2. Probability.

Introduction: The term probability refers to the study of randomness and uncertainty. In any situation in which one of a number of possible outcomes may occur, the discipline of probability provides methods for quantifying the chances, or likelihoods, associated with the various outcomes. The language of probability is constantly used in an informal manner in both written and spoken contexts. Examples include such statements as "There is a 50-50 chance that the incumbent will seek reelection," "There will probably be at least one section of that course offered next year," and "It is expected that at least 20000 concert tickets will be sold."

In this chapter, we introduce some elementary probability concepts, indicate how probabilities can be interpreted and show how the rules of ~~prob~~ probability can be applied to compute the probabilities of many interesting events. The methodology of probability will then permit us to express in precise language such informal statements as those given above.

2.1 Sample spaces and events

Random Experiment: An experiment or observation which may be repeated a large number of times under very nearly identical conditions and the possible outcome of any particular observations is unpredictable but all possible outcomes can be described prior to its performance, is known as Random Experiment.

for example, The experiment of tossing a coin is a random experiment, as The possible outcomes are 'tails' or 'heads' but The outcome of a particular tossing cannot be predicted.

Sample points / Event points :

The outcomes of a Random experiment are called sample points or event points.
for example, The sample points in The experiment 'tossing a coin' are 'Head' and 'Tail', in symbol H and T.

Sample space / Event space :

The set of all sample points i.e. The set of all possible outcomes of a Random Experiment is called the sample space or event space. It is denoted by S.
for example, if we Throw two coins once, then $S = \{HH, HT, TH, TT\}$; if we roll a die once, we have $S = \{1, 2, 3, 4, 5, 6\}$ as The event space.

Event: Any subset of The sample space S of a random experiment is called an event.
for example, in The experiment of 'throwing two coins', $A = \{TH, HT\}$ is an event because $A \subset S$.
An event is simple if it consists of exactly one outcome and compound if it consists of more than one outcome.

When a random experiment is performed, a particular event A is said to occur if the resulting experimental outcome is contained in A. In general, exactly one simple event will occur, but many compound events will occur simultaneously.

Ex. 1 Consider an experiment in which each of three vehicles taking a particular freeway exit turns left (L) or right (R) at the end of the exit ramp. The eight possible outcomes that comprise the sample space are LLL, RLL, LRL, LLR, LRR, RLR, RRL and RRR. Thus there are eight simple events, among which are $E_1 = \{LLL\}$ and $E_5 = \{LRR\}$.

Some compound events include

$A = \{RLL, LRL, LLR\}$ = the event that exactly one of the three vehicles turns right.

$B = \{LLL, RLL, LRL, LLR\}$ = the event that at most one of the vehicles turns right.

$C = \{LLL, RRR\}$ = the event that all three vehicles turn in the same direction.

Suppose that when the random experiment is performed, the outcome is LLL. Then the simple event E_1 has occurred and so also have the compound events B and C (but not A).

Ex.2 Example for infinite sample space:

A reasonably large percentage of C++ programs written at a particular company compile on the first run, but some do not. Suppose an experiment consists of selecting and compiling C++ programs at this location one by one until encountering a program that compiles on the first run. Denote a program that compiles on the first run by S (for success) and one that doesn't do so by F (for failure). Although it may not be very likely, a possible outcome of this experiment is that the first 5 (or 10 or 20 or ...) are F's and the next one is an S. That is, for any positive integer n, we may have to examine n programs before seeing the first S. The sample ~~space~~ space is $S = \{S, FS, FFS, \ldots\}$, which contains an infinite number of possible outcomes. The same abbreviated form of the sample space is appropriate for an experiment in which, starting at a specified time, the gender of each newborn infant is recorded until the birth of a male is observed.

Ex.3 (Ex.2 continued).

The sample space for the program compilation experiment contains an infinite number of outcomes, so there are an infinite number of simple events. Compound events include

$A = \{S, FS, FFS\}$ = the event that atmost three programs are examined.

$E = \{FS, FFFS, FFFFFS, \ldots\}$ = The event that an even number of programs are examined.

<u>certain event</u>:
Since every set is also a subset of itself, so the sample space is a subset of itself. So This is an event. This event is called a certain event.

<u>Impossible event</u>:
An event that contains no sample points is called impossible event and is denoted by Φ. for example in The experiment ' rolling a die The event ' face 7' $= \Phi$.

<u>Complementary event</u>: The <u>complement</u> of an event A, denoted by A' or $\bar{A}$ or A^c, is the set of all outcomes in S that are not contained in A.

For example, if $A = \{TH, HT\}$ where $S = \{HH, TT, TH, HT\}$, Then $\bar{A} = \{HH, TT\}$.

Note. $\bar{S} = \Phi$; $\bar{\Phi} = S$; $\overline{(\bar{A})} = A$.

Simultaneous occurence of two events.
The set $A_1 \cap A_2$ represents the simultaneous occurence of the two events A_1 and A_2. It also denoted by $A_1 A_2$.

For example, in the experiment " rolling a die" let A_1 = ' Even face', A_2 = ' Multiple of Three'. Then $A_1 \cap A_2 = \{6\}$ is The event whose occurence shows the simultaneous occurrence of A_1 and A_2.

At least one of two events:

The set $A_1 \cup A_2$ represents 'at least one of A_1 and A_2'. This event is also denoted by $A_1 + A_2$.

For example, in the experiment rolling a die let A_1 = Even face $= \{2, 4, 6\}$, A_2 = Multiple of Three $= \{3, 6\}$. Then $A_1 \cup A_2 = \{2, 3, 4, 6\}$ is the event whose occurrence shows the occurrence of at least one of 'even face' and 'Multiple of 3'.

