Calculus I

Explanations, *Problems* & *Solutions* (*Solutions*)

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Preface

Welcome to **Calculus I**, a rigorous journey into the mathematics of change and motion that forms the backbone of modern scientific analysis.

About This Textbook

This textbook has been carefully structured to guide you through the essential pillars of single variable calculus, summarizing the core concepts and techniques of the course:

Limits and Continuity — Establishing the precise mathematical foundation required to analyze functions at their boundaries, defining the critical concepts of instantaneous velocity and continuous motion.

The Derivative and Its Applications — Mastering the tools to measure instantaneous rates of change, using differentiation to solve complex optimization problems, approximate linear behaviors, and sketch the precise shapes of curves.

Integration — Exploring the accumulation of quantities to determine net change, calculating complex areas and volumes, and bridging the gap between slopes and areas through the Fundamental Theorem of Calculus.

Transcendental Functions and Advanced Techniques — Extending calculus to exponential, logarithmic, and inverse trigonometric functions, and developing sophisticated strategies such as integration by parts and trigonometric substitution to solve intricate problems.

How to Use This Book

To facilitate a deep and structured understanding of the material, this book utilizes specific block types to organize information:

Definition blocks clarify fundamental concepts, ensuring you grasp the precise language of calculus.

Example blocks demonstrate standard problems derived from the syllabus, illustrating how to apply theoretical concepts.

Solution blocks offer detailed, step-by-step walkthroughs to model the logical flow required for your own proofs and calculations.

Theorem blocks highlight the pivotal mathematical truths, such as the Mean Value Theorem and the Fundamental Theorem of Calculus.

Proof blocks guide you through the logical derivations of theorems, fostering the analytical reasoning skills emphasized in this course.

Note blocks provide crucial insights, common calculation pitfalls, and connections between differentiation and integration.

A Note on Learning

Calculus is more than a set of rules for manipulating symbols; it is a way of thinking about the world. True mastery requires **active engagement** with the material. We encourage you to:

- Attempt to solve the examples yourself before revealing the solution blocks.
- Focus on the geometric and physical intuition behind the formulas.
- Understand not just **how** to differentiate or integrate, but **when** and **why** these tools are used.
- Collaborate with peers to discuss concepts and refine your logical argumentation.

The notes provided here are designed to support your learning journey, clarifying the lecture material and helping you build the confidence to solve real-world problems mathematically.

Looking Ahead

The concepts you master in **Calculus I**—limits, derivatives, and integrals—are the indispensable tools of the future. They are the language used to describe the motion of planets, the flow of fluids, the growth of populations, and the fluctuations of markets. Whether you pursue physics, engineering, economics, or computer science, the analytical framework you build here will serve as the foundation for your advanced studies.

We hope this book serves as a clear and supportive guide as you learn to see the world through the lens of calculus.

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Chapter 00

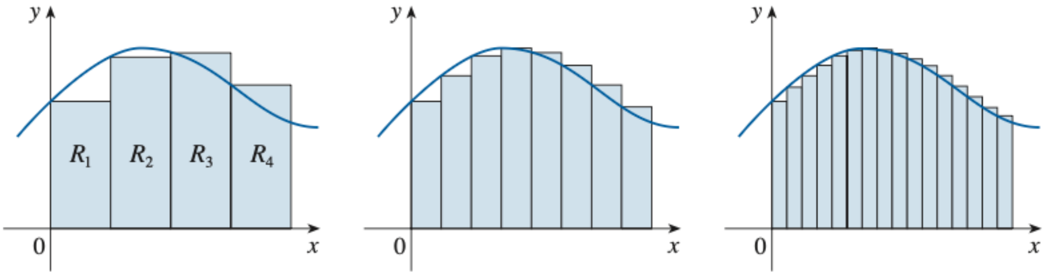
Preview of Calculus

This section bridges the gap between static pre-calculus mathematics and the dynamic nature of calculus by addressing two fundamental problems: determining the slope of a tangent line to a curve and calculating the instantaneous velocity of a moving object. Both problems are solved using the same limiting process, where approximations—such as the slopes of secant lines or average velocities over shrinking time intervals—converge to a single, precise value. This concept of the "limit" serves as the foundation for the entire course, establishing the mathematical tools necessary to analyze instantaneous rates of change.

Chapter 00.01

Preview of Calculus

What is calculus?

- We have two main problems that we deal in calculus: The area problem & the tangent problem.
  - The area problem is about finding the area under a curve.
  - The tangent problem is about finding the slope of a curve at a given point.
  - 

  - Approximation of an area using rectangles.

