4.4 : The exponential and gamma distributions :

The exponential distribution :

The family of exponential distributions provides probability models that are very widely used in engineering and science disciplines.

Defn: X is said to have an exponential distn with parameter λ ($\lambda > 0$) if the pdf of X is

$$f(x) = \begin{cases} \lambda e^{-\lambda x}, & \text{if } x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$$E(X) = \int_0^\infty x \lambda e^{-\lambda x} dx = \frac{1}{\lambda},$$

$$V(X) = \frac{1}{\lambda^2}$$

CDF : $F(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-\lambda x} & x \geq 0 \end{cases}$

The Gamma distribution :

$$f(x; \alpha, \beta) = \begin{cases} \frac{1}{\beta^\alpha \Gamma(\alpha)} x^{\alpha-1} e^{-x/\beta}, & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

$, \alpha > 0, \beta > 0.$

The standard gamma distribution has $\beta = 1$, so the pdf of a standard gamma rv is given by

$$f(x; \alpha) = \begin{cases} \frac{e^{-x} x^{\alpha-1}}{\Gamma(\alpha)} & x \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

The mean and variance of a random variable X having the gamma distribution $f(x; \alpha, \beta)$ are

$$E(X) = \mu = \alpha\beta, \quad V(X) = \sigma^2 = \alpha\beta^2.$$

Prob. 59 (Hard copy):

Ans: The rv X has an exponential distribution with $\lambda = 1$.

$$\therefore f(x) = \begin{cases} e^{-x}, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

CDF:

$$F(x) = \begin{cases} 0, & x < 0 \\ 1 - e^{-x}, & x \geq 0. \end{cases}$$

(a) $E(X) = \frac{1}{\lambda} = \frac{1}{1} = 1.$

(b) $\sigma_X = \frac{1}{\lambda} = 1.$

(c) $P(X \leq 4) = F(4) = 1 - e^{-4} = 0.9817$

(d) $P(2 \leq X \leq 5) = (1 - e^{-5}) - (1 - e^{-2})$

$$= e^{-2} - e^{-5} = 0.9933 - 0.8647$$

$$= 0.1286.$$

60. (Hard copy & soft copy)

$$f(x) = \begin{cases} 0.01386\, e^{-0.01386x} & \text{if } x \geq 0, \\ 0 & \text{if } x < 0 \end{cases}$$

$$\therefore F(x) = \begin{cases} 0, & x \leq 0 \\ 1 - e^{-0.01386x}, & x > 0 \end{cases}$$

(a) $P(X \leq 100) = F(100) = 1 - e^{-0.01386 \times 100}$

$$= 0.7499$$

$P(X \leq 200) = 0.9377$

$P(100 \leq X \leq 200) = 0.1878$

$\mu = \frac{1}{\lambda} = 72.15, \quad \sigma = \frac{1}{\lambda} = 72.15$

(11)

67. $X \to$ lifetime of a transistor (in weeks)

X has gamma distribution with :

$\mu = 24$

$\sigma = 12$

As $\mu = \alpha\beta = 24$ —— ①.

$\sigma^2 = \alpha\beta^2 = (12)^2 = 144$ — ②.

, $\alpha > 0$, $\beta > 0$.

$\therefore \beta = \frac{\alpha\beta^2}{\alpha\beta} = \frac{\cancel{144}^{\cancel{12}\,6}}{\cancel{24}_{2}} = 6.$

$\therefore \alpha = 4.$

Part ⓐ $P(12 < X < 24) = 0.424$

Part ⓑ $P(X \leq 24) = 0.567.$

Part ⓒ $P(X \leq \eta_{0.99}) = 0.99.$

It can be found that $\eta_{0.99} = 60.$

Chapter 5: Joint probability distributions and random samples.