Disjoint or Mutually Exclusive Events:

If two events A_1, A_2 have no common sample points i.e. if $A_1 \cap A_2 = \phi$, They are called Mutually Exclusive Events.

For example, in a previous example if $A_1 = \{HH, TT\}$ and $A_2 = \{HT, TH\}$, then $A_1 \cap A_2 = \phi$. So, A_1 and A_2 are mutually exclusive events. Two mutually exclusive (m.e) events cannot occur simultaneously.

Pairwise disjoint Events:

Let $A_1, A_2, \ldots, A_n$ be n number of events. Events A_i $(i = 1, 2, \ldots, n)$ are said to be pairwise disjoint if no two of them have any common events points i.e. if $A_i \cap A_j = \phi$, $i \neq j$ and $i, j = 1, 2, \ldots, n$.

Exhaustive events: Two or more events are said to be exhaustive if at least one of them necessarily occurs or in other words the events $A_1, A_2, \ldots$ are exhaustive if $A_1 \cup A_2 \cup A_3 \cup \cdots = S$.

for example, in the experiment of throwing two coins once, the events $A_1 = \{HH\}$, $A_2 = \{TT\}$ and $A_3 = \{HT, TH\}$ are exhaustive.

Equally likely sample points:

The sample points of a sample space are said to be equally likely if one of them may not be expected rather than the other.

2.2 Axioms, ~~interpretations~~ Interpretations, and Properties of Probability:

Given an experiment and a sample space S, the objective of probability is to assign to each event A a number $P(A)$, called the probability of the event A, which will give a precise measure of the chance that A will occur. To ensure that the probability assignments will be consistent with our intuitive notions of probability, all assignments should satisfy the following axioms of probability:

Axiom 1: For any event A, $P(A) \geq 0$.

Axiom 2: $P(S) = 1$.

Axiom 3: If $A_1, A_2, A_3, \ldots$ is an infinite collection of disjoint events, then

$$P(A_1 \cup A_2 \cup A_3 \cup \cdots) = \sum_{i=1}^{\infty} P(A_i).$$

Proposition: / Result: $P(\Phi) = 0$, when Φ is the null event.

Proposition: For any event A, $P(A) + P(A') = 1$, from which $P(A) = 1 - P(A')$.

Proposition: For any event A, $P(A) \leq 1$.

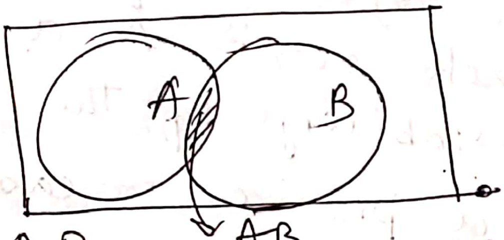
Proof: $1 = P(A) + P(A') \geq P(A)$ since $P(A') \geq 0$.

When events A and B are mutually exclusive, $P(A \cup B) = P(A) + P(B)$. For events, that are not mutually exclusive, adding $P(A)$ and $P(B)$ results in "double-counting" outcomes in the intersection.

Proposition: For any two events A and B,

$$P(A \cup B) = P(A) + P(B) - P(A \cup B)$$

Proof:



$A = A - AB + AB$, where $A - AB$ and AB are mutually exclusive.

$\therefore P(A) = P(A - AB) + P(AB)$ —— ① [by Axiom 3]

Similarly $P(B) = P(B - AB) + P(AB)$ —— ②.

Now, $A + B = A - AB + AB + B - AB$, where $A - AB$, AB and $B - AB$ are mutually exclusive.

$\therefore P(A+B) = P(A - AB) + P(AB) + P(B - AB)$

$= P(A) - P(AB) + \cancel{P(AB)} + P(B) - \cancel{P(AB)}$, using ① & ②.

$= P(A) + P(B) - P(AB)$.

Result: For any three events A, B and C,

$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC)$.

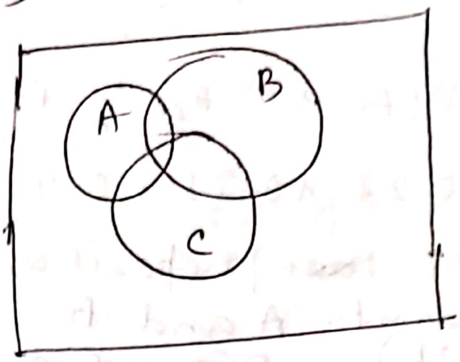


figure $A \cup B \cup C$

Proof:

$P(A + B + C) = P(A) + P(B+C) - P(A(B+C))$

$= P(A) + P(B) + P(C) - P(BC) - P(AB + AC)$

[using above proposition]

$= P(A) + P(B) + P(C) - P(BC) - P(AB) - P(AC) + P(ABC)$ $[AB \cap AC = ABC]$

$= P(A) + P(B) + P(C) - P(AB) - P(BC) - P(CA) + P(ABC)$.

Exercises: Section 2.2.

Selected problems:

(13) A computer consulting firm presently has bids out on three projects. Let $A_i = \{$awarded project $i\}$, for $i = 1, 2, 3$, and suppose that $P(A_1) = .22$, $P(A_2) = .25$, $P(A_3) = .28$, $P(A_1 \cap A_2) = .11$, $P(A_1 \cap A_3) = .05$, $P(A_2 \cap A_3) = .07$, $P(A_1 \cap A_2 \cap A_3) = .01$. Compute the probability of each event:

(18)

a. $A_1 \cup A_2$; b. $A_1' \cap A_2'$; c. $A_1 \cup A_2 \cup A_3$

d. $A_1' \cap A_2' \cap A_3'$; e. $A_1' \cap A_2' \cap A_3$; f. $(A_1' \cap A_2') \cup A_3$

Ans: We have that

$$A_i = \{\text{awarded project } i\}, i = 1, 2, 3.$$

We

(a) $P(A_1 \cup A_2) = P(A_1) + P(A_2) - P(A_1 \cap A_2)$

$= 0.22 + 0.25 - 0.11 = 0.36.$

~~here~~ [Using the ~~...~~ propositions:
For every two events A and B
$P(A \cup B) = P(A) + P(B) - P(AB)$.

