## 3.6 The Poisson probability Distribution

**Defn:** A discrete random variable X is said to have a Poisson distribution with parameter $\mu$ ($\mu > 0$) if the pmf of X is

$$p(x;\mu) = \frac{e^{-\mu}\mu^{x}}{x!}, \quad x = 0, 1, 2, 3, \ldots$$

Here $p(x) \geq 0$ for all $x$ and

$$\sum_{x=0}^{\infty} p(x) = \sum_{x=0}^{\infty} \frac{e^{-\mu}\mu^{x}}{x!} = e^{-\mu}\sum_{x=0}^{\infty}\frac{\mu^{x}}{x!}$$

$$= e^{-\mu}\left[1 + \frac{\mu}{1!} + \frac{\mu^2}{2!} + \cdots\right]$$

$$= e^{-\mu} \cdot e^{\mu} = e^{0} = 1.$$

**Ex. 3.39** Let X denote the number of creatures of a particular type captured in a trap during a given time period. Suppose X has a Poisson distribution with $\mu = 4.5$, so on ~~ab~~ average traps will contain 4.5 creatures. The probability that a trap contains exactly 5 creatures is

$$P(X = 5) = \frac{e^{-4.5}(4.5)^{5}}{5!} = 0.1708.$$

The probability that a trap has at most five creatures is

$$P(X \le 5) = \sum_{x=0}^{5} e^{-4.5} \frac{(4.5)^x}{x!}$$

$$= e^{-4.5}\left[1 + 4.5 + \frac{(4.5)^2}{2!} + \frac{(4.5)^3}{3!} + \frac{(4.5)^4}{4!} + \frac{(4.5)^5}{5!}\right]$$

$$= 0.7029.$$

The Poisson distribution as a limit:

Proposition: Suppose that in a binomial pmf $b(x; n, p)$, we let $n \to \infty$ and $p \to 0$ in such a way that $np$ approaches a value $\mu > 0$. Then $b(x; n, p) \to p(x; \mu)$.

According to this proposition, in any binomial experiment in which $n$ is large and $p$ is small,

$$b(x; n, p) \approx p(x; \mu), \text{ where } \mu = np.$$

As a rule of thumb, this approximation can safely be applied if $n > 50$ and $np < 5$.

Ex. 3.40 If a publisher of nontechnical books takes great pains to ensure that its books are free of typographical errors, so that the prob.ty of any given page containing at least one such error is .005 and errors are independent from page to page, what is the probability that one of its 400-page novels will contain exactly one page with errors? At most three pages with errors?

With S denoting a page containing at least one error and F an error-free page, the number X of pages containing at least one error is a binomial rv with $n = 400$ and $p = 0.005$, so $np = 2$. We wish

$$P(X=1) = b(1;\, 400,\, 0.005)$$

$$\simeq p(1;\, 2) = e^{-2}\frac{(2)^1}{1!} = 0.27067$$

The binomial value is $b(1;\, 400, 0.005) = .270669$, so the approximation is very good.

Similarly,

$$P(X \le 3) \simeq \sum_{x=0}^{3} p(x;\, 2)$$

$$= e^{-2}\sum_{x=0}^{3}\frac{2^x}{x!} = 0.8571$$

and this again is quite close to the binomial value $P(X \le 3) = 0.8576$.

The mean and variance of X:

Since $b(x;\, n,\, p) \longrightarrow p(x;\, \mu)$ as $n \to \infty$, $p \to 0$, $np \to \mu$, the mean and variance of a binomial variable should approach those of a Poisson variable. These limits are $np \to \mu$ and $np(1-p) \longrightarrow \mu$. $(\because p \to 0, \therefore np^2 \to 0)$

**Proposition**: If X has a Poisson distribution with parameter $\mu$, then $E(X) = V(X) = \mu$.

**Proof**: $E(X) = \sum_{x=0}^{\infty} x\,p(x;\, \mu)$

$$= \sum_{x=0}^{\infty} x \, e^{-\mu} \frac{\mu^x}{x!} \qquad (\mu > 0)$$

$$= e^{-\mu} \sum_{x=1}^{\infty} \frac{x \, \mu^x}{x!} \qquad [\because \text{ for } x=0, \text{ the 1st term is } 0]$$

$$= e^{-\mu} \sum_{x=1}^{\infty} \frac{\mu \cdot \mu^{x-1}}{(x-1)!} \qquad [\because x \geq 1, \text{ we have } \frac{x}{x!} = \frac{1}{(x-1)!}]$$

$$= \mu e^{-\mu} \sum_{k=0}^{\infty} \frac{\mu^k}{k!} \qquad [\text{Put } k = x-1. \text{ When } x=1, k=0. \text{ When } x=\infty, k=\infty]$$

$$= \mu e^{-\mu} \cdot e^{\mu} \qquad [\because e^{\mu} = 1 + \mu + \frac{\mu^2}{2!} + \cdots = \sum_{k=0}^{\infty} \frac{\mu^k}{k!}]$$

$$= \mu \cdot e^0 = \mu \cdot 1 = \mu.$$

To find $V(X)$:

We use the shortcut formula

$$V(X) = E(X(X-1)) - E(X)(E(X)-1)$$

$$E(X(X-1)) = \sum_{x=0}^{\infty} x(x-1) \, e^{-\mu} \frac{\mu^x}{x!}$$

$$= \sum_{x=2}^{\infty} e^{-\mu} \cdot \mu^2 \cdot \frac{\mu^{x-2}}{(x-2)!}$$

$$= \mu^2 e^{-\mu} \sum_{x=2}^{\infty} \frac{\mu^{x-2}}{(x-2)!}$$

$$= \mu^2 e^{-\mu} \sum_{k=0}^{\infty} \frac{\mu^k}{k!} \qquad [\text{Put } k = x-2 \;\; \because \text{ Limits are } 0 \text{ and } \infty]$$

$$= \mu^2 e^{-\mu} \cdot e^{\mu}$$

$$= \mu^2 e^{-\mu+\mu} = \mu^2 \cdot e^0 = \mu^2$$

Also $E(X) = \mu$.

