Bayesian Models of Brains, Minds, & Behaviors

DRCMR · Copenhagen · May 2025

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Lecture 1: Preamble & housekeeping

Roadmap

What this course is about

Materials: GitHub, Binder, Book…

Schedule, social, and how the week works

Group project

Vibe & expectations

How the week will go

A mix of lectures, interactive demos, case studies, discussion & group work

You'll build and present a research project with your group

Lectures & demos lay the conceptual groundwork

You build the research project sequentially as you learn more

Morning - lectures and demos

Afternoons - self-paced demos and/or group work

Materials

GitHub: slides, code, schedule, everything…

Binder: run interactive notebooks in browser, no install

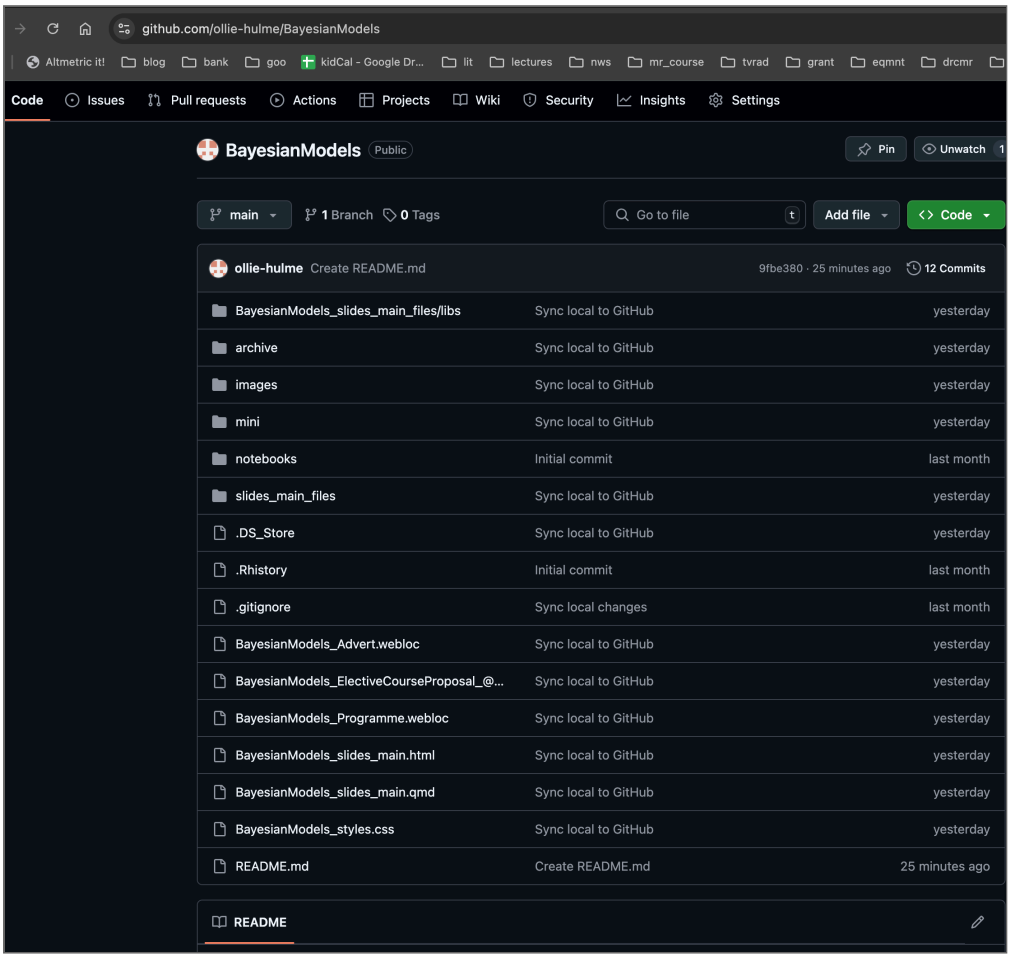
Book: key reference, especially early in the week

Important links: GitHub `README.md`

GitHub

Main hub for slides, code, notebooks, and schedule etc.

link in footer 👇



Binder

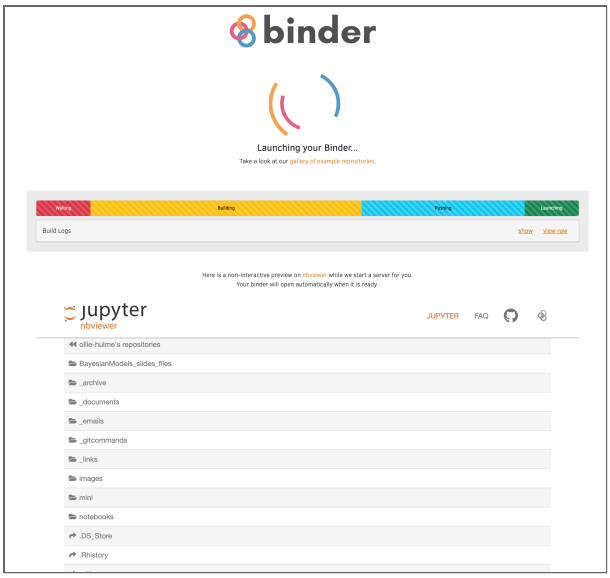
Launch notebooks in your browser

No installation needed

Be patient — it takes time to load

Keep your Binder tab open

It can time out so best to use continuously in longer sessions

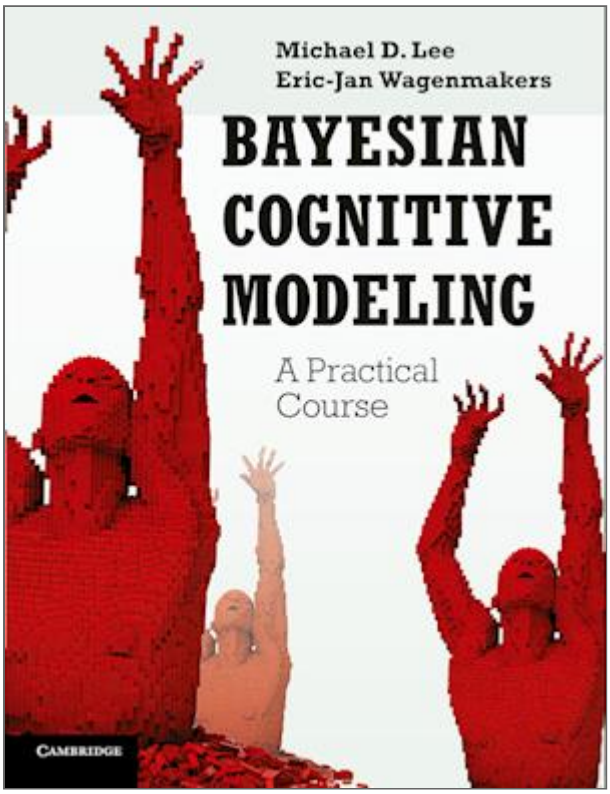

binder
Launching your Binder...
Take a look at our gallery of example repositories.
Building
Launching
Build Logs
show
view raw
Here is a non-interactive preview on nbviewer while we start a server for you.
Your binder will open automatically when it is ready.
jupyter
nbviewer
JUPYTER
FAQ
ollie-hulme's repositories
BayesianModels_slides_files
_archive
_documents
_emails
_gitcommands
_links
images
mini
notebooks
.DS_Store
.Rhistory

Book

Core reference for models & theory

This course roughly goes through it in order

Especially useful Mon–Wed



Schedule

Each day blends concepts, demos, & group work

Schedule

Up to date on GitHub

Note different room on wed & thurs

Time	Mon 5 May	Tue 6 May	Wed 7 May	Thu 8 May	Fri 9 May
Venue	Aud. 3–4	Aud. 3–4	MR Conf. Room	MR Conf. Room	Aud. 3–4
9–10	*Ollie Hulme*	*Ollie Hulme*	Inter. gen. models – *Meder*	fMRI – *Bühler*	Clinics / Group work
10–11	*Ollie Hulme*	*Ollie Hulme*	Model comparison – *Kangavari*	EEG – *Kit Larsen*	Clinics / Group work
11–11.15	Coffee break	Coffee break	Coffee break	Coffee break	Coffee break
11.15–12.15	*Ollie Hulme*	*Ollie Hulme*	Param. recovery – *Steinkamp*	Structural MRI – *Demnitz*	Group presentations
12.15–1	Lunch	Lunch	Lunch	Lunch	Lunch
1–2	*Ollie Hulme*	*Ollie Hulme*	Model recovery – *Steinkamp*	Clinic: Exp. design – *Instructors*	More presentations
2–2.15	Coffee break	Coffee break	Coffee break	Coffee break	Coffee break
2.15–4	*Ollie Hulme*	*Ollie Hulme*	Clinic: Model design – *Instructors*	Clinic: Exp. design – *Instructors*	Wrap-up & goodbye

Course overview

Basic modelling → Get started (Mon-Tues)

Intermediate modelling → Go deeper (Wed)

Neural data integration → Link models to brain (Thurs)

Group project → Model, analyze, present (Fri)

Course schematic

Your group project will follow this workflow

Group project

Form a small group

Pick a cognitive question

Design an experiment (behavioral + neural)

Build models to test your hypotheses

Present your work on Friday (~15 min)

See google doc on the Github README

Supervision & Support

Janine: talk to her if you are shy to ask in class

Ollie & David: logistics, organisation, schedule

Simon: anything technical, Binder, GitHub, Python

Group work: you will have many supervisors depending on the day

Social

Join the WhatsApp group (link on GitHub)

There you can ask informal questions, comment, idle thoughts

Friday bar to celebrate

Our expectations of you

Ask questions — don’t nod and fake it

Disrupt — curiosity is good

You deserve to understand this

Things we love to hear

“I might have missed this, but…”

“Can I ask a stupid question?”

