equal to a fixed constant times h . Thus, at least for uniform grids, both schemes can be considered to be of the first-order in terms of the spatial grid size only.

3.5 Complements on the Equation of the Vibrating String (VSE)

We conclude this chapter with some material related to the vibrating string equation (3.3) that comes in complement to what we saw in Section 3.2. More particularly, we study basic explicit schemes both for the equation's standard second-order form and for the equivalent first-order system (equation (3.5)).

3.5.1 The Lax Scheme to Solve the VS First-order System

We already know that all the schemes studied in Section 3.2 can be used to solve equation (3.3) rewritten in the form of the first-order two-unknown system (cf. equation (3.5)). For example, denoting by $(v_j^k; w_j^k)$ the approximation of the pair $(v; w)$ at grid point jh at the k th time step, starting from $(v_j^0; w_j^0) = (v(jh,0); w(jh,0))$ with $v(jh,0) = -[u_0]'(jh)$ and $w(jh,0) = u_1(jh)/o$, and setting $w_0^k = w_n^k = 0$, for $j = 1, 2, \ldots, n-1$, we approximate the system (equation (3.5)) by

The Lax scheme for the VS first-order system at inner grid points

$$\boxed{\begin{array}{l} \text{For } j = 1, \ldots, n-1: \\ v_j^k = (v_{j+1}^{k-1} + v_{j-1}^{k-1})/2 - o\tau[(w_{j+1}^{k-1} - w_{j-1}^{k-1})/(2h)]; \\ w_j^k = (w_{j+1}^{k-1} + w_{j-1}^{k-1})/2 - o\tau[(v_{j+1}^{k-1} - v_{j-1}^{k-1})/(2h)] + \tau f(jh,[k-1])/o] \end{array}} \quad (3.66)$$

Scheme (3.66) can be labeled as incomplete because expressions to determine v_j^k for $j = 0$ and $j = n$ are missing. As already pointed out in the introduction of this chapter, Neumann boundary conditions for v at both ends of the string can be applied. Recalling the approximation of these boundary conditions using fictitious grid points, considered in Section 1.2, we are led to the definitions of $v_{-1}^{k-1} := v_1^{k-1} - 2hf(0,[k-1]\tau)/p$ and $v_{n+1}^{k-1} := v_{n-1}^{k-1} + 2hf(L,[k-1]\tau)/p$. On the other hand, we note that it is not possible to define in a similar manner w_{-1}^{k-1} and w_{n+1}^{k-1} . Therefore, we cannot use a centred FD to approximate $\partial_x w$ at $x = 0$ and $x = L$. Taking these remarks into account, we add the following expressions to equation (3.66):

End point equations to complete Lax scheme (3.66)

$$\boxed{\begin{array}{l} v_0^k = (v_1^{k-1} + v_{-1}^{k-1})/2 - o\tau[(w_1^{k-1} - w_0^{k-1})/h]; \\ v_n^k = (v_{n+1}^{k-1} + v_{n-1}^{k-1})/2 - o\tau[(w_n^{k-1} - w_{n-1}^{k-1})/h], \\ \text{where} \\ v_{-1}^{k-1} := v_1^{k-1} - 2hpf(0, [k-1]\tau); \\ v_{n+1}^{k-1} := v_{n-1}^{k-1} + 2hpf(L, [k-1]\tau). \end{array}} \quad (3.67)$$

Similarly to the case of equation (3.6), it can be shown that the complete scheme (equations (3.66)–(3.67)) is stable in the maximum norm, provided $|o|\tau \leq h$ (cf. [67])

3.5.2 Example 3.2: Numerical Study of Schemes for the VS First-order System

We consider the application of numerical schemes to solve the following system of the form of (equation (3.5)) in $(0,1) \times (0,1)$:

$$\begin{cases} \partial_t v + \partial_x w = 0 \ \forall (x,t); \\ \partial_t w + \partial_x v = 0 \ \forall (x,t); \\ w(x,0) = u_1(x) \ \forall x; \\ v(x,0) = -u_0'(x) \ \forall x; \\ w(0,t) = w(1,t) = 0 \ \forall t. \end{cases}$$

with $u_0(x) = 2\sin(\pi x)$ and $u_1(x) \equiv 0$. The exact solution is given by $v = -\partial u/\partial x$ and $w = \partial u/\partial t$, where $u(x,t) = \sin[\pi(x+t)] + \sin[\pi(x-t)]$.

