Mathematics II

Explanations, problems & solutions (Solutions)

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Preface

Welcome to **Mathematics II**, a comprehensive exploration of advanced mathematical concepts that will expand your understanding of how mathematics describes and analyzes the world around us.

About This Textbook

This textbook has been thoughtfully designed to guide you through four fundamental areas of advanced mathematics:

Polar Coordinates, Parametric Equations, and Vectors — Moving beyond the rectangular coordinate system to explore dynamic representations of position, motion, and force in multi-dimensional space.

Matrices — Discovering powerful tools for organizing data, solving linear systems, and transforming geometric space through elegant algebraic operations.

Conic Sections — Investigating the beautiful curves that emerge from intersecting planes with cones, revealing the geometric patterns that govern planetary orbits and architectural designs.

Probability and Statistics — Transitioning from deterministic mathematics to the realm of uncertainty, developing tools to quantify chance and extract meaning from complex datasets.

How to Use This Book

This textbook follows a structured learning approach designed to build deep understanding. You will encounter various block types throughout:

Definition blocks introduce key concepts and terminology with precise mathematical language.

Example blocks present problems that illustrate the application of concepts, followed by detailed step-by-step solutions.

Solution blocks provide complete, worked-out answers with clear reasoning at each step.

Theorem blocks state important mathematical results that you can rely upon in your work.

Proof blocks demonstrate the logical reasoning behind theorems, helping you understand why results are true.

Note blocks offer helpful insights, common pitfalls to avoid, and connections between ideas.

A Note on Learning

Mathematics is not a spectator sport. While reading through definitions and examples is important, true understanding comes from *active engagement*. We encourage you to:

- Work through each example *before* looking at the solution
- Try to understand *why* each step follows from the previous one
- Connect new concepts to what you already know
- Practice additional problems to reinforce your understanding
- Ask questions to your instructor when something is unclear

The solutions provided are detailed and show the complete thought process, but they should serve as a guide and verification tool, not a substitute for your own problem-solving efforts.

Looking Ahead

The mathematics you encounter in this course is not merely *abstract theory*—it forms the foundation for advanced work in *physics*, *engineering*, *computer science*, *economics*, and countless other fields. From the elliptical orbits of satellites to the encryption protecting digital communications, from statistical analysis of scientific data to the rendering of 3D computer graphics, these mathematical tools are actively shaping our modern world.

We hope this textbook serves as both a rigorous introduction to these powerful ideas and an invitation to see mathematics as a lens through which to understand the elegant patterns underlying our universe.

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Chapter 08

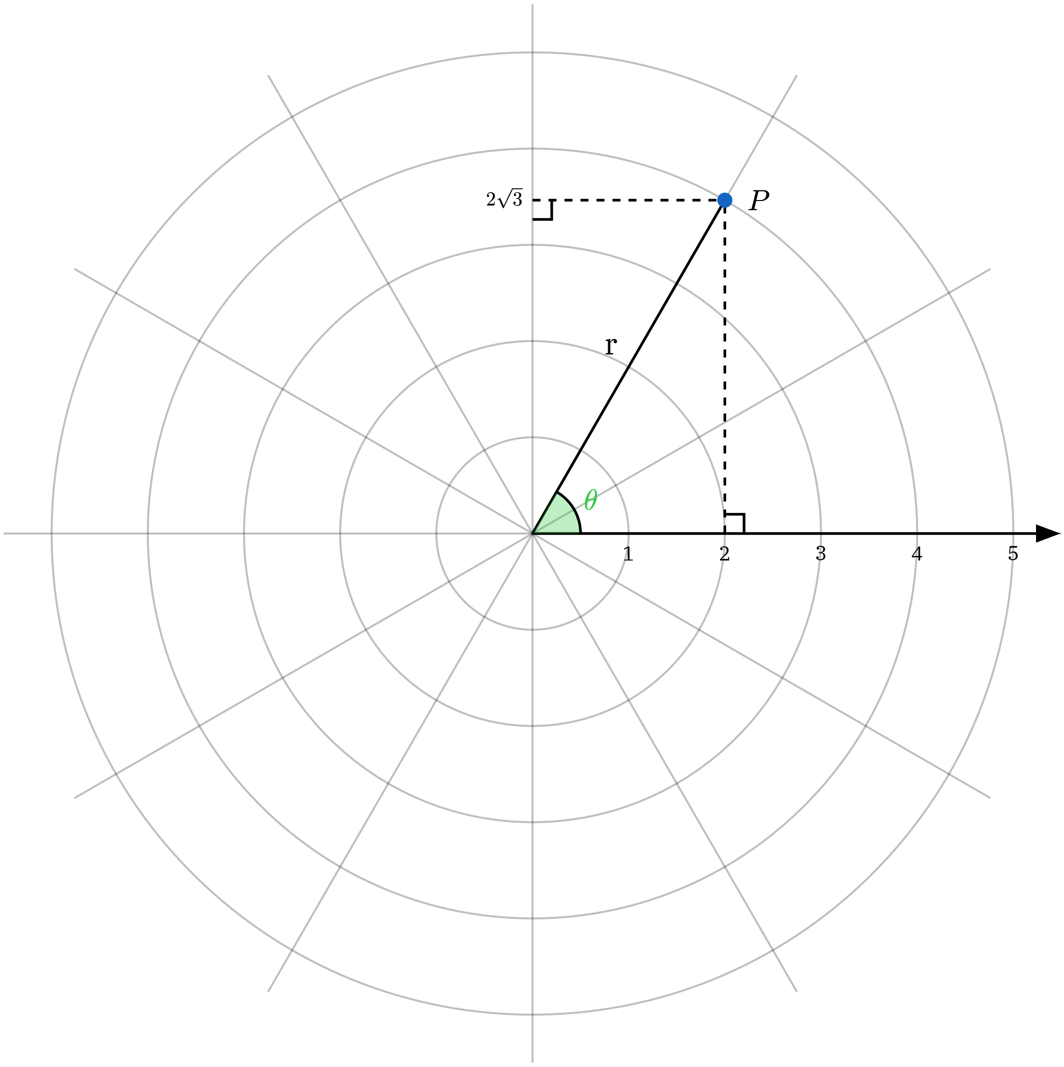
Polar Coordinates, Parametric Equations & Vectors

