Chapter – 4: Continuous random variables and probability distribution

Introduction:

Chapter 3 concentrated on the development of probability distributions for discrete random variables. In this chapter, we consider the second general type of random variable that arises in many applied problems.

4.1 Probability density functions:

A random variable X is continuous if (1) possible values comprise either a sigle interval on the number line or a union of disjoint intervals, and
(2) $P(X = c) = 0$ for any number c that is a possible value of X.

Ex. 4.1 If in the study of the ecology of a lake, we make depth measurements at randomly chosen locations, then X = the depth at such a location is a continuous r.v. We can write $A < x < B$, where A is the minimum depth in the region being sampled, and B is the maximum depth.

Ex. 4.2 If a chemical compound is randomly selected and its pH X is determined, then X is a continuous r.v because any pH value between 0 and 14 is possible. If more is known about the compound selected for analysis, then the set of ~~values~~ possible values might be a subinterval of [0, 14], such as $5.5 \le x \le 6.5$, but X would still be continuous.

Ex. 4.3 follow.

Defn: Let X be a continuous r.v. Then a probability distribution or probability density function (pdf) of X is a function f(x) such that for any two numbers a and b with $a \le b$,

$$P(a \le X \le b) = \int_a^b f(x)\,dx.$$

For f(x) to be a legitimate pdf, it must satisfy the following two conditions:

1. $f(x) \geq 0$ for all x

2. $\int_{-\infty}^{\infty} f(x)\,dx = 1$

Ex. 4.4 One possible pdf for X is

$$f(x) = \begin{cases} \frac{1}{360}, & 0 \le x \le 360 \\ 0, & \text{otherwise.} \end{cases}$$

Here (1) clearly $f(x) \geq 0$ for all x

and (2) $\int_0^{360} f(x)\,dx = \frac{1}{360} |x|_0^{360} = \frac{360}{360} = 1.$

Defn. A continuous r.v X is said to have a uniform distribution on the interval [A, B] if the pdf of X is

$$f(x; A, B) = \begin{cases} \frac{1}{B-A}, & A \le x \le B \\ 0, & \text{otherwise} \end{cases}$$

$$P(X=c) = \int_c^c f(x)\,dx = \lim_{\epsilon \to 0} \int_{c-\epsilon}^{c+\epsilon} f(x)\,dx = 0.$$

Note:

$$P(a \le X \le b) = P(a < X < b) = P(a < X \le b)$$
$$= P(a \le X < b).$$

Exercises 4.1

1. The current in a certain circuit as measured by an ammeter is a continuous random variable X with the following density function:

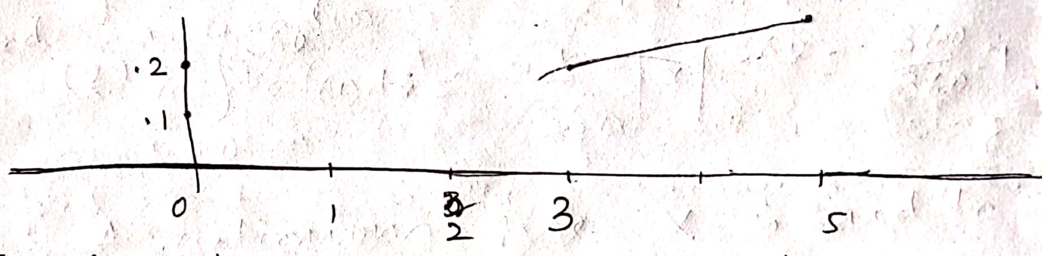
$$f(x) = \begin{cases} 0.075x + 0.2, & 3 \le x \le 5 \\ 0, & \text{otherwise} \end{cases}$$

a. Graph the pdf and verify that the total area under the density curve is indeed 1.

b. Calculate $P(X \le 4)$. How does this probability ~~con~~ compare to $P(X < 4)$?

c. Calculate $P(3.5 \le X \le 4.5)$ and also $P(4.5 < X)$.