(b) Describing in words event $A_1' \cap A_2'$ is

$A_1' \cap A_2' = \{\text{awarded project neither 1 or 2}\}.$

$P(A_1' \cap A_2') = P((A_1 \cup A_2)')$

$= 1 - P(A_1 \cup A_2) = 1 - 0.36 = 0.64$

[$\because$ ~~P(A~~ for any event A, we know
$P(A) + P(A') = 1$].

(c) $P(A_1 \cup A_2 \cup A_3) = P(A_1) + P(A_2) + P(A_3) - P(A_1 \cap A_2)$

$- P(A_1 \cap A_3) - P(A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_3).$

$= 0.22 + 0.25 + 0.28 - 0.11 - 0.05 - 0.07 + 0.01$

$= 0.75 - 0.23 + 0.01 = 0.76 - 0.23 = 0.53$

(d) $A_1' \cap A_2' \cap A_3' = \{\text{none of the three project awarded}\}.$

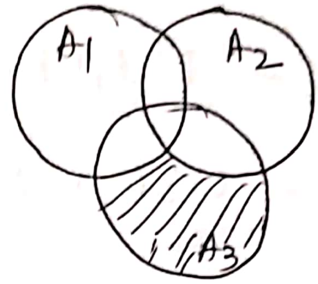
$P(A_1' \cap A_2' \cap A_3') = P((A_1 \cup A_2 \cup A_3)')$

$= 1 - P(A_1 \cup A_2 \cup A_3)$

$= 1 - 0.53 = 0.47.$

(e) $A_1' \cap A_2' \cap A_3 = \{$ awarded project 3 and neither awarded project 1 nor project 2$\}$.

We compute the probability as follows



Venn Diagram

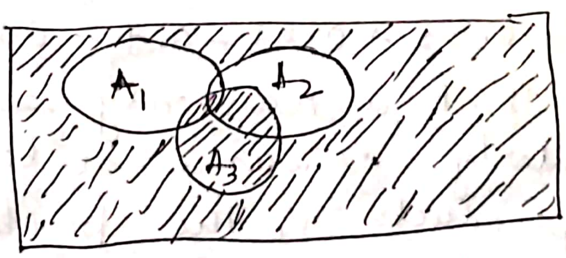
$$P(A_1' \cap A_2' \cap A_3)$$
$$= P(A_3) - P(A_1 \cap A_3) - P(A_2 \cap A_3) + P(A_1 \cap A_2 \cap A_3)$$
$$= 0.28 - 0.05 - 0.07 + 0.01$$
$$= 0.17$$

[Here we add the last term (The intersection of the three events) because we subtracted it twice from the probability of event A_3, once when we subtracted intersection of A_1 and A_3 and once when we subtracted intersection of A_2 and A_3. See the Venn Diagram above for better understanding.

(f) $(A_1' \cap A_2') \cup A_3 = \{$ awarded neither project 1 nor 2, or awarded project 3$\}$.

$$P\left((A_1' \cap A_2') \cup A_3\right)$$
$$= P(A_1' \cap A_2' \cap A_3') + P(A_3) = 0.47 + 0.28 = 0.75.$$

Here we use the fact that the event described in (f) is the sum of event described in (d) [none ~~awarded~~ awarded] plus the disjoint event that project 3 was awared. See the venn diagram given below.



Another method (using formula and previous result)

$P((A_1' \cap A_2') \cup A_3)$

$= P(A_1' \cap A_2') + P(A_3) - P(A_1' \cap A_2' \cap A_3)$

$= 1 - P(A_1 \cup A_2) + P(A_3) - P(A_1' \cap A_2' \cap A_3)$

$= 1 - 0.36 + 0.28 - 0.17 = 1.28 - 0.53 = 0.75.$

(14) Suppose that 55% of all adults regularly consume coffee, 45% regularly consume carbonated soda and 70% regularly consume at least one of these two products.

a. What is the probability that a randomly selected adult regularly consumes both coffee and soda?

b. What is the probability that a randomly selected adult doesn't regularly consume at least one of these two products?

Ans: Let events A, B and C be

$A = \{$Adult regularly consumes coffee$\}$;

$B = \{$Adult regularly consumes carbonated soda$\}$;

$C = \{$Adult regularly consumes coffee, soda or both$\}$;

and also $P(A) = 0.55$; $P(B) = 0.45$; $P(C) = 0.70$.

(a) $P(A \cap B) = P(A) + P(B) - P(A \cup B)$

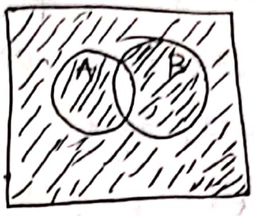
$= 0.55 + 0.45 - 0.7 = 0.3$.

