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Energy cost optimisation in two-machine Bernoulli serial lines under time-of-use pricing

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ABSTRACT

Energy cost optimisation in manufacturing systems has gained more and more attention. Although there are many papers about energy consumption optimisation in serial production lines, energy cost optimisation in serial production lines has rarely been focused. In this paper, we formulate an energy cost optimisation problem in two-machine Bernoulli serial line under time-of-use pricing. We analyse the structural characteristics of the problem and transform the problem into optimally allocating the production rate among the time periods of different electricity rates. A definition of the extreme allocation is proposed and completed, and the optimal allocation is proved to be one of the extreme allocations. Using the property, an efficient method to solve the optimal allocation is proposed. With the help of the method, the multi-electricity-rate problem is transformed into several single-electricity-rate problems, which has been solved in the literature.

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Production lines; time-of-use electricity pricing; energy-intensive enterprises; energy cost optimisation; production rate allocation

1. Introduction

The manufacturing systems consume a huge amount of energy. It is estimated that in China, the second industry consumes up to 67.4% energy in the first eleven months of 2020 (China Electricity Council 2020). With the ever-increasing energy cost and environmental and social concerns, reducing the total energy consumption or energy cost attracts more and more attention from manufacturing enterprises, especially from the energy-intensive enterprises.

In manufacturing systems, the production mode of most energy-intensive enterprises is mass production. The typical energy-intensive mass production manufacturing systems are, for example, assembly shops and paint shops in automobile general assembly plants (Kolta 1992; Galitsky and Worrell 2008). As shown in practical plants, the paint shops and assembly shops are usually serial lines and assembly systems, respectively (Li and Meerkov 2009), both of which are called production systems. In these production systems, machines are usually assumed to be unreliable due to random breakdowns. The goal of this paper is to reduce the total energy cost in such production systems.

In the past several years, there has been a growing research interest in the energy consumption optimisation in production systems. The main concept in these researches is that by optimising efficiencies of the unreliable machines, the total energy consumption of the system is minimised when a required production rate is ensured. The seminal work in this area is Su et al. (2016), where the energy consumption optimisation problem in the two-machine Bernoulli serial line is formulated as a non-linear programming. For this problem with machine efficiencies confined to (0,1], the relationships between the optimal solution and the system parameters are qualitatively analysed based on extensive simulations when neither quantitative results nor algorithms are developed. To make up for this deficiency, Yan et al. (2020a) analyse the characteristics of the problem and properties of the objective function in the feasible region and develop an effective algorithm to numerically optimally solve the problem. Based on the results obtained in Yan et al. (2020a), the energy consumption optimisation problem with general bounds on machine efficiencies (i.e. machine efficiencies are confined to a subset of (0,1]) is formulated and solved in Yan (2021). Besides, the analysis method and results obtained are extended to solve the problem under time-of-use (TOU) electricity pricing (Cheng, Yan, and Gao 2019) and the one in two-machine geometric serial lines (Yan and Liu 2020).

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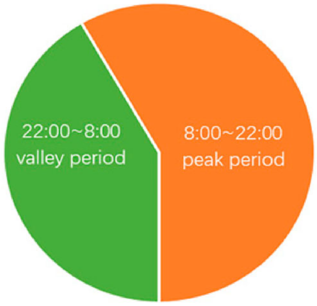

Figure 1. TOU electricity tariffs.

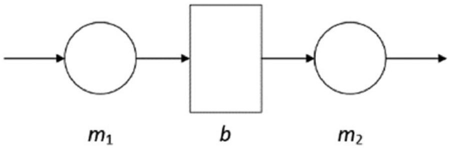

Figure 2. Two-machine Bernoulli serial line.

In addition to the two-machine serial lines, the energy consumption optimisation problem in production lines with more than two machines is studied as well. Su et al. (2017) investigate the problem in long Bernoulli serial lines. For small-size systems (i.e. with three or four machines and buffer capacities one or two), this paper suggests to solve the problem by commercial softwares, whereas for larger systems, it proposes a heuristic algorithm. To optimally solve this problem, Yan and Zheng (2020a) propose an iterative recursive algorithm, which is developed based on the aggregation method in Li and Meerkov (2009) and the solution method in Yan et al. (2020a) for two-machine lines. Although this method can (numerically) optimally solve the energy consumption optimisation problem in long Bernoulli lines, it is very time-consuming, especially for lines with more than seven machines. In this case, Yan and Zheng (2020b) propose a divide-and-conquer method, which decomposes the original problem into a series of small-size ones (most are two-machine problems), to efficiently solve it.

From the above literature review, one can see that except Cheng, Yan, and Gao (2019), almost all of the above publications focus on the reduction of the total energy consumption. Certainly, reducing the total energy consumption is beneficial for the manufacturing enterprises to reduce their energy cost. However, it is far from enough. To further reduce the energy cost, the electricity price should be considered in the optimisation model. Since the TOU electricity pricing is widely implemented in lots of countries around the world, in this paper, the energy cost reduction in energy-intensive production systems is investigated under the TOU tariffs. An example of the TOU electricity pricing is shown in Figure 1, where there are two electricity rates and the one during the peak period is higher.

The TOU pricing is a typical method for controlling the energy use from the supplier side. Since the electricity price varies from time to time under TOU tariffs, in general, the manufacturing enterprises (i.e. the demand side) would shift their production from time periods of high electricity rate to those of low rate. In other words, by appropriately responding to the TOU policy, the energy-intensive manufacturing enterprises can reduce their energy cost.

Although the TOU has been extensively investigated in buildings (Sechilariu, Wang, and Locment 2013; Xu et al. 2012; Wang et al. 2018) and electric vehicles (Davis and Bradley 2012; Dubey et al. 2015; Yang et al. 2015), in manufacturing systems, most publications focus on the scheduling, e.g. single machine scheduling (Sin and Chung 2020), parallel machine scheduling (Ding et al. 2016; Che, Zhang, and Wu 2017; Cheng, Chu, and Zhou 2018), and flow shop scheduling (Zhao, Grossmann, and Tang 2018; Zheng et al. 2020). To the best of our knowledge, except Cheng, Yan, and Gao (2019), no publications investigating the reduction of the energy cost in production systems under the TOU tariffs are available. The goal of this paper is to contribute to this end.

In this paper, the energy cost reduction in the two-machine Bernoulli serial line under TOU tariffs is studied. Specifically, the line is shown in Figure 2, where circles and the rectangle represent machines and the intermediate buffer, respectively. In contrast to Cheng, Yan, and Gao (2019), the current paper not only formulates and solves the problem under two electricity rates but also investigates the problem under any number of electricity rates.

The contribution of this paper is as follows. From the theoretical aspect, the energy cost optimisation problem in the two-machine Bernoulli serial line under TOU tariffs is formulated, analysed, and optimally solved. From the managerial aspect, by appropriately allocating the required productivity to time periods of different electricity rates, the enterprises can operate their systems with the minimal energy cost while maintaining the required productivity.

The rest of the paper is structured as follows. First, the energy cost minimisation problem investigated is formulated in Section 2. The structural characteristics of the problem are analysed, and based on these characteristics, the problem is solved in Section 3. Then, in Section 4, a property of the optimal solution is analysed, and thus, the efficiency of the solution method is improved; furthermore, a simple method is proposed for a special kind of problems. Numerical results are shown in Section 5. Finally, conclusions and future work are presented in Section 6. Proofs of theorems are provided in the Appendix.

2. Production system modelling and problem formulation

2.1. Production system model

To formalise the model of the production line shown in Figure 2, we make the following assumptions:

(i) The system consists of two machines m_1 and m_2, and a buffer b in between. The buffer capacity is N, where $0 < N < \infty$.
(ii) There are K electricity rates corresponding to K time periods. The duration and electricity price of the kth period are T_k and c_k, where $k = 1, 2, \ldots, K$.
(iii) Machines m_1 and m_2 have identical cycle time, which is denoted by τ. Without loss of generality, τ is set as 1 min.
(iv) Machine m_i, $i = 1, 2$, obeys the Bernoulli reliability model, and its efficiency in time period k is $p_{i,k}$, where $k = 1, 2, \ldots, K$. Specifically, in period k, during a cycle time, m_i is up with probability $p_{i,k}$ and down with $1 - p_{i,k}$. Herein, $p_{i,k}$ can be selected in [0, 1].
(v) The time is slotted with duration τ. The status of a machine (i.e. up or down) is determined at the beginning of each time slot and cannot change state until the time slot ends. The buffer occupancy is determined at the end of each time slot.
(vi) The blocking before service is assumed. That is to say, at the beginning of a time slot, if the buffer is empty, then m_2 is starved; if the buffer is full and m_2 fails to take a part from it, then m_1 is blocked. Machine m_1 is never starved and m_2 is never blocked.
(vii) When machine m_i, $i = 1, 2,$, is up, its electrical power is P_i; when m_i is down, its electrical power is 0. Herein, $0 < P_i < \infty$.
(viii) Transients of the production line between different electricity rates are ignored.

Remark 2.1: Assumption (viii) makes sense because from the results of settling time analysis for two-machine Bernoulli lines in Li and Meerkov (2009), it follows that the settling time of the production rate is several minutes. It should be pointed out that (1) this result is obtained for initial buffer occupancy being empty and (2) the settling time would be shorter when the initial buffer occupancy is non-empty. In other words, compared with the time period of electricity rates being measured in hours, the transients of the production system are ignorable. In the current paper, the main performance metric is long-term throughput (see Equation (3) below). From the point of view of ignoring the impact of transients on the performance metric, Assumption (viii) is necessary as well.

Remark 2.2: The current paper is intended to reduce the total energy cost of the two-machine Bernoulli line by optimising the machine efficiencies under TOU tariffs. This concept makes sense because to some extent, the machine efficiency is controllable. In practice, machines in most production lines obey continuous, e.g. exponential, reliability models. In other words, the uptime and downtime of each machine are characterised by continuous probability distributions and machine efficiency $e = \frac{T_{up}}{T_{up}+T_{down}}$, where T_{up} and T_{down} are average uptime and downtime, respectively. Although the uptime cannot be intervened (since machine's random breakdown is its intrinsic attribute), the downtime (i.e. the repair time) and, thus, the machine efficiency is controllable. To optimise the energy consumption, the exponential line is first transformed into a Bernoulli line using the exponential-to-Bernoulli transformation method in Li and Meerkov (2009), then optimal efficiencies of the Bernoulli machines are solved using the method developed in the current paper and the average downtime of the machine in the original exponential line can be obtained.

2.2. Problem formulation

In this subsection, the energy consumption optimisation problem of the two-machine Bernoulli serial line in K electricity rates is formulated. Specifically, by optimising machine efficiencies in different time periods ($p_{1,1}$, $p_{2,1}$, $p_{1,2}$, $p_{2,2}$, ... ,$p_{1,K}$, $p_{2,K}$), this paper optimises the total energy cost of the production line and maintains the total production rate not less than a required production rate (PR_{req}). Mathematically,

$$\text{(P1-}K\text{)} \min z = \sum_{k=1}^{K}\sum_{i=1}^{2} P_i \cdot p_{i,k} \cdot T_k \cdot c_k, \tag{1}$$

$$\text{s.t.}: PR \geq PR_{req} \tag{2}$$

$$PR = \frac{\sum_{k=1}^{K} T_k \cdot PR_k}{\sum_{k=1}^{K} T_k} \tag{3}$$

$$PR_k = p_{2,k} \cdot [1 - Q(p_{1,k}, p_{2,k}, N)], \ \ k = 1, 2, \ldots, K \tag{4}$$

$$0 \leq p_{i,k} \leq 1, \ \ i = 1, 2, \ \ k = 1, 2, \ldots, K \tag{5}$$

From (P1-K), one can see that the objective is to minimise the energy cost of machines during K electricity rates. Constraint (3) means that the weighted sum of production rates of K time periods should not be less than the required production rate. Constraint (4) is the expression of the production rate of the line in time period k (Li and Meerkov 2009), where

$$Q(x,y,N) = \begin{cases} \dfrac{1-x}{N+1-x}, & \text{if } x = y, \\ \dfrac{(1-x)\cdot(1-\alpha)}{1-\dfrac{x}{y}\cdot\alpha^N}, & \text{if } x \neq y, \end{cases} \tag{6}$$

$$\alpha = \frac{x\cdot(1-y)}{y\cdot(1-x)}. \tag{7}$$

Constraint (5) indicates that decision variables $p_{i,k}$ can take any value in [0, 1].