Ans: a. The pdf is the straight-line function graphed below on $[3, 5]$.



The function is clearly non-negative. To verify its integral equals 1, compute:

$$\int_3^5 (0.075x + 0.2)\,dx = \left| 0.075\frac{x^2}{2} + 0.2x \right|_3^5$$

$$= \frac{0.075}{2}(25-9) + 0.2(5-3)$$

$$= 0.075 \times 8 + 0.4 = 0.6 + 0.4 = 1.00$$

b. $$P(X \le 4) = \int_{-\infty}^{4} f(x)\,dx = \int_{-\infty}^{3} f(x)\,dx + \int_3^4 f(x)\,dx$$

$$= 0 + \int_3^4 (0.075x + 0.2)\,dx$$

$= \left| 0.0375x^2 + .2x \right|_3^4$

$= 0.4625.$

Since X is a continuous r.v,

$P(X < 4) = P(X \leq 4) = 0.4625$ as well.

c. $P(3.5 \leq X \leq 4.5)$

$= \left| 0.0375x^2 + 0.2x \right|_{3.5}^{4.5}$

$= 0.5$

$P(4.5 < X)$

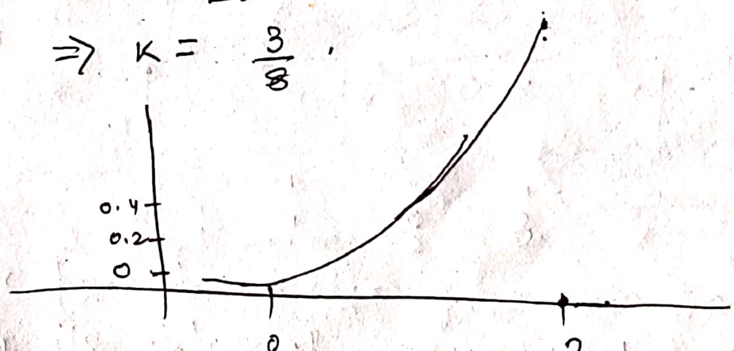
$= P(X \geq 4.5)$

$= \left| 0.0375x^2 + 0.2x \right|_{4.5}^{5} = 0.278125.$

(5) A college professor never finishes his lecture

Ans: a. $1 = \int_{-\infty}^{\infty} f(x)dx = \int_0^2 kx^2 dx = \left.\frac{kx^3}{3}\right]_0^2 = \frac{8k}{3}$

$\Rightarrow k = \frac{3}{8}.$



b. $P(0 \leq X \leq 1) = \int_0^1 \frac{3}{8}x^2 dx = \left.\frac{1}{8}x^3\right]_0^1 = 0.125$

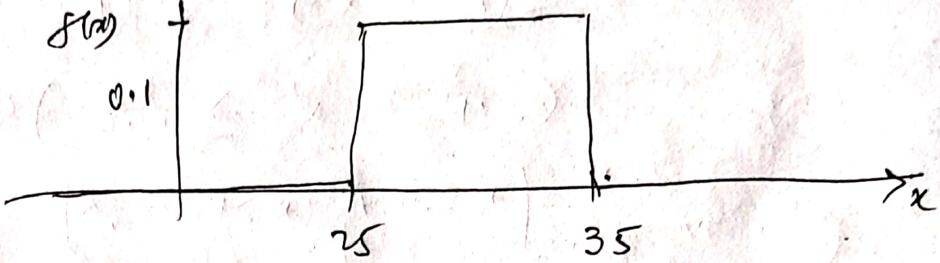
c. $P(1 \leq X \leq 1.5) = \int_1^{1.5} \frac{3}{8}x^2 dx = \left.\frac{x^3}{8}\right]_1^{1.5} = \frac{19}{64} = 0.297$

d. $P(X \geq 1.5) = \int_{1.5}^{2} \frac{3}{8}x^2 dx = \left.\frac{x^3}{8}\right]_{1.5}^{2} = 0.578.$