(b) Probability that a randomly selected adult doesn't regularly consume at least one of these two products is the probability that the selected adult consumes only coffee or consumes only carbonated soda or doesn't consume both products which is ~~actually~~

~~$(A \cap B)'$.~~

~~$P((A \cap B)') = 1 - P(A \cap B)$~~

~~$= 1 - 0.3 = 0.70$.~~



~~$P(A \cap B' + B \cap A' + (A \cup B)')$~~

$P(A \cap B' + B \cap A' + (A \cup B)')$

$= P(A \cap B') + P(B \cap A') + P((A \cup B)')$ [$\because$ $A \cap B'$, $B \cap A'$ & $(A \cup B)'$ are mutually exclusive]

$= P(A) - P(AB) + P(B) - P(AB)$

$+ 1 - P(A \cup B)$.

$= (0.55 - .3) + (0.45 - 0.3) + (1 - 0.7)$

$= 1.0 - 0.3 = 0.7$.

(15) Consider The type of clothes dryer (gas or electric) purchased by each of five different customers at a certain store.

a. If The probability That at most one of these purchases an electric dryer is 0.428, what is the probability That at least two purchase an electric dryer?

b. If P(all five purchase gas) = 0.116 and P(all five purchase electric) = 0.005, what is the probability That at least one of each type is purchased?

Ans a. Let E be The event that at most one purchases an electric dryer. Then E' is The event That atleast two purchase electric dryers, and

$P(E') = 1 - P(E) = 1 - 0.428 = 0.572$.

b. Let A be the event that all five purchase gas dryer and let B be the event that all five purchase electric. All other possible outcomes are those in which at least one of each type of clothes dryers is purchased.

Let E = { At least one of each type is purchased. }

$\therefore E' = A \cup B$ (where A and B are disjoint events).

$\therefore E = 1 - P(E') = 1 - P(A \cup B)$

$= 1 - [P(A) + P(B)]$

[$\because$ A, B are disjoint events]

$= 1 - (0.116 + 0.005) = 0.879.$

(16) An individual is presented with three different glasses of cola, labeled C, D and P. He is asked to taste all three and then list them in order of preference. Suppose the same cola has actually been put into all three glasses.

a. What are the simple events in this ranking experiment, and what probability would you assign to each one?

b. What is the probability that C is ranked first?

c. What is the prob.^{ty} that C is ranked first and D is ranked last?

Ans: a. $S = \{ CDP, CPD, DCP, DPC, PCD, PDC \}$

$\downarrow$

6 simple events.

Probability of each one $= \frac{1}{6}$ (As we have the same cola in all glasses, therefore it would be completely random order of how an individual chooses preferences).

b. P(C is ranked first) = P({CDP, CPD}) $= \frac{2}{6} = \frac{1}{3}$.

c. $P(\{CPD\}) = \frac{1}{6}$.

17. Let A denote the event that the next request for assistance from a statistical software consultant relates to the SPSS package and let B be the event that the next request is for help with SAS. Suppose that $P(A) = 0.30$ and $P(B) = 0.50$.
a. Why is it not the case that $P(A) + P(B) = 1$?
b. Calculate $P(A')$.
c. Calculate $P(A \cup B)$.
d. Calculate $P(A' \cap B')$.

Ans: a. The probabilities do not add to 1 because there are other software packages besides SPSS and SAS for which requests could be made.

b. $P(A') = 1 - P(A) = 1 - 0.3 = 0.7$.

c. Since A and B are mutually exclusive events,
$P(A \cup B) = P(A) + P(B) = 0.3 + 0.5 = 0.8$.

d. By de Morgan's law, $P(A' \cap B') = P((A \cup B)')$

$= 1 - P(A \cup B) = 1 - 0.8 = 0.2$

In this example, de Morgan's ~~la~~ law says the event "neither A nor B" is the complement of the event "either A or B." (That's true regardless of whether they ~~are~~ are mutually exclusive.)

23. ~~Two~~ The computers of six faculty members in a certain department are to be replaced. Two of the faculty members have selected laptop ~~and~~ machines and the other four have chosen desktop machines. Suppose that only two of the setups can be done ~~a~~ on a particular day, and the ~~other four have choosen~~ two computers to be set up are randomly selected from the six (implying 15 equally likely outcomes; if the computers are numbered 1, 2, ..., 6, then one outcome consists of computers 1 and 2, another consists of 1 and 3, and so on).

a. What is the probability that both selected setups are for laptop computers?

b. What is the probability that both selected setups are desktop machines?

c. What is the probability that at least one selected set up is for a desktop computer?

d. What is the probability that at least one computer of each type is chosen for setup?

Ans: Assume that the computers are numbered 1–6 as described and that computers 1 and 2 are the two laptops. There are 15 possible outcomes $[({}^6C_2 = 15)]$: $(1,2), (1,3), (1,4), \ldots (5,6)$.

a. $P(\text{both are laptops}) = P(\{(1,2)\}) = \frac{1}{15} = 0.067$.

b. $P(\text{both are desktops}) = P(\{(3,4),(3,5),(3,6),(4,5),(4,6),(5,6)\})$

$= \frac{6}{15} = 0.40$.

c. $P(\text{at least one desktop}) = 1 - P(\text{no desktop})$
$= 1 - P(\text{both are laptops})$
$= 1 - \frac{1}{15} = \frac{14}{15} = 0.933$.

d. $P(\text{at least one of each type})$

$= 1 - P(\text{both are the same}) = 1 - [P(\text{both are laptops}) + P(\text{both are desktops})]$

$= 1 - [0.067 + 0.40] = 0.533$.

