ML
LECTURE-23

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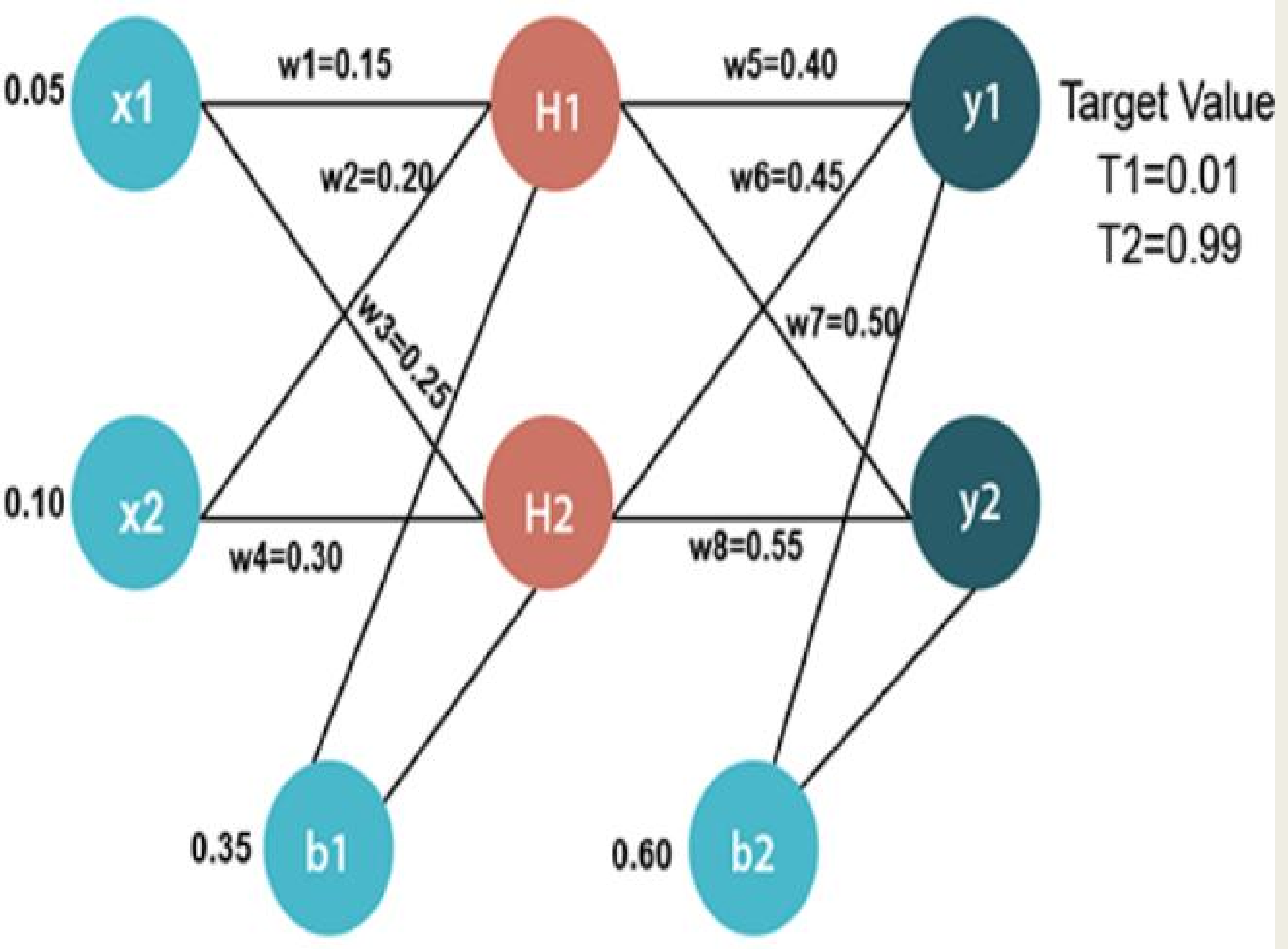
Backpropagation