One can see that the model (P1-K) is non-linear, and the involved function Q is complicated. Furthermore, for problem (P1-K), we proposed Theorem 2.1.

Theorem 2.3: *The feasible region of problem* (P1−K) *is non-convex.*

Proof: See the Appendix.

From Theorem 2.1, it follows that (P1-K) is non-convex, which implies that it is not easy to solve (P1-K). In the following, the structural characteristics of (P1-K) are explored, and based on these structural characteristics, a method for solving (P1-K) is developed.

To investigate the structural characteristics of (P1-K), we introduce the following problem:

$$(\text{P2-}K)\ \min z = \sum_{k=1}^{K}\sum_{i=1}^{2} P_i \cdot p_{i,k}\cdot T_k \cdot c_k, \tag{8}$$

$$\text{s.t.}: \frac{\sum_{k=1}^{K} T_k \cdot PR_k}{\sum_{k=1}^{K} T_k} = PR_{req} \tag{9}$$

$$PR_k = p_{2,k}\cdot[1 - Q(p_{1,k}, p_{2,k}, N)],\ \ k = 1,2 \tag{10}$$

$$0 \le p_{i,k} \le 1,\ \ i = 1,2,\ \ k = 1,2,\ldots,K \tag{11}$$

Note that (P2-K) is the same as (P1-K) except for the production rate constraint. For this problem, we proposed Theorem 2.2

Theorem 2.4: *The optimal value, z^*, of* (P2−K) *is strictly increasing in PR_{req}.*

Proof: See the Appendix.

Assume that the optimal value of (P1-K) is z_1^* and the optimal value of (P2-K) is z_2^*. Let PR^* denote the optimal production rate of (P1-K), i.e. the production rate of the optimal solution of (P1-K). When $PR^* > PR_{req}$, according to Theorem 2.2, we have $z_1^* > z_2^*$. However, z_2^* is also a feasible solution of (P1-K), which contradicts the conclusion $z_1^* > z_2^*$. Therefore, we have $PR^* = PR_{req}$. In other words, constraint (2) is equivalent to constraint (9), and (P1-K) can be transformed to (P2-K).

Corollary 2.5: *The feasible region of* (P2−K) *is non-convex.*

This corollary can be proved by the same example in the Proof of Theorem 2.1. To illustrate the feasible region of (P2-K), an example with two electricity rates is presented in Figure 3, where $PR_{req} = 0.4$, $N = 5$, and $T_1 = T_2 = 1$.

Although (P1-K) is transformed to (P2-K) by specifying the optimal production rate, the latter is still a non-linear programming problem and hard to solve. In the following, we investigate properties of the optimal solution of (P2-K), and based on them, transform (P2-K) into a model that could be effectively solved.

3. Solution methodology

In this section, a solution method is developed to solve optimisation problem (P2-K) formulated in Section 2.2. Specifically, the idea of this method is as follows: first, the optimal production rate (i.e. the production rate of the optimal solution) in each time period is solved, and thus, the problem is decoupled from the electricity rate; then, each of the decoupled single-electricity-rate problems with their optimal production rates allocated is solved. Since the single-electricity-rate problem, i.e.

$$(\text{P0})\ \min z^s = \sum_{i=1}^{2} P_i \cdot p_i \tag{12}$$

$$\text{s.t.}: PR_{req} = p_2\cdot[1 - Q(p_1, p_2, N)] \tag{13}$$

$$0 \le p_i \le, i = 1,2 \tag{14}$$

can be (numerically) optimally solved by the method developed in Yan et al. (2020a). In fact, the production rate allocation depends not only on the electricity rates but also on the durations of time periods (see the Proof of Theorem 3.1 for details). In other words, the time period with higher electricity rate may be assigned more workload, which makes the production rate allocation non-trivial. In the following, we focus on developing a method to optimally allocate the production rate for each time period. In Section 3.1, the optimal allocation is obtained for (P2-2), i.e. the energy cost optimisation problem with two electricity rates. Then, this result is extended to obtain the optimal allocation for (P2-3) and (P2-K) for general K in Sections 3.2 and 3.3, respectively.

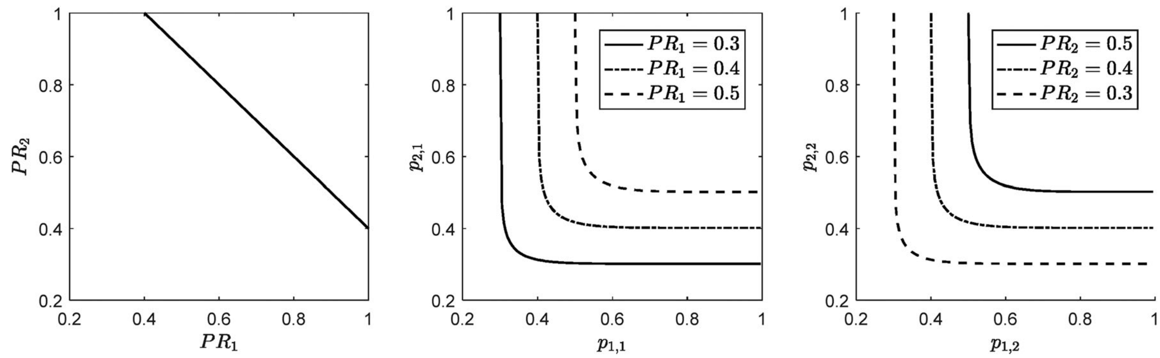

Figure 3. Feasible region of two-electricity-rate problem.

Table 1. Extreme allocations of two-electricity-rate problem.

Extreme allocations	Feasibility
$(PR_{1\min}, PR_{2\max})$	$T_1 \cdot PR_{1\min} + T_2 \cdot PR_{2\max} = (T_1 + T_2) \cdot PR_{req}$
$(PR_{1\max}, PR_{2\min})$	$T_1 \cdot PR_{1\max} + T_2 \cdot PR_{2\min} = (T_1 + T_2) \cdot PR_{req}$

3.1. Two-electricity-rate problem

When $K = 2$, (P2-K) becomes (P2-2). In the following, the optimal production rates, PR_1^* and PR_2^*, of the production line in time periods 1 and 2, respectively, are solved. From (9), it follows that solutions with production rates satisfying $PR_1 \cdot T_1 + PR_2 \cdot T_2 = PR_{req} \cdot (T_1 + T_2)$ are all feasible solutions of (P2-2).

Let $PR_{k\max}$ and $PR_{k\min}$ denote the maximum and minimum attainable production rate in time period k, $k = 1, 2$, respectively. Since both PR_1 and PR_2 are within [0,1], the maximum and minimum production rate allocated to period k are $\frac{(T_1+T_2)PR_{req}}{T_k}$ and $\frac{(T_1+T_2)PR_{req}}{T_k} - \frac{T_{3-k}}{T_k}$, respectively. Thus, we have:

$$PR_{k\max} = \min\left(1, \frac{(T_1 + T_2)PR_{req}}{T_k}\right), \quad k = 1, 2 \tag{15}$$

$$PR_{k\min} = \max\left(0, \frac{(T_1 + T_2)PR_{req}}{T_k} - \frac{T_{3-k}}{T_k}\right), \quad k = 1, 2 \tag{16}$$

Based on (15) and (16), it is easy to check that no matter which one of terms $\frac{(T_1+T_2)PR_{req}}{T_k}$ and 1 in (15) (correspondingly, $\frac{(T_1+T_2)PR_{req}}{T_k} - \frac{T_{3-k}}{T_k}$ and 0 in (16)), is greater, we have $PR_{k\max} \cdot T_k + PR_{(3-k)\min} \cdot T_{3-k} = (T_1 + T_2)PR_{req}$. In other words, as shown in Table 1, the extreme allocations $(PR_{1\min}, PR_{2\max})$ and $(PR_{1\max}, PR_{2\min})$ are two feasible allocations of (P2-2).

As a result, we proposed Theorem 3.1.

Theorem 3.1: *The optimal allocation of the production rates* (PR_1^*, PR_2^*) *to* (P2 − 2) *is either* $(PR_{1\min}, PR_{2\max})$ *or* $(PR_{1\max}, PR_{2\min})$.

Proof: See the Appendix.

Theorem 3.1 indicates that to solve (P2-2), we only need to solve the following two problems and compare their optimal objective values:

$$\text{(P3-2}')\ \min z = \sum_{k=1}^{2}\sum_{i=1}^{2} P_i \cdot p_{i,k} \cdot T_k \cdot c_k, \tag{17}$$

$$\text{s.t.}: PR_{1\max} = p_{2,1} \cdot [1 - Q(p_{1,1}, p_{2,1}, N)] \tag{18}$$

$$PR_{2\min} = p_{2,2} \cdot [1 - Q(p_{1,2}, p_{2,2}, N)] \tag{19}$$

$$0 \le p_{i,k} \le 1, \quad i = 1, 2, \quad k = 1, 2 \tag{20}$$

$$\text{(P3-2}'')\ \min z = \sum_{k=1}^{2}\sum_{i=1}^{2} P_i \cdot p_{i,k} \cdot T_k \cdot c_k, \tag{21}$$

$$\text{s.t.}: PR_{1\min} = p_{2,1} \cdot [1 - Q(p_{1,1}, p_{2,1}, N)] \tag{22}$$

$$PR_{2\max} = p_{2,2} \cdot [1 - Q(p_{1,2}, p_{2,2}, N)] \tag{23}$$

$$0 \le p_{i,k} \le 1, \quad i = 1, 2, \quad k = 1, 2 \tag{24}$$

Note that (P3-2′) and (P3-2″) are both separable with respect to the electricity rate. That is to say, by solving two single-electricity-rate problems, each of them can be solved. Thus, to solve (P2-2), we need to solve four single-electricity-rate problems, i.e. (P0) with production rate $PR_{1\min}$, $PR_{1\max}$, $PR_{2\min}$, $PR_{2\max}$, respectively. In fact, for these four optimal production rates, we proposed Theorem 3.2.

Theorem 3.2: *For feasible allocation* $(PR_{1\min}, PR_{2\max})$ *(correspondingly, for* $(PR_{1\max}, PR_{2\min})$*), one of the production rates equals to* 0 *or* 1.

Proof: See the Appendix.

According to Theorem 3.2, one of the extreme production rates is 0 or 1 in an extreme allocation. Then, the solution of the corresponding single-electricity-rate problem is 0 or $P_1 + P_2$. In other words, two of the four single-electricity-rate problems, i.e. (P0) with the optimal production rate $PR_{1\min}$, $PR_{1\max}$ and $PR_{2\min}$, $PR_{2\max}$, respectively, are trivial. Hence, solving two-electricity-rate problem (P2-2) can be transformed into solving only two single-electricity-rate problems, which can be effectively solved in Yan et al. (2020a).

