Prob. set 5.1

1. (Hard copy & soft copy)
Let X and Y be discrete random variables. The joint probability mass function $p(x,y)$ is

$$p(x,y) = P(X=x, Y=y) \text{ for every}$$

pair (x,y), where $p(x,y) \geq 0$ and $\sum_x \sum_y p(x,y) = 1$.

(a) From the definition given above, the following is true:
$P(X=1 \text{ and } Y=1) = p(1,1) = 0.2$, where we can find the value in the given table.

(b) $P(X \leq 1 \text{ and } Y \leq 1)$

$= p(0,0) + p(0,1) + p(1,0) + p(1,1)$

$= 0.1 + 0.04 + 0.08 + 0.2 = 0.42$.

(c) Since both random variables are not equal to zero, this means that at least one of the hoses is used on both islands.
The following is the probability of the given event

$P(X \neq 0 \text{ and } Y \neq 0) = p(1,1) + p(1,2) + p(2,1) + p(2,2)$

$= 0.2 + 0.06 + 0.14 + 0.3$

$= 0.7$.

(d) The marginal prob.ty mass function of discrete random variable X is

$$p_X(x) = \sum_y p(x,y), \text{ for every } x.$$

The marginal prob. mass function of discrete random variable Y is

$$p_Y(y) = \sum_x p(x,y), \text{ for every } y.$$

By the definition, for $x \in \{0,1,2\}$ the

marginal prob. is the sum of a particular row.
Therefor, we have,

$$p_X(0) = 0.10 + 0.04 + 0.020 = \cancel{0.34}\ 0.16$$

$$p_X(1) = 0.08 + 0.2 + 0.06 = 0.34$$

$$p_X(2) = 0.06 + 0.14 + 0.30 = 0.\cancel{4}50$$

which determine the marginal pmf of X.
Similarly, for marginal pmf of Y, we get

$$p_Y(0) = 0.1 + 0.08 + 0.06 = 0.24$$

$$p_Y(1) = 0.04 + 0.20 + 0.14 = 0.38$$

$$p_Y(2) = 0.02 + 0.06 + 0.30 = 0.38,$$

where we sum the probabilities in the particular column.

To summarize, we have the marginal ~~pdf~~ pmf of X and the marginal pmf of Y, respectively, given with

$$p_X(0) = 0.16,$$

$$p_X(1) = 0.34$$

$$p_X(2) = 0.5,$$

$$p_X(x) = 0, \quad x \notin \{0,1,2\},$$

$$p_Y(0) = 0.24$$

$$p_Y(1) = 0.38$$

$$p_Y(2) = 0.38$$

$$p_Y(y) = 0, \quad y \notin \{0,1,2\}.$$

Using the marginal pmf of X, we get

$$P(X \le 1) = p_X(0) + p_X(1) = 0.16 + 0.34 = 0.5$$

(e) Two random variables X and Y are independent if and only if

1. $p(x,y) = p_X(x) \cdot p_Y(y)$, for every (x,y) and when X and Y discrete rv's,

2. $f(x,y) = f_X(x) \cdot f_Y(y)$ for every (x,y) and when X and Y continuous rv's,

otherwise They are dependent.

Notice That

$$p_X(2) \cdot p_Y(2) = 0.5 \times 0.38 = 0.19,$$

$$p(2,2) = 0.3,$$

from which we can conclude That for pair $(2,2)$ The following holds

$$p(2,2) = 0.3 \neq 0.19 = p_X(2) \cdot p_Y(2).$$

Therefore, The random variables are dependent.

Prob. 9 (both hard copy & soft copy)

Let X and Y be continuous random variables. The joint prob. density function $f(x,y)$ is a function that is non-negative and for which

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) = 1.$$

Also, $P[(X,Y) \in A] = \iint_A f(x,y)\,dx\,dy.$

(a) From relation,

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$$

We can find value of K.

