Now $P(A_1|B) = \dfrac{P(B|A_1)\,P(A_1)}{P(B)}$

$= \dfrac{.00099}{.02097} = 0.047.$

Exercises Section 2.4

(45) The population of a particular country consists of three ethnic groups. Each individual belongs to one of the four major blood groups. The accompanying joint probability table gives the proportions of individuals in the various ethnic group - blood group combinations.

		Blood group			
		O	A	B	AB
Ethnic Group	1	.082	.106	.008	.004
	2	.135	.141	.018	.006
	3	.215	.200	.065	.020

Suppose that an individual is randomly selected from the population, and define events by $A = \{\text{type A selected}\}$, $B = \{\text{type B selected}\}$, and $C = \{\text{ethnic group 3 selected}\}$.

a. Calculate $P(A)$, $P(C)$, and $P(A \cap C)$.

b. Calculate both $P(A|C)$ and $P(C|A)$, and explain in context what each of these probabilities represents.

c. If the selected individual does not have type B blood, what is the probability that he or she is from ethnic group 1?

Ans: a. $P(A) = .106 + .141 + .200 = .447$,

$P(C) = .215 + .200 + .065 + .020 = .500$

and $P(A \cap C) = .200$

b. $P(A|C) = \frac{P(A \cap C)}{P(C)} = \frac{.200}{.500} = .400$.

If we know that the individual came from ethnic group 3, the probability that he has type A blood is .40.

$P(C|A) = \frac{P(C \cap A)}{P(A)} = \frac{.200}{.447} = .447$.

If a person has Type A blood, the probability that he or she is from ethnic group 3 is .447.

c. Define D = "ethnic group 1 selected." We are asked for $P(D|B')$. From the table, $P(D \cap B') = .082 + .106 + .004$

$= .192$ and $P(B') = 1 - P(B)$

$= 1 - [.008 + .018 + .065] = .909$.

So, the desired probability is

$P(D|B') = \frac{P(D \cap B')}{P(B')} = \frac{.192}{.909} = .211$.

47. Consider randomly selecting a student at a large university, and let A be the event that the selected student has a Visa card and B be the analogous event for MasterCard. Suppose that $P(A) = .6$ and $P(B) = .4$.

further, let C be the event that the selected student has an American Express card. Given that $P(A \cap B) = .3$, $P(C) = .2$, $P(A \cap C) = .15$, $P(B \cap C) = .1$, and $P(A \cap B \cap C) = .08$.

a. What is the prob^{ty} that the selected student has ~~both~~ at least one of the three types of cards?

b. What is the prob^{ty} that the selected student has both a Visa card and a MasterCard but not an American Express card?

c. Calculate and interpret $P(B|A)$ and also $P(A|B)$!

d. If we learn that the selected student has an American Express card, what is the prob^{ty} that she or he also has both a Visa card and a MasterCard?

e. Given that the selected student has an American Express card, what is the prob^{ty} that she or he has at least ~~of~~ one of the other two types of cards?

Ans: a. Apply the addition rule for three events:

$$P(A \cup B \cup C) = .6 + .4 + .2 - .3 - .15 - .1 + .08 = .73.$$

b. The event is $A \cap B \cap C'$.

$$\therefore P(A \cap B \cap C') = P(A \cap B) - P(A \cap B \cap C)$$

$= .3 - .08 = .22.$

c. $P(B|A) = \dfrac{P(A \cap B)}{P(A)} = \dfrac{.3}{.6} = .50$ and

$P(A|B) = \dfrac{P(A \cap B)}{P(B)} = \dfrac{.3}{.4} = .75.$

Half of the students with Visa cards also have a MasterCard, while three-quarters of students with a MasterCard also have a Visa card.