- The backpropagation algorithm was **discovered in 1985-86.**
- Here is an outline of the algorithm.
- 1. **Initially the weights are assigned at random.**
- 2. Then the algorithm **iterates through many cycles of two processes until a stopping criterion is reached. Each cycle is known as an epoch**. Each epoch includes:
- (a) A **forward phase** in which the **neurons are activated in sequence from the input layer to the output layer, applying each neuron's weights and activation function along the way**. Upon reaching the final layer, an output signal is produced.
- (b) A **backward phase** in which the network's output signal resulting from the forward phase is compared to the true target value in the training data. **The difference between the network's output signal and the true value results in an error that is propagated backwards in the network to modify the connection weights** between neurons and reduce future errors.
- 3. The technique used to determine how much a weight should be changed is known as **gradient descent method.** At every stage of the computation, the error is a function of the weights. **If we plot the error against the wights, we get a higher dimensional analog of something like a curve or surface. At any point on this surface, the gradient suggests how steeply the error will be reduced or increased for a change in the weight. The algorithm will attempt to change the weights that result in the greatest reduction in error.**

Backpropagation Example

- ❖ **Input values**
- ❖ X1=0.05
- ❖ X2=0.10
- ❖ **Initial weight**
- ❖ W1=0.15 w5=0.40
- ❖ W2=0.20 w6=0.45
- ❖ W3=0.25 w7=0.50
- ❖ W4=0.30 w8=0.55
- ❖ **Bias Values**
- ❖ b1=0.35
- ❖ b2=0.60
- ❖ **Target Values**
- ❖ T1=0.01
- ❖ T2=0.99



Backpropagation Example Solution

- **Forward Pass**
- To find the value of H1 we first multiply the input value from the weights as
- H1 = x1×w1+x2×w2+b1 = 0.05×0.15+0.10×0.20+0.35 = 0.3775
- To calculate the final result of H1, we performed the sigmoid function as

$$H1_{final} = \frac{1}{1 + \frac{1}{e^{H1}}}$$

$$H1_{final} = \frac{1}{1 + \frac{1}{e^{0.3775}}}$$

$$\mathbf{H1_{final} = 0.593269992}$$

- We will calculate the value of H2 in the same way as H1
- H2 = x1×w3+x2×w4+b1 = 0.05×0.25+0.10×0.30+0.35 = 0.3925
- To calculate the final result of H1, we performed the sigmoid function as

$$H2_{final} = \frac{1}{1 + \frac{1}{e^{H2}}}$$

$$H2_{final} = \frac{1}{1 + \frac{1}{e^{0.3925}}}$$

$$\mathbf{H2_{final} = 0.596884378}$$

Backpropagation Example Solution

- Now, we calculate the values of y1 and y2 in the same way as we calculate the H1 and H2.
- To find value of y1, we first multiply the input value i.e., the outcome of H1 and H2 from the weights as
- $y1=H1\times w5+H2\times w6+b2=0.593269992\times 0.40+0.596884378\times 0.45+0.60=1.10590597$
- To calculate the final result of y1 we performed the sigmoid function as

$$y1_{\text{final}} = \frac{1}{1+\frac{1}{e^{y1}}}$$

$$y1_{\text{final}} = \frac{1}{1+\frac{1}{e^{1.10590597}}}$$

$$y1_{\text{final}} = 0.75136507$$

- We will calculate the value of y2 in the same way as y1
- $y2=H1\times w7+H2\times w8+b2=0.593269992\times 0.50+0.596884378\times 0.55+0.60=1.2249214$
- To calculate the final result of y2, we performed the sigmoid function as

$$y2_{\text{final}} = \frac{1}{1+\frac{1}{e^{y2}}}$$

$$y2_{\text{final}} = \frac{1}{1+\frac{1}{e^{1.2249214}}}$$

$$y2_{\text{final}} = 0.772928465$$

Backpropagation Example Solution

- Our target values are 0.01 and 0.99. Our y1 and y2 value is not matched with our target values T1 and T2.
- Now, we will find the total error, which is simply the difference between the outputs from the target outputs. The total error is calculated as

$$E_{total} = \sum \frac{1}{2}(\text{target} - \text{output})^2$$

- So, the total error is

$$= \frac{1}{2}(t1 - y1_{final})^2 + \frac{1}{2}(T2 - y2_{final})^2$$

$$= \frac{1}{2}(0.01 - 0.75136507)^2 + \frac{1}{2}(0.99 - 0.772928465)^2$$

$$= 0.274811084 + 0.0235600257$$

$$E_{total} = 0.29837111$$

- Now, we will backpropagate this error to update the weights using a backward pass.

