5. GRONWALL'S INEQUALITY

5.1 Theorem Let μ be a Borel measure on $[0, \infty)$, let $\varepsilon \geq 0$, and let f be a Borel measurable function that is bounded on bounded intervals and satisfies

(5.1) $$0 \leq f(t) \leq \varepsilon + \int_{[0, t)} f(s)\mu(ds), \qquad t \geq 0.$$

Then

(5.2) $$f(t) \leq \varepsilon e^{\mu[0, t)}, \qquad t \geq 0.$$

In particular, if $M > 0$ and

(5.3) $$0 \leq f(t) \leq \varepsilon + M \int_0^t f(s)\, ds, \qquad t \geq 0,$$

then

(5.4) $$f(t) \leq \varepsilon e^{Mt}, \qquad t \geq 0.$$

Source: Pg 498 of Markov processes characterisation and Convergence, by Ethier & Kurtz, 2005, WILEY.

Lemma A.1: $p \geq \frac{2B}{B-1}$, $m \in \mathbb{N}$.

The 1st ineq. holds since (well $t \geq \tau_N$) is obvious, and $t \geq \tau_N \Rightarrow t - k\Delta \leq t \in \tau_N$, & $t_n \leq t < \tau_N$.

Next, note

$$\left| \sum_{i=1}^{n} \{X_i\} \right|^p \leq n^{p-1} \sum_{i=1}^{n} |X_i|^p$$

Thus

$$E|Y^N|^p \leq C|X_0|^p + CE\left|\sum c_j \dots\right|^p$$

$$+ CE| \quad |^p \dots$$

Note $MAX(c_j e^{-\gamma_j k\Delta})$ bounded unif in n. Thus takin outside and then use Jensens (keep p outside). Note the $\|\cdot\|_1$ norm. The result follows (eq of norms).

For the BDG inequality note that

$$* \equiv \int_{t-k\Delta}^{t} K(t-s)\,\sigma(s_n^-, \hat{X}^N_{M,n}(s_n^-))\,dW_s = \int_0^t \mathbb{1}_{(s \geq t}$$

Define then for $s \in [0,t]$ (t fixed now)

$$\varphi_s \equiv \int_0^s \mathbb{1}_{(u \geq t-k\Delta)} K(t-u)\,\sigma(u_n^-, \hat{X}^N_{M,n}(u_n^-$$

which is then a local martingale (Zero

Note that

$$\varphi_t = \int_{t-k\Delta}^{t} K(t-s)\,\sigma(s_n^-, \hat{X}^N_{M,n}(s_n^-))\,d$$

Now see that

$$* \leq E\Big(\sup_{0 \leq s \leq t} |\varphi_s|^p\Big) \overset{\text{BDG}}{\leq} C_p E$$

$$= C_p E\Big(\Big[\int_{t-k\Delta}^{t} K^2(t-s)\,\sigma^2(s_n^-$$

$$\equiv **$$

~~Recall now linear growth $\sigma^2(t,x) \leq$~~

~~Then~~

Recall now Hölder:

$$\|fg\|_1 \le \|f\|_q \|g\|_p \quad , \quad p^{-1} + q^{-1} = 1 \quad , \quad p, q \ge 1.$$

Then

$$\int_{t-n\Delta}^{t} K^2(t-s)\,\sigma^2(\cdot,\cdot)\,ds \le \left(\int_{t-n\Delta}^{t} |K|^{2\beta}(t-s)\,ds\right)^{1/\beta} \Big(\underbrace{\int |\sigma|^{2\beta^*}(\,)}_{\text{Finite too?}} \Big)^{1/\beta^*}$$

Here we have used K is $L^{2\beta}$ integrable and $\beta^{-1} + (\beta^*)^{-1} = 1$.

