Calculus I

Explanations, Problems & Solutions (Solutions)

Written by :

Sihoo Lee

KOREA SCIENCE ACADEMY OF KAIST

2025-12-30

Preface

Welcome to **Calculus I**, a rigorous journey into the mathematics of change and motion that forms the backbone of modern scientific analysis.

1 About This Textbook

This textbook has been carefully structured to guide you through the essential pillars of single variable calculus, summarizing the core concepts and techniques of the course:

Limits and Continuity — Establishing the precise mathematical foundation required to analyze functions at their boundaries, defining the critical concepts of instantaneous velocity and continuous motion.

The Derivative and Its Applications — Mastering the tools to measure instantaneous rates of change, using differentiation to solve complex optimization problems, approximate linear behaviors, and sketch the precise shapes of curves.

Integration — Exploring the accumulation of quantities to determine net change, calculating complex areas and volumes, and bridging the gap between slopes and areas through the Fundamental Theorem of Calculus.

Transcendental Functions and Advanced Techniques — Extending calculus to exponential, logarithmic, and inverse trigonometric functions, and developing sophisticated strategies such as integration by parts and trigonometric substitution to solve intricate problems.

2 How to Use This Book

To facilitate a deep and structured understanding of the material, this book utilizes specific block types to organize information:

Definition blocks clarify fundamental concepts, ensuring you grasp the precise language of calculus.

Example blocks demonstrate standard problems derived from the syllabus, illustrating how to apply theoretical concepts.

Solution blocks offer detailed, step-by-step walkthroughs to model the logical flow required for your own proofs and calculations.

Theorem blocks highlight the pivotal mathematical truths, such as the Mean Value Theorem and the Fundamental Theorem of Calculus.

Proof blocks guide you through the logical derivations of theorems, fostering the analytical reasoning skills emphasized in this course.

Note blocks provide crucial insights, common calculation pitfalls, and connections between differentiation and integration.

3 A Note on Learning

Calculus is more than a set of rules for manipulating symbols; it is a way of thinking about the world. True mastery requires **active engagement** with the material. We encourage you to:

- Attempt to solve the examples yourself before revealing the solution blocks.
- Focus on the geometric and physical intuition behind the formulas.
- Understand not just **how** to differentiate or integrate, but **when** and **why** these tools are used.
- Collaborate with peers to discuss concepts and refine your logical argumentation.

The notes provided here are designed to support your learning journey, clarifying the lecture material and helping you build the confidence to solve real-world problems mathematically.

4 Looking Ahead

The concepts you master in **Calculus I**—limits, derivatives, and integrals—are the indispensable tools of the future. They are the language used to describe the motion of planets, the flow of fluids, the growth of populations, and the fluctuations of markets. Whether you pursue physics, engineering, economics, or computer science, the analytical framework you build here will serve as the foundation for your advanced studies.

We hope this book serves as a clear and supportive guide as you learn to see the world through the lens of calculus.

Sihoo Lee
Korea Science Academy of KAIST

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Chapter 00

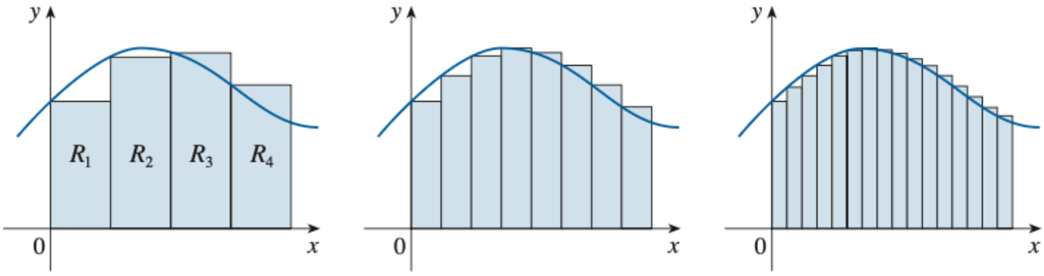
Preview of Calculus

This section bridges the gap between static pre-calculus mathematics and the dynamic nature of calculus by addressing two fundamental problems: determining the slope of a tangent line to a curve and calculating the instantaneous velocity of a moving object. Both problems are solved using the same limiting process, where approximations—such as the slopes of secant lines or average velocities over shrinking time intervals—converge to a single, precise value. This concept of the "limit" serves as the foundation for the entire course, establishing the mathematical tools necessary to analyze instantaneous rates of change.

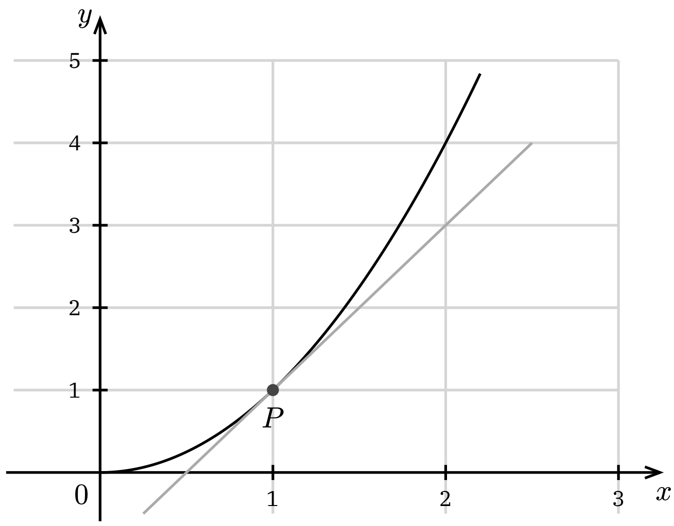
Chapter 00.00

Preview of Calculus

1 What is calculus?

- We have two main problems that we deal in calculus: The area problem & the tangent problem.
  - The area problem is about finding the area under a curve.
  - The tangent problem is about finding the slope of a curve at a given point.
  - 

  - Approximation of an area using rectangles.



Notation | The Area Problem

We can define the area under the curve as

$$A = \lim_{n \to \infty} A_n$$

Here, A_i represents sum of each area block equally dividing the given range into n parts.

NOTATION | The Tangent Problem

The slope of a curve at a given point a can be defined as

$$m_a = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

Here, m represents the slope of the tangent line to the curve at point x. Or, alternatively, we can write it as :

$$m_a = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

NOTE | So what's the relation?

Both problems involve limits. The area problem uses a limit to find the exact area under a curve. The tangent problem uses a limit to find the exact slope of a tangent line. In the end, solving one of them leads to solving the other. We call this the **Fundamental Theorem of Calculus**.

Chapter 01

Functions and Limits

This chapter formalizes the concept of the limit, moving from intuitive approximations to rigorous mathematical definitions including the precise Epsilon-Delta definition. It provides the algebraic rules (Limit Laws) necessary to calculate limits without graphing, methods for handling indeterminate forms, and defines continuity to describe functions without breaks or holes. These concepts culminate in the Intermediate Value Theorem, a powerful tool used to prove the existence of roots and solutions within specific intervals, setting the stage for the definition of the derivative.

Chapter 01.04

The Tangent & Speed Problem

1 The Tangent Problem

- What does tangent mean in calculus?

Definition | Tangency

A tangent line to a curve at a given point is a straight line that just "touches" the curve at that point. In calculus, the tangent line represents the instantaneous rate of change of the function at that point, which is given by the derivative of the function.

Example | Tangency Basics

Say we have the function $f(x) = x^2$. Let us deifne the base point of our tangent slope calculation as $x = 1$. Here, our slope calculation equation would become :

$$m_{\mathrm{pq}} = \frac{p_x^2 - q_x^2}{p_x - q_x}$$

where $q_x = 1$. If we plot in values from $p_x = 2$ to $p_x \approx 1$, we get the following table:

p_x	m_pq
2	3
1.5	2.5
1.1	2.1
1.01	2.01
1.001	2.001

As we can see from the table, as p_x approaches 1, the slope m_{pq} approaches 2. Therefore, the slope of the tangent line to the curve at the point where $x = 1$ is 2. Thus, after a bit of calculation, we can find that the equation of the tangent line at the point $(1, f(1)) = (1, 1)$ is:

$$y - 1 = 2(x - 1)$$

or simplified,

$$y = 2x - 1$$

.

Note | Approach from L/R

Note from the example above that there are two directions a line can approach a curve: from the left side (as p_x approaches 1 from values less than 1) and from the right side (as p_x approaches 1 from values greater than 1).

- Normally, this doesnt pose that much of a problem, but if the function is **severed** at a point, the left-hand limit and right-hand limit may not be equal, leading to different tangent slopes from each side.

2 The Velocity Problem

- Take a car. How do we define its **speed**?
- We know from middle school knowledge that speed is defined by the following equation :

$$v = \frac{\text{total distance travelled}}{\text{total time spent}}$$

- Note here that this is the **exact same form** as the slope equation we used in the tangent problem!

Definition | Average Velocity

Average velocity over a time interval is defined as the total displacement divided by the total time taken. Mathematically, if a car moves from position $s(a)$ at time a to position $s(b)$ at time b, the average velocity v_{avg} over the interval $[a, b]$ is given by:

$$v_{\text{avg}} = \frac{s(b) - s(a)}{b - a}$$

Definition | Instantaneous Velocity

A bit off course, but we can define instantaneous velocity using the later-defined idea of the limit.

Instantaneous velocity at a specific time is the limit of the average velocity as the time interval approaches zero. It represents the velocity of the car at a precise moment in time. Mathematically, the instantaneous velocity $v(t)$ at time t is given by:

$$v(t) = \lim_{\Delta \mathrm{t}} \to 0 \frac{s(t + \Delta \mathrm{t}) - s(t)}{\Delta \mathrm{t}}$$

Chapter 01.05

The Limit of a Function

1 The limit

- In the previous section, whilst discussing tangents and velocity, we kept encountering a problem: how do we define the slope of a curve at a single point, or the velocity of a car at a precise moment in time?
- Both of these problems can be solved using the concept of **limits**.

Definition | Intuitive Definition of the Limit

Suppose $f(x)$ is defined whilst x is near a. Then we write

$$l\,\text{im}_{\{xtoa\}}\, f(x) = L$$

We read this as "the limit of $f(x)$ as x approaches a is L". This claim must be preceded by the statement that as $f(x)$ gets arbitrarily closer and closer to L as x gets closer and closer to a.

Example | Limit Basics

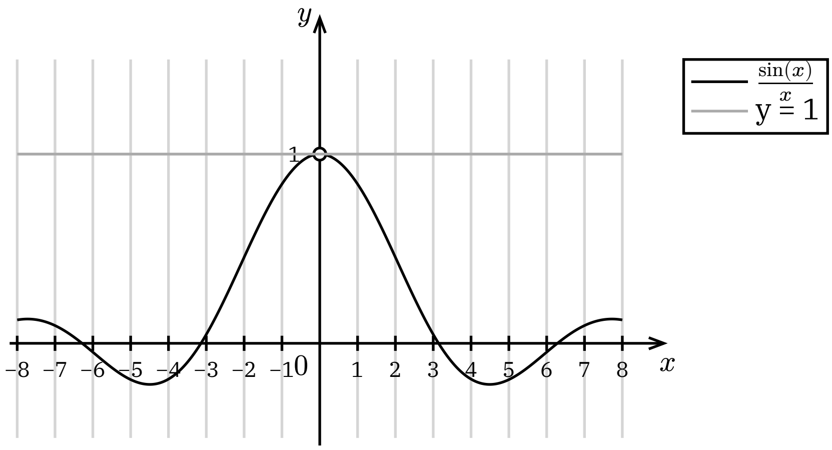
Consider the function $f(x) = \frac{\sin(x)}{x}$. Evaluate $f(x)$ at values of x that get closer and closer to 0:

Solution 1 |

x	$f(x)$
1	0.84147098
0.1	0.99833417
0.01	0.99998333
0.001	0.99999983
0.0001	0.99999998

By the intuitive definition of the limit, we can see that as x approaches 0, $f(x)$ approaches 1. Therefore, we can conclude that:

$$\lim_{x \to 0} \frac{\sin(x)}{x} = 1$$



- Above, we briefly mentioned that limits may differ when approached from different sides. This can be expanded to the fact that limits may not exist at all from some sides in some cases.

Definition | Intuitive Definition of One-Sided Limits

Suppose $f(x)$ is defined whilst x is near a. Then we write

$$l\,\text{im}_{\{xtoa^{+}\}} f(x) = L_1$$

and

$$l\,\text{im}_{\{xtoa^{-}\}} f(x) = L_2$$

We read these as "the limit of $f(x)$ as x approaches a from the right is L_1" and "the limit of $f(x)$ as x approaches a from the left is L_2". This claim must be preceded by the statement that as $f(x)$ gets arbitrarily closer and closer to L_1 as x gets closer and closer to a from the right, and similarly for L_2 from the left. Here, we call L_1 as the **right-hand limit** and L_2 as the **left-hand limit**.

