3.3 Expected values

The expected value of X

Def^n: Let X be a discrete r.v. with set of possible values D and pmf $p(x)$. The expected value or mean value of X, denoted by $E(X)$ or μ_X or just μ, is

$$E(X) = \mu_X = \sum_{x \in D} x \cdot p(x)$$

Ex. 3.16 Consider a random variable X whose probability distribution is given in the following table.

x	1	2	3	4	5	6	7
$p(x)$	.01	.03	.13	.25	.39	.17	.02

$$\mu = 1 \cdot p(1) + 2 \cdot p(2) + \cdots + 7 \cdot p(7)$$

$$= (1)(.01) + 2(.03) + \cdots + 7 \cdot (.02)$$

$$= .01 + .06 + .39 + 1.00 + 1.95 + 1.02 + .14$$

$$= 4.57.$$

Ex. 3.19, The general form for the pmf of X = number of children born up to and including the first boy is

$$p(x) = \begin{cases} p(1-p)^{x-1} & x = 1, 2, 3, \ldots \\ 0 & \text{otherwise} \end{cases}$$

$$E(X) = \sum_{D} x\, p(x) = \sum_{x=1}^{\infty} x\, p\, (1-p)^{x-1}$$

$$= p \sum_{x=1}^{\infty} \left[-\frac{d}{dp} (1-p)^{x} \right]$$

$$= -p \frac{d}{dp}\left[\sum_{x=1}^{\infty} (1-p)^x\right]$$

$$= -p \frac{d}{dp}\left[\frac{(1-p)}{1-(1-p)}\right], \quad 0<1-p<1$$

$$= -p \frac{d}{dp}\left(\frac{1}{p}-1\right) = -p \times -\frac{1}{p^2} = \frac{1}{p}.$$

If p is near 1, we expect to see a boy very soon, whereas if p is near 0, we expect many births before the 1st boy. for $p = .5$, $E(X) = 2$.

The expected value of a function:
Sometimes interest will focus on the expected value of some function $h(X)$ rather than on just $E(X)$.

Ex. 3.21 Suppose a bookstore purchases ten copies of a book at \$6.00 each to sell at \$12.00 with the understanding that at the end of a 3-month period any unsold copies can be redeemed for \$2.00. If X = the no. of copies sold, then net revenue

$= h(X) = 12X + (10-X)2 - 60 = 10X - 40.$

What then is the expected net revenue?

Defn. $$E[h(X)] = \sum_{x \in D} h(x)\, p(x)$$

Rules of expected value:

Th. $$E(aX+b) = aE(X)+b$$

(Or, using alternative notation, $\mu_{aX+b} = a\mu_X + b$.

Proof:

$$E(aX+b) = \sum_D (ax+b)\, p(x)$$

$$= a\sum_D x\, p(x) + b\sum_D p(x) = aE(X)+b.$$

Results: 1. For any constant a, $E(a) = a$, $E(aX) = aE(X)$

2. For any constant b, $E(X+b) = E(X)+b$.

The Variance of X:

Let X have pmf $p(x)$ and expected value μ. Then the <u>variance</u> of X, denoted by $V(X)$ or σ_X^2, or just σ^2, is

$$V(X) = \sum_D (x-\mu)^2 \cdot p(x)$$

$$= E\left[(X-\mu)^2\right] = \sigma_X^2.$$

The <u>standard deviation</u> (SD) of X is

$$\sigma_X = \sqrt{\sigma_X^2}.$$

The quantity $h(X) = (X-\mu)^2$ is the squared deviation of X from its mean, and σ^2 is the expected squared deviation – i.e., the weighted average of squared deviations, where the weights are probabilities from the distribution. If most of the prob.ty dis.n is close to μ, then σ^2 will be relatively small. However, if there are x values far from μ that have large $p(x)$, then σ^2 will be quite large. Very roughly, σ can be interpreted as the size of a representative deviation from the mean value μ. So if $\sigma = 10$, then in a long sequence of observed X values, some will deviate from μ by more than 10 while others will be closer to the mean than that – a typical deviation from the mean will be something on the order of 10.