Then

$$\ldots \le \left(\int |K|^{2\beta}(\,)\right)^{p/2\beta} \left(\int |\sigma|^{2\beta^*}(\,)\right)^{p/2\beta^*}$$

Now use $|\sigma|^2 \le C + C|x|^2$ and ~~$\beta^* \ge 1$~~ and $p \ge 2\beta^* = \frac{2\beta}{\beta-1}$.

$$\ldots \le \underbrace{\left(\int |K|^{2\beta}\right)^{p/2\beta}}_{\le C} \left(\int (C + C|x|^2)^{\beta^*}\right)^{p/2\beta^*}$$

$$\le C \left(\int (C + C|x|^2)^{p/2}\right)$$

$$= C\left(\int |x|^p\right)$$

~~because~~

~~$p \ge 2\beta$~~

and $\beta^* > 1$

as $\beta > 1$.

and $p \ge 2$.

That $L^p \Rightarrow L^q$ follows from Jensens inequality. Makes sense if finite measure space. Lower ~~$\|\cdot\|_{L^p}$~~ part est by $\mu([0,1]) = 1$.

$1 \leq p < q \leq \infty$, $L^p(\mu) \supseteq L^q(\mu)$ if $\mu(X) < \infty$.
Thus it holds

For the middle term in A5:

$C E \| \int_0^t \gamma^M \cdot \hat{U}^N_{M,n}(s_n^-)\, ds \|^p$ can treat single entry

if max norm is considered. ~~Thus, replacing $|\cdot|$ abs val, then~~ just use every norm is convex, thus may take it inside

$$\leq C\, E\left(\left(\int_0^t \|\gamma^M \cdot \hat{U}^N_{M,n}(s_n^-)\|\, ds\right)^p\right)$$

Jensens $\leq C E\left(\int_0^t \underbrace{\|\gamma^M \cdot \hat{U}^N_{M,n}(s_n^-)\|^p}_{\text{views as } \|\cdot\|, \text{ then take max } |\gamma| \text{ outside}} ds\right)$

$\leq C E \int_0^t \|\hat{U}^N(s_n)\|^p ds$ Done!

The argument for bounding (A2) works

Importantly integrable

For (A5):

Jensens

$$E\|\hat{U}_{M,n}(t)\|^p \leq CE\|\int_0^t b(\ )ds\|^p + CE\|$$

$$+CE\|\int_0^t \sigma(\ )ds\|^p.$$

So why is Jensen's valid for non
Could use the triangle inequality first
w. p'th exponent.

The first b-term can be est
of single entry. Same with σ-te
are then est. as before using L
also BDG. For the middle term s
Outside. Then we get $E\|\hat{U}_{M,n}$

In the above we did <u>not</u> cover
Note here that

$$f_N(t) = \sup_{0 \le s \le t} E|\hat{X}^N_m(s)|^p \ldots$$

then from

$$E|X^N(t)|^p \le C + C\int_0^t \underbrace{E|X^N(s_n^-)|^p ds}_{\text{note increasing } t} + C\int$$

$\Rightarrow$

$$\sup_{s \le t} E|X^N(s)|^p \le C \ldots$$

$$\Rightarrow \underbrace{\text{— '' —}}_{= f_N(t)} \le C + C\int_0^t f_N(s)\,ds + \ldots$$

Similarly

$$\Rightarrow \quad g_N(t) \le C + C\int f_N + \int g_N$$

$$\Rightarrow \quad h_N(t) \le C + C\int g h_N(s)\,ds$$

Gronwall, Let $M > 0$ and $(\varepsilon \geq 0)$

$$0 \leq f(t) \leq \varepsilon + M \int_0^t f(s)ds, \ t \geq 0$$

w. f bounded on bounded intervals ~~k~~ then

$$f(t) \leq \varepsilon e^{Mt}.$$

Note in our case that

$$\mathrm{MAX}(|X^N(t)|, \|\hat{U}^N_{mn}(t)\|) \leq N$$

for all t thus so will p expectations and even w. sup in front. Thus, f_N, g_N and thus h_N is unif. bounded. Then

$$\|h_N(t)\|_\infty \leq C \quad \forall t \text{ bounded.}$$

<u>FATOU:</u> f_n non neg. (M) then
$\int_X f d\mu \leq \liminf \int_X f_n d\mu$

In our case

$$\mathrm{MAX}(f_N(t), g_N(t)) = \mathrm{MAX}(\sup_{s \leq t} E|X^N_{mn}(s)|^p,$$

$$\sup_{s \leq t} E\|\hat{U}^N_{mn}(s)\|^p)$$

Then

liminf MAX(. . .) (since always exist for both entries) right?