Example | The Heaveside Function

Consider the Heaveside function $H(x)$ defined as follows:

$$H(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1 & \text{if } x \geq 0 \end{cases}$$

Evaluate $H(x)$ at values of x that get closer and closer to 0 from both sides:

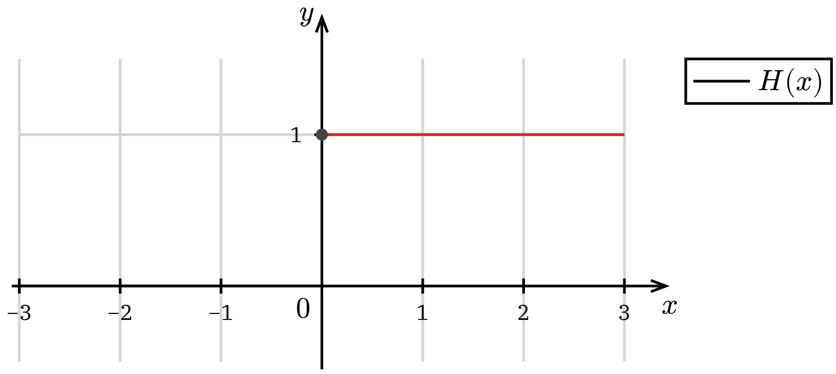
Solution 1 | More Evaluation...

x	$H(x)$
−1	0
−0.1	0
−0.01	0
0.01	1
0.1	1
1	1

By the intuitive definition of one-sided limits, we can see that as x approaches 0 from the left, $H(x)$ approaches 0, and as x approaches 0 from the right, $H(x)$ approaches 1. Therefore, we can conclude that:

$$\lim_{x \to 0^-} H(x) = 0$$

$$\lim_{x \to 0^+} H(x) = 1$$



- With the basic idea, we can now consider the bigger problem : can limits **fail** to exist?

Example | Uh Oh

Investigate the following :

$$\lim_{x \to 0} \sin\left(\frac{\pi}{x}\right)$$

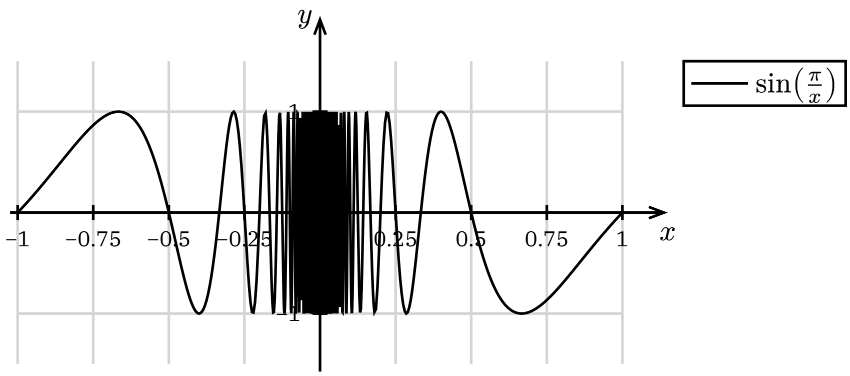
Solution 1 |

x	$\sin\left(\frac{\pi}{x}\right)$
1	0
0.1	0.58778525
0.01	0.95105652
0.001	0.30901699
0.0001	0.98768834

By evaluating $\sin\left(\frac{\pi}{x}\right)$ at values of x that get closer and closer to 0, we can see that the function does not approach a single value. Instead, it oscillates between -1 and 1. Therefore, we can conclude that:

$$\lim_{x\to 0} \sin\left(\frac{\pi}{x}\right)$$

does not exist.



Example | Uh Oh 2

Find

$$\lim_{x\to 0}\left(\frac{1}{x^2}\right)$$

if it exists

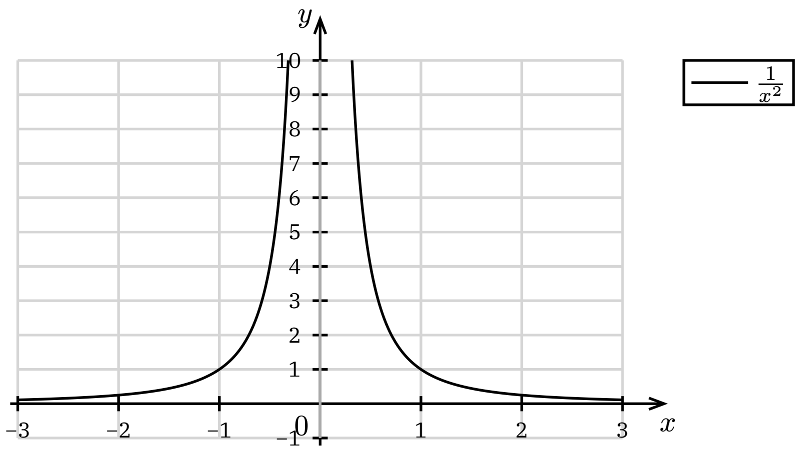
Solution 1 |

x	$\frac{1}{x^2}$
1	1
0.1	100
0.01	10000
0.001	1000000
0.0001	10000000

By evaluating $\frac{1}{x^2}$ at values of x that get closer and closer to 0, we can see that the function grows without bound. Therefore, we can conclude that:

$$\lim_{x \to 0} \left(\frac{1}{x^2} \right) = \infty$$

(which mathematically means that the limit does not exist in the real number system)



Definition | Intuitive Definition of an Infinite Limit

Suppose $f(x)$ is defined whilst x is near a on both sides, except at a itself. Then we write

$$\lim_{x \to a} f(x) = \infty$$

which means that the values of $f(x)$ can be made arbitrarily large by taking x sufficiently close but not equal to a.

- Of course, we can do the same for negative infinity.

Definition | Intuitive Definition of a Negative Infinity Limit

Suppose $f(x)$ is defined whilst x is near a on both sides, except at a itself. Then we write

$$\lim_{x \to a} f(x) = -\infty$$

which means that the values of $f(x)$ can be made arbitrarily small (negatively large) by taking x sufficiently close but not equal to a.

- At these points, we can define a **vertical asymptote**.

Definition | Vertical Asymptote

A vertical asymptote is a vertical line $x = a$ when the function $f(x)$ suffices at least one of the 6 conditions :

- $\lim_{x \to a^+} f(x) = \infty$
- $\lim_{x \to a^+} f(x) = -\infty$
- $\lim_{x \to a^-} f(x) = \infty$
- $\lim_{x \to a^-} f(x) = -\infty$
- $\lim_{x \to a} f(x) = \infty$
- $\lim_{x \to a} f(x) = -\infty$

In simpler terms, a vertical asymptote is a vertical line where the function grows without bound as it approaches the line from at least one side.

Example | Vertical Asymptote Example

Investigate if the function $y = \frac{2x}{x-3}$ have a vertical asymptote.

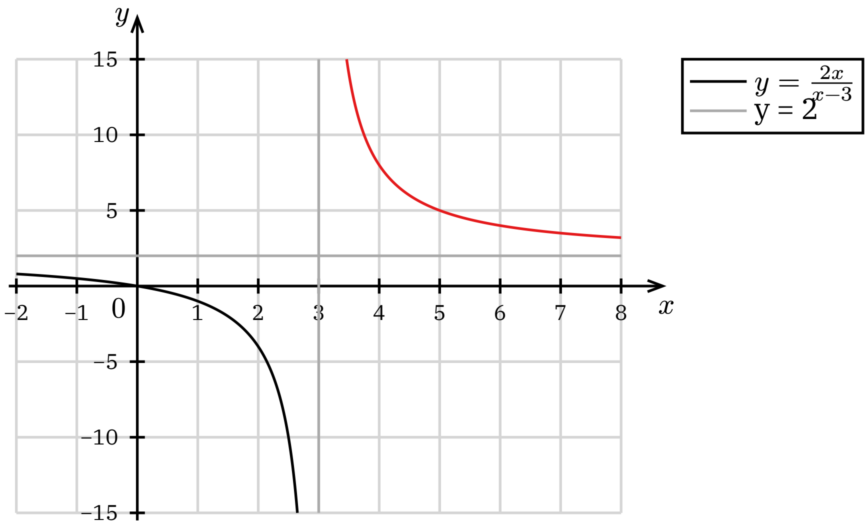
Solution 1 |

To find vertical asymptotes, we need to look for values of x that make the denominator equal to 0. Here, the denominator $x - 3$ equals 0 when $x = 3$. Next, we need to evaluate the limits as x approaches 3 from both sides:

$$\lim_{x \to 3^+} \frac{2x}{x - 3} = \infty$$

$$\lim_{x \to 3^-} \frac{2x}{x - 3} = -\infty$$

Since both one-sided limits approach infinity (one positive, one negative), we can conclude that the function has a vertical asymptote at $x = 3$.



Chapter 01.06

Calculating Limits using the Limit Laws

- Limits Have some rules that we must comply to.

THEOREM | The Laws of Limits

Suppose that c i s a constant and the limits $\lim_{x \to a} f(x)$ and $\lim_{x \to a} g(x)$ exist. Then, the following properties hold:
1. $\lim_{x \to a}[f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x)$
2. $\lim_{x \to a}[f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x)$
3. $\lim_{x \to a}[c * f(x)] = c * \lim_{x \to a} f(x)$
4. $\lim_{x \to a}[f(x) * g(x)] = \lim_{x \to a} f(x) * \lim_{x \to a} g(x)$
5. $\lim_{x \to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)}$, provided that $\lim_{x \to a} g(x) \neq 0$

Using these laws, we can derive a few more :

THEOREM | The Power Law of Limits

Suppose that $\lim_{x \to a} f(x) = L$ and n is a positive integer. Then, $\lim_{x \to a} [f(x)]^n = [\lim_{x \to a} f(x)]^n = L^n$

THEOREM | The Root Law of Limits

Suppose that $\lim_{x \to a} f(x) = L$ and n is a positive integer. Then, $\lim_{x \to a} [f(x)]^{\frac{1}{n}} = [\lim_{x \to a} f(x)]^{\frac{1}{n}} = L^{\frac{1}{n}}$, provided that if n is even, then $L \geq 0$

also, some special limits :

THEOREM | The Constant Function Law

For any constant c, $\lim_{x \to a} c = c$

Written differently, $\lim_{x \to a} = a$

Example | Limit Law Basics

Evaluate $\lim_{x\to 5}(2x^2 - 3x + 4)$.

Solution 1 |

$$\lim_{x\to 5}(2x^2 - 3x + 4) = \lim_{x\to 5} 2x^2 - \lim_{x\to 5} 3x + \lim_{x\to 5} 4$$

($\because$ the Sum and Difference Law)

$$= 2 * \lim_{x\to 5} x^2 - 3 * \lim_{x\to 5} x + 4$$

($\because$ the Constant Multiple Law)

$$= 2 * 5^2 - 3 * 5 + 4$$

($\because$ the Power Law and Constant Function Law)

$$= 50 - 15 + 4 = 39$$

Above, we have discussed that $\lim_{x\to a} x = a$. What if we generalize this into any function $f(x)$?

Theorem | The Direct Substitution Property

If f is a polynomial/rational function and a is in the domain of f, at $x = a$, then

$$\lim_{x\to a} f(x) = f(a)$$

Example | Direct Substitution Example

Evaluate $\lim_{x\to 2} \frac{x^3-4x+1}{x^2+3}$.

Solution 1 |

Since the function is a rational function and 2 is in the domain of the function, we can use the Direct Substitution Property.

$\lim_{x\to 2} \frac{x^3-4x+1}{x^2+3} = \frac{2^3-4*2+1}{2^2+3} = \frac{8-8+1}{4+3} = \frac{1}{7}$

Example | Laws of Limits General

Evaluate $\lim_{x \to 1} \frac{x^2-1}{x-1}$.

Solution 1 |

We cannot directly substitute 1 into the function, since it would create a division by zero. However, we can simplify the expression first.

$$\frac{x^2-1}{x-1} = \frac{(x-1)(x+1)}{x-1} = x+1 \because x \neq 1$$

.

Now we can find the limit:

$$\lim_{x \to 1} (x+1) = 1+1 = 2$$

From the intuitive definition of left/right limits, we can know the following :

Theorem | Relationship between One-Sided Limits and Two-Sided Limits

Let $f(x)$ be defined whilst x is near a. Then,

$$\lim_{x \to a} f(x) = L$$

if and only if both

$$\lim_{x \to a^+} f(x) = L$$

and

$$\lim_{x \to a^-} f(x) = L$$

1 The Squeeze Theorem

- Sometimes, it is hard to find the limit of a function directly. However, if we can find two other functions that "squeeze" the function we are interested in, we can use the Squeeze Theorem to find the limit.

