Chapter 1

Random processes

Let us define firstly random processes

Definition 1.0.1. Random variables which depend on time t are called random processes.

Thus random processes are random variables changing with time.

1.1 Markov chains

Let us study random processes called as Markov chains.

Let us have random attempt which has mostly countable possible results $E_1, E_2, ..., E_n, ...$ By repeating the experiment, we get a random list of test results. For example E_2, E_3, E_7, E_1, E_1 etc. Every such experiment we can handle as time step. Let us call these step as states. That is random process X_t with possible values $E_1, E_2, ..., E_i, ...$ Index t denotes the time moment.

If X_t depends only on X_{t-1} then the random process is called as Markov chain.

Definition 1.1.1. The sequence of random variables $\{X_t\}$ where $t = 0, 1, 2, ...$ is called Markov chain if in the case of arbitrary $E_j, E_{k_1}, E_{k_2}, ..., E_{k_{t-1}}$ the conditional probability

$$P(X_t = E_j | X_1 = E_{k_1}, ..., X_{t-2} = E_{k_{t-2}}, X_{t-1} = E_{k_{t-1}}) =$$

$$= P(X_t = E_j | X_{t-1} = E_{k_{t-1}}).$$

Definition 1.1.1 can characterize with following sentence: For any present, the future does not depend on the past. In other words Markov chains are random processes where the history will be forgotten. These are processes without memory.

Markov chain can describe by means of transition matrix

$$\mathbf{P} = \begin{pmatrix} p_{11} & \cdots & p_{1j} & \cdots & p_{1n} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ p_{i1} & \cdots & p_{ij} & \cdots & p_{in} \\ \vdots & \ddots & \vdots & \ddots & \vdots \\ p_{n1} & \cdots & p_{nj} & \cdots & p_{nn} \end{pmatrix},$$

where the row denotes state where we are and the column the state where we will go. Every element of this matrix $p_{ij} \leq 1$ and for every row

$$\sum_{j=1}^{n} p_{ij} = 1.$$

The element of matrix $\mathbf{P} : n \times n$ is conditional probability

$$p_{ij} = P(X_t = E_j \mid X_{t-1} = E_i)$$

or probability to go with 1 time step from state E_i to state E_j. If matrix $\mathbf{P}$ does not depend on time t then Markov chain is called homogeneous. How we can go in the case of homogeneous Markov chain with m steps from state E_i to state E_j? The answer gives us the equality

$$p_{ij}(m) = \sum_{l=1}^{n} p_{il}(k) p_{lj}(m-k). \tag{1.1}$$

The equality (1.1) is called as Chapman-Kolmogorov formula. In the matrix form we can express Chapman-Kolmogorov formula as

$$\mathbf{P}(m) = \mathbf{P}(k)\mathbf{P}(m-k),$$

where $\mathbf{P}(1) = \mathbf{P}$. Thus

$$\mathbf{P}(m) = \mathbf{P}^m.$$

Let the distribution of states $E_1, E_2, ..., E_n$ at time moment t is

$$\pi(t) = \begin{pmatrix} \pi_1(t) & \pi_2(t) & \cdots & \pi_n(t) \end{pmatrix},$$

where $\sum_{i=1}^{n} \pi_i(t) = 1$. The distribution $\pi(0)$ is called initial distribution of Markov chain. According to Chapman-Kolmogorov formula we get

$$\pi(t+m) = \pi(t)\mathbf{P}^m$$

or

$$\pi(1) = \pi(0)\mathbf{P}.$$

Definition 1.1.2. The distribution of sates π^* is called stationary if

$$\pi^* = \pi^*\mathbf{P}.$$

The random process (for example Markov chain) is called to be stationary if its distribution does not depend on time. According to the Definition 1.1.2 we have to solve the system of equations

$$\pi_i = \sum_{j=1}^{n} \pi_j p_{ij}, \quad i = 1, 2, ..., n,$$

at condition $\pi_1 + \pi_2 + ... + \pi_n = 1$ to find the stationary distribution $\pi = \begin{pmatrix} \pi_1 & \pi_2 & \cdots & \pi_n \end{pmatrix}$.

Let us demonstrate functioning of Markov chains on the base of following examples.