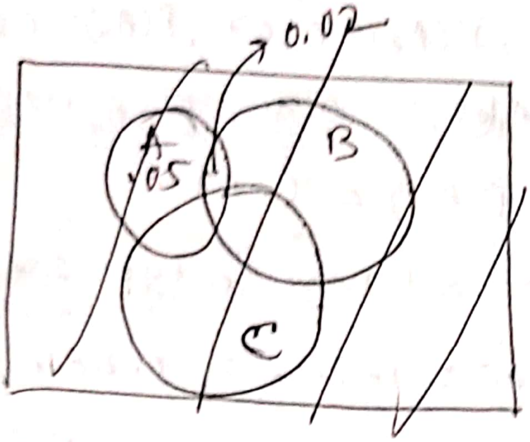
25. $P(AB) = P(A) + P(B) - P(AB) = 0.4 + 0.55 - 0.63 = 0.32$.

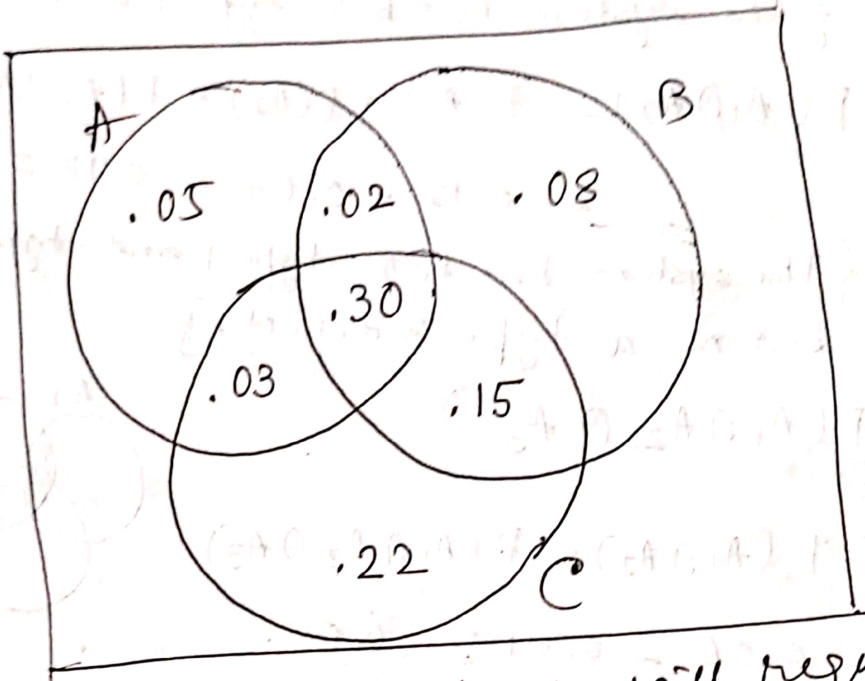
$P(AC) = .4 + 0.7 - 0.77 = 0.33$ and $P(BC) = .55 + 0.7 - 0.8 = 0.45$.

Finally, $P(ABC) = P(A+B+C) - P(A) - P(B) - P(C) + P(AB) + P(BC) + P(AC)$

$= 0.85 - 0.4 - 0.55 - 0.7 + 0.32 + 0.33 + 0.45 = 0.30$.

These probabilities are reflected in the Venn diagram, in the next page.





a. P { The next purchaser will request at least one of the three options }

= P(A ∪ B ∪ C) = 0.85.

b. P { The next purchaser will select none of the three options. }

= 1 − P { at least one selected } = 1 − P(A ∪ B ∪ C)

= 1 − 0.85 = 0.15.

c. From the Venn diagram,
P (only automatic transmission selected) = 0.22.

d. From the Venn diagram, P (exactly one of the three) = 0.05 + 0.08 + 0.22 = 0.35.

(26) Given $P(A_1) = 0.12$, $P(A_2) = 0.07$, $P(A_3) = 0.05$

$P(A_1 \cup A_2) = 0.13$, $P(A_1 \cup A_3) = 0.14$, $P(A_2 \cup A_3) = 0.10$,

~~$P(A_1 \cup A_2$~~ $P(A_1 \cap A_2 \cap A_3) = 0.01$

(a) P{The system does not have a type 1 defect.}

$= P(A_1') = 1 - P(A_1) = 1 - 0.12 = 0.88$.

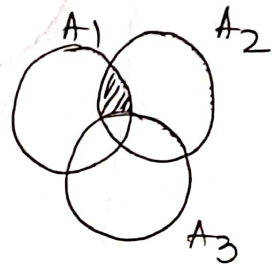
(b) P{The system has both type 1 and type 2 defects.}

$= P(A_1 \cap A_2) = P(A_1) + P(A_2) - P(A_1 \cup A_2)$

$= 0.12 + 0.07 - 0.13 = 0.06$.

(c) P{The system has both type 1 and type 2 defects but not a type 3 defect.}

$= P(A_1 \cap A_2 \cap A_3')$



$= P(A_1 \cap A_2) - P(A_1 \cap A_2 \cap A_3)$

$= 0.06 - 0.01 = 0.05$.

(d) P{The system has at most two of these defects.}

$= 1 - P(A_1 \cap A_2 \cap A_3)$

$= 1 - 0.01 = 0.99$.

(27) There are 10 equally likely outcomes: {A, B} {A, Co} {A, Cr} {A, f} {B, Co} {B, Cr} {B, f} {Co, Cr} {Co, f} and {Cr, f}.