Backpropagation Example Solution

- **Backward pass at the output layer**
- To update the weight, we calculate the error correspond to each weight with the help of a total error. The error on weight w is calculated by differentiating total error with respect to w.

$$\mathrm{Error}_w = \frac{\partial E_{total}}{\partial w}$$

- We perform backward process so first consider the last weight w5 as

$$\mathrm{Error}_{w5} = \frac{\partial E_{total}}{\partial w5} \ldots\ldots\ldots..(1)$$

$$E_{total} = \frac{1}{2}(T1 - y1_{final})^2 + \frac{1}{2}(T2 - y2_{final})^2 \ldots\ldots\ldots..(2)$$

- From equation two, it is clear that we cannot partially differentiate it with respect to w5 because there is no any w5. We split equation one into multiple terms so that we can easily differentiate it with respect to w5 as

$$\frac{\partial E_{total}}{\partial w5} = \frac{\partial E_{total}}{\partial y1_{final}} \times \frac{\partial y1\_final}{\partial y1} \times \frac{\partial y1}{\partial w5} \ldots\ldots\ldots\ldots (3)$$

Backpropagation Example Solution

❖ Now, we calculate each term one by one to differentiate Etotal with respect to w5 as

$$\frac{\partial E_{total}}{\partial y1_{final}} = \frac{\partial(\frac{1}{2}(T1 - y1_{final})^2 + \frac{1}{2}(T2 - y2_{final})^2)}{\partial y1_{final}}$$

$$= 2 \times \frac{1}{2} \times (T1 - y1_{final})^{2-1} \times (-1) + 0$$

$$= -(T1 - y1_{final})$$

$$= -(0.01 - 0.75136507)$$

$$\boldsymbol{\frac{\partial E_{total}}{\partial y1_{final}} = 0.74136507 \ldots\ldots\ldots (4)}$$

$$y1_{final} = \frac{1}{1 + e^{-y1}} \ldots\ldots\ldots\ldots\ldots\ldots (5)$$

$$\frac{\partial y1_{final}}{\partial y1} = \frac{\partial(\frac{1}{1 + e^{-y1}})}{\partial y1}$$

$$= \frac{e^{-y1}}{(1 + e^{-y1})^2}$$

$$= e^{-y1} \times (y1_{final})^2 \ldots\ldots\ldots\ldots\ldots (6)$$

$$y1_{final} = \frac{1}{1 + e^{-y1}}$$

$$e^{-y1} = \frac{1 - y1_{final}}{y1_{final}} \ldots\ldots\ldots\ldots\ldots (7)$$

Backpropagation Example Solution

- Putting the value of e^{-y} in equation (5)

$$= \frac{1 - y1_{final}}{y1_{final}} \times (y1_{final})^2$$

$$= y1_{final} \times (1 - y1_{final})$$

$$= 0.75136507 \times (1 - 0.75136507)$$

$$\frac{\partial y1_{final}}{\partial y1} = 0.186815602 \ldots\ldots\ldots.(8)$$

$$y1 = H1_{final} \times w5 + H2_{final} \times w6 + b2 \ldots\ldots\ldots\ldots\ldots\ldots \ (9)$$

$$\frac{\partial y1}{\partial w5} = \frac{\partial(H1_{final} \times w5 + H2_{final} \times w6 + b2)}{\partial w5}$$

$$= H1_{final}$$

$$\frac{\partial y1}{\partial w5} = 0.596884378 \ldots\ldots\ldots(10)$$

- So, we put the values of $\frac{\partial E_{total}}{\partial y1_{final}}, \frac{\partial y1_{final}}{\partial y1}$, and $\frac{\partial y1}{\partial w5}$ in equation no (3) to find the final result.