5.1 Jointly distributed random variables:

Two discrete random variables:

The Joint probability mass function

$$p(x, y) = P(X = x \text{ and } Y = y)$$

It must be the case that $p(x,y) \geq 0$

and

$$\sum_x \sum_y p(x,y) = 1.$$

Now let A be any set consisting of pairs of (x, y) values (e.g., $A = \{(x,y) : x+y = 5\}$ or $\{(x,y) : \max(x,y) \leq 3\}$). Then the probability $P[(X,Y) \in A]$ is obtained by summing the joint pmf over pairs in A:

$$P[(X,Y) \in A] = \sum_{(x,y) \in A} \sum p(x,y).$$

Ex. 5.1 A Large insurance agency services a number of customers who have purchased both a homeowner's ~~poly~~ policy and an automobile policy from the agency. For each type of policy, a deductable amount must be specified. For an automobile policy, the choices are \$100, and \$250, whereas for a homeowner's policy, the choices are 0, \$100, and \$200. Suppose an individual with both types of policy is selected at random from the agency's files.

Let X = The deductable amount on the auto policy. and Y = The deductable amount on the home owner's policy. Possible (X, Y) pairs are then $(100, 0)$, $(100, 100)$, $(100, 200)$, $(250, 0)$, $(250, 100)$ and $(250, 200)$; the joint pmf specifies the probability associated with each one of these pairs, with any other pair having probability zero. suppose the joint pmf is given in the accompanying joint probability table:

		y	
$p(x,y)$	0	100	200
100	.20	.10	.20
x 250	.05	.15	.30

Then $p(100,100) = P(X=100 \text{ and } Y=100)$

$= P(\$100 \text{ deductable on both policies}) = .10.$

The probability $P(Y \geq 100)$ is computed by summing probabilities of all (x,y) pairs for which $y \geq 100$:

$$P(Y \geq 100) = p(100,100) + p(250,100)$$
$$+ p(100,200) + p(250,200).$$
$$= .10 + .20 + .15 + 0.30$$
$$= 0.75.$$

Once the joint pmf of these two variables X and Y is available, it is in principle straight-forward to obtain the distribution of just one of these variables. As an example, let X and Y be the number of statistics and mathematics courses, respectively, currently being taken by a randomly slected statistics major. Suppose that we wish the distribution of X, and that when X = 2, the only possible values of Y are 0, 1, and 2. Then

$$p_X(2) = P(X=2) = P[(X,Y) = (2,0) \text{ or } (2,1) \text{ or } (2,2)]$$

$$= p(2,0) + p(2,1) + p(2,2).$$

Defn: The marginal probability mass function of X, denoted by $p_X(x)$, is given by

$$p_X(x) = \sum_{y:\, p(x,y)>0} p(x,y) \quad \text{for each possible value } x.$$

Similarly, the marginal probability mass function of Y, is

$$p_Y(y) = \sum_{x:\, p(x,y)>0} p(x,y) \quad \text{for each possible value } y.$$

~~Ex-5.2~~ ~~[illegible]~~

Two continuous random variables:

The probability that the pair (X, Y) of continuous rv's falls in a two-dimensional set A (such as a rectangle) is obtained by integrating a function called the joint density function.

Defn: Let X and Y be continuous rv's. A joint prob^{ty}. density fun^n $f(x,y)$ for these two variables is a function satisfying $f(x,y) \geq 0$ and $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\,dx\,dy = 1$.

Then for any two dimensional set A

$$P[(X, Y) \in A] = \iint_A f(x,y)\, dx\, dy$$

If, in particular, $A = \{(x,y) : a \le x \le b,\ c \le y \le d\}$, then

$$P[(X, Y) \in A] = P(a \le X \le b,\ c \le Y \le d)$$

$$= \int_a^b \int_c^d f(x,y)\, dy\, dx$$

Ex. 5.3

A bank operates both a drive-up facility and a walk-up window. On a randomly selected day, let X = the proportion of time that the drive-up facility is in use (at least one customer is being served or waiting to be served) and Y = the proportion of time that the walk-up window is in use. Then the set of possible values for (X, Y) is the rectangle

$$D = \{(x, y) : 0 \le x \le 1,\ 0 \le y \le 1\}.$$

Suppose the joint pdf of (X, Y) is given by

$$f(x,y) = \begin{cases} \frac{6}{5}(x+y^2) & 0 \le x \le 1,\ 0 \le y \le 1, \\ 0 & \text{otherwise}. \end{cases}$$

