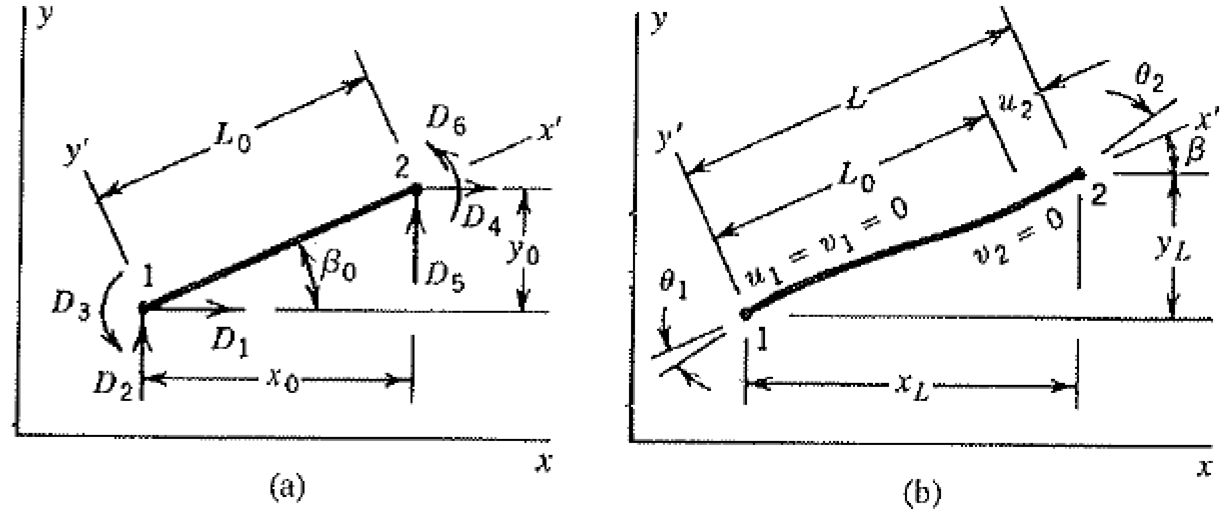


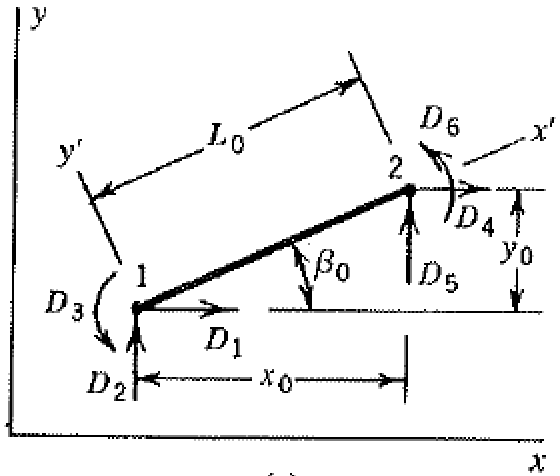
Element length projections

$$x_L = x_0 + (D_4 - D_1) = x_0 + D_{41}$$

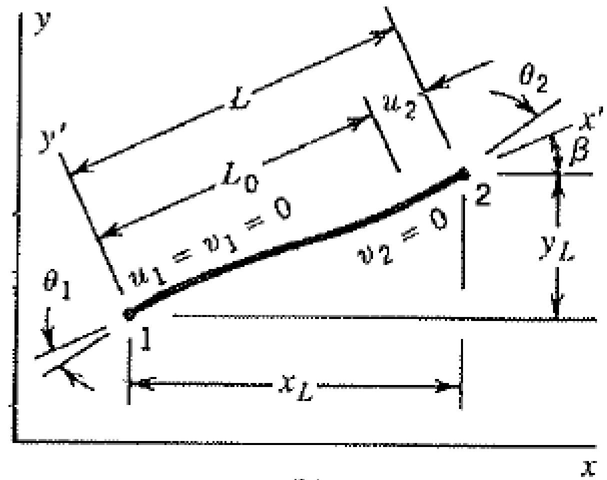
$$y_L = y_0 + (D_5 - D_2) = y_0 + D_{51}$$