Notation | The Area Problem

We can define the area under the curve as

$$A = \lim_{n \to \infty} A_n$$

Here, A_i represents sum of each area block equally dividing the given range into n parts.

Notation | The Tangent Problem

The slope of a curve at a given point a can be defined as

$$m_a = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Here, m represents the slope of the tangent line to the curve at point x. Or, alternatively, we can write it as :

$$m_a = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

Note | So what's the relation?

Both problems involve limits. The area problem uses a limit to find the exact area under a curve. The tangent problem uses a limit to find the exact slope of a tangent line. In the end, solving one of them leads to solving the other. We call this the **Fundamental Theorem of Calculus**.

Chapter 01

Functions and Limits

This chapter formalizes the concept of the limit, moving from intuitive approximations to rigorous mathematical definitions including the precise Epsilon-Delta definition. It provides the algebraic rules (Limit Laws) necessary to calculate limits without graphing, methods for handling indeterminate forms, and defines continuity to describe functions without breaks or holes. These concepts culminate in the Intermediate Value Theorem, a powerful tool used to prove the existence of roots and solutions within specific intervals, setting the stage for the definition of the derivative.

Chapter 01.04

The Tangent & Speed Problem

The Tangent Problem

- What does tangent mean in calculus?

Definition | Tangency

A tangent line to a curve at a given point is a straight line that just "touches" the curve at that point. In calculus, the tangent line represents the instantaneous rate of change of the function at that point, which is given by the derivative of the function.

Example | Tangency Basics

Say we have the function $f(x) = x^2$. Let us deifne the base point of our tangent slope calculation as $x = 1$. Here, our slope calculation equation would become :

$$m_{\text{pq}} = \frac{p_x^2 - q_x^2}{p_x - q_x}$$

where $q_x = 1$. If we plot in values from $p_x = 2$ to $p_x \approx 1$, we get the following table:

p_x	m_pq
2	3
1.5	2.5
1.1	2.1
1.01	2.01
1.001	2.001

As we can see from the table, as p_x approaches 1, the slope m_{pq} approaches 2. Therefore, the slope of the tangent line to the curve at the point where $x = 1$ is 2. Thus, after a bit of calculation, we can find that the equation of the tangent line at the point $(1, f(1)) = (1, 1)$ is:

$$y - 1 = 2(x - 1)$$

or simplified,

$$y = 2x - 1$$

.

Note | Approach from L/R

Note from the example above that there are two directions a line can approach a curve: from the left side (as p_x approaches 1 from values less than 1) and from the right side (as p_x approaches 1 from values greater than 1).

- Normally, this doesnt pose that much of a problem, but if the function is **severed** at a point, the left-hand limit and right-hand limit may not be equal, leading to different tangent slopes from each side.

The Velocity Problem

- Take a car. How do we define its **speed**?
- We know from middle school knowledge that speed is defined by the following equation :

$$v = \frac{\text{total distance travelled}}{\text{total time spent}}$$

- Note here that this is the **exact same form** as the slope equation we used in the tangent problem!

Definition | Average Velocity

Average velocity over a time interval is defined as the total displacement divided by the total time taken. Mathematically, if a car moves from position $s(a)$ at time a to position $s(b)$ at time b, the average velocity v_{avg} over the interval $[a, b]$ is given by:

$$v_{\text{avg}} = \frac{s(b) - s(a)}{b - a}$$

Definition | Instantaneous Velocity

A bit off course, but we can define instantaneous velocity using the later-defined idea of the limit.

Instantaneous velocity at a specific time is the limit of the average velocity as the time interval approaches zero. It represents the velocity of the car at a precise moment in time. Mathematically, the instantaneous velocity $v(t)$ at time t is given by:

$$v(t) = \lim_{\Delta\mathrm{t}} \to 0 \frac{s(t + \Delta\mathrm{t}) - s(t)}{\Delta\mathrm{t}}$$

Chapter 01.05

The Limit of a Function

The limit

- In the previous section, whilst discussing tangents and velocity, we kept encountering a problem: how do we define the slope of a curve at a single point, or the velocity of a car at a precise moment in time?
- Both of these problems can be solved using the concept of **limits**.

Definition | Intuitive Definition of the Limit

Suppose $f(x)$ is defined whilst x is near a. Then we write

$$l\,\mathrm{im}_{\{xtoa\}}\, f(x) = L$$

We read this as "the limit of $f(x)$ as x approaches a is L". This claim must be preceded by the statement that as $f(x)$ gets arbitrarily closer and closer to L as x gets closer and closer to a.

