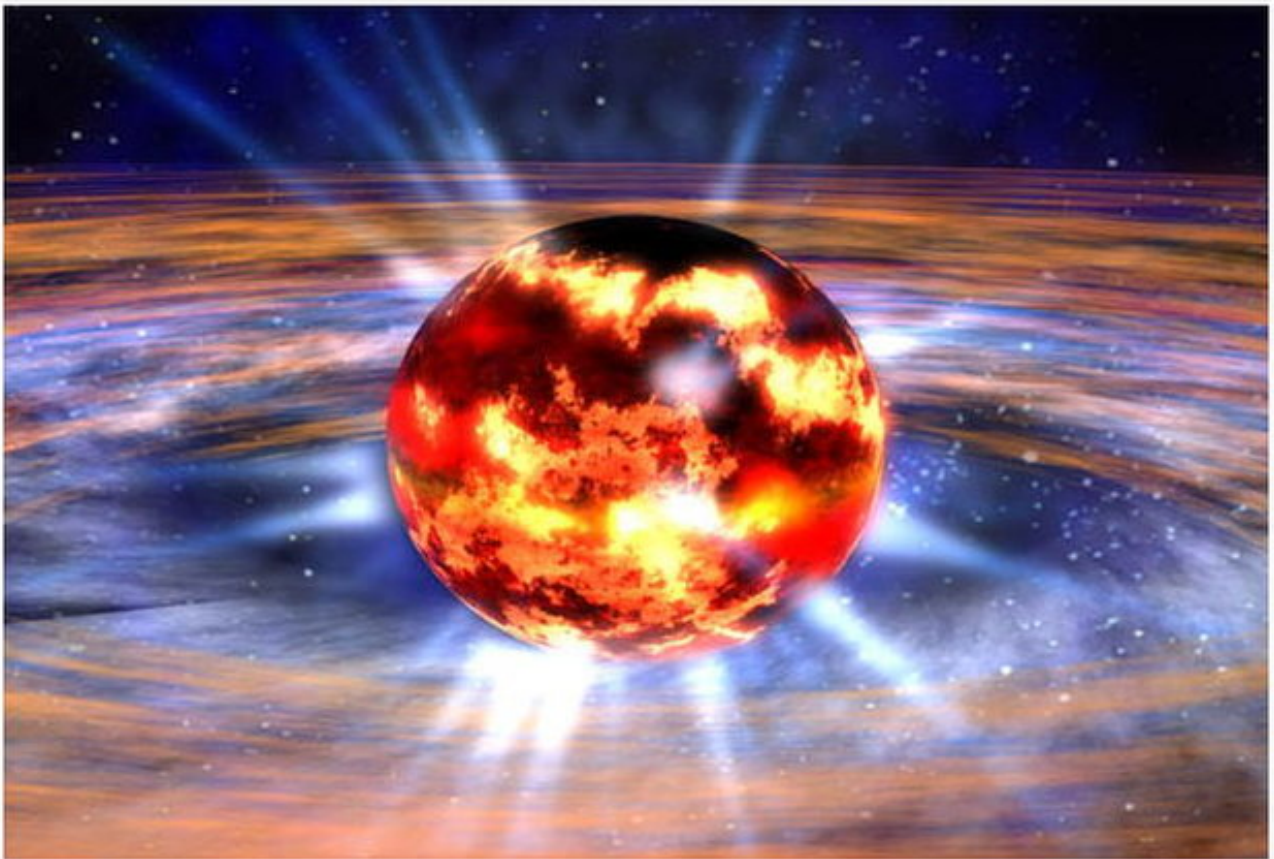# AST2000 Lecture Notes

Part 1G
Quantum gases

Questions to ponder before the lecture

1. If hydrostatic equilibrium in a star is lost and gravitational forces are stronger than pressure, what will happen with the star?