3.2. Three-electricity-rate problem

When $K = 3$, (P2-K) becomes (P2-3). In this subsection, the optimal allocation of the three-electricity-rate problem is solved.

Similar to the two-electricity-rate problem, if production rates of the three-electricity-rate problem are optimally allocated, then (P2-3) can be solved by solving three simple problems. In the two-electricity-rate problem, the optimal allocation of production rates is either $(PR_{1\min}, PR_{2\max})$ or $(PR_{1\max}, PR_{2\min})$. The three-electricity-rate problem may have a similar property. It is obvious that solutions satisfying $PR_1 \cdot T_1 + PR_2 \cdot T_2 + PR_3 \cdot T_3 = PR_{req} \cdot (T_1 + T_2 + T_3)$ are all feasible solutions of (P2-3). Let the minimal production rate of time period k $PR_{k\min} = \max(0, \frac{\sum_{i=1}^{3} T_i}{T_k} PR_{req} - \frac{\sum_{i=1}^{3} T_i - T_k}{T_k})$ and the maximal production rate of time period k $PR_{k\max} = \min(1, \frac{\sum_{i=1}^{3} T_i}{T_k} PR_{req})$. To obtain extreme allocations of the three-electricity-rate problem, we reduce the three-electricity-rate problem to a master two-electricity-rate problem and a sub-problem. We regard two time periods (take periods 2 and 3 as an example) as an aggregate period. The aggregate period and the third period (period 1) compose the master problem. The aggregated two periods compose the sub-problem.

For the master problem, there are two cases:

(i) When time period 1 takes its minimal production rate, $PR_{1\min} = \max(0, \frac{T_1+T_2+T_3}{T_1} PR_{req} - \frac{T_2+T_3}{T_1})$. Then, allocate the rest production rate, $(PR_{req} - \frac{T_1}{T_1+T_2+T_3} PR_{1\min})$, to periods 2 and 3. There are also two cases:
- When time period 3 takes its maximal production rate, $PR'_{3\max} = \min(1, \frac{T_1+T_2+T_3}{T_3} PR_{req} - \frac{T_1}{T_3} PR_{1\min})$. Since period 1 already took the minimal production rate, period 2 may not be able to take the global minimal production rate. We define this production rate as the sub-minimal production rate of period 2 when period 1 takes the minimal production rate and uses $PR_{2submin,1}$ to denote it. The formula is $PR_{2\text{sub}\min,1} = \max(0, \frac{T_1+T_2+T_3}{T_2} PR_{req} - \frac{T_1}{T_2} PR_{1\min} - \frac{T_3}{T_2})$. Noticed that in the formula of $PR_{2submin,1}$, the second item is $\frac{T_1+T_2+T_3}{T_2} PR_{req} - \frac{T_1}{T_2} PR_{1\min} - \frac{T_3}{T_2}$ instead of $\frac{T_1+T_2+T_3}{T_2} PR_{req} - \frac{T_1+T_3}{T_2}$, which is the second item of PR_{2min}. That explains why the sub-minimal production rate is not global minima. We define this allocation as extreme allocation $(PR_{1\min}, PR_{2submin,1}, PR'_{3\max})$.
- When time period 2 takes its maximal production rate, $PR'_{2\max} = \min(1, \frac{T_1+T_2+T_3}{T_2} PR_{req} - \frac{T_1}{T_2} PR_{1\min})$. In this case, period 3 takes the sub-minimal production rate $PR_{3\text{sub}\min,1} = \max(0, \frac{T_1+T_2+T_3}{T_3} PR_{req} - \frac{T_1}{T_3} PR_{1\min} - \frac{T_2}{T_3})$. We define this allocation as extreme allocation $(PR_{1\min}, PR'_{2\max}, PR_{3submin,1})$.

(ii) When time period 1 takes its maximal production rate, $PR_{1\max} = \min(1, \frac{T_1+T_2+T_3}{T_1} PR_{req})$. Then, allocate the rest production rate to periods 2 and 3. There are also two cases:
- When time period 3 takes its minimal production rate, $PR'_{3\min} = \max(0, \frac{T_1+T_2+T_3}{T_3} PR_{req} - \frac{T_1}{T_3} PR_{1\max} - \frac{T_2}{T_3})$. Similar to case (i), period 2 may not be able to take the global maximal production rate because period 1 already took the maximal production rate. Therefore, we define this production rate as the sub-maximal production rate of period 2 when period 1 takes the minimal production rate and uses $PR_{2sub\max,1}$ to denote it. The formula is $PR_{2\text{sub}\max,1} = \min(1, \frac{T_1+T_2+T_3}{T_2} PR_{req} - \frac{T_1}{T_2} PR_{1\max})$.We define this allocation as extreme allocation $(PR_{1\max}, PR_{2sub\max,1}, PR'_{3\min})$.
- When time period 2 takes its minimal production rate, $PR'_{2\min} = \max(0, \frac{T_1+T_2+T_3}{T_2} PR_{req} - \frac{T_1}{T_2} PR_{1\max} - \frac{T_3}{T_2})$. In this case, period 3 takes the sub-maximal production rate $PR_{3\text{sub}\max,1} = \min(1, \frac{T_1+T_2+T_3}{T_3} PR_{req} - \frac{T_1}{T_3} PR_{1\max})$. We define this allocation as extreme allocation $(PR_{1\max}, PR'_{2\min}, PR_{3sub\max,1})$.

Table 2. Extreme allocations.

Extreme allocations	
$(PR_{1\min}, PR_{2sub\min,1}, PR'_{3\max})$	$(PR'_{1\min}, PR_{2sub\max,3}, PR_{3\max})$
$(PR_{1\min}, PR'_{2\max}, PR_{3sub\min,1})$	$(PR'_{1\min}, PR_{2max}, PR_{3sub\max,2})$
$(PR_{1\text{sub}\min,2}, PR_{2min}, PR'_{3\max})$	$(PR_{1\text{sub}\max,3}, PR'_{2\min}, PR_{3\max})$
$(PR_{1\text{sub}\min,3}, PR'_{2\max}, PR_{3min})$	$(PR_{1\text{sub}\max,2}, PR_{2max}, PR'_{3\min})$
$(PR'_{1\max}, PR_{2min}, PR_{3sub\min,2})$	$(PR_{1\max}, PR'_{2\min}, PR_{3sub\max,1})$
$(PR'_{1\max}, PR_{2sub\min,3}, PR_{3min})$	$(PR_{1\max}, PR_{2sub\max,1}, PR'_{3\min})$

One can see from above, that when we regard time periods 2 and 3 as an aggregate period, there are four extreme allocations. Apparently, when we regard periods 1 and 2 (1 and 3) as an aggregate rate, there are also four extreme allocations. In summary, for the three-electricity-rate problem, there are 12 extreme allocations as shown in Table 2.

The extreme allocations in the left column are similar to the right column except for the sub-minimal one and the sub-maximal one. We compare the formulas of $(PR_{1\,\min}, PR_{2sub\,\min,1}, PR'_{3\,\max})$ and $(PR'_{1\,\min}, PR_{2sub\,\max,3}, PR_{3\,\max})$.

$$PR'_{1\,\min} = \max\left(0, \frac{T_1+T_2+T_3}{T_1}PR_{req} - \frac{T_3}{T_1}PR_{3\,\max} - \frac{T_2}{T_1}\right) \tag{25}$$

$$PR_{2\mathrm{sub}\,\min,1} = \max\left(0, \frac{T_1+T_2+T_3}{T_2}PR_{req} - \frac{T_1}{T_2}PR_{1\,\min} - \frac{T_3}{T_2}\right) \tag{26}$$

$$PR_{3\,\max} = \min\left(1, \frac{T_1+T_2+T_3}{T_3}PR_{req}\right) \tag{27}$$

$$PR_{1\,\min} = \max\left(0, \frac{T_1+T_2+T_3}{T_1}PR_{req} - \frac{T_1+T_3}{T_1}\right) \tag{28}$$

$$PR_{2\mathrm{sub}\,\max,3} = \min\left(1, \frac{T_1+T_2+T_3}{T_2}PR_{req} - \frac{T_3}{T_2}PR_{3\,\max}{}'\right) \tag{29}$$

$$PR'_{3\,\max} = \min\left(1, \frac{T_1+T_2+T_3}{T_3}PR_{req} - \frac{T_1}{T_3}PR_{1\,\min}{}'\right) \tag{30}$$

Substitute (27) into (25), (25) can be rewritten as $PR'_{1\,\min} = \max(0, \frac{T_1+T_2+T_3}{T_1}PR_{req} - \frac{T_2+T_3}{T_1}, -\frac{T_2}{T_1})$. Apparently, $-\frac{T_2}{T_1} < 0$. Therefore, we have $PR'_{1\,\min} = \max(0, \frac{T_1+T_2+T_3}{T_1}PR_{req} - \frac{T_2+T_3}{T_1}, -\frac{T_2}{T_1}) = \max(0, \frac{T_1+T_2+T_3}{T_1}PR_{req} - \frac{T_2+T_3}{T_1}) = PR_{1\,\min}$.

Substitute (28) into (30), (30) can be rewritten as $PR'_{3\,\max} = \min(1, \frac{T_1+T_2+T_3}{T_3}PR_{req}, \frac{T_1+T_3}{T_3})$. Apparently, $\frac{T_1+T_3}{T_1} > 1$. Therefore, we have $PR'_{3\,\max} = \min(1, \frac{T_1+T_2+T_3}{T_3}PR_{req}, \frac{T_1+T_3}{T_3}) = \min(1, \frac{T_1+T_2+T_3}{T_3}PR_{req}) = PR_{3\,\max}$.

Since $PR_{1\,\min} = PR'_{1\,\min}$ and $PR'_{3\,\max} = PR_{3\,\max}$, according to (9), we have $PR_{2\mathrm{sub}\,\min,1} = PR_{2\mathrm{sub}\,\max,3}$. Therefore, $(PR_{1\,\min}, PR_{2sub\,\min,1}, PR'_{3\,\max})$ is the same as $(PR'_{1\,\min}, PR_{2sub\,\max,3}, PR_{3\,\max})$. That is to say, there are only six extreme allocations as shown in Table 3. We define the extreme allocation as $(PR_{k_1\,\max}, PR_{k_2 sub,k_1}, PR_{k_3 min})$ in summary, and the formulas are

$$PR_{k_1\,\max} = \min\left(1, \frac{\sum_{k=1}^{3} T_k}{T_{k_1}}PR_{req}\right) \tag{31}$$

$$PR_{k_2 sub,k_1} = \min\left(1, \frac{\sum_{k=1}^{3} T_k}{T_{k_2}}PR_{req} - \frac{T_{k_1}}{T_{k_2}}PR_{k_1\,\max}\right) \tag{32}$$

$$PR_{k_3\,\min} = \max\left(0, \frac{\sum_{k=1}^{3} T_k}{T_{k_3}}PR_{req} - \frac{T_{k_1}+T_{k_2}}{T_{k_3}}\right) \tag{33}$$

We can see from above that the three-electricity-rate problem has 12 extreme allocations, but 6 of them are repetitive. Then to obtain the optimal allocation of the three-electricity-rate problem, Theorem 3.3 is proposed.

Theorem 3.3: *The optimal allocation of the production rates (PR_1^*, PR_2^*, PR_3^*) to (P2-3) is one of the extreme allocations.*

Proof: See the Appendix.

