Sequential Estimation

Online learning is very important in machine learning as it allows for the inclusion of new data samples without having to recalculate model parameters for the rest of the data. The aim of this exercise is to explore this concept.

Section 1

We will now look into online estimation of a mean vector. The objective is to apply the following formula for estimating a mean *(see Bishop Section 2.3.5)*:

$$ \mu_{ML}^{N} = \mu_{ML}^{N-1} + \frac{1}{N}(x_n - \mu_{ML}^{N-1}) $$

Section 1.1

Let's first create a data generator. Create a function `gen_data(n, k, mean, var)` which returns a $n\times k$ array, X. This X contains a sequence of n vectors of dimension k. Each vector x_i in X should be $x_i \sim N_k(\mu, \sigma^2I_k)$ where:

- $N_k()$ is the k-variate normal distribution
- μ (or `mean`) is the mean vector of dimension k
- σ (or `var`) is the variance.
- I_k is the identity matrix

You should use `np.random.multivariate_normal` for this.

Example inputs and outputs (these examples use `np.random.seed(1234)`):

1. `gen_data(2, 3, np.array([0, 1, -1]), 1.3)`

```
[[ 0.61286571, -0.5482684 ,  0.86251906],
 [-0.40644746,  0.06323465,  0.15331182]]
```

2. `gen_data(5, 1, np.array([0.5]), 0.5)`

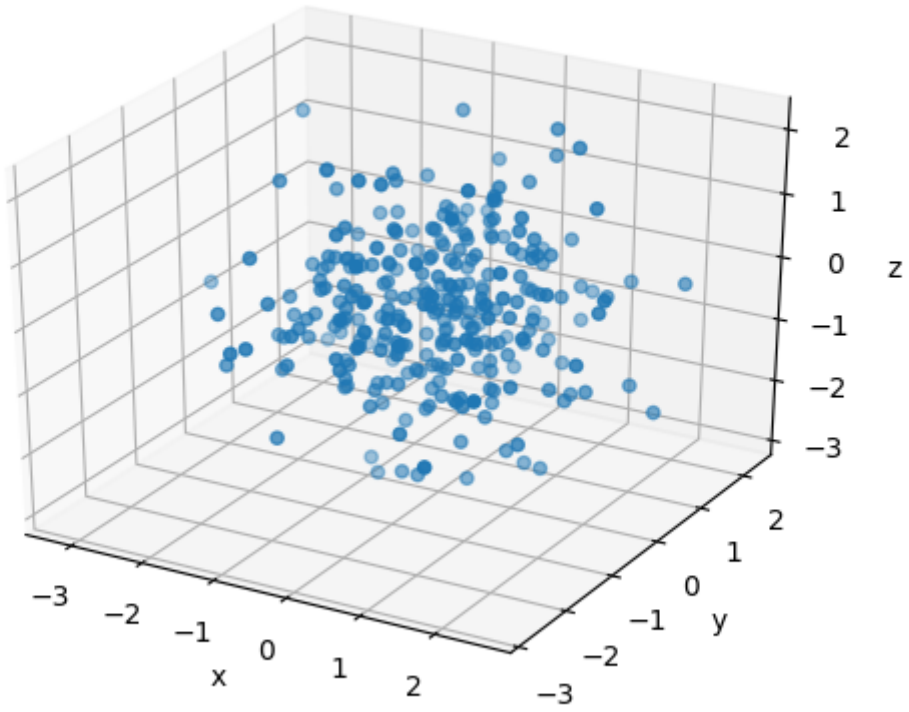
```
[[ 0.73571758],
 [-0.09548785],
 [ 1.21635348],
 [ 0.34367405],
 [ 0.13970563]]
```

