Useful results:

$$Cov(X,Y) = E(XY) - \mu_X \cdot \mu_Y .$$

Proof:

$$Cov(X,Y) = E[(X-\mu_X)(Y-\mu_Y)].$$

$$= E[XY - \mu_X Y - X\mu_Y + \mu_X\mu_Y]$$

$$= E(XY) - \mu_X E(Y) - \mu_Y E(X) + \mu_X\mu_Y$$

$$= E(XY) - \mu_X \mu_Y - \cancel{\mu_Y \mu_X} + \cancel{\mu_X \mu_Y}$$

$$= E(XY) - \mu_X \mu_Y .$$

Ex. 5.16 The joint and marginal pdf's of X = amount of almonds and Y = amount of cashews were

$$f(x,y) = \begin{cases} 24xy & 0 \le x \le 1,\ 0 \le y \le 1,\ x+y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_X(x) = \begin{cases} 12x(1-x)^2 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$f_Y(y)$ is obtained by replacing $x$ by $y$ in $f_X(x)$. It is easily verified that $\mu_X = \mu_Y = 2/5$, and

$$E(XY) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} xy\, f(x,y)\,dx\,dy$$

$$= \int_0^1 \int_0^{1-x} xy \cdot 24xy\, dy\, dx$$

$$= 8\int_0^1 x^2(1-x)^3\,dx = \frac{2}{15}.$$

Thus $\text{Cov}(X,Y) = \frac{2}{15} - \left(\frac{2}{5}\right)\left(\frac{2}{5}\right) = \frac{2}{15} - \frac{4}{25} = -\frac{2}{75}$

A negative covariance is reasonable here because more almonds in the can implies fewer cashews.

Correlation:

The correlation coefficient of $X$ and $Y$, denoted by $\text{Corr}(X,Y)$, $\rho_{X,Y}$, or just $\rho$, is defined by

$$\rho_{X,Y} = \frac{\text{Cov}(X,Y)}{\sigma_X \cdot \sigma_Y}$$

Result: 1. If $a$ and $c$ are either both positive or both negative,

$$\text{Corr}(aX+b, cY+d) = \text{Corr}(X,Y).$$

2. For any two rv's $X$ and $Y$,

$$-1 \le \text{Corr}(X,Y) \le 1.$$

3. If $X$ and $Y$ are independent, then $\rho = 0$, but $\rho = 0$ does not imply independence.

4. $\rho = 1$ or $-1$ iff $Y = aX + b$ for some numbers $a$ and $b$ with $a \neq 0$,

When $\rho = 0$, $X$ and $Y$ are said to be uncorrelated. Two variables could be uncorrelated yet highly dependent.

(22) ( Soft copy & Hard copy ):

An instructor has given a short quiz consisting of two parts. For a randomly selected students, let X = the number of points earned on the first part and Y = the number of points earned on the 2nd part. Suppose that the joint pmf of X and Y is given in the accompanying table.

| $p(x,y)$ | | y: 0 | 5 | 10 | 15 |
|---|---|---|---|---|---|
| x | 0 | 0.02 | 0.06 | .02 | .10 |
| | 5 | .04 | .15 | .20 | .10 |
| | 10 | .01 | .15 | .14 | .01 |

a) If the score recorded in the grade book is the total no. of points earned on the two parts, what is the expected recorded score $E(X+Y)$?

(b) If the maximum of the two scores is recorded, what is the expected recorded score?

Solutn: We are given joint pmf of X and Y.

(a) : The expected value (mean value) of a random variable $g(X,Y)$, where $g(\cdot)$ is a function, denoted as $E[g(X,Y)]$ is given by

$$E[g(X,Y)] = \begin{cases} \sum_x \sum_y g(x,y)\, p(x,y), & X \text{ and } Y \text{ discrete} \\ \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} g(x,y)\, f(x,y)\,dx\,dy, & \text{cont's case.} \end{cases}$$

where $p(x,y)$ is pmf and $f(x,y)$ pdf.

Here $g(X,Y) = X+Y$ and the rv's are discrete.

$$\therefore E(X+Y) = \sum_x \sum_y g(x,y)\, p(x,y)$$

$$= (0+0)\times 0.02 + (0+5)\cdot 0.06 + \cdots + (10+10)\times 0.14 + (10+15)\times 0.01 = 14.1$$

(b): We are interested in expectation of random variable $g(x, Y) = \max(X, Y)$.

Therefore,

$$E[\max(X,Y)] = \sum_x \sum_y g(x,y)\, p(x,y)$$

$$= \max(0,0) \times 0.02 + \max(0,5) \times 0.06 + \ldots$$

$$+ \max(10,10) \times 0.14 + \max(10,15) \times 0.01 = 9.6.$$

~~40 (Hand copy)~~ 30 (soft copy):

(a). Compute the covariance for X and Y in exercise 22.

