Questions to discuss during the lectures for part 1B

Important: You also need the lecture notes for part 1B to read together with this guide.
Also note: The lecture guides are made for use during the lectures. They will not make much sense if you are not present at the lecture, in this case, go directly to the lecture notes.

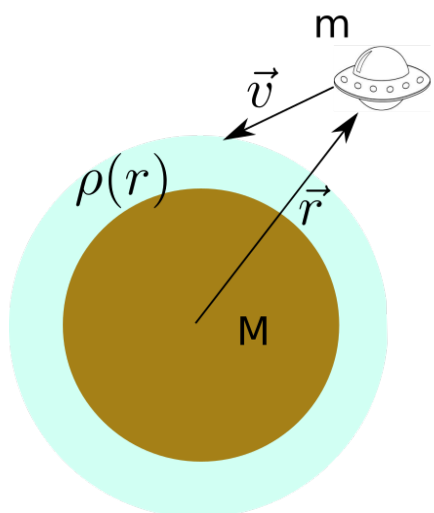

Figure 1: A spaceship with no engines landing on a planet

1. In figure 1 you can see a spaceship approaching the surface of a planet. The space ship has no engine. The position of the space ship is given by $\vec{r}$ pointing from the centre of the planet. Atmospheric friction forces are braking the spaceship as described by the formula in exercise 1B.8. Write the equation of motion of the spaceship: you may only use $\vec{r}$ and its time derivatives as well as the mass m of the spaceship, the mass M of the planet, the density of the atmosphere as a function of distance from the center of the planet, $\rho(r)$, the surface area of the spaceship A as well as natural constants.

2. Rewrite the equation
$$m\frac{d^2\vec{r}}{dt^2} = \vec{F}(\vec{r})$$
as a first order differential equation using the velocity $\vec{v}$.

3. Rewrite $d\vec{v}$ as a finite difference $\Delta\vec{v}$ and dt as a finite time interval Δt. Reorder the equation and use this to find the velocity of the spaceship a time Δt after which it had a velocity $\vec{v}_0$. In addition to the quantities you were allowed to use in question 1, your expression may also include Δt and $\vec{v}_0$.

4. In the same manner, write the definition of $\vec{v}$ in terms of the time derivative of $\vec{r}$ and then rewrite the change in position as a finite change $\Delta\vec{r}$ and express the new position of the spaceship a time Δt after which it was in the position $\vec{r}_0$. Your expression for $\vec{r}$ may contain $\vec{r}_0$, Δt and $\vec{v}$.

5. When implementing this as a numerical iterative scheme to solve the position of the spaceship as a function of the time, you need to write the vector $\vec{r}$ and its derivatives in a vector basis. One possibility is to use the x- and y-coordinate unite vectors $\vec{e}_x$ and $\vec{e}_y$. Another possibility is to use the unit vector $\vec{e}_r$ pointing along $\vec{r}$ as well as the vector $\vec{e}_\theta$ pointing orthgonal to $\vec{r}$. The two possibilities are shown in figure 2 in lecture notes 1B.2. You are probably familiar with the first possibility, so let's discuss the second: The position of the spaceship is given by $r = |\vec{r}|$ as well as the angel θ. Use this to find the constants a and b below:

$$\vec{r} = a\vec{e}_r + b\vec{e}_\theta$$

6. What are the advantages and disadvantages for using either $\vec{e}_x$, $\vec{e}_y$ or $\vec{e}_r$, $\vec{e}_\theta$ when solving for the motion of the spaceship numerically?

7. If we now keep to polar coordinate bases ($\vec{e}_r$ and $\vec{e}_\theta$), we can write the velocity as

$$\vec{v} = v_r\vec{e}_r + v_\theta\vec{e}_\theta.$$

   Given the expression you found for $\vec{r}$ in polar coordinates as well as the definition for $\vec{v}$, how would you go about to find the radial velocity component v_r and the tangential velocity component v_θ expressed only in terms of r, θ and their time derivatives? (you should not yet do the calculation, only find out how you would do it)

8. You have discovered that you need to calculate

$$\dot{\vec{e}}_r = \frac{d\vec{e}_r}{dt}$$

   In order to solve this, you might need to express $\vec{e}_r$ in terms of the fixed vectors $\vec{e}_x$ and $\vec{e}_y$

$$\vec{e}_r = c\vec{e}_x + d\vec{e}_y$$

   find c and d expressed in terms of r and/or θ and use this to find $\dot{\vec{e}}_r$.

9. Show that

$$\vec{v} = \dot{r}\vec{e}_r + \dot{\theta}r\vec{e}_\theta$$

10. Could you have used pure physical reasoning and some geometry to arrive at the expression for v_θ? **HINT:** Use figure 2: looking only at the movement along the θ-direction, suppose the spaceship moves a distance Δs along the arc during the time Δt. How can you use this to express v_θ and how can Δs be written in terms of r and $\Delta\theta$? Assume that $\Delta\theta$ and Δs are tiny.