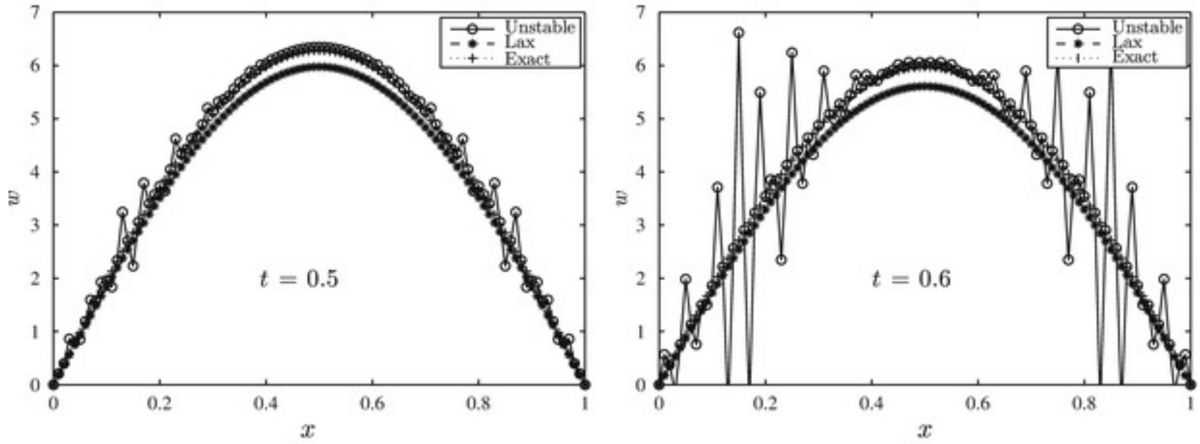
For solving the above problem, we applied the Lax scheme (equations (3.66)–(3.67)) with $h = 1/n$ and τ such that $1/\tau$ is an integer m . We also tested the following counterpart of the unstable scheme (equation (3.19)):

Starting from $(v_j^0; w_j^0) = (v[jh,0]; w[jh,0])$ with $v[jh,0] = -u_0'(jh) = -2\pi\cos(\pi x)$ and $w([jh,0] = u_1(jh) = 0$ for $0 \leq j \leq n$ and setting $w_0^k = w_n^k = 0$ for all k , determine $(v_j^k; w_j^k)$ for $k = 1, \dots, m$ by

$$\begin{cases} \text{For } j = 1, \dots, n-1: \\ v_j^k = v_j^{k-1} - \tau[(w_{j+1}^{k-1} - w_{j-1}^{k-1})/(2h)]; \\ w_j^k = w_j^{k-1} - \tau[(v_{j+1}^{k-1} - v_{j-1}^{k-1})/(2h)]; \\ \\ v_0^k = v_0^{k-1} - \tau[(w_1^{k-1} - w_0^{k-1})/h]; \\ v_n^k = v_n^{k-1} - \tau[(w_n^{k-1} - w_{n-1}^{k-1})/h]. \end{cases}$$

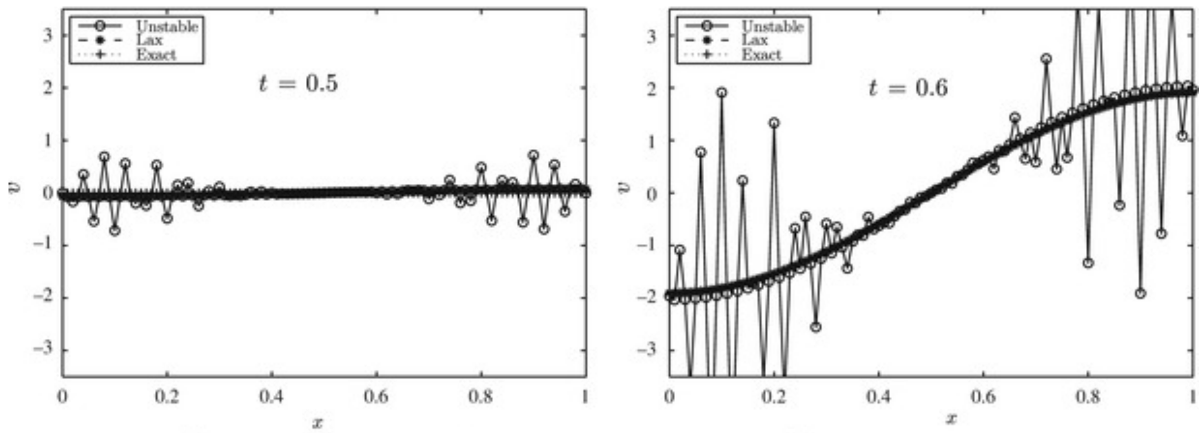
We took $n = 100$ and $m = 250$, which implies that h and τ satisfy the CFL condition $\tau \leq h$ for the above problem. However, we operated significantly apart from the limiting case $\tau = h$.