According to Theorem 3.3, (P2-3) can be solved by solving 18 single-electricity-rate problems. To obtain extreme allocations, we proposed Theorem 3.4.

Theorem 3.4: *For any extreme allocation of* (P2-3), *among the three production rates for three electricity periods, two of them are equal to either* 0 *or* 1.

Theorem 3.4 can be proved similarly to Theorem 3.2.

According to Theorem 3.4, the optimal values of two single-electricity-rate problems are 0 or $P_1 + P_2$ in every extreme allocation. In other words, 12 of the 18 single-electricity-rate problems are trivial. Therefore, optimising the three-electricity-rate problem can be transformed into optimising 6 single-electricity-rate problems at most.

To further illustrate that the production rate allocation is non-trivial, a three-electricity-rate problem is exemplified. Let $PR_{req} = 0.25$, $P_1 = P_2 = 1$, $N = 1$, $T_1 = 18$, $T_2 = 4$, $T_3 = 2$, $c_1 = 8$, $c_2 = 10$, and $c_3 = 14$. If the production rate is allocated to time periods in terms of the low-electricity-rate first rule, then the allocation should be $(PR_{1\,\max}, PR_{2\mathrm{sub},1}, PR_{3\,\min})$ and its objective value $z = 143.9568$. However, the optimal allocation solved by the above method is $(PR_{1\,\min}, PR_{2\,\max}, PR_{3\mathrm{sub},2})$, and its optimal value is $z^* = 136$. In other words, the low-electricity-rate first rule does not work in this example.

3.3. K-electricity-rate problem

Similar to the two-electricity-rate problem and the three-electricity-rate problem, the optimal allocation of the K-electricity-rate problem is still one of the extreme allocations.

To obtain the number of extreme allocations of the K-electricity-rate problem, we start with the four-electricity-rate problem. We choose one time period and

Table 3. Extreme allocations.

Extreme allocations	Feasibility
$(PR_{1\min}, PR_{2sub,3}, PR_{3\max})$	$T_1 \cdot PR_{1\min} + T_2 \cdot PR_{2sub,3} + T_3 \cdot PR_{3\max} = (T_1 + T_2 + T_3) \cdot PR_{req}$
$(PR_{1\min}, PR_{2max}, PR_{3sub,2})$	$T_1 \cdot PR_{1\min} + T_2 \cdot PR_{2\max} + T_3 \cdot PR_{3sub,2} = (T_1 + T_2 + T_3) \cdot PR_{req}$
$(PR_{1sub,3}, PR_{2min}, PR_{3\max})$	$T_1 \cdot PR_{1sub,3} + T_2 \cdot PR_{2\min} + T_3 \cdot PR_{3\max} = (T_1 + T_2 + T_3) \cdot PR_{req}$
$(PR_{1sub,2}, PR_{2max}, PR_{3min})$	$T_1 \cdot PR_{1sub,2} + T_2 \cdot PR_{2\max} + T_3 \cdot PR_{3\min} = (T_1 + T_2 + T_3) \cdot PR_{req}$
$(PR_{1\max}, PR_{2min}, PR_{3sub,1})$	$T_1 \cdot PR_{1\max} + T_2 \cdot PR_{2\min} + T_3 \cdot PR_{3sub,1} = (T_1 + T_2 + T_3) \cdot PR_{req}$
$(PR_{1\max}, PR_{2sub,1}, PR_{3min})$	$T_1 \cdot PR_{1\max} + T_2 \cdot PR_{2sub,1} + T_3 \cdot PR_{3\min} = (T_1 + T_2 + T_3) \cdot PR_{req}$

regard the rest three periods as an aggregate period. Then, we allocate the maximal production rate to the chosen period and allocate the minimal production rate to the aggregate period. The allocation problem of the aggregate period can be regarded as a three-electricity-rate problem. As shown in Section 3.2, this problem has 12 extreme allocations. Apparently, when we allocate the minimal production rate to the chosen period, there are also 12 extreme allocations. Since every period can be the chosen period, the four-electricity-rate problem has 96 extreme allocations at most. This conclusion can be extended to the K-electricity-rate problem that the K-electricity-rate model has $2^{K-2} \cdot K!$ extreme allocations. However, as we discuss in Section 3.2, plenty of the extreme allocations are redundant.

To avoid redundancy, a new method to obtain extreme allocations of the K-electricity-rate problem is proposed. We select time period k_1 and regard the rest $K-1$ periods as one period. Allocate the production rate between period k_1 and the rest periods. Let period k_1 take the maximal production rate. Then, select period k_2 and regard the rest $K-2$ periods as one period. Let period k_2 take the maximal production rate. Due to period k_1 already taking the maximal production rate, period k_2 may not be able to take the global maximal production rate. We define this production rate as the 2nd-maximal production rate of period k_2. Then let period k_3 take the 3rd-maximal production rate and the rest can be deduced analogically. The last period takes the minimal production rate. The formulas of $PR_{k\max}$, PR_{kmin} are as follows:

$$PR_{k\max} = \min\left(1, \frac{\sum_{i=1}^{K} T_i}{T_k} PR_{req}\right) \tag{34}$$

$$PR_{kmin} = \max\left(0, \frac{\sum_{i=1}^{K} T_i}{T_k} PR_{req} - \frac{\sum_{i=1}^{K} T_i - T_k}{T_k}\right) \tag{35}$$

The formulas of production rates of the rest $K-2$ time periods can be summarised as $PR_{k_i i^{th}-\max,S_{k_i}} = \min(1, \frac{\sum_{i=1}^{K} T_i}{T_k} PR_{req} - \frac{\sum_{x\in S} T_{k_x} PR_{k_x x^{th}-\max,S_{k_x}}}{T_{k_i}})$, in which S_{k_i} represents to the ordered list of time periods which are allocated the production rate prior to period k_i.

Table 4. Number of extreme allocations.

K	Number of extreme allocations	Number of extreme allocations without repetitive ones
3	12	6
6	11,520	720
12	4.90e+11	4.79e+8
24	2.60e+30	6.20e+23

After pruning the redundant allocations, the number of extreme allocations of the K-electricity-rate problem is cut down from $2^{K-2} \cdot K!$ to $K!$ as shown in Table 4.

Especially, when $K = 2$, the extreme allocation is either time period 1 or time period 2 taking the maximal production; when $K = 3$, the extreme allocations are one time period taking the minimal production rate, another period taking the 2nd-maximal production rate (which is the same as the sub-maximal production rate), and the last one taking the maximal production rate. One can see that when $K = 2, 3$, the extreme allocations come out with the new method are the same as the extreme allocations in Sections 3.1 and 3.2. To prove the optimal allocation of the K-electricity-rate problem, which is still one of the extreme allocations, we propose Theorem 3.5.

Theorem 3.5: *The optimal allocation of the production rates ($PR_1^*, PR_2^*, \ldots, PR_K^*$) to* (P2−$K$) *is one of the extreme allocations.*

Theorem 3.5 can be proved similarly to Theorem 3.3.

The K-electricity-rate problem has $K!$ extreme allocations, which means that we need to optimise $K \cdot K!$ single-electricity-rate problems to find the optimal solution. To further decrease the number of single-electricity-rate problems that need to be optimised, we propose Theorem 3.6.

Theorem 3.6: *For any extreme allocation of (P2-K), among the K production rates for the time periods, $K-1$ of them is equal to* 0 *or* 1.

Theorem 3.6 can be proved similarly to Theorem 3.2.

One can see from Theorem 3.6 that the production rate of only one time period is not constant in an extreme allocation. We take the production rate of period k_1 is not constant as an example. The rest $K-1$ periods can take 0

Table 5. Number of single-electricity-rate problems under different K.

K	Number of single-electricity-rate problems before reduction	Number of single-electricity-rate problems after reduction
3	18	6
6	4320	60
12	5.75e+9	5544
24	6.20e+23	3.24e+7

or $PR_{\max}$. We determine an allocation of production rates of the rest $K-1$ periods, then we can get a corresponding production rate of period k_1. Therefore, obtaining the number of possible production rates of period k_1 can be regarded as a combinatorial problem. The maximal number of possible production rates of period k_1 is $C_{K-1}^{[(K-1)/2]}$. Noticed that not every allocation of production rates of the rest $K-1$ periods can elicit a possible value of production rate of the period k_1, since constraint (9) need to be satisfied. Besides, every period can be period k_1. Therefore, the maximal number of possible production rates is $K \cdot C_{K-1}^{[(K-1)/2]}$. In other words, the number of single-electricity-rate problems need to be solved, which is cut down from $K \cdot K!$ to $K \cdot C_{K-1}^{[(K-1)/2]}$ as shown in Table 5.

4. Further analysis

4.1. Solution methodology improvement

In Section 3, we transform optimising the K-electricity-rate problem into optimising $K \cdot C_{K-1}^{[(K-1)/2]}$ single-electricity-rate problems. However, the number of single-electricity-rate problems increases with K dramatically as shown in Table 5. Therefore, we further work on improving the solution methodology.

Let K_p represent the number of time periods that need to work in an allocation. According to Theorem 3.6, the optimal allocation is $K_p - 1$ periods taking 1 and one period (mark as k_r) taking the rest production rate. Define the set of the $K_p - 1$ periods as S_{full} and the set of the rest period k_r as $S_{partial}$. The production rate of k_r is $\frac{\sum_{x=1}^{K} T_x}{T_{k_r}} PR_{req} - \frac{\sum_{x\in S_{full}} T_x}{T_{k_r}}$. To improve the methodology of obtaining the optimal allocation, we need to obtain the priorities of electricity periods in the optimal allocation. Substituting one period in the allocation with another and comparing the electricity cost is a method to obtain the priorities of periods. However, since durations of time periods may be different, substituting time periods can lead to many different conditions. To simplify the analysis, we classify allocations by $S_{partial}$. Specifically, we let every period be $S_{partial}$ respectively and obtain the optimal allocation of every $S_{partial}$. Then, we compare the allocations and obtain the global optimal allocation. To obtain the optimal allocation of every $S_{partial}$, we need to obtain the priorities of periods that belong to S_{full}. We substitute one period that belongs to S_{full} with another period that does not belong to S_{full} and obtain Theorem 4.1.

Theorem 4.1: *Assume that there are two periods k_1 and k_2 satisfying $k_1 \notin S_{full}$, $k_2 \in S_{full}$, $T_{k_1}c_{k_1} \le T_{k_2}c_{k_2}$. When $T_{k_1} \ge T_{k_2}$, substituting k_2 with k_1 will decrease the optimal value; when $T_{k_1} < T_{k_2}$ and $\frac{T_{k_r}\cdot c_{k_r}}{z^{s*}(PR_{\max})} \ge \frac{T_{k_1}c_{k_1} - T_{k_2}c_{k_2}}{z^{s*}(PR_{k_r}) - z^{s*}(PR_{k_r} - \frac{T_{k_2}-T_{k_1}}{T_{k_r}} PR_{\max})}$, substituting k_2 with k_1 will decrease the optimal value, and vice versa.*

***Proof*:** See the Appendix.

Lemma 4.1: *Assume that there are two groups of periods S_1 and S_2 satisfying $S_1 \not\subset S_{full}$, $S_2 \subseteq S_{full}$, $\sum_{x\in S_1} T_x c_x \le \sum_{x\in S_2} T_x c_x$. When $\sum_{x\in S_1} T_x \ge \sum_{x\in S_2} T_x$, substituting S_2 with S_1 will decrease the optimal value; when $\sum_{x\in S_1} T_x < \sum_{x\in S_2} T_x$ and $\frac{T_{k_r}\cdot c_{k_r}}{z^{s*}(PR_{\max})} \ge \frac{\sum_{x\in S_1} T_x c_x - \sum_{x\in S_2} T_x c_x}{z^{s*}(PR_{k_r}) - z^{s*}(PR_{k_r} - \frac{\sum_{x\in S_2} T_x - \sum_{x\in S_1} T_x}{T_{k_r}} PR_{\max})}$, substituting S_2 with S_1 will decrease optimal value, and vice versa.*

According to Theorem 4.1, one can see that the period with lower Tc has higher priority in most cases. Therefore, we use Tc as a preliminary evaluation criterion. Since the period with larger Tc and longer T may be better in some cases, the preliminary evaluation criterion is not completely credible. In that case, we use a method similar to bubble sort to compare the rest periods with the periods in the preliminary allocation and obtain the optimal allocation of every $S_{partial}$. The algorithm is shown below.