2. Do you know what a degenerate gas is?

A neutron star consists of degenerate quantum gas (Illustration: NASA/Dana Berry)

AST2000 Lecture Notes

Part 1G
Quantum gases

We will finish part 1 where we started: statistics and gas dynamics. We have so far looked at and used statistics for ideal gases. In part 3 we will learn the different stages in stellar evolution. In the final stages of stellar evolution, the stellar core (and sometimes shells around the core) becomes extremely compact with densities of the order $10^9 \mathrm{kg/m^3}$ and more. At these extreme densities, the ideal gas approximation is not valid. Strange phenomena caused by quantum mechanical processes appear and the gas aquires properties which makes it behave very different from a normal ideal gas. The first purpose of this chapter is to give you a rough introduction to the exciting world of quantum physics by extending the statistics we learned in part1A. The second purpose is to deduce the equations governing gases at these extreme densities that we will need for part 3.

1 Distribution functions

In part 1A, we learned the distribution function $P(v)$ for ideal gas giving the probability for a gas particle to have a velocity v. We expressed this also as the number density of particles per velocity, $n(v) = P(v)n$. Finally, we transformed this, using $v = p/m$, to number density per momentum, $n(p)$ giving the number of particles per volume with a given momentum. In order to get the number of particles per volume within a tiny momentum interval $[p, p + dp]$ you need to multiply $n(p)$ by the momentum interval dp. Before reading on, go back to part 1A and make sure you remember how these are defined and used.

By making the substitution $E = p^2/(2m)$ we can obtain the distribution function $n(E)$ giving the number density of particles having a certain energy E. Make sure here that the total number density for a given interval is conserved, i.e.

$$n(p)dp = n(E)dE.$$

It is important to include the intervals dp and dE here when making the transformation otherwise total number density is not conserved: we are looking at exactly the same particles in both cases, we just change the description from p to E and in order to work with actual particle number densities, we have learned that we need to multiply $n(p)$ and $n(E)$ by dp and dE. This has its origin in the probability distribution which needs to be multiplied with these intervals in order to represent actual probabilities (go back to part 1A now if this is not clear). We therefore need to express dp in terms of dE and then insert $p = \sqrt{2mE}$ everywhere. Taking the derivative

$$\frac{dE}{dp} = \frac{p}{m},$$

we get $dp = \frac{m}{p}dE$. We therefore arrive at

$$n(E)dE = \frac{2n}{\sqrt{\pi}(kT)^{3/2}} E^{1/2} e^{-\frac{E}{kT}} dE,$$

Check that you manage to arrive at this equation yourself.

In part 1A, we used the distribution functions $n(\vec{v})$ and $n(\vec{p})$ for the velocity and momentum vectors as well as the distribution functions $n(v)$ and $n(p)$ for the absolute value of velocity and momentum. We will start this part by deducing how you can from the vector distribution to the distribution for the absolute value.
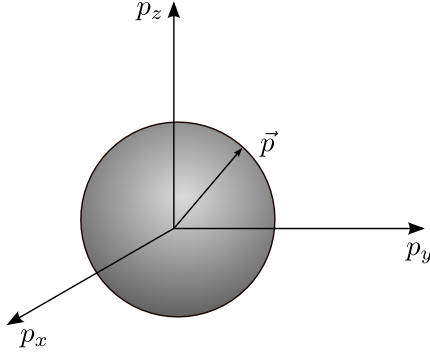
Figure 1: Momentum space: All particles with $|p|$ within $[p, p + \Delta p]$ are located on the thin shell of thickness Δp at radius $|p|$.

The Maxwell-Boltzmann distribution function for the momentum vector of particles in an ideal gas is

$$n(\vec{p}) = n \left(\frac{1}{2\pi mkT} \right)^{3/2} e^{-p^2/(2mkT)},$$

where n is the total number of particles per volume. This is the density $n(\vec{p})$ of particles with momentum $\vec{p}$. We will use this expression to find the number density $n(p)$ of particles with absolute momenta p. This corresponds to all particles with a momentum vector $\vec{p}$ such that the length of the vector is p and with an arbitrary direction. We therefore need to integrate over all particles with all possible angles of the vector $\vec{p}$ keeping the absolute value fixed.