(a) $P(\{A, B\}) = \frac{1}{10} = 0.1$. $\left[\frac{{}^2C_2}{{}^5C_2} = 0.1\right]$

(b) P{at least one C} = P({A, Co} or {A, Cr} or {B, Co} or {B, Co} or {B, Cr} or {Co, Cr} or {Co, f} or {Cr, f})

$= \frac{7}{10} = 0.7$. $\left[1 - \frac{{}^3C_2}{{}^5C_2}\right] = 1 - \frac{3}{10} = \frac{7}{10} = .7]$

(c) Replacing each person with his/her years of experience, P(at least 15 years)

$= P(\{3, 14\}$ or $\{6, 10\}$ or $\{6, 14\}$ or $\{7, 10\}$ or $\{7, 14\}$ or $\{10, 14\}) = \frac{6}{10} = 0.6$, $\left[\frac{{}^1C_1 \times {}^4C_1 + {}^1C_1 \times {}^2C_1}{{}^5C_2} = \frac{4+2}{10} = 0.6\right]$.

2.4 Conditional probability

In this section, we examine how the information "an event B has occurred" affects the prob. assigned to A.

Ex. 2.24 Complex components are assembled in a plant that uses two different assembly lines, A and A'. Line A uses older equipment than A', so it is somewhat slower and less reliable. Suppose on a given day line A has assembled 8 components, of which 2 ~~have~~ have been identified as defective (B) and 6 as nondefective (B'), whereas A' has produced 1 defective and 9 nondefective components. This information is summarized in the accompanying table.

Condition

		B	B'
Line	A	2	6
	A'	1	9

Unaware of this information, the sales manager randomly selects 1 of these 18 components for a demonstration. Prior to the demonstration

P(line A component selected)

$= P(A) = \frac{N(A)}{N} = \frac{8}{18} = 0.44$.

However, if the chosen component turns out to be defective then the event B has occurred, so the component must have been 1 of the 3 in the B column of the table. Since these 3 components are equally likely among themselves after B has ~~occurred~~ occurred,

$P(A|B) = \frac{2}{3} = \frac{2/18}{3/18} = \frac{P(A \cap B)}{P(B)}$.

The definition of conditional probability

For any two events A and B with $P(B) > 0$, the conditional probability of A given that B has occurred is defined by

$$P(A|B) = \frac{P(AB)}{P(B)}$$

Ex. 2.25 Suppose that of all individuals buying a certain digital camera, 60% include an optional memory card in their purchase, 40% include an extra battery, and 30% include both a card and battery. Consider randomly selecting a buyer and let A = {memory card purchased} and B = {battery purchased}. Then $P(A) = 0.6$, $P(B) = 0.4$, and $P(\text{both purchased}) = .30$.

Given that the selected individual purchased an extra battery, the probability that an optional card was also purchased is

$$P(A|B) = \frac{P(A \cap B)}{P(B)} = \frac{0.30}{0.40} = 0.75.$$

That is, of all those purchasing an extra battery, 75% purchased ~~a~~ an optional memory card. Similarly,

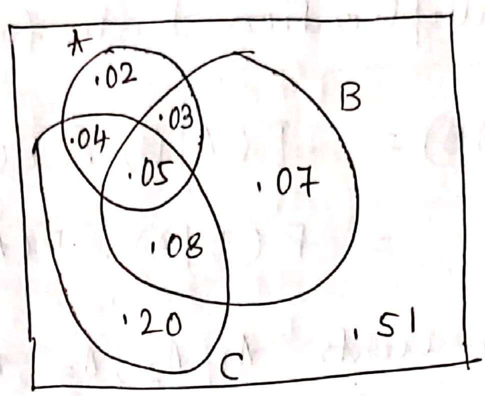
$$P(\text{battery} \mid \text{memory card}) = P(B|A) = \frac{P(B \cap A)}{P(A)} = \frac{0.3}{0.6} = 0.5$$

Notice that $P(A|B) \neq P(A)$ and $P(B|A) \neq P(B)$.

~~The event whose probability~~

Ex. 2.26 A news magazine publishes three columns entitled "Art" (A), "Books" (B), and "Cinema" (C). Reading habits of a randomly selected reader with respect to these columns are

Read regularly	A	B	C	$A \cap B$	$A \cap C$	$B \cap C$	$A \cap B \cap C$
Probability	.14	.23	.37	.08	.09	.13	.05



Venn diagram for ~~ex~~ Ex. 2.26

Consider the following four conditional probabilities:

(i) $P(A|B) = \dfrac{P(AB)}{P(B)} = \dfrac{.08}{.23} = .348$

(ii) The probability that the selected individual regularly reads the Art column given that he or she regularly reads at least one of the other two columns is

$$P(A|B \cup C) = \frac{P(A \cap (B \cup C))}{P(B \cup C)} = \frac{P(A \cap (B \cup C))}{P(B \cup C)}$$

$$= \frac{.04 + .05 + .03}{.47} = \frac{.12}{.47} = .255$$

(iii) $P(A|\text{ reads at least one}) = P(A|A \cup B \cup C)$

$$= \frac{P(A \cap (A \cup B \cup C))}{P(A \cup B \cup C)} = \frac{P(A)}{P(A \cup B \cup C)}$$

$$= \frac{.14}{.49} = .286.$$

(iv) The probability that the selected individuals reads at least one of the first two columns given that he or she reads the Cinema column is

$$P(A \cup B \mid C) = \frac{P((A \cup B) \cap C)}{P(C)}$$

$$= \frac{.04 + .05 + .08}{.37} = .459$$

The Multiplication Rule for $P(A \cap B)$

$$P(A \cap B) = P(A \mid B) \cdot P(B)$$

$$= P(B \mid A) \cdot P(A).$$

For three events A_1, A_2, A_3, the multiplication rule:

$$P(A_1 A_2 A_3) = P((A_1 A_2) A_3)$$

$$= P(A_3 \mid A_1 A_2) \; P(A_1 A_2)$$

$$= P(A_3 \mid A_1 A_2) \; P(A_1) \; P(A_2 \mid A_1)$$

$$= P(A_1) \; P(A_2 \mid A_1) \; P(A_3 \mid A_1 A_2)$$

Follow Ex. 2.27, Ex. 2.28

Ex. 2.29

An electronics store sells three different brands of DVD Players. Of its DVD player sales, 50 % are brand 1 (the least expensive), 30% are brand 2, and 20 % are brand 3. Each manufacturer offers a 1-year warranty on parts and labor. It is known that 25% of brand 1's DVD players require warranty repair work, whereas the corresponding percentages for brands 2 and 3 are 20% and 10% respectively.