Example 1.1.1. Let us have two states which characterize the weather: E_1 – rainy day and E_2 – sunny day. Let rainy and sunny days change according to the rules of homogeneous Markov chains. Let transition matrix

$$\mathbf{P} = \begin{pmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{pmatrix}$$

where first column is for rainy and second for sunny day. Let t be the time and initial distribution of rainy and sunny days

$$\pi(0) = \begin{pmatrix} 0.5 & 0.5 \end{pmatrix}.$$

Let us find that distribution in the next day. We get

$$\pi(1) = \pi(0)P = \begin{pmatrix} 0.5 & 0.5 \end{pmatrix} \begin{pmatrix} 0.7 & 0.3 \\ 0.2 & 0.8 \end{pmatrix} =$$

$$= \begin{pmatrix} 0.5 \cdot 0.7 + 0.5 \cdot 0.2 & 0.5 \cdot 0.3 + 0.5 \cdot 0.8 \end{pmatrix} = \begin{pmatrix} 0.45 & 0.55 \end{pmatrix}.$$

What is stationary distribution of sunny and rainy days. According to the Definition mstats we get system of equations

$$\begin{cases} 0.7\pi_1 + 0.2\pi_2 = \pi_1 \\ 0.3\pi_1 + 0.8\pi_2 = \pi_2 \end{cases}$$

at condition $\pi + \pi_2 = 1$. We get that $\pi_1 = \frac{2}{5}$ and $\pi_2 = \frac{3}{5}$. According to this we have 40% rainy and 60% sunny days.

Example 1.1.2. Let us have wiring diagram which life time has exponential distribution. Let us have 3 different parameters which characterize its reliability: E_1 – low reliability, E_2 – medium reliability and E_3 – high reliability. Let the time step be twenty-four and the transition matrix

$$\mathbf{P} = \begin{pmatrix} 0.6 & 0.4 & 0 \\ 0.2 & 0,6 & 0.2 \\ 0.1 & 0.2 & 0.7 \end{pmatrix}.$$

Let us find the stationary distribution of reliability. We have to solve the system

$$\begin{cases} 0.6\pi_1 + 0.2\pi_2 + 0.3\pi_3 = \pi_1 \\ 0.4\pi_1 + 0.6\pi_2 + 0.3\pi_3 = \pi_2 \\ 0.2\pi_2 + 0.7\pi_3 = \pi_3 \end{cases}$$

at condition $\pi_1 + \pi_2 + \pi_3 = 1$. We get

$$\pi_1 = \frac{2}{7}, \pi_2 = \frac{3}{7} \text{ ja } \pi_3 = \frac{2}{7}.$$

Example 1.1.3. Third example of Markov chain is about gambling. Let us have a player who begins with k units of money. In every game is the probability to win 1 unit p and probability to lose 1 unit is q. A player will finish the game after he (or she) has gotten $n \geq k$ units or ruined with 0 unit. Let the states $E_0, E_1, ..., E_k, ..., E_n$ denote k units of money. Let X_m be the situation of player after game number m. Because this situation depends only on the situation X_{m-1} and on the result of game number m then that random process is Markov chain. Its transition matrix

$$\mathbf{P} = \begin{pmatrix} 1 & 0 & 0 & 0 & \cdots & 0 \\ q & 0 & p & 0 & \cdots & 0 \\ 0 & q & 0 & p & \cdots & 0 \\ 0 & 0 & q & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & 1 \end{pmatrix}.$$

If have solved the equation of stationarity (1.1.2)then we get $\pi_0 + \pi_n = 1$. That mens it exists 2 stationary states:

1) We have won n units;

2) We have ruined with 0 units.

1.2 Poisson processes

In this section we study countable random processes. Let us denote that process as $N(t)$.

Definition 1.2.1. The random process $N(t)$ is said to be countable if its values are $0, 1, 2, ..., n, ...$.

Definition 1.2.2. The countable random process $N(t)$ is called as Poisson process if

1° $N(0) = 0$.

2° $N(s) \geq N(t)$ if $s > t$.

3° All $N(t_2 - t_1)$ and $N(t_4 - t_3)$ are independent, $t_4 > t_3 > t_2 > t_1$..

4° Number of events in time interval t has Poisson distribution with mean value λt. That means

$$P(N(t+s) - N(s) = n) = \frac{(\nu t)^n}{n!} e^{-\nu t}$$

where $n = 0, 1, 2, ...$.