We can imagine that we have a momentum space, i.e. a three dimensional space with axes p_x, p_y and p_z (see figure 1). All possible momentum vectors $\vec{p}$ are vectors pointing to a coordinate (p_x, p_y, p_z) in this momentum space. All particles which have an absolute value p of their momentum $\vec{p}$ are located on a spherical shell at distance p from the origin in this momentum space. Thus we may imagine a particle to have a position in the six dimensional position-momentum space $(\vec{x}, \vec{p})$. All particles have a position in real space (x, y, z) and a position in momentum space (p_x, p_y, p_z). In this 6-dimensional space, the distribution $n(\vec{p})$ is a 6-dimensional number density: the number of particles per real space and momentum space volume.

All particles with momentum between p and $p+dp$ are located on a thin shell of thickness dp at a dis-

tance p from the origin in momentum space. The total volume of this shell is $4\pi p^2 dp$ (where did this expression come from?). Thus, to obtain the total number of particles within this momentum range, we need to multiply the distribution with the momentum space volume $4\pi p^2 dp$,

$$n(p)dp = n \left(\frac{1}{2\pi mkT} \right)^{3/2} e^{-p^2/(2mkT)} 4\pi p^2 dp.$$

This is the distribution function for absolute momenta p that we already know. Note that whereas $n(\vec{p})$ has dimensions number density per real volume and momentum space volume (6-dimensional density), $n(p)dp$ has dimensions number density per real volume and is therefore the normal 3-dimensional number density. The latter follows from the fact that we have simply multiplied $n(\vec{p})$ with a volume in momentum space $(4\pi p^2 dp)$ to obtain $n(p)dp$.

We have, without knowing it, encountered another distribution function in this course. The Planck distribution. The Planck distribution is the number density of photons within a given frequency range

$$B(\nu) = \frac{2h\nu^3}{c^2} \frac{1}{e^{h\nu/(kT)} - 1}.$$

When you have taken courses in quantum mechanics and thermodynamics you will deduce two more general distribution functions. When taking quantum mechanical effects into account it can be shown that the distribution function for fermions (from part1F we know that fermions are particle with half integer quantum spin like the electron, proton or neutron) and bosons (again, from part 1F we know that bosons are particles with integer quantum spin like the photon) can be written generally as

$$n(E) = \frac{g(E)}{e^{(E-\mu_C)/(kT)} \pm 1}, \qquad (1)$$

where μ_C is the *chemical potential* and $g(E)$ is the *density of states* which we will come back to later. Here the minus sign is for bosons and the plus sign for fermions. In the limit of low densities it turns out (we will not show it here) that the exponential part dominates and the distribution function

becomes equal for fermions and bosons. In this case the chemical potential has such a form that we get back the Maxwell-Boltzmann distribution function (compare with the above expression for E). Note that the expression for bosons resembles the Planck function (can you see this?): the Planck function can be derived from the distribution function for bosons (you will do this in later courses).

2 Degenerate gases

In the core of stars, the fermions, i.e. the electrons, is the dominating species. Therefore we will here study the distribution function for fermions and use the $+$ sign in the above equation. We will look at an approximation of the distribution function for an electron gas at low temperature. Of course, the temperature in the core of a star is not particularly low, but we will later show that the same approximation and results are valid even for high temperatures provided we are in the high density limit.