d. $P(A \cap B|C) = \dfrac{P([A \cap B] \cap C)}{P(C)}$

$= \dfrac{P(A \cap B \cap C)}{P(C)} = \dfrac{.08}{.2} = .40.$

e. $P(A \cup B|C) = \dfrac{P([A \cup B] \cap C)}{P(C)}$

$= \dfrac{P([A \cap C] \cup [B \cap C])}{P(C)}$ [Using Distributive Law]

$= \dfrac{P(A \cap C) + P(B \cap C) - P([A \cap C] \cap [B \cap C])}{P(C)}$

$= \dfrac{P(A \cap C) + P(B \cap C) - P(A \cap B \cap C)}{P(C)}$

$= \dfrac{.15 + .1 - .08}{.2} = .85.$

50) A department store sells sport shirts in three sizes (small, medium, and large), three patterns (plaid, print, and stripe), and two sleeve lengths (long and short). The accompanying tables give the proportions of shirts sold in the various category combinations.

Short sleeved

Size	Pattern Pl	Pr	St
S	.04	.02	.05
M	.08	.07	.12
L	.03	.07	.08

Long-sleeved

Size	Pattern Pl	Pr	St
S	.03	.02	.03
M	.10	.05	.07
L	.04	.02	.08

a. What is the probability that the next shirt sold is a medium, long-sleeved, print shirt?

b. What is the prob[ty] that the next shirt sold is a medium print shirt?

c. What is the probability that the next shirt sold is a short-sleeved shirt? A long-sleeved shirt?

d. What is the prob^{ty} that the size of the next shirt sold is medium? That the pattern of the next shirt sold is a print?

e. Given that the shirt just sold was a short-sleeved plaid, what is the prob^{ty} that its size was medium?

f. Given that the shirt just sold was a medium plaid, what is the prob^{ty} that it was short-sleeved? Long-sleeved?

Ans: Denote events

$M = \{$sold shirt is a medium$\}$;

$LS = \{$sold shirt is a long-sleeved shirt$\}$;

$SS = \{$sold shirt is a short-sleeved shirt$\}$;

$P_r = \{$sold shirt is a print shirt$\}$;

$P_L = \{$sold shirt is a plaid shirt$\}$;

(a) $P(M \cap LS \cap P_r) = .05$

(b) $P(M \cap P_r \cap (LS \cup SS))$

$= P(M \cap P_r \cap LS) + P(M \cap P_r \cap SS)$

$= 0.05 + 0.07 = .12$

c. $P(SS) = .04 + .08 + .03 + .02 + .07 + .07 + .05 + .12 + .08 = .56$

$\therefore P(LS) = 1 - .56 = .44$

[Using $P(A) + P(A') = 1$].

d. $P(M) = .08 + .07 + .12 + .10 + .05 + .07$

$= .49$

$P(P_R) = .02 + .07 + .07 + .02 + .05 + .02 = .25$

e. $P(M \mid \{SS \cap PL\})$

$= \frac{.08}{.04+.08+.03} = \frac{.08}{.15} = .533.$

f. $P(SS \mid \{M \cap PL\})$

$= \frac{.08}{.08+.10} = \frac{.08}{.18} = \frac{8}{18} = \frac{4}{9} = .444,$

54. In Exercise 13, $A_i = \{$awarded project $i\}$, for $i = 1, 2, 3$. Given That $P(A_1) = .22$, $P(A_2) = .25$, $P(A_3) = .28$, $P(A_1 \cap A_2) = .11$, $P(A_1 A_3) = .05$, $P(A_2 A_3) = .07$, $P(A_1 A_2 A_3) = .01$.