Theorem | The Squeeze Theorem

Let $f(x)$, $g(x)$, and $h(x)$ be functions defined on an open interval containing a, except possibly at a itself. If for every x in this interval, except possibly at a, we have

$$g(x) \leq f(x) \leq h(x)$$

and if

$$\lim_{x \to a} g(x) = \lim_{x \to a} h(x) = L$$

then,

$$\lim_{x \to a} f(x) = L$$

Example | Squeeze Theorem Example

Evaluate

$$\lim_{x \to 0} x^2 \sin\left(\frac{1}{x}\right)$$

Solution 1 |

We know that

$$-1 \leq \sin\left(\frac{1}{x}\right) \leq 1$$

for all $x \neq 0$. Multiplying all parts of the inequality by x^2 (which is always non-negative), we get :

$$-x^2 \leq x^2 * \sin\left(\frac{1}{x}\right) \leq x^2$$

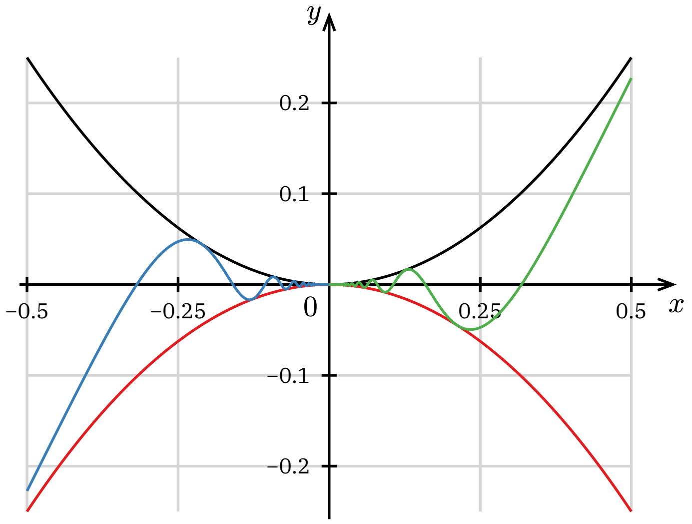
Now, we can find the limits of the two outer functions as x approaches 0:

$$\lim_{x \to 0} -x^2 = 0$$

$$\lim_{x \to 0} x^2 = 0$$

Since both limits are equal to 0, by the Squeeze Theorem, we have :

$$\lim_{x - u003e0} x^2 * \sin\left(\frac{1}{x}\right) = 0$$



y
0.2
0.1
−0.5
−0.25
0
0.25
0.5
x
−0.1
−0.2

Chapter 01.07

The Precise Definition of a Limit

1 The "Precise" Definition of a Limit

- Till now, we have been using *intuitive* methods to define the limits of functions. However, in order to be more *precise*, we need a more rigorous definition of limits.

Example | A Weird Function

Consider the following function : $\begin{cases} 2x-1 & \text{if } x \neq 3 \\ 6 & \text{if } x = 3 \end{cases}$. What is $f(3)$? What is $\lim_{x \to 3} f(x)$?

Solution 1 |

$f(3) = 6$, since when $x = 3$, the function outputs 6. However, to find $\lim_{x \to 3} f(x)$, we need to see what value $f(x)$ approaches as x gets closer and closer to 3. As x approaches 3, $f(x)$ approaches $2(3) - 1 = 5$. Therefore, $\lim_{x \to 3} f(x) = 5$.

- Consider the example above. How can we more "accurately" define the limit of a function at a point, without relying on intuition or graphs?
- In te problem above, consider the following question.
  - "How close to 3 does x need to be, so that $f(x)$ is within 0.1 of 5?"
  - In other words, we want to find some value δ s.t.

$$\|f(x) - 5\| < 0.1 \quad \text{if} \quad \|x - 3\| < \delta \quad \text{but } x \neq 3$$

  If $\|x - 3\| > 0$, then $x \neq 3$, so we can simplify the problem into the following :
  - "How close to 3 does x need to be, so that $f(x)$ is within 0.1 of 5, given that $x \neq 3$

$$\|f(x) - 5\| < 0.1 \quad \text{if} \quad 0 < \|x - 3\| < \delta$$

  Here, we call 0.1 the "Error Tolerance". Since we are going to send this to 0, let us replace this with ε for the time being. So the question becomes :

$$\|f(x) - 5\| < \varepsilon \quad \text{if} \quad 0 < \|x - 3\| < \delta$$

  Now, we finally have the tools to define the limit of a function more "precisely".

Definition | Precise Definition of a Limit

Let f be a function defined on an open interval containing a, except possibly at a itself. We say that $\lim_{x \to a} f(x) = L$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\|f(x) - L\| < \varepsilon \qquad \text{if} \qquad 0 < \|x - a\| < \delta$$

To simplify even more, we can write like this(in style):

$$\forall \varepsilon > 0, \exists \delta > 0, \text{ s.t. } 0 < \|x - a\| < \delta \Longrightarrow \|f(x) - L\| < \varepsilon$$

Since we have to divide them later anyway, we can do it now :

Definition | Precise definition of a left-hand limit

Let f be a function defined on an open interval containing a, except possibly at a itself. We say that $\lim_{x \to a^-} f(x) = L$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\|f(x) - L\| < \varepsilon \qquad \text{if} \qquad a - \delta < x < a$$

To simplify even more, we can write like this:

$$\forall \varepsilon > 0, \exists \delta > 0, \text{ s.t. } a - \delta < x < a \Longrightarrow \|f(x) - L\| < \varepsilon$$

Definition | Precise definition of a right-hand limit

Let f be a function defined on an open interval containing a, except possibly at a itself. We say that $\lim_{x \to a^+} f(x) = L$ if for every $\varepsilon > 0$, there exists a $\delta > 0$ such that

$$\|f(x) - L\| < \varepsilon \qquad \text{if} \qquad a < x < a + \delta$$

To simplify even more, we can write like this:

$$\forall \varepsilon > 0, \exists \delta > 0, \text{ s.t. } a < x < a + \delta \Longrightarrow \|f(x) - L\| < \varepsilon$$

One good way to understand the epsilon-delta method is by assuming we chose a "wrong" limit value "L'". In this case, we must be able to find a δ for any arbitrary ε given. Since we chose a wrong limit value, there will always be some x values within the δ-neighborhood of a that make $f(x)$ be outside the ε-neighborhood of "L'". This means that no matter how small we choose δ, there will always be some x values that break the condition. Thus, we can conclude that "L'" is not the correct limit value.

Note | Usage of the epsilon-delta method

The epsilon-delta method can only be used to **"prove"** that a certain limit value is correct. It cannot be used to **"find"** the limit value itself.

Now lets utilize this to solve some problems.

Example | Epsilon-Delta Basics

Prove that $\lim_{x\to 3}(4x-5)=7$ using the epsilon-delta definition of a limit.

Solution 1 |

Let $\varepsilon > 0$ be given. We need to find a $\delta > 0$ such that

$$\|(4x-5)-7\| < \varepsilon \quad \text{if} \quad 0 < \|x-3\| < \delta$$

1. Preliminary Analysis :

Simplifying the left side, we have :

$$\|4x-12\| = 4 * \|x-3\|$$

So we want to ensure that

$$4 * \|x-3\| < \varepsilon \quad \text{if} \quad 0 < \|x-3\| < \delta$$

To achieve this, we can choose $\delta = \frac{\varepsilon}{4}$. Then, if $0 < \|x-3\| < \delta$, we have

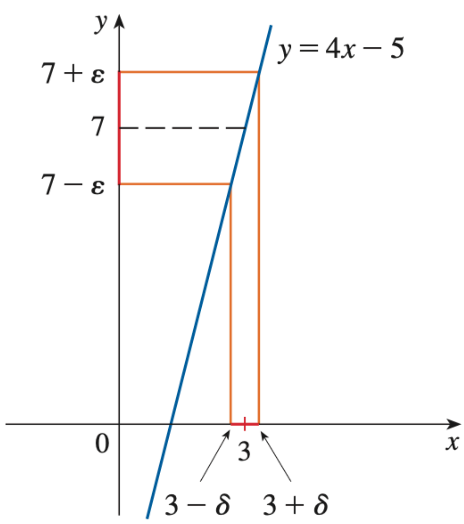
$$\|(4x-5)-7\| = 4 * \|x-3\| < 4 * \delta = \varepsilon$$

2. Formal Proof :

Let $\varepsilon > 0$ be given. Choose $\delta = \frac{\varepsilon}{4}$. Then, if $0 < \|x-3\| < \delta$, we have

$$\|(4x-5)-7\| = 4 * \|x-3\| < 4 * \delta = \varepsilon$$

Thus, by the epsilon-delta definition of a limit, we conclude that $\lim_{x\to 3}(4x-5)=7$.



Example | Epsilon-Delta Basics 2

Describe $\lim_{x \to a} x^2 = a^2$ using the epsilon-delta definition of a limit.

Solution 1 |

Let $\varepsilon > 0$ be given. We need to find a $\delta > 0$ such that

$$\left\|x^2 - a^2\right\| < \varepsilon \quad \text{if} \quad 0 < \|x - a\| < \delta$$

1. Preliminary Analysis :

Simplifying the left side, we have :

$$\left\|x^2 - a^2\right\| = \|(x - a)(x + a)\| = \|x - a\| * \|x + a\|$$

To control $\|x + a\|$, we can restrict δ to be less than 1(an arbitrary number). This means that if $0 < \|x - a\| < \delta < 1$, then

$$\|x - a\| < 1 \Longrightarrow \|x\| < \|a\| + 1$$

Thus, we have

$$\|x + a\| \leq \|x\| + \|a\| < (\|a\| + 1) + \|a\| = 2 * \|a\| + 1$$

Therefore, we want to ensure that

$$\|x - a\| * (2 * \|a\| + 1) < \varepsilon \quad \text{if} \quad 0 < \|x - a\| < \delta < 1$$

To achieve this, we can choose $\delta = \min\left(1, \frac{\varepsilon}{2*\|a\|+1}\right)$. Then, if $0 < \|x - a\| < \delta$, we have

$$\left\|x^2 - a^2\right\| = \|x - a\| * \|x + a\| < \delta * (2 * \|a\| + 1) \leq \left(\frac{\varepsilon}{2 * \|a\| + 1}\right) * (2 * \|a\| + 1) = \varepsilon$$

2. Formal Proof :

Let $\varepsilon > 0$ be given. Choose $\delta = \min\left(1, \frac{\varepsilon}{2*\|a\|+1}\right)$. Then, if $0 < \texttt{norm}\ (x - a) < \delta$, we have

$$\left\|x^2 - a^2\right\| = \|x - a\| * \|x + a\| < \delta * (2 * \|a\| + 1) \leq \left(\frac{\varepsilon}{2 * \|a\| + 1}\right) * (2 * \|a\| + 1) = \varepsilon$$

Thus, by the epsilon-delta definition of a limit, we conclude that $\lim_{x \to a} x^2 = a^2$.

Note | Why can we assign an arbitrary number to delta?

In the preliminary analysis, we restricted δ to be less than 1 to control the value of $\|x + a\|$. This is a common technique in epsilon-delta proofs to ensure that certain expressions remain bounded. By choosing δ to be the minimum of 1 and another expression, we can effectively manage the behavior of the function within the desired neighborhood around the point a. To say intuitively, we are "bounding" the function in the neighborhood of a once more outside of delta.

Example | Epsilon-Delta Basics 3

Describe $\lim_{x\to 1} \frac{x^2-1}{x-1} = 2$ using the epsilon-delta definition of a limit.

Solution 1 |

Let $\varepsilon > 0$ be given. We need to find a $\delta > 0$ such that

$$\left\| \frac{x^2 - 1}{x - 1} - 2 \right\| < \varepsilon \qquad \text{if} \qquad 0 < \|x - 1\| < \delta$$

1. Preliminary Analysis :

Simplifying the left side, we have :

$$\left\| \frac{x^2 - 1}{x - 1} - 2 \right\| = \left\| (x-1)\frac{x + 1}{x - 1} - 2 \right\| = \|x + 1 - 2\| = \|x - 1\|$$

Therefore, we want to ensure that

$$\|x - 1\| < \varepsilon \qquad \text{if} \qquad 0 < \|x - 1\| < \delta$$

To achieve this, we can choose $\delta = \varepsilon$. Then, if $0 < \|x - 1\| < \delta$, we have

$$\left\| \frac{x^2 - 1}{x - 1} - 2 \right\| = \|x - 1\| < \delta = \varepsilon$$

2. Formal Proof :

Let $\varepsilon > 0$ be given. Choose $\delta = \varepsilon$. Then, if $0 < \|x - 1\| < \delta$, we have

$$\left\| \frac{x^2 - 1}{x - 1} - 2 \right\| = \|x - 1\| < \delta = \varepsilon$$

Thus, by the epsilon-delta definition of a limit, we conclude that $\lim_{x \to 1} \frac{x^2-1}{x-1} = 2$.