In the low temperature limit it can be shown that the chemical potential μ_C equals the so-called *Fermi energy* E_F. We will later find an expression and physical interpretation for the Fermi energy, but for now consider it a number which has units Joule. We will first consider the distribution function for fermions (in our case, electrons) in this low temperature limit for which expression 1 reduces to

$$n(E) = \frac{g(E)}{e^{(E-E_F)/(kT)} + 1} \tag{2}$$

where

$$g(E) = 4\pi \left(\frac{2m_e}{h^2}\right)^{3/2} E^{1/2}, \tag{3}$$

where m_e is the electron mass. The number of electrons per volume with an energy between E and $E + dE$ in a gas with temperature T is now given by $n(E)dE$. The energy E of an electron may be larger or smaller than the Fermi energy E_F.

We will now measure the energy of an electron in the gas in units of the Fermi energy. We define $x = E/E_F$ such that $x < 1$ when the energy of an electron is less than the Fermi energy and $x > 1$

when the energy is larger than the Fermi energy. The distribution function as a function of x, the energy in units of the Fermi energy, can thus be written

$$n(x) = \frac{g(x)}{e^{(x-1)E_F/(kT)} + 1}.$$

In the very low temperature limit, $T \to 0$, the factor $E_F/(kT)$ is a very large quantity. The energy x defines whether the number in the exponential is a large positive or a large negative quantity.

For electrons with $x > 1$, i.e. that the energy is larger than the Fermi energy, then the number in the exponential is a large positive number and $n(x) \to 0$. There are therefore almost no electrons with $E > E_F$ for very low temperatures. For electrons with $x < 1$, i.e. the energy is less than the Fermi energy, the number in the exponential is a large negative number. Thus the exponential goes to zero and $n(x) \to g(x)$. So for very low temperatures, there is a sharp limit at $x = 1$. For $E < E_F$ we find $n(x) = g(x)$ whereas for $E > E_F$ we find $n(x) = 0$. In figure 2 we show $n(x)/g(x)$ for lower and lower temperatures.
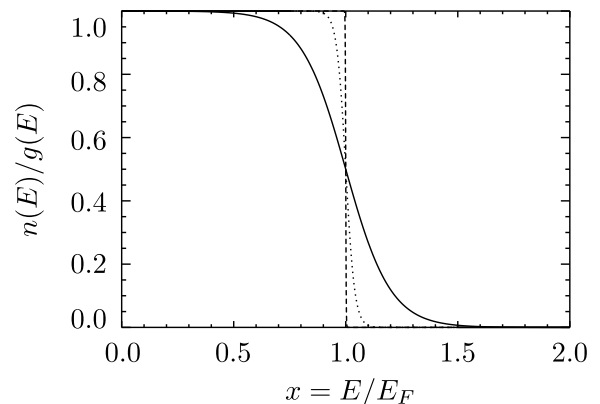

Figure 2: The number of electrons $n(E)$ divided by $g(E)$ for different energies E. The solid line is for a gas at temperature $T = 10$ K, the dotted line for a gas at temperature $T = 2$ K and the dashed line for $T = 0.1$ K. When the temperature approaches zero, there are less and less electrons with energy larger than the Fermi energy E_F.

The physical meaning of this is that for very low temperatures, all the electrons have energies up to the Fermi energy whereas no electrons have energies larger than the Fermi energy. The Fermi energy is a low temperature energy limit for the electrons. Even if we cool an electron gas down to zero temperature, there will still be electrons

having energies all the way up to the Fermi energy.

But if the temperature is zero, why do not all electrons have an energy close to zero? Why don't all electrons go and occupy the lowest possible energy state allowed by quantum mechanics (in quantum mechanics, a particle cannot have zero energy)? The reason for this is hidden in quantum physics: at low temperatures the gas of electrons start to behave like a quantum gas, a gas where quantum mechanical effects are important. The quantum mechanical effect which we see on play here is the Pauli exclusion principle: Two fermions cannot occupy the same energy state.