7. a. $f(x) = \frac{1}{B-A} = \frac{1}{35-25} = \frac{1}{10} = 0.1$

for $25 \le x \le 35$

and $= 0$ otherwise.



b. $P(x > 33) = \int_{33}^{35} \frac{1}{10}\, dx = \frac{2}{10}$ $= 0.02$

c. $\mu = \frac{35+25}{2} = \frac{60}{2} = 30.$

$\therefore$ P(that the preparation time is within 2 min of the mean time)

$= P(28 < x < 32) = \int_{28}^{32} \frac{1}{10}\, dx = \frac{4}{10}$ $= 0.4.$

d. $\int_{a}^{a+2} \frac{1}{10}\, dx = \frac{a+2-a}{10} = \frac{2}{10} = 0.2.$

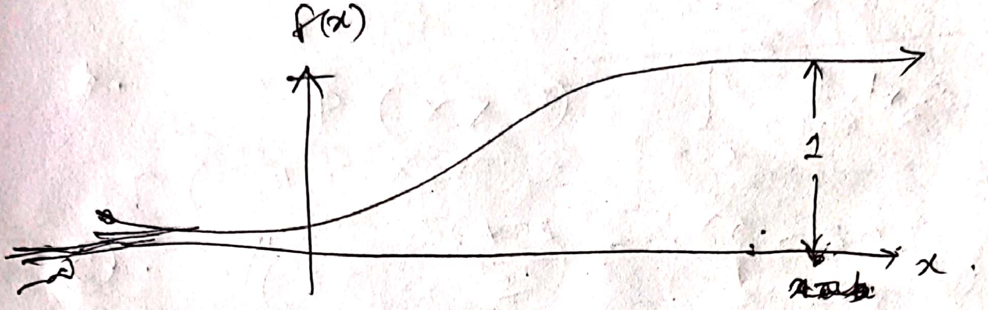
where, $25 < a < a+2 < 35.$

4.2 Cummulative distribution functions and expected values :

The Cumulative Distribution function :

Def^n: The cumulative distribution function (cdf) $F(x)$ for a continuous r.v X is defined for every number x by

$$F(x) = P(X \leq x) = \int_{-\infty}^{x} f(y)\,dy$$



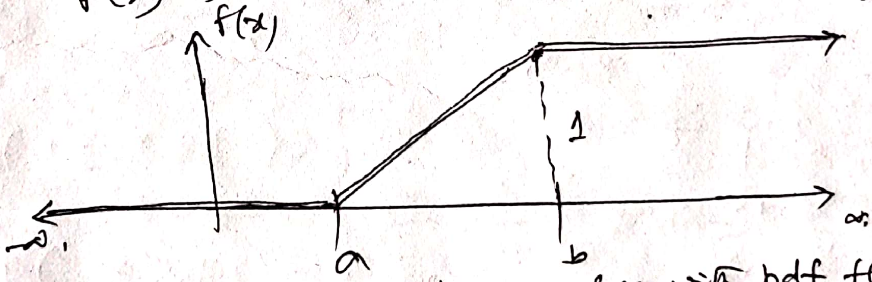
Ex. for uniform distribution

$$f(x) = \begin{cases} \frac{1}{b-a}, & a < x < b \\ 0, & \text{otherwise}, \end{cases}$$

$$F(x) = \begin{cases} 0, & x < a \\ \frac{x-a}{b-a}, & a \leq x \leq b \\ 1, & x \geq b. \end{cases}$$

$F(x)$ is a continuous function.

Also $F(x)$ is a ~~d~~ non-decreasing function.