Algorithm:

Assume k_r is $S_{partial}$.

Calculate the range $[T_{\min}, T_{\max}]$ of total length of S_{full} based on $S_{partial}$.

Arrange periods in an ascending order according to Tc

Allocate production rate for periods sequence from small to large until the total length T_{full} is higher than $T_{\min}$.

Assume k_a is the last period that has been allocated.

Initialise the compare set $S_{comp} = \emptyset$, the full set $S_{full} = \{1, 2, \ldots, k_a\}$, and the zero set $S_{zero} = \{k_a + 1, k_a + 2, \ldots, K - 1\}$.

1: for $x = 1$ to $length[S_{zero}]$ do
2: $\quad S_{comp} = S_{zero}(x)$
3: $\quad$ for $y = length[S_{full}]$ to 1 do

4: if $\sum_{S_{comp}} T > T_{S_{full}(y)}$ and $T_{full} - T_{S_{full}(y)} + \sum_{S_{comp}} T < T_{\max}$
5: if exchanging period $S_{zero}(x)$ with periods in S_{comp} reduces electricity cost
6: Switch S_{comp} with $S_{full}(y)$
7: $T_{full} = \sum_{S_{full}} T$
8: end if
9: end if
10: end
11: end

Theorem 4.2: *Substituting any periods belonging to S_{full} with periods that have been weeded out by the algorithm cannot reduce the electricity cost.*

***Proof*:** See the Appendix.

Theorem 4.2 confirms that the allocation obtained through the algorithm is the optimal allocation of $S_{partial}$. We randomly generated 1000 6-electricity-rate problems, and the numerical experience confirms that the allocation obtained through the methodology is the optimal allocation.

4.2. Sufficient conditions of the optimal allocation

In this subsection, we further analyse the impacts of duration, electricity price, and required production rate on optimal allocation. Using the conclusions, we propose some sufficient conditions of the optimal allocation.

It is shown in Section 3 that the optimal allocation is one of the extreme allocations. To obtain the priority of periods, three basic differences of allocations are divided.

(i) S_{full} is different.

Assume that there are two allocations satisfying $S^1_{full} - S^2_{full} = k_1$, $S^2_{full} - S^1_{full} = k_2$, and $S^1_{partial} = S^2_{partial} = k_r$. The optimal values' difference of this condition can be written as follows:

$$\begin{aligned}\Delta z &= z^{s*}(PR_{\max})T_{k_1}c_{k_1} + z^{s*}(PR_{k_r})T_{k_r}\cdot c_{k_r} \\ &\quad - z^{s*}(PR_{\max})T_{k_2}c_{k_2} - z^{s*}(PR_{k_r}{}')T_{k_r}\cdot c_{k_r} \\ &= z^{s*}(PR_{\max})T_{k_1}c_{k_1} + z^{s*}\left(\frac{\sum_{i=1}^{k_r} T_i}{T_{k_r}}PR_{req} - \frac{\sum_{i\in S^1_{full}} T_i}{T_{k_r}}PR_{\max}\right) T_{k_r}\cdot c_{k_r} - z^{s*}(PR_{\max})T_{k_2}c_{k_2} \\ &\quad - z^{s*}\left(\frac{\sum_{i=1}^{k_r} T_i}{T_{k_r}}PR_{req} - \frac{\sum_{i\in S^2_{full}} T_i}{T_{k_r}}PR_{\max}\right) \\ &\quad \times T_{k_r}\cdot c_{k_r},\end{aligned}$$

When $T_{k_1}c_{k_1} < T_{k_2}c_{k_2}$ and $T_{k_1} \geq T_{k_2}$, we have $T_{k_r}(PR_{k_r} - PR_{k_r}{}') \leq 0$, which means $PR_{k_r} \leq PR_{k_r}{}'$. Therefore, we have $\Delta z < 0$, which means allocation 1 is better.

(ii) $S_{partial}$ is different.

Assume that there are two allocations satisfying $S^1_{full} = S^2_{full}$, $S^1_{partial} = k_1$, and $S^2_{partial} = k_2$. The electricity cost difference of two allocations is $\Delta z = z^{s*}(PR_{k_1})T_{k_1}\cdot c_{k_1} - z^{s*}(PR_{k_2})T_{k_2}\cdot c_{k_2}$. Production rates of k_1 and k_2 satisfy $T_{k_1}PR_{k_1} = T_{k_2}PR_{k_2}$.

When $T_{k_1}c_{k_1} < T_{k_2}c_{k_2}$ and $T_{k_1} \geq T_{k_2}$, we have $PR_{k_1} \leq PR_{k_2}$. Therefore, we have $\Delta z < 0$, which means allocation 1 is better.

(iii) A period belonging to S_{full} switches with the period belonging to $S_{partial}$.

Assume that there are two allocations satisfying $S^1_{full} - S^2_{full} = k_1$, $S^1_{partial} = k_2$, and $S^2_{full} - S^1_{full} = k_2$, $S^2_{partial} = k_1$. The electricity cost difference of two allocations is

$$\begin{aligned}\Delta z &= z^{s*}(PR_{\max})T_{k_1}c_{k_1} + z^{s*}(PR_{k_2})T_{k_2}c_{k_2} \\ &\quad - z^{s*}(PR_{k_1})T_{k_1}c_{k_1} - z^{s*}(PR_{\max})T_{k_2}c_{k_2} \\ &= [z^{s*}(PR_{\max}) - z^{s*}(PR_{k_1})]T_{k_1}c_{k_1} \\ &\quad - [z^{s*}(PR_{\max}) - z^{s*}(PR_{k_2})]T_{k_2}c_{k_2},\end{aligned}$$

in which $T_{k_1}PR_{\max} + T_{k_2}PR_{k_2} = T_{k_1}PR_{k_1} + T_{k_2}PR_{\max}$.

When $T_{k_1}c_{k_1} < T_{k_2}c_{k_2}$ and $T_{k_1} \geq T_{k_2}$, which means $PR_{k_1} \geq PR_{k_2}$, we have $z^{s*}(PR_{\max}) - z^{s*}(PR_{k_1}) \leq z^{s*}(PR_{\max}) - z^{s*}(PR_{k_2})$. Therefore, we have $\Delta z < 0$, which means allocation 1 is better.

We give a numerical example in which $PR_r = 0.8$, $T_1c_1 = 10$, and $T_2c_2 = 10.1$. Let T_2 increase from $0.3\cdot T_1$ to T_1. One can see from Figure 4 that the results always satisfy $z^{s*}(PR_{1\max}) + z^{s*}(PR_{2\min}) < z^{s*}(PR_{1\min}) + z^{s*}(PR_{2\max})$ in the condition $T_1 \geq T_2$.

In summary, the sufficient condition of the optimal allocation can be concluded as when $T_{k_1}c_{k_1} < T_{k_2}c_{k_2}$ and $T_{k_1} \geq T_{k_2}$, allocating production rate to period k_1 is better than period k_2. It is obvious that in this condition, we have $c_{k_1} < c_{k_2}$. We can conclude a more general conclusion that is when durations of periods are equal, allocating production rate to the period with lower electricity price is better. The TOU tariffs of provinces and cities of China are listed in Table 6. One can see that TOU tariffs of most provinces and cities satisfy the sufficient condition.

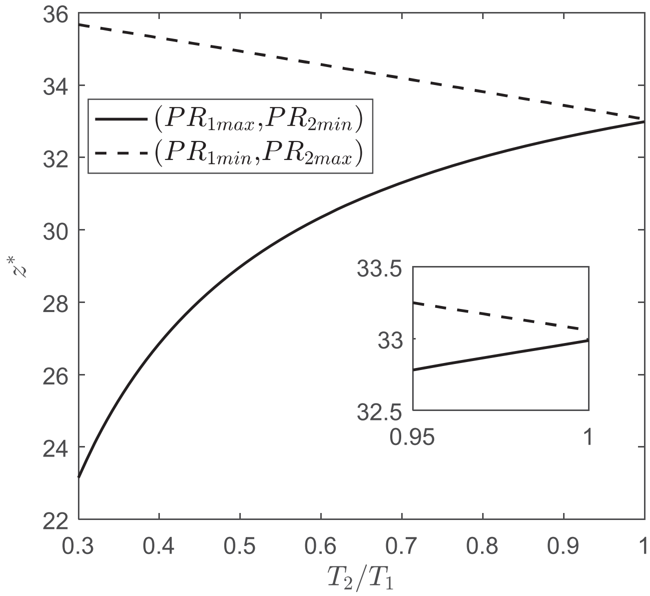

Figure 4. The optimal value comparison between $(PR_{1\,\max}, PR_{2\,\min})$ and $(PR_{1\,\min}, PR_{2\,\max})$.

5. Numerical results

In this section, we describe experiments corresponding to Sections 3 and 4 to verify the conclusions.

5.1. Optimal allocation and optimal solution of a 3-electricity-rate problem

In this subsection, we give some numerical results of a 3-electricity-rate problem. Let $c_1 = 6.7, c_2 = 10.4, c_3 = 12.4, T_1 = 12, T_2 = 8, T_3 = 4, P_1 = P_2 = 1, N = 5$. We use an enumeration method to calculate the electricity cost of all extreme allocations and obtain optimal allocations and optimal solutions under different PR_{req} as shown in Table 7.

Table 7. The optimal allocations and optimal solutions under different PR_{req}.

PR_{req}	PR_1^*	$p_{1,1}^*$	$p_{1,2}^*$	PR_2^*	$p_{2,1}^*$	$p_{2,2}^*$	PR_3^*	$p_{3,1}^*$	$p_{3,2}^*$
0.1	0.2	0.23	0.23	0	0	0	0	0	0
0.3	0.6	0.64	0.64	0	0	0	0	0	0
0.5	1	1	1	0	0	0	0	0	0
0.7	1	1	1	0.6	0.64	0.64	0	0	0
0.9	1	1	1	1	1	1	0.4	0.44	0.44

The relationship between optimal production rates is shown in Figure 5.

One can see from the curve that when $PR_{req} \le \frac{1}{2}$, the optimal allocation is $PR_2^* = 0$, $PR_3^* = 0$, and allocating all production rate in period 1. When $\frac{1}{2} < PR_{req} \le \frac{5}{6}$, the optimal allocation is $PR_1^* = 1$, $PR_3^* = 0$, and allocating the rest production rate in period 2. When $PR_{req} > \frac{5}{6}$, the optimal allocation is $PR_1^* = 1$, $PR_2^* = 1$, and allocating the rest production rate in period 3. That is to say, the optimal allocation in this case is always one of the extreme allocations. Besides, for the extreme allocation, two of the elements equal to either 0 or 1, which follows our conclusions. Therefore, we can obtain the optimal solution of the three-electricity-rate problem by optimising 6 single-electricity-rate problems instead of 36 single-electricity-rate problems.