11. Use the expression for v_θ to show that the spin per mass h, of the spaceship is given by $h = r^2\dot{\theta}$. Spin per mass is defined as the spin divided by the mass of the object

$$h = \frac{\vec{r} \times \vec{p}}{m}$$

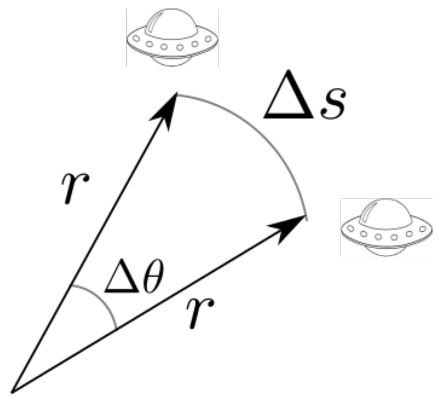

Figure 2: A tiny movement along the θ-direction

12. Going back to the coordinate bases $\vec{e}_x$ and $\vec{e}_y$, how can you calculate the force vector $\vec{F}$ numerically, given that you have defined a vector $\vec{r} = (x, y)$? (exactly how would you code this in python) Assume that you already have defined values for `m`, `M`, `A`, `x`, `y`, `vx`, `vy` (x- and y- components of the velocity) and all natural constants. You also already have a function for `rho(r)`.

13. Do you remember Kepler's 3 laws for the motion of the planets? What about the formula for an ellipse?

14. Discuss how you would solve exercise 1B.1 (the project students have similar questions in challenge B1 in part 2)

15. Discuss how you would solve exercise 1B.2, question 1 (the project students have similar questions in challenge B1 in part 2)

16. Discuss how you would solve exercise 1B.2, question 3 (the project students have similar questions in challenge B1 in part 2)

17. Discuss how you would solve exercise 1B.3, question 1.

18. Kepler's 1st law tells us that, seen from the star, a planet will move in an elliptical orbit with the star as the focus. The opposite will therefore also be true: Seen from the system of reference of the planet, the star will move in an elliptical orbit with the planet as the focus. We know that both bodies exert forces on each other. Given a fixed coordinate system outside of the star or the planet, how will this look? Can you guess what will be the movement of the star and the planet? Or how you could find it out?

19. Given two masses m_1 and m_2 with position vectors $\vec{r}_1$ and $\vec{r}_2$. Find an expression for the position vector $\vec{R}$ of the center of mass.

20. Now generalize your expression for $\vec{R}$ to a system of N bodies with masses m_i and position vectors $\vec{r}_i$ for $i = [1, N]$.

21. We will now derive Newton's 2nd law for an N-body system. Assume that the external forces on all the bodies is $\vec{F}_{\text{ext}}$ and that the gravitational force from body j on body i is $\vec{f}_{ij}$. Use Newton's 2nd law for all bodies to find

$$\vec{F} = \Sigma\Sigma\vec{f} + \vec{F}_{\text{ext}} = \Sigma m\ddot{\vec{r}}$$

and insert indexes on the sums, on the $\vec{f}$, m and $\vec{r}$. Make sure the indexes are correct, in particular the range over which the sums are running, it might not be quite as straight forward as you think.

22. Show that this gives

$$M\ddot{\vec{R}} = \vec{F}_{\text{ext}}$$

(remember the definition for the center of mass position). Can you interpret this answer?

23. We will now move the origin of our coordinate system from a random point in space to the center of mass. We consider again the 2-body system (i.e. a star and a planet). Can you show that the position vectors $\vec{R}^{\text{CM}}$, $\vec{r}_1^{\text{CM}}$ and $\vec{r}_2^{\text{CM}}$ can be written as

$$\begin{aligned}\vec{R}^{\text{CM}} &= 0\\ \vec{r}_1^{\text{CM}} &= \frac{\hat{\mu}}{m_1}\vec{r}\\ \vec{r}_2^{\text{CM}} &= \frac{\hat{\mu}}{m_2}\vec{r}\end{aligned}$$

where the reduced mass $\hat{\mu} = m_1 m_2/(m_1 + m_2)$ and $\vec{r} = \vec{r}_2 - \vec{r}_1$

24. Take the absolute value of these equations to show that the two bodies move in ellipses with the center of mass as the focus. Express a_1 and a_2. the semimajor axis of the ellipses of the two bodies arount the center of mass, by a, the semimajor axis of the ellipse of the star around the planet (or vice versa).**HINT:** use that you have an expression for $r(f)$.

25. The velocity $\vec{v} = \dot{\vec{r}}$ is the velocity of the star around the planet (or vice versa), where the origin of the position vectors is in the star or the planet. Suppose your point of view is from the star where $\vec{r}$ points to the planet. Check if this is the correct expression for the total energy of the system:

$$E = \frac{1}{2}mv^2 - G\frac{mM}{r}$$

26. Discuss how you would solve 1B.4.

27. We have arrived at

$$p = \frac{\hat{\mu}m}{2E}(e^2 - 1)$$

discuss how you can use this to determine if an orbit is an ellipse, a parabola or a hyperbola.

28. Discuss how you would solve exercise 1B.5 (project students do not have an explicit challenge where you need this, but you might still need to do this in the project to test your results).