(b) Compute $\rho$ for X and Y ~~is~~ for this X and Y.

Solution:

Given

| $p(x,y)$ | | y: 0 | 5 | 10 | 15 |
|---|---|---|---|---|---|
| | 0 | 0.02 | 0.06 | 0.02 | 0.10 |
| x | 5 | 0.04 | 0.15 | 0.20 | 0.10 |
| | 10 | 0.01 | 0.15 | 0.14 | 0.01 |

(a) $E(X) = \sum x\, P_X(x)$

$$= 0(0.02+0.06+0.02+0.10)+5(0.04+0.15+0.20+0.10)$$
$$+10(0.01+0.15+0.14+0.01) = 5.55$$

$$E(Y) = 0(0.02+0.04+0.01) + 5(0.06+0.15+0.15)$$
$$+10(0.02+0.20+0.14) + 15(0.10+0.10+0.01) = 8.55$$

$\therefore \mu_X = 5.55, \quad \mu_Y = 8.55.$

Now $E(XY) = \sum\sum xy\, p(x,y)$

$$= 0(0)(0.02) + 0(5)(0.06) + 0(10)(0.02) + 0(15)(0.10)$$
$$+5(0)(0.04) + 5(5)(0.15) + 5(10)(0.20) + 5(15)(0.10)$$
$$+10(0)(0.01) + 10(5)(0.15) + 10(10)(0.14) + 10(15)(0.01) = 44.25$$

$Cov(X,Y) = E(XY) - \mu_X \mu_Y$

$= 44.25 - 5.55\,(8.55) = -3.2025.$

(b) $\sigma_X^2 = \sum_x (x-\mu_X)^2\, P_X(x)$

$= (0-5.55)^2 \times (0.02+0.06+0.02+0.10)$

$+ (5-5.55)^2 \times (0.04+0.15+0.20+0.10)$

$+ (10-5.55)^2 \times (0.01+0.15+0.14+0.01)$

$= 12.4475.$

$\sigma_Y^2 = \sum_y (y-\mu_Y)^2\, P_Y(y)$

$= (0-8.55)^2 \times (0.02+0.04+0.01)$

$+ (5-8.55)^2\,(0.06+0.15+0.15)$

$+ (10-8.55)^2\,(0.02+0.20+0.14) + (15-8.55)^2\,(0.10+0.10+0.01)$

$= 19.1475.$

$\sigma_X = \sqrt{\sigma_X^2} = \sqrt{12.4475}$

$\sigma_Y = \sqrt{19.1475}$

$\therefore \rho = Corr(X,Y) = \dfrac{Cov(X,Y)}{\sigma_X \sigma_Y} = \dfrac{-3.2025}{\sqrt{12.4475}\,\sqrt{19.1475}}$

$\approx -0.2074$

(23). (Hard copy) (31). (soft copy). (a) Compute the covariance between X and Y in Exercise 9. (b) Compute the correlation coefficient $\rho$ for this X and Y.

Soln:- (a). $Cov(X,Y) = E(XY) - E(X).E(Y)$.

$$E(Y) = E(X) = \int_{20}^{30} x\, f_X(x)\, dx = \int_{20}^{30} x(10k\, x^2 + 0.05)\, dx$$

$$= 10k \left.\frac{x^3}{3}\right|_{20}^{30} + 0.05 \left.\frac{x^2}{2}\right|_{20}^{30}$$

$$= 25.33$$

$$E(XY) = \int_{20}^{30}\int_{20}^{30} xy\, k(x^2+y^2)\, dx\, dy$$

$$= k\int_{20}^{30}\int_{20}^{30} x^3 y\, dy\, dx + k\int_{20}^{30}\int_{20}^{30} y^3 x\, dx\, dy$$

$$= k\int_{20}^{30} x^3 \left(\left.\frac{y^2}{2}\right|_{20}^{30}\right) dx + k\int_{20}^{30} y^3 \left(\left.\frac{x^2}{2}\right|_{20}^{30}\right) dy$$

$$= \left(\left.\frac{y^2}{2}\right|_{20}^{30}\right) k\left(\left.\frac{x^4}{4}\right|_{20}^{30}\right) + k\left(\left.\frac{x^2}{2}\right|_{20}^{30}\right)\left(\left.\frac{y^4}{4}\right|_{20}^{30}\right)$$

$$= 641.447.$$

$$\therefore Cov(X,Y) = E(XY) - E(X)\, E(Y)$$

$$= 641.447 - (25.33)^2$$

$$= -0.1619.$$

(b)

$$\rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y}.$$

$\sigma_X = \sqrt{V(X)}$, $\sigma_Y = \sqrt{V(Y)}$,

$V(X) = E(X^2) - (\mu_X)^2$

$E(Y^2) = E(X^2) = \int_{20}^{30} x^2 f_X(x)\, dx$

$= \int_{20}^{30} x^2 (10k\, x^2 + 0.05)\, dx$

$= 10k \left( \frac{x^5}{5} \Big|_{20}^{30} \right) + 0.05 \frac{x^3}{3} \Big|_{20}^{30}$

$= 649.825$.