From Definition 1.2.2 we can conclude that

$$E(N(t)) = V(N(t)) = \nu t.$$

Let us denote

$$P_n(t) = P(N(t) = n).$$

Next we generalize Poisson processes to the non-homogeneous case. That means the events intensity $\nu(s)$ depends on time s. We get that expected number of events

$$\lambda = \int_s^{t+s} \nu(\tau) d\tau.$$

In homogeneous case

$$\lambda = \int_s^{t+s} \nu d\tau = \nu t.$$

Example 1.2.1. Let the moment 11 AM be as time $t = 0$. The intensity of pubs visitors per 1 hour follows between 11 AM and 11 PM the statement

$$\nu(t) = e^{0.2t}.$$

Find the probability that number of pubs visitors between 6 PM and 8 PM is at least 12.

We have to find integral

$$\int_7^9 e^{0.2t} dt = \frac{1}{0.2}(e^{9\cdot 0.2} - e^{7\cdot 0.2}) \approx 9.97 = \lambda.$$

So the expected value of visitors between 6 PM and 8 PM is almost 10 and the probability

$$P(\text{"Number of visitors between 6 PM and 8 PM is at least 12"}) =$$

$$= \sum_{k=0}^{12} \frac{\lambda^k}{k!} e^{-\lambda} \approx 0.794.$$

1.2.1 Distributions of waiting time and moments of events

Let us have Poisson process $N(t)$ with intensity ν. Let $S_1, S_2, ...$ be the moments of events and $T_1, T_2 = S_2 - S_1, T_3 = S_3 - S_2, ...$ length of waiting times between events.

Distribution of waiting times between events

The event $\{T_1 > t\}$ is equal with event $\{N(t) = 0\}$. So we get

$$P(T_1 > 1) = P(N(t) = 0) = e^{-\nu t}.$$

The distribution function

$$F(t) = P(T_1 \leq t) = 1 - P(T_1 > t) = 1 - e^{-\nu t}.$$

That means T_1 has exponential distribution with parameter ν. Exponential distribution is without memory. That means

$$P(T > s + t \mid T > s) = \frac{P(T > s + t)}{P(T > s)} = \frac{e^{-\nu(s+t)}}{e^{-\nu s}} = e^{-\nu t} = P(T > t).$$

From this follows that all waiting times $T_1, T_2, T_3, ...$ have exponential distributions with parameter ν.

Distribution of moments

Let us go that events $\{S_n > t\}$ and $\{N(t) < n\}$ are equal. Thus

$$P(S_n > t) = P(N(t) < n) = \sum_{i=0}^{n-1} P_n(t) = \sum_{n=0}^{n-1} \frac{(\nu t)^k}{k!} e^{-\nu t}.$$

For S_n we get distribution function

$$F(t) = P(S_n \leq t) = 1 - P(S_n > t) = 1 - \sum_{n=0}^{n-1} \frac{(\nu t)^k}{k!} e^{-\nu t}.$$

After differentiation follows that density function

$$f(t) = \nu \frac{(\nu t)^{n-1}}{(n-1)!} e^{-\nu t}, t \geq 0.$$

That means the time moment of nth event has gamma distribution with parameters n and ν. The expectation $E(S_n) = \frac{n}{\nu}$ and the variance $V(S_n) = \frac{n}{\nu^2}$.

1.3 Markov chains with continuous time

Markov chains with continuous time can handle as generalization of Poisson processes.

Definition 1.3.1. Random process $X(t)$ is called as Markov chain with conitinuous time if for every if in the case of arbitrary real numbers $s, t > 0$ and natural numbers $i, j, x(u)$ the probability

$$P(X(t+s) = j \mid X(s) = i, X(u) = x(u), 0 \leq u < s) =$$

$$= P(X(t+s) = j \mid X(s) = i).$$

That means the distribution of the future states $X(t+s)$ in the case of arbitrary present $X(s)$ does not depend on the process of past $X(u) = x(u)$, $0 \leq u < s$.

Markov chain with continuous time can characterize as follows.

1) Arriving to the state i the chain will be in this state time period T_i which has exponential distribution with parameter λ_i. All time periods are independent.