Example | Limit Basics

Consider the function $f(x) = \frac{\sin(x)}{x}$. Evaluate $f(x)$ at values of x that get closer and closer to 0:

Solution 1 |

x	$f(x)$
1	0.84147098
0.1	0.99833417
0.01	0.99998333
0.001	0.99999983
0.0001	0.99999998

By the intuitive definition of the limit, we can see that as x approaches 0, $f(x)$ approaches 1. Therefore, we can conclude that:

- Above, we briefly mentioned that limits may differ when approached from different sides. This can be expanded to the fact that limits may not exist at all from some sides in some cases.

DEFINITION | Intuitive Definition of One-Sided Limits

Suppose $f(x)$ is defined whilst x is near a. Then we write

$$l\,\mathrm{im}_{\{xtoa^{+}\}}\, f(x) = L_1$$

and

$$l\,\mathrm{im}_{\{xtoa^{-}\}}\, f(x) = L_2$$

We read these as "the limit of $f(x)$ as x approaches a from the right is L_1" and "the limit of $f(x)$ as x approaches a from the left is L_2". This claim must be preceded by the statement that as $f(x)$ gets arbitrarily closer and closer to L_1 as x gets closer and closer to a from the right, and similarly for L_2 from the left. Here, we call L_1 as the **right-hand limit** and L_2 as the **left-hand limit**.

EXAMPLE | The Heaveside Function

Consider the Heaveside function $H(x)$ defined as follows:

$$H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

Evaluate $H(x)$ at values of x that get closer and closer to 0 from both sides:

Solution 1 | More Evaluation...

x	$H(x)$
−1	0
−0.1	0
−0.01	0
0.01	1
0.1	1
1	1

By the intuitive definition of one-sided limits, we can see that as x approaches 0 from the left, $H(x)$ approaches 0, and as x approaches 0 from the right, $H(x)$ approaches 1. Therefore, we can conclude that:

$$\lim_{x \to 0^-} H(x) = 0$$

$$\lim_{x \to 0^+} H(x) = 1$$

- With the basic idea, we can now consider the bigger problem : can limits **fail** to exist?

Example | Uh Oh

Investigate the following :

$$\lim_{x \to 0} \sin\left(\frac{\pi}{x}\right)$$

Solution 1 |

x	$\sin(\frac{\pi}{x})$
1	0
0.1	0.58778525
0.01	0.95105652
0.001	0.30901699
0.0001	0.98768834

By evaluating $\sin(\frac{\pi}{x})$ at values of x that get closer and closer to 0, we can see that the function does not approach a single value. Instead, it oscillates between -1 and 1. Therefore, we can conclude that:

$$\lim_{x \to 0} \sin\left(\frac{\pi}{x}\right)$$

does not exist.

Example | Uh Oh 2

Find

$$\lim_{x \to 0} \left(\frac{1}{x^2} \right)$$

if it exists

Solution 1 |

x	$\frac{1}{x^2}$
1	1
0.1	100
0.01	10000
0.001	1000000
0.0001	10000000

By evaluating $\frac{1}{x^2}$ at values of x that get closer and closer to 0, we can see that the function grows without bound. Therefore, we can conclude that:

$$\lim_{x \to 0} \left(\frac{1}{x^2} \right) = \infty$$

(which mathemtaically means that the limit does not exist in the real number system)

Definition | Intuitive Definition of an Infinite Limit

Suppose $f(x)$ is defined whilst x is near a on both sides, except at a itself. Then we write

$$\lim_{x \to a} f(x) = \infty$$

which means that the values of $f(x)$ can be made arbitrarily large by taking x sufficiently close but not equal to a.

- Of course, we can do the same for negative infinity.

Definition | Intuitive Definition of a Negative Infinity Limit

Suppose $f(x)$ is defined whilst x is near a on both sides, except at a itself. Then we write

$$\lim_{x \to a} f(x) = -\infty$$

which means that the values of $f(x)$ can be made arbitrarily small (negatively large) by taking x sufficiently close but not equal to a.

- At these points, we can define a **vertial asymptote**.

Definition | Vertical Asymptote

A vertical asymptote is a vertical line $x = a$ when the function $f(x)$ suffices at least one of the 6 conditions :

- $\lim_{x \to a^+} f(x) = \infty$
- $\lim_{x \to a^+} f(x) = -\infty$
- $\lim_{x \to a^-} f(x) = \infty$
- $\lim_{x \to a^-} f(x) = -\infty$
- $\lim_{x \to a} f(x) = \infty$
- $\lim_{x \to a} f(x) = -\infty$

In simpler terms, a vertical asymptote is a vertical line where the function grows without bound as it approaches the line from at least one side.

Example | Vertical Asymptote Example

Investigate if the function $y = \frac{2x}{x-3}$ have a vertical asymptote.