Section 1.2

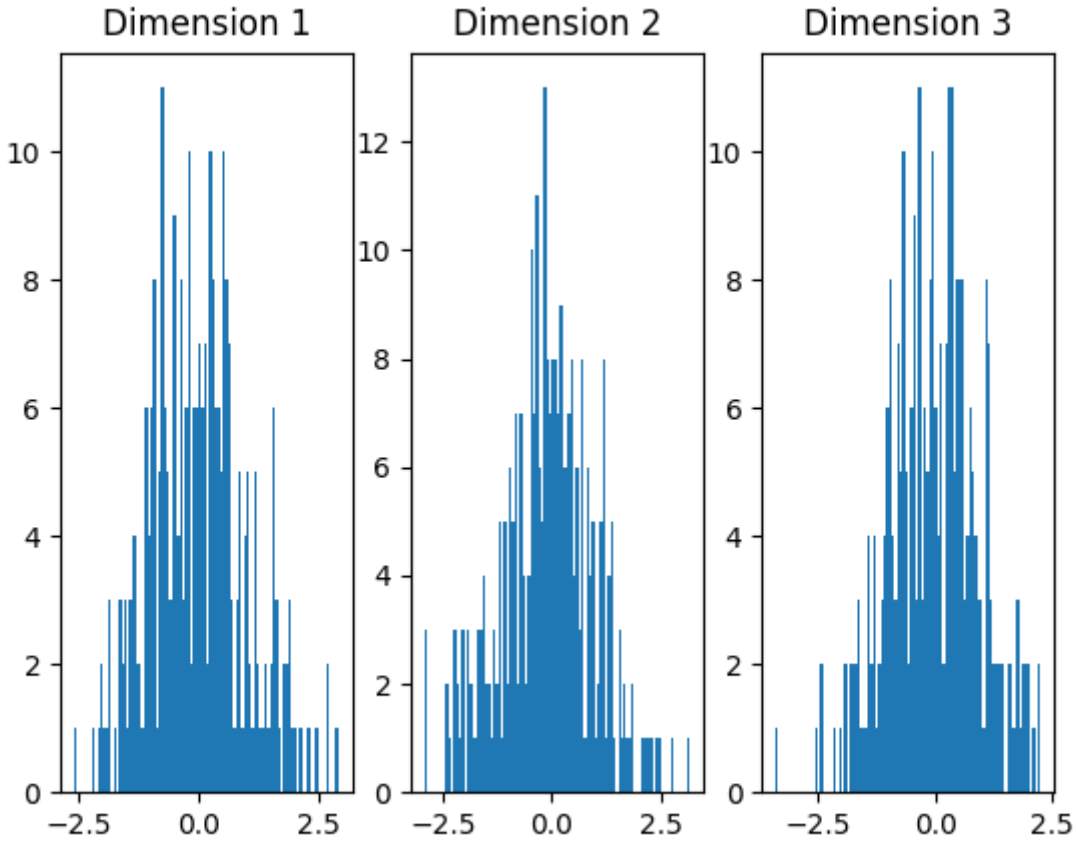
Answer this question in a pdf (you can use the same pdf as for the independend section)

Lets create some data X. Create 300 3-dimensional data points sampled from $N_3([0, 1, -1], \sqrt{3})$

You can visualize your data using `tools.scatter_3d_data` to get a plot similar to the following



You can also use `tools.bar_per_axis` to visualize the distribution of the data per dimension:



Do you expect the batch estimate to be exactly $(0, 1, -1)$? Which two parameters can be used to make this estimate more accurate?

Section 1.4

We will now implement the sequential estimate.

We want a function that returns N number of sequential estimates of the mean vector where we feed one vector from X at a time into the function. We start by implementing the update equation above.

Create a function `update_sequence_mean(mu, x, n)` which performs the update in the equation above.

Example inputs and outputs:

```
mean = np.mean(X, 0)
new_x = gen_data(1, 3, np.array([0, 0, 0]), 1)
update_sequence_mean(mean, new_x, X.shape[0])
```

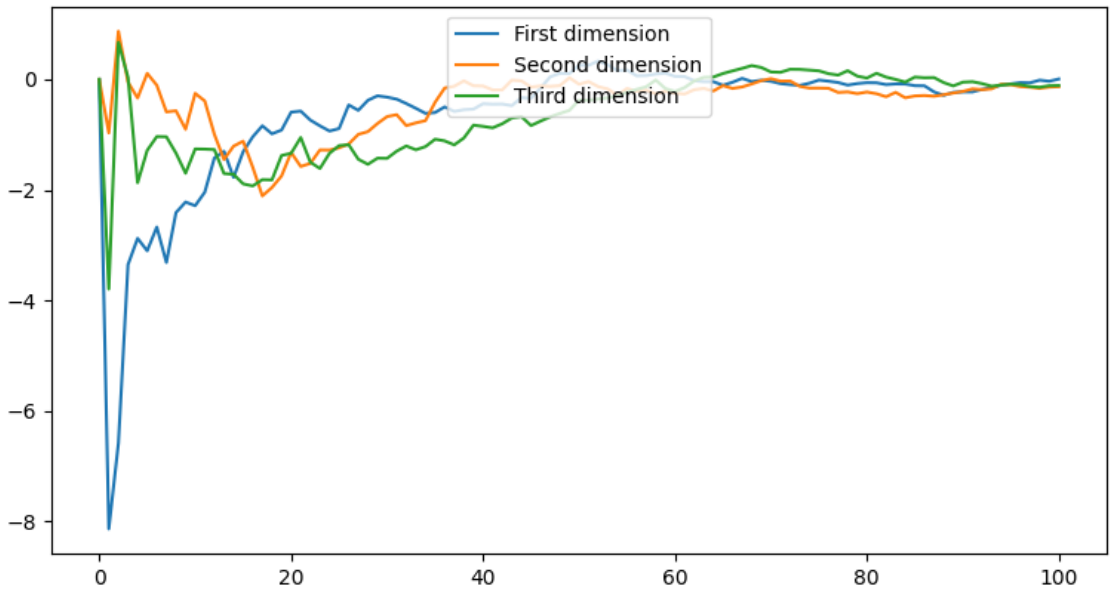
Results in an array, similar to `[[-0.21653761 -0.00721158 -0.15876203]]` (since we're using random numbers, the values you get will probably not be exactly the same).