To verify that this is a legitimate pdf, note that $f(x,y) \ge 0$ and

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y)\, dx\, dy = \int_0^1\int_0^1 \frac{6}{5}(x+y^2)\, dx\, dy$$

$$= \int_0^1\int_0^1 \frac{6}{5} x\, dx\, dy + \int_0^1\int_0^1 \frac{6}{5} y^2\, dx\, dy$$

$$= \int_0^1 \frac{6}{5} x\, dx + \int_0^1 \frac{6}{5} y^2\, dy = \frac{6}{10} + \frac{6}{15} = 1$$

The probability that neither facility is busy more than one-quater of the time is

$$P(0 \le X \le \tfrac{1}{4},\ 0 \le Y \le \tfrac{1}{4})$$

$$= \int_0^{\frac{1}{4}} \int_0^{1/4} \frac{6}{5}(x+y^2)\,dx\,dy.$$

$$= \frac{6}{5}\int_0^{1/4}\int_0^{1/4} x\,dx\,dy + \frac{6}{5}\int_0^{1/4}\int_0^{1/4} y^2\,dx\,dy$$

$$= \frac{6}{20}\int_0^{1/4} x\,dx + \frac{6}{20}\int_0^{1/4} y^2\,dy$$

$$= \frac{6}{20}\cdot \frac{x^2}{2}\Big|_{x=0}^{x=1/4} + \frac{6}{20}\cdot\frac{y^3}{3}\Big|_{y=0}^{y=1/4} = \frac{7}{640} = 0{,}0109.$$

The marginal pdf of each variable can be obtained in a manner analogous to what we did in the case of two discrete variables.

Defn. The marginal probability density functions of X and Y, denoted by $f_X(x)$ and $f_Y(y)$, respectively, are given by

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy, \quad -\infty < x < \infty$$

$$f_Y(y) = \int_{-\infty}^{\infty} f(x,y)\,dx, \quad -\infty < y < \infty$$

Ex. 5.4 (Example 5.3 continued) The marginal pdf of X, which gives the probability distribution of busy time for the drive-up facility without reference to the walk-up window, is

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \begin{cases} \int_0^1 \frac{6}{5}(x+y^2)\,dy & , 0\le x\le 1 \\ 0 & \text{otherwise} \end{cases}$$

$$= \begin{cases} \frac{6}{5}x + \frac{2}{5} & , 0\le x\le 1 \\ 0 & , \text{otherwise}. \end{cases}$$

The marginal pdf of Y is

$$f_Y(y) = \begin{cases} \frac{6}{5}y^2 + \frac{3}{5} & , 0\le y\le 1 \\ 0 & , \text{otherwise} \end{cases}$$

Then

$$P\left(\tfrac{1}{4}\le Y\le \tfrac{3}{4}\right) = \int_{1/4}^{3/4} f_Y(y)\,dy$$

$$= \int_{1/4}^{3/4}\left(\frac{6}{5}y^2 + 3/5\right)dy = \frac{\cancel{6}^{2}}{5}\cdot\frac{y^3}{\cancel{3}}\Big|_{1/4}^{3/4} + \frac{3}{5}\cdot y\Big|_{1/4}^{3/4}$$

$$= \frac{\cancel{2}}{5}\cdot\frac{\cancel{26}^{13}}{\cancel{64}_{\cancel{32}\,16}} + \frac{3}{5}\cdot\frac{1}{2} = \frac{13}{80} + \frac{24}{80} = \frac{37}{80} = 0.4625$$

Ans.

5.5 A nut company markets cans of deluxe mixed nuts containing almonds, cashews, and peanuts. Suppose the net weight of each can is exactly 1 lb, but the weight contribution of each type of nut is random. Because the three weights sum to 1, a joint probability model for any two gives all necessary information about the weight of the third type. Let X = the weight of almonds in a selected can and Y = the weight of cashews. Then the region of positive density is $D = \{(x,y) : 0\le x\le 1, 0\le y\le 1, x+y\le 1\}$, the shaded region pictured in the figure-1, given below.