Compute the following probabilities and.

a. $P(A_2 \mid A_1)$, b. $P(A_2 A_3 \mid A_1)$

c. $P(A_2 + A_3 \mid A_1)$ d. $P(A_1 A_2 A_3 \mid A_1 + A_2 + A_3)$

Ans: a. $P(A_2 \mid A_1) = \frac{P(A_2 A_1)}{P(A_1)} = \frac{.11}{.22} = .5$

b. $P(A_2 A_3 \mid A_1) = \frac{P(A_2 A_3 A_1)}{P(A_1)}$

$= \frac{.01}{.22} = \frac{1}{22} = 0.0455.$

c. $P(A_2 + A_3 \mid A_1)$

$= \frac{P((A_2 + A_3) A_1)}{P(A_1)} = \frac{P(A_2 A_1 + A_3 A_1)}{P(A_1)}$

$= \frac{P(A_1 A_2) + P(A_1 A_3) - P(A_1 A_2 A_3)}{P(A_1)}$

$= \frac{.11 + .05 - .01}{.22} = \frac{15}{22} = 0.682.$

d. $P(A_1A_2A_3 \mid A_1+A_2+A_3)$

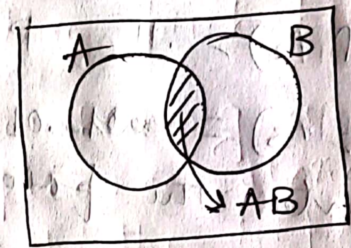
$$= \frac{P\big((A_1A_2A_3)\cap(A_1+A_2+A_3)\big)}{P(A_1+A_2+A_3)}$$

$$= \frac{P(A_1A_2A_3)}{P(A_1+A_2+A_3)} = \frac{.01}{.53} = 0.0189$$

56. For any events A and B with $P(B)>0$, show that

$$P(A\mid B) + P(A'\mid B) = 1$$

Proof:



$$A'\cap B = B - AB$$

Now, $B = (B-AB)\cup AB$

$\therefore P(B) = P(B-AB) + P(AB)$ [$\because$ $(B-AB)$ and AB are mutually exclusive]

$\therefore P(B-AB) = P(B) - P(AB)$ — (1)

Hence, $P(A\mid B) + P(A'\mid B)$

$$= \frac{P(AB)}{P(B)} + \frac{P(A'B)}{P(B)} \quad [\text{Given, } P(B)>0 \text{ given}]$$

$$= \frac{P(AB)+P(A'B)}{P(B)}$$

$$= \frac{P(\cancel{AB}) + P(B) - P(\cancel{AB})}{P(B)} \quad [\text{using (1)}]$$

$$= \frac{P(B)}{P(B)} = 1.$$

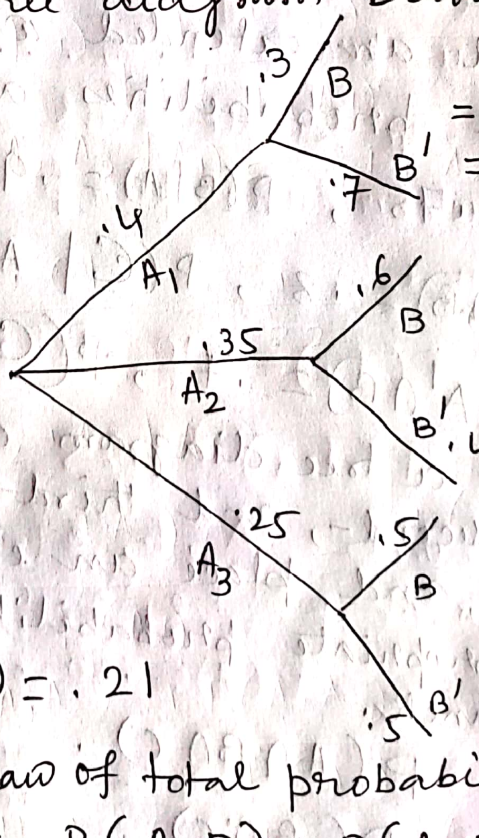
59. At a certain gas station, 40% of the customers use regular gas (A_1), 35% use plus gas (A_2), and 25% use premium (A_3). Of those customers using regular gas, only 30% fill their tanks (event B). Of those customers using plus, 60% fill their tanks, whereas of those using premium, 50% fill their ~~tanks~~ tanks.

a. What is the prob^ty that the next customer will request plus gas and fill the tank?

b. What is the probability that the next customer fill the tank?

c. If the next customer fills the tank, what is the probability that regular gas is requested? ~~Plus~~ Plus? Premium?