As a result, the Lax scheme, though stable, tends to generate numerical values a little away from the exact solution here and there, as shown in Figure 3.5 (resp. 3.6). The exact values of w (resp. v) at times $t = 0.5$ (label a) and $t = 0.6$ (label b) are displayed in Figures 3.5 and 3.6, together with the corresponding approximate values obtained with the Lax scheme and the unstable scheme. Figures 3.5a and 3.6a indicate that the latter is already showing up some instabilities at time $t = 0.5$, and from Figures 3.5b and 3.6b one can infer that after just a little more time, the numerical solution obtained with this scheme becomes completely chaotic.



(a) Plots at time $t = 0.5$ (b) Plots at time $t = 0.6$

Figure 3.5 Function $w = \partial u/\partial t$ approximated by the Lax scheme and an unstable scheme



(a) Plots at time $t = 0.5$ (b) Plots at time $t = 0.6$

Figure 3.6 Function $v = -\partial u/\partial x$ approximated by the Lax scheme and an unstable scheme

3.5.3 A Natural Explicit Scheme for the VSE

The solution u of equation (3.3) can be successively approximated at grid points $(jh, k\tau)$ for $j = 1, \dots, n-1$, $k = 1, \dots, m$, through a post-processing of the values w_j^k given by the Lax scheme (equations (3.66)–(3.67)) to solve the equivalent first-order system (3.5). In this aim, one of the following time numerical integration formulae can be employed, starting from $u_j^0 = u_0(jh)$:

- Forward formula: $u_j^k = u_j^{k-1} + \tau w_j^{k-1}$;
- Centred formula: $u_j^k = u_j^{k-1} + \tau(w_j^{k-1} + w_j^k)/2$;
- Backward formula: $u_j^k = u_j^{k-1} + \tau w_j^k$.

Unfortunately, in all cases the accuracy will be damaged by the accumulation of errors from at least two sources: the numerical time integration itself and the approximation of $\partial u/\partial t$ at the grid points by the w_j^k s. When better approximations of u are required, a good alternative is to use a higher order scheme to solve equation (3.5) such as the **Lax–Wendroff scheme** (see e.g. [67]) and a more accurate time integration formula to determine approximations of u . However, we refrain from further elaborating on the numerical treatment of system (3.5), and for more details on the subject we refer the reader to reference [67].

In view of the above introduction, among other reasons it might be interesting to treat directly the second-order equation (3.3), rather than the first-order system (3.5). That is why we conclude this chapter with a brief presentation of a popular space–time discretisation of equation (3.3) by the FDM. In doing so, we note beforehand that the extension of the resulting scheme can be easily adapted to the FVM or the FEM space discretisations, keeping unchanged the way the time discretisation is performed.

Denoting by u_j^k the approximation of $u(jh, k\tau)$, the main difference from the heat equation lies on the treatment of the time dependence. Since the time derivative is $\partial_{tt}u$, quite naturally we can apply the centred FD operator

$$D_{j,\tau}^{k,2} := (-2u_j^k + u_j^{k+1} + u_j^{k-1})/\tau^2$$

to approximate $\partial_{tt}u$ at point $(jh, k\tau)$ for $k \geq 1$ and $1 \leq j \leq n-1$.

If, like in the case of the first-order equation, we adopt an explicit approach, we can centre the scheme about point $(jh, [k-1]\tau)$, by adjusting the FD operator $D^{k,2}_{j,\tau}$ to the $(k-1)$ th step. Then, we proceed to

A two-level time marching scheme for the vibrating string equation

$$\boxed{\begin{array}{l}\text{Starting from } u_j^0 \text{ and } u_j^1, \text{for} 1 \leq j \leq n-1, \text{determine for } k = 2, 3, \ldots, l, \\ u_j^k = -p\tau^2(2u_j^{k-1} - u_{j+1}^{k-1} - u_{j-1}^{k-1})/h^2 + 2u_j^{k-1} - u_j^{k-2} + \tau^2 f(jh, [k-1]\tau),\end{array}} \quad (3.68)$$

Notice that a new difficulty arises, since we cannot launch the calculations without knowing the values u_j^1 in advance. However, somehow we still have to satisfy the second initial condition $\partial_t u(x,0) = u_1(x)$ for $x \in [0, L]$. Although this is not the best thing to do in terms of accuracy, for the sake of simplicity we consider that this condition is applied to determine the u_j^1 s in the most straightforward manner, that is, by

$$u_j^1 = u_j^0 + \tau u_1(jh) \text{ for } j = 1, 2, \ldots, n-1, \quad (3.69)$$

Indeed, by a standard Taylor expansion up to the first-order derivative, $u(jh, \tau) = u(jh, 0) + \tau \partial_t u(jh, \tau_j)$ for a certain $\tau_j \in [0, \tau]$. This means that we approximate $\partial_t u(jh, \tau_j)$ by the closest value at our disposal, namely, $u_1(jh) = \partial_t u(jh, 0)$ for $0 \leq j \leq n-1$.