“Do you have an intuition for why…”

“I’m confused” 🤔

Things not to do to yourself

Don’t pretend to get it

Don’t assume you’re the only one confused

Don’t sit in silence out of self-doubt

Overarching aim

Use probability theory to explain minds, brains, and behavior

Upgrade your scientific reasoning

Simple: intuitive, powerful tools

Universal: same ideas apply across science

From description → explanation

Specific aims

Build and test cognitive models

Link cognitive models to neural data

Be hands-on, interactive, exploratory

Upcoming lectures

Basics of Bayesian thinking

Modeling cognition as a binary process

Modeling mixtures of processes

Selecting models

Bayes factors and posterior odds

Lecture 2: Bayesian basics

Roadmap

The spirit of Bayesian thinking

What is Bayesian modeling?

Principles of Bayesian inference

Observable vs. latent variables

Beliefs and evidence

Estimation methods

Why Bayesian methods?

The spirit of Bayesian thinking

“Probability theory is nothing but common sense reduced to calculation.” — Laplace (1814)

“The rules of probability are the rules of consistent reasoning.” — Jaynes (2003)

“Bayesian methods are not a special brand of inference; they are the only logically consistent rules for inference that are known.” — Jaynes (2003)

Bayesianism as the calculus of common sense

Bayesianism is just probability theory applied to inference. — Jaynes (2003)

Probability as rational consistency

Bayesian modeling follows the rules of probability.

Probability is logic.

Logic is consistency.

Consistency is rationality.

And rationality is just thinking clearly.

So what is “Bayesian Modeling of Minds, Brains, and Behavior”?

This course is about **thinking clearly about minds, brains, and behavior**.

It’s about testing theories rationally, using the evidence provided by data.

Bayesian modeling offers a principled, rational way to update beliefs based on evidence.

Ta-da!

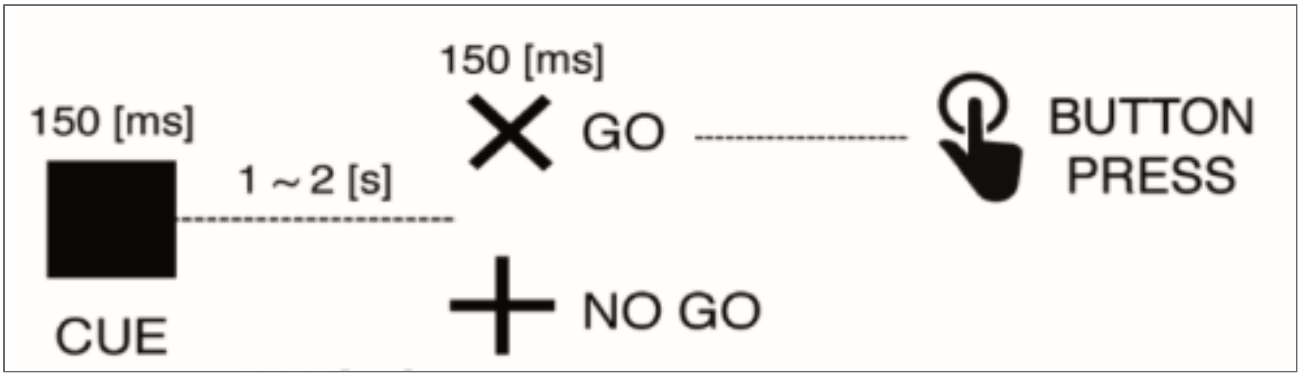
Bayesian Updating in a Nutshell

Prior belief → Evidence → Posterior belief

Thats it.

A cognitive task

The go-nogo task



You respond (Go) to frequent stimuli…

…and withhold response (No-Go) to infrequent ones.

Measures response inhibition, impulse control, and attention.

Commonly used in motor neuroscience / ADHD / addiction / tests of frontal lobe function…

Estimating ability from behavioral data

10 trials of equal difficulty

Binary outcomes, correct or incorrect

We want to estimate ability θ from behavior

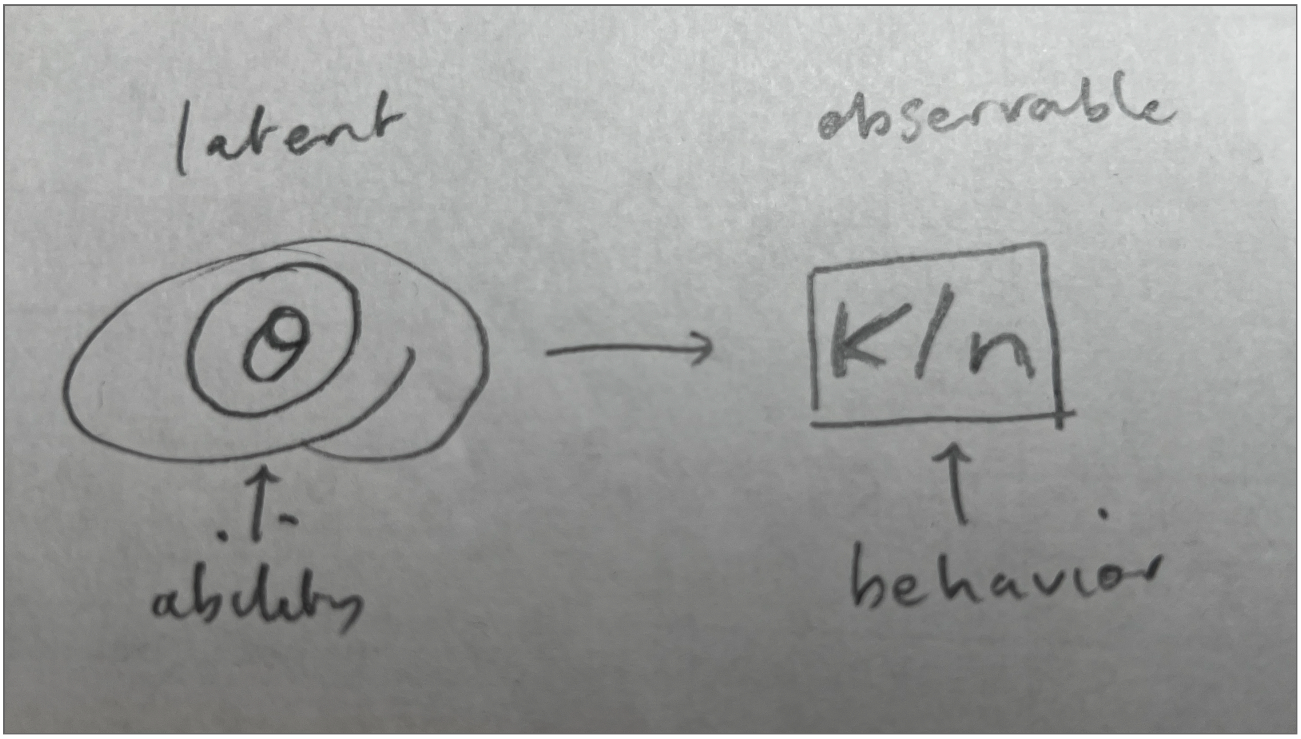
θ is *latent* which means it is hidden

Data are *observed* which means we can see it

e.g., correct responses $k = 8$ out of $n = 10$

We will use the same symbols & letters throughout

Latent vs. observed



Why latent variables?

We want to explain, not just describe

Descriptive: e.g. “What did the subject score?”

Explanatory: e.g. “What ability caused that score?”

Scientitific questions pertain to the latent

Do parkinsons patients differ in *risk taking* on and off medication?

Does serotonin change *empathy*?

Does alpha waves cause *memory consolidation*?

Does ozempic improve *cognitive flexibility*?