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = \int_{20}^{30}\int_{20}^{30} K(x^2+y^2)\,dx\,dy$$

$$= K\int_{20}^{30}\int_{20}^{30} x^2 dy\, dx + K\int_{20}^{30}\int_{20}^{30} y^2 dx\, dy$$

$$= K\int_{20}^{30} x^2 \left(y\Big|_{20}^{30}\right) dx + K\int_{20}^{30} y^2\left(x\Big|_{20}^{30}\right) dy$$

$$= 10K \cdot \frac{x^3}{3}\Big|_{20}^{30} + 10K \cdot \frac{y^3}{3}\Big|_{20}^{30}$$

$$= \frac{20K}{3}\left(30^3 - 20^3\right) = \frac{380000}{3} K = 1$$

Hence, we have $K = \dfrac{3}{380\,000}$.

(b) The underfill limit is 26 psi, therefore we need the following prob.

$P(X < 26 \text{ and } Y < 26)$

$$= P[(X,Y) \in A] = \iint_A f(x,y)\, dx\, dy$$

$$= \int_{20}^{26}\int_{20}^{26} K(x^2+y^2)\, dx\, dy$$

$$= K\int_{20}^{26}\int_{20}^{26} x^2 dy\, dx + K\int_{20}^{26}\int_{20}^{26} y^2 dx\, dy$$

$$= 6K\int_{20}^{26} x^2 dx + 6K\int_{20}^{26} y^2 dy$$

$$= {}^{2}\cancel{6}K \cdot \frac{2}{\cancel{3}}\left[26^3 - 20^3\right]$$

$$= 4 \cdot \frac{3}{380\,000}\left(26^3 - 20^3\right) = 0.3024.$$

(c) We need to find subset of

$$20 \le x \le 30, \quad 20 \le y \le 30$$

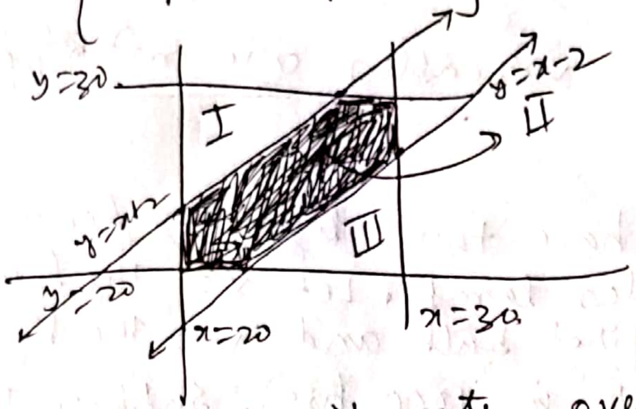
for which the difference of X and Y is <u>at most</u> 2. We can represent the distance between two points x and y as $|x-y|$.

Therefore, we need to find prob. of event

$$\{ |X-Y| \le 2 \}.$$

When $X \geq Y$,
$|X-Y| = (X-Y)$.
$X - Y \le 2 \Rightarrow Y \geq X-2$
i.e., $y \geq x-2$

When $X \le Y$,
$|X-Y| = Y-X$
$Y-X \le 2 \Rightarrow y \le x+2$



We need to integrate over region II.

$$P(|X-Y| \le 2) = \iint_{II} f(x,y)\,dx\,dy$$

$$= 1 - \iint_{I} f(x,y)\,dx\,dy - \iint_{III} f(x,y)\,dx\,dy$$

$$= 1 - \underbrace{\int_{20}^{28} \int_{x+2}^{30} f(x,y)\,dy\,dx}_{I_1} - \underbrace{\int_{22}^{30} \int_{20}^{x-2} f(x,y)\,dy\,dx}_{I_2}$$

$I_1 = 0.3203$, $I_2 = 0.3203$

$\therefore P(|X-Y| \le 2) = 0.3594.$

(d) $$f_X(x) = \int_{20}^{30} K(x^2+y^2)\,dy = 10K\,x^2 + K \left.\frac{y^3}{3}\right|_{20}^{30}$$

$$= \begin{cases} 10K\,x^2 + 0.05, & 20 \le x \le 30, \\ 0, & x \notin [20,30]. \end{cases}$$

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We would get the same marginal distribution for Y if we substitute x with y.