Example | Epsilon-Delta Basics 4

Describe $\lim_{x \to 0+} \sqrt{x} = 0$ using the epsilon-delta definition of a limit.

Solution 1 |

Let $\varepsilon > 0$ be given. We need to find a $\delta > 0$ such that

$$\left\| \sqrt{x} - 0 \right\| < \varepsilon \quad \text{if} \quad 0 < x < \delta$$

1. Preliminary Analysis :

Simplifying the left side, we have :

$$\left\| \sqrt{x} - 0 \right\| = \sqrt{x}$$

Therefore, we want to ensure that

$$\sqrt{x} < \varepsilon \quad \text{if} \quad 0 < x < \delta$$

To achieve this, we can choose $\delta = \varepsilon^2$. Then, if $0 < x < \delta$, we have

$$\left\| \sqrt{x} - 0 \right\| = \sqrt{x} < \sqrt{\delta} = \sqrt{\varepsilon^2} = \varepsilon$$

2. Formal Proof :

Let $\varepsilon > 0$ be given. Choose $\delta = \varepsilon^2$. Then, if $0 < x < \delta$, we have

$$\left\| \sqrt{x} - 0 \right\| = \sqrt{x} < \sqrt{\delta} = \sqrt{\varepsilon^2} = \varepsilon$$

Thus, by the epsilon-delta definition of a limit, we conclude that $\lim_{x \to 0+} \sqrt{x} = 0$.

Chapter 01.08

Continuity

1 Continuity

Intuitively, continuity can be simply defined as “a function that is connected”...but we dont deal with intuition here. We need a more “precise” definition of continuity.

Definition | Continuity at a Point

A function f is said to be continuous at a point $x = a$ if the following three conditions are satisfied :
1. $f(a)$ is defined.
2. $\lim_{x \to a} f(x)$ exists.
3. $\lim_{x \to a} f(x) = f(a)$

If these conditions are not met, we call the function is “discontinuous” at $x = a$, or has a “discontinuity” at $x = a$.

Example | Continuity Example

Identify if the functions are discontinuous.

(a) $f(x) = \frac{x^2-x-2}{x-2}$

(b) $f(x) = \begin{cases} \frac{x^2-x-2}{x-2} & \text{if} \quad x \neq 2 \\ 1 & \text{if} \quad x = 2 \end{cases}$

Solution (a) |

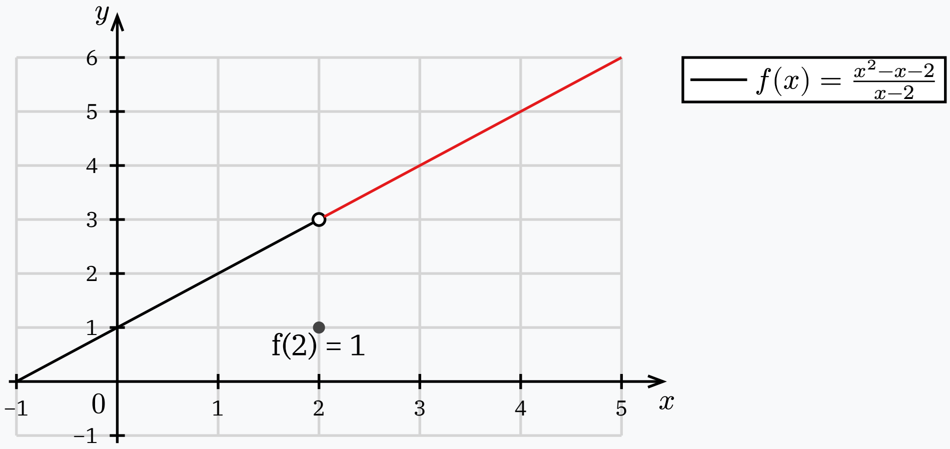
The function is discontinuous at $x = 2$. This is because $f(2)$ is not defined, since the denominator becomes 0 at this point. Therefore, the first condition for continuity is not satisfied.

Solution (b) |

The function is continuous at $x = 2$. Let’s check the three conditions:
1. $f(2) = 1$, so the first condition is satisfied.
2. To find $\lim_{x \to 2} f(x)$, we simplify the expression for $x \neq 2 : f(x) = \frac{x^2-x-2}{x-2} = \frac{(x-2)(x+1)}{x-2} = x + 1$ for $x \neq 2$. Therefore, $\lim_{x \to 2} f(x) = \lim_{x \to 2}(x + 1) = 3$. So the second condition is satisfied.

3. However, $\lim_{x \to 2} f(x) = 3$ and $f(2) = 1$, so the third condition is not satisfied.

Thus, the function is discontinuous at $x = 2$.



As we did with limits, we can define the left and right continuities for a function too.

Definition | Left Continuity at a Point

A function f is said to be left continuous at a point $x = a$ if the following three conditions are satisfied :
1. $f(a)$ is defined.
2. $\lim_{x \to a^-} f(x)$ exists.
3. $\lim_{x \to a^-} f(x) = f(a)$

Definition | Right Continuity at a Point

A function f is said to be right continuous at a point $x = a$ if the following three conditions are satisfied :
1. $f(a)$ is defined.
2. $\lim_{x \to a^+} f(x)$ exists.
3. $\lim_{x \to a^+} f(x) = f(a)$

Using this info, we can create the following definition :

Definition | Continuity on an Interval

A function f is said to be continuous on an interval I if it is continuous at every point in I.

- If I is a closed interval $[a, b]$, then we also need to check that f is left continuous at $x = b$ and right continuous at $x = a$.

Example | Continuity on an Interval Example

Show that the function $f(x) = 1 - \sqrt{1 - x^2}$ is continuous in the interval $[-1, 1]$.

Solution 1 |

If $-1 < a < 1$, the using the Limit Laws from Secton 1.6, we have :

$$\lim_{x \to a} f(x) = \lim_{x \to a} \left(1 - \sqrt{1 - x^2}\right) = 1 - \lim_{x \to a} \sqrt{1 - x^2} = 1 - \sqrt{1 - a^2} = f(a)$$

. Thus, f is continuous at every point in the open interval $(-1, 1)$. Now, we need to check the endpoints $x = -1$ and $x = 1$.

- At $x = -1$:

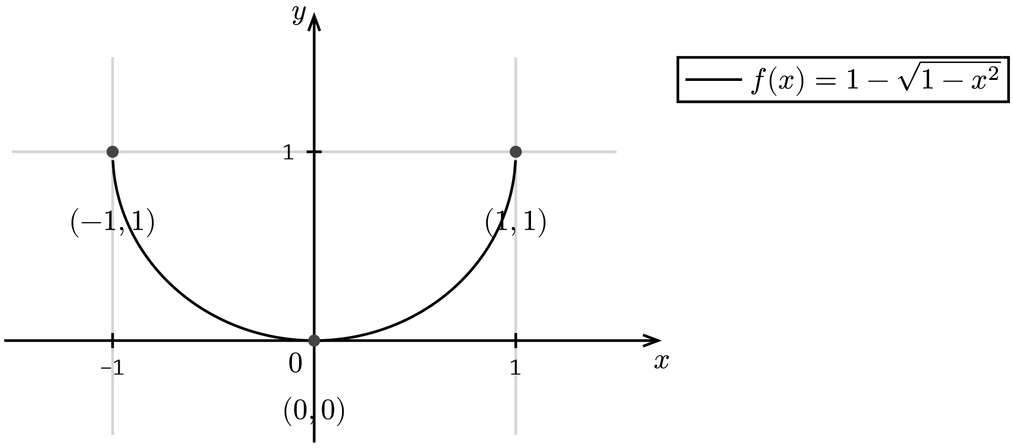
$$\begin{aligned} \lim_{x \to -1^+} f(x) &= \lim_{x \to -1^+} \left(1 - \sqrt{1 - x^2}\right) \\ &= 1 - \lim_{x \to -1^+} \sqrt{1 - x^2} \\ &= 1 - \sqrt{1 - (-1)^2} \\ &= 1 - 0 \\ &= 1 = f(-1) \end{aligned}$$

  . Thus, f is right continuous at $x = -1$.

- At $x = 1$:

$$\begin{aligned} \lim_{x \to 1^-} f(x) &= \lim_{x \to 1^-} \left(1 - \sqrt{1 - x^2}\right) \\ &= 1 - \lim_{x \to 1^-} \sqrt{1 - x^2} \\ &= 1 - \sqrt{1 - 1^2} \\ &= 1 - 0 \\ &= 1 = f(1) \end{aligned}$$

  . Thus, f is left continuous at $x = 1$. Therefore, f is continuous on the closed interval $[-1, 1]$.



2 Properties of Continous Functions

Just like limits, continuous functions have some properties that we can use to solve problems more easily.

Theorem | Properties of Continuous Functions

If f and g are continuous at $x = a$, then the following functions are also continuous at $x = a$:

1. $f + g$
2. $f - g$
3. cf, where c is any constant
4. fg
5. $\frac{f}{g}$, provided that $g(a) \neq 0$
6. $[f(x)]^n$, where n is a positive integer (derived from rule 4)

Proof | Rule 1

$$\lim_{x \to a}[f(x) + g(x)] = \lim_{x \to a} f(x) + \lim_{x \to a} g(x) = f(a) + g(a) = (f + g)(a)$$

Thus, since $f(x)$ and $g(x)$ are continuous at $x = a$, $f + g$ is also continuous at $x = a$.

Proof | Rule 2

$$\lim_{x \to a}[f(x) - g(x)] = \lim_{x \to a} f(x) - \lim_{x \to a} g(x) = f(a) - g(a) = (f - g)(a)$$

Thus, since $f(x)$ and $g(x)$ are continuous at $x = a$, $f - g$ is also continuous at $x = a$.

Proof | Rule 3

$$\lim_{x \to a}[cf(x)] = c * \lim_{x \to a} f(x) = c * f(a) = (cf)(a)$$

Thus, since $f(x)$ is continuous at $x = a$, cf is also continuous at $x = a$.

Proof | Rule 4

$$\lim_{x \to a}[f(x)g(x)] = \lim_{x \to a} f(x) * \lim_{x \to a} g(x) = f(a) * g(a) = (fg)(a)$$

Thus, since $f(x)$ and $g(x)$ are continuous at $x = a$, fg is also continuous at $x = a$.

Proof | Rule 5

$$\lim_{x \to a}\left[\frac{f(x)}{g(x)}\right] = \frac{\lim_{x \to a} f(x)}{\lim_{x \to a} g(x)} = \frac{f(a)}{g(a)} = \left(\frac{f}{g}\right)(a)$$

Thus, since $f(x)$ and $g(x)$ are continuous at $x = a$ and $g(a) \neq 0$, $\frac{f}{g}$ is also continuous at $x = a$.

Proof | Rule 6

This rule is derived from Rule 4. Since $f(x)$ is continuous at $x = a$, using Rule 4 repeatedly, we can show that $[f(x)]^n$ is also continuous at $x = a$.

You may have noticed that we didnt even care about continuity during polynomials. This is because of the following theorem :

Theorem | Continuity of Polynomials and Rational Functions

1. Every polynomial function is continuous for all real numbers; that is, it is continuous on $\mathbb{R} = (-\infty, \infty)$
2. Every rational function is continuous at every point in its domain; that is, it is continuous on its domain.

Proof |

1. Let $P(x)$ be a polynomial function. Since polynomials are formed by adding, subtracting, and multiplying constant functions and the identity function $f(x) = x$, and both of these functions are continuous everywhere, by the Properties of Continuous Functions theorem, $P(x)$ is continuous for all real numbers.
2. Let $R(x) = \frac{P(x)}{Q(x)}$ be a rational function, where $P(x)$ and $Q(x)$ are polynomial functions. Since both $P(x)$ and $Q(x)$ are continuous everywhere (from part 1), and provided that $Q(a) \neq 0$, by the Properties of Continuous Functions theorem, $R(x)$ is continuous at every point in its domain.