$$\frac{\partial E_{total}}{\partial w5} = \frac{\partial E_{total}}{\partial y1_{final}} \times \frac{\partial y1_{final}}{\partial y1} \times \frac{\partial y1}{\partial w5}$$

$$= 0.74136507 \times 0.186815602 \times 0.593269992$$

$$\mathbf{Error_{w5}} = \frac{\partial E_{total}}{\partial w5} = 0.0821670407 \ldots\ldots\ldots.(11)$$

Backpropagation Example Solution

- Now, we will calculate the updated weight w5new with the help of the following formula

$$w5_{new} = w5 - \eta \times \frac{\partial E_{total}}{\partial w5} \text{ Here, } \eta = \text{learning rate} = 0.5$$

$$= 0.4 - 0.5 \times 0.0821670407$$

$$\mathbf{w5_{new} = 0.35891648 \ldots \ldots \ldots (12)}$$

- In the same way, we calculate $w6_{new}$,$w7_{new}$, and $w8_{new}$ and this will give us the following values
- $w5_{new}$=0.35891648
- $w6_{new}$=408666186
- $w7_{new}$=0.511301270
- $w8_{new}$=0.561370121

Backpropagation Example Solution

- **Backward pass at Hidden layer**
- Now, we will backpropagate to our hidden layer and update the weight w1, w2, w3, and w4 as we have done with w5, w6, w7, and w8 weights.
- We will calculate the error at w1 as

$$\text{Error}_{w1} = \frac{\partial E_{total}}{\partial w1}$$

$$E_{total} = \frac{1}{2}(T1 - y1_{final})^2 + \frac{1}{2}(T2 - y2_{final})^2$$

- From eqn (2), it is clear that we cannot partially differentiate it with respect to w1 because there is no any w1. We split equation (1) into multiple terms so that we can easily differentiate it with respect to w1 as

$$\frac{\partial E_{total}}{\partial w1} = \frac{\partial E_{total}}{\partial H1_{final}} \times \frac{\partial H1\_final}{\partial H1} \times \frac{\partial H1}{\partial w1} \dots\dots\dots\dots (13)$$

- Now, we calculate each term one by one to differentiate E_{total} with respect to w1 as

$$\frac{\partial E_{total}}{\partial H1_{final}} = \frac{\partial(\frac{1}{2}(T1 - y1_{final})^2 + \frac{1}{2}(T2 - y2_{final})^2)}{\partial H1} \dots\dots\dots\dots (14)$$

- We again split this because there is no any $H1_{final}$ term in E_{toatal} as

$$\frac{\partial E_{total}}{\partial H1_{final}} = \frac{\partial E_1}{\partial H1_{final}} + \frac{\partial E_2}{\partial H1_{final}} \dots\dots\dots (15)$$

Backpropagation Example Solution

- ❖ $\frac{\partial E_1}{\partial H1_{final}}$ and $\frac{\partial E_2}{\partial H1_{final}}$ will again split because in E1 and E2 there is no H1 term. Splitting is done as

$$\frac{\partial E_1}{\partial H1_{final}} = \frac{\partial E_1}{\partial y1} \times \frac{\partial y1}{\partial H1_{final}} \ldots\ldots\ldots (16)$$

$$\frac{\partial E_2}{\partial H1_{final}} = \frac{\partial E_2}{\partial y2} \times \frac{\partial y2}{\partial H1\_final} \ldots\ldots\ldots (17)$$

- ❖ We again Split both $\frac{\partial E_1}{\partial y1}$ and $\frac{\partial E_2}{\partial y2}$ because there is no any y1 and y2 term in E1 and E2. We split it as

$$\frac{\partial E_1}{\partial y1} = \frac{\partial E_1}{\partial y1_{fianl}} \times \frac{\partial y1_{final}}{\partial y1} \ldots\ldots\ldots (18)$$

$$\frac{\partial E_2}{\partial y2} = \frac{\partial E_2}{\partial y2_{fianl}} \times \frac{\partial y2_{final}}{\partial y2} \ldots\ldots\ldots (19)$$