1. What is the probability that a randomly selected purchaser has bought a brand 1 DVD player that will need repair while under warranty?
2. What is the probability that a randomly selected purchaser has a DVD player that will need repair while under warranty?
3. If a customer returns to the store with a DVD player that needs warranty repair work, what is the probability that it is a brand 1 DVD player? A brand 2 DVD player? A brand 3 DVD player?

Soln: First stage:

Let $A_i = \{$ brand i is purchased $\}$, $i=1,2,3$.

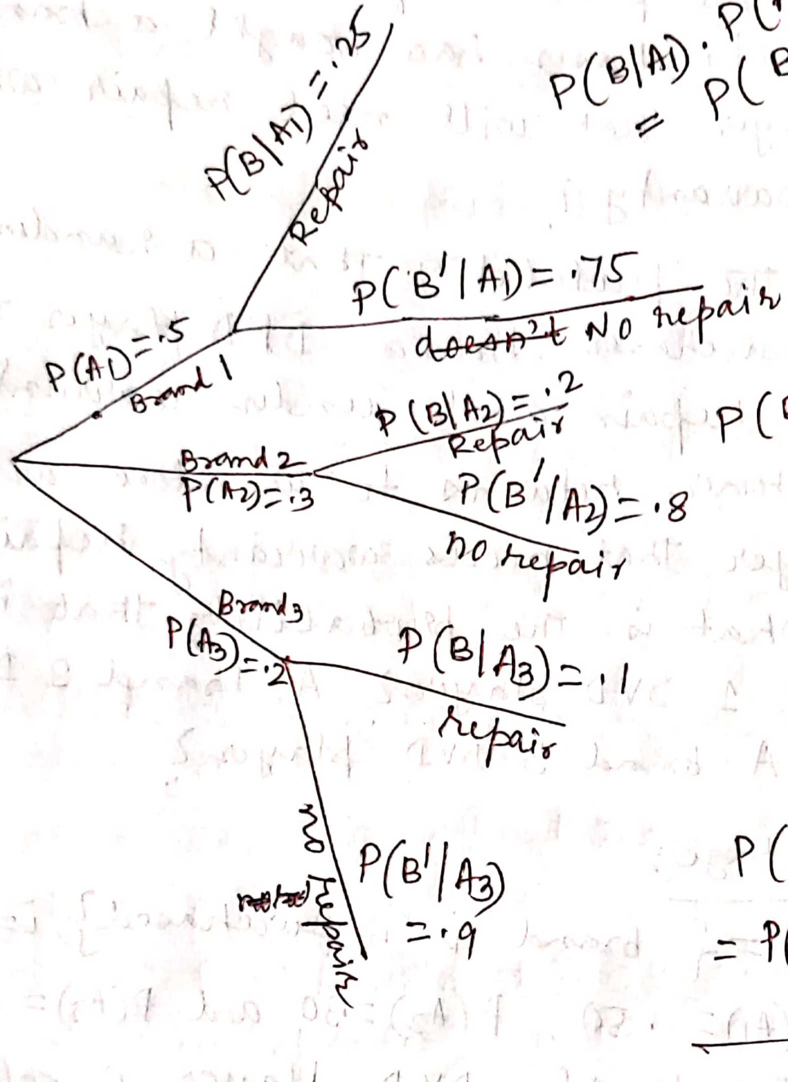
Then $P(A_1) = .50$, $P(A_2) = .30$ and $P(A_3) = .20$.

Once a brand of DVD player is selected, the 2nd stage involves observing whether the selected DVD player needs warranty repair. With $B = \{$ needs repair $\}$ and $B' = \{$ ~~needs~~ doesn't need repair $\}$, the given information implies that

$P(B|A_1) = .25$, $P(B|A_2) = .20$, $P(B|A_3) = .10$

The tree diagram representing this experimental situation is shown in the following figure.

The initial branches corresponds to different brands of DVD players; There are two second-generation branches emanating from the tip of each initial branch, one for "needs repair" and the other for "doesn't need repair."



$P(B|A_1) \cdot P(A_1) = P(B \cap A_1) = .125$

$P(B|A_2)\, P(A_2) = .06 = P(B \cap A_2)$

$P(B|A_3) \cdot P(A_3) = P(B \cap A_3) = .02$

$P(B) = .205$

Tree diagram for example 2.29

1. $P(A_1 \cap B) = P(B|A_1) \cdot P(A_1) = 0.125$

2. $P(B) = P$[(brand 1 and repair) or (brand 2 and repair) or (brand 3 and repair)]

$= P(A_1 \cap B) + P(A_2 \cap B) + P(A_3 \cap B)$

$= .125 + 0.060 + 0.020 = .205$

3. finally,

$$P(A_1|B) = \frac{P(A_1 \cap B)}{P(B)} = \frac{.125}{.205} = .61$$

$$P(A_2|B) = \frac{P(A_2 B)}{P(B)} = \frac{.060}{.205} = .29$$

and

$$P(A_3|B) = 1 - P(A_1|B) - P(A_2|B) = .10.$$

The initial or prior probability of brand 1 is .50. Once it is known that the selected DVD player needed repair, the posterior probability of brand 1 increases to .61.