Ans: The required probabilities appear in the tree diagram below.



$P(A_1 \cap B) = \cancel{P(A_1)P(B|A_1)}$
$= P(A_1)\,P(B|A_1)$
$= .4 \times .3 = .12$

$P(A_2 B) = .35 \times .6$
$= .21$

$P(A_3 B) = .25 \times .5$
$= .125$

a. $P(A_2 B) = .21$

b. By the law of total probability,

$$P(B) = P(A_1 B) + P(A_2 B) + P(A_3 B)$$
$$= .12 + .21 + .125 = .455.$$

c. $P(A_1 | B) = \dfrac{P(A_1 B)}{P(B)}$

$= \dfrac{P(A_1)\, P(B | A_1)}{P(B)}$

$= \dfrac{.12}{.455} = .264$;

$P(A_2 | B) = \cancel{\dfrac{.21}{.264}}\ \dfrac{.21}{.455} = .462$;

$P(A_3 | B) = 1 - .264 - .462 = .274$.

Notice the three probabilities sum to 1.

63. For customers purchasing a refrigerator at a certain appliance store, let A be the event that the refrigerator was manufactured in the U.S., B be the event that the refrigerator had an icemaker, and C be the event that the customer purchased an extended warranty. Relevant probabilities are

$P(A) = .75$, $P(B|A) = .9$, $P(B|A') = .8$,
$P(C| A \cap B) = .8$, $P(C| A \cap B') = .6$,
$P(C| A' \cap B) = .7$, $P(C | A' \cap B') = .3$

a. Construct a tree diagram consisting of first-, second-, and third-generation branches, and place an event label and appropriate probability next to each branch.

b. Compute $P(A \cap B \cap C)$;

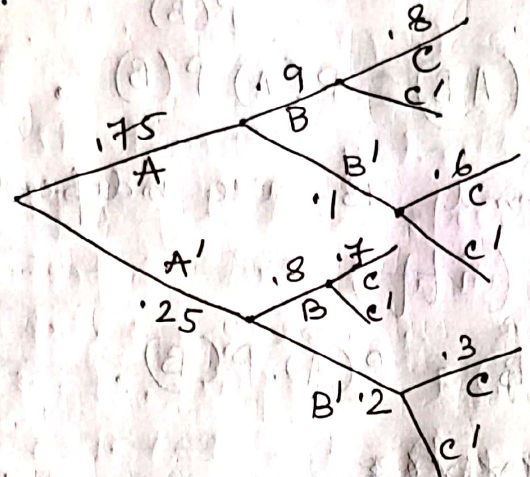
c. compute $P(B \cap C)$.

d. Compute $P(C)$.

(39)

e. Compute $P(A \mid B \cap C)$, the probability of a U.S. purchase given that an icemaker and extended warranty are also purchased.

Ans: a.



b. $P(A \cap B \cap C) = P(A)\, P(B \mid A)\, P(C \mid A \cap B)$

$= .75 \times .9 \times .8 = .54$

c. Event $B \cap C$ occurs twice on the diagram: $P(B \cap C) = P(A \cap B \cap C) + P(A' \cap B \cap C)$

$= .54 + (.25)(.8)(.7) = .68$

d. $P(C) = P(A \cap B \cap C) + P(A' \cap B \cap C) + P(A \cap B' \cap C)$

$+ P(A' \cap B' \cap C)$

$= .54 + .14 + (.75) \times (.1) \times (.6)$

$+ (.25)(.2)(.3)$

$= .54 + .14 + .045 + .015 = .74.$

e. $P(A \mid B \cap C) = \dfrac{P(A \cap B \cap C)}{P(B \cap C)}$

$= \dfrac{.54}{.68} = .7941.$

x

2.5 Independence

Def^n: Two events A and B are independent if $P(A|B) = P(A)$ and are dependent otherwise.