Until now, our study of functions has been confined to the static grid of the Rectangular coordinate system. However, the dynamic physical universe—filled with spiraling galaxies and directional forces—rarely conforms to such rigid constraints. In this chapter, we expand our mathematical lexicon to include Polar Coordinates, Parametric Equations, and Vectors, freeing us from the limitations of Cartesian graphs. By redefining position through distance and angle and introducing time as a driving parameter, we gain the sophisticated framework necessary to analyze motion, force, and form in a complex, multi-dimensional world.

Chapter 08.01

Polar Coordinates

Polar Coordinates



The cartesian coordinate of dot P is $\left(2, 2\sqrt{3}\right)$.
In polar coordinate system, the coordinate of dot P is $\left(4, \frac{\pi}{3}\right)$.

Definition | Polar Coordinates

In this section, we will study the polar coordinate system (**극좌표계**). The polar coordinate system uses distances and directions to specify the location of a point in the plane.

To set up this system, we choose a fixed point O in the plane called the **pole** (or origin, **극점**, **원점**) and draw from O a half-line called the **polar axis** (**극축**). Then each point P can be assigned polar coordinates $P(r,\theta)$, where
- r is the distance from O to P,
- θ is the angle between the polar axis and the segment OP.

We use the convention that θ is positive if measured in a counterclockwise direction from the polar axis or negative if measured in a clockwise direction. If r is negative, then $P(r,\theta)$ is defined to be the point that lies $|r|$ units from the pole in the direction opposite to that given by θ.

Example | Plotting Points

Plot the points whose polar coordinates are given.
(a) $\left(1, \frac{3}{4}\pi\right)$
(b) $\left(3, -\frac{\pi}{6}\right)$
(c) $(3, 3\pi)$
(d) $\left(-4, \frac{\pi}{4}\right)$

Solution 1 |

(a) Point at distance 1, angle 135°.
(b) Point at distance 3, angle -30°.
(c) Point at distance 3, angle $540^\circ \equiv 180^\circ$.
(d) Point at distance 4, angle $45^\circ + 180^\circ = 225^\circ$ (opposite direction).

Example | Exercise

Explain why any polar coordinate $P(r,\theta)$ represents the same point as

$$P(r, \theta + 2n\pi) \quad \text{and} \quad P(-r, \theta + (2n+1)\pi), \quad n \in \mathbb{Z}.$$

Solution 1 |

First representation: $P(r, \theta + 2n\pi)$

Since angles are measured from the polar axis, adding $2n\pi$ (where $n \in \mathbb{Z}$) represents a complete rotation(s) around the origin. After completing full rotation(s), we return to the same direction. The distance r from the pole remains unchanged, so the point is identical to $P(r,\theta)$.

Second representation: $P(-r, \theta + (2n+1)\pi)$

The term $(2n+1)\pi$ represents rotation by an odd multiple of π, which means rotating by $\pi, 3\pi, 5\pi, \ldots$ radians. Each of these rotations places us in the **opposite direction** from θ.