Science cares about the latent

We observe data, but we want to infer about latent variables

Cognitive & brain sciences are ultimately about latent variables

Bayesian modeling connects observables to latent processes

Back to go-nogo

Observed: behavior → number correct k/n

Latent: ability → θ

There is always uncertainty

The same ability can result in different behavior

Different ability can result in the same behavior

Bayesian inference accounts for this uncertainty

Beliefs as Distributions

Probability distributions encode **beliefs**

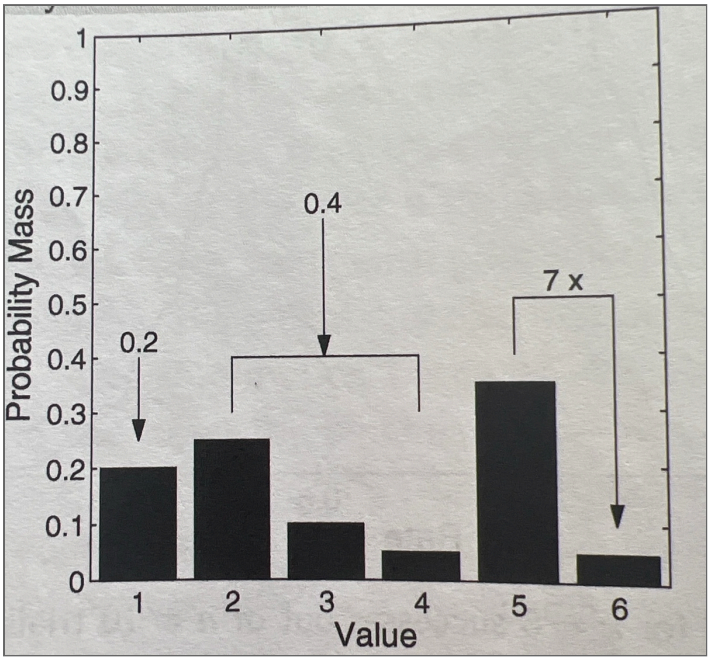
The **center** = most likely value

The **spread** = uncertainty

`notebooks/probability_distributions.ipynb`

see “Beliefs as distributions”

Probability mass functions for discrete variables

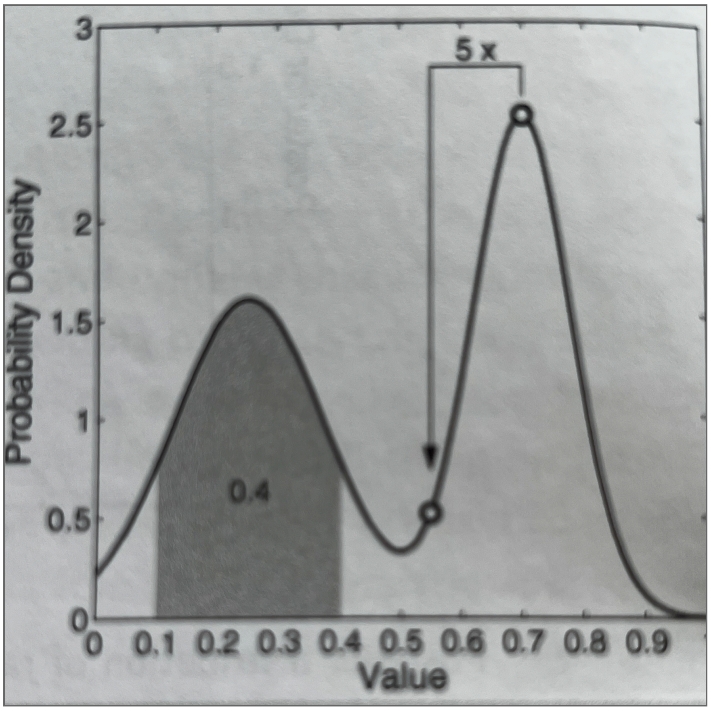


Total mass sums to 1: $\sum_x p(x) = 1$

Range sums: $p(2 \leq x \leq 4) = 0.4$

Odds: $\frac{p(5)}{p(7)} = 7$

Probability density functions for continuous variables



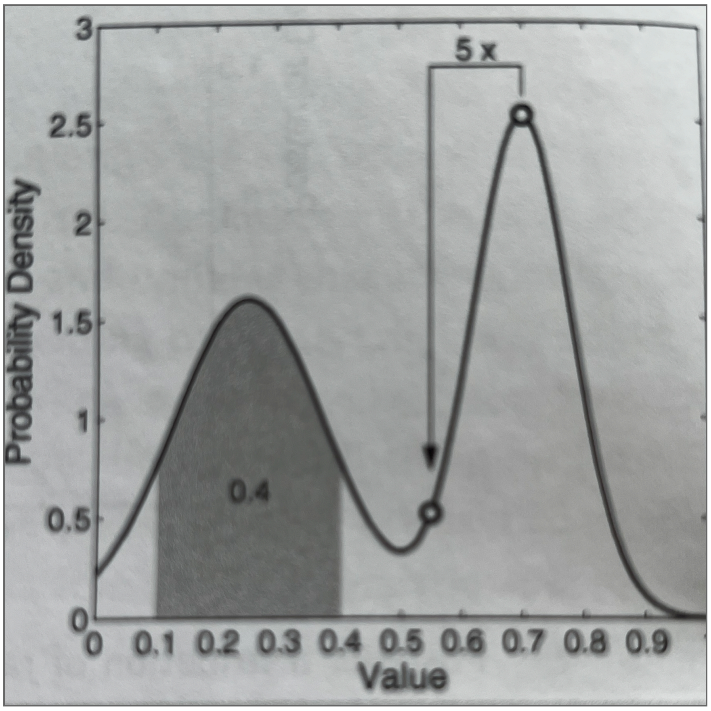
All probability density integrates to 1

(area under the curve is 1)

Densities can exceed 1

Ratios make sense: The value “5.5” is 5 times less likely than “0.7”

Probability Density Functions for continuous variables



Total area under curve: $\int p(x)\,dx = 1$

Densities can > 1, but only area matters.

Likelihood ratios make sense: e.g., $p(5.5)/p(0.7) = 1/5$

Interpreting probability distributions

It’s important to read and reason with probabilty distributions.

`notebooks/probability_distributions.ipynb`

see “Interpreting probability distributions”

Bayes' rule

$$p(\theta \mid D) = \frac{p(D|\theta) \cdot p(\theta)}{p(D)}$$

$$\text{Posterior} = \frac{\text{Likelihood} \cdot \text{Prior}}{\text{Marginal likelihood}}$$

θ is a parameter, here "ability" on go-nogo task

D is data, here it is the correct performance, k successes out of n trials

This tells us how our beliefs about ability are updated by the evidence provided by the data.

Prior

$$p(\theta \mid D) = \frac{p(D|\theta) \cdot p(\theta)}{p(D)} \quad \text{Posterior} = \frac{\text{Likelihood} \cdot \text{Prior}}{\text{Marginal likelihood}}$$

Prior is what we believe about θ before seeing the data.

It reflects our prior assumptions or knowledge.

Likelihood

$$p(\theta \mid D) = \frac{p(D|\theta) \cdot p(\theta)}{p(D)} \quad \text{Posterior} = \frac{\text{Likelihood} \cdot \text{Prior}}{\text{Marginal likelihood}}$$

Likelihood is the probability of data D given a value of θ.

It tells us how well each θ explains the data

It also tells us how to update our beliefs about each value of θ

Higher likelihood → stronger belief in θ

Lower likelihood → weaker belief in θ

Marginal likelihood

$$p(\theta \mid D) = \frac{p(D|\theta) \cdot p(\theta)}{p(D)} \quad \text{Posterior} = \frac{\text{Likelihood} \cdot \text{Prior}}{\text{Marginal likelihood}}$$

Marginal likelihood is the total probability of the data averaged over all possible values of θ.

It represents how good the model is at predicting the data.

(It also normalizes the posterior so it is a valid probability distribution)

Posterior

$$p(\theta \mid D) = \frac{p(D|\theta) \cdot p(\theta)}{p(D)} \qquad \text{Posterior} = \frac{\text{Likelihood} \cdot \text{Prior}}{\text{Marginal likelihood}}$$

Posterior is what we believe about θ *after* seeing the data.

It's the prior that has been updated by the evidence provided by the data.

Recap in context of go-nogo

The *prior* is our prior belief about ability $p(\theta)$

The *likelihood* is how likely the behavior is under each ability $p(data|\theta)$

The *posterior* is our new belief about ability after observing the behavior $p(\theta|data)$

The *marginal likelihood* is how good in general this model predicts the behaivor $p(data)$

Updating beliefs with data

We start with a **prior** belief $p(\theta)$

Observe **data**: e.g. (k = 9) correct out of (n = 10)

Bayes updates the belief:

$$p(\theta \mid D) = \frac{p(D|\theta) \cdot p(\theta)}{p(D)}$$

Intuition behind belief updating

The more likely the data is for a given θ

The more we believe in that value of θ after seeing the data.