Table 6. TOU tariffs of provinces in China.

	Period (hour)			Price (CNY/$kw \cdot h$)			
Province/city	T_1	T_2	T_3	c_1	c_2	c_3	Pairs of periods that satisfy sufficient condition
Shenzhen	8	9	7	0.2286	0.715	0.855	(1, 3), (2, 3)
Jiangsu	8	16		0.305	0.735		None
Zhejiang	10	14		0.376	0.975		None
Anhui	9	7	8	0.238	0.438	0.679	(1, 2), (1, 3)
Fujian	8	16		0.435	0.863		None
Hebei	8	8	8	0.277	0.659	0.978	All
Hubei	8	10	6	0.366	0.689	1.185	(1, 3), (2, 3)
Guangxi	8	8	8	0.354	0.745	0.823	All
Tianjin	8	8	8	0.342	0.71	1.11	All
Jiangxi	6	12	6	0.49	0.949	1.408	None
Shaanxi	8	8	8	0.267	0.49	0.713	All
Jilin	7	9	8	0.264	0.486	0.709	(2, 3)
Shanghai	8	8	8	0.302	0.691	1.014	All
Beijing	8	8	8	0.3053	0.7275	1.1753	All
Shandong	8	8	8	0.308	0.712	1.116	All
Shanxi	8	8	8	0.336	0.732	1.149	All
Sichuan	8	8	8	0.357	0.833	1.308	All
Hunan	8	16		0.711	0.993		None
Liaoning	7	9	8	0.438	0.836	1.235	(2, 3)
Guangzhou	8	8	8	0.411	0.801	1.253	All
Henan	8	8	8	0.303	0.582	0.862	All
Chongqing	8	8	8	0.29	0.611	0.965	All
Yunnan	8	8	8	0.356	0.535	0.714	All

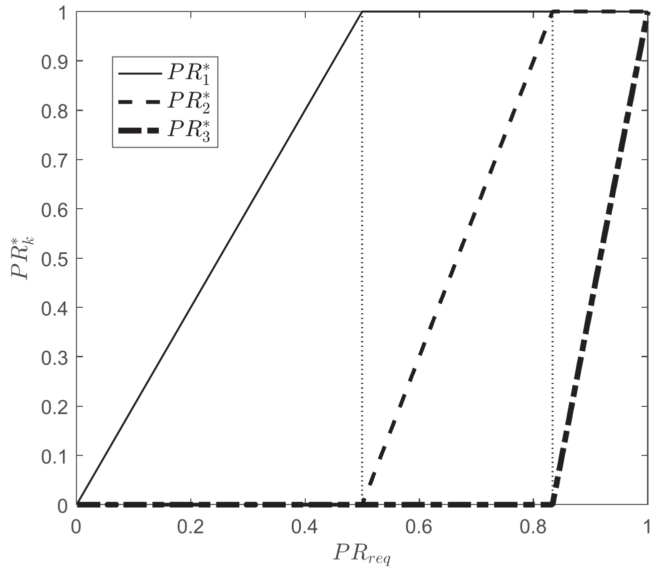

Figure 5. The relationship between optimal production rates.

5.2. Optimal allocation and optimal solution of a 6-electricity-rate problem

In this subsection, we give some numerical results of a problem with more electricity rates. Let $c_1 = 0.29$, $c_2 = 0.34$, $c_3 = 0.49$,$c_4 = 0.54$, $c_5 = 0.59$, $c_6 = 0.84$, $T_1 = 6$, $T_2 = 6$, $T_3 = 4$, $T_4 = 4$, $T_5 = 2$, $T_6 = 2$, $P_1 = P_2 = 1$, and $N = 5$. The optimal allocations under different PR_{req} are shown in Table 8.

One can see that the optimal allocation in this case is still always one of the extreme allocations. Besides, for the extreme allocation, five of the elements are equal to either 0 or 1, which follows our conclusions. Therefore, we can obtain the optimal solution of the six-electricity-rate problem by optimising 60 single-electricity-rate problems instead of 69120 single-electricity-rate problems.

Table 8. The optimal allocations under different PR_{req}.

PR_{req}	PR_1^*	PR_2^*	PR_3^*	PR_4^*	PR_5^*	PR_6^*
0.1	0.4	0	0	0	0	0
0.3	1	0.2	0	0	0	0
0.5	1	1	0	0	0	0
0.7	1	1	1	0.2	0	0
0.9	1	1	1	1	0.8	0

5.3. Sufficient condition verification

In this subsection, we give some numerical results based on Table 6 to verify the sufficient condition. We choose Hebei and Anhui as examples. One can see from the table that all pairs of periods of Hebei satisfy the sufficient condition. Pairs of periods (1, 2) and (1, 3) of Anhui satisfy the sufficient condition.

Let $P_1 = P_2 = 1$ and $N = 5$. The relationships between the optimal objective value and PR_{req} under several allocations are shown in Figure 6, in which $(1, 2, 3)$ represents to allocating production rate to period 1 first and to period 3 last.

One can see from Figure 6 that the optimal value of allocation (1,2,3) is always the lowest, which coincides with the sufficient condition. Allocations (1,2,3) and (1,3,2) show allocating production rate to period 2 instead of period 3, which will reduce the electricity cost. Allocations (2,1,3) and (2,3,1) show the same result between periods 1 and 3.

Different from Hebei, the duration of time periods of Anhui is not the same. The electricity cost of allocation (1,2) is always lower than (2,1), which coincides with the sufficient condition. Allocations (1,3) and (3,1) show that a same result exists between periods 1 and 3.

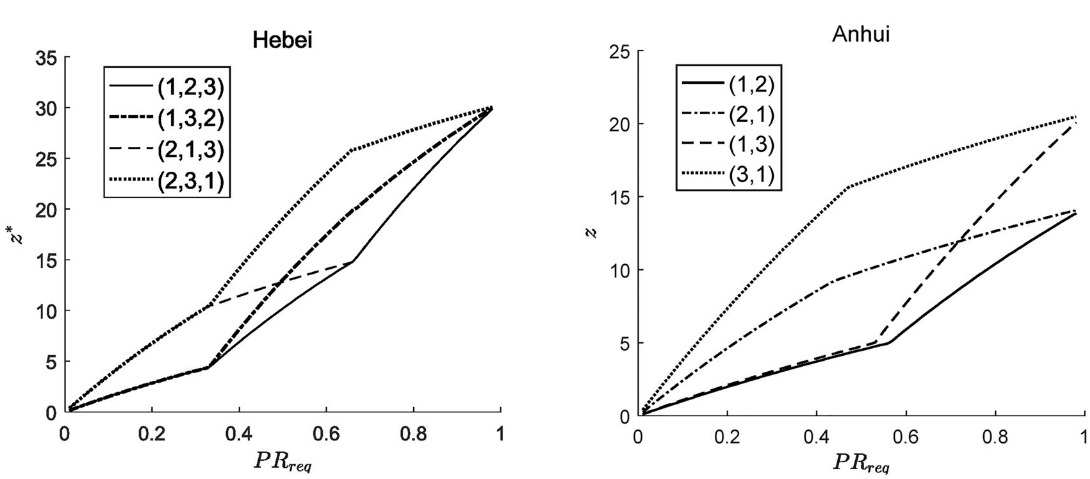

Figure 6. The optimal objective value as a function of PR_{req}.

6. Conclusions

In this paper, a non-linear energy cost optimisation problem in two-machine Bernoulli serial lines under the TOU pricing is formulated. By analysing the structural characteristics of the problem and the properties of the optimal allocation, the non-linear optimisation problem is solved by solving $K \cdot C_{K-1}^{[(K-1)/2]}$ single-electricity-rate problems. The potential efficiency improvement of the solution method is investigated as well for large-scale problems.

Although the two-machine Bernoulli serial line problem under K electricity rates is solved, there are lots of problems worthwhile to be further investigated. In the future, we will focus on:

- Investigating the problem in Bernoulli serial lines with more machines.
- Investigating the problem in systems with geometric and exponential machine reliability models.
- Investigating the problem in systems with non-Markovian (i.e. Weibull, gamma, and log-normal) machine reliability models.
- Investigating the problem in systems with more complex structures, e.g. each stage consisting of multiple parallel machines (Diamantidis et al. (2020)) and/or multiple consecutive series machines (Yan and Zhao (2013) and Yan and Zhao (2018)).
- Investigating the problem under other electricity pricing tariffs (i.e. real-time pricing and day-ahead pricing).

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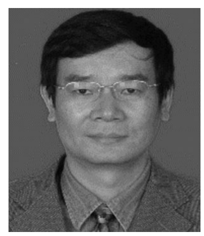

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Appendix

Proof of Theorem 2.1

Proof: We prove it by constructing an example, of which the feasible region is non-convex. Take the two-period problem as an example. Assume $PR_{req} = 0.5$ and $T_1 = T_2$, and choose two solutions $(p_{1,1}, p_{2,1}, p_{1,2}, p_{2,2}) = (0, 0, 1, 1)$ and $(p_{1,1}, p_{2,1}, p_{1,2}, p_{2,2}) = (1, 1, 0, 0)$. Clearly, the production rates of both solutions are 0.5 since for the former, the production rate in the first time period is 0 and in the second one is 1, whereas for the latter, vice versa. In other words, all constraints (especially constraint (2)) are satisfied and both solutions are feasible. For the midpoint of these two solutions, i.e. $(p_{1,1}, p_{2,1}, p_{1,2}, p_{2,2}) = (0.5, 0.5, 0.5, 0.5)$, the production rate in both time periods is less than 0.5, and thus, its production rate is less than 0.5, which indicates that constraint (2) is not satisfied, implying that the feasible region is non-convex. ■

Proof of Theorem 2.2

Proof: Assume that allocation $(PR_1, PR_2, \ldots, PR_K)$ is the optimal one of PR_{req} and its optimal value is $z^*(PR_{req}) = \sum_{k=1}^{K} z^{s*}(PR_k)T_k c_k$, where $z^{s*}(PR_k)$ represents the optimal value of single-period problem (P0), which is shown in Section 3, with required throughput PR_k. Then, for $PR_{req}{}' < PR_{req}$, there exists an allocation $(PR_1, PR_2, \ldots, PR_K{}')$ satisfying $\sum_{k=1}^{K} T_k PR_{req}{}' = \sum_{k=1}^{K-1} T_k PR_k + T_K PR_K{}'$. Due to

$PR_{req}{'} < PR_{req}$, we have $PR_K{'} < PR_K$. The value of (PR_1, $PR_2, \ldots, PR_K{'}$) is $z(PR_1, PR_2, \ldots, PR_k{'}) = \sum_{k=1}^{K-1} z^{s*}(PR_k)T_k c_k + z^{s*}(PR_K{'})T_K c_K$. From Proposition 3 in Yan et al. (2020a) (the proof is provided in Yan et al. (2020b)), it follows that $z^{s*}(PR_k)$ is strictly increasing in PR_k. Therefore, we have $z^{s*}(PR_K{'}) < z^{s*})PR_K)$, which means $z(PR_1, PR_2, \ldots, PR_k{'}) < z^*(PR_{req})$. Since $z(PR_1, PR_2, \ldots, PR_k{'})$ is a feasible solution of $PR_{req}{'}$, we have $z^*(PR_{req}{'}) \leq z(PR_1, PR_2, \ldots, PR_k{'}) < z^*(PR_{req})$. In conclusion, for any $PR_{req}{'} < PR_{req}$, we have $z^*(PR_{req}{'}) < z^*(PR_{req})$, which completes the proof. ■