- ❖ Now, we find the value of $\frac{\partial E_1}{\partial y1}$ and $\frac{\partial E_2}{\partial y2}$ by putting values in equation (18) and (19) as
- ❖ From equation (18)

$$\frac{\partial E_1}{\partial y1} = \frac{\partial E_1}{\partial y1_{fianl}} \times \frac{\partial y1_{final}}{\partial y1}$$

$$= \frac{\partial(\frac{1}{2}(T1 - y1_{final})^2)}{\partial y1_{final}} \times \frac{\partial y1_{final}}{\partial y1}$$

$$= 2 \times \frac{1}{2}(T1 - y1_{final}) \times (-1) \times \frac{\partial y1_{final}}{\partial y1}$$

Backpropagation Example Solution

- From equation (8)

$$= 2 \times \frac{1}{2}(0.01 - 0.75136507) \times (-1) \times \ 0.186815602$$

$$\frac{\partial \mathbf{E_1}}{\partial \mathbf{y1}} = \mathbf{0.138498562} \dots \dots \dots (20)$$

- From equation (19)

$$\frac{\partial E_2}{\partial y2} = \frac{\partial E_2}{\partial y2_{fianl}} \times \frac{\partial y2_{final}}{\partial y2}$$

$$= \frac{\partial(\frac{1}{2}(T2 - y2_{final})^2)}{\partial y2_{final}} \times \frac{\partial y2_{final}}{\partial y2}$$

$$= 2 \times \frac{1}{2}(T2 - y2_{final}) \times (-1) \times \frac{\partial y2_{final}}{\partial y2} \dots \dots \dots (21)$$

$$y2_{final} = \frac{1}{1 + e^{-y2}} \dots \dots \dots \dots \dots \dots \ (22)$$

$$\frac{\partial y2_{fianl}}{\partial y2} = \frac{\partial(\frac{1}{1 + e^{-y2}})}{\partial y2}$$

$$= \frac{e^{-y2}}{(1 + e^{-y2})^2}$$

$$= e^{-y2} \times (y2_{final})^2 \dots \dots \dots \dots \dots (23)$$

$$y2_{final} = \frac{1}{1 + e^{-y2}}$$

$$e^{-y2} = \frac{1 - y2_{final}}{y2_{final}} \dots \dots \dots \dots \dots (24)$$

Backpropagation Example Solution

Putting the value of e^{-y2} in equation (23)

$$= \frac{1 - \mathbf{y2_{final}}}{\mathbf{y2_{final}}} \times (\mathbf{y2_{final}})^2$$

$$= \mathbf{y2_{final}} \times (1 - \mathbf{y2_{final}})$$

$$= 0.772928465 \times (1 - 0.772928465)$$

$$\frac{\partial \mathbf{y2_{fianl}}}{\partial \mathbf{y2}} = \mathbf{0.175510053 \ldots\ldots\ldots (25)}$$

From equation (21)

$$= 2 \times \frac{1}{2}(0.99 - 0.772928465) \times (-1) \times 0.175510053$$

$$\frac{\partial \mathbf{E_1}}{\partial \mathbf{y1}} = \mathbf{-0.0380982366126414 \ldots\ldots\ldots (26)}$$

Backpropagation Example Solution

- Now from equation (16) and (17)

$$\frac{\partial E_1}{\partial H1_{final}} = \frac{\partial E_1}{\partial y1} \times \frac{\partial y1}{\partial H1_{final}}$$

$$= 0.138498562 \times \frac{\partial(H1_{final} \times w_5 + H2_{final} \times w_6 + b2)}{\partial H1_{final}}$$

$$= 0.138498562 \times \frac{\partial(H1_{final} \times w_5 + H2_{final} \times w_6 + b2)}{\partial H1_{final}}$$

$$= 0.138498562 \times w5$$

$$= 0.138498562 \times 0.40$$

$$\boldsymbol{\frac{\partial E_1}{\partial H1_{final}} = 0.0553994248 \ldots \ldots \ldots (27)}$$

$$\frac{\partial E_2}{\partial H1_{final}} = \frac{\partial E_2}{\partial y2} \times \frac{\partial y2}{\partial H1\_final}$$

$$= -0.0380982366126414 \times \frac{\partial(H1_{final} \times w_7 + H2_{final} \times w_8 + b2)}{\partial H1_{final}}$$

$$= -0.0380982366126414 \times w7$$

$$= -0.0380982366126414 \times 0.50$$

$$\boldsymbol{\frac{\partial E_2}{\partial H1_{final}} = -0.0190491183063207 \ldots \ldots \ldots (28)}$$