To understand this principle we need to dig even deeper into the quantum theory. According to quantum mechanics momentum is quantized. This means that a particle cannot have an arbitrary momentum. The momentum in any direction can be written as

$$p_x = N_x \cdot p_0,$$

where N_x is an integer quantum number and p_0 is the lowest possible momentum. Thus, an electron can only have x-momenta p_0, $2p_0$, $3p_0$ etc. No values in between are allowed. So the total momentum of an electron (or any particle) can be written

$$p^2 = p_x^2 + p_y^2 + p_z^2 = p_0^2(N_x^2 + N_y^2 + N_z^2) \equiv p_0^2 N^2,$$

where (N_x, N_y, N_z) are the three quantum numbers defining the state of the electron. According to the Pauli exclusion principle only one electron can occupy the quantum state (N_x, N_y, N_z). No other electrons can have exactly the same combination of quantum numbers (this is actually a strong simplification, two electrons (but maximum two) may occupy the same state if they have different quantum spin, but we will neglect this here to make the description simpler).

We go back to the above image of a momentum space where a particle has a position (p_x, p_y, p_z) in a three dimensional momentum space in addition to a position in normal space. We can now write this position in terms of quantum numbers as $(p_x, p_y, p_z) = p_0(N_x, N_y, N_z)$. Since only one electron can have a given momentum $p_0(N_x, N_y, N_z)$, one could imagine the momentum space filled

with boxes of volume $p_0 \times p_0 \times p_0$. Only one electron fits into each box. We remember that all electrons with momentum lower than a given momentum p is within a sphere with radius p in this momentum space. All electrons with a higher momentum p are outside of this sphere. But inside the sphere of radius p, there is only room for $4/3\pi N^3$ boxes of size p_0^3 (total volume of the momentum space sphere $(4/3)\pi p^3 = (4/3)\pi p_0^3 N^3$ divided by volume of box p_0^3). If all these boxes are filled, no more electron may settle on a position inside this sphere, it has to remain outside of the sphere.

When you lower the temperature of an electron gas, the electrons loose momentum and start to occupy the lowest possible momentum states, i.e. they all start to move towards the origin $(0, 0, 0)$ in momentum space. But when all start to move towards the origin in momentum space (see again figure 1), all the boxes around the origin are soon occupied, so the electrons need to remain with higher momenta at larger distances p from the origin. But if they need to remain with larger momenta, this means that they also have larger energy: The same argument therefore applies to energy. The energy states of the electrons are quantized so not all electrons may occupy the lowest energy state. For this reason we see that the distribution function for electrons at low temperatures is a step function: All electrons try to occupy the lowest possible energy state. The lowest energy states are filled up to the Fermi energy. If we call p_F the Fermi momentum, the momentum corresponding to the Fermi energy we can imagine that all electrons start to gather around the origin in momentum space out to the radius p_F. All electrons are packed together inside a sphere of radius p_F in momentum space. When you add more electrons to the gas, i.e. the density of electrons increases, the sphere in momentum space inside which all the electrons are packed also needs to expand and the Fermi momentum p_F increases. Thus the Fermi momentum and the Fermi energy are functions of the electron density n_e.

Having learned that for very low temperatures, the electrons are packed together in momentum space in a sphere of radius p_F we can find the total number density (per real space volume) n_e

of electrons in the gas by summing up all the boxes of size p_0^3 inside this sphere. We know that all these boxes are occupied by one electron and that no electrons are outside this sphere (this is completely true only for $T = 0$). First we need to know the fermion distribution function $n(\vec{p})$ in terms of momentum rather than in terms of energy which we used above. The fermion distribution function for momentum can be written in the low temperature limit as

$$ n(\vec{p}) = \frac{1}{e^{(p^2 - p_F^2)/(2mkT)} + 1} \frac{2}{h^3}. $$

This is the number density per volume in real space per volume in momentum space. Considering again the low temperature case, we see, using the same arguments as before, that $n(\vec{p}) \to 0$ for $p > p_F$ and $n(\vec{p}) \to 2/h^3$ for $p < p_F$. Thus $n(\vec{p})$ is a constant for $p < p_F$ and zero for $p > p_F$. In order to obtain the number density of electrons per real space volume we need to integrate this expression over the momentum space volume. So for $T \to 0$