Orientation of the local axis x'

$$\beta = \arctan(y_L / x_L)$$



(a)



(b)

Element's d.o.f. in the local CSYS

$$\{\boldsymbol{d}'\} = [0 \quad 0 \quad \theta_1 \quad u_2 \quad 0 \quad \theta_2]^T$$

$$\theta_1 = D_3 - (\beta - \beta_0)$$

$$\theta_2 = D_6 - (\beta - \beta_0)$$

$$u_2 = L - L_0 = (L - L_0)\frac{(L + L_0)}{(L + L_0)} = \frac{L^2 - L_0^2}{L + L_0} =$$

$$= \frac{1}{L + L_0}\left[(2x_0 + D_{41})D_{41} + (2y_0 + D_{52})D_{52}\right] = \frac{1}{2L_0}\cdots$$

$$u \ll L_0$$

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Nodal forces $\{R_{ext}\}$ must equilibrate the elastic reaction at nodes (nodal loads) in each element

$$\{\mathbf{r}'\} = [\mathbf{k}']\{\mathbf{d}'\}$$

$$[\mathbf{k}] = [\mathbf{T}]^T[\mathbf{k}'][\mathbf{T}] \quad \text{and} \quad \{\mathbf{r}\} = [\mathbf{T}]^T\{\mathbf{r}'\}$$

[T], depends on β

$$[k'] = \begin{array}{c} \left[\begin{array}{cccccc} X & 0 & 0 & -X & 0 & 0 \\ 0 & Y_1 & Y_2 & 0 & -Y_1 & Y_2 \\ 0 & Y_2 & Y_3 & 0 & -Y_2 & Y_4 \\ -X & 0 & 0 & X & 0 & 0 \\ 0 & -Y_1 & -Y_2 & 0 & Y_1 & -Y_2 \\ 0 & Y_2 & Y_4 & 0 & -Y_2 & Y_3 \end{array}\right] \end{array} \begin{array}{c} u_1 \\ v_1 \\ \theta_{z1} \\ u_2 \\ v_2 \\ \theta_{z2} \end{array}$$

Suppose we have reached equilibrium after a previous iteration

$$[\mathbf{K}_t] = \sum[\mathbf{k}] \quad \text{and} \quad \{\mathbf{R}^{\text{int}}\} = \sum\{\mathbf{r}\}$$

Now, a new load vector $\{R^{ext}\}$ is applied.

Iterations continue until the load imbalance is driven toward zero

$$\{\Delta\mathbf{D}\}_{i+1} = [\mathbf{K}_t]_i^{-1}\left(\{\mathbf{R}^{\text{ext}}\} - \{\mathbf{R}^{\text{int}}\}_i\right)$$

At each iteration, a tangent stiffness matrix $[K_t]$ is calculated.

$$\{\Delta \mathbf{D}\}_{i+1} = [\mathbf{K}_t]_i^{-1}\left(\{\mathbf{R}^{\text{ext}}\} - \{\mathbf{R}^{\text{int}}\}_i\right) \qquad \{\mathbf{D}\}_{i+1} = \{\mathbf{D}\}_i + \{\Delta \mathbf{D}\}_{i+1}$$

$$[\mathbf{K}_t] = \sum[\mathbf{k}] \quad \text{and} \quad \{\mathbf{R}^{\text{int}}\} = \sum\{\mathbf{r}\}$$

At each iteration, we seek to reduce the load imbalance, until convergence is achieved within the load increment step ([K] is the displacement dependent stiffness matrix)

$$\{\mathbf{e}_R\} = \{\mathbf{R}\} - [\mathbf{K}]\{\mathbf{D}\}$$

Force convergence: $\|\mathbf{e}_R\| < \varepsilon_R \|\mathbf{R}\|$

Displacement convergence: $\|\Delta \mathbf{D}\| < \varepsilon_D \|\Delta \mathbf{D}_0\|$

$$\|\mathbf{R}\| = \sqrt{\{\mathbf{R}\}^T\{\mathbf{R}\}}$$

$\|\Delta \mathbf{D}_0\|$ is the initial increment, at the beginning of the step

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