According to celebrated text books (see e.g. [139]), the stability of schemes (3.68)–(3.69) is ensured under the same condition relating the time step τ to the grid size h , that is, the CFL condition. We do not intend to treat this case in detail. Instead, we next carry out a **Von Neumann stability analysis** [41], following arguments similar to those of Subsection 3.2.1. Our goal is to derive at least a **necessary condition** for stability of this scheme. Before starting, it is instructive to point out that every solution to equation (3.3) with $f = 0$ is necessarily of the form $v(x + ot)$ or $v(x - ot)$, where o is the positive square root of p and v is a twice-differentiable function. For example, if $v(y) = \sin(m\pi y/L)$ for a certain nonzero integer m , the function $u(x,t) = v(x + ot) + v(x - ot)$ also satisfies the Dirichlet boundary conditions $u(0,t) = u(L,t) = 0 \; \forall t \in (0, \Theta]$. Without any loss of essential aspects, we could take $L = 1$ and examine what happens if we approximate a solution of the form $u(x,t) = \sin(m\pi[x + ot]) + \sin(m\pi[x - ot])$ by means of schemes (3.68)–(3.69). Notice that such a solution corresponds to the case where the string is maintained at a position defined by

$u_0(x) = 2\sin(m\pi x)$ for a certain nonzero integer m before the motion starts. Then, by freely releasing it, in the absence of forces f and for an initial velocity $u_1 = 0$, the string will vibrate to successively occupy the position $u(x,t)$ given above.

Although we are dealing with real solutions, the Von Neumann stability analysis consists of handling initial data u_0, which are linear combinations with complex coefficients of functions of the form $e^{\pm im\pi x}$ (notice that in the above test problem, $u_0(x) = -i[e^{im\pi x} - e^{-im\pi x}]$).

If, under these conditions, we apply scheme (3.68) with a suitable modification to determine the u_j^1 s that we decline to specify, we obtain for $j = 1, 2, \ldots, n-1$,

$$u_j^2 = -p\tau^2(2u_j^1 - u_{j+1}^1 - u_{j-1}^1)/h^2 + 2u_j^1 - u_j^0$$

Since $u_j^0 = e^{\pm im\pi j/n}$ for m positive or negative, say with $|m| \leq n$, we assume that the starting step yields $u_j^1 = \gamma u_j^0$ where γ is a certain function of m, τ, h and p but not of j. Plugging these values into the above expression, by means of some elementary calculations, similarly to Section 3.3, we obtain for every j, $u_j^2 = (2\gamma\beta - 1)e^{\pm im\pi j/n}$, where $\beta = 1 - p\tau^2[1 - \cos(m\pi/n)]/h^2$. Hence, provided $\gamma^2 = 2\beta\gamma - 1$, we have $u_j^2 = \gamma^2 e^{\pm im\pi j/n}$. Now, assume that for a certain $k > 2$, we have $u_j^l = \gamma^l e^{\pm im\pi j/n}$ for every j, with $l = k$ and $l = k - 1$. Then, plugging these values into equation (3.68) and replacing k with $k+1$, by the same calculations, we easily derive

$$u_j^{k+1} = -2p\tau^2\gamma^k e^{\pm im\pi j/n}[1 - \cos(m\pi/n)]/h^2 + (2\gamma^k - \gamma^{k-1})e^{\pm im\pi j/n}.$$