The values of θ that are better supported by the data, are more believed in after experiencing the data

We have updated our beliefs according to the data

Proportional form of Bayes rule

Since:

$$p(\theta \mid D) = \frac{p(D|\theta) \cdot p(\theta)}{p(D)}$$

The Marginal likelihood $p(D)$ doesn't depend on θ

So we can rewrite as:

$$p(\theta \mid D) \propto p(D \mid \theta) \cdot p(\theta)$$

Posterior $\propto$ Likelihood $\cdot$ Prior

"*Posterior is proportional to likelihood times prior*""

Prior beliefs for theta

`notebooks/probability distributions.ipynb`

see “Prior beliefs for theta”

Multiplying prior and likelihood

`notebooks/probability distributions.ipynb`

see “Multiplying prior and likelihood”

Is the likelihood a probability distribution?

`notebooks/probability distributions.ipynb`

see “Is the likelihood a probability distribution?”

It is if you plot it appropriately.

e.g $p(data|theta = 0.5)$

It is if you plot it over the data rather than theta

Summarising the posterior

`notebooks/probability distributions.ipynb`

see “Bayesian credibility intervals”

see “Summarising the posterior”

Compute the posterior for the beta

Here is an easy way to calculate the posterior

We start with a flat prior: $p(\theta) \sim \text{Beta}(1, 1)$

Observe some data - k = correct, n = total trials

Posterior is then: $p(\theta \mid D) \sim \text{Beta}(1 + k, 1 + n - k)$

Simple, tractable update rule for binary outcomes

`notebooks/probability distributions.ipynb`

see “Computing posterior for the beta”

Sequential updating of posterior

Bayesian inference is consistent across steps

One-step:

Prior → Combined data1 & data2 → Posterior

Two-step:

Prior → Data1 → Intermediate Posterior → Data2 → Final Posterior

Final result is identical

`notebooks/probability distributions.ipynb`

see “Sequential updating”

Why sequential updating of posterior matters

You can peak at your data, it’s ok!

Enables inference as data rolls in.

Optional stopping: stop early, or extend data collection

Efficient: time, money, resources

Ethical: minimises animals, humans, unnecessary treatment etc.

Optional stopping

Frequentist methods don’t allow this

Frequentist inference assumes a fixed sample size and plan

Stopping early or collecting more data invalidates p-values

Counterfactual policies: what you would have done, if the data had turned out different impacts on your p-values.

Not widely understood.

Conjugate priors

If Prior and posterior from the same distribution family → *conjugate*

For example:

$p(\theta) : \text{Beta}(\alpha, \beta)$

$p(\theta|data) : \text{Beta}(\alpha + k, \beta + n - k)$

The posterior can be computed by plugging data directly into an equation

Conjugacy allows for *analytic updates*

When conjugacy isnt possible: Sampling

Conjugacy is relatively rare in real world cases

In cases where conjugacy is not available sampling solutions are possible

Commonly MCMC - Markov Chain Monte Carlo

Works even when no closed-form solution

Approximates the posterior via sampling

Analytic vs. Sampling

Analytic: Exact, requires conjugacy, rare

MCMC: Approximate, doesnt require conjugacy, more flexible, common

MCMC in Practice

Red pill: Learn how MCMC really works 🤓

Intuitive guide to MCMC internals

Metropolis-Hastings explained simply

These methods will keep evolving — expect newer algorithms, faster sampling, and better approximations.

Blue pill: Trust the method and use it (but know how to spot when it's broken) 😄

We'll demonstrate and visualise MCMC in practice.

MCMC demo

An easy way to sample the posterior.

With enough samples you can get as close to the true posterior as you want.

Works for conjugacy cases too.

Go to “MCMC”

`notebooks/probability distributions.ipynb`

Why Bayesian Methods?

Principled reasoning: Probabilistic logic = consistent thinking

Uncertainty-aware: Fully models uncertainty

Latent variables: Model hidden causes, not just outcomes

Sequential updating: Updating is the same whether data is sequential or all-in-one-go.

Explanations: From describing data to explaining via theory

Flexibility: Hierarchical, generative, extensible

Simple: Same principle always. Learn it once. Apply it forever.

Lecture 3: Modeling a Binary Process

Roadmap

Cogntive tasks with binary outcomes

From observed data to hidden probabilities

Graphical models and their notation

Bayesian inference via sampling in JAGS

Convergence diagnostics and posterior checks

Modeling a Binary Process

Start simple: focus on binary outcomes

e.g., *Success/Failure*, *Correct/Incorrect*, *Yes/No*

Common examples:

coin flips, true/false questions, detection tasks, motor responses

In our go/no-go example, each of the n trials results in either k successes

Binary processes are foundational for modeling cognition

Getting Started

Our goal is to infer ability in a go-no-go task

We estimate a rate — the hidden probability θ that a response is correct

We represent our uncertainty about θ as a probability distribution

Many cognitive tasks can be modeled this way

Binary Tasks in Cognitive Science

Go/no-go, stop-signal, 2AFC, task switching

Recognition memory, Stroop, Flanker, oddball detection

Visual search, discrimination tasks, and so on.

From Binary Outcomes to a Hidden Rate

Observe k successes in n trials → compute $\frac{k}{n}$

But our interest is in the *underlying* success rate θ

Model: $k \sim \text{Binomial}(\theta, n)$

$p(k \mid \theta, n) = \binom{n}{k} \theta^k (1-\theta)^{n-k}$

For a given θ and n, this is the probability distribtion for k

Assumes independent, identically distributed (i.i.d.) trials — no history effects

Try it Yourself

`notebooks/probability distributions.ipynb`

see “Binomial distribution”

Graphical Models

Graphical models represent probabilistic structure visually

Nodes represent variables; edges represent dependencies

Child nodes are conditionally dependent on parent nodes

This is a simple model of the go-nogo task:

Graphical Notation

Circular nodes: continuous variables

Square nodes: discrete variables

Shaded nodes: observed

Unshaded nodes: hidden

Single border: stochastic variable

Double border: deterministic relationship derived from others

Graphical Notation Reference

Graphical Model Quiz

Upper node: What type of variable is this?

Answer: Continuous, stochastic, and observed

Lower node: What type of variable is this?

Answer: Discrete, deterministic, and unobserved

Sampling via JAGS

```
model {               # Define the model
  theta ~ dbeta(1,1) # Prior: theta follows a uniform beta distributio
  k ~ dbin(theta,n)  # Likelihood: k follows a binomial distribution
                      # with parameters theta and n
}                     # End of model
```

Interpreting the graphical model for the code

Theta is latent, and continuous

n is observed and discrete

k is observed and discrete

Both feed n and k feed in to the likelihood to generate k

R-hat as a convergence check

It's important to check that the sampling has converged to the stationary distribution.

One heuristic is the R-hat statistic:

$$\hat{R} = \frac{\text{var(within-chain)}}{\text{var(across-chain)}}$$

Rule of thumb: $\hat{R}$ should be between 1.00 and 1.01 for convergence.

Inspecting the chains

the chains of samples should look like *hairy catapillars*

like this:

Try it yourself

go to “MCMC convergence checks**

📂 `notebooks/probability distributions.ipynb`

Difference between two rates

Suppose we observe two processes, each producing successes out of trials:

Process 1: k_1 successes out of n_1 trials

Process 2: k_2 successes out of n_2 trials

We assume each is governed by a different underlying rate: θ_1 and θ_2.

Estimating the difference

We want to model each rate with a posterior Beta distribution, and we are interested in the difference:

$\delta = \theta_1 - \theta_2$

This tells us how much more (or less) likely success is in one group compared to the other.

Examples and intuition

Examples:

📈 Effect of a drug on performance (θ_1 = treated, θ_2 = control)

👶 Performance difference between age groups

🧪 Comparison of two algorithms on success rate

A positive δ means group 1 is better; a negative δ means group 2 is better.

Graphical model for inferring differences

Why is delta double boundary?

Because it is completely determined by the two thetas

JAGS code for infferring differences in rates

```
model{
  k1 ~ dbin(theta1,n1)
  k2 ~ dbin(theta2,n2)
  theta1 ~ dbeta(1,1)
  theta2 ~ dbeta(1,1)
  delta <-theta1-theta2
}                  # End of model
```

Try it yourself

go to “Inferring the difference between two rates”

`notebooks/probability distributions.ipynb`

Interpret the posterior

What is the approximate probability that the difference in rates (δ) is below 0?

Inferring a common rate

In some cases we want to infer a common rate for 2 different processes

e.g. *same subject & task, two different sessions*

e.g. *same group, different subjects*

e.g. *same subject, different tasks*

Here we would model a single θ

Same model with plate notation

note only one theta, but multiple processes indexed by i

JAGS code for inferring a common rate

```
 model{
  k1 ~ dbin(theta,n1)
  k2 ~ dbin(theta,n2)
  theta ~ dbeta(1,1)
}
```

Only one θ is modelling the two sets of data $k1$ and $k2$

Try it out

go to “Inferring a common rate”

`notebooks/probability distributions.ipynb`

Predictions

In Bayesian modeling, everything is about prediction.

There are two fundamental axes:

Are we predicting **parameters** or **data**?

Are we predicting **before** or **after** observing data?