$$\therefore V(X) = \mu^2 - \mu(\mu-1)$$

$$= \cancel{\mu^2} - \cancel{\mu^2} + \mu = \mu.$$

$$\sigma = \sigma_X = \sqrt{V(X)} = \sqrt{\mu} \; (>0).$$

The Poisson Process :

A very important application of the Poisson distribution arises in connection with the occurrence of events of some type over time. Events of interests might be visits to a particular website, pulses of some sort recorded by a counter, email messages sent to a particular address, accidents in an industrial facility, or cosmic ray showers observed by astronomers at a particular observatory. We make the following assumptions about the way in which the events of interest occur:

1. There exists a parameter $\alpha > 0$ such that for any short time interval of length $\Delta t$, the probability that exactly one event occurs is $\alpha \cdot \Delta t + o(\Delta t)$.

2. The probability of more than one event occurring during $\Delta t$ is $o(\Delta t)$ [which, along with Assumption 1, implies that the prob.$^{ty}$ of no events during $\Delta t$ is $1 - \alpha \cdot \Delta t - o(\Delta t)$].

3. The number of events occurring during the time interval $\Delta t$ is independent of the number that occurs prior to this time interval.

A quantity is $o(\Delta t)$ (read "little o of delta t) if, as $\Delta t \to 0$, so does $o(\Delta t)/\Delta t$. That is $o(\Delta t)$ is even more negligible (approaches 0 faster) than $\Delta t$ itself. The quantity $(\Delta t)^2$ has this property, but $\sin(\Delta t)$ does not.

Let $P_K(t)$ denote the probability that $K$ events will be observed during any particular time interval of length $t$.

Proposition:

$$P_K(t) = e^{-\alpha t} \frac{(\alpha t)^K}{K!}$$

, so that the number of events during a time interval of length $t$ is a Poisson rv with parameter $\mu = \alpha t$. The expected number of events during any such time interval is then $\alpha t$, so the expected number during a unit interval of time is $\alpha$.

The occurrence of events over time as described is called a Poisson process; the parameter $\alpha$ specifies the rate for the process.

Ex. 3.42 Suppose pulses arrive at a counter at an average rate of six per minute, so that $\alpha = 6$. To find the prob.$^{ty}$ that in a .5-min interval at least one pulse is received, note that the number of pulses in such an interval has a Poisson distribution with parameter $\alpha t = 6(0.5) = 3$. Then with $X$ = the number of pulses received in the 30-sec interval,

$$P(X \geq 1) = 1 - P(X = 0)$$

$$= 1 - e^{-3} \frac{3^0}{0!} = 0.950.$$

Exercises Section 3.6

(84) $\mu = \frac{0.1}{\cancel{100}} \times 10\cancel{000} = \frac{1}{10} \times 100 = 10.$

(a) $E(X) = \mu = 10$

$V(X) = \mu = 10.$

$\therefore \sigma_X = \sqrt{V(X)} = \sqrt{10} = 3.16.$

(b) $P(X > 10) = 1 - P(X \leq 10) = 1 - 0.5826 = 0.4174.$

$\left[P(X \leq 10) = \cancel{P(X} \; p(0) + p(1) + p(2) + \cdots + p(10)\right.$

$$= \sum_{x=0}^{10} e^{-10} \frac{10^x}{x!} = 0.5826.]$$

(c) $P(X = 0) = e^{-10} \frac{10^0}{0!} = 0.0000454.$

86 (a) $\mu = 5$ per hour.

$P(x=4) = e^{-5} \frac{5^4}{4!} \simeq 0.1755$

(b) $P(x \geq 4) = 1 - P(x < 4)$

$= 1 - P(x \leq 3)$

$= 1 - [p(0) + p(1) + p(2) + p(3)]$

$= 1 - 0.2650 = 0.7350$.

[$p(0) = P(x=0) = e^{-5} \frac{5^0}{0!} \simeq 0.0067$

$p(1) = P(x=1) = e^{-5} \frac{5^1}{1!} \simeq 0.0337$

$p(2) = P(x=2) = e^{-5} \frac{5^2}{2!} = 0.0842$

$p(3) = P(x=3) = e^{-5} \frac{5^3}{3!} \simeq 0.1404$.

$\therefore P(x \leq 3) = 0.2650$ ]

(c) Now $t = 3/4$ hour.

$\therefore \mu = 5 \times \frac{3}{4} = \frac{15}{4} = 3.75$.

Thus we expect 3.75 people to arrive during a 45 minute period.

[time period 2 hours]

87 (a) $\mu = 4 \times 2 = 8$.

$P(x=10) = e^{-8} \frac{8^{10}}{10!} = 0.099$.

(b) $\mu = 4 \times \frac{1}{2} = 2$ $\quad [t = \frac{1}{2}$ hour$]$

~~P~~ Do not miss any calls for assistance means $x = 0$.

$\therefore P(x=0) = \dfrac{e^{-2}\, 2^{0}}{0!} = e^{-2} = 0.135.$

(c) The expected number of calls during break

$= \mu = 2.$

~~9.1 (a) For a quarter-acre (.25 acre) plot, the mean parameter $\mu = 80 \times (.25)$ $= 20$, so $P(x \le 16) = F(16; 20)$~~

~~$= .221.$~~

X