Solution 1 |

To find vertical asymptotes, we need to look for values of x that make the denominator equal to 0. Here, the denominator $x - 3$ equals 0 when $x = 3$. Next, we need to evaluate the limits as x approaches 3 from both sides:

$$\lim_{x \to 3^+} \frac{2x}{x-3} = \infty$$

$$\lim_{x \to 3^-} \frac{2x}{x-3} = -\infty$$

Since both one-sided limits approach infinity (one positive, one negative), we can conclude that the function has a vertical asymptote at $x = 3$.

Chapter 01.06

Calculating Limits using the Limit Laws

- Limits Have some rules that we must comply to.

Theorem | The Laws of Limits

Suppose that c i s a constant and the limits $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist. Then, the following properties hold:

1. $\lim_{x \to a}[f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
2. $\lim_{x \to a}[f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$
3. $\lim_{x \to a}[c * f(x)] = c * \lim_{x \to a} f(x)$
4. $\lim_{x \to a}[f(x) * g(x)] = \lim_{x \to a} f(x) * \lim_{x \to a} g(x)$
5. $\lim_{x \to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, provided that $\lim_{x \to a} g(x) \neq 0$

Using these laws, we can derive a few more :

Theorem | The Power Law of Limits

Suppose that $\lim_{x \to a} f(x) = L$ and n is a positive integer. Then, $\lim_{x \to a} \left[f(x)\right]^n = \left[\lim_{x \to a} f(x)\right]^n = L^n$

Theorem | The Root Law of Limits

Suppose that $\lim_{x \to a} f(x) = L$ and n is a positive integer. Then, $\lim_{x \to a} \left[f(x)\right]^{\frac{1}{n}} = \left[\lim_{x \to a} f(x)\right]^{\frac{1}{n}} = L^{\frac{1}{n}}$, provided that if n is even, then $L \geq 0$

also, some special limits :

Theorem | The Constant Function Law

For any constant c, $\lim_{x \to a} c = c$

Written differently, $\lim_{x \to a} = a$

Example | Limit Law Basics

Evaluate $\lim_{x \to 5}(2x^2 - 3x + 4)$.

Solution 1 |

$$\lim_{x \to 5}(2x^2 - 3x + 4) = \lim_{x \to 5} 2x^2 - \lim_{x \to 5} 3x + \lim_{x \to 5} 4$$

($\because$ the Sum and Difference Law)

$$= 2 * \lim_{x \to 5} x^2 - 3 * \lim_{x \to 5} x + 4$$

($\because$ the Constant Multiple Law)

$$= 2 * 5^2 - 3 * 5 + 4$$

($\because$ the Power Law and Constant Function Law)

$$= 50 - 15 + 4 = 39$$

Above, we have discussed that $\lim_{x \to a} x = a$. What if we generalize this into any function $f(x)$?

Theorem | The Direct Substitution Property

If f is a polynomial/rational function and a is in the domain of f, at $x = a$, then

$$\lim_{x \to a} f(x) = f(a)$$

Example | Direct Substitution Example

Evaluate $\lim_{x \to 2} \frac{x^3 - 4x + 1}{x^2 + 3}$.

Solution 1 |

Since the function is a rational function and 2 is in the domain of the function, we can use the Direct Substitution Property.

$\lim_{x \to 2} \frac{x^3 - 4x + 1}{x^2 + 3} = \frac{2^3 - 4 * 2 + 1}{2^2 + 3} = \frac{8 - 8 + 1}{4 + 3} = \frac{1}{7}$

Example | Laws of Limits General

Evaluate $\lim_{x\to 1} \frac{x^2-1}{x-1}$.

Solution 1 |

We cannot directly substitute 1 into the function, since it would create a division by zero. However, we can simplify the expression first.

$$\frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1} = x+1 \because x \neq 1$$

.

Now we can find the limit:

$$\lim_{x\to 1}(x+1) = 1+1 = 2$$

Chapter 01.07

The Precise Definition of a Limit

The “Precise” Definition of a Limit

- Till now, we have been using *intuitive* methods to define the limits of functions. However, in order to be more *precise*, we need a more rigorous definition of limits.

Example | A Weird Function

Consider the following function : $\begin{cases} 2x-1 & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases}$. What is $f(3)$? What is $\lim_{x\to 3} f(x)$?

Solution 1 |

$f(3) = 6$, since when $x = 3$, the function outputs 6. However, to find $\lim_{x\to 3} f(x)$, we need to see what value $f(x)$ approaches as x gets closer and closer to 3. As x approaches 3, $f(x)$ approaches $2(3) - 1 = 5$. Therefore, $\lim_{x\to 3} f(x) = 5$.