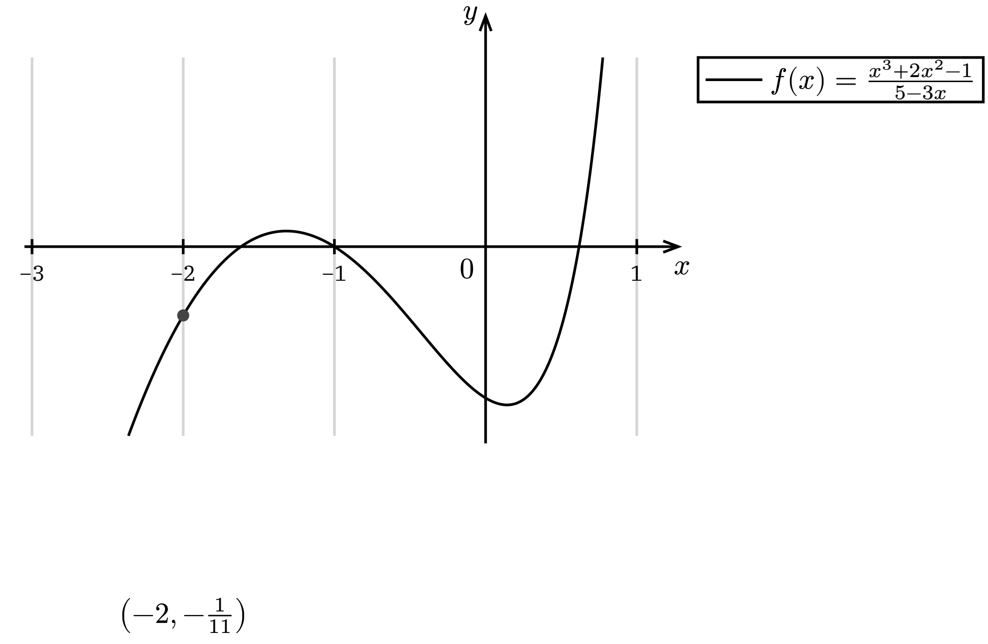
Example | Continuity of a Rational Function

Find $\lim_{x\to -2} \frac{x^3+2x^2-1}{5-3x}$.

Solution 1 |

Since the function is a rational function and 2 is in the domain of the function, we can use the Continuity of Rational Functions theorem.

$\lim_{x\to -2} \frac{x^3+2x^2-1}{5-3x} = \frac{-2^3+2*(-2)^2-1}{5-3*(-2)} = \frac{-8+8-1}{5+6} = -\frac{1}{11}$



3 Intermediate Value Theorem

The Intermediate Value Theorem is a very useful theorem that utilizes the property of continuous functions.

Theorem | Intermediate Value Theorem

Let f be a function that is continuous on the closed interval $[a, b]$, and let N be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then, there exists at least one number c in the open interval (a, b) such that $f(c) = N$.

Proof |

Since f is continuous on the closed interval $[a, b]$, by the definition of continuity, for every x in $[a, b]$, $f(x)$ takes on every value between $f(a)$ and

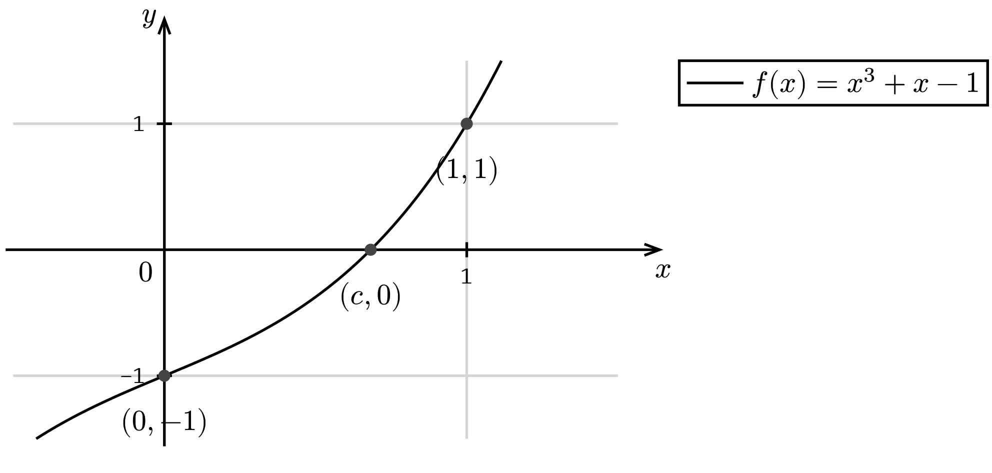
$f(b)$. Therefore, for any number N between $f(a)$ and $f(b)$, there must exist at least one number c in the open interval (a, b) such that $f(c) = N$.

Example | Intermediate Value Theorem Example

Show that the equation $x^3 + x - 1 = 0$ has a solution in the interval $[0, 1]$.

Solution 1 |

Let $f(x) = x^3 + x - 1$. We need to show that there exists a number c in the interval $(0, 1)$ such that $f(c) = 0$. First, we check the values of f at the endpoints of the interval: $f(0) = 0^3 + 0 - 1 = -1$ $f(1) = 1^3 + 1 - 1 = 1$ Since $f(0) = -1$ and $f(1) = 1$, and 0 is between -1 and 1, by the Intermediate Value Theorem, there exists at least one number c in the interval $(0, 1)$ such that $f(c) = 0$. Therefore, the equation $x^3 + x - 1 = 0$ has a solution in the interval $[0, 1]$.



Chapter 02

Derivatives

Chapter 02.01

Derivatives and Rates of Change

We now should discuss the elephant in the room, **derivatives**.

1 Introduction to Derivatives

- So far, we have discussed limits and how they can be used to find the slope of tangents and instantaneous velocity. However, we have not yet formally defined derivatives.
- Derivatives are a way to represent the rate of change of a function at a specific point. In simpler terms, it tells us how quickly the function's output is changing as we make small changes to its input.

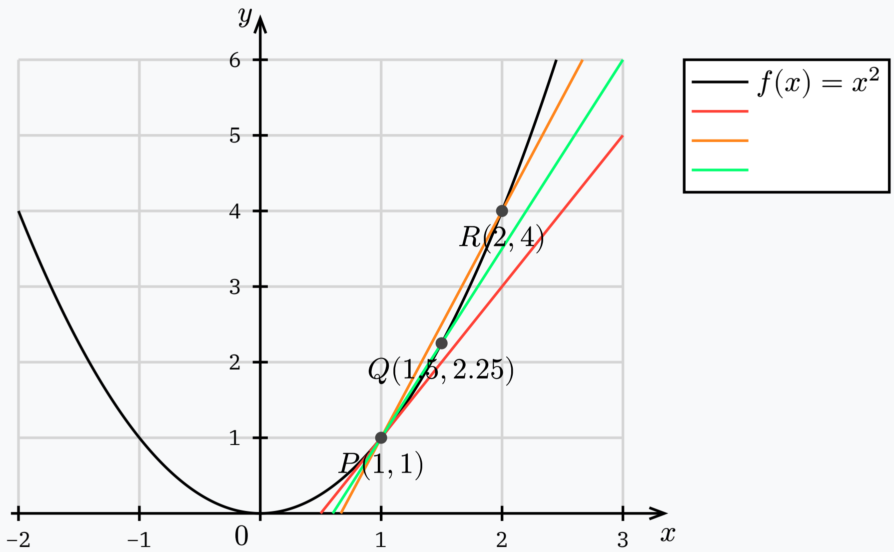
Definition | The Tangent Line

The tangent line to the curve $y = f(x)$ at the point $P(a, f(a))$ is the line through P with the slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

. provided that this limit exists.

As we move the limit closer to P, the line more accurately represents the slope of the curve at that point.



Example | Derivative Basics

Find the equation of the tangent line to the curve $y = x^2$ at the point $P(2, 4)$.

Solution 1 |

First, we need to find the slope of the tangent line at point P. We can use the definition of the tangent line:

$$m = \lim_{x \to 2} \frac{f(x) - f(2)}{x - 2}$$

where $f(x) = x^2$ and $f(2) = 4$.

So,

$$m = \lim_{x \to 2} \frac{x^2 - 4}{x - 2}$$

We can factor the numerator:

$$m = \lim_{x \to 2} \frac{(x - 2)(x + 2)}{x - 2}$$

For $x \neq 2$, we can cancel out $(x - 2)$:

$$m = \lim_{x \to 2} (x + 2)$$

Now, we can directly substitute $x = 2$:

$$m = 2 + 2 = 4$$

Now that we have the slope of the tangent line, we can use the point-slope form of a line to find its equation:

$$y - y_1 = m(x - x_1)$$

where $(x_1, y_1) = (2, 4)$ and $m = 4$.

Substituting these values in, we get:

$$y - 4 = 4(x - 2)$$

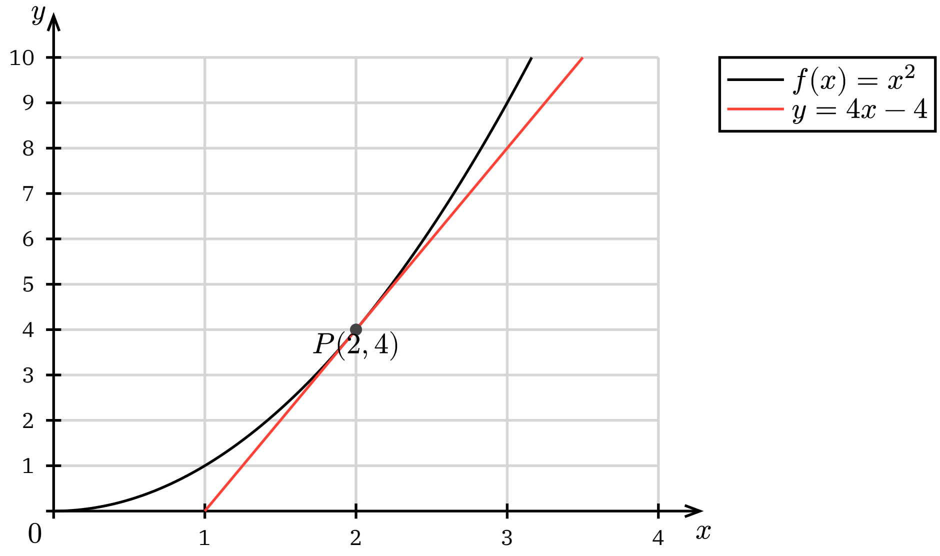
Simplifying this, we get:

$$y - 4 = 4x - 8$$

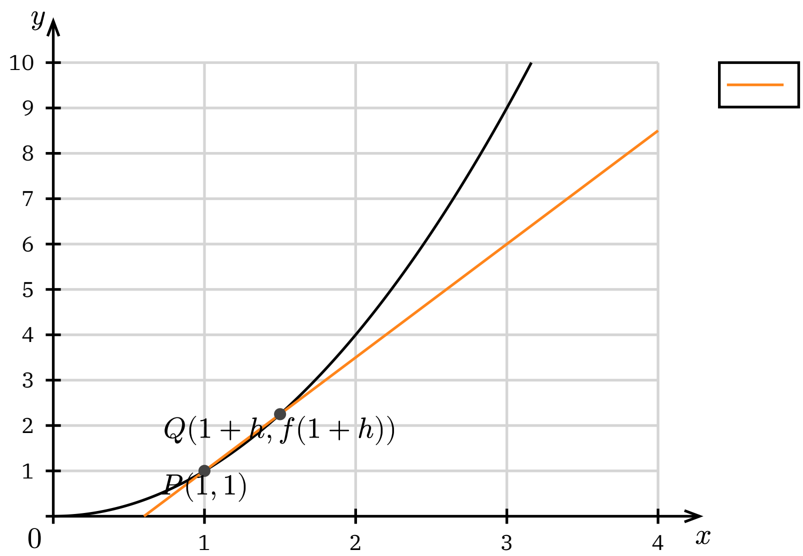
$$y = 4x - 4$$

Therefore, the equation of the tangent line to the curve $y = x^2$ at point $P(2, 4)$ is:

$$y = 4x - 4$$



However, we can have a alternate method for defining derivatives. Imageine we have a points P and Q, seperated by a small distance h on the curve of $f(x)$.



DEFINITION | Alternative Definition of the Derivative

The derivative of a function $f(x)$ at a point a is defined as :

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

provided that this limit exists.

Do you remember the velocity problem we discussed in chapter 0? We can use this definition to find the instantaneous velocity of an object at a specific time.

Definition | Instantaneous Velocity

The instantaneous velocity of an object at time $t = a$ is defined as :

$$v(a) = \lim_{h \to 0} \frac{s(a+h) - s(a)}{h}$$

where $s(t)$ is the position function of the object at time t.

Note | Let's go a bit more...

We can connect a few concepts of basic Physics here. For instance, acceleration can be defined as the derivative of velocity with respect to time. Thus, we have :

$$a(t) = v'(t) = s''(t)$$

To write it out in the long method, we have :

$$a(t) = \lim_{h \to 0} \frac{v(t+h) - v(t)}{h} = \lim_{h \to 0} \frac{\frac{s(t+h)-s(t)}{h} - v(t)}{h}$$

2 Rate of Change

Derivatives can also be used to find the rate of change of one quantity with respect to another. For example, if we have a function that describes the relationship between distance and time, we can use derivatives to find the rate of change of distance with respect to time, which is velocity.

