Calculus Final - Cheat Sheet

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1 Basic Math Skills

Simplifying Expressions

$$(a+b)^2 = a^2 + 2ab + b^2 \tag{1}$$
$$(a-b)^2 = a^2 - 2ab + b^2 \tag{2}$$
$$(a+b)(a-b) = a^2 - b^2 \tag{3}$$

Binomial Theorem

$$(x+y)^n = \sum_{k=0}^{n} \binom{n}{k} x^{n-k} y^k \tag{4}$$

$$\binom{n}{k} = \frac{n!}{k!(n-k)!} \tag{5}$$

Annoying Inequalities

$$|x| \leq a \iff -a \leq x \leq a \tag{6}$$
$$|x| \geq a \iff x \leq -a \text{ or } x \geq a \tag{7}$$

$$(a+b)^2 - (a+b) \geq 0 \tag{8}$$
$$\implies (a+b-1)(a+b) \geq 0 \tag{9}$$

Exponents

$$a^0 = 1 \tag{10}$$
$$a^m a^n = a^{m+n} \tag{11}$$
$$(a^m)^n = a^{mn} \tag{12}$$
$$(ab)^n = a^n b^n \tag{13}$$

Logarithms

$$\log_a b = c \iff a^c = b \tag{14}$$
$$\log_a b + \log_a c = \log_a bc \tag{15}$$
$$\log_a b - \log_a c = \log_a \frac{b}{c} \tag{16}$$
$$\log_a b^n = n \log_a b \tag{17}$$

Complex Numbers

cartesian form: $z = a + bi$

Convert to polar form:

$$r = \sqrt{a^2 + b^2} \tag{18}$$
$$\theta = \arctan\left(\frac{b}{a}\right) \tag{19}$$
$$z = r(\cos\theta + i\sin\theta) \tag{20}$$
$$= (r, \theta) \quad \text{polar coordinates} \tag{21}$$

Polar Form: $z = r(\cos\theta + i\sin\theta)$

Convert to cartesian form:

$$a = r\cos\theta \tag{22}$$
$$b = r\sin\theta \tag{23}$$
$$z = a + bi \tag{24}$$

Euler's Formula:

$$e^{i\theta} = \cos\theta + i\sin\theta \tag{25}$$

De Moivre's Theorem:

$$z^n = (r(\cos\theta + i\sin\theta))^n = r^n(\cos n\theta + i\sin n\theta) \tag{26}$$

Roots of Unity:

$$z^n = 1 \implies z = e^{2\pi ik/n} \quad \text{for } k = 0, 1, 2, \ldots, n-1 \tag{27}$$

nth Roots of a Complex Number

Given a complex number $z = r(\cos\theta + i\sin\theta)$, the nth roots are:

$$\sqrt[n]{z} = \sqrt[n]{r}\left(\cos\left(\frac{\theta + 2\pi k}{n}\right) + i\sin\left(\frac{\theta + 2\pi k}{n}\right)\right) \quad \text{for } k = 0, 1, 2, \ldots, n-1 \tag{28}$$

2 Trigonometry

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Table of Basic Trigonometric Values

Co-function Identities

$$\sin\theta = \cos\left(\frac{\pi}{2} - \theta\right) \tag{29}$$
$$\cos\theta = \sin\left(\frac{\pi}{2} - \theta\right) \tag{30}$$
$$\tan\theta = \cot\left(\frac{\pi}{2} - \theta\right) \tag{31}$$
$$\cot\theta = \tan\left(\frac{\pi}{2} - \theta\right) \tag{32}$$
$$\sec\theta = \csc\left(\frac{\pi}{2} - \theta\right) \tag{33}$$
$$\csc\theta = \sec\left(\frac{\pi}{2} - \theta\right) \tag{34}$$