- Consider the example above. How can we more “accurately” define the limit of a function at a point, without relying on intuition or graphs?
- In te problem above, consider the following question.
  - “How close to 3 does x need to be, so that $f(x)$ is within 0.1 of 5?”
  - In other words, we want to find some value δ s.t.

$$\|f(x) - 5\| < 0.1 \quad \text{if} \quad \|x - 3\| < \delta \quad \text{but } x \neq 3$$

If $\|x - 3\| > 0$, then $x \neq 3$, so we can simplify the problem into the following :

- “How close to 3 does x need to be, so that $f(x)$ is within 0.1 of 5, given that $x \neq 3$

$$\|f(x) - 5\| < 0.1 \quad \text{if} \quad 0 < \|x - 3\| < \delta$$

Here, we call 0.1 the “Error Tolerance”. Since we are going to send this to 0, let us replace this with ε for the time being. So the question becomes :

$$\|f(x) - 5\| < \varepsilon \quad \text{if} \quad 0 < \|x - 3\| < \delta$$

Now, we finally have the tools to define the limit of a function more “precisely”.

Definition | Precise Definition of a Limit

Let f be a function defined on an open interval containing a, except possibly at a itself. We say that $\lim_{x \to a} f(x) = L$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\|f(x) - L\| < \varepsilon \qquad \text{if} \qquad 0 < \|x - a\| < \delta$$

To simplify even more, we can write like this(in style):

$$\forall \varepsilon > 0, \exists \delta > 0, \text{ s.t. } 0 < \|x - a\| < \delta \Longrightarrow \|f(x) - L\| < \varepsilon$$

Since we have to divide them later anyway, we can do it now :

Definition | Precise definition of a left-hand limit

Let f be a function defined on an open interval containing a, except possibly at a itself. We say that $\lim_{x \to a^-} f(x) = L$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\|f(x) - L\| < \varepsilon \qquad \text{if} \qquad a - \delta < x < a$$

To simplify even more, we can write like this:

$$\forall \varepsilon > 0, \exists \delta > 0, \text{ s.t. } a - \delta < x < a \Longrightarrow \|f(x) - L\| < \varepsilon$$

Definition | Precise definition of a right-hand limit

Let f be a function defined on an open interval containing a, except possibly at a itself. We say that $\lim_{x \to a^+} f(x) = L$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\|f(x) - L\| < \varepsilon \qquad \text{if} \qquad a < x < a + \delta$$

To simplify even more, we can write like this:

$$\forall \varepsilon > 0, \exists \delta > 0, \text{ s.t. } a < x < a + \delta \Longrightarrow \|f(x) - L\| < \varepsilon$$

One good way to understand the epsilon-delta method is by assuming we chose a "wrong" limit value "L'". In this case, we must be able to find a δ for any arbitrary ε given. Since we chose a wrong limit value, there will always be some x values within the δ-neighborhood of a that make $f(x)$ be outside the ε-neighborhood of "L'". This means that no matter how small we choose δ, there will always be some x values that break the condition. Thus, we can conclude that "L'" is not the correct limit value.

Note | Usage of the epsilon-delta method

The epsilon-delta method can only be used to **"prove"** that a certain limit value is correct. It cannot be used to **"find"** the limit value itself.

Now lets utilize this to solve some problems.

EXAMPLE | Epsilon-Delta Basics

Prove that $\lim_{x\to 3}(4x-5) = 7$ using the epsilon-delta definition of a limit.

Solution 1 |

Let $\varepsilon > 0$ be given. We need to find a $\delta > 0$ such that

$$\|(4x-5)-7\| < \varepsilon \quad \text{if} \quad 0 < \|x-3\| < \delta$$

1. Preliminary Analysis :

Simplifying the left side, we have :

$$\|4x-12\| = 4 * \|x-3\|$$

So we want to ensure that

$$4 * \|x-3\| < \varepsilon \quad \text{if} \quad 0 < \|x-3\| < \delta$$

To achieve this, we can choose $\delta = \frac{\varepsilon}{4}$. Then, if $0 < \|x-3\| < \delta$, we have

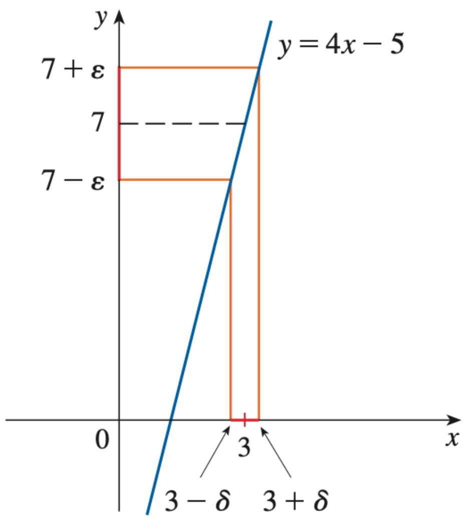
$$\|(4x-5)-7\| = 4 * \|x-3\| < 4 * \delta = \varepsilon$$