2) Leaving from the state i the chain enters to the state j with probability p_{ij}. The next conditions are fulfilled

$$p_{ii} = 0 \text{ and } \sum_j p_{ij} = 1.$$

Let us denote

$$p_{ij} := P(X(t+s) = j \mid X(s) = i).$$

Theorem 1.3.1. The next statements are valid

$$\lim_{h\to 0} \frac{1 - p_{ii}(h)}{h} = \lambda_i \tag{1.2}$$

$$\lim_{h\to 0} \frac{p_{ij}(h)}{h} = \lambda_i p_{ij} \text{ if } i \neq j. \tag{1.3}$$

Chapman-Koomogorov equation in continuous case can express as follows

$$p_{ij}(t+s) = \sum_{k=0}^{\infty} p_{ik}(t) p_{kj}(s). \tag{1.4}$$

1.3.1 Kolmogorov forward and backward equations

Let us derive differential equations finding probability $p_{ij}(t)$. These equations are based on Theorem 1.3.1. Using statements 1.2-1.3 and equation 1.4 we get

$$p_{ij}(t+h) - p_{ij}(t) = \sum_{k=0}^{\infty} p_{ik}(h) p_{kj}(t) - p_{ij}(t) =$$

$$= \sum_{k \neq i} p_{ik}(h) p_{kj}(t) - [1 - p_{ii}(h)] p_{ij}(t).$$

Dividing it by h we see that

$$\lim_{h\to 0} \frac{p_{ij}(t+h) - p_{ij}(t)}{h} = \lim_{h\to 0} \sum_{k\neq i} \frac{p_{ik}(h)}{h} p_{kj}(t) - \lambda_i p_{ij}(t).$$

We have gotten the next statement

Theorem 1.3.2. (Kolmogorov backward equation) For every i, j and $t \geq 0$

$$p'_{ij}(t) = \lambda_i \sum_{k \neq i} p_{ik} p_{kj}(t) - \lambda_i p_{ij}(t). \tag{1.5}$$

Almost in the same way we get in some conditions

Theorem 1.3.3. (Kolmogorov forward equation) For every i, j and $t \geq 0$

$$p'_{ij}(t) = \sum_{k \neq j} \lambda_k p_{kj} p_{ik}(t) - \lambda_j p_{ij}(t). \tag{1.6}$$

In some cases we can solve differential equations 1.5-1.6 analytically. Mostly that is impossible.

Birth and death processes

Let us study Markov chains with continuous time where in every $t \geq 0$

$$p_{ij}(t) = 0 \text{ if } \mid i - j \mid \geq 2. \tag{1.7}$$

These processes are called birth and death processes. The process $n \to n+1$ is called as birth and process $n \to n-1$ as death. Let μ_n be intensity of birth and ν_n intensity of death. According to the condition 1.7 we get

$$p_{n,n+1} + p_{n,n-1} = 1.$$

Intensity or parameter of exponential distribution λ_n can express as

$$\lambda_n = \mu_n + \nu_n.$$

Addends

$$\mu_n = \lambda_n p_{n,n+1}, n \geq 0,$$
$$\nu_n = \lambda_n p_{n,n-1}, n \geq 1$$

are called birth intensity and death intensity respectively. Let us denote that $\nu_0 = 0$.

Equations of equilibrium

Firstly we derive from statement 1.6 equation equations of equilibrium. Then we apply this equation on birth and death processes. If equilibrium is achieved that means $t \to \infty$ then

$$p_{ij}(t) \to p_j \text{ for every } i,$$

$$p'_{ij}(t) \to 0.$$

The probability

$$p_j = P(\text{"We are in the state } j\text{"}).$$

Applying conditions of equilibrium on equation 1.6 we get

$$\lambda_j p_j = \sum_{k=0, k\neq j}^{n} \lambda_k p_{kj} p_k, j = 0, 1, 2, ... \tag{1.8}$$

which is called equation of equilibrium.

In the case of birth and death processes $p_{kk} = 0$. So the equation 1.8 reduces to the form

$$\lambda_j p_j = \lambda_{j-1} p_{j-1,j} p_{j-1} + \lambda_{j+1} p_{j+1,j} p_{j+1}.$$

Because the intensity

$$\lambda_j = \mu_j + \nu_j$$

where $\mu_j = \lambda_j p_{j,j+1}$, $\nu_j = \lambda_j p_{j,j-1}$ and $\nu_0 = 0$ we get