Definition | Rate of Change

The rate of change of a function $f(x)$ with respect to x at a point a is given by the derivative $f'(a)$. We can represent it as :

$$\text{rate of change} = \frac{\Delta x}{\Delta y} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}$$

Example | Rate Of Change Example

A car travels along a straight road, and its position at time t (in seconds) is given by the function $s(t) = 3t^2 + 2t$. Find the car’s velocity at time $t = 4$ seconds.

Solution 1 |

To find the car's velocity at time $t = 4$ seconds, we need to find the derivative of the position function $s(t)$, which gives us the velocity function $v(t)$.

The position function is:

$$s(t) = 3t^2 + 2t$$

We can find the derivative $s'(t)$ using the power rule:

$$s'(t) = \frac{d}{dt}(3t^2) + \frac{d}{dt}(2t)$$

$$s'(t) = 6t + 2$$

Therefore, the velocity function is:

$$v(t) = 6t + 2$$

Now, we can find the velocity at time $t = 4$ seconds by substituting $t = 4$ into the velocity function:

$$v(4) = 6(4) + 2$$

$$v(4) = 24 + 2$$

$$v(4) = 26$$

Thus, the car's velocity at time $t = 4$ seconds is 26 units per second.

Chapter 02.02

The Derivative as a Function

1 Derivatives

As we did in the previous section, we can define derivatives using limits.

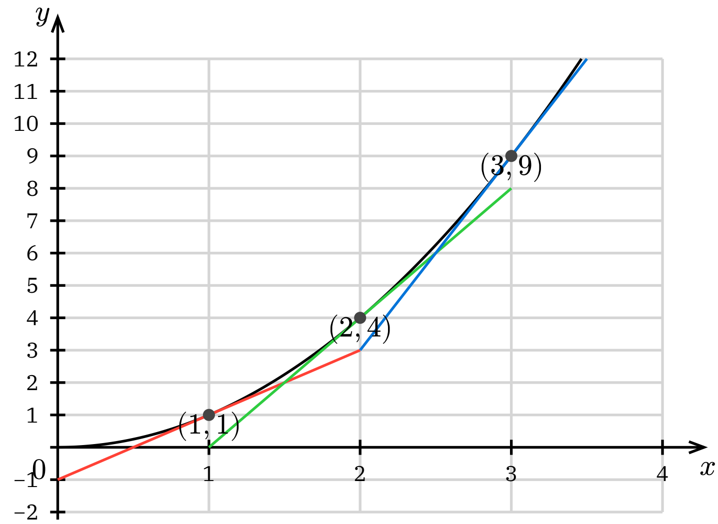
DEFINITION | The Derivative at a Point

The derivative of the function $f(x)$ at the point $x = a$ is defined as:

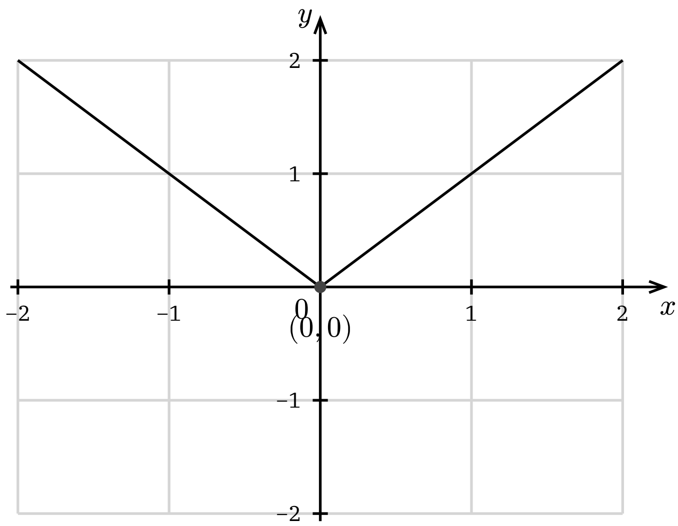
$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

This limit represents the slope of the tangent line to the curve at the point $(a, f(a))$.

We can define a derivative on any point of a function as long as the limit exists.



But, Are there functions that we can't define a derivative on? Yes! For example, consider the function $f(x) = |x|$ at the point $x = 0$.



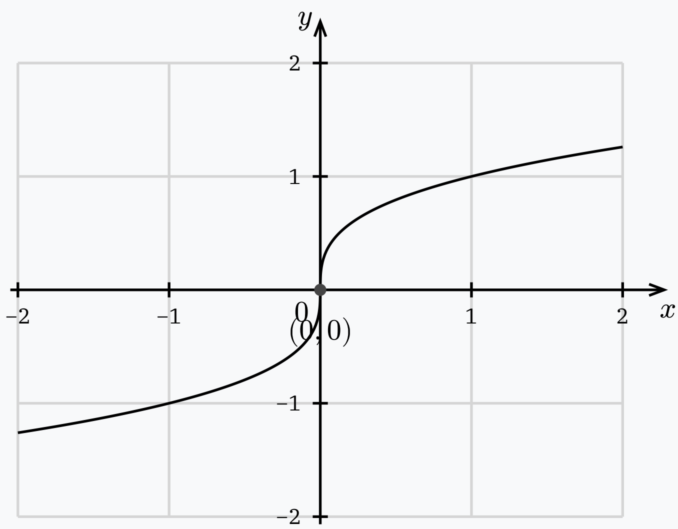
Since there is a sharp corner at $x = 0$, we cannot define a unique tangent line, and thus the derivative does not exist at that point.

Definition | Non-Differentiable Point

A point $x = a$ is called a non-differentiable point of the function $f(x)$ if the derivative $f'(a)$ does not exist. The opposite is called a differentiable point, where the derivative does exist.

We also call a function differentiable on an interval if it is differentiable at every point in an open interval.

Then, what makes a function non-differentiable at a point? There are three common reasons:
1. The function has a sharp corner or cusp at that point (like $f(x) = |x|$ at $x = 0$).
2. The function has a vertical tangent line at that point (like $f(x) = x^{\frac{1}{3}}$ at $x = 0$).
3. The function is not continuous at that point (like $f(x) = \frac{1}{x}$ at $x = 0$).



2 Other Notations

There are several notations for derivatives. The most common ones are:

- Lagrange's notation: $f'(x)$, $f''(x)$, $f'''(x)$ for the first, second, and third derivatives respectively.
- Leibniz's notation: In Leibniz's notation, the derivative of y with respect to x is denoted as

$$\left.\frac{dy}{dx}\right|_{x=a} \quad \text{or} \quad \left.\frac{dy}{dx}\right]_{x=a}$$

We have noted earlier that if f is differentiable at $x = a$, a limit at that point must exist, and also the function value must be defined at that point. This shows resemblance to the definition of continuity.

Theorem | Differentiability Implies Continuity

If a function f is differentiable at a point $x = a$, then it is also continuous at that point.

Proof |

Since f is differentiable at $x = a$, the limit

$$\lim_{h\to 0} \frac{f(a+h) - f(a)}{h}$$

exists.

This implies that

$$\lim_{h\to 0}(f(a+h) - f(a)) = 0$$

because if the limit of the difference quotient exists, the numerator must approach zero as h approaches zero.

Therefore,

$$\lim_{h\to 0} f(a+h) = f(a)$$

which is the definition of continuity at the point $x = a$.

Note | Converse

The converse is not necessarily true; a function can be continuous at a point but not differentiable there. For example, the function $f(x) = |x|$ is continuous at $x = 0$ but not differentiable there.

3 Higher Derivatives.

Till now, we only dealt with the first derivative of a function. However, we can take derivatives of derivatives, leading to higher-order derivatives. The second derivative, denoted as $f''(x)$ or $\frac{d^2y}{(dx)^2}$, represents the rate of change of the first derivative and provides information about the concavity of the function. Similarly, the third derivative, denoted as $f'''(x)$ or $\frac{d^3y}{(dx)^3}$, represents the rate of change of the second derivative.

Chapter 02.03

Differentiation Formulas

1 Differentiation Formulas

- Let's start with the simplest

Definition | Derivative of a Constant Function

If $f(x) = c$, where c is a constant, then $\frac{d}{dx}(c) = 0$.

- Now, lets look at functions with powers of a certain number. Thus, this form : $f(x) = x^n$. What is the derivative of this function?

Definition | Derivate of Power Functions

If $f(x) = x^n$, where n is a real number, then $\frac{d}{dx}(x^n) = n * x^{n-1}$.

Proof |

We can prove this using the definition of the derivative.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Substituting $f(x) = x^n$, we get:

$$f'(x) = \lim_{h \to 0} \frac{(x+h)^n - x^n}{h}$$

Using the Binomial Theorem to expand $(x+h)^n$, we have:

$$(x+h)^n = \sum_{k=0}^{n} \left(\binom{n}{k} \right) * x^{n-k} * h^k$$

Therefore,

$$f'(x) = \lim_{h \to 0} \frac{\sum_{k=0}^{n} \left(\binom{n}{k} \right) * x^{n-k} * h^k - x^n}{h}$$

Simplifying, we get:

$$f'(x) = \lim_{h \to 0} \frac{\sum_{k=1}^{n} \left(\binom{n}{k} \right) * x^{n-k} * h^k}{h}$$

$$= \lim_{h \to 0} \left(\sum_{k=1}^{n} \left(\binom{n}{k} \right) * x^{n-k} * h^{k-1} \right)$$

As h approaches 0, all terms with $k > 1$ vanish, leaving us with:

$$f'(x) = n * x^{n-1}$$

From this, we can derive the power rule,

Theorem | Power Rule

If $f(x) = x^n$, where n is a real number, then $\frac{d}{dx}(x^n) = n * x^{n-1}$.

Proof | Method 1

The Formula

$$x^n - a^n = (x - a) * \left(x^{n-1} + x^{n-2} * a + ... + x * a^{n-2} + a^{n-1}\right)$$

can be used to find the derivative of $f(x) = x^n$.

$$\begin{aligned} f'(a) &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \\ &= \lim_{x \to a} \left(\frac{x^n - a^n}{x - a} \right) \\ &= \lim_{x \to a} \frac{(x - a) * \left(x^{n-1} + x^{n-2} * a + ... + x * a^{n-2} + a^{n-1}\right)}{x - a} \\ &= \lim_{x \to a} \left(x^{n-1} + x^{n-2} * a + ... + x * a^{n-2} + a^{n-1}\right) \\ &= a^{n-1} + a^{n-2} * a + ... + a * a^{n-2} + a^{n-1} = n * a^{n-1} \end{aligned}$$

Proof | Method 2

We can use the binomial theorem to prove this.

$$(x + h)^n = \sum_{k=0}^{n} \left(\binom{n}{k} \right) * x^{n-k} * h^k$$

Substituting $f(x) = x^n$, we get:

$$f'(x) = \lim_{h \to 0} \frac{(x + h)^n - x^n}{h}$$

Using the Binomial Theorem to expand $(x+h)^n$, we have:

$$(x+h)^n = \sum_{k=0}^{n} \left(\binom{n}{k} \right) * x^{n-k} * h^k$$

Therefore,

$$f'(x) = \lim_{h \to 0} \frac{\sum_{k=0}^{n} \left(\binom{n}{k} \right) * x^{n-k} * h^k - x^n}{h}$$

Simplifying, we get:

$$\begin{aligned} f'(x) &= \lim_{h \to 0} \frac{\sum_{k=1}^{n} \left(\binom{n}{k} \right) * x^{n-k} * h^k}{h} \\ &= \lim_{h \to 0} \left(\sum_{k=1}^{n} \left(\binom{n}{k} \right) * x^{n-k} * h^{k-1} \right) \end{aligned}$$

As h approaches 0, all terms with $k > 1$ vanish, leaving us with:

$$f'(x) = n * x^{n-1}$$

Note | What if $n < 0$?

We can extend the power rule to work for negative powers.

Proof |

$$\begin{aligned} \frac{d}{dx}(x^{-n}) &= \frac{d}{dx}\left(\frac{1}{x^n}\right) \\ &= \frac{x^n \frac{d}{dx}(1) - 1 \frac{d}{dx}(x^n)}{x^{2n}} \\ &= \left(-n \frac{x^{n-1}}{x^{2n}}\right) = -n x^{-n-1} \end{aligned}$$

Then how to we take care of coefficients? For that, we have the constant multiple rule :

Theorem | Constant Multiple Rule

If $f(x) = c * g(x)$, where c is a constant and $g(x)$ is a differentiable function, then $f'(x) = c * g'(x)$.

Proof |

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

Substituting $f(x) = c * g(x)$, we get:

$$f'(x) = \lim_{h \to 0} \frac{c * g(x+h) - c * g(x)}{h}$$

Factoring out c, we get:

$$f'(x) = c * \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

Since $g(x)$ is differentiable, we can use the definition of the derivative to write:

$$f'(x) = c * g'(x)$$

As we talked about during limit laws and continuity, we need to deal with the arithemetic of function derivatives.