Different types of prediction

	Before data	After data
Parameters	Prior distribution: $p(\theta)$	Posterior distribution: $p(\theta \mid d_{\text{obs}})$
Data	Prior predictive distribution: $p(d_{\text{new}})$	Posterior predictive distribution: $p(d_{\text{new}} \mid d_{\text{obs}})$

Predicting parameters

The **prior distribution** $p(\theta)$ is our prediction about the parameter before seeing data.

The **posterior distribution** $p(\theta \mid d_{\text{obs}})$ is our updated prediction after observing data.

Predicting data.

The **prior predictive distribution** $p(d_{\text{new}})$ tells us what data we expect based on our prior belief about θ.

The **posterior predictive distribution** $p(d_{\text{new}} \mid d_{\text{obs}})$ predicts new data based on our updated belief.

Todays posterior is tomorrows prior

This means that any posterior can always become a prior for a future prediction, and so on.

Prior and posterior prediction

```
model {
  # Prior distribution
  thetaprior ~ dbeta(1,1)

  # Prior predictive distribtion
  priorpredk ~ dbin(thetapior,n)

  # Posterior distribution
  theta ~ dbeta(1,1) # theta becomes the posterior distribution of
  k ~ dbin(theta,n) #likelihood for updating prior to posterior

  # Posterior predictive distribution
  postpredk ~ dbin(theta,n) # posterior predictive distibution
}
```

Samples of the four distributions

Prior and posterior distributions are in the space of the parameter

Prior and posterior predictive distributions are in space of the data k out of n trials.

Comparing data to the posterior predictive distribution

If we estimate the model along with its predictive distributions

We can see how this compares to the actual data

Descriptive adequacy

…means how well does the model describe the data

Posterior for this data: k1 = 0, n1 =10 & k2=10, n2 = 10

Looks ok, but does it have descriptive adequacy?

Check posterior predictive distribution against data

X marks the observed data, square size indicates probability

The model has poor *descriptive adequacy*. Why?

Its a common rate model, so it predicts the same rate, but the data clearly is better modelled with different rates.

Prediction forward and backward in time

Prediction usually targets the future — but can also fill in the past

When data are missing, we infer what might have happened

Cognitive models use observed data to uncover hidden causes of past behavior

Inference helps us predict both outcomes and hidden history

Bayesian inference works backwards and forwards in time

Latent mixture models

What are they?

How to use them to model mixtures of cognitive processes or traits

How to use them to model compare models

Latent mixture models

…allow you to model data as coming from a *mixture* of *latent* processes.

This could be a mixture of cognitive processes, e.g. *guessing and trying, attending and not attending, remembering and forgetting.*

Or mixture of states or traits, e.g. *depresssed vs. healthy, parkinsons vs. healthy, sleepy vs. awake*

An indicator variable estimates which mixture of processes generated the data.

You can also use these mixture models to compare different models

Example: Latent mixture model of cognitive test

"Tryers" have an ability that determines their rate of correct responses.

"Guessers" score at chance level (e.g. 50%).

Each participant belongs to one of the two groups.

An *indicator* variable z_i models *guesser* or *tryer*.

Latent mixture model vs. simpler model

Its the same model, z just allows a mix of two different processes that can set the rate θ.

Simples.

Latent mixture graphical model

z_i is an indicator variable for participant i.

$z_i = 0 \rightarrow$ guesser; $z_i = 1 \rightarrow$ tryer.

$\psi = 0.5$ is the known chance performance level.

ϕ is the tryer's ability (same as θ).

Posterior of z tells us the probability of each subject being either a guessers vs. a tryer.

JAGS code for latent mixture

```
model {
  for (i in 1:p) {
    z[i] ~ dbern(0.5)
  }

  psi <- 0.5
  phi ~ dbeta(1, 1) I(0.5, 1)

  for (i in 1:p) {
    theta[i] <- equals(z[i], 0) * psi + equals(z[i], 1) * phi
    k[i] ~ dbin(theta[i], n)
  }
}
```

Interactive demo

`notebooks/probability distributions.ipynb`

Let’s simulate data first

go to “Simulate guessers and tryers”

Then we can run inference on each subject

go to “Infer guessers and tryers”

Model selection

Why we need to compare models

Simplicity vs. complexity

Marginal likelihood as a comparison tool

Predictive performance and prior bets

Bayesian model comparison intuition

Why compare models?

Exept for the last section, we've mainly focused on single models

Science advances by comparing competing explanations

"This theory is good" → compared to what?

We want to know which theory explains the data *better*

This requires comparing models

Simplicity vs. complexity

“Explain phenomena by the simplest hypothesis that works” — Ptolemy

“Avoid unnecessary plurality” — Occam’s razor

“Complexity must pay for itself” — Hinton

“Minimize free energy” — Friston

These ideas all reflect the same principle: balance fit and complexity

The Bayesian solution

Many methods try to balance fit and complexity

Bayesian methods do it naturally

Bayes gives us a single number for model quality: marginal likelihood

It rewards accuracy…

…and penalizes wasted complexity

Conditioning on a model

Bayes' rule always assumes *a model*

But we can easily imagine different models, with different priors or structures

Science is filled with competing explanations and models after all

We now ask: which model better predicts the data?

This leads us to the *marginal likelihood*

It's the probability of the data, *given the model*

Marginal likelihood

$p(D \mid M)$ = average predictive performance of model M

It's a single number:

How well the model predicted the data, on average

It accounts for *all* parameter values, weighted by their prior

Analogy to betting

Think of it like your model is betting on which parameter values best predict the data

The better your bets, the higher your model's score

The prior is the placing of the bets, and the marginal likelihood is how good those bets paid off.

Marginal likelihood in words

How probable was the data under this model M?

Did the model concentrate its predictions where the data actually were?

Priors spread out predictive mass

Bad priors waste predictions on wrong areas

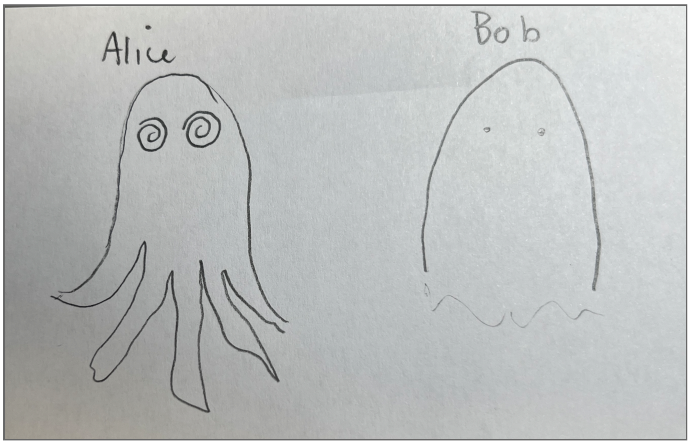
Good priors focus predictions where the data land

Octopus example

There are two ~~octopi~~ octopusses.

Both claim to be paranormal.

It’s 1970, so they are both working for the CIA



Octopus predictions

Where is the Russian sub?

Alice: northern hemisphere

Bob: Baltic Sea

Data: Off the coast of Stockholm

Alice was vaguely right

Bob was more precisely right → higher marginal likelihood

How is marginal likelihood calculated?

It’s the expected likelihood under the predictions of the prior:

$$p(D \mid M) = \int p(D \mid \theta, M)\, p(\theta \mid M)\, d\theta$$

For discrete parameters:

$$p(D \mid M) = \sum p(D \mid \xi_i)\, p(\xi_i)$$

It’s a weighted average across all parameter settings

Try it out

Lets compare alice and bob

`notebooks/probability distributions.ipynb`

“Compare octopusses via marginal likelihood”

Worked example

Suppose a model has three parameter values: $\theta = \{0, 0.5, 1\}$

Prior probabilities: $p(\theta_1) = 0.6$, $p(\theta_2) = 0.3$, $p(\theta_3) = 0.1$

Likelihoods: $p(D \mid \theta_1) = 0.1$, $p(D \mid \theta_2) = 0.4$,
$p(D \mid \theta_3) = 0.6$