* Then note

$$\liminf \sup_{s \le t} E|\hat{X}^N(s)|^p$$

How does liminf and sup go together?

Note $f_n(x) \le \sup_{x \in A} f_n(x)$

$$\Rightarrow \liminf_{n \to \infty} f_n(x) \le \liminf \underbrace{\sup_{x \in A} f_n(x)}_{\text{Indep of } x!}$$

$$\Rightarrow \sup_{x \in A} \liminf f_n(x)$$

So bound down by taking inside!

...

$$\ge \sup_{s \le t} \liminf E|\ |^p$$

FATOU $\ge \sup E(\liminf |\ |^p)$ use $\|\cdot\|_1$ norm for U!

Since $1_{()} \to 1$ a.s as num. sol finite $\to = \sup E|\hat{X}_{min}(t)|^p$

Lemma A.2:

Sensus is dun
(+ trian.) then use elem by elem. rules.
then LG and moment bounds indp of
n, t.

Focus now on $E|\int h \circ dW|^p$ est:

* BDG, then do $\sigma^2(\,) \le C + C|x|^2$ and take $p/2$ into each term. Then:

Use Hölder w. (β^*, β) on K^2 and 1!

$$\dots \le C \, \mathbb{E}\left[\left(\int K^{2\beta} ds\right)^{1/\beta} \left(\int ds\right)^{1/\beta^*}\right]^{p/2}$$

Similarly

$$\left(\int K^2 X^2 ds\right)^{p/2} = \left[\left(\int K^{2\beta} ds\right)^{1/\beta} \left(\int X^{2\beta^*} ds\right)^{1/\beta^*}\right]^{p/2}$$

$$\le C \left(\int K^{2\beta} ds\right)^{p/2\beta} \left(\int ds\right)^{p/2\beta^*}$$

$$+ C \left(\int K^{2\beta} ds\right)^{p/2\beta} \left(\int X^{2\beta^*} ds\right)^{p/2\beta^*}$$

common

Note also

$$\frac{p}{2\beta^*} \ge 1 ?$$

and

$$\frac{p}{\beta^*} \ge 1 ?$$

$$p \ge \frac{2\beta}{\beta - 1} = 2\beta^*$$

by ass. So indeed

$p/2\beta^* \ge 1$ of course then

$$\frac{p}{\beta^*} \ge 2 \ge 1$$

<u>Thm A.3:</u> Should I change K_m in the definition? Did define earlier on though... No wait is ok actually!

<u>(A8) - (A9):</u>

$$\left|\frac{\sum x_i}{n}\right|^p \le \frac{\sum |x_i|^p}{n}$$

$$\Rightarrow \left|\sum x_i\right|^p \le n^{p-1} \sum |x_i|^p \quad (n=2 \text{ here})$$

~~Then note~~ To I_3, note

~~$E|I_3|^p$~~ $|K_{m,n} - K| \le |K - K_m|$ is obvious by construction

<u>For I_4:</u>

Why $\|K_{m,n}\|_{L^{2p}} \le C$?

Similar guarantees for I_1, I_2: Use $L_{2p} \Rightarrow L_p$ (also wrt. conv.) Only thing beg that we end up with

$\|K - K_{m,n}\|_{L^p}^p$ but this is then

bounded by $\|K_{m,n} - K\|_{L^{2p}}^p$ so ok.

~~Before~~ Gronwall in A.10 ok since $E X_i^q, E X_{m}^p$ is sup bounded by lemma!

Now turn to term in (A.9) i.e.

$$E | X_{m,n}(t) - \hat{X}_{m,n}(t) |^p$$

Start by looking at U difference!

note C_2 dep. on m.