In otherwords, $P(A|B) = \frac{P(AB)}{P(B)} = P(A), P(B) > 0$.

i.e, $P(AB) = P(A)\,P(B)$.

Proposition: A and B are independent if and only if (iff)

$$P(A \cap B) = P(A) \cdot P(B).$$

~~Also,~~ ~~P(A'B)~~

Also, A and B are independent iff A' and B are independent, iff A and B' are independent, and iff A' and B' are independent.

$$\left[P(AB') = P(A - AB) = P(A) - P(AB) = P(A) - P(A)P(B) = P(A)(1 - P(B)) = P(A)\,P(B') \right] \leftarrow \text{Proof of 1st case.}$$

Ex. 2.32 Consider a gas station with six pumps numbered $1, 2, \ldots, 6$, and let E_i denote the simple event that a randomly selected customer uses pump i $(i = 1, \ldots, 6)$. Suppose that

$$P(E_1) = P(E_6) = .10, \; P(E_2) = P(E_5) = .15$$

$$P(E_3) = P(E_4) = .25$$

Define events A, B, C by

$$A = \{2, 4, 6\}, \; B = \{1, 2, 3\}, \; C = \{2, 3, 4, 5\}.$$

We then have $P(A) = .50$, $P(A|B) = \frac{P(A \cap B)}{P(B)}$

$$= \frac{P(E_2)}{.50} = \frac{.15}{.50} = \frac{30}{100} = .30.$$

$$P(A|C) = \frac{.40}{.80} = .50.$$

that is, events A and B are dependent, whereas events A and C are independent.

Ex. 2.33 Let A and B be any two mutually exclusive events with $P(A) > 0$. For example, for a randomly chosen automobile, let $A = \{$the car has a four cylinder engine$\}$ and $B = \{$the car has a six cylinder engine$\}$. Since the events are mutually exclusive, if B occurs, then A cannot possibly have occurred, so $P(A|B) = 0 \neq P(A)$. The message here is that if two events are mutually exclusive, they cannot be independent.

Ex. 2.34 It is known that 30% of a certain company's washing machines require service while under warranty, whereas only 10% of its dryers need such service. If someone purchases both a washer and a dryer made by this company, what is the prob.^{ty} that both machines will need warranty service?

Let A denote the event that the washer needs service while under warranty, and let B be defined analogously for the dryer. Then $P(A) = .3$ and $P(B) = .1$. Assuming that the two machines will function independently of one another, the desired probability is