When r is negative, we move $|r|$ units in the direction **opposite** to the angle. So $P(-r, \theta + (2n+1)\pi)$ means:

- Start at angle $\theta + (2n+1)\pi$ (which is opposite to θ)
- Move $|r|$ units in the opposite direction of that angle
- This brings us to the direction of θ at distance $|r| = r$

Therefore, both representations describe the same point as $P(r, \theta)$.

Example | Finding other representations

Find two other polar coordinate representations of $P\left(2, \frac{\pi}{3}\right)$ with $r > 0$ and two with $r < 0$.

Solution 1 |

$r > 0$:

$$\ldots, \left(2, -11\frac{\pi}{3}\right), \left(2, -5\frac{\pi}{3}\right), \left(2, 7\frac{\pi}{3}\right), \left(2, 13\frac{\pi}{3}\right), \ldots$$

$r < 0$:

$$\ldots, \left(-2, -8\frac{\pi}{3}\right), \left(-2, -2\frac{\pi}{3}\right), \left(-2, 4\frac{\pi}{3}\right), \left(-2, 10\frac{\pi}{3}\right), \ldots$$

Relationship Between Polar and Rectangular Coordinates

1. To change from polar to rectangular coordinates, use the formulas

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

2. To change from rectangular to polar coordinates, use the formulas

$$r^2 = x^2 + y^2, \quad \tan\theta = \frac{y}{x}(x \neq 0)$$

or

$$r^2 = x^2 + y^2, \quad \cos\theta = \frac{x}{r} \ \text{ and } \ \sin\theta = \frac{y}{r}.$$

Example | Coordinate Conversion

(a) Find rectangular coordinates for the point that has polar coordinates $\left(4, 2\frac{\pi}{3}\right)$.
(b) Find polar coordinates for the point that has rectangular coordinates $(2, -2)$.

Solution 1 |

(a)

$$x = 4\cos\left(2\frac{\pi}{3}\right) = 4 \cdot \left(-\frac{1}{2}\right) = -2$$

$$y = 4\sin\left(2\frac{\pi}{3}\right) = 4 \cdot \left(\frac{\sqrt{3}}{2}\right) = 2\sqrt{3}$$

$$\therefore \left(-2, 2\sqrt{3}\right)$$

(b)

$$r = \pm\sqrt{2^2 + (-2)^2} = \pm 2\sqrt{2}$$

$$\tan\theta = -\frac{2}{2} = -1$$

$$\theta = 3\frac{\pi}{4} + n\pi, \left(-\frac{\pi}{4}\right) + n\pi, \quad n \in \mathbb{Z}$$

$$\therefore \left(2\sqrt{2}, -\frac{\pi}{4} + 2n\pi\right), \left(-2\sqrt{2}, 3\frac{\pi}{4} + 2n\pi\right), \quad n \in \mathbb{Z}$$

Note | Remarks

1. In most cases, we assume $r > 0$ and $0 \leq \theta < 2\pi$ in polar coordinates. Under these conditions, each point corresponds uniquely to a single polar coordinate, establishing a one-to-one correspondence with Cartesian coordinates. This convention is often preferred.
2. The origin $(0, 0)$ corresponds to all polar coordinates of the form $(0, \theta)$ for any angle θ. However, it is generally preferred not to consider the origin explicitly in polar coordinates.
3. Allowing $r \leq 0$ is mainly useful when sketching graphs in polar coordinates, as it simplifies the representation of certain curves.

Polar Equations

Definition | Polar Equation

A **polar equation** (tr. 극방정식) is an equation in the polar coordinates r and θ; similarly, a rectangular equation is an equation in the rectangular coordinates x and y.

Example | Converting to Polar Equation

Express the equation $x^2 = 4y$ in polar coordinates.

Solution 1 |

(Case 1) $r \neq 0$ Since $r^2 = x^2 + y^2 = 4y + y^2$ and $y \geq 0$, we have $y \neq 0$ and therefore $x \neq 0$, which implies $\cos\theta \neq 0$. Then we get

$$x^2 = 4y$$
$$(r\cos\theta)^2 = 4r\sin\theta$$
$$r\cos^2\theta = 4\sin\theta$$
$$r = \frac{4\sin\theta}{\cos^2\theta} = 4\sec\theta\tan\theta$$