Proposition: Let X be a continuous r.v with pdf $f(x)$ and cdf $F(X)$. Then for any number a,

$$P(X > a) = 1 - F(a)$$

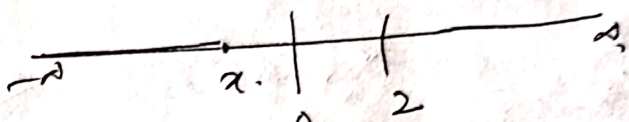
and for any two numbers a and b with $a < b$,

$$P(a \leq X \leq b) = F(b) - F(a).$$

Ex. 4.7. Suppose the pdf of the magnitude X of a dynamic load, on a bridge (in newtons) is given by

$$f(x) = \begin{cases} \frac{1}{8} + \frac{3}{8}x & 0 \le x \le 2 \\ 0 & \text{otherwise} \end{cases}$$

Thus for any number $x \le 0$,



$$F(x) = P(X \le x) = \int_{-\infty}^{x} 0 \, dx = 0,$$

For any number x between 0 and 2,

$$F(x) = \int_{-\infty}^{x} f(y)\,dy = \int_{0}^{x} \left(\frac{1}{8} + \frac{3}{8}y\right) dy$$

$$= \frac{x}{8} + \frac{3}{16}x^2.$$

For any number $x > 2$,

$$F(x) = \int_{-\infty}^{x} f(y)\,dy = F(2) + \int_{2}^{\infty} f(z)\,dz$$

$$= F(2) + 0 = \frac{2}{8} + \frac{3}{16} \cdot 4 = \frac{4}{4} = 1.$$

Thus,

$$F(x) = \begin{cases} 0 & x < 0 \\ \frac{x}{8} + \frac{3}{16}x^2 & 0 \le x \le 2 \\ 1 & x > 2 \end{cases}$$

$$P(1 \le X \le 1.5) = F(1.5) - F(1)$$

$$= \left[\frac{1}{8}(1.5) + \frac{3}{16}(1.5)^2\right] - \left[\frac{1}{8} + \frac{3}{16}(1)^2\right]$$

$$= 0.297.$$

$P(X>1) = 1 - P(X \le 1) = 1 - F(1)$

$= 1 - \left[\frac{1}{8}(1) + \frac{3}{16}(1)^2\right] = 0.688$

Obtaining $f(x)$ from $F(x)$:

Proposition: If X is a continuous rv with pdf $f(x)$ and cdf $F(x)$, then at every x at which $F'(x)$ exists, $F'(x) = f(x)$.

Ex. 4.8 When X has a uniform distribution, $F(x)$ is differentiable except at $x = A$ and $x = B$, where the graph $F(x)$ has sharp corners. Since $F(x) = 0$ for $x < A$ and $F(x) = 1$ for $x > B$, $F'(x) = 0 = f(x)$ for such x, for $A < x < B$,

$$F'(x) = \frac{d}{dx}\left(\frac{x-A}{B-A}\right) = \frac{1}{B-A} = f(x).$$

Defn A point $x = a$ is the median of a continuous rv, if

$$F(a) = \frac{1}{2}.$$

Expected values:

Def. The expected value or mean value of a continuous rv X with pdf $f(x)$ is

$$\mu_X = E(X) = \int_{-\infty}^{\infty} x f(x)\,dx.$$

Ex. 4.10 Let $f(x) = \begin{cases} \frac{3}{2}(1-x^2), & 0 \le x \le 1 \\ 0, & \text{Otherwise.} \end{cases}$

$$E(X) = \int_0^1 x \cdot \frac{3}{2}(1-x^2)\,dx = \int_0^1 \frac{3}{2}(x - x^3)\,dx$$

$$= \frac{3}{2}\left(\frac{x^2}{2} - \frac{x^4}{4}\right)\Big]_0^1 = \frac{3}{2}\left(\frac{1}{2} - \frac{1}{4}\right) = \frac{3}{8}.$$

<u>Proposition</u>: If X is a continuous rv with pdf $f(x)$ and $h(X)$ is any function of X, then

$$E(h(X)) = \mu_{h(X)} = \int_{-\infty}^{\infty} h(x) \cdot f(x)\,dx$$

<u>Corro.</u> If $h(x) = ax + b$, Then

$$E(ax+b) = a\,E(x) + b.$$

<u>Def^n</u> The <u>variance</u> of a continuous random variable X with pdf $f(x)$ and mean value μ is

$$\sigma_X^2 = V(x) = \int_{-\infty}^{\infty} (x-\mu)^2 \cdot f(x)\,dx$$

$$= E[(X-\mu)^2]$$

The standard deviation (SD) of X is $\sigma_X = \sqrt{V(X)}$ (>0).