$$P(A \cap B) = P(A) \cdot P(B) = (.3)(.1) = .03.$$

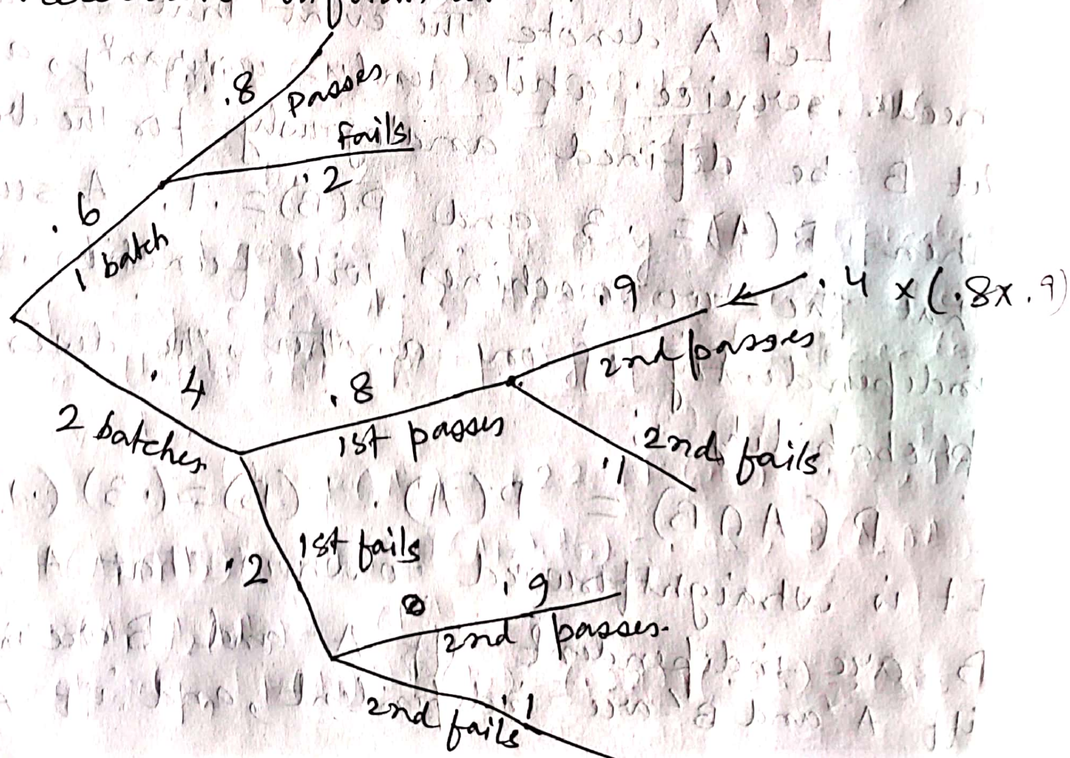
It is straightforward to show that A and B are independent iff A' and B are independent, iff A and B' are independent, and iff A' and

B' are independent. Thus in Ex. 2.34, the probability that neither machine needs service is

$$P(A' \cap B') = P(A') \cdot P(B')$$

$$= (.7)(.9) = .63.$$

Ex. 2.35 Each day, Monday through Friday, a batch of components sent by a first supplier arrives at a certain inspection facility. Two days a week, a batch also arrives from a second supplier. Eighty percent of all supplier 1's batches pass inspection and 90% of supplier 2's do likewise. What is the prob'ty that on a randomly selected day, two batches pass inspection? We will answer this assuming that on days when two batches are tested, whether the 1st batch passes is independent of whether the second batch does so. Figure, given below, displays the relevant information.



$$P(\text{two passes}) = P(\text{two received} \cap \text{both pass})$$
$$= P(\text{both pass} \mid \text{two received}) \cdot P(\text{two received})$$
$$= [(.8)(.9)](.4)$$
$$= .288.$$

Independence of more than two events:

Def^n: Events $A_1, A_2, \ldots, A_n$ are ~~mutually~~ mutually independent if for every k ($k = 2, 3, \ldots, n$) and every subset of indices $i_1, i_2, \ldots, i_k$,

$$P(A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}) = P(A_{i_1}) \cdot P(A_{i_2}) \cdots P(A_{i_k}).$$

Exercises Section 2.5

71. An oil exploration company currently has two active projects, one in Asia and other in Europe. Let A be the event that the Asian project is successful and B be the event that the European project is successful. Suppse that A and B are independent events with $P(A) = .4$ and $P(B) = .7$.

a. If the Asian ~~po~~ project is not successful, what is the probability that the European project is also not successful? Explain your reasoning.

b. What is the prob^{ty} that at least one of the two projects will be successful?

c. Given that at least one of the two projects is successful, what is the prob^{ty} that only the Asian project is successful?

Ans: (a) Since the events are independent, then A' and B' are independent, too.

Thus, $P(B'|A') = P(B') = 1-0.7 = 0.3$.