$$ n_e = \int_0^\infty n(\vec{p})4\pi p^2 dp = \int_0^{p_F} \frac{2}{h^3} 4\pi p^2 dp = \frac{8\pi}{3h^3} p_F^3 $$

where we integrate over the sphere in momentum space in shells of thickness dp out to the Fermi momentum p_F where $n(\vec{p})$ is a constant $(2/h^3)$ for $p < p_F$ and is zero for $p > p_F$. Make sure you understood this derivation! We use this result to obtain an expression for the Fermi momentum

$$ p_F = \left(\frac{3h^3 n_e}{8\pi} \right)^{1/3}. \tag{4} $$

Using the non-relativistic expression for energy we can now find the Fermi energy expressed in terms of the electron number density n_e

$$ E_F = \frac{p_F^2}{2m_e} = \frac{h^2}{8m_e} \left(\frac{3n_e}{\pi} \right)^{2/3}. \tag{5} $$

As we anticipated, the Fermi energy depends on the density of electrons. The higher the density, the larger the Fermi energy and the Fermi momentum in order to have space for all the electrons within the sphere of radius p_F. A gas where all particles are packed within this sphere so that

the particles are fighting for a box in momentum space among the lowest energy states is called a *degenerate gas*. A partially degenerate gas is a gas where there are still a few vacant boxes among the lowest energy states such that some particles have energies larger than the Fermi energy. We now need to find a criterion for when a gas is degenerate.

When the temperature of a gas is high and the density low, the distribution function is the Maxwell-Boltzmann distribution function. We have previously learned that for a gas following the Maxwell-Boltzmann distribution function, the mean energy per particle is $< E > = (3/2)kT$. The gas starts to become degenerate when most of the particles have energies below the Fermi energy. The gas therefore starts to be degenerate when the mean energy of the particles go below the Fermi energy. For an electron gas we thus have the criterion

$$ \frac{3}{2}kT < E_F = \frac{h^2}{8m_e} \left(\frac{3n_e}{\pi} \right)^{2/3}, $$

or

$$ \frac{T}{n_e^{2/3}} < \frac{h^2}{12m_e k} \left(\frac{3}{\pi} \right)^{2/3}. \tag{6} $$

As discussed above, this criterion is satisfied for very low temperatures, but we now see that it is also satisfied for very high densities. In the lectures on stellar evolution you will estimate what kind of densities are needed in the stellar cores for the core to be degenerate.

Now take a deep breath, close your eyes and try to find out how much you have understood from this section. Then if this is not the 3rd time you read it, go back and read again with the goal of understanding a little bit more this time.

3 The pressure of a degenerate electron gas

When the density of electrons in the stellar core becomes high enough, most electrons have energies below the Fermi energy and the above criterion for degeneracy is satisfied. The core is electron degenerate. Now we will study the properties

of a degenerate gas. The equation of state, the equation for the pressure as a function of density and temperature, is one of the most important properties describing how a gas behaves.

In order to find the pressure, we need to evaluate the pressure integral (see part 1A) for the degenerate gas. First we need the density $n(p)dp$ of electrons per volume with momentum p in the interval $[p, p + dp]$. By now we have learned that $n(\vec{p})4\pi p^2 dp = n(p)dp$ such that for $p < p_F$ we have $n(p)dp = (2/h^3)4\pi p^2 dp$ and for $p > p_F$ we have $n(p) = 0$.

$$P = \frac{1}{3}\int_0^\infty p\, v\, n(p)\, dp = \frac{1}{3}\int_0^{p_F} \frac{p^2}{m_e}\frac{2}{h^3}4\pi p^2 dp$$
$$= \frac{8\pi}{3m_e h^3}\frac{1}{5}p_F^5.$$