The variance and standard deviation give quantitative measures of how much spread there is in the distribution or population of x values. Again σ is roughly the size of a typical deviation from μ.

<u>Proposition</u>: $V(X) = E(X^2) - (E(X))^2$.

<u>Ex. 4.12</u>
(Ex. 4.10 continued)

$$E(X) = \frac{3}{8}$$

$$E(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f(x)\,dx = \int_0^1 x^2 \cdot \frac{3}{2}(1-x^2)\,dx$$

$$= \int_0^1 \frac{3}{2}(x^2 - x^4)\,dx = \frac{1}{5}.$$

$$V(X) = \frac{1}{5} - \left(\frac{3}{8}\right)^2 = \frac{19}{320} = 0.059$$

and $\sigma_X = 0.244$.

When $h(x) = ax + b$,

$$E[h(X)] = aE(X) + b = a\mu + b$$

$$V[h(X)] = a^2 V(X) = a^2\sigma^2.$$

<u>Exercises (Section 4.2):</u>

11. a. $P(X \le 1) = F(1) = \frac{(1)^2}{4} = \frac{1}{4} = 0.25$

b. $P(.5 \le X \le 1) = F(1) - F(0.5)$

$$= \frac{1}{4} - \frac{(.5)^2}{4} = \frac{1}{4}(1 - 0.25) = \frac{1}{4}\cdot\frac{3}{4}$$
$$= \frac{3}{16} = 0.1875$$

c. $P(X > 1.5) = 1 - P(X \le 1.5)$

$$= 1 - F\left(\frac{3}{2}\right) = 1 - \frac{(3/2)^2}{4}$$

$$= \frac{7}{16} = 0.4375$$

d. To find median $\tilde{\mu}$, we solve:

$$F(\tilde{\mu}) = 0.5$$

$$\therefore \frac{\tilde{\mu}^2}{4} = 0.5 \qquad \boxed{0 \le \tilde{\mu} < 2}$$

$$\Rightarrow \tilde{\mu}^2 = 2 \Rightarrow \tilde{\mu} = \sqrt{2}\ (>0).$$
$$= 1.4142$$

e. $F'(x)$ to obtain the density function $f(x)$.

$$F'(x) = f(x) = \begin{cases} 0, & x \le 0, \\ \frac{x}{2} & 0 \le x < 2 \\ 0 & , x \ge 2 \end{cases}$$

$\cancel{f.\ E(X) = \int_0^2 x \cdot \frac{x^2}{2}\,dx = \left[\frac{x^4}{16}\right]_0^2 = 1}$

$\cancel{g.\ E(X^2) = \int_0^2 x^2 \cdot \frac{x^2}{4}\,dx = \frac{x^5}{20}\Big|_0^2 = \frac{32}{20} = \frac{8}{5} = 1.6}$

$\therefore V(x) = E(x^2) - (E(x))^2$

$= \frac{8}{5} - (1)^2 = \frac{3}{5}.$

$\therefore \sigma_x = \sqrt{\frac{3}{5}}$

h. $E(x^2) = 1.6$

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f. $E(x) = \int_{-\infty}^{\infty} x\, f(x)\, dx = \int_0^2 x \cdot \frac{x}{2}\, dx$

$= \frac{x^3}{6}\Big|_0^2 = \frac{8}{6} = \frac{4}{3} = 1.33.$

g. $E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx = \int_0^2 x^2 \cdot \frac{x}{2}\, dx$