(b) Using the addition rule,

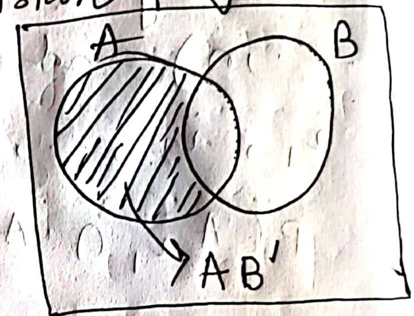
$$P(A\cup B) = P(A)+P(B)-P(A\cap B)$$

$$= P(A)+P(B)-P(A)P(B) \quad [\because A \text{ and } B \text{ are independent}]$$

$$= 0.4+0.7-(0.4)(0.7)$$

$$= 0.82$$

(c) The event "only the Asian project is successful" is AB'.



Therefore, the required probability is

$$P(AB' \mid A\cup B)$$

$$= \frac{P(AB' \cap (A\cup B))}{P(A\cup B)}$$

$$= \frac{P(AB')}{P(A\cup B)} = \frac{P(A)\,P(B')}{P(A\cup B)}$$

$$= \frac{(0.4)(1-0.7)}{0.82} = \frac{0.12}{0.82} = 0.146.$$

72. In Exercise 13, is any A_i independent of any other A_j? Answer using the multiplication property for independent events.

Ans: Given, $P(A_1) = .22$, $P(A_2) = .25$, $P(A_3) = .28$, $P(A_1\cap A_2) = .11$, $P(A_1\cap A_3) = .05$, $P(A_2\cap A_3) = .07$, $P(A_1\cap A_2\cap A_3) = .01$.

Clearly, $P(A_1).P(A_2) = (.22)(.25) = 0.055 \neq P(A$

$P(A_1)P(A_3) = (.22)(.28) = 0.0616 \neq P(A_1 \cap A_3)$;

$P(A_2)P(A_3) = (.25)(.28) = 0.07 = P(A_2 \cap A_3)$.

So, only A_2 and A_3 are independent of each other.

Events A_1 and A_2 are dependent.

Events A_1 and A_3 are dependent.

(78) A boiler has five identical relief valves. The probability that any particular valve will open on demand is .96. Assuming independent operation of the valves, calculate P(at least one valve opens) and P(at least one valve fails to open).

Ans: Let A be event "valve opens".

$\therefore$ A' is event "valve doesn't open".

$P(A) = .96$, $P(A') = .04$.

Let X be number of valves which open.

Let Y be number of valves which don't open.

The probability that at least one valve opens is

$$P(X \geq 1) = 1 - P(X = 0)$$
$$= 1 - P(Y = 5)$$
$$= 1 - (.04)^5 \approx 1$$

The probability that at least one valve fails to open is

$$P(Y \geq 1) = 1 - P(Y = 0) = 1 - P(X = 5)$$
$$= 1 - (.96)^5 = 1 - 0.8154$$
$$= 0.1846.$$

83. Components arriving at a distributor are checked for defects by two different inspectors (each component is checked by both inspectors). The first inspector detects 90% of all defectives that are present, and the second inspector does likewise. At least one inspector does not detect a defect on 20% of all defective components. What is the prob that the following occur?

a) A defective component will be detected only by the first inspector? By exactly one of the two inspectors?

b) All three defective components in a batch escape detection by both inspectors (assuming inspections of different components are independent of one another)?

Ans: ~~the first~~ P(both inspectors detect the defect) = 1 − P(at least one doesn't)

$= 1 - (.2) = 0.8$

(a) P(1st detects ∩ 2nd doesn't)

= P(1st detects) − P(1st does ∩ 2nd does)

$= .9 - .8 = .1$

Similarly, P(1st doesn't ∩ 2nd does)

$= .1$

∴ P(exactly one ~~do~~ does)

$= .1 + .1 = .2$

(b) P(neither detects a defect)

$= 1 - [P(\text{both do}) + P(\text{exactly 1 does})]$

$= 1 - [.8 + .2] = 0.$

That is, under this model there is a 0% probability neither inspector detects a defect. As a result,

P(all 3 escapes) $= (0)(0)(0) = 0.$

(80) Let $A = 1 \cup 2$ and $B = 3 \cap 5$.

