Electromagnetic braking: a simple quantitative model

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A calculation is presented which quantitatively accounts for the terminal velocity of a cylindrical magnet falling through a long copper or aluminum pipe. The experiment and the theory are a dramatic illustration of the Faraday's and Lenz's laws and are bound to capture student's attention in any electricity and magnetism course.

I. INTRODUCTION

Take a long metal pipe made of a non-ferromagnetic material such as copper or aluminum, hold it vertically with respect to the ground and place a small magnet at its top aperture. The question is: when the magnet is released will it fall faster, slower, or at the same rate as a non magnetic object of the same mass and shape? The answer is a dramatic demonstration of the Lenz's law, which never ceases to amaze students and professors alike. The magnet takes much more time to reach the ground than a non-magnetic object. In fact we find that for a copper pipe of length $L = 1.7$m, the magnet takes more than 20s to fall to the ground, while a non-magnetic object covers the same distance in less than a second! Furthermore, when various magnets are stuck together and then dropped through the pipe, the time of passage varies non-monotonically with the number of magnets in the chain. This is contrary to the prediction of the point dipole approximation which is commonly used to explain the slowness of the falling magnets[1,2]. The easy availability of powerful rare earth magnets, which can now be purchased in any toy store, make this demonstration a "must" in any electricity and magnetism course[1,2,3,4].

In this paper we will go beyond a qualitative discussion of the dynamics of the falling magnetic and present a theory which quantitatively accounts for all the experimental findings. The theory is sufficiently simple that it should be easily accessible to students with only an intermediate exposure to the Maxwell's equations in their integral form.

II. THEORY

Consider a long vertical copper pipe of internal radius a and wall thickness w. A cylindrical magnet of cross-sectional radius r, height d, and mass m is held over its top aperture, see figure 1. It is convenient to imagine that the pipe is uniformly subdivided
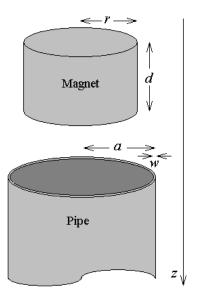

FIG. 1: The magnet and the pipe used in the experiment

into parallel rings of width l. When the magnet is released, the magnetic flux in each one of the rings begins to change. This, in accordance with the Faraday's law, induces an electromotive force and an electric current inside the ring. The magnitude of the current will depend on the distance of each ring from the falling magnet as well as on the magnet's speed. On the other hand, the law of Biot-Savart states that an electric current produces its own magnetic field which, according to the Lenz's law, must oppose the action that induced it in the first place i.e. the motion of the magnet. Thus, if the magnet is moving away from a given ring the induced field will try to attract it back, while if it is moving towards a ring the induced field will tend to repel it. The net force on the magnet can be calculated by summing the magnetic interaction with all the rings. The electromagnetic force is an increasing function of the velocity and will decelerate the falling magnet. When the fall velocity reaches the value at which the magnetic force completely compensates gravity, acceleration will go to zero and the magnet will continue falling at a constant terminal velocity v. For a sufficiently strong magnet, the terminal velocity is reached very quickly.

It is interesting to consider the motion of the magnet from the point of view of energy conservation. When an object free falls in a gravitational field, its potential energy is converted into kinetic energy. In the case of a falling magnet inside a copper pipe, the situation is quite different. Since the magnet moves at a constant velocity, its kinetic energy does not change and the gravitational potential energy must be transformed into something else. This "something else" is the ohmic heating of the copper pipe. The gravitational energy must, therefore, be dissipated by the eddy currents induced inside the pipe. In the steady state the rate at which the magnet looses its gravitational energy is equal to the rate of the energy dissipation by the ohmic resistance,