$= \frac{1}{2} \cdot \frac{x^4}{4}\Big|_0^2 = \frac{1}{8} \cdot 16 = 2.$

$V(x) = E(x^2) - \mu^2$

$= 2 - \left(\frac{4}{3}\right)^2 = \frac{18-16}{9} = \frac{2}{9} = 0.222$

$\sigma_x = \sqrt{V(x)} = \sqrt{0.222} = 0.471.$

h. $E(x^2) = 2.$

12. a. $P(X < 0) = F(0) = \frac{1}{2} + \frac{3}{32}\left(4 \times 0 - \frac{(0)^3}{3}\right)$

$= \frac{1}{2}.$

b. $P(-1 < X < 1) = F(1) - F(-1)$

$= \left\{\frac{1}{2} + \frac{3}{32}\left(4 - \frac{1}{3}\right)\right\} - \left\{\frac{1}{2} + \frac{3}{32}\left(-4 + \frac{1}{3}\right)\right\}$

$= \frac{3}{32}\left[8 - \frac{2}{3}\right] = \frac{3}{32} \cdot \frac{22}{3} = \frac{11}{16}.$

c. $P(.5 < X) = 1 - P(X \le .5) = 1 - F(0.5).$

$$= 1 - \left[\frac{1}{2} + \frac{3}{32}\left(4 \times 0.5 - \frac{(0.5)^3}{3}\right)\right]$$

$$= 1 - \left[\frac{1}{2} + \frac{3}{32}\left(2 - \frac{0.125}{3}\right)\right]$$

$$= 1 - \left[\frac{1}{2} + \frac{\cancel{3}}{32} \times \frac{5.875}{\cancel{3}}\right]$$

$$= 1 - \frac{1}{2} - \frac{5.875}{32}$$

$$= \frac{1}{2} - 0.1836 = 0.5 - 0.1836 = 0.3164.$$

d. $$f(x) = F'(x) = \begin{cases} 0, & x < -2 \\ \frac{3}{32}(4-x^2), & -2 \le x \le 2 \\ 0, & x \ge 2 \end{cases}$$

e. We see that $f(0) = \frac{1}{2}$.

$\therefore \tilde{\mu} = 0$.

15. a. Since X is limit to the interval $(0,1)$, $F(x) = 0$ for $x \le 0$ and $F(x) = 1$ for $x \ge 1$.

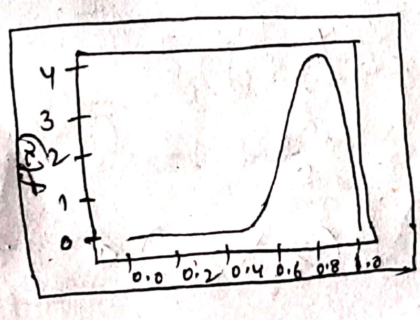
for $0 < x < 1$,

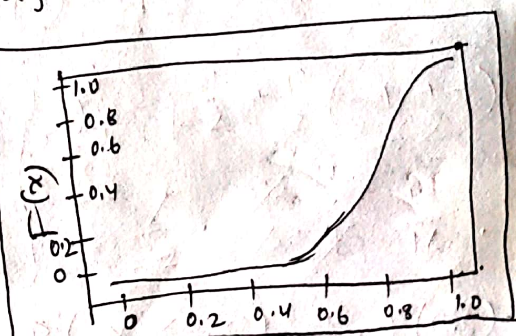
$$F(x) = \int_{\cancel{-\infty}\,0}^{x} f(y)\,dy$$

$$= \int_0^x 90\,y^8(1-y)\,dy = \int_0^x (90y^8 - 90y^9)\,dy$$

$$= 10y^9 - 9y^{10}\Big]_0^x = 10x^9 - 9x^{10}.$$

The graphs of the pdf and cdf appear below:





b. $F(0.5) = 10(.5)^9 - 9(0.5)^{10} = 0.0107.$

c. $P(0.25 \leq X \leq 0.5)$

$= F(0.5) - f(0.25)$

$= 0.0107 - [10(.25)^9 - 9(0.25)^{10}]$

$= 0.0107 - 0.0000 = 0.0107.$

d. The 75^{th} percentile is the value of x for which $f(x) = 0.75$:

$10x^9 - 9x^{10} = 0.75$

$\Rightarrow x = 0.9036$ (using software).

e. $E(x) = \int_{-\infty}^{\infty} x\, f(x)\, dx = \int_0^1 x \cdot 90x^8(1-x)\, dx$

$= \int_0^1 (90x^9 - 90x^{10})\, dx = \left[9x^{10} - \frac{90}{11} x^{11}\right]_0^1$

$= 9 - \frac{90}{11} = \frac{9}{11} = 0.8182.$

Similarly $E(x^2) = \int_{-\infty}^{\infty} x^2 f(x)\, dx$

$= \int_0^1 x^2 \cdot 90x^8(1-x)\, dx = 0.6818$, from which $V(x) = 0.6818 - (0.8182)^2 = 0.0124.$

$\sigma_x = 0.11134.$

f. $\mu \pm \sigma = (0.7068, .9295).$

Thus $P(\mu - \sigma \leq X \leq \mu + \sigma)$

$= F(0.9295) - F(0.7068)$

$= 0.8465 - 0.1602 = 0.6863$ and the probability X is more than 1 standard deviation from its mean value equals $1 - 0.6863 = 0.3137.$

21. $E(\text{area}) = E(\pi R^2)$

$= \int_{-\infty}^{\infty} \pi r^2 f(r)dr = \int_9^{11} \frac{3}{4}\pi r^2 (1 - (10-r)^2)dr$

$= \frac{501}{5}\pi = 314.79 \ m^2.$

23. $°F = 1.8 °C + 32.$
With X = temperature in °C, the temperature in °F equals $1.8X + 32$, so the mean and standard deviation in °F are
$1.8\mu_X + 32 = 1.8(120) + 32 = 248°f$ and
$|1.8|\sigma_X = 1.8(2) = 3.6 °f.$ ~~Notice~~

25. a. $P(Y \le 1.8\tilde{\mu} + 32) = P(1.8X + 32 \le 1.8\tilde{\mu} + 32)$
$= P(X \le \tilde{\mu}) = 0.5$ since $\tilde{\mu}$ is the median of X. This shows that $1.8\tilde{\mu} + 32$ is the median of Y.

b. The 90^{th} percentile for Y equals $1.8\eta(0.9)+32$, where $\eta(0.9)$ is the 90^{th} percentile for X. To see this, $P(Y \le 1.8\eta(.9) + 32) = P(1.8X + 32 \le 1.8\eta(.9) + 32) = P(X \le \eta(.9)) = .9$, since $\eta(.9)$ is the 90^{th} percentile of X. This shows that $1.8\eta(.9) + 32$ is the 90^{th} percentile of Y.

c. When $Y = aX + b$ (i.e., a linear transformation of X) and the $(100p)^{th}$ percentile of the X distribution is $\eta(p)$, then the corresponding $(100p)^{th}$ percentile of the Y distribution is $a\,\eta(p)+b$.

27. Since X is uniform on [0, 360],

$E(X) = \frac{0+360}{2} = 180^{\circ}$ and $\sigma_X = \frac{360-0}{\sqrt{12}}$

$= 103.82^{\circ}$. Using the suggested linear representation of Y, ~~E(Y)~~

$$E(Y) = \frac{2\pi}{360}\mu_X - \pi$$

$$= \frac{2\pi}{360}(180) - \pi = 0 \text{ radians},$$

and $\sigma_Y = \frac{2\pi}{360}\sigma_X = 1.814$ radians.