Marginal likelihood:
$p(D) = 0.6 \cdot 0.1 + 0.3 \cdot 0.4 + 0.1 \cdot 0.6$

Step by step:
$= 0.06 + 0.12 + 0.06 = \boxed{0.24}$

Complexity and spread

More complex models spread their predictions widely

This lowers the average likelihood

Even if they include the truth, they may assign low probability to it

Marginal likelihood punishes this

Broad priors = wasted predictions = lower score = lower marginal likelihood

Complexity is not just parameter count

A model with *many* narrow priors can be simple

A model with *one* vague prior can be complex

Complexity = how broadly a model spreads its predictions

Narrow predictive distributions = simpler models

Wide, uncertain predictions = complex models

Misconception - sidenote

AIC, BIC penalize complexity by counting parameters

But parameter count ≠ true complexity

Complexity = how broadly a model spreads its predictions

Bayesian marginal likelihood captures this automatically

Example: prior vagueness

$\theta \sim \text{Uniform}(0, 1)$ → vague → complex

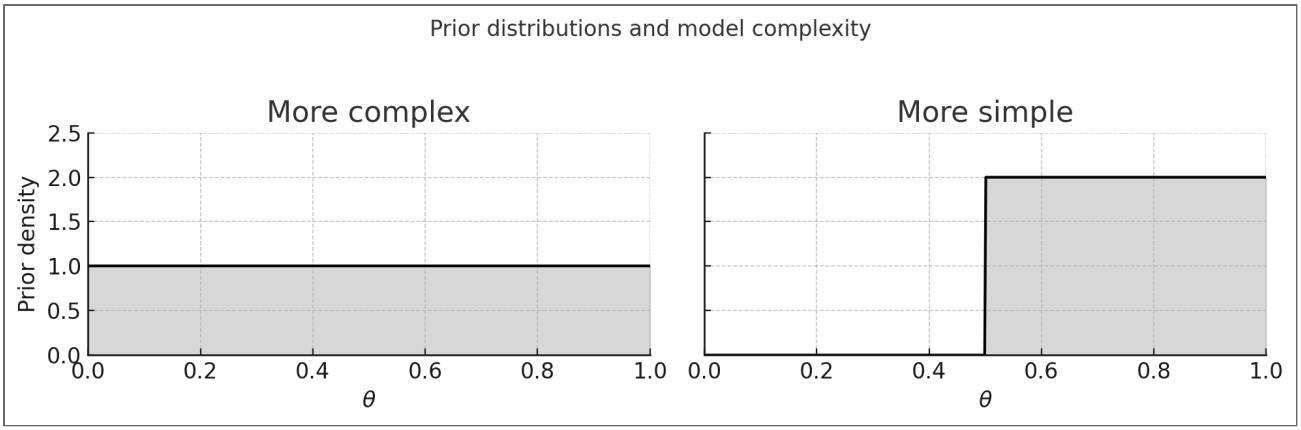
$\theta \sim \text{Uniform}(0.5, 1)$ → tighter → simpler

Both models have one parameter

But they differ in how much of the prediction space they cover

Complexity depends on how much ground a model tries to cover

Example: prior vagueness



Why marginal likelihood matters

It’s the most important quantity in Bayesian model comparison

It unifies inference in brain, behavior, science

Maximizing it means best average predictive performance

It is the quantity that arguably everything else is trying to approximate: variational Bayes, free energy minimisation, ELBO, predictive coding, predictive processing

Why marginal likelihood matters

Can even argue it is a *unique universal maximandum* for all physical and adaptive systems

Woah man, that’s like, deep.

The Bayes factor

Compared to what?

From marginal likelihood to relative evidence

Interpreting Bayes factors

Worked example: guessing vs. non-guessing

Pitfalls and philosophical notes

Why compare models?

A model with high marginal likelihood is good — but only *relative* to alternatives

Absolute goodness is rarely meaningful on its own

The key question is: compared to what?

We want relative evidence — which model explains the data better

The Bayes factor gives us that answer

Compared to what?

Set a compared-to-what alarm in your brain

Listen out for one-sided superiority / inferiority claims

“this theory explains…”

“this model predicts the data poorly…”

“this data is unlikely under the null hypothesis…”

Ding! → Compared to what?

Bayes factor definition

Marginal likelihood: *average predictive performance of a model*

Bayes factor compares this between two models

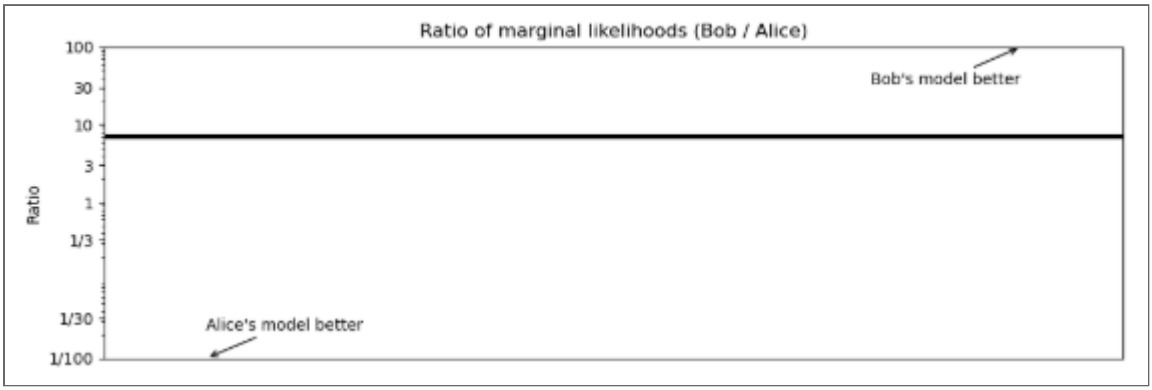
Defined as the ratio of marginal likelihoods:

$$BF_{12} = \frac{p(D|M_1)}{p(D|M_2)}$$

Quantifies how much more likely the data is under M_1 than M_2

We already plotted Bayes factors

Sneakly i didnt tell you they were bayes factors.



Interpreting the Bayes factor

$BF_{12} > 1$ → data favors M_1

$BF_{12} < 1$ → data favors M_2

$BF_{12} = 5$ → data is 5× more likely under M_1

$BF_{12} = \frac{1}{5}$ → data is 5× more likely under M_2

Strength of evidence depends on how far from 1 the ratio is

Jeffreys' scale

BF_{12}	**Interpretation**
>100	Extreme evidence for M_1
30–100	Very strong evidence for M_1
10–30	Strong evidence for M_1
3–10	Moderate evidence for M_1
1–3	Anecdotal evidence for M_1
1	No preference
1/3–1	Anecdotal evidence for M_2
1/10–1/3	Moderate evidence for M_2
1/30–1/10	Strong evidence for M_2
1/100–1/30	Very strong evidence for M_2
<1/100	Extreme evidence for M_2

Try it out

see “Bayes factor scale interpretation”

`notebooks/probability distributions.ipynb`

Example: guessing vs. non-guessing

9 out of 10 trials correct → $k = 9, n = 10$

M_1: unknown ability → $\theta \sim \text{Uniform}(0, 1)$

M_2: guessing → $\theta = 0.5$

Compute marginal likelihood under each model

Compare with a Bayes factor

Bayes factor calculation

For M_1: $p(D \mid M_1) = \frac{1}{1+n} = \frac{1}{11} \approx 0.0909$

For M_2: $p(D \mid M_2) = \binom{10}{9}(0.5)^{10} = 10 \cdot 0.000976 = 0.0098$

Bayes factor:

$BF_{12} = \frac{0.0909}{0.0098} \approx 9.3$

Data is about 9× more likely under M_1 than M_2

Try it out

see “Bayes factor calculation step by step”

`notebooks/probability distributions.ipynb`

Flipping the BF

If $BF_{12} < 1 \ldots$

…take the reciprocal: $BF_{21} = \frac{1}{BF_{12}}$

Keeps interpretation intuitive: how many times more likely is the data?

Example: $BF_{12} = 0.2 \rightarrow BF_{21} = 5$

Now we say: data is 5 × more likely under M_2

Same info, more digestible

Bayes vs. Fisher

Bayes compares models

Fisherian methods tests a single null hypothesis

p-values ask: “how unlikely is this data under H_0?”

Bayes factors ask: “which model better explains the data?”

Evidence is always comparative: $p(\text{data} \mid \text{A})$ vs. $p(\text{data} \mid \text{B})$

Critiques

Criticism: *Bayes factors are sensitive to prior choice*

Answer: This is a **feature** — priors are part of the model.

Criticism: *But I don't want my conclusions to depend so much on the prior*

Answer: Then use **sensitivity analysis** to check robustness.

Criticism: *BFs be high if one bad model is much worse than another*

Answer: True. Check **descriptive adequacy** — look at **posterior predictive distributions**, simulate data, or compare out-of-sample predictions.

The arc of civilisation

Oppposable thumbs, fire, the wheel, writing, zero, the printing press, Newtonian physics, germ theory, the steam engine, the combustion engine, the Moon landing, In Rainbows, the internet, CRISPR, Bayes factors

I’m being satirical (kinda).