2. Formal Proof :

Let $\varepsilon > 0$ be given. Choose $\delta = \frac{\varepsilon}{4}$. Then, if $0 < \|x-3\| < \delta$, we have

$$\|(4x-5)-7\| = 4 * \|x-3\| < 4 * \delta = \varepsilon$$

Thus, by the epsilon-delta definition of a limit, we conclude that $\lim_{x\to 3}(4x-5) = 7$.



Example | Epsilon-Delta Basics 2

Describe $\lim_{x\to a} x^2 = a^2$ using the epsilon-delta definition of a limit.

Solution 1 |

Let $\varepsilon > 0$ be given. We need to find a $\delta > 0$ such that

$$\left\|x^2 - a^2\right\| < \varepsilon \qquad \text{if} \qquad 0 < \|x - a\| < \delta$$

1. Preliminary Analysis :

Simplifying the left side, we have :

$$\left\|x^2 - a^2\right\| = \|(x - a)(x + a)\| = \|x - a\| * \|x + a\|$$

To control $\|x + a\|$, we can restrict δ to be less than 1(an arbitrary number). This means that if $0 < \|x - a\| < \delta < 1$, then

$$\|x - a\| < 1 \Longrightarrow \|x\| < \|a\| + 1$$

Thus, we have

$$\|x + a\| \leq \|x\| + \|a\| < (\|a\| + 1) + \|a\| = 2 * \|a\| + 1$$

Therefore, we want to ensure that

$$\|x - a\| * (2 * \|a\| + 1) < \varepsilon \qquad \text{if} \qquad 0 < \|x - a\| < \delta < 1$$

To achieve this, we can choose $\delta = \min\left(1, \frac{\varepsilon}{2*\|a\|+1}\right)$. Then, if $0 < \|x - a\| < \delta$, we have

$$\left\|x^2 - a^2\right\| = \|x - a\| * \|x + a\| < \delta * (2 * \|a\| + 1) \leq \left(\frac{\varepsilon}{2 * \|a\| + 1}\right) * (2 * \|a\| + 1) = \varepsilon$$

2. Formal Proof :

Let $\varepsilon > 0$ be given. Choose $\delta = \min\left(1, \frac{\varepsilon}{2*\|a\|+1}\right)$. Then, if $0 < \texttt{norm}\ (x - a) < \delta$, we have

$$\left\|x^2 - a^2\right\| = \|x - a\| * \|x + a\| < \delta * (2 * \|a\| + 1) \leq \left(\frac{\varepsilon}{2 * \|a\| + 1}\right) * (2 * \|a\| + 1) = \varepsilon$$

Thus, by the epsilon-delta definition of a limit, we conclude that $\lim_{x\to a} x^2 = a^2$.

Note | Why can we assign an arbitrary number to delta?

In the preliminary analysis, we restricted δ to be less than 1 to control the value of $\|x + a\|$. This is a common technique in epsilon-delta proofs to ensure that certain expressions remain bounded. By choosing δ to be the minimum of 1 and another expression, we can effectively manage the behavior of the function within the desired neighborhood around the point a. To say intuitively, we are "bounding" the function in the neighborhood of a once more outside of delta.

Example | Epsilon-Delta Basics 3

Describe $\lim_{x \to 1} \frac{x^2-1}{x-1} = 2$ using the epsilon-delta definition of a limit.

Solution 1 |

Let $\varepsilon > 0$ be given. We need to find a $\delta > 0$ such that

$$\left\| \frac{x^2 - 1}{x - 1} - 2 \right\| < \varepsilon \qquad \text{if} \qquad 0 < \|x - 1\| < \delta$$

1. Preliminary Analysis :

Simplifying the left side, we have :

$$\left\| \frac{x^2 - 1}{x - 1} - 2 \right\| = \left\| (x-1)\frac{x + 1}{x - 1} - 2 \right\| = \|x + 1 - 2\| = \|x - 1\|$$

Therefore, we want to ensure that

$$\|x - 1\| < \varepsilon \qquad \text{if} \qquad 0 < \|x - 1\| < \delta$$

To achieve this, we can choose $\delta = \varepsilon$. Then, if $0 < \|x - 1\| < \delta$, we have

$$\left\| \frac{x^2 - 1}{x - 1} - 2 \right\| = \|x - 1\| < \delta = \varepsilon$$

2. Formal Proof :

Let $\varepsilon > 0$ be given. Choose $\delta = \varepsilon$. Then, if $0 < \|x - 1\| < \delta$, we have

$$\left\| \frac{x^2 - 1}{x - 1} - 2 \right\| = \|x - 1\| < \delta = \varepsilon$$

Thus, by the epsilon-delta definition of a limit, we conclude that $\lim_{x \to 1} \frac{x^2-1}{x-1} = 2$.