Proof of Theorem 3.1

Proof: Let c_k denote electricity price of period k. Without loss of generality, assume $c_2 > c_1$. From (9), it follows that when PR_1 increases to $PR_1 + \Delta PR$, PR_2 will decrease to $PR_2 - \frac{T_1}{T_2}\Delta PR$. Thus,

$$
\begin{aligned}
& z\left(PR_1 + \Delta PR, PR_2 - \frac{T_1}{T_2}\Delta PR\right) - z(PR_1, PR_2) \\
&= z^{s*}(PR_1 + \Delta PR)T_1 c_1 + z^{s*}\left(PR_2 - \frac{T_1}{T_2}\Delta PR\right)T_2 c_2 \\
&- z^{s*}(PR_1)T_1 c_1 - z^{s*}(PR_2)T_2 c_2 \\
&= \left.\frac{dz^{s*}}{dPR_{req}}\right|_{PR_{req}=PR_1} \cdot T_1 c_1 \Delta PR - \left.\frac{dz^{s*}}{dPR_{req}}\right|_{PR_{req}=PR_2} \\
&\cdot T_2 c_2 \frac{T_1}{T_2}\Delta PR \\
&= \left(\left.\frac{dz^{s*}}{dPR_{req}}\right|_{PR_{req}=PR_1} \cdot c_1 - \left.\frac{dz^{s*}}{dPR_{req}}\right|_{PR_{req}=PR_2} \cdot c_2\right) \\
&\times T_1 \Delta PR
\end{aligned} \tag{36}
$$

Where $z^{s*}(PR)$ is the optimal value of the single-electricity-rate problem with required production rate PR. Clearly, Equation (36) can be rewritten as follows:

$$
\begin{aligned}
& \frac{dz(PR_1, PR_2)}{dPR_1} \\
&= \left(\left.\frac{dz^{s*}}{dPR_{req}}\right|_{PR_{req}=PR_1} \cdot c_1 - \left.\frac{dz^{s*}}{dPR_{req}}\right|_{PR_{req}=PR_2} \cdot c_2\right) \cdot T_1
\end{aligned} \tag{37}
$$

From Proposition 3 in Yan et al. (2020a) (the proof is provided in Yan et al. (2020b)), it follows that $\left.\frac{dz^{s*}}{dPR_{req}}\right|_{PR_{req}=PR_1}$ is positive and strictly decreasing in PR_1, which implies that $\frac{dz(PR_1,PR_2)}{dPR_1}$ is strictly decreasing in PR_1.

To prove the theorem, the sign of $\frac{dz(PR_1,PR_2)}{dPR_1}$ for all $PR_1 \in [PR_{1\min}, PR_{1\max}]$ should be analysed. In fact, based on the signs of $\left.\frac{dz(PR_1,PR_2)}{dPR_1}\right|_{PR_1=PR_{1\min}}$ and $\left.\frac{dz(PR_1,PR_2)}{dPR_1}\right|_{PR_1=PR_{1\max}}$, the optimal allocation of the production rate can be solved. Obviously, if $PR_1 = PR_2 = PR_{req}$, $\left.\frac{dz(PR_1,PR_2)}{dPR_1}\right|_{PR_1=PR_2} = \left(\left.\frac{dz^{s*}}{dPR_{req}}\right|_{PR_{req}=PR_1} \cdot c_1 - \left.\frac{dz^{s*}}{dPR_{req}}\right|_{PR_{req}=PR_2} \cdot c_2\right) \cdot T_1 < 0$. Considering that $\frac{dz^{s*}}{dPR_{req}}$ is positive and strictly decreasing in PR_1 (Yan et al. 2020a) and taking into account $PR_{1\max} \geq PR_{req}$, we have $\left.\frac{dz(PR_1,PR_2)}{dPR_1}\right|_{PR_1=PR_{1\max}} < 0$.

The sign of $\left.\frac{dz(PR_1,PR_2)}{dPR_1}\right|_{PR_1=PR_{1\min}}$ is either positive or negative, so we need to discuss it in different cases. As for the sign of $\left.\frac{dz(PR_1,PR_2)}{dPR_1}\right|_{PR_1=PR_{1\min}}$, we have:

(i) If $\left.\frac{dz(PR_1,PR_2)}{dPR_1}\right|_{PR_1=PR_{1\min}} \leq 0$, since $\frac{dz(PR_1,PR_2)}{dPR_1}$ is strictly decreasing in PR_1, $\frac{dz(PR_1,PR_2)}{dPR_1}$ is always negative, which implies that $z(PR_1, PR_2)$ is strictly decreasing in PR_1. The behaviours of $\frac{dz(PR_1,PR_2)}{dPR_1}$ and $z(PR_1, PR_2)$ are shown in Figure 7. In this case, the optimal value $z^* = z(PR_{1\max}, PR_{2\min})$ and the optimal allocation is ($PR_{1\max}$, $PR_{2\min}$).

(ii) If $\frac{dz(PR_1,PR_2)}{dPR_1} > 0$, then as PR_1 increases, $z(PR_1, PR_2)$ increases at the beginning and then turns to decreasing. The behaviours of $\frac{dz(PR_1,PR_2)}{dPR_1}$ and $z(PR_1, PR_2)$ are shown in Figure 8. In this case, the optimal value $z^* = \min(z(PR_{1\max}, PR_{2\min}),\ z(PR_{1\min}, PR_{2\max}))$ and the optimal allocation is either $(PR_{1\min}, PR_{2\max})$ or $(PR_{1\max}, PR_{2\min})$.

Based on the above, one can see that in both cases, the optimal allocation can only take from ($PR_{1\min}$, $PR_{2\max}$) and ($PR_{1\max}$, $PR_{2\min}$), which completes the proof. ■

Proof of Theorem 3.2

Proof: Assume that $PR_{k\max} < 1$ and $PR_{(3-k)\min} > 0$. According to (P0), this assumption means $z^{s*}(PR_{k\max}) < P_1 + P_2$ and $z^{s*}(PR_{(3-k)\min}) > 0$. According to (15), we have $\frac{(T_1+T_2)PR_{req}}{T_k} < 1$. However, according to (16), we have $0 < \frac{(T_1+T_2)PR_{req}}{T_{3-k}} - \frac{T_k}{T_{3-k}}$, which can be rewritten as $\frac{(T_1+T_2)PR_{req}}{T_k} > 1$ and results in contradiction. Hence, $PR_{k\max} < 1$ ($z^{s*}(PR_{k\max}) < P_1 + P_2$) and $PR_{(3-k)\min} > 0$ ($z^{s*}(PR_{(3-k)\min}) > 0$) cannot both be true, which completes the proof. ■

Proof of Theorem 3.3

Proof: The energy cost of allocation (PR_1, PR_2, PR_3) is $z = z^{s*}(PR_1)T_1c_1 + z^{s*}(PR_2)T_2c_2 + z^{s*}(PR_3)T_3c_3$. Let $z_1 = z^{s*}(PR_1)T_1c_1 + z^{s*}(PR_2)T_2c_2$, z_1 can be seen as a two-electricity-rate problem. According to Theorem 3.1, the optimal allocations of z_1 are periods 1 and 2, which take maximal or minimal production rate. To distinguish from extreme allocation of z, let ($PR_{1\widetilde{\min}}$, $PR_{2\widetilde{\max}}$) and ($PR_{1\widetilde{\max}}$, $PR_{2\widetilde{\min}}$) denote extreme allocations of z_1. We have:

$$
\begin{aligned}
z_1^* &= z^{s*}(PR_{1\widetilde{\min}})T_1c_1 + z^{s*}(PR_{2\widetilde{\max}})T_2c_2 \\
&\leq z^{s*}(PR_1)T_1c_1 + z^{s*}(PR_2)T_2c_2 = z_1
\end{aligned} \tag{38}
$$

or

$$
\begin{aligned}
z_1^* &= z^{s*}(PR_{1\widetilde{\max}})T_1c_1 + z^{s*}(PR_{2\widetilde{\min}})T_2c_2 \leq z^{s*}(PR_1)T_1c_1 \\
&+ z^{s*}(PR_2)T_2c_2 = z_1
\end{aligned} \tag{39}
$$

When z_1^* satisfying (38), since PR_1 takes the minimal allocation, PR_2 takes the maximal allocation. According

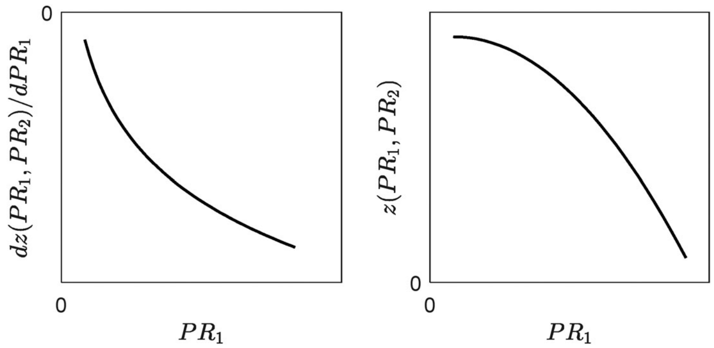

Figure 7. The behaviours of $\frac{dz(PR_1,PR_2)}{dPR_1}$ and $z(PR_1, PR_2)$ when $\frac{dz(PR_1,PR_2)}{dPR_1} \le 0$.

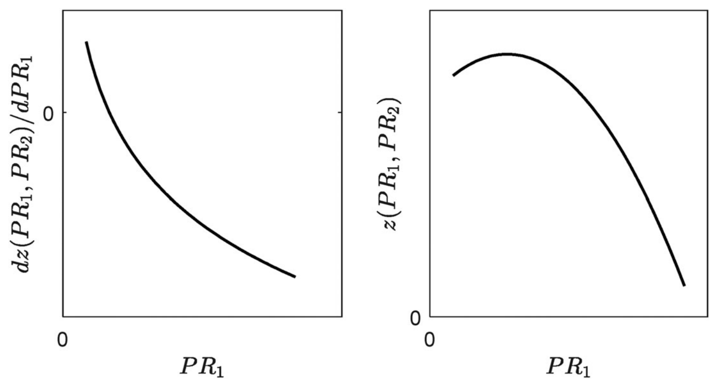

Figure 8. The behaviours of $\frac{dz(PR_1,PR_2)}{dPR_1}$ and $z(PR_1, PR_2)$ when $\frac{dz(PR_1,PR_2)}{dPR_1} > 0$.

to Theorem 3.2, there are two scenarios: $PR_{1\widetilde{\min}} = 0$ or $PR_{2\widetilde{\max}} = 1$

(i) If $PR_{1\widetilde{\min}} = 0 = PR_{1\,\min}$

Let $z_2 = z^{s*}(PR_{2\widetilde{\max}})T_2c_2 + z^{s*}(PR_3)T_3c_3$, according to Theorem 3.1, we have:

$$\begin{aligned} z_2^* &= z^{s*}(PR_{2\,\max})T_2c_2 + z^{s*}(PR_{3\text{sub},2})T_3c_3 \\ &\le z^{s*}(PR_{2\widetilde{\max}})T_2c_2 + z^{s*}(PR_3)T_3c_3 = z_2 \end{aligned} \tag{40}$$

or

$$\begin{aligned} z_2^* &= z^{s*}(PR_{2\text{sub},3})T_2c_2 + z^{s*}(PR_{3\,\max})T_3c_3 \\ &\le z^{s*}(PR_{2\widetilde{\max}})T_2c_2 + z^{s*}(PR_3)T_3c_3 = z_2 \end{aligned} \tag{41}$$