(e) $f(x,y) = K(x^2+y^2)$

$f_X(x) \cdot f_Y(y) = (10K x^2 + 0.05)(10K y^2 + 0.05)$

Clearly $f(x,y) \neq f_X(x) f_Y(y)$.

$\therefore$ the random variables are not ~~dependent~~ independent.

$\boxed{\text{(19) (Hard copy)} \quad \text{(13) (Soft copy)}}$ You have two lightbulbs for a particular lamp. Let X = the lifetime of the first bulb and Y = the lifetime of the 2nd bulb (both in 1000 hrs). Suppose that X and Y are independent and that each has an exponential distribution with parameter $\lambda = 1$.

(a) What is the joint pdf of X and Y?

(b) What is the prob^{ty} that each bulb lasts at most 1000 hrs (ie, $X \le 1$ and $Y \le 1$)?

(c) What is the prob^{ty} that the total lifetime of the two bulbs is at most 2?

(d) What is the prob^{ty} that the total lifetime is between 1 and 2?

Solution: (a). Random variable X with pdf

$$f_X(x) = \begin{cases} \lambda e^{-\lambda x} & , x \geq 0, \\ 0 & , x < 0 \end{cases}$$

is said to have exponential distr^n with parameter $\lambda = 1$.

$\therefore\; f_X(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0, \end{cases}$

and $f_Y(y) = \begin{cases} e^{-y}, & y \geq 0. \\ 0, & y < 0. \end{cases}$

(a) ~~Two~~ Given That X and Y are independent.
Since X & Y have continuous distribution, The joint pdf is given by

$$f(x,y) = f_X(x)\, f_Y(y)$$

$$= \begin{cases} e^{-(x+y)}, & x \geq 0,\ y \geq 0. \\ 0, & \text{otherwise} \end{cases}$$

(b) $P(X \leq 1 \text{ and } Y \leq 1)$

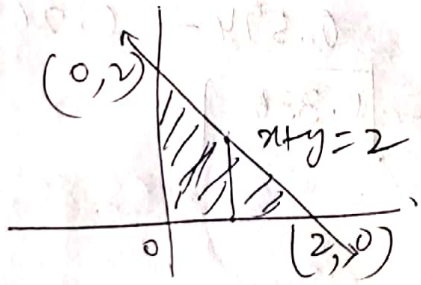
$= P(X \leq 1) \cdot P(Y \leq 1)$ [from the multiplication property of two events A and B].

$= (1 - e^{-1})(1 - e^{-1}) = 0.4.$

$\hookrightarrow$ [cdf of exponentially distributed random variable is

$F_X(x) = P(X \leq x) = 1 - e^{-\lambda x},\ x \geq 0$]

(c)



$$P(X+Y \leq 2) = \int_0^2 \int_0^{2-x} f(x,y)\, dy\, dx$$

$$= \int_0^2 \int_0^{2-x} e^{-x-y}\, dy\, dx = \int_0^2 \int_0^{2-x} e^{-x} \cdot e^{-y}\, dy\, dx.$$

$$= \int_0^2 e^{-x} \left(-e^{-y} \Big|_0^{2-x} \right) dx$$

$$= \int_0^2 e^{-x} (1 - e^{x-2})\, dx = 1 - e^{-2} - 2e^{-2} = 0.594$$

(d) The following is true:

$P(1 \le X+Y \le 2)$

$= P(X+Y \le 2) - P(X+Y \le 1)$

$P(X+Y \le 2) = 0.594$ [found in part (c)]

$$P(X+Y \le 1) = \int_0^1 \int_0^{1-x} e^{-x-y}\,dy\,dx.$$

$$= \int_0^1 \int_0^{1-x} e^{-x} e^{-y}\,dy\,dx$$

$$= \int_0^1 e^{-x} \left(-e^{-y}\Big|_0^{1-x} \right) dx$$

$$= \int_0^1 e^{-x}\,dx - \int_0^1 e^{-1}\,dx$$

$$= -e^{-x}\Big|_0^1 - e^{-1}$$

$$= 1 - e^{-1} - e^{-1} = 0.264.$$

$\therefore P(1 \le X+Y \le 2) =$

$0.594 - 0.264$

$= \boxed{0.330}.$

5.2 Expected values, covariance, and correlation:

Let X and Y be jointly distributed rv's with pmf $p(x,y)$ or pdf $f(x,y)$ according to whether the variables are discrete or continuous. Then the expected value of a function $h(X,Y)$, denoted by $E[h(X,Y)]$ or $\mu_{h(X,Y)}$, is given by

$$E[h(X,Y)] = \begin{cases} \sum_x \sum_y h(x,y)\, p(x,y) & \text{if } X \text{ and } Y \text{ are discrete} \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} h(x,y)\, f(x,y)\, dx\, dy ; & \text{if } X \text{ and } Y \text{ are cont's.} \end{cases}$$

Ex. 5.13: Five friends have purchased tickets to a certain concert. If the tickets are for seats 1–5 in a particular row and the tickets are randomly distributed among the five, what is the expected number of seats separating any particular two of the five? Let X and Y denote the seat numbers of the 1st and 2nd individuals, respectively. Possible (X,Y) pairs are $\{(1,2), (1,3), \ldots (5,4)\}$ and the joint pmf of (X,Y) is

$$p(x,y) = \begin{cases} \frac{1}{20}, & x = 1,2,\ldots,5;\ y = 1,\ldots,5;\ x \neq y \\ 0 & \text{otherwise} \end{cases}$$

The number of seats separating the two individuals is $h(X,Y) = |X-Y| - 1$. The accompanying table gives $h(x,y)$ for each possible (x,y) pair.

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$h(x,y)$	x: 1	2	3	4	5
y: 1	–	0	1	2	3
2	0	–	0	1	2
3	1	0	–	0	1
4	2	1	0	–	0
5	3	2	1	0	–

Thus

$$E[h(X,Y)] = \sum_x \sum_y h(x,y)\, p(x,y)$$

$$= \sum_{x=1}^{5} \sum_{\substack{y=1 \\ x \neq y}}^{5} \left(|x-y| - 1 \right) \cdot \frac{1}{20} = 1.$$

<u>5.14</u> In example 5.5, the joint pdf of the amount X of almonds and amount Y of cashews in a 1-lb can of nuts was

$$f(x,y) = \begin{cases} 24xy & 0 \le x \le 1,\ 0 \le y \le 1,\ x+y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

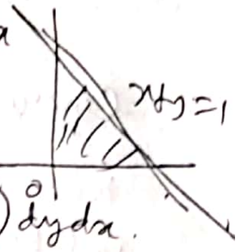
If 1 lb of almonds costs the company \$1.00, 1 lb of cashews costs \$1.50, and 1 lb of peanuts costs \$0.50, then the total cost of the contents of a can is

$$h(X,Y) = (1)X + (1.5)Y + 0.5(1 - X - Y)$$
$$= 0.5 + 0.5X + Y.$$

The expected total cost is

$$E[h(X,Y)] = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} h(x,y)\, f(x,y)\, dx\, dy$$

$$= \int_0^1 \int_0^{1-x} (.5 + .5x + y)\, 24xy\, dy\, dx$$



$$= \int_0^1 \int_0^{1-x} (12xy + 12x^2y + 24xy^2)\, dy\, dx.$$

$$= \int_0^1 \left(\left| \frac{12xy^2}{2} + 12x^2\frac{y^2}{2} + \frac{\cancel{24}^{\,8}xy^3}{\cancel{3}} \right|_{y=0}^{y=1-x} \right) dx.$$

