Exercise 1: Derive field of magnetic dipole

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In this exercise we are going to derive the magnetic field from a magnetic dipole. First we are going to consider the field from an electric dipole. Lets say we have two oppositely charged particles a distance d from eachother. We place origin between the two particles and the particles are placed along the z-axis.

a) Find the electric field in $\mathbf{r} = (x, y, z)$, where $|\mathbf{r}| \gg d$.

Solution. We know the electric potential for a positive charge in origin to be

$$V(x,y,z) = \frac{Q}{4\pi\epsilon\sqrt{x^2+y^2+z^2}} \tag{1}$$

Using this we can write down the electric potential from the particles.

$$V(x,y,z) = \frac{1}{4\pi\epsilon}\left[\frac{q}{\sqrt{x^2+y^2+\left(z-d/2\right)^2}} + \frac{-q}{\sqrt{x^2+y^2+\left(z+d/2\right)^2}}\right] \tag{2}$$

Because we are looking at the positions where $|\mathbf{r}| \gg d$ We can simplify by inserting $(z-d/2)^2 = z^2 + zd + d^2 \approx z^2 - zd$. This is called a binomial expansion.

$$\frac{1}{\sqrt{x^2+y^2+\left(z-d/2\right)^2}} \approx \frac{1}{\sqrt{z^2-zd+x^2+y^2}} = \frac{1}{\sqrt{r^2-zd}} = \frac{1}{r}\frac{1}{\sqrt{1-\frac{zd}{r^2}}} \tag{3}$$

Using binomial expansion once again we can simplify the term further

$$\frac{1}{r}\frac{1}{\sqrt{1-\frac{zd}{r^2}}} \approx \frac{1}{r}\frac{1}{\sqrt{\left(1-\frac{zd}{2r^2}\right)^2}} = \frac{1}{r}\frac{1}{1-\frac{zd}{2r^2}} \tag{4}$$

Lastly we can Taylor expansion the term to get

$$\frac{1}{r}\frac{1}{1-\frac{zd}{2r^2}} \approx \frac{1}{r}\left(1+\frac{zd}{2r^2}\right) \tag{5}$$

Doing this for both terms we get a easy term for the potential

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0}\frac{z}{|\mathbf{r}|^3}qd \tag{6}$$

Now to get that term to look like the ordinary electric dipole term, we need to insert some new variables. Notice that the electric dipole is defined as $\mathbf{p} = qd\hat{\mathbf{z}} = q\mathbf{d}$, where $\mathbf{d}$ is the vector pointing between the two particles. Also $z/|\mathbf{r}| = cos(\phi)$ where ϕ is the angle between the y-axis and $\mathbf{r}$. This gives us the term

$$\frac{1}{4\pi\epsilon_0}\frac{|\mathbf{p}|cos(\phi)}{|\mathbf{r}|^2} \tag{7}$$

Last but not least we can use the definition of the dot product to insert $|\mathbf{p}|cos(\phi) = \mathbf{p} \cdot (\mathbf{r}/|\mathbf{r}|)$, giving us the potential of a electric dipole where $|\mathbf{r}| \gg d$.

$$V(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \frac{\mathbf{p} \cdot \mathbf{r}}{|\mathbf{r}|^3} \tag{8}$$

b) Use the potential to find the electric field $\mathbf{E}(\mathbf{r})$.

Solution. We need to solve the equation $\mathbf{E} = -\nabla V$. Then it is easier to work with the expression

$$V(\mathbf{r}) = \frac{|\mathbf{p}|}{4\pi\epsilon_0} \frac{z}{|\mathbf{r}|^3} \tag{9}$$

Working through the math, gives us the solution

$$E_x = \frac{|\mathbf{p}|}{4\pi\epsilon_0} \frac{3zx}{|\mathbf{r}|^5} \hat{\mathbf{x}} \quad \wedge \quad E_y = \frac{|\mathbf{p}|}{4\pi\epsilon_0} \frac{3zy}{|\mathbf{r}|^5} \hat{\mathbf{y}} \quad \wedge \quad E_z = \frac{|\mathbf{p}|}{4\pi\epsilon_0} \left(\frac{1}{|\mathbf{r}|^3} - \frac{3z^2}{|\mathbf{r}|^5} \right) \hat{\mathbf{z}} \tag{10}$$

Combining the terms we get our final term

$$\mathbf{E}(\mathbf{r}) = \frac{3(\mathbf{p} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \mathbf{p}}{4\pi\epsilon_0 |\mathbf{r}|^3}, \qquad \hat{\mathbf{r}} = \mathbf{r}/|\mathbf{r}| \tag{11}$$

c) It turns out that the magnetic field for a dipole has the same as the field for a electric dipole, if you do the necessary substitutions. Do the substitutions and find the magnetic field $\mathbf{B}$.

Solution. You only need to make two substitutions

$$\mathbf{p} = \frac{\boldsymbol{\mu}}{c^2} \qquad \wedge \qquad \epsilon_0 = \frac{1}{\mu_0 c^2} \tag{12}$$

Where $\boldsymbol{\mu}$ is the magnetic momen and c the speed of light. This gives us the final solution

$$\mathbf{B}(\mathbf{r}) = \frac{\mu_0}{4\pi} \frac{3(\boldsymbol{\mu} \cdot \hat{\mathbf{r}})\hat{\mathbf{r}} - \boldsymbol{\mu}}{|\mathbf{r}|^3} \tag{13}$$

It is also possible to find the field through calculating the field fram a charge following a small loop, but this requires a bit more work.