Theorem | The Sum and Difference Rules

If f and g are both differentiable, then

$$(f(x) \pm g(x))' = f'(x) \pm g'(x)$$

Proof |

Let $l(x) = f(x) \pm g(x)$. Then,

$$\begin{aligned} l'(x) &= \lim_{h \to 0} \frac{l(x+h) - l(x)}{h} \\ &= \lim_{h \to 0} \frac{f(x+h) \pm g(x+h) - f(x) \mp g(x)}{h} \\ &= \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} \pm \lim_{h \to 0} \frac{g(x+h) - g(x)}{h} \\ &= f'(x) \pm g'(x) \end{aligned}$$

Theorem | The Product Rule

If f and g are both differentiable, then

$$(f(x) * g(x))' = f'(x) * g(x) + f(x) * g'(x)$$

Proof |

Let $h(x) = f(x) * g(x)$. Then,

$$h'(x) = \lim_{h \to 0} \frac{h(x+h) - h(x)}{h}$$

Substituting $h(x) = f(x) * g(x)$, we get:

$$h'(x) = \lim_{h \to 0} \frac{f(x+h) * g(x+h) - f(x) * g(x)}{h}$$

We can add and subtract the same value, so lets add and subtract $f(x + h)g(x)$.

$$h'(x) = \lim_{h \to 0} \frac{f(x+h) * g(x+h) - f(x+h) * g(x) + f(x+h) * g(x) - f(x) * g(x)}{h}$$

Factoring out $g(x)$, we get:

$$h'(x) = \lim_{h \to 0} \left(g(x) * \frac{f(x+h) - f(x)}{h} + f(x) * \frac{g(x+h) - g(x)}{h} \right)$$

Since $f(x)$ and $g(x)$ have no relation with h, we can pull it out of the limit using the constant multiple rule.

$$h'(x) = g(x) * \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} + f(x) * \lim_{h \to 0} \frac{g(x+h) - g(x)}{h}$$

Using the definition of the derivative, we get:

$$h'(x) = g(x) * f'(x) + f(x) * g'(x)$$

Theorem | The Quotient Rule

If f and g are both differentiable, then

$$\left(\frac{f(x)}{g(x)} \right)' = \frac{f'(x) * g(x) - f(x) * g'(x)}{g(x)^2}$$

Proof |

Let $h(x) = \frac{f(x)}{g(x)}$. Then,

$$h'(x) = \lim_{h \to 0} \frac{h(x+h) - h(x)}{h}$$

Substituting $h(x) = \frac{f(x)}{g(x)}$, we get:

$$h'(x) = \lim_{h \to 0} \frac{\frac{f(x+h)}{g(x+h)} - \frac{f(x)}{g(x)}}{h}$$

$$= \lim_{h \to 0} \frac{f(x+h)g(x) - f(x)g(x+h)}{hg(x)g(x+h)}$$

Now lets add and subtract $f(x)g(x)$ to the numerator.

$$h'(x) = \lim_{h \to 0} \frac{f(x+h)g(x) - f(x)g(x+h) + f(x)g(x) - f(x)g(x)}{hg(x)g(x+h)}$$

Group into useful groups,

$$h'(x) = \lim_{h \to 0} \frac{f(x+h)g(x) - f(x)g(x+h) + f(x)g(x) - f(x)g(x)}{hg(x)g(x+h)}$$

$$= \lim_{h \to 0} \frac{g(x)\frac{f(x+h)-g(x)}{h} + \frac{f(x)g(x)-g(x+h)}{h}}{g(x)g(x+h)})$$

$$= \frac{g(x)f'(x) - f(x)g'(x)}{g(x)^2}$$

in the end, the full list of differntial formulas is:

$$\frac{d}{dx}(c) = 0$$

,

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

,

$$(c * f(x))' = c * f'(x)$$

,

$$(f(x) \pm g(x))' = f'(x) \pm g'(x)$$

,

$$(f(x) * g(x))' = f'(x) * g(x) + f(x) * g'(x)$$

,

$$\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x) * g(x) - f(x) * g'(x)}{g(x)^2}$$

Chapter 02.04

Derivative of Trigonometric Functions

1 Trigonometric Functions...

We dealed with normal polynomials in the last section, so now we need to deal with trigonometric functions.

Definition | Derivative of Sine Function

If $f(x) = \sin(x)$, then $f'(x) = \cos(x)$.

Proof |

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x+h) - \sin(x)}{h} \\
&= \lim_{h\to 0} \frac{\sin(x)\cos(h) + \cos(x)\sin(h) - \sin(x)}{h} \\
&= \lim_{h\to 0} \left[\frac{\sin(x)\cos(h) - \sin(x)}{h} + \frac{\cos(x)\sin(h)}{h}\right] \\
&= \lim_{h\to 0} \left[\sin(x)\left(\frac{\cos(h) - 1}{h}\right) + \cos(x)\left(\frac{\sin(h)}{h}\right)\right] \\
&= \lim_{h\to 0} \sin(x) \cdot \lim_{h\to 0} \frac{\cos(h) - 1}{h} + \lim_{h\to 0} \cos(x) \cdot \lim_{h\to 0} \frac{\sin(h)}{h}
\end{aligned}$$

Since we treat x as a constant during limits with h, we can take $\sin(x)$ and $\cos(x)$ out of the limits.

$$= \sin(x) \cdot \lim_{h\to 0} \frac{\cos(h) - 1}{h} + \cos(x) \cdot \lim_{h\to 0} \frac{\sin(h)}{h}$$

We'll later prove that $\lim_{h\to 0} \frac{\sin(h)}{h} = 1$, and $\lim_{h\to 0} \frac{\cos(h)-1}{h} = 1$.

Thus, we get $\cos(x)$.

Using the exact same method, we can prove the derivative of cosine.

Definition | Derivative of Cosine Function

If $f(x) = \cos(x)$, then $f'(x) = -\sin(x)$.

Proof |

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\cos(x+h) - \cos(x)}{h} \\
&= \lim_{h\to 0} \frac{\cos(x)\cos(h) - \sin(x)\sin(h) - \cos(x)}{h} \\
&= \lim_{h\to 0} \left[\frac{\cos(x)\cos(h) - \cos(x)}{h} - \frac{\sin(x)\sin(h)}{h}\right] \\
&= \lim_{h\to 0} \left[\cos(x)\left(\frac{\cos(h) - 1}{h}\right) - \sin(x)\left(\frac{\sin(h)}{h}\right)\right] \\
&= \lim_{h\to 0} \cos(x) \cdot \lim_{h\to 0} \frac{\cos(h) - 1}{h} - \lim_{h\to 0} \sin(x) \cdot \lim_{h\to 0} \frac{\sin(h)}{h}
\end{aligned}$$

Since we treat x as a constant during limits with h, we can take $\sin(x)$ and $\cos(x)$ out of the limits.

$$= \cos(x) \cdot \lim_{h\to 0} \frac{\cos(h) - 1}{h} - \sin(x) \cdot \lim_{h\to 0} \frac{\sin(h)}{h}$$

We'll later prove that $\lim_{h\to 0} \frac{\sin(h)}{h} = 1$, and $\lim_{h\to 0} \frac{\cos(h)-1}{h} = 1$.

Thus, we get $-\sin(x)$.

Since we know the derivatives of sine and cosine, we can easily think of the derivative of the tangent function, since $\tan(x) = \frac{\sin(x)}{\cos(x)}$.

Definition | Derivative of Tangent Function

If $f(x) = \tan(x)$, then $f'(x) = \sec^2(x)$.

Proof |

Using the Quotient Rule :

$$
\begin{aligned}
f'(x) &= \frac{\cos(x) \cdot \frac{d}{dx}[\sin(x)] - \sin(x) \cdot \frac{d}{dx}[\cos(x)]}{(\cos(x))^2} \\
&= \frac{\cos(x) \cdot \cos(x) - \sin(x) \cdot (-\sin(x))}{(\cos(x))^2} \\
&= \frac{\cos^2(x) + \sin^2(x)}{(\cos(x))^2} \\
&= \frac{1}{(\cos(x))^2} \\
&= \sec^2(x)
\end{aligned}
$$

The derivatives of $\csc(x), \sec(x), \cot(x)$ can be derived using a similar method. In the end, we have the following results :

$$\frac{d}{dx}(\sin(x)) = \cos(x)$$

$$\frac{d}{dx}(\cos(x)) = -\sin(x)$$

$$\frac{d}{dx}(\tan(x)) = \sec^2(x)$$

$$\frac{d}{dx}(\csc(x)) = -\csc(x)\cot(x)$$

$$\frac{d}{dx}(\sec(x)) = \sec(x)\tan(x)$$

$$\frac{d}{dx}(\cot(x)) = -\csc^2(x)$$

Example | Trigonometric Derivative Example

An object fastened to the end of a vertical string stretched 4cm from rest is releasted at time $t = 0$. Its position at time t is

$$s = f(t) = 4\cos(t)$$

Find the velocity and acceleration of the object at time t.

Solution 1 |

The velocity is the derivative of the position function:

$$v(t) = f'(t) = \frac{d}{dt}[4\cos(t)] = -4\sin(t)$$

The acceleration is the derivative of the velocity function:

$$a(t) = v'(t) = \frac{d}{dt}[-4\sin(t)] = -4\cos(t)$$

2 Special Trigonometric Limits

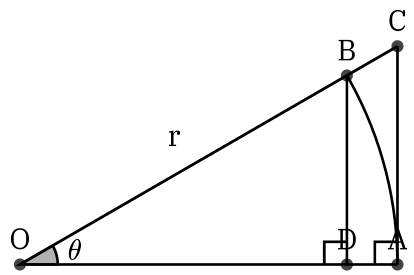
- There are two special trigonometric limits that are very useful in calculus.

Theorem | Special Trigonometric Limits

1. $$\lim_{x\to 0}\left(\frac{\sin(x)}{x}\right) = 1$$
2. $$\lim_{x\to 0}\left(\frac{1-\cos(x)}{x}\right) = 0$$

Proof | 1

We will utilize the Squeeze Theorem to prove this.



Here, we know that the inequality

$$\overline{BD} \leq \overset{\frown}{AB} \leq \overline{AC}$$

If we take the limit $\theta \to 0$, we get

$$\lim_{\theta\to 0}\overline{BD} \leq \lim_{\theta\to 0}\overset{\frown}{AB} \leq \lim_{\theta\to 0}\overline{AC}$$

$$\lim_{\theta\to 0}\overline{OB}\sin(\theta) \leq \lim_{\theta\to 0} r\theta \leq \lim_{\theta\to 0}\overline{OA}\tan(\theta)$$

Divide both sides by $\sin(\theta)$.

$$= \lim_{\theta \to 0} \cancel{r} \leq \lim_{\theta \to 0} \frac{\cancel{r}\theta}{\sin(\theta)} \leq \lim_{\theta \to 0} \frac{\cancel{r}}{\cos(\theta)}$$

Take the inverse of each fraction.

$$\lim_{\theta \to 0} \cos(\theta) \leq \lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} \leq \lim_{\theta \to 0} 1$$

Since $\lim_{\theta \to 0} \cos(\theta) = 1$, we get

$$1 \leq \lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} \leq 1$$

By the Squeeze Theorem, we get

$$\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} = 1$$

Proof | 2

Multiply the numerator and denominator by $\cos(\theta) + 1$ to turn the numerator into a nice form.

$$\begin{aligned}
\lim_{\theta \to 0} \frac{\cos(\theta) - 1}{\theta} &= \lim_{\theta \to 0} \frac{\cancel{\cos(\theta)^2 - 1} - \sin(\theta)^2}{\theta(\cos(\theta) + 1)} \\
&= -\lim_{\theta \to 0} \frac{\sin(\theta)}{\theta} \cdot \lim_{\theta \to 0} \frac{\sin(\theta)}{\cos(\theta) + 1} \\
&= -1 * 0 = 0
\end{aligned}$$

Chapter 02.05

The Chain Rule

1 The Chain Rule

Say we want to differentiate $f(x) = \sqrt{\sin(x)}$. How do we do this? The chain rule is a powerful tool to solve the derivative of composite functions.

Definition | Chain Rule

If $f(x) = a(b(x))$, then $f'(x) = a'(b(x))b'(x)$.