$$= \int_0^1 \left[6x(1-x)^2 + 6x^2(1-x)^2 + 8x(1-x)^3\right] dx.$$

$$= 6\int_0^1 x(1-x)^2 dx + 6\int_0^1 x^2(1-x)^2 dx + 8\int_0^1 x(1-x)^3 dx.$$

$$\int_0^1 x(1-x)^2 dx = -x\frac{(1-x)^3}{3}\Big|_0^1 - \frac{(1-x)^4}{12}\Big|_0^1$$

$$= \frac{1}{12}$$

$$\int_0^1 x^2(1-x)^2 dx = \int_0^1 x \cdot x(1-x)^2 dx$$

$$= x\left(-x\frac{(1-x)^3}{3} - \frac{(1-x)^4}{12}\right)\Big|_0^1$$

$$- \int_0^1 \left[-x\frac{(1-x)^3}{3} - \frac{(1-x)^4}{12}\right] dx.$$

$$= +\frac{1}{3}\int_0^1 x(1-x)^3 dx - \left|\frac{(1-x)^5}{60}\right|_0^1$$

$$= \frac{1}{3}\left[0 - \frac{(1-x)^5}{20}\Big|_0^1\right] + \frac{1}{60} = \frac{2}{60} = \frac{1}{30}.$$

$$\int_0^1 x(1-x)^3 dx = \frac{1}{20}.$$

$\therefore E[h(X,Y)] = \frac{6}{12} + 6 \times \frac{1}{30} + \cancel{6}^{2} \times \frac{1}{\cancel{15}\,5}$

$= \frac{1}{2} + \frac{1}{5} + \frac{2}{5} = \frac{1}{2} + \frac{3}{5} = \frac{11}{10}$ ₹1.1

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Covariance :

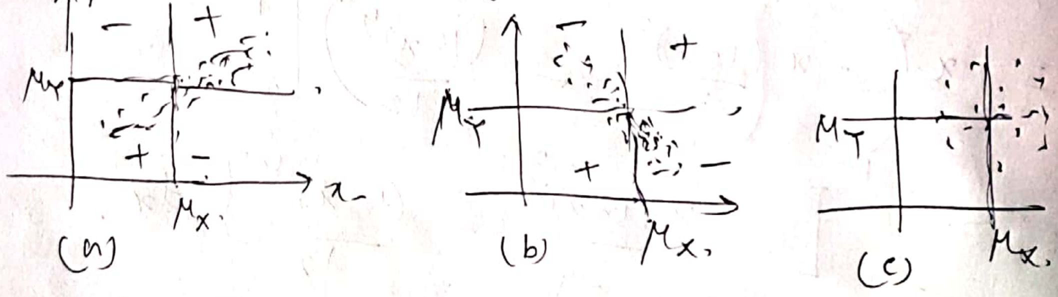
When two random variables X and Y are not independent, it is frequently of interest to assess how strongly they are related to ~~each other~~ one another.

<u>Defn.</u> The covariance between two rv's X and Y is

$$\mathrm{Cov}(X,Y) = E[(X-\mu_X)(Y-\mu_Y)]$$

$$= \begin{cases} \sum_x \sum_y (x-\mu_x)(y-\mu_y)\, p(x,y), & X, Y \text{ discrete} \\ \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} (x-\mu_x)(y-\mu_y) f(x,y)\, dx\, dy, & X, Y \text{ continuous} \end{cases}$$

That is, since $X - \mu_X$ and $Y - \mu_Y$ are the deviations of the two variables from their respective mean values, the covariance is the expected product of deviations.



(a) → positive covariance

(b) → −ve covariance

(c) → covariance near zero.

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