Backpropagation Example Solution

Put the value of $\frac{\partial E_1}{\partial H1_{final}}$ and $\frac{\partial E_2}{\partial H1_{final}}$ in equation (15) as

$$\frac{\partial E_{total}}{\partial H1_{final}} = \frac{\partial E_1}{\partial H1_{final}} + \frac{\partial E_2}{\partial H1_{final}}$$

$$= 0.0553994248 + (-0.0190491183063207)$$

$$\frac{\partial \mathbf{E_{total}}}{\partial \mathbf{H1_{final}}} = \mathbf{0.0364908241736793} \dots\dots\dots (29)$$

We have $\frac{\partial E_{total}}{\partial H1_{final}}$; we need to figure out $\frac{\partial H1\_final}{\partial H1}$, $\frac{\partial H1}{\partial w1}$ as

$$\frac{\partial H1_{final}}{\partial H1} = \frac{\partial\left(\frac{1}{1+e^{-H1}}\right)}{\partial H1}$$

$$= \frac{e^{-H1}}{(1+e^{-H1})^2}$$

$$e^{-H1} \times (H1_{final})^2 \dots\dots\dots (30)$$

$$H1_{final} = \frac{1}{1+e^{-H1}}$$

$$e^{-H1} = \frac{1 - H1_{final}}{H1_{final}} \dots\dots\dots\dots\dots (31)$$

Backpropagation Example Solution

Putting the value of e^{-H1} in equation (30)

$$= \frac{1 - H1_{final}}{H1_{final}} \times (H1_{final})^2$$

$$= H1_{final} \times (1 - H1_{final})$$

$$= 0.593269992 \times (1 - 0.593269992)$$

$$\frac{\partial \mathbf{H1_{final}}}{\partial \mathbf{H1}} = \mathbf{0.2413007085923199}$$

We calculate the partial derivative of the total net input to H1 with respect to w1 the same as we did for the output neuron:

$$H1 = H1_{final} \times w5 + H2_{final} \times w6 + b2 \dots\dots\dots\dots\dots\dots (32)$$

$$\frac{\partial y1}{\partial w1} = \frac{\partial(x1 \times w1 + x2 \times w3 + b1 \times 1)}{\partial w1}$$

$$= x1$$

$$\frac{\partial \mathbf{H1}}{\partial \mathbf{w1}} = \mathbf{0.05} \dots\dots\dots (\mathbf{33})$$

Backpropagation Example Solution

So, we put the values of $\frac{\partial E_{total}}{\partial H1_{final}}$, $\frac{\partial H1_{final}}{\partial H1}$, and $\frac{\partial H1}{\partial w1}$ in equation (13) to find the final result.

$$\frac{\partial E_{total}}{\partial w1} = \frac{\partial E_{total}}{\partial H1_{final}} \times \frac{\partial H1_{final}}{\partial H1} \times \frac{\partial H1}{\partial w1}$$

$$= 0.0364908241736793 \times 0.2413007085923199 \times 0.05$$

$$\mathbf{Error_{w1}} = \frac{\partial \mathbf{E_{total}}}{\partial \mathbf{w1}} = \mathbf{0.000438568} \ldots\ldots\ldots.(34)$$

Now, we will calculate the updated weight $w1_{new}$ with the help of the following formula

$$w1_{new} = w1 - \eta \times \frac{\partial E_{total}}{\partial w1} \quad \text{Here } \eta = \text{learning rate} = 0.5$$

$$= 0.15 - 0.5 \times 0.000438568$$

$$\mathbf{w1_{new}} = \mathbf{0.149780716} \ldots\ldots\ldots(35)$$

In the same way, we calculate $w2_{new}$, $w3_{new}$, and w4 and this will give us the following values