$\therefore$ P(system works)

$= P(A \cup B)$ [$\because$ A and B are parallel.]

$= P(A) + P(B) - P(AB)$

$= P(A) + P(B) - P(A)P(B)$ [$\because$ both the systems A and B are independent of each other].

Now $P(A)$

$= P(1 \cup 2)$ [1 and 2 are in parallel connection]

$= P(1) + P(2) - P(1 \cap 2)$

$= P(1) + P(2) - P(1)P(2)$ [All components ~~equipments~~ are independent]

$= .9 + .9 - (.9)(.9)$

$= .99$

$P(B) = P(3 \cap 4)$ [3 & 4 are in series connection]

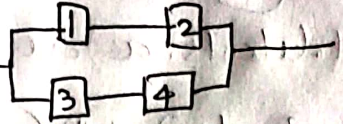
$= P(3)\,P(4)$ [3, 4 are independent]

$= .8 \times .8 = .64$

$\therefore$ P(system works) $= .99 + .64 - (.99)(.64)$

$= .9964$

(81) (48)



Let $P(A_i) = p$, $i = 1, 2, 3, 4$;

Let $x = p^2$ [Using hints.]

P(system lifetime exceeds t_0)

$= p^2 + p^2 - p^4 = 2p^2 - p^4 = 2x - x^2$.

Now, we set

$2x - x^2 = .99$

$\Rightarrow x^2 - 2x + .99 = 0$

$\Rightarrow x = 0.9$ or $1.1 \Rightarrow p = 0.9487$ or 1.049.

Since the value we want is a probability and cannot exceed 1, the correct answer is $p = 0.9487$.

(87) Given, $P(A_1) = .55$, $P(A_2) = .65$, $P(A_3) = .70$, $P(A_1 \cup A_2) = .8$, $P(A_2 \cap A_3) = .40$ and $P(A_1 \cup A_2 \cup A_3) = .88$.

(a) $P(A_1 \cap A_2) = P(A_1) + P(A_2) - P(A_1 \cup A_2)$ [from addition rule]

$= .55 + .65 - .8 = .4$

(b) By definition,

$$P(A_2 | A_3) = \frac{P(A_2 \cap A_3)}{P(A_3)} = \frac{.40}{.70} = .5714.$$

If a person likes vehicle #3, there's a 57.14% chance s/he will also like vehicle #2.

(c) No. from (b), $P(A_2 | A_3) = .5714 \neq P(A_2) = .65$. Therefore A_2 and A_3 are not independent. Alternatively, $P(A_2 \cap A_3) = .40 \neq P(A_2) P(A_3) = (.65)(.70) = .455$.

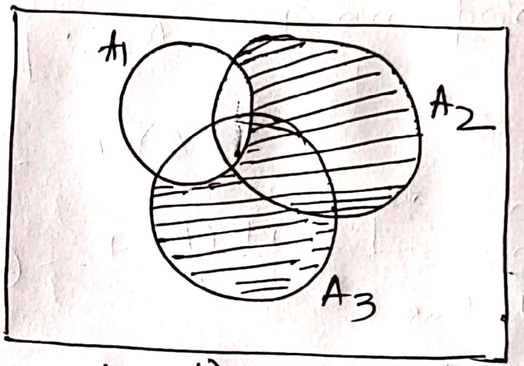
(d) The goal is to find $P(A_2 \cup A_3 \mid A_1')$, i.e.

$$\frac{P([A_2 \cup A_3] \cap A_1')}{P(A_1')}.$$

The denominator $P(A_1') = 1 - .55 = .45$

There are several ways to calculate the numerator; the simplest approach using the information provided is to draw a Venn diagram and observe that

$$P([A_2 \cup A_3] \cap A_1') = P(A_1 \cup A_2 \cup A_3) - P(A_1)$$
$$= .88 - .55 = .33.$$



Hence $P(A_2 \cup A_3 \mid A_1') = \dfrac{.33}{.45} = .7333.$

— x —