$$mgv = \sum_z I(z)^2 R . \tag{1}$$

In the above equation z is the coordinate along the pipe length, $I(z)$ is the current induced in the ring located at some position z, and R is the resistance of the ring. Since the time scales associated with the speed of the falling magnet are much larger than the ones associated with the decay of eddy currents[1,5], almost all the variation in electric current through a given ring results from the changing flux due to magnet's motion. The self-induction effects can thus be safely ignored. Our goal, now, is to calculate the distribution of currents in each ring, $I(z)$. To achieve this we first study the rate of change of the magnetic flux through one ring as the magnet moves through the pipe. Before proceeding, however, we must first address the question of the functional form of the magnetic field produced by a stationary magnet. Since the magnetic permeability of copper and aluminum is very close to that of vacuum, the magnetic field inside the pipe is practically identical to the one produced by the same magnet in vacuum. Normally this field is approximated by that of a point dipole. This approximation is sufficient as long as one wants to study the far field properties of the magnetic field. For a magnet confined to a pipe whose radius is comparable to its size, this approximation is no longer valid. Since a large portion of the energy dissipation occurs in the near field, one would have to resum all of the magnetic moments to correctly account for the field in the magnet's vicinity. Clearly this is more work than can be done in a classroom demonstration. We shall, therefore, take a different road. Let us suppose that the magnet has a uniform magnetization $\mathbf{M} = M\hat{\mathbf{z}}$. In this case the magnetic charge density inside the magnet is zero, while on the top and the bottom of the magnet there is a uniform magnetic surface charge density $\sigma_M = M$ and $-\sigma_M = -M$ respectively. The flux produced by a cylindrical magnet can, therefore, be approximated by a field of two disks, each of radius r separated by a distance d. Even this, however, is not an easy calculation, since the magnetic field of a charged disk is a complicated quadrature involving Bessel functions. We shall, therefore, make a further approximation and replace the charged disks by point monopoles of the same net charge $q_m = \pi r^2 \sigma_M$. The flux through a ring produced by the two monopoles can now be easily calculated

$$\Phi(z) = \frac{\mu_0 q_m}{2} \left[\frac{z+d}{\sqrt{(z+d)^2 + a^2}} - \frac{z}{\sqrt{z^2 + a^2}} \right] , \tag{2}$$

where μ_0 is the permeability of vacuum and z is the distance from the nearest monopole, which we take to be the positively charged one, to the center of the ring. As the magnet falls, the flux through the ring changes, which results in an electromotive force given by the Faraday's law,

$$\mathcal{E}(z) = -\frac{\mathrm{d}\Phi(z)}{\mathrm{d}t} \tag{3}$$

and an electric current

$$I(z) = \frac{\mu_0 q_m a^2 v}{2R} \left[\frac{1}{(z^2 + a^2)^{3/2}} - \frac{1}{[(z+d)^2 + a^2]^{3/2}} \right] . \tag{4}$$

The rate of ohmic dissipation can now be calculated by evaluating the sum on the right hand side of Eq. (1). Passing to the continuum limit, we find the power dissipated to be

$$P = \frac{\mu_0^2 q_m^2 a^4 v^2}{4R} \int_{-\infty}^{\infty} \frac{\mathrm{d}z}{l} \left[\frac{1}{(z^2 + a^2)^{3/2}} - \frac{1}{[(z+d)^2 + a^2]^{3/2}} \right]^2 . \tag{5}$$

Since most of the energy dissipation takes place near the magnet, we have explicitly extended the limits of integration to infinity. The resistance of each ring is $R = 2\pi a\rho/(wl)$, where ρ is the electrical resistivity. Eq. (5) can now be rewritten as

$$P = \frac{\mu_0^2 q_m^2 v^2 w}{8\pi\rho a^2} f\left(\frac{d}{a}\right) , \tag{6}$$

where $f(x)$ is a scaling function defined as

$$f(x) = \int_{-\infty}^{\infty} dy \left[\frac{1}{(y^2+1)^{3/2}} - \frac{1}{[(y+x)^2+1]^{3/2}}\right]^2 . \tag{7}$$

Substituting Eq. (6) into Eq. (1), the terminal velocity of a falling magnet is found to be

$$v = \frac{8\pi mg\rho a^2}{\mu_0^2 q_m^2 w f\left(\frac{d}{a}\right)} . \tag{8}$$

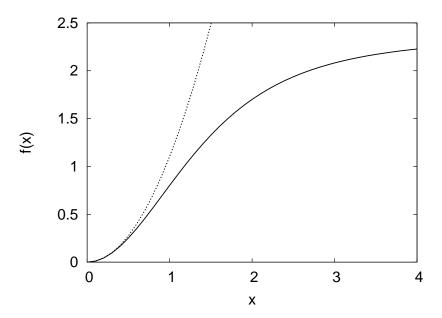In figure 2 we plot the scaling function $f(x)$. For small x