Example | Epsilon-Delta Basics 4

Describe $\lim_{x \to 0+} \sqrt{x} = 0$ using the epsilon-delta definition of a limit.

Solution 1 |

Let $\varepsilon > 0$ be given. We need to find a $\delta > 0$ such that

$$\|\sqrt{x} - 0\| < \varepsilon \quad \text{if} \quad 0 < x < \delta$$

1. Preliminary Analysis :

Simplifying the left side, we have :

$$\|\sqrt{x} - 0\| = \sqrt{x}$$

Therefore, we want to ensure that

$$\sqrt{x} < \varepsilon \quad \text{if} \quad 0 < x < \delta$$

To achieve this, we can choose $\delta = \varepsilon^2$. Then, if $0 < x < \delta$, we have

$$\|\sqrt{x} - 0\| = \sqrt{x} < \sqrt{\delta} = \sqrt{\varepsilon^2} = \varepsilon$$

2. Formal Proof :

Let $\varepsilon > 0$ be given. Choose $\delta = \varepsilon^2$. Then, if $0 < x < \delta$, we have

$$\|\sqrt{x} - 0\| = \sqrt{x} < \sqrt{\delta} = \sqrt{\varepsilon^2} = \varepsilon$$

Thus, by the epsilon-delta definition of a limit, we conclude that $\lim_{x \to 0+} \sqrt{x} = 0$.

Chapter 01.08

Continuity

Continuity

Intuitively, continuity can be simply defined as "a function that is connected"...but we dont deal with intuition here. We need a more "precise" definition of continuity.

DEFINITION | Continuity at a Point

A function f is said to be continuous at a point $x = a$ if the following three conditions are satisfied :

1. $f(a)$ is defined.
2. $\lim_{x \to a} f(x)$ exists.
3. $\lim_{x \to a} f(x) = f(a)$

If these conditions are not met, we call the function is "discontinuous" at $x = a$, or has a "discontinuity" at $x = a$.

EXAMPLE | Continuity Example

Identify if the functions are discontinuous.

(a) $f(x) = \frac{x^2-x-2}{x-2}$

(b) $f(x) = \begin{cases} \frac{x^2-x-2}{x-2} & \text{if} \quad x \neq 2 \\ 1 & \text{if} \quad x = 2 \end{cases})$

Solution (a) |

The function is discontinuous at $x = 2$. This is because $f(2)$ is not defined, since the denominator becomes 0 at this point. Therefore, the first condition for continuity is not satisfied.

Solution (b) |

(b) The function is continuous at $x = 2$. Let's check the three conditions:

1. $f(2) = 1$, so the first condition is satisfied.
2. To find $\lim_{x \to 2} f(x)$, we simplify the expression for $x \neq 2 : f(x) = \frac{x^2-x-2}{x-2} = \frac{(x-2)(x+1)}{x-2} = x + 1$ for $x \neq 2$. Therefore, $\lim_{x \to 2} f(x) = \lim_{x \to 2}(x + 1) = 3$. So the second condition is satisfied.

3. However, $\lim_{x \to 2} f(x) = 3$ and $f(2) = 1$, so the third condition is not satisfied.

Thus, the function is discontinuous at $x = 2$.

As we did with limits, we can define the left and right continuities for a function too.

Definition | Left Continuity at a Point

A function f is said to be left continuous at a point $x = a$ if the following three conditions are satisfied :

1. $f(a)$ is defined.
2. $\lim_{x \to a^-} f(x)$ exists.
3. $\lim_{x \to a^-} f(x) = f(a)$

Definition | Right Continuity at a Point

A function f is said to be right continuous at a point $x = a$ if the following three conditions are satisfied :

1. $f(a)$ is defined.
2. $\lim_{x \to a^+} f(x)$ exists.
3. $\lim_{x \to a^+} f(x) = f(a)$

Using this info, we can create the following definition :

Definition | Continuity on an Interval

A function f is said to be continuous on an interval I if it is continuous at every point in I.

- If I is a closed interval $[a, b]$, then we also need to check that f is left continuous at $x = b$ and right continuous at $x = a$.

Example | Continuity on an Interval Example

Show that the function $f(x) = 1 - \sqrt{1 - x^2}$ is continuous in the interval $[-1, 1]$.

