Markov Inequality

If X is any random variable with mean $\mu = E(X)$ where X is nonnegative and μ is finite, then for any $t > 0$,

$$P[X \geq t] \leq \frac{\mu}{t}$$

Proof For a fixed $t > 0$, define the r.v. Y as
$$Y = \begin{cases} 0 & \text{if } X < t \\ t & \text{if } X \geq t. \end{cases}$$

then Y is a discrete r.v. with pmf

$$p_Y(0) = P[X < t]$$

and $p_Y(t) = P[X \geq t]$. where $Y = \{0, t\}$

Thus
$$\text{mean}(Y) = E(Y) = \sum_y y\, p_Y(y)$$
$$= 0 \cdot p_Y(0) + t\, p_Y(t)$$
$$= t\, P[X \geq t]$$

Since $X \geq Y$, we have

$$E(X) \geq E(Y)$$
$$\geq t\, P[X \geq t]$$
$$\Rightarrow \mu \geq t\, P[X \geq t]$$
$$\Rightarrow P[X \geq t] \leq \frac{\mu}{t}$$ proved

Note ① Taking $Y = \begin{cases} 0 & \text{if } X < t \\ t^k & \text{if } X \geq t, \end{cases}$

we get $P[X^k \geq t^k] \leq \frac{E(X^k)}{t^k}$, $k = 1, 2, \ldots$

and 0 otherwise

② Replacing X by $X - \mu$, we have

$$P[(X-\mu)^k \geq t^k] \leq \frac{E(X-\mu)^k}{t^k}, \quad k = 1, 2, \ldots$$

and 0 otherwise

Chebyshev Inequality

Let μ and σ^2 be the mean and variance of any r.v. X, then

$$P[|X-\mu| \geq t] \leq \frac{\sigma^2}{t^2}, \quad t > 0$$

Proof By Markov inequality, we have

$$P[(X-\mu)^2 \geq t^2] \leq \frac{E(X-\mu)^2}{t^2} = \sigma^2/t^2 \qquad \boxed{\because \sigma^2 = E(X-\mu)^2}$$

Since $[(X-\mu)^2 \geq t^2] = [|X-\mu| \geq t]$,

so we have

$$P[|X-\mu| \geq t] \leq \frac{\sigma^2}{t^2}, \quad t > 0$$

proved

Note

Replacing t by $K\sigma$, $K \geq 1$, we have

$$P[|X-\mu| \geq K\sigma] \leq \frac{\sigma^2}{K^2\sigma^2} = \frac{1}{K^2}$$

Q. Let μ & σ be mean & SD of any r.v. X then prove that $P[|X-\mu| \geq K\sigma] = \frac{1}{K^2}$ for any $K \geq 1$

Ex The r.v. Y has the probabilities given in the accompanying table

y	45	46	47	48	49	50	51	52	53	54	55
$p(y)$	0.05	0.10	0.12	0.14	0.2[illegible]	0.17	0.06	0.05	0.03	0.02	0.01

Find $E(Y)$, $SD(Y)$ and $P(|Y-\mu| \geq t)$ for $t = 26, 36$

Sol^n Expectation of $Y = E(Y) = \sum_{y=45}^{55} y\,p(y)$

$= 48.84$

Variance $= V(Y) = E(Y^2) - (E(Y))^2 = \sum_{y=45}^{55} y^2 p(y) - (48.84)^2$

$= 2389.84 - (48.84)^2$

$\Rightarrow \sigma^2 = 4.4944$

$\Rightarrow \sigma = 2.12$

By Chebyshev inequality, we have

$$P[|Y-\mu| \geq t] \leq \frac{\sigma^2}{t^2}$$

a) For $t = 26$, $P[|Y-\mu| \geq 26] \leq \frac{\sigma^2}{(26)^2} = \frac{1}{4}$

$P[|Y-\mu| \geq 26] = P[Y-\mu \geq 26] + P[Y-\mu \leq -26]$

$= P[Y \geq \mu+26] + P[Y \leq \mu-26]$

$= P[Y \geq 53.08] + P[Y \leq 44.60]$

$= 1 - P[Y < 53.08] + P[Y \leq 44]$

$= 1 - P[Y \leq 53] + P[Y \leq 44]$

$= 1 - (F(53) - F(44))$

$= 1 - (0.97 - 0) = 0.03$

(b) Find $P[|Y-\mu| \geq 36]$ (Task)