$V(Y) = V(X) = E(X^2) - (E(X))^2$

$= 649.8285 - (25.33)^2 = 8.2161$.

Finally, the correlation co-efficient is

$\rho_{X,Y} = \frac{Cov(X,Y)}{\sigma_X \sigma_Y} = \frac{-0.1619}{\sqrt{8.2161}\sqrt{8.2161}}$

$= -0.0024$

(28) (Hand copy & soft copy)

Show that if $X$ and $Y$ are indipendent r.v's, then $E(XY) = E(X) \cdot E(Y)$.

Sln $E(X) = \mu_X = \int_{-\infty}^{\infty} x f(x)\, dx$

$E(X) \cdot E(Y) = \int_{-\infty}^{\infty} x\, f_X(x)\, dx \cdot \int_{-\infty}^{\infty} y \cdot f_Y(y)\, dy$

$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy\, f_X(x)\, f_Y(y)\, dx\, dy$

$= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} xy\, f(x,y)\, dx\, dy$. [$\because X$ & $Y$ are indipendent $f(x,y) = f_X(x) \cdot f_Y(y)$]

$= E(XY)$.

$\left\{\begin{array}{l} 31.\ \text{(Hard copy)} \\ 27.\ \text{(Soft copy)} \end{array}\right\}$ : Annie and Alvie have agreed to meet for lunch between noon (0.00 PM) and 1.00 P.M. Denote Annie's arrival time by X, Alvie's by Y, and suppose X and Y are independent with pdf's

$$f_X(x) = \begin{cases} 3x^2 & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$f_Y(y) = \begin{cases} 2y & 0 \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

What is the expected amount of time that the one who arrives first must wait for the other person?

Soln: Depending on who arrives first we look at different random variables. for example, if Annie (X) arrives first, then she waits $Y - X$ minutes, but if Alvie (Y) arrives first, then he ~~arrives~~ waits $X - Y$ minutes. We can represent both random variables as one $g(X, Y) = |X - Y|$.

We do not have the joint pdf. From the independence given in the exercise we can compute the joint pdf (conti's case).

The joint pd

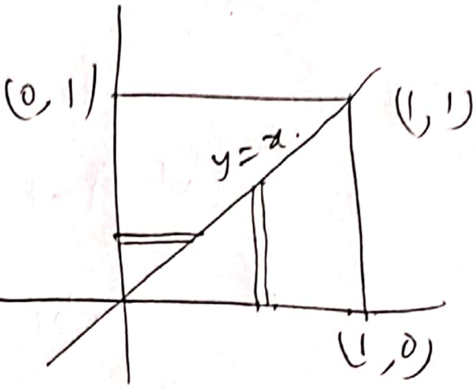
$$f(x,y) = f_X(x)\, f_Y(y) = \begin{cases} 3x^2 \cdot 2y, & 0 \le x, y \le 1 \\ 0, & \text{otherwise}, \end{cases}$$

Now,

$$E(|X-Y|) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} |x-y|\, f(x,y)\, dx\, dy$$

$$= \int_0^1 \int_0^1 |x-y|\, 6x^2 y\, dx\, dy$$

$$= \int_0^1 \int_0^x (x-y)\, 6x^2 y \, dy\, dx + \int_0^1 \int_x^1 (y-x)\, 6x^2 y \, dy\, dx$$




$$= \int_0^1 \int_0^x 6x^3 y \, dy\, dx - \int_0^1 \int_0^x 6x^2 y^2 \, dy\, dx + \int_0^1 \int_x^1 6x^2 y^2 \, dy\, dx - \int_0^1 \int_x^1 6x^3 y \, dy\, dx$$

$$= \int_0^1 6x^3 \left( \frac{y^2}{2} \Big|_0^x \right) dx - \int_0^1 6x^2 \left( \frac{y^3}{3} \Big|_0^x \right) dx + \int_0^1 6x^2 \left( \frac{y^3}{3} \Big|_x^1 \right) dx - \int_0^1 6x^3 \left( \frac{y^2}{2} \Big|_x^1 \right) dx.$$

$$= \int_0^1 3x^5 \, dx - \int_0^1 2x^5 \, dx + \int_0^1 2x^2 (1-x^3)\, dx - \int_0^1 3x^3 (1-x^2)\, dx =$$

$$= \int_0^1 3x^5 \, dx - \int_0^1 2x^5 \, dx + \int_0^1 2x^2 \, dx + \int_0^1 x^5 \, dx - \int_0^1 3x^3 \, dx$$

$$= \frac{1}{6} + \frac{2}{3} - \frac{3}{4} + \frac{1}{6} = \frac{1}{6} + \frac{1}{12} = \frac{1}{4}.$$

— x —