Bayes' Theorem :

~~The second~~ The Law of Total Probability

Let $A_1, A_2, \ldots, A_K$ be mutually exclusive and exhaustive events. Then for any other event B,

$$P(B) = P(B|A_1)\,P(A_1) + \cdots + P(B|A_K)\,P(A_K)$$

$$= \sum_{i=1}^{K} P(B|A_i)\,P(A_i)$$

Proof: Since A_i's are mutually exclusive

$$A_i A_j = \phi, \quad i \neq j, \quad i,j = 1, \ldots, K.$$

Also A_i's are exhaustive.

$$\therefore A_1 + A_2 + \cdots + A_K = S.$$

$$\therefore B = B \cap S = B \cap (A_1 + A_2 + \cdots + A_K)$$

$$= B \cap A_1 + B \cap A_2 + \cdots + B \cap A_K$$

$BA_1, BA_2, \ldots, BA_K$ are again mutually exclusive, for

$(BA_i) \cap (BA_j) = B(A_i A_j) = B\phi = \phi$

$, i \neq j, \ i, j = 1, 2, \cdots K.$

$\therefore$ using 3rd axiom of axiomatic definition of probability

$$P(B) = P(BA_1) + P(BA_2) + \cdots + P(BA_K)$$

$$= P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + \cdots + P(B|A_K)P(A_K)$$

$$= \sum_{i=1}^{K} P(B|A_i) P(A_i) \quad \text{(Proved)}$$

Ex. 2.30

An individual has 3 different email accounts. Most of her messages, in fact 70%, come into account #1, whereas 20% come into account #2 and the remaining 10% into account #3. Of the messages into account #1, only 1% are spam, whereas the corresponding percentages for accounts #2 and #3 are 2% and 5% respectively. What is the probability that a randomly selected message is spam?

Ans: Let us first establish some notation:

$A_i = \{\text{message is from account } \# i\}$

for $i = 1, 2, 3$, $B = \{\text{message is spam}\}$.

Then the given percentages imply that

$P(A_1) = 0.70$, $P(A_2) = 0.20$, $P(A_3) = 0.10$

$P(B|A_1) = 0.01$, $P(B|A_2) = 0.02$, $P(B|A_3) = 0.05$

∴ from law of ~~total~~ total probability,

$$P(B) = P(B|A_1)P(A_1) + P(B|A_2)P(A_2) + P(B|A_3)P(A_3)$$

$$= 0.01 \times 0.70 + 0.02 \times 0.20 + 0.05 \times 0.10$$

$$= .007 + 0.004 + 0.005 = 0.016.$$

In the long run, 1.6 % of this individual's messages will be spam.

Bayes' Theorem

Let $A_1, A_2, \ldots, A_K$ be a collection of K mutually exclusive and exhaustive events with prior probabilities $P(A_i)$ $(i = 1, \ldots, K)$. Then for any other event B for which $P(B) > 0$, the posterior probability of A_j given that B has occurred is

$$P(A_j|B) = \frac{P(A_j \cap B)}{P(B)}$$

$$= \frac{P(B|A_j)\,P(A_j)}{\sum_{j=1}^{K} P(B|A_j)\,P(A_j)} \quad [\text{using multiplication rule}], \; j = 1, \ldots, K$$

Ex. 2.31 Incidence of a rare disease. Only 1 in 1000 adults is afflicted with a rare disease for which a diagonastic test has been developed. ~~Th~~ The test is such that when an individual actually has the disease, a positive result will occur 99% of the time, where as an individual without the disease, a positive will show a positive test result only 2% of the time. If a randomly selected individual is tested and the result is positive, what is the probability that the individual has the disease?

Ans: To use Bayes' theorem, let

A_1 = individual has the disease,

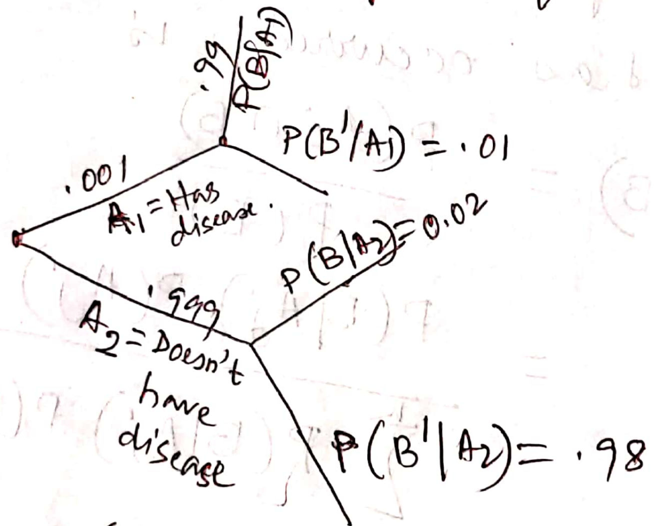
A_2 = individual doesn't have the disease,

and B = positive test result.

Then $P(A_1) = 0.001$, $P(A_2) = .999$,

$P(B|A_1) = .99$ and $P(B|A_2) = .02$.

The tree diagram for this problem:



$\therefore P(B) = P(B|A_1)P(A_1) + P(B|A_2)P(A_2)$

$= \cancel{.99 \times 0.001 + .02 \times .999}$

$= .99 \times .001 + .02 \times .999$

$= .00099 + .01998$

$= .02097$

Now $P(A_1|B) = \dfrac{P(B|A_1)\,P(A_1)}{P(B)}$

$= \dfrac{.00099}{.02097} = 0.047.$