Inserting the expression for the Fermi momentum (equation 4), we find

$$P = \left(\frac{3}{\pi}\right)^{2/3}\frac{h^2}{20m_e}n_e^{5/3} \qquad (7)$$

We see that the pressure of a degenerate gas does not depend on the temperature. If the temperature increases or decreases, the pressure does not change! This is very different from a normal gas. It means that the degenerate stellar core will not expand or contract as the temperature changes. The only exception being when the temperature increases so much that the condition (6) for degeneracy is no longer valid and the degeneracy is broken. In this case, the electrons have gained so much energy that they are not packed in the sphere of the lowest momentum states in momentum space. The gas is no longer degenerate and a normal equation of state which depends on the temperature needs to be used.

We have deduced the pressure of a degenerate gas using the non-relativistic expressions for energy. The temperature in the stellar cores are often so high that the velocities of the particles are relativistic. Repeating the above deductions using the relativistic expression, we would obtain

$$P = \frac{hc}{8}\left(\frac{3}{\pi}\right)^{1/3}n_e^{4/3}. \qquad (8)$$

4   Summary

We have seen that if we compress a gas of fermions sufficiently, so that the degneracy condition (equation 6) is fulfilled, the fermions are packed together inside a sphere of radius p_F in momentum space. All the lowest energy states of the fermions are occupied up to the Fermi energy E_F. This typically happens when the temperature is very low so that the fermions fall down to the lowest possible energy states in momentum space. It might also happen for high temperatures if the density is high enough: In this case there are so many fermions present within a volume so all fermion states up to E_F are occupied even if the temperature is not particularly low.

A degenerate fermion gas has a degeneracy pressure which is independent of the temperature of the gas given by equation (7) for a non-relativistic gas (the particles have non-relativistic velocities) and by equation (8) for a relativistic gas. This pressure originates from the resistance against being squeezed further together in real and momentum space and only depends on the density of the gas. We obtained the expression for the pressure by inserting the distribution function for a degenerate gas in the pressure integral (part 1A). The distribution function for a degenerate gas took on a particular form: It is a step function being constant for energies below the Fermi energy and zero above. This was simply a consequence of the Pauli exclusion principle, one energy state cannot be occupied by two fermions at the same time. When the quantum states of lowest energy are occupied, the fermions need to occupy states of higher energy. For a completely degenerate gas, the Fermi energy E_F gives within which energy there is room for all fermions at a given density.

If the temperature increases sufficiently, the fermions gain enough energy to occupy states well outside the sphere of radius p_F in momentum space. Then there will be vacant low energy states, the condition of degeneracy is no longer fulfilled and the gas has become non-degenerate following a normal temperature-dependent equation of state.

5   Exercises

Exercise 1G.1 The number density per real space volume per momentum space volume of particles with momentum $\vec{p}$ is given by $n(\vec{p})$ found in the text. In order to find the number density per real space volume of particles with absolute momentum p we multiplied $n(\vec{p})$ with an infinitely small volume element $4\pi p^2 dp$ and obtained $n(p)dp$. Go back to the text and make sure that you understand this transition.

1. Now we will try to find the number density per real space volume of particles with energy E using the non-relativistic formula for energy $E = p^2/2m$. Start with $n(\vec{p})$ for fermions at temperature $T = 0$ (you have to look through the text to find the correct expression), arrive at $n(p)dp$ and then make the substitution to energy to show that you arrive at equation 2 with $g(E)$ looking like equation 3.

2. In the exercises in part 1A, we found that the mean kinetic energy of a particle in an ideal gas is $(3/2)kT$. Now we will try to find the mean kinetic energy in a degenerate gas. First of all, repeat what you did in part 1A. Now you will repeat the same procedure, but use $n(E)$ and E directly,

$$< E > = \int_0^\infty P(E) E\, dE.$$

You will need to find out how $P(E)$ looks like. The answer is

$$< E > = \frac{3}{5} E_F.$$

Hint: Assume a very degenerate gas at very high density.