Solution 1 |

If $-1 < a < 1$, the using the Limit Laws from Secton 1.6, we have :

$$\lim_{x \to a} f(x) = \lim_{x \to a}\left(1 - \sqrt{1 - x^2}\right) = 1 - \lim_{x \to a} \sqrt{1 - x^2} = 1 - \sqrt{1 - a^2} = f(a)$$

. Thus, f is continuous at every point in the open interval $(-1, 1)$. Now, we need to check the endpoints $x = -1$ and $x = 1$.

- At $x = -1$:

$$\begin{aligned} \lim_{x \to -1^+} f(x) &= \lim_{x \to -1^+}\left(1 - \sqrt{1 - x^2}\right) \\ &= 1 - \lim_{x \to -1^+} \sqrt{1 - x^2} \\ &= 1 - \sqrt{1 - (-1)^2} \\ &= 1 - 0 \\ &= 1 = f(-1) \end{aligned}$$

  . Thus, f is right continuous at $x = -1$.
- At $x = 1$:

$$\begin{aligned} \lim_{x \to 1^-} f(x) &= \lim_{x \to 1^-}\left(1 - \sqrt{1 - x^2}\right) \\ &= 1 - \lim_{x \to 1^-} \sqrt{1 - x^2} \\ &= 1 - \sqrt{1 - 1^2} \\ &= 1 - 0 \\ &= 1 = f(1) \end{aligned}$$

  . Thus, f is left continuous at $x = 1$. Therefore, f is continuous on the closed interval $[-1, 1]$.

Properties of Continous Functions

Just like limits, continuous functions have some properties that we can use to solve problems more easily.

THEOREM | Properties of Continuous Functions

If f and g are continuous at $x = a$, then the following functions are also continuous at $x = a$:

1. $f + g$
2. $f - g$
3. cf, where c is any constant
4. fg
5. $\frac{f}{g}$, provided that $g(a) \neq 0$
6. $[f(x)]^n$, where n is a positive integer (derived from rule 4)

Proof | Rule 1

$$\lim_{x \to a}[f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = f(a) + g(a) = (f + g)(a)$$

Thus, since $f(x)$ and $g(x)$ are continuous at $x = a$, $f + g$ is also continuous at $x = a$.

Proof | Rule 2

$$\lim_{x \to a}[f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = f(a) - g(a) = (f - g)(a)$$

Thus, since $f(x)$ and $g(x)$ are continuous at $x = a$, $f - g$ is also continuous at $x = a$.

Proof | Rule 3

$$\lim_{x \to a}[cf(x)] = c * \lim_{x \to a} f(x) = c * f(a) = (cf)(a)$$

Thus, since $f(x)$ is continuous at $x = a$, cf is also continuous at $x = a$.

Proof | Rule 4

$$\lim_{x \to a}[f(x)g(x)] = \lim_{x \to a} f(x) * \lim_{x \to a} g(x) = f(a) * g(a) = (fg)(a)$$

Thus, since $f(x)$ and $g(x)$ are continuous at $x = a$, fg is also continuous at $x = a$.

Proof | Rule 5

$$\lim_{x \to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{f(a)}{g(a)} = \left(\frac{f}{g}\right)(a)$$

Thus, since $f(x)$ and $g(x)$ are continuous at $x = a$ and $g(a) \neq 0$, $\frac{f}{g}$ is also continuous at $x = a$.

Proof | Rule 6

This rule is derived from Rule 4. Since $f(x)$ is continuous at $x = a$, using Rule 4 repeatedly, we can show that $[f(x)]^n$ is also continuous at $x = a$.

You may have noticed that we didnt even care about continuity during polynomials. This is because of the following theorem :

Theorem | Continuity of Polynomials and Rational Functions

1. Every polynomial function is continuous for all real numbers; that is, it is continuous on $\mathbb{R} = (-\infty, \infty)$
2. Every rational function is continuous at every point in its domain; that is, it is continuous on its domain.

Proof |

1. Let $P(x)$ be a polynomial function. Since polynomials are formed by adding, subtracting, and multiplying constant functions and the identity function $f(x) = x$, and both of these functions are continuous everywhere, by the Properties of Continuous Functions theorem, $P(x)$ is continuous for all real numbers.
2. Let $R(x) = \frac{P(x)}{Q(x)}$ be a rational function, where $P(x)$ and $Q(x)$ are polynomial functions. Since both $P(x)$ and $Q(x)$ are continuous everywhere (from part 1), and provided that $Q(a) \neq 0$, by the Properties of Continuous Functions theorem, $R(x)$ is continuous at every point in its domain.