Lecture 7 Model probabilities

Roadmap

Model probabilities

Prior model probability

Marginal likelihood

Posterior model probability

Odds and model comparison

From prior odds to posterior odds

Conceptual and computational challenges

Model probabilities

Often we want to know which model to believe in

As Bayesians, we assign prior probabilities to models

Then update these into posterior model probabilities

Updating is based on how well each model predicts the data

Prior model probability

Assign prior probabilities to models: $p(m_1)$, $p(m_2)$ …

Reflects belief before seeing any data

Must sum to 1 across the model space M

Can be informed by prior knowledge or data…

…or set uniformly if agnostic

Marginal likelihood

Measures average predictive performance of a model

e.g. for a model m_1:

$$p(D \mid m_1) = \int p(D \mid \theta, m_1)\, p(\theta \mid m_1)\, d\theta$$

Rewards models that predict the data well

Penalizes models that spread probability too widely

Posterior model probability

Probability of model after seeing data

Bayes' rule for models:

$$p(m_1 \mid D) = \frac{p(D \mid m_1)\ p(m_1)}{p(D \mid M)}$$

Note: The denominator is the marginal likelihood of the model space:

$$p(D \mid M) = \sum_{m \in M}\ p(D \mid m)\ p(m)$$

Posterior model probability

Probability of model after seeing data

Bayes' rule for models:

$$p(m_1 \mid D) = \frac{p(D \mid m_1)\, p(m_1)}{p(D \mid M)}$$

Note: The denominator is the marginal likelihood of the model space M:

$$p(D \mid M) = \sum_{m \in M} p(D \mid m)\, p(m)$$

Odds

Odds express the ratio of one probability relative to another

If probability of success is p = 0.75

…and probability of failure is then p = 0.25 then:

$$\text{odds} = \frac{0.75}{.25} = 3$$

“Three to one” odds means success is three times more likely than failure

Prior odds

Compares plausibility of two models before seeing data

Defined as: $\text{Prior odds} = \frac{p(m_1)}{p(m_2)}$

Encodes how much you believe in m_1 relative to m_2 a priori

Posterior odds

Compares plausibility of two models after seeing data

Defined as: $\text{Posterior odds} = \frac{p(m_1 | D)}{p(m_2 | D)}$

Reflects relative belief in one model over another after the data

From prior odds to posterior odds

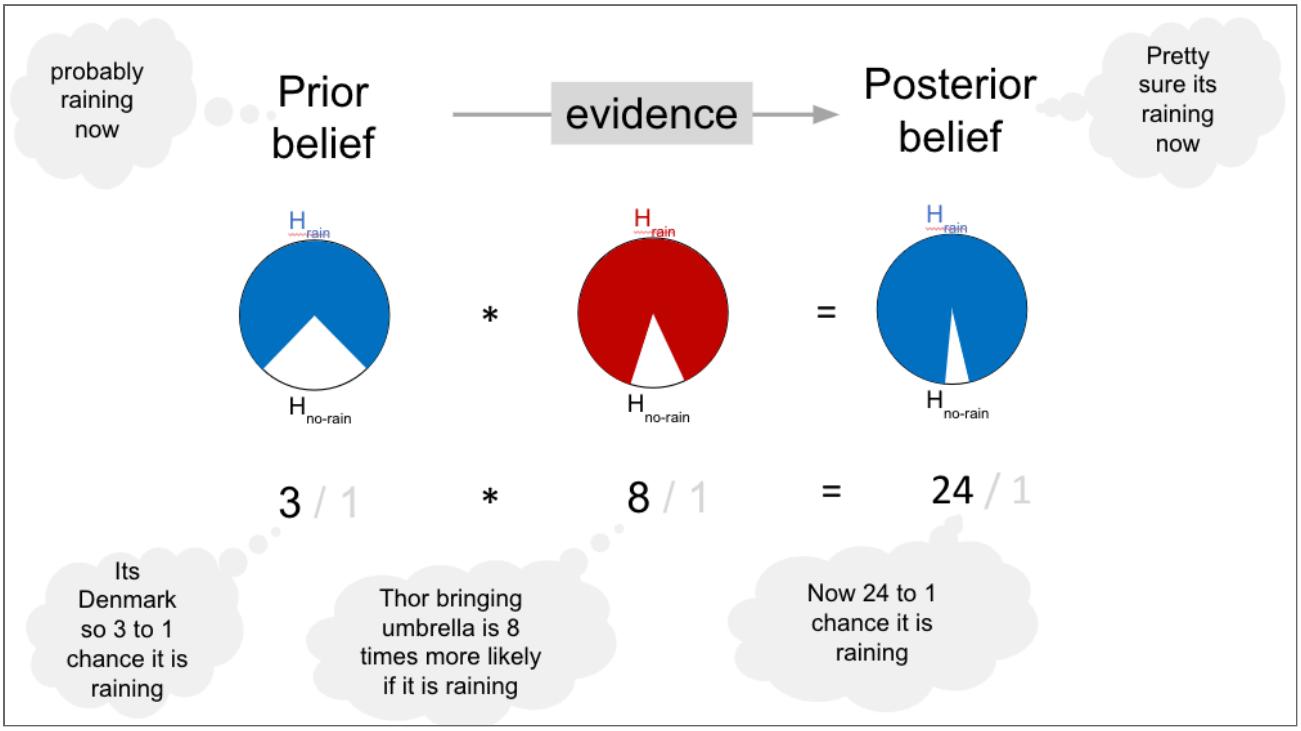
Bayes' rule for model comparison (in odds form):

$$\frac{p(m_1)}{p(m_2)} \times \frac{p(D \mid m_1)}{p(D \mid m_2)} = \frac{p(m_1 \mid D)}{p(m_2 \mid D)}$$

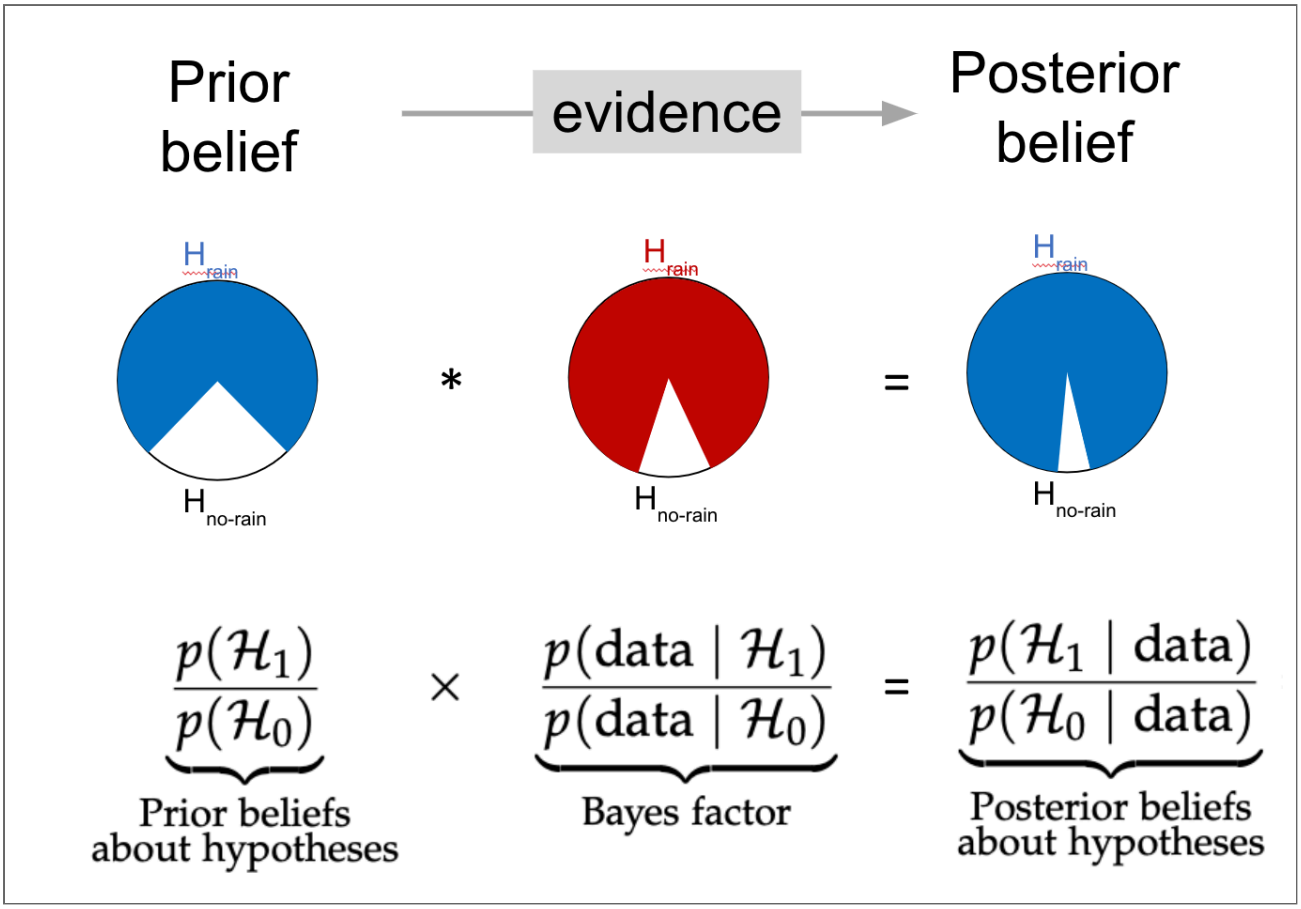
Prior odds × Bayes factor = Posterior odds

Simples!