θ	$\sin\theta$	$\cos\theta$	$\tan\theta$	$\cot\theta$	$\sec\theta$	$\csc\theta$
0	0	1	0	∞	1	∞
$\frac{\pi}{6}$	$\frac{1}{2}$	$\frac{\sqrt{3}}{2}$	$\frac{\sqrt{3}}{3}$	$\sqrt{3}$	$\frac{2\sqrt{3}}{3}$	2
$\frac{\pi}{4}$	$\frac{\sqrt{2}}{2}$	$\frac{\sqrt{2}}{2}$	1	1	$\sqrt{2}$	$\sqrt{2}$
$\frac{\pi}{3}$	$\frac{\sqrt{3}}{2}$	$\frac{1}{2}$	$\sqrt{3}$	$\frac{\sqrt{3}}{3}$	2	$\frac{2\sqrt{3}}{3}$
$\frac{\pi}{2}$	1	0	∞	0	∞	1

Reciprocal Identities

$$\csc\theta = \frac{1}{\sin\theta} \tag{35}$$
$$\sec\theta = \frac{1}{\cos\theta} \tag{36}$$
$$\cot\theta = \frac{1}{\tan\theta} \tag{37}$$

Pythagorean Identities

$$\sin^2\theta + \cos^2\theta = 1 \tag{38}$$
$$1 + \tan^2\theta = \sec^2\theta \tag{39}$$
$$1 + \cot^2\theta = \csc^2\theta \tag{40}$$

Sum and Difference Formulas

$$\sin(A \pm B) = \sin A\cos B \pm \cos A\sin B \tag{41}$$
$$\cos(A \pm B) = \cos A\cos B \mp \sin A\sin B \tag{42}$$
$$\tan(A \pm B) = \frac{\tan A \pm \tan B}{1 \mp \tan A\tan B} \tag{43}$$

Double Angle Formulas

$$\sin 2\theta = 2\sin\theta\cos\theta \tag{44}$$
$$\cos 2\theta = \cos^2\theta - \sin^2\theta \tag{45}$$
$$= 2\cos^2\theta - 1 \tag{46}$$
$$= 1 - 2\sin^2\theta \tag{47}$$
$$\tan 2\theta = \frac{2\tan\theta}{1 - \tan^2\theta} \tag{48}$$

3 Polynomials

Quadratic Formula

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \tag{49}$$

Vieta's Formulas

For a quadratic equation $ax^2 + bx + c = 0$, the roots x_1 and x_2 satisfy:

$$x_1 + x_2 = -\frac{b}{a} \tag{50}$$
$$x_1 x_2 = \frac{c}{a} \tag{51}$$

Remainder Theorem

If a polynomial $P(x)$ is divided by $x - a$, then the remainder is $P(a)$.

Factor Theorem

If $P(a) = 0$, then $x - a$ is a factor of $P(x)$.

Taylor Series

The Taylor series of a function $f(x)$ about $x = a$ is:

$$f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n \tag{52}$$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)}{2!}(x-a)^2 + \dots \tag{53}$$

4 Limits

Properties of Limits

$$\lim_{x\to a}(f(x) + g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x) \tag{54}$$
$$\lim_{x\to a}(f(x) - g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x) \tag{55}$$
$$\lim_{x\to a}(f(x)g(x)) = \lim_{x\to a} f(x) \lim_{x\to a} g(x) \tag{56}$$
$$\lim_{x\to a}\left(\frac{f(x)}{g(x)}\right) = \frac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)} \quad \text{if } \lim_{x\to a} g(x) \neq 0 \tag{57}$$

$$\lim_{x\to a^+} f(x) = \lim_{x\to a^-} f(x) = L \implies \lim_{x\to a} f(x) = L \tag{58}$$

L'Hopital's Rule

$$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)} \tag{59}$$

Euler's Limit

$$\lim_{n\to\infty}\left(1 + \frac{x}{n}\right)^n = e^x \tag{60}$$

Squeeze Theorem

If $f(x) \leq g(x) \leq h(x)$ for all x near a (except possibly at a) and $\lim_{x\to a} f(x) = \lim_{x\to a} h(x) = L$, then $\lim_{x\to a} g(x) = L$.