When z_2^* satisfying (40), we have:

$$\begin{aligned} z &= z^{s*}(PR_1)T_1c_1 + z^{s*}(PR_2)T_2c_2 + z^{s*}(PR_3)T_3c_3 \\ &\ge z^{s*}(PR_{1\,\min})T_1c_1 + z^{s*}(PR_{2\widetilde{\max}})T_2c_2 + z^{s*}(PR_3)T_3c_3 \\ &\ge z^{s*}(PR_{1\,\min})T_1c_1 + z^{s*}(PR_{2\,\max})T_2c_2 + z^{s*}(PR_{3\text{sub},2})T_3c_3. \end{aligned}$$

When z_2^* satisfying (41), we have:

$$\begin{aligned} z &= z^{s*}(PR_1)T_1c_1 + z^{s*}(PR_2)T_2c_2 + z^{s*}(PR_3)T_3c_3 \\ &\ge z^{s*}(PR_{1\,\min})T_1c_1 + z^{s*}(PR_{2\widetilde{\max}})T_2c_2 + z^{s*}(PR_3)T_3c_3 \\ &\ge z^{s*}(PR_{1\,\min})T_1c_1 + z^{s*}(PR_{2\text{sub},3})T_2c_2 + z^{s*}(PR_{3\,\max})T_3c_3 \end{aligned}$$

In other words, allocation (PR_1, PR_2, PR_3) is not better than allocation ($PR_{1\,\min}$, $PR_{2\,\max}$, $PR_{3\text{sub},2}$) or allocation ($PR_{1\,\min}$,$PR_{2\text{sub},3}$, $PR_{3\,\max}$).

(ii) If $PR_{2\widetilde{\max}} = PR_{\max} = PR_{2\,\max}$

Let $z_2 = z^{s*}(PR_{1\widetilde{\min}})T_1c_1 + z^{s*}(PR_3)T_3c_3$, according to Theorem 4.1, we have:

$$\begin{aligned} z_2^* &= z^{s*}(PR_{1\,\min})T_1c_1 + z^{s*}(PR_{3\text{sub},2})T_3c_3 \\ &\le z^{s*}(PR_{1\widetilde{\min}})T_1c_1 + z^{s*}(PR_3)T_3c_3 = z_2 \end{aligned} \tag{42}$$

or

$$\begin{aligned} z_2^* &= z^{s*}(PR_{1\text{sub},2})T_1c_1 + z^{s*}(PR_{3\,\min})T_3c_3 \\ &\le z^{s*}(PR_{1\widetilde{\min}})T_1c_1 + z^{s*}(PR_3)T_3c_3 = z_2 \end{aligned} \tag{43}$$

When z_2^* satisfying (42), we have:

$$\begin{aligned} z &= z^{s*}(PR_1)T_1c_1 + z^{s*}(PR_2)T_2c_2 + z^{s*}(PR_3)T_3c_3 \\ &\ge z^{s*}(PR_{1\widetilde{\min}})T_1c_1 + z^{s*}(PR_{2\,\max})T_2c_2 + z^{s*}(PR_3)T_3c_3 \\ &\ge z^{s*}(PR_{1\,\min})T_1c_1 + z^{s*}(PR_{2\,\max})T_2c_2 + z^{s*}(PR_{3\text{sub},2})T_3c_3 \end{aligned}$$

When z_2^* satisfying (43), we have:

$$\begin{aligned} z &= z^{s*}(PR_1)T_1c_1 + z^{s*}(PR_2)T_2c_2 + z^{s*}(PR_3)T_3c_3 \\ &\ge z^{s*}(PR_{1\widetilde{\min}})T_1c_1 + z^{s*}(PR_{2\,\max})T_2c_2 + z^{s*}(PR_3)T_3c_3 \\ &\ge z^{s*}(PR_{1\text{sub},2})T_1c_1 + z^{s*}(PR_{2\,\max})T_2c_2 + z^{s*}(PR_{3\,\min})T_3c_3 \end{aligned}$$

In other words, allocation (PR_1, PR_2, PR_3) is not better than allocation ($PR_{1\,\min}$, $PR_{2\,\max}$, $PR_{3\text{sub},2}$) or allocation ($PR_{1\text{sub},2}$,$PR_{2\,\max}$, $PR_{3\,\min}$). As discussed in Section 3.2, for extreme allocation ($PR_{1\,\min}$, $PR_{2\,\max}$, $PR_{3\text{sub},2}$), letting period 1 take maximal production rate first and letting period 2 take minimal production rate first make no difference. This explains why cases (i) and (ii) have a common extreme allocation $(PR_{1\,\min}, PR_{2\,\max}, PR_{3\mathrm{sub},2})$.

Similarly, when z_1^* satisfying (39), we can prove that allocation (PR_1, PR_2, PR_3) is not better than allocation $(PR_{1\,\max}, PR_{2\,\min}, PR_{3\mathrm{sub},1})$, allocation $(PR_{1\,\max}, PR_{2\mathrm{sub},1}, PR_{3\,\min})$, or allocation $(PR_{1\mathrm{sub},3}, PR_{2\,\min}, PR_{3\,\max})$. One can see that all six extreme allocations have corresponding situations in our proof. Therefore, for any allocation (PR_1, PR_2, PR_3), we can use the upper method to find an extreme allocation that is not worse than the allocation, which completes the proof. ■

Proof of Theorem 4.1

Proof: Assume that there are two allocations satisfying $S_{full} - S_{full}' = \{k_1\}$, $S_{full}' - S_{full} = \{k_2\}$, and $S_{partial} = S_{partial}' = \{k_r\}$. Two electricity periods satisfy $T_{k_1}c_{k_1} \le T_{k_2}c_{k_2}$. The electricity cost difference of two allocations is $\Delta z = z^{s*}(PR_{\max})T_{k_1}c_{k_1} + z^{s*}(PR_{k_r})T_{k_r} \cdot c_{k_r} - z^{s*}(PR_{\max})T_{k_2}c_{k_2} - z^{s*}(PR_{k_r}')T_{k_r} \cdot c_{k_r}$, in which PR_{k_r} and PR_{k_r}' satisfy $T_{k_r}(PR_{k_r} - PR_{k_r}') = (T_{k_2} - T_{k_1})PR_{\max}$.

(i) When $T_{k_1} \ge T_{k_2}$, $T_{k_r}(PR_{k_r} - PR_{k_r}') \le 0$, which means $PR_{k_r} \le PR_{k_r}'$.

Therefore, $\Delta z = z^{s*}(PR_{\max})(T_{k_1}c_{k_1} - T_{k_2}c_{k_2}) + [z^{s*}(PR_{k_r}) - z^{s*}(PR_{k_r}')]T_{k_r} \cdot c_{k_r} \le 0$. In this condition, the electricity cost of allocation 1 is lower.

(ii) When $T_{k_1} < T_{k_2}$, $T_{k_r}(PR_{k_r} - PR_{k_r}') > 0$, which means $PR_{k_r} > PR_{k_r}'$.

Therefore, $\Delta z = z^{s*}(PR_{\max})(T_{k_1}c_{k_1} - T_{k_2}c_{k_2}) + [z^{s*}(PR_{k_r}) - z^{s*}(PR_{k_r}')]T_{k_r} \cdot c_{k_r}$.

When $\frac{T_{k_1}c_{k_1} - T_{k_2}c_{k_2}}{z^{s*}(PR_{k_r}) - z^{s*}(PR_{k_r} - \frac{T_{k_2}-T_{k_1}}{T_{k_r}}PR_{\max})} \le \frac{T_{k_r} \cdot c_{k_r}}{z^{s*}(PR_{\max})}$, the electricity cost of allocation 1 is lower, and vice versa. ■

Proof of Theorem 4.2

Proof: To simplify the analysis, we give a small-size problem as an example. Let four time periods k_1, k_2, k_3, and k_4 satisfy $T_{k_1}c_{k_1} < T_{k_2}c_{k_2} < T_{k_3}c_{k_3}$ and $T_{k_1} < T_{k_2} < T_{k_3}$. Let $S_{partial} = \{k_4\}$, $S_{full} = \{k_1, k_2\}$, and $S_{zero} = \{k_3\}$.

(i) Assume that substituting k_2 by k_3 or substituting k_1 by k_3 cannot reduce electricity cost. In this condition, k_3, a period in S_{zero}, is the period wiped out by the algorithm. Apparently, the period that has been wiped out by the algorithm in S_{zero} cannot substitute any period in S_{full}.

(ii) Assume that substituting k_2 by k_3 reduces electricity cost, which means $z^{s*}(1)(T_{k_3}c_{k_3} - T_{k_2}c_{k_2}) > T_{k_4}c_{k_4}(z^{s*}(PR_{k_4}) - z^{s*}(PR_{k_4} - \frac{T_{k_3}-T_{k_2}}{T_{k_4}}))$. Then, assume that substituting k_1 by k_2 reduces electricity cost, which means $z^{s*}(1)(T_{k_2}c_{k_2} - T_{k_1}c_{k_1}) > T_{k_4}c_{k_4}(z^{s*}(PR_{k_4} - \frac{T_{k_3}-T_{k_2}}{T_{k_4}}) - z^{s*}(PR_{k_4} - \frac{T_{k_3}-T_{k_1}}{T_{k_4}}))$. In this condition, k_1, a period in S_{full}, is the period that has been wiped out by the algorithm and the new $S_{full}' = \{k_2, k_3\}$. Add two inequations together, we have $z^{s*}(1)(T_{k_3}c_{k_3} - T_{k_1}c_{k_1}) > T_{k_4}c_{k_4}(z^{s*}(PR_{k_4}) - z^{s*}(PR_{k_4} - \frac{T_{k_3}-T_{k_1}}{T_{k_4}}))$, which means substituting k_3 by k_1 cannot reduce electricity cost. In other words, the period that has been wiped out by the algorithm in S_{full} cannot substitute the period, from S_{zero}, in S_{full}'.

(iii) Assume that substituting k_2 by k_3 cannot reduce electricity cost, which means $z^{s*}(1)(T_{k_3}c_{k_3} - T_{k_2}c_{k_2}) \le T_{k_4}c_{k_4}(z^{s*}(PR_{k_4}) - z^{s*}(PR_{k_4} - \frac{T_{k_3}-T_{k_2}}{T_{k_4}}))$. Assume that substituting k_1 by k_3 reduces electricity cost, which means $z^{s*}(1)(T_{k_3}c_{k_3} - T_{k_1}c_{k_1}) > T_{k_4}c_{k_4}(z^{s*}(PR_{k_4}) - z^{s*}(PR_{k_4} - \frac{T_{k_3}-T_{k_1}}{T_{k_4}}PR_{\max}))$. Use second inequation minus first inequation, we have $z^{s*}(1)(T_{k_2}c_{k_2} - T_{k_1}c_{k_1}) > T_{k_4}c_{k_4}(z^{s*}(PR_{k_4} - \frac{T_{k_3}-T_{k_2}}{T_{k_4}}) - z^{s*}(PR_{k_4} - \frac{T_{k_3}-T_{k_1}}{T_{k_4}}))$, which means substituting k_2 by k_1 cannot reduce electricity cost. In other words, the period that has been wiped out by the algorithm in S_{full} cannot substitute the period, from S_{full}, in S_{full}'.

From above, we can come out with the conclusion that any period that has been wiped out by the algorithm cannot substitute any period in S_{full}, which complete the proof. ■