Posterior odds example



Posterior odds schematic



Try it out

Demo brings it all together

See: `notebooks/probability distributions.ipynb`
→ “Putting everything together into one massive beast of a demo kind of thing”

Conceptual and computational challenges

Priors affect inference — vague ≠ safe

Marginal likelihoods are often hard to compute

Complex models often need approximations

Specifying model priors

Model priors should reflect genuine uncertainty

Two strategies:

Subjective (expert-informed)

Objective (e.g., unit-information)

Objective priors support reproducibility

Subjective priors offer flexibility when justified

Common confusion about priors

Model priors ≠ parameter priors

Be clear what you are talking about with “priors”

What priors influence what

Model priors → Prior odds

Since prior odds are composed of model priors

Parameter priors → Bayes factor

Parameter priors infuence how good each model predicts data

Both model AND parameter priors → posterior odds

Since Posterior odds = prior odds BF*

Prior sensitivity

Different priors can lead to different conclusions based on the posterior

This is not a flaw — it’s a feature of honest uncertainty

Sensitivity analysis tests how robust your conclusions are

Try wider/narrower priors and compare results

Some models are robust; others are fragile

Belief updating for models vs. parameters

Posterior probability of parameters (within a model, m):

$$p(\theta \mid D, m) = \frac{p(D \mid \theta, m) \cdot p(\theta \mid m)}{p(D \mid m)}$$

Posterior probability of models m (within a model space, M):

$$p(m \mid D) = \frac{p(D \mid m) \cdot p(m)}{p(D \mid M)}$$

Note: First denominator is for a single model m, second sums over the full space of possible models, M {m1, m2…}.

Computational solutions

Marginal likelihoods are rarely available in closed form

Approximate inference methods include:

Variational Bayes

Free energy minimisation

MCMC (like we've used)

Savage-Dickkey method of model comparison

In this method two models are compared:

Null hypothesis (H_0): fixes parameter to a specific value, e.g., $\phi = \phi_0$

Alternative hypothesis (H_1): parameter free to vary, e.g., $\phi \neq \phi_0$

H_0 is nested within H_1 (by constraining parameter).

Classical null hypothesis usually sharp (point-null).

Savage–Dickey Density Ratio

Defines Bayes factor for nested models:

$$BF_{01} = \frac{p(D|H_0)}{p(D|H_1)} = \frac{p(\phi=\phi_0|D,H_1)}{p(\phi=\phi_0|H_1)}$$

Simply the ratio of posterior to prior densities at the point of interest ϕ_0 under the alternative hypothesis.

Example: Binomial Scenario

Binomial scenario: θ parameter, observing 9 correct and 1 incorrect response.

Null hypothesis (H_0): $\theta = 0.5$

Alternative hypothesis (H_1): θ free to vary, prior $\theta \sim Beta(1, 1)$

Bayes factor is the ratio of posterior and prior densities at $\theta = 0.5$

Visual Interpretation of Savage-Dickey

Prior (uniform) and posterior distributions shown.

Density ratio at $\theta = 0.5$ gives Bayes factor.

MCMC-Based Estimation for Savage-Dickey

When analytical solutions are difficult, use MCMC: -

Posterior and prior estimated from MCMC samples.

Heights of posterior and prior at the null point give Bayes factor.

Advantages of Savage–Dickey

Direct interpretation as density ratio.

Simplifies computation —no separate marginal likelihood calculation needed.

Works well for nested models.

Commpare Gaussian means

Common task: test if two Gaussian means differ

Example: does glucose improve detection performance

Focus: test claim that glucose boost has larger effect in summer

Data

Season	N	Mean	SD
Winter	41	0.11	0.15
Summer	41	0.07	0.23

Difference not significant

t = 0.79, p = 0.44

p-values and the Null Hypothesis

“From a null result, we cannot conclude that no difference exists…”

p = 0.44 does not support H_0

It just means data are not incompatible with H_0

Need a Bayes factor to quantify support for H_0

Bayes Factor Overview

Bayes factor compares posterior vs prior odds

Quantifies evidence for or against H_0

Unlike p-values, can support H_0

One-Sample Comparison Model

Test standardized difference scores (e.g., winter - summer)

Assume:

$\delta \sim \text{Cauchy}(0,1)$

$x_i \sim \text{Gaussian}(\mu, 1/\sigma^2)$

$\mu = \delta\sigma$

One-Sample Graphical Model

Fig 8.1

Prior on δ: Cauchy(0,1)

Prior on σ: Half-Cauchy

Estimate posterior with MCMC

Posterior vs Prior

Figure 8.2

Posterior peaks near $\delta = 0$

Bayes Factor ≈ 5:1 in favor of H_0

Order-Restricted Model

SMM predicts **$\delta < 0$**

Use order-restricted prior:

δ ~ Cauchy(0,1) truncated to $(-\infty, 0)$

Updated Bayes Factor

Figure 8.4

Stronger evidence for H_0: BF ≈ 10:1

Summary

p-values can’t confirm H_0

Bayes factors can

“Evidence of absence” of support for SMM’s prediction

Two-Sample Comparison

Compare oxygenated vs plain water on memory

Two independent groups

Two-Sample Model Structure

Figure 8.5

Shared variance σ^2

$\delta = \alpha / \sigma$

$\alpha = \mu_x - \mu_\gamma$

Large Effect Example

Group	N	Mean	SD
Plain Water	20	68.35	6.38
Oxygenated	20	76.65	4.06

$t(38) = 4.47$, $p < .01$

Two-Sample Bayes Factor

-

- Posterior moves away from 0 - BF ≈ 447:1 in favor of H_1
- Decisive evidence for oxygenated water effect

Comparing binomial rates

We will naturally compute binomial rates for different groups or conditions

And ask which is larger?

We thus need to compare binomial rates and test hypotheses about which is bigger etc.

Bayesian graphical model

Figure 9.1

graphical model for comparing two proportions

Bayesian model

We model the observed counts using binomial likelihoods and assign uniform Beta priors: s1 ~ Binomial(theta1, n1)
s2 ~ Binomial(theta2, n2)
theta1 ~ Beta(1, 1)
theta2 ~ Beta(1, 1)
delta <- theta1 - theta2

theta1:

theta2:

delta = theta1 - theta2: difference in proportions

We are interested in the posterior distribution of delta.

Model Code

Here is the model used for posterior simulation: model { theta1 ~ dbeta(1,1) theta2 ~ dbeta(1,1) delta <- theta1 - theta2 s1 ~ dbin(theta1, n1) s2 ~ dbin(theta2, n2) theta1prior ~ dbeta(1,1) theta2prior ~ dbeta(1,1) deltaprior <- theta1prior - theta2prior } This allows us to compare the prior and posterior density of delta at zero.

Prior and Posterior Distributions

Figure 9.2

We estimate the posterior distribution for the rate difference delta = theta1 - theta2 using Bayesian inference.

The left plot shows prior and posterior distributions for delta across its full range.

The right plot zooms in near delta = 0.

This is used in the Savage–Dickey density ratio to compute the Bayes factor.

The Savage–Dickey method compares: BF_01 = prior density at delta = 0 / posterior density at delta = 0

Interpreting the Bayes Factor

The posterior density at delta = 0 is about half the prior density.

This gives a Bayes factor ≈ 2 in favor of the alternative hypothesis H1: delta ≠ 0.

The 95% credible interval for delta is approximately [-0.09, 0.01], which does not include 0.

Interpretation: - There is only modest evidence that one rate is higher than another - The Bayes factor penalizes H1 for spreading prior mass over implausible values.

Inferences with Gaussians

Due to central limit theorem, data and parameters are frequently Gaussian

Gaussians have 2 parameters, a mean and a measure of their spread

Spread can be expressed as a variance, std, or a precision (1/var)

Graphical model for Gaussians

Simple model for inferring Gaussian with unknown mean and std

Interactive demo of Gaussian

Jupyter notebook - interactive plotting of gaussian

Sampling model for inferring Gaussians

model { for (i in 1:n){ x[i]~dnorm(mu,lambda) }
mu~dnorm(0,0.001) sigma~dunif(0,10)
lambda<-1/pow(sigma,2) }

Repeated measures of IQ

Imagine taking a cognitive test like IQ multiple times

The mean is your IQ, and the spread models fluctuations in your performance. e.g. attention, fatigue, emotion, venus orbitting satturn

We can model this as a Gaussian for each person

Graphical model for IQ

What parameter is common to all subjects?

No index on the std. This means it is fixed.

Is this justified?

How to change it?

Sampling code for IQ

```
model{ for (i in 1:n) { for (j in 1:m) {
x[i,j]~dnorm(mu[i],lambda) } } sigma~dunif(0,100) lambda
<-1/pow(sigma,2) for (i in 1:n) { mu([i]~ dunif(0,300)) }
}
```

tinyurl.com/bayesBinderRepo tinyurl.com/bayesGithub

Speaker notes