Ex. 3.24

x	1	2	3	4	5	6
$p(x)$	.30	.25	.15	.05	.10	.15

$E(X) = \mu = 2.85$. The variance of X is then

$$V(X) = \sigma^2 = \sum_{x=1}^{6} (x - 2.85)^2 \cdot p(x)$$

$$= (1-2.85)^2(.30) + (2-2.85)^2(.25) + \cdots + (6-2.85)^2(.15)$$

$= 3.2275$.

The standard deviation of X is σ

$= \sqrt{3.2275} = 1.800$.

* Shortcut formulae for σ^2

Th. $V(X) = \sigma^2 = E(X^2) - (E(X))^2$

Proof: $V(X) = E[(X-\mu)^2]$

$= E[X^2 - 2\mu X + \mu^2]$

$= E(X^2) - 2\mu E(X) + E(\mu^2)$

$= E(X^2) - 2\mu \cdot \mu + \mu^2$

$= E(X^2) - 2\mu^2 + \mu^2$

$= E(X^2) - \mu^2 = E(X^2) - [E(X)]^2$.

Th. $V(X) = E[X(X-1)] - [E(X)][E(X) - 1]$

$= E[X(X-1)] - \mu(\mu - 1)$ $[\mu = E(X)]$.

Proof. $V(X) = E[(X-\mu)^2]$

$= E[X^2 - 2\mu X + \mu^2]$

$= E[X^2 - X + X - 2\mu X + \mu^2]$

$= E[X(X-1)] + E(X) - 2\mu E(X) + E(\mu^2)$

$$= E[X(X-1)] + \mu - 2\mu \cdot \mu + \mu^2$$

$$= E[X(X-1)] + \mu - 2\mu^2 + \mu^2$$

$$= E[X(X-1)] + \mu(1-\mu)$$

$$= E[X(X-1)] - \mu(\mu-1).$$

$$= E[X(X-1)] - E(X)[E(X)-1].$$

Rules of variance:

Th. $V(aX+b) = a^2 V(X) = a^2 \sigma^2_{aX+b}$

$$\sigma_{aX+b} = |a| \sigma_X.$$

Proof. $V(aX+b) = \sigma^2_{aX+b}$

$$= E[aX+b - E(aX+b)]^2$$

$$= E[aX+\cancel{b} - a\mu - \cancel{b}]^2$$

$$= E[a(X-\mu)]^2$$

$$= a^2 E[(X-\mu)^2]$$

$$= a^2 \sigma_X^2$$

$$\therefore \sigma^2_{aX+b} = a^2 \sigma_X^2$$

$$\therefore \sigma_{aX+b} = |a| \sigma_X \quad \left[\because \text{'a' may be -ve number also.} \; \therefore \sqrt{a^2} = |a| \right]$$

The absolute value is necessary because 'a' might be negative, yet a standard deviation cannot be.

Ex. 3.26 (you follow).

Exercises 3.3

29 a. $E(X) = \sum_{\text{all } x} x\,p(x) = 1(.05)+2(.10)+4(.35)+8(.40)+16(.10) = 6.45 \text{ GB} = \mu$

b. $V(X) = \sum_{\text{all } x} (x-\mu)^2 p(x)$

$= (1-6.45)^2(.05) + (2-6.45)^2(.10) + (4-6.45)^2(.35) + (8-6.45)^2(.40) + (16-6.45)^2(.10)$

$= 15.6475$

c. $\sigma = \sqrt{V(X)} = \sqrt{15.6475} = 3.956 \text{ GB}$.

d. $E(X^2) = \sum_{\text{all } x} x^2 p(x)$

$= (1)^2(.05) + 2^2(.10) + 4^2(.35) + 8^2(.40) + 16^2(.10)$

$= 57.25$,

Using the shortcut formula,

$V(X) = E(X^2) - \mu^2 = 57.25 - (6.45)^2$

$= 15.6475$

30 a. $E(Y) = 0(.6) + 1(.25) + 2(.1) + 3(.05)$

$= 0.6$

b. $E(100Y^2)$

$= 100 \sum_{\text{all } y} y^2 p(y)$

$= 100\left[0(.6) + 1(.25) + 2^2(.1) + 3^2(.05)\right]$

$= 110$.