5 Continuity

Properties of Continuous Functions

1. $f(x)$ is continuous at $x = a$ if $\lim_{x\to a} f(x) = f(a)$.
2. If $f(x)$ and $g(x)$ are continuous at $x = a$, then $f(x) + g(x)$, $f(x) - g(x)$, $f(x)g(x)$, and $\frac{f(x)}{g(x)}$ are continuous at $x = a$.
3. If $f(g(x))$ is continuous at $x = a$ and $g(x)$ is continuous at $x = b$, then $f(g(x))$ is continuous at $x = b$.

6 Derivatives and Integrals

The Table of Derivatives and Integrals

Integration by Parts

$$\int udv = uv - \int vdu \tag{61}$$

Integration by Substitution

$$\int f(g(x))g'(x)dx = \int f(u)du \tag{62}$$

Partial Fractions

If $P(x)$ and $Q(x)$ are polynomials and $\deg P < \deg Q$, then

$$\frac{P(x)}{Q(x)} = \frac{A}{x-a} + \frac{B}{x-b} + \dots \tag{63}$$

Improper Integrals

$$\int_a^\infty f(x)dx = \lim_{b\to\infty} \int_a^b f(x)dx \tag{64}$$

$$\int_{-\infty}^\infty f(x)dx = \lim_{a\to-\infty} \lim_{b\to\infty} \int_a^b f(x)dx \tag{65}$$

7 Differential Equations

Variation of Constants

Given a differential equation of the form $y' + p(x)y = q(x)$.

1. Find the general solution to the homogeneous equation $y' + p(x)y = 0$.
2. Assume the solution to the original equation is of the form $y(x) = u(x)y_h(x)$.

3. Find $u'(x)$ and substitute into the original equation.
4. Solve for $u(x)$ and integrate to find $y(x)$.

Separation of Variables

Given a differential equation of the form $\frac{dy}{dx} = f(x)g(y)$.

1. Separate the variables and integrate.
2. Solve for y.

Method of Undetermined Coefficients

Given a differential equation of the form $y'' + ay' + by = f(x)$.

1. Find the general solution to the homogeneous equation $y'' + ay' + by = 0$. (Use the characteristic equation: $r^2 + ar + b = 0$)
   (a) If the roots are real and distinct, the general solution is $y_h(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$.
   (b) If the roots are real and repeated, the general solution is $y_h(x) = c_1 e^{rx} + c_2 x e^{rx}$.
   (c) If the roots are complex, the general solution is $y_h(x) = e^{ax}(c_1 \cos bx + c_2 \sin bx)$.
2. Make an educated guess for the particular solution $y_p(x)$.
   (a) if $f(x) = e^{kx} P_n(x)$, $P_n(x)$ is a polynomial of degree n, then $y_p(x) = e^{kx} Q_n(x)$, where $Q_n(x)$ is a polynomial of the same degree.
   (b) if $f(x) = e^{kx} P_n(x) \sin mx$ or $e^{kx} P_n(x) \cos mx$, then $y_p(x) = e^{kx} Q_n(x) \sin mx + e^{kx} R_n(x) \cos mx$, where $Q_n(x)$ and $R_n(x)$ are polynomials of the same degree as $P_n(x)$.
   (c) Note: if any term in $y_p(x)$ is a solution to the homogeneous equation, multiply by x (or sometimes x^2) to get a linearly independent solution.
3. Substitute $y(x) = y_h(x) + y_p(x)$ into the original equation and solve for the coefficients.