Section 1.5

Lets plot the estimates on all dimensions as the sequence estimate gets updated. You can use `_plot_sequence_estimate()` as a template. You should:

- Generate 100 3-dimensional points with the same mean and variance as above.
- Set the initial estimate as $(0, 0, 0)$
- And perform `update_sequence_mean` for each point in the set.
- Collect the estimates as you go

For a different set of points this plot looks like the following:



Turn in your plot as `1_5_1.png`

Section 1.6

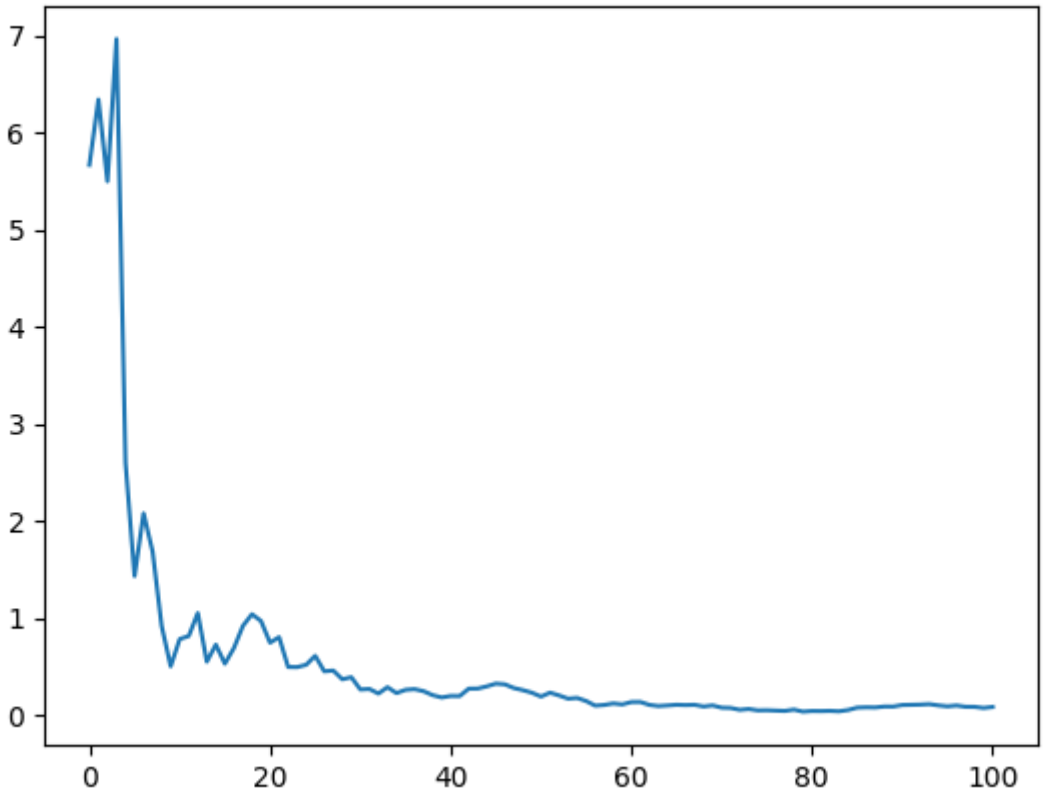
Lets now plot the squared error between the estimate and the actual mean after every update.

The squared error between e.g. a ground truth y and a prediction $\hat{y}$ is $(y-\hat{y})^2$.

Of course our data will be 3-dimensional so after calculating the squared error you will have a 3-dimensional error. Take the mean of those three values to get the average error across all three dimensions and plot those values.

You can use `_plot_square_error` and `_square_error` for this.

For a different distribution this plot looks like the following:



Turn in your plot as `1_6_1.png`

Independent Section

What happens if the mean value changes (perhaps slowly) with time? What if $\mu =(0,1,-1)$ moves to $\mu=(1,-1,0)$ in 500 time ticks? How would we track the mean? Some sort of a forgetting could be added to the update equation. How would that be done?

Create this type of data and formulate a method for tracking the mean.

Plot the estimate of all dimensions and the mean squared error over all three dimensions. Turn in these plots as `indep_1.png` and `indep_2.png`.

Write a short summary how your method works.