34 $p(x) = \begin{cases} c/x^3, & x = 1, 2, 3, \ldots \\ 0, & \text{otherwise} \end{cases}$

$$E(x) = \sum_{\text{all } x} x\, p(x)$$

$$= c \sum_{x=1}^{\infty} \frac{1}{x^2} < \infty$$

because the infinite series

$$\sum_{x=1}^{\infty} \frac{1}{x^2}$$

converges (~~it~~ is a well known result).

So, $E(x)$ is finite.

38 $E(h(X)) = \sum_{x=1}^{6} h(x)\, p(x)$

$p(x) = \begin{cases} \frac{1}{6}, & x = 1, 2, 3, 4, 5, 6 \\ 0 & \text{otherwise} \end{cases}$

$$= \sum_{x=1}^{6} \frac{1}{x} \cdot \frac{1}{6}$$

$$= \frac{1}{6}\left[1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \frac{1}{5} + \frac{1}{6}\right]$$

$$= 0.408333 > 1/3.5$$

I would gamble.

$\boxed{E(1/X) \neq \frac{1}{E(X)} \text{ in general.}}$

39. $E(X) = 2.3$, $V(X) = E(X^2) - \mu^2$

$$= 6.1 - (2.3)^2$$

$$= .81.$$

Each lot weighs 5 lbs, so the number of lbs left $= 100 - 5X$.

Thus the expected weight left is $E(100 - 5X) = 100 - 5E(X)$

$$= 100 - 5 \times 2.3$$

$$= 88.5 \text{ lbs}$$

81

The variance of the weight left is

$V(100-5X) = 25\,V(X) = 20.25$

31 (Hints) From the table in Prob. 12,

$$E(Y) = 48.84$$

$$V(Y) = 4.4944$$

$$\sigma_Y = \sqrt{4.4944} = 2.12$$

One standard deviation from the mean value of Y gives 48.84 ± 2.12

$= 46.72$ to 50.96.

So, the probability Y is within one standard deviation of its mean value equals $P(46.72 < Y < 50.96)$

$= P(Y = 47, 48, 49, 50)$

$= 0.12 + 0.14 + 0.25 + 0.17 = 0.68$.

45 Given, $a \le X \le b$

ie, $a \le x \le b$

$\therefore\ a\,p(x) \le x\,p(x) \le b\,p(x)$ $[\because p(x) \ge 0]$

Taking sum over all x, we get

$$a\sum_{\text{all } x} p(x) \le \sum_{\text{all } x} x\,p(x) \le b\sum_{\text{all } x} p(x).$$

$\Rightarrow a \cdot 1 \le E(X) \le b \cdot 1$

$\Rightarrow a \le E(X) \le b.$

3.4 The Binomial probability distribution

~~There are many~~ There are many experiments that conform either exactly or ~~appropriately~~ approximately to the following list of requirements:

1. The experiment consists of a sequence of n smaller experiments called trials, where n is fixed in advance of the experiment.
2. Each trial can result in one of the same two possible outcomes, which we ~~generally~~ generically denote by success (S) and failure (F).
3. The trials are independent, so that the outcome on any particular trial does not influence the outcome on any other trial.
4. The ~~prob~~ probability of success $P(S)$ is constant from trial to trial; we denote this probability by p.

Defn. An experiment for which conditions 1 – 4 are satisfied is called a binomial experiment.

~~Defn:~~ The Binomial Random variable and Distribution.

Defn: The binomial random variable X associated with a binomial experiment consisting of n trials is defined as

$X =$ The number of S's among the n trials.