8 Exam Strategies

Funky limits

1. Try direct substitution.
2. Try factoring.
3. Try to rationalize the numerator/denominator (multiply by the conjugate).
4. Try L'Hopital's Rule.
5. Try Squeeze Theorem.

sin, cos integrals

Given an integral of the form $\int \sin^n x \cos^m x dx$

1. If n is odd, use $\sin^{2k+1} x = \sin^{2k} x \sin x$ and $\sin^2 x = 1 - \cos^2 x$, $u = \cos x$.
2. If m is odd, use $\cos^{2k+1} x = \cos^{2k} x \cos x$ and $\cos^2 x = 1 - \sin^2 x$, $u = \sin x$.
3. if n and m are both odd, use either of the above.
4. If n and m are both even, use $\sin^2 x = \frac{1-\cos 2x}{2}$ and $\cos^2 x = \frac{1+\cos 2x}{2}$, $u = \tan x$. Sometimes, also use $\sin 2x = 2 \sin x \cos x$.

tan, sec integrals

Given an integral of the form $\int \tan^n x \sec^m x dx$

1. If n is odd, use $\tan^{2k+1} x = \tan^{2k} x \tan x$ and $\tan^2 x = \sec^2 x - 1$, $u = \sec x$.
2. If m even, use $\sec^2 x = \tan^2 x + 1$, $u = \tan x$.
3. If n odd and m even, use either of the above.
4. If n even and m odd, good luck, you're on your own :)

Funky integrals

1. Try substitution.
2. Try integration by parts.
3. Try partial fractions.
4. Try trigonometric substitution.
5. Try to simplify the integrand.

Function	Derivative	Integral
$f(x)$	$f'(x)$	$\int f(x)dx$
x^n	nx^{n-1}	$\frac{x^{n+1}}{n+1} + C$
e^x	e^x	$e^x + C$
$\ln x$	$\frac{1}{x}$	$x \ln x - x + C$
$\log_a x$	$\frac{1}{x \ln a}$	$x \log_a x - x + C$
$\sin x$	$\cos x$	$-\cos x + C$
$\cos x$	$-\sin x$	$\sin x + C$
$\tan x$	$\sec^2 x$	$\log \lvert \sec x \rvert + C$
$\cot x$	$-\csc^2 x$	$\log \lvert \sin x \rvert + C$
$\sec x$	$\sec x \tan x$	$\log \lvert \sec x + \tan x \rvert + C$
$\csc x$	$-\csc x \cot x$	$\log \lvert \csc x - \cot x \rvert + C$
$\sin^{-1} x$	$\frac{1}{\sqrt{1-x^2}}$	$x \sin^{-1} x + \sqrt{1-x^2} + C$
$\cos^{-1} x$	$-\frac{1}{\sqrt{1-x^2}}$	$x \cos^{-1} x + \sqrt{1-x^2} + C$
$\tan^{-1} x$	$\frac{1}{1+x^2}$	$x \tan^{-1} x - \frac{1}{2} \log(1+x^2) + C$
$\cot^{-1} x$	$-\frac{1}{1+x^2}$	$x \cot^{-1} x + \frac{1}{2} \log(1+x^2) + C$
$\sec^{-1} x$	$\frac{1}{\lvert x \rvert \sqrt{x^2-1}}$	$x \sec^{-1} x + \sqrt{x^2-1} + C$
$\csc^{-1} x$	$-\frac{1}{\lvert x \rvert \sqrt{x^2-1}}$	$x \csc^{-1} x + \sqrt{x^2-1} + C$
$\sinh x$	$\cosh x$	$\cosh x + C$
$\cosh x$	$\sinh x$	$\sinh x + C$
$\tanh x$	$\operatorname{sech}^2 x$	$\log \lvert \cosh x \rvert + C$
$\coth x$	$-\operatorname{csch}^2 x$	$\log \lvert \sinh x \rvert + C$
$\operatorname{sech} x$	$-\operatorname{sech} x \tanh x$	$2 \tan^{-1} e^x + C$
$\operatorname{csch} x$	$-\operatorname{csch} x \coth x$	$\log \lvert \operatorname{csch} x - \coth x \rvert + C$