$$\mu_0 p_0 = \nu_1 p_1$$

$$(\mu_1 + \nu_1) p_1 = \mu_0 p_0 + \nu_2 p_2$$

$$(\mu_2 + \nu_2) p_2 = \mu_1 p_1 + \nu_3 p_3$$

$$\vdots$$

$$(\mu_k + \nu_k) p_k = \mu_{k-1} p_{k-1} \nu_{k+1} p_{k+1}$$

$$\vdots$$

Our purpose is to express probabilities p_k. We get

$$p_1 = \frac{\mu_0 p_0}{\nu_1}$$

$$p_2 = \frac{\mu_1 p_1}{\nu_2} = \frac{\mu_0 \mu_1 p_0}{\nu_1 \nu_2}$$

$$p_3 = \frac{\mu_0 \mu_1 \mu_2}{\nu_1 \nu_2 \nu_3} p_0$$

$$\vdots$$

In general form

$$p_k = \prod_{i=1}^{k} \frac{\mu_{i-1}}{\nu_i} p_0, k \geq 1. \tag{1.9}$$

Taking into account that $\sum_{k=0}^{\infty} p_k = 1$ we get

$$p_0 = \frac{1}{1 + \sum_{k=1}^{\infty} \prod_{i=1}^{k} \frac{\mu_{i-1}}{\nu_i}}. \tag{1.10}$$

Next we give some examples applying equations 1.9-1.10.

Example 1.3.1. Erlang formulas. Let us study service system with losses. This kind of the system can handle as a pub where are n tables. If a client enter to this pub and a table is free then he (or she) stay in this pub. If all tables are occupied then this client leave this pub. So

$$P(\text{"Client will be lost"}) = P(\text{"All } n \text{ tables all occupied"}) = p_n.$$

Let $\mu_k = \mu$, $k = 0, 1, ..., n-1$ be the intensity of clients entering to this pub. Then $\mu_k = 0$ if $k \geq n$. Let every client spend in this pub time T which has exponential distribution with parameter ν. Then death intensity $\nu_k = k\nu$, $k = 0, 1, ..., n$ and $\nu_k = 0$ if $k > n$. Let $\rho = \frac{\mu}{\nu}$ be the system load. Using formula 1.9 we get

$$p_k = \prod_{i=1}^{k} \frac{\mu_{i-1}}{\nu_i} p_0 = \prod_{i=1}^{k} \frac{\mu}{i\nu} p_0 = \frac{\rho^k}{k!} p_0.$$

Then the lost probability

$$p_n = \frac{\rho^n}{n!} p_0 \tag{1.11}$$

where

$$p_0 = \frac{1}{\sum_{k=0}^{n} \frac{\rho^k}{k!}}. \tag{1.12}$$

Equations 1.11-1.12 are called as Erlang formulas.

In Tartu is pub München. In this pub are 10 tables. In Friday evening visit this pub 8 clients per hour. If a client see that all 10 tables are occupied then he (or she) leaves pub München. Let spending time in pub München has exponential distribution. The expected time of being in this pub is 1 hour. In these conditions

$$p_{10} = P(\text{"All 10 tables are occupied"}) \approx 0.1385.$$

Example 1.3.2. Queuing service systems. Let us examine a service system with n servants. Let the time to complete the service has exponential distribution with parameter ν. Let the costumers arrive to the system according to the rules of Poisson process with intensity μ. If all the servants are busy then the customer will be queued. Let

$$p_k = P(\text{"In the service system are } k \text{ customers"}).$$

The birth intensity $\mu_k = \mu$ if $k \geq 0$ and the death intensity

$$\nu_k = \begin{cases} k\nu, & \text{if } 0 \leq k \leq n \\ n\nu, & \text{if } k > n. \end{cases}$$

Applying equations 1.9-1.10 we get

$$p_k = \begin{cases} \frac{\rho^k}{k!} p_0, & \text{if } 0 \leq k \leq n \\ \frac{\rho^k}{n! n^{k-n}} p_0, & \text{if } k > n \end{cases}$$

and

$$p_0 = \frac{1}{\sum_{k=0}^{n} \frac{\rho^k}{k!} + \frac{\rho^n}{n!} \sum_{k=n+1}^{\infty} (\frac{\rho}{n})^{k-n}}.$$

The series $\sum_{K=n+1}^{\infty} (\frac{\rho}{n})^{k-n}$ is a geometric progression which converge if $\frac{\rho}{n} < 1$. If $\rho \geq n$ then the service system is overloaded. Then $p_k = 0$ if $k > 1$. That means $X(t) \to \infty$ (the queue grows unlimited) and conditions of equation 1.8 are not fulfilled. If system is overloaded then equilibrium will never be achieved.