This means that

$$u_j^k = (2\gamma\beta - 1)\gamma^{k-1}e^{\pm im\pi j/n}.$$

Since $\gamma^2 = 2\gamma\beta - 1$, we have $u_j^{k+1} = \gamma^{k+1}e^{\pm im\pi j/n}$, and by mathematical induction we can state that the numerical solution is given by $u_j^k = \gamma^k e^{\pm im\pi j/n}$ for all j and k as long as γ is a root of the quadratic function $x^2 - 2\beta x + 1$. These roots are given by $\gamma_1 = \beta + \sqrt{\beta^2 - 1}$ and $\gamma_2 = \beta - \sqrt{\beta^2 - 1}$. It is clear that if $\beta < -1$ (resp. $\beta > 1$), then both γ_1 and γ_2 are real numbers satisfying $\gamma_2 < 0$ and $|\gamma_2| > 1$ (resp. $\gamma_1 > 1$), whereas $|\gamma_1| < 1$ (resp. $0 < \gamma_2 < 1$). Now, by linearity, initial data corresponding to $u_j^0 = e^{im\pi j/n} - e^{-im\pi j/n}$ and $u_j^1 = \gamma_1 e^{im\pi j/n} - \gamma_2 e^{-im\pi j/n}$ will yield at the k th step $u_j^k = \gamma_1^k e^{im\pi j/n} - \gamma_2^k e^{-im\pi j/n}$.
On the one hand, the first (resp. second) term will tend to zero as k increases, the absolute value

of the second (resp. first) term will indefinitely grow and we will gradually diverge from the expected values of the solution to our test problem. If we take, for instance, β just a little less than -1, we observe that at fixed grid points such as $(1/2;\Theta)$, the numerical solution will tend to infinity as k goes to infinity, whatever m (in order to systematically attain $x=1/2$ as h goes to zero, it suffices to take an increasing even n). Take, for example, $\beta=-1-\tau$ and $\tau=\Theta/k$ arbitrarily small (i.e. k arbitrarily large). We have $|\gamma_2|^k=(-\gamma_2)^{\Theta/\tau}=[1+\tau+\sqrt{(1+\tau)^2-1}]^{\Theta/\tau}=\{[1+\sigma]^{1/\sigma}\}^{\Theta(1+\sqrt{2/\tau+1})}$ with $\sigma=\tau+\sqrt{2\tau+\tau^2}$. Notice that σ tends to zero like $\sqrt{\tau}$ as τ goes to zero. Thus, the limit of $|\gamma_2|^k$ as k goes to infinity is $+\infty$, since it equals the limit of $(1+\sigma)^{1/\sigma}$ to a power $c\Theta/\sigma$, for a suitable constant $c>0$ as σ goes to zero (by a similar argument, one can verify that the limit of $|\gamma_1|^k$ as k goes to infinity is zero).

Incidentally, $\beta=1-2p\tau^2 sin^2[m\pi/(2n)]/h^2$, and taking the most unfavorable case, we readily see that $\beta<-1$ implies that $p\tau^2>h^2$ and the CFL condition $\sqrt{p}\tau\leq h$ is violated.

On the other hand, the largest moduli values of the solution occur for $m=n/2$ if n is even. But, for such a value of m, we have $1\geq\beta\geq 0$ if $(o^2\tau/h)^2\leq 1$, which is nothing but the CFL condition. Actually the scheme is stable in the maximum norm in this case. In contrast the natural implicit counterpart of equations (3.68)–(3.69) is stable whatever τ and h. The reader may check this assertion as Exercise 3.9.

Finally, the case $-1\leq\beta<0$ is more tricky, because the roots are complex conjugate with moduli equal to one. This certainly rules out any uncontrolled solution growth, even though stability in norm cannot be ensured.

Summarizing, the CFL condition is necessary for the Von Neumann stability of schemes (3.68)–(3.69). It is also sufficient for stability in norm. We refer to reference [139] for a more comprehensive analysis of FD schemes for equation (3.3).

Remark 3.3

To date, several efficient FD- or FV-based schemes for the numerical treatment of hyperbolic equations not studied in this chapter can be found in the literature. The main principle of such techniques is to simulate as accurately as possible conservation laws of physics modelled by a system of PDEs, which is usually of the hyperbolic type. Most equations in these systems are non-linear and can even have discontinuous solutions. This is the main reason for not addressing them in this book. We refer readers to references [125] or [194] for a comprehensive study of these methods. As an outstanding example of FDM to handle discontinuous solutions to hyperbolic systems, we quote Godunov's celebrated scheme [91], which can be exported to a FV framework.

3.6 Exercises

3.1 Introduce the necessary modifications in the matrix forms (3.12)–(3.13) and (3.15)–(3.13) of the Backward Euler and the Crank–Nicolson FD schemes for the heat equation, in order to accommodate inhomogeneous Dirichlet boundary conditions. More specifically, use the $(n-1) \times (n+1)$ extension A_h^+ of the $(n-1) \times (n-1)$ matrix A_h whose non-zero entries are defined in equation (3.13).

3.2 Show that both equations (3.27) and (3.29) enjoy the property of avoiding unphysical oscillations and solution uncontrolled growth, as long as $\tau|o| \leq ch$, where