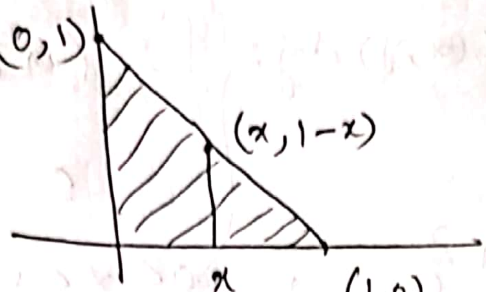

Fig. 1: Region of positive density for example 5.5.

Now let the joint pdf for (X, Y) be

$$f(x,y) = \begin{cases} 24xy, & 0 \le x \le 1,\ 0 \le y \le 1,\ x+y \le 1 \\ 0 & \text{otherwise} \end{cases}$$

Clearly $f(x,y) \geq 0$. Also

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x,y) = \iint_D f(x,y)\,dx\,dy$$

$$= \int_0^1 \left\{ \int_0^{1-x} 24xy\,dy \right\} dx$$

$$= \int_0^1 24x \left\{ \frac{y^2}{2}\Big|_{y=0}^{y=1-x} \right\} dx$$

$$= \int_0^1 12x\,(1-x^2)\,dx = 1.$$

Hence, the 2nd condition on a joint pdf is verified.

To compute the probability that the two types of nuts together make up at most 50% of the can, let $A = \{(x,y) : 0 \le x \le 1,\ 0 \le y \le 1,$ and $x+y \le 0.5\}$.

$$P[(X,Y) \in A] = \iint_A f(x,y)\,dx\,dy = \int_0^{0.5}\int_0^{0.5-x} 24xy\,dx\,dy$$

$$= 0.0625.$$

The marginal pdf for almonds is obtained by holding X fixed at x and integrating the joint pdf $f(x,y)$ along the vertical line through x:

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)\,dy = \begin{cases} \int_0^{1-x} 24xy\,dy = 12x(1-x)^2, & 0 \le x \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

By symmetry of $f(x,y)$ and the region D, the marginal pdf of Y is obtained by replacing x and X in $f_X(x)$ by y and Y, respectively.

<u>Independent random variables</u>:

<u>Defn</u> Two random variables X and Y are said to be independent if for every pair of x and y values

$$p(x,y) = p_X(x) \cdot p_Y(y) \quad \text{when X and Y are discrete}$$

or

$$f(x,y) = f_X(x) \cdot f_Y(y) \quad \text{when X and Y are continuous}$$

If the above conditions are not satisfied for all (x,y), then X and Y are said to be dependent.

<u>Conditional distributions</u>:

<u>Defn</u>. Let X and Y be two continuous rv's with joint pdf $f(x,y)$ and marginal X pdf $f_X(x)$. Then for any X value x for which $f_X(x) > 0$ the conditional probability density function of Y given that $X = x$ is

$$f_{Y|X}(y|x) = \frac{f(x,y)}{f_X(x)}, \quad -\infty < y < \infty$$

If X and Y are discrete, replacing pdf's by pmf's in this definition gives the conditional probability mass function of Y when X = x.

Ex. 5.12: Reconsider the situation of Examples 5.3 and 5.4 involving X = the proportion of time that a bank's drive-up facility is busy and Y = the analogous proportion for the walk-up window. The conditional pdf of Y given that X = 0.8 is

$$f_{Y|X}(y|0.8) = \frac{f(0.8, y)}{f_X(0.8)} = \frac{1.2(.8 + y^2)}{1.2(0.8) + 0.4}$$

$$= \frac{1}{34}(24 + 30y^2), \quad 0 < y < 1.$$

The probability that the walk-up facility is busy at most half the time given that X = .8 is then

$$P(Y \le 0.5 \mid X = 0.8) = \int_{-\infty}^{0.5} f_{Y|X}(y|.8)\,dy$$

$$= \int_0^{0.5} \frac{1}{34}(24 + 30y^2)\,dy = 0.390$$

Also

$$P(Y \le 0.5) = 0.350.$$

$$E(Y) = 0.6$$

The expected proportion of time that the walk-up facility is busy given that X = 0.8 (a conditional expectation) is

$$E(Y|X=.8) = \int_{-\infty}^{\infty} y \cdot f_{Y|X}(y|.8)\,dy = \frac{1}{34}\int_0^1 y(24+30y^2)\,dy$$
$$= 0.574.$$