Example 1.3.3. Serving of machines. Let us have n machines (for example computers) which don't need intervention of operator if they are working normally. Let $r < n$ be the number of operators. Let The working time without failure T be for machine exponentially distributed and let $E(T) = \frac{1}{\mu}$. Let the working time of operators has exponential distribution with parameter ν. Let

$$p_k = P(\text{"Number of failed machines is } k\text{"}).$$

Birth intensities

$$\mu_k = \begin{cases} (n-k)\mu, & \text{if } 0 \leq k \leq n \\ 0, & \text{if } k > n \end{cases}$$

and death intensities

$$\nu_k = \begin{cases} 0, & \text{if } k = 0 \\ k\nu, & \text{if } 0 \leq k \leq r \\ r\nu, & \text{if } k > r. \end{cases}$$

Applying equations 1.9-1.10 we get

$$p_k = \begin{cases} \frac{n!}{k!(n-k)!}\rho^k p_0, & \text{if } 1 \le k \le r \\ \frac{n!}{r^{n-k} r!(n-k)!}\rho^k p_0, & \text{if } r < k \le n \end{cases}$$

and

$$p_0 = \frac{1}{\sum_{k=0}^{r} \frac{n!}{k!(n-k)!}\rho^k + \sum_{k=r+1}^{n} \frac{n!}{r^{n-k} r!(n-k)!}\rho^k}.$$

1.4 Markov processes

Let us generalize Markov chains to the case where set of states is not countable. That means we can use instead of states E_i and E_j real numbers x and y. These real numbers are realizations of random variables X and Y. In this case we can define transition probability as

$$P_{xy} = P(X_{t+1} \le y \mid X_t = x).$$

If P_{xy} is continuous in relation of argument y then we get conditional density function

$$p_{xy} = \frac{\partial P_{xy}}{\partial y} = f(y \mid x).$$

The conditional density $f(y \mid x)$ is called as transition kernel of Markov chain. The transition probability with m steps is defined as

$$P_{xy}(m) = P(X_{t+m} \le y \mid X_t = x) = P(X_m \le y \mid X_0 = x).$$

If set of states is continuous then

$$P_{xy}(k+m) = \int_{-\infty}^{\infty} p_{xs}(k) p_{sy}(m) ds.$$

The change of states distribution π_t in 1 time step can find as follows:

$$\pi_{t+1}(y) = \int_{-\infty}^{\infty} p_{xy} \pi_t(x) dx.$$

In similar way we can find random vectors component marginal density by joint density of random vector. In or case are states distributions $\pi_t(x)$ and $p_{t+1}(y)$ as marginal densities of random variables X and Y. Transition kernel p_{xy} is as conditional density. If distribution π is stationary then

$$\pi(y) = \int_{-\infty}^{\infty} p_{xy} \pi(x) dx.$$

Example 1.4.1. Let us study process which is called as random walk in continuous case. Then

$$X_t = X_{t-1} + W_t$$

where random variable W_t has continuous density function $f(w)$. Random variable X_t can express as the sum of independent random variables

$$X_t = X_0 + W_1 + W_2 + ... + W_t.$$

Thus X_t is the Markov chain where in every time step it can move with random length to the left or right. This movements depends on the value of random variable W_t. The transition probabilities

$$P_{xy} = P(X_{t+1} \leq y \mid X_t = x) = P(X_t + W_{t+1} \leq y \mid X_t = x) =$$
$$= P(W_{t+1} \leq y - X_t \mid X_t = x) = P(W_{t+1} \leq y - x \mid X_t = x) =$$
$$= \int_{-\infty}^{y-x} f(w)dw.$$

The transition kernel

$$p_{xy} = \frac{\partial}{\partial y} \int_{-\infty}^{y-x} f(w)dw = f(y - x).$$

Let initial distribution $\pi(0)$ be normal distribution $\mathcal{N}(a, \sigma_0)$ and let $W_t \sim \mathcal{N}(0, \sigma)$. Then the Markov chain (random walk) has normal distribution $\mathcal{N}(a, \sqrt{\sigma_0^2 + t\sigma^2})$.

Let W_t, $t \geq 0$, has the following conditions:

1° $W_0 = 0$.

2° For every $t > 0$ random variable $W_t \sim \mathcal{N}(0, \sigma\sqrt{t})$ where $\sigma > 0$ is constant.

3° The changes of random variables W_t are stationary and independent (don't depend on time t).

4° The trajectories of W_t are continuous.

Then we have gotten random process W_t, $t \geq 0$, which is called Brownian motion (or Wiener process).

In recent decades, Markov chains have been increasingly used in statistical modeling. These applications are based on the Bayesian theorem which are related with prior and posterior distributions. to the front and back distributions of parameters.