(Case 2) $r = 0$ This corresponds to the origin $(0, 0)$, which also satisfies the equation $x^2 = 4y$. However, substituting $\theta = 0$ into the expression derived in Case 1 yields $r = 0$, which satisfies the condition in Case 2 as well. Therefore, we conclude that

$$r = 4\sec\theta\tan\theta.$$

Example | Converting to Rectangular Equation

Express each polar equation in rectangular coordinates. (a) $r = 5\sec\theta$ (b) $r = 2\sin\theta$ (c) $r = 2 + 2\cos\theta$

Solution 1 |

(a) A vertical line.

$$r = 5\sec\theta \Rightarrow r\cos\theta = 5 \Rightarrow x = 5$$

(b) A circle.

$$r = 2\sin\theta \Rightarrow r^2 = 2r\sin\theta \Rightarrow x^2 + y^2 = 2y \Rightarrow x^2 + (y-1)^2 = 1$$

(c) A "Cardioid".

$$r = 2 + 2\cos\theta \Rightarrow r^2 = 2r + 2r\cos\theta \Rightarrow x^2 + y^2 = 2r + 2x$$

$$\left(x^2 + y^2 - 2x\right)^2 = 4r^2 = 4(x^2 + y^2)$$

Example | Distance Formula

Prove that the distance between the polar points (r_1, θ_1) and (r_2, θ_2) is

$$d = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_2 - \theta_1)}.$$

Solution 1 |

The corresponding points in rectangular coordinates for (r_1, θ_1) and (r_2, θ_2) are $(r_1\cos\theta_1, r_1\sin\theta_1)$ and $(r_2\cos\theta_2, r_2\sin\theta_2)$, respectively. Then the distance between them is

$$\begin{aligned} d &= \sqrt{\left(r_2\cos\theta_2 - r_1\cos\theta_1\right)^2 + \left(r_2\sin\theta_2 - r_1\sin\theta_1\right)^2} \\ &= \sqrt{r_1^2 + r_2^2 - 2r_1 r_2(\cos\theta_2\cos\theta_1 + \sin\theta_2\sin\theta_1)} \\ &= \sqrt{r_1^2 + r_2^2 - 2r_1 r_2\cos(\theta_2 - \theta_1)}. \end{aligned}$$

Example | Exercise

(a) Use the Law of Cosines to prove the formula above.
(b) Find the distance between the points whose polar coordinates are $(3, 3\frac{\pi}{4})$ and $(-1, 7\frac{\pi}{6})$.

Solution 1 |

(a) Consider the triangle formed by the origin O and the two points $P_1(r_1, \theta_1)$ and $P_2(r_2, \theta_2)$.

The three sides of this triangle have lengths:

- From O to P_1: length r_1
- From O to P_2: length r_2
- From P_1 to P_2: length d (what we want to find)

The angle at the origin between the two radii is $\theta_2 - \theta_1$.

By the Law of Cosines:

$$d^2 = r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_2 - \theta_1)$$

Taking the square root:

$$d = \sqrt{r_1^2 + r_2^2 - 2r_1 r_2 \cos(\theta_2 - \theta_1)}$$

(b) Given: $P_1(3, 3\frac{\pi}{4})$ and $P_2\left(-1, 7\frac{\pi}{6}\right)$

First, note that $P_2\left(-1, 7\frac{\pi}{6}\right)$ with negative r is equivalent to $P_2\left(1, 7\frac{\pi}{6} + \pi\right) = P_2\left(1, 13\frac{\pi}{6}\right)$.

For simplicity, we can also write this as $P_2\left(1, \frac{\pi}{6}\right)$ (since $13\frac{\pi}{6} - 2\pi = \frac{\pi}{6}$).

Using the distance formula:

$$d = \sqrt{3^2 + 1^2 - 2(3)(1)\cos\left(\frac{\pi}{6} - 3\frac{\pi}{4}\right)} \backslash = \sqrt{9 + 1 - 6\cos\left(-7\frac{\pi}{12}\right)} \backslash = \sqrt{10 - 6\cos\left(7\frac{\pi}{12}\right)}$$

Since $\cos(7\frac{\pi}{12}) = \cos(105^\circ) = -\sin(15^\circ) = -\frac{\sqrt{6}-\sqrt{2}}{4}$:

$$d = \sqrt{10 - 6 \cdot \left(-\frac{\sqrt{6} - \sqrt{2}}{4}\right)} \backslash = \sqrt{10 + \frac{3\left(\sqrt{6} - \sqrt{2}\right)}{2}} \backslash = \sqrt{10 + \frac{3\sqrt{6} - 3\sqrt{2}}{2}}$$

Numerically: $d \approx \sqrt{10 + 1.421} \approx \sqrt{11.421} \approx 3.38$