$w1_{new}$=0.149780716

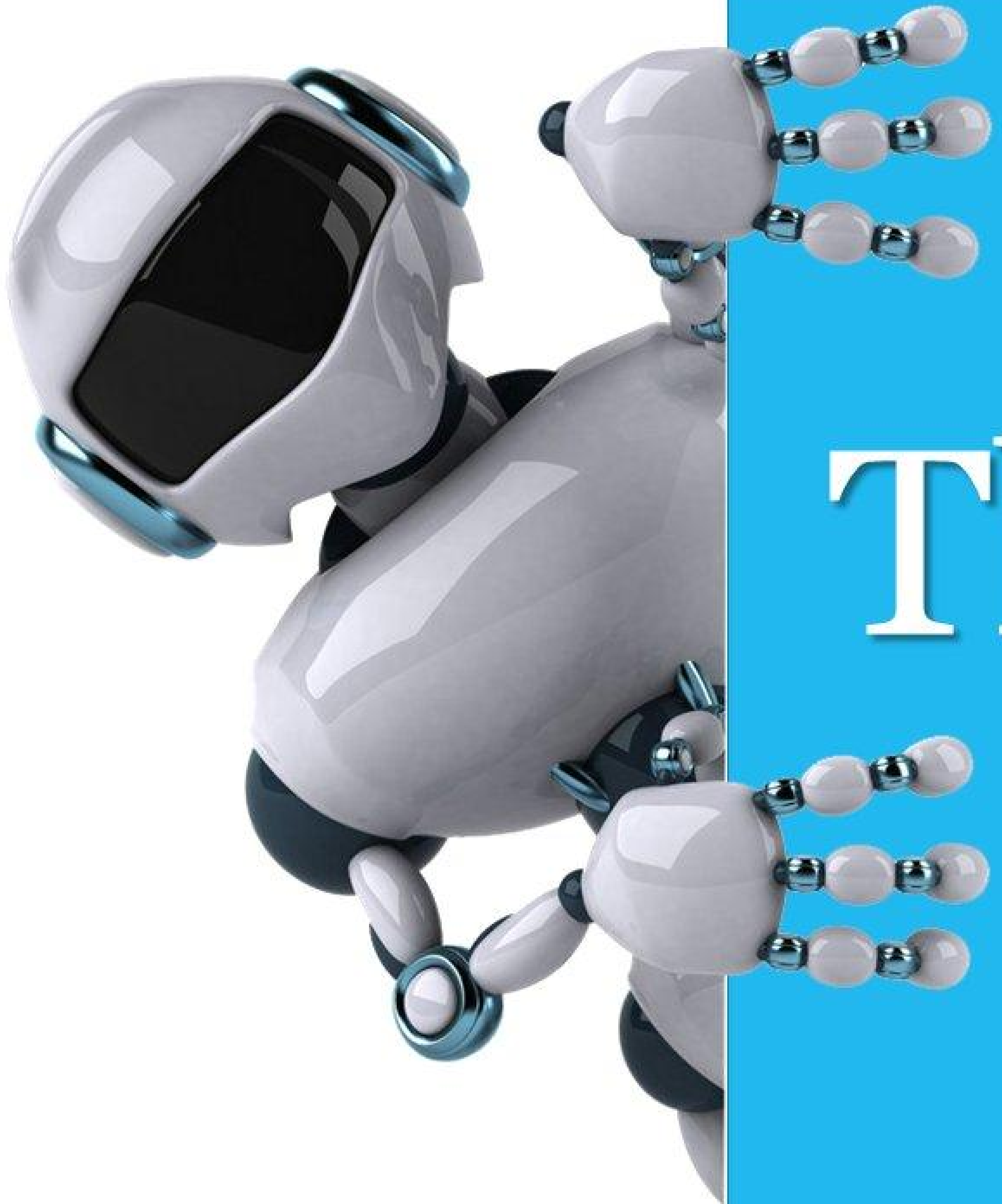
$w2_{new}$=0.19956143

$w3_{new}$=0.24975114

$w4_{new}$=0.29950229

Backpropagation Example Solution

- We have updated all the weights.
- We found the error 0.298371109 on the network when we fed forward the 0.05 and 0.1 inputs.
- In the first round of Backpropagation, the total error is down to 0.291027924.
- **After repeating this process 10,000, the total error is down to 0.0000351085.**
- At this point, the outputs neurons generate 0.159121960 and 0.984065734 i.e., nearby our target value when we feed forward the 0.05 and 0.1.



Thank you