Proof |

$$\begin{aligned} f'(x) &= \lim_{h\to 0} \frac{f(x+h)-f(x)}{h} = \lim_{h\to 0} \frac{a(b(x+h))-a(b(x))}{h} \\ &= \lim_{h\to 0} \frac{a(b(x+h))-a(b(x))}{b(x+h)-b(x)} \cdot \frac{b(x+h)-b(x)}{h} \\ &= a'(b(x)) \cdot b'(x) \end{aligned}$$

However, if $b(x)$ is a constant function, or for sake has a place where the denominator is 0, it directly leads to a divide by zero error. Thus, people found a better method for that :

Proof | Foolproof

We must know a small theorem before we continue :

Theorem |

If a function is differentiable, the actual change is equal to the linear approximation at the point plus a small error.

$$\Delta y = \underbrace{f'(a)\Delta x}_{\text{Linear Projection}} + \underbrace{\varepsilon \Delta x}_{\text{Error Term}}$$

Let's utilize this. Let $b = g(a)$. If we have $u = g(x)$ and $y = f(u)$, we can derive the two equations :

$$\Delta u = g'(a)\Delta x + \varepsilon_1 \Delta x = [g'(a) + \varepsilon_1]\Delta x$$

$$\Delta y = f'(b)\Delta u + \varepsilon_2 \Delta u = [f'(b) + \varepsilon_2]\Delta u$$

Plug in Δu to the lower equation, we get

$$\frac{\Delta y}{\Delta x} = [f'(b) + \varepsilon_2][g'(a) + \varepsilon_1]\cancel{\Delta x}$$

Now if we take the limit of both sides, we get :

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{\Delta x \to 0} [f'(b) + \varepsilon_2][g'(a) + \varepsilon_1]$$

Since the error will near zero once $\lim_{\Delta x \to 0}$, we get $f'(b)g'(a) = f'(g(a))g'(a)$.

Note | Leibniz Notation

If you prefer the leibniz notation, you can write

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx}$$

.

Example | Chain Rule Basics

Find $F'(x)$ if $F = \sqrt{x^2 + 1}$.

Solution 1 |

Using the chain rule, we get $F'(x) = \left(\frac{1}{2}\right)\left(x^2 + 1\right)^{-\frac{1}{2}} \cdot 2x = \frac{x}{\sqrt{x^2+1}}$

Example | Chain Rule with Trigonometric Functions

Find $F'(x)$ if $F = \sin(\sqrt{x})$.

Solution 1 |

Using the chain rule, we get $F'(x) = \cos(\sqrt{x}) \cdot \left(\frac{1}{2}\right)\left(x^2 + 1\right)^{-\frac{1}{2}} \cdot 2x = \frac{\cos(\sqrt{x})}{2 \cdot \sqrt{x^2+1}}$.

One useful form to note is the Power Rule combined with the Chain Rule :

DEFINITION | Power Rule with Chain Rule

If $f(x) = (g(x))^n$, then $f'(x) = n(g(x))^{n-1}g'(x)$.

EXAMPLE | Power Rule with Chain Rule

Differentiate $y = (x^3 - 1)^{100}$

Solution 1 |

Using the power rule with chain rule, we get

$$y' = 100(x^3 - 1)^{99} \cdot 3x^2 = 300x^2(x^3 - 1)^{99}$$

Chapter 02.06

Implicit Differentiation

1 Implicit Function Differentiation

Implicit functions are functions that dont take the form of $y = f(x)$. Some examples include $x^2 + y^2 = 1$ and $\frac{x^2}{4} - \frac{y^2}{9} = 1$.

How do we find the drerivative of these functions? We can use a very cool method called **Implicit Differentiation**.

Definition | Implicit Differentiation

Implicit Differentatiation means to take $\frac{d}{dx}$ of both sides in order to relieve the final result in the form of $\frac{dy}{dx}$.

Example | Implicit Differentiation

Find $\frac{dy}{dx}$ if $x^2 + y^2 = 25$. Then find the equation of the tangent to the circle $x^2 + y^2 = 25$ at point $(3, 4)$.

Solution 1 |

Take $\frac{d}{dx}$ on both sides.

$$\frac{d}{dx}(x^2 + y^2) = \frac{d}{dx}25$$

$$\frac{d}{dx}x^2 + \frac{d}{dx}y^2 = 0$$

Since y is a function of x, we can use the chain rule.

$$2x + 2\frac{y(dy)}{dx} = 0$$

$$\frac{dy}{dx} = -\frac{x}{y}$$

Therefore, at the point $(3, 4)$, we get $\frac{dy}{dx} = -\frac{3}{4}$.

2 Second derivatives

Example | Second Derivative of Implicit Functions

Find y'' if $x^4 + y^4 = 16$.

Solution 1 |

Differentiate once to get $4x^3 + 4y^3y' = 0$. Solving for y' gives $y' = -\frac{x^3}{y^3}$.
Differentiate again to get

$$y'' = \frac{d}{dx}\left(-\frac{x^3}{y^3}\right) = -\frac{y^3 \frac{d}{dx}x^3 - x^3 \frac{d}{dx}y^3}{y^6}$$

$$= -\frac{y^3 \cdot 3x^2 - x^3 \cdot 3y^2y'}{y^6}$$

this can be simplified into

$$y'' = -48\frac{x^2}{y^7}$$

Chapter 02.07

Rates of Change in Science and Society

1 Rate of Change in Science

1.1 Physics

- We already briefly discussed this before.
- $s' = v, v' = a$
- Average rate of change : $\frac{\Delta y}{\Delta x}$
- Instantaneous rate of change : $\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x}$
- Average linear density : $\frac{\Delta m}{\Delta x}$
- Linear Density : $\rho = \lim_{\Delta x \to 0} \frac{\Delta m}{\Delta x}$
- Average current = $\frac{\Delta Q}{\Delta t}$
- Instantaneous current = $I = \lim_{\Delta t \to 0} \frac{\Delta Q}{(\Delta t}$

1.2 Chemistry

- Say we have the following reaction :

$$A + B \longrightarrow C$$

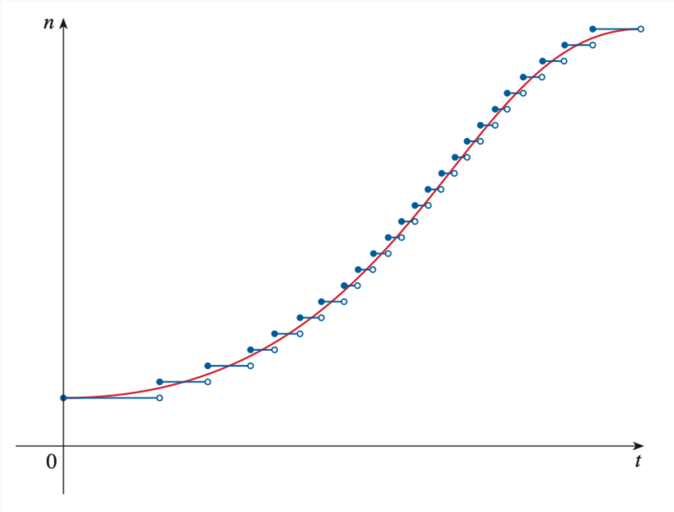
- Average rate of reaction : $\frac{\Delta [C]}{\Delta t}$
- Instantaneous rate of reaction : $r = \lim_{\Delta t \to 0} \frac{\Delta [C]}{\Delta t}$
- Or, if we write in terms of reactants, $r = -\frac{d[B]}{\Delta t} = -\frac{d[A]}{\Delta t}$
- Compressibility : $-\beta = -\frac{1}{V} \frac{\Delta V}{\Delta P}$

1.3 Biology

- Average rate of growth : $\frac{\Delta N}{\Delta t}$
- Instantaneous rate of growth : $\lim_{\Delta t \to 0} \frac{\Delta N}{\Delta t} = \frac{dn}{dt}$

Note | population growth

Since populations grow in integers, it cannot be smooth. However, using the function above, we can approximate like such :



- Inside blood vessels, the flow of blood is described by the law of laminar flow. The law states that :

$$v = \frac{P}{4\eta l}(R^2 - r^2)$$

Thus, finding $\frac{\Delta v}{\Delta r}$ gives us the velocity gradient of blood flow. Using limits, we get :

$$\frac{dv}{dr} = \frac{P}{4\eta l} \cdot 2r = \frac{\Pr}{2\eta l}$$

2 Economics

- Marginal cost is the change of cost relative to the number of items produced.
- thus,

$$C_m = \lim_{\Delta x \to 0} \frac{\Delta C}{\Delta x} = \frac{dC}{dx}$$

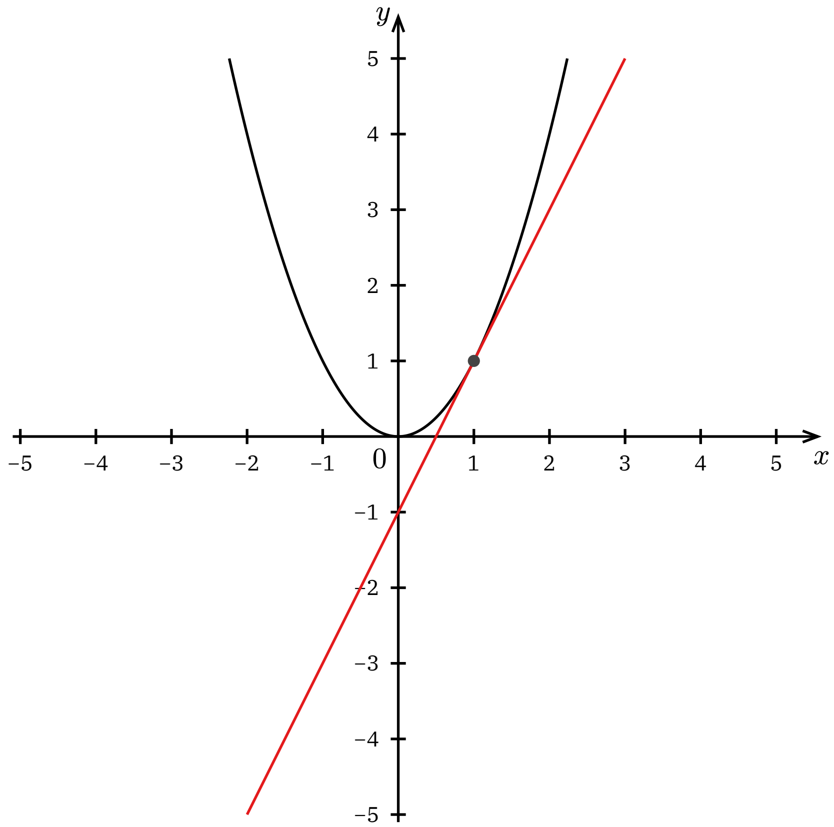
Chapter 02.09

Linear Approximation & Differentials

1 Linear approximation

Remember our proof of the **chain rule** a few sections back? That was our preview of the topic of **linear approximation**. We can define a tangent line at a point $(a, f(a))$ as

$$L(x) = f(a) + f'(a)(x - a)$$



We can use this linearized function to predict a value near a. For example, if we want to find $f(1.1)$, we can use the linearized function to predict $f(1.1) = 2.1$

Definition | Linear Approximation

If $f(x)$ is differentiable at a, then the linear approximation of $f(x)$ at a is $L(x) = f(a) + f'(a)(x - a)$. We can use this to predict $f(x)$ near a.

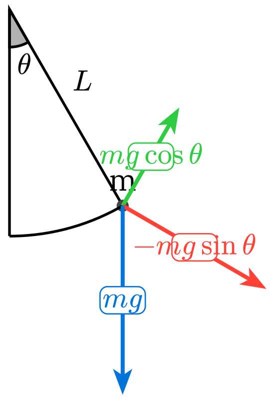
$$f(x) \approx L(x) = f(a) + f'(a)(x - a)$$

So where is this used? It is a famous thing that $\sin(\theta) \approx \theta$ near 0. This is used numerously for physics problems, such as finding the period of a pendulum.

Example | Period of a Pendulum

Find the period of a pendulum with length L.

Solution 1 |



The equation of motion for a pendulum is derived from Newton's second law:

$$mL\frac{d^2\theta}{dt^2} = -mg\sin(\theta)$$

This simplifies to:

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\sin(\theta)$$

For small angles, we use the linear approximation $\sin(\theta) \approx \theta$ (since $\sin'(0) = 1$):

$$\frac{d^2\theta}{dt^2} = -\frac{g}{L}\theta$$

This matches the standard form of simple harmonic motion:

$$\frac{d^2\theta}{dt^2} = -\omega^2\theta$$

Comparing the two equations, we identify:

$$\omega^2 = \frac{g}{L} \quad \Rightarrow \quad \omega = \sqrt{\frac{g}{L}}$$

The period is therefore:

$$T = \frac{2\pi}{\omega} = 2\pi\sqrt{\frac{L}{g}}$$

2 Differentials

- We dealed with $\frac{dy}{dx}$ a few times before. However, we can actually use this to obtain dy and dx.

Definition | Differential

If $f(x)$ is differentiable at x, then $dy = f'(x)dx$.