FIG. 2: The scaling function $f(x)$ (solid curve) and the limiting form, Eq. (9) (dotted curve). Note the strong deviation from the parabola (point dipole approximation) when $x > 1$.

$$f(x) \approx \frac{45\pi}{128} x^2 , \tag{9}$$

and the terminal velocity reduces to[1,2]

$$v = \frac{1024}{45} \frac{mg\rho a^4}{\mu_0^2 p^2 w} , \tag{10}$$

where $p = q_m d$ is the dipole moment of the falling magnet. We see, however, that as soon as the length of the magnet becomes comparable to the radius of the pipe, the point dipole approximation fails. In fact, for a realistic cylindrical magnets used in most demonstrations, one is always outside the the point dipole approximation limit, and the full expression (8) must be used.

III. DEMONSTRATION AND DISCUSSION

In our demonstrations we use a copper pipe (conductivity $\rho = 1.75 \times 10^{-8}\Omega$m)[6] of length $L = 1.7$m, radius $a = 7.85$mm, and wall thickness $w = 1.9$mm; three neodymium cylindrical magnets of mass 6g each, radius $r = 6.35$mm, and height

$d = 6.35$mm; a stop watch; and a teslameter. We start by dropping one magnet into the pipe and measure its time of passage — $T = 22.9$s. For two magnets stuck together the time of passage increases to $T = 26.7$s. Note that if the point dipole approximation would be valid, the time of passage would increase by a factor of two, which is clearly not the case (within point dipole approximation the time of passage is directly proportional to p^2 and inversely proportional to the mass, Eq. (10), sticking two magnets together increases both the dipole moment and the mass of the magnet by a factor of two). Furthermore, when all three magnets are stuck together the time of passage drops to $T = 23.7$s. Since the terminal velocity is reached very quickly, a constant speed of fall approximation is justified for the whole length of the pipe. In Table 1 we present the values for the measured velocity $v = L/T$. We next compare this measurements with the predictions of the theory. First, however, we have to obtain the value of q_m for the magnet. To do this we measure the magnetic field at the *center* of one of the *flat surfaces* of the magnet using the digital teslameter *Phywe* (the probe of teslameter is brought in direct contact with the surface of the magnet). Within our uniform magnetization approximation this field is produced by two parallel disks of radius r and magnetic surface charge $\pm\sigma_M$, separated by distance d,

$$ B = \frac{\mu_0 \sigma_m}{2} \frac{d}{\sqrt{d^2 + r^2}} \ . \tag{11} $$

The magnetic charge is, therefore,

$$ q_m = \frac{2\pi B r^2 \sqrt{d^2 + r^2}}{\mu_0 d} \ . \tag{12} $$

For n magnets associated in series $d \to h = nd$ and one can check using the values of the measured magnetic field presented in the Table 1 that q_m is invariant of n, up to experimental error, justifying our uniform magnetization approximation. Rewriting Eq. (8) in terms of the measured magnetic field for a combination of n magnets, we arrive at

$$ v = \frac{2Mg\rho a^2 h^2}{\pi B^2 r^4 w(h^2 + r^2) f\left(\frac{h}{a}\right)} \ , \tag{13} $$

where $M = nm$ and $h = nd$. In Table 1 we compare the values of the measured and the calculated terminal velocities.

TABLE I: Experimental and theoretical values of the terminal velocity

n magnets	B(mT)	v_{exp} (10^{-2} m/s)	v_{theory} (10^{-2} m/s)
1	393	7.4	7.3
2	501	6.4	5.8
3	516	7.2	6.9

Considering the complexity of the problem, the simple theory presented above accounts quite well for all the experimental findings. In particular, the theory correctly predicts that two magnets stuck together fall slower than either one magnet separately or all three magnets together. For each pipe there is, therefore, an optimum magnetic size which falls the slowest.

IV.   CONCLUSIONS

We have presented a simple theory which accounts for the electromagnetic braking of a magnet falling through a conducting pipe. The experiment is a dramatic illustration of the Faraday's and Lenz's law. Perhaps surprisingly, a quantitative discussion of the experiment is possible with only a basic knowledge of electrodynamics. Furthermore, the only specialized equipment necessary for performing the measurements is a teslameter, which is usually present in any physics laboratory. The demonstration and the calculations presented in this paper should, therefore, be easily adoptable to almost any electricity and magnetism course.

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