$E \| I, \sigma \|^p$

~~But wait, should it not be $n^{-\lambda}$ just? yep!~~

~~$E | X - X |^p \leq C n^{-1}$ C indep of n, t, t'~~

Well Hölder cont part is ok

$$(A.21) \quad E | \underbrace{Z^x_{m,n}(t)}_{= X_{m,n} - \hat{X}_{m,n}} |^p \leq C E \| U_{mn} - \hat{U}_{mn} \|^p$$

$$+ E \| \int b K(\ ) (b(\ )_{mn} - \hat{b}(\ )_{mn}) ds \|$$

$$+ E \| \quad n$$

Do need to handle K by Hölder right? But both uses K_{mn} so that is ok right? Or rather just K actually...

$\boxed{\text{then too Lipschitz.}}$

$$Z^X_{M,n} = \sum_j c_j e^{\cdot\cdot} \left(\hat{U}_{m}(t-n\Delta) - U_m(t-n\Delta) \right)$$

$$+ \int_0^t K(t-s) \big(b(s_n^-, \hat{X}_m(s_n^-))$$

$$- b(s, X_m(s)) \big) ds$$

$$+ \int_0^t K(t-s) \big(\sigma(s_n^-, \hat{X}_m(s_n^-))$$

$$- \sigma(s, X_m(s)) \big) dw_s$$

Is it the Same θ? (May change?)

Gronwall ok as (?) boundedness of Z^X obvious by Lemma. ~~tw~~. above and lemma

Then decomp. in both time and VAR. Then use ~~Lipschitz~~ Hölder drives on A.1.

K, Lipschitz and Hölder cont and previous lemmas (need BDG too).

Similarly, θ by Lemma A.1 for $U_{m,n}$ note that (what?).

Just say θ may change too?

Constants obviously still dont depend on n!

Why Gronwall ok? Dont need Fatou though.

Rearrange so

$$|X_m - \int ds - \int dw| = |\sum U \ldots|$$

Yeah, it is not obvious?... However is ok using BDG and Moment bound of X and LG. (yes!). (Circular no?)

no since $U = \int e^{-\gamma(t-s)} (ds + dw)$ so is ok.

Remove otherstuff? ~7 lines!

Reintro a couple of eqn?

Remove (3.1)?

No...

Remove eqn w

local Gaussian approx?

Double check Thm 4.1!

Remove Blank space under plot.

How L_p vs. L_q ok in our proof? Need to be precise exercise says

$$C_q \| u \|_{L_q} \leq C_p \| u \|_{L_p} \qquad 1 \leq q < p < \infty.$$

Specifically we have $X(t) - \hat{X}_{M,N}(t) \in L^p$, $p \geq \frac{2\beta}{\beta-1}$

Then

$$\underbrace{C_q \left(E \left| X(t) - \hat{X}_{M,N}(t) \right|^{q} \right)^{1/q}}_{\|\cdot\|_{L_q}} \leq C_p \left(E \left| X(t) - \hat{X}_{M,N}(t) \right|^{p} \right)^{1/p}$$

Then take sup on both sides and take limits the result follows.

Consider a finite $\mu(X) < \infty$ measure space

apply Hölder to $|f|^q \times 1$ w. exponents

p/q , $p/(p-q)$ to get

$$\| |f|^q \cdot 1 \|_1 \le \cancel{\|f\|_{p/q}} \left(\int |f|^p d\mu \right)^{\cancel{p/q}\, q/p}$$

$\cancel{\text{R}}$ $= \|f\|_q^q$

$$\|f\|_p \cdot \underbrace{\mu(X)^{\frac{p-q}{q}}}_{< \infty}$$

then it follows

Comment on Kloeden:

see pg 577 ch 14

Use word "can".

Why does her update scheme looks different?

$$dU = (-\gamma U - \lambda V)dt + \nu\sqrt{V}dW$$

Why the fuck?

Added to g_0? (Think sooo)

yep, indeed!