of τ reproduce such a mathematically and physically acceptable temperature evolution. Notice, however, that the smaller the τ, the slower the temperature will decrease, which is logical since larger values of τ correspond to time discretisations not so close to reality.

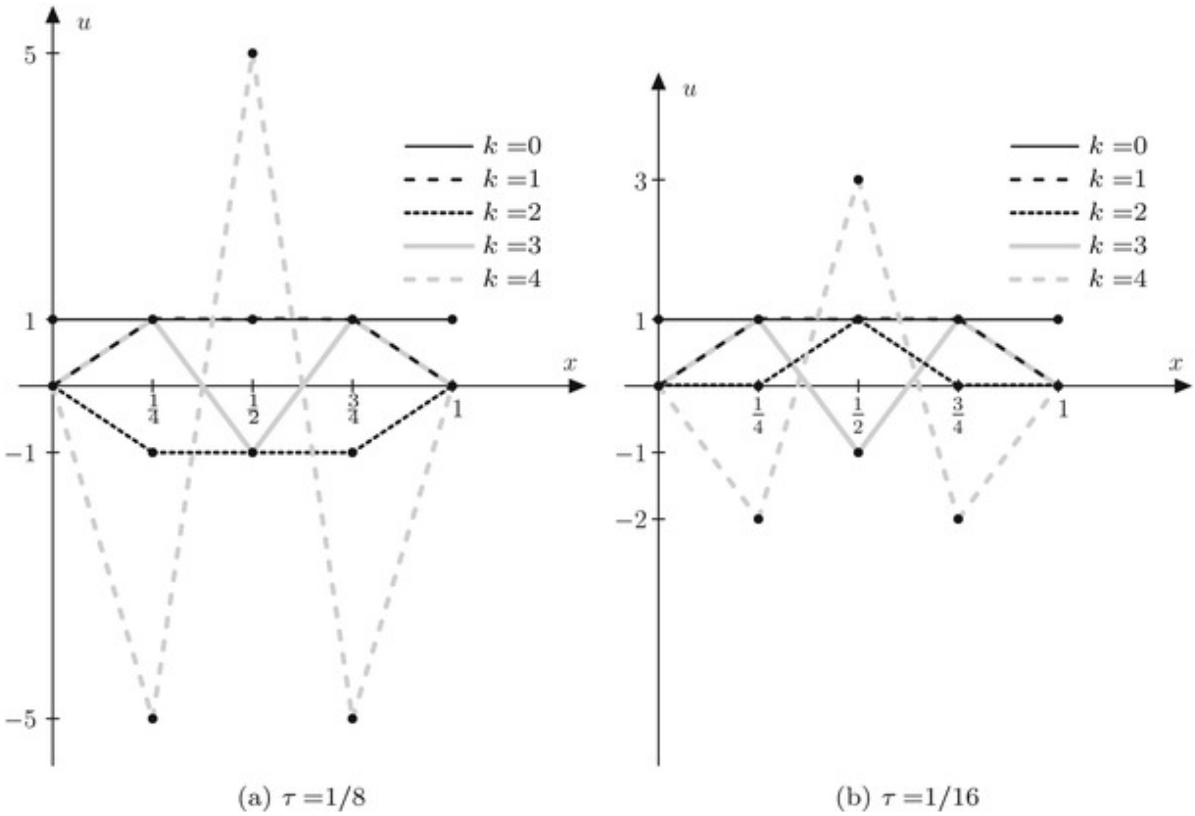

Figure 3.1 Unstable solution of the heat equation by the Forward Euler scheme for $h = 1/4$

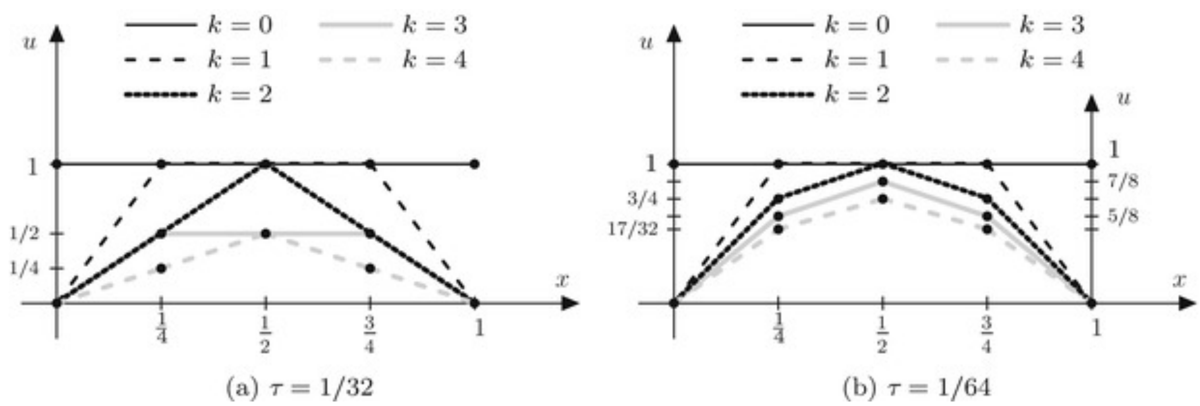

Figure 3.2 Stable solution of the heat equation by the Forward Euler scheme for $h = 1/4$

Summarizing, we have showed in this example that scheme (3.17) is unreliable if $\tau > h^2/(2p)$. The other way around, according to the study to be conducted in Section 3.3, it is always reliable as long as τ is bounded by $h^2/(2p)$. However, this implies that the time step must be very small if one uses fine spatial grids. As a consequence, in this case the numerical resolution will advance very slowly in time. As previously announced, this is the price to pay for using an explicit scheme.

3.2 Numerical Solution of the Transport Equation

From the numerical point of view, considering equations (3.6)–(3.7) will definitively lead to conclusions typical of hyperbolic equations, including second-order ones, such as equations (3.3)–(3.4). That is why we confine ourselves to studying the (first-order) transport equation in its simplest form. This equation carries this name because it governs the transport with velocity o of a quantity, whose initial distribution along the x -axis is u_0 (possibly satisfying $u_0(0) = 0$ if $o > 0$ or $u_0(L) = 0$ if $o < 0$). In case the space domain is extended to the whole real axis, and at least as long as u_0 is differentiable in $\Re$, the unique solution of equation (3.6) is given by $u(x,t) = u_0(x - ot)$, as one can easily check. This means that at time $t > 0$, the value of u_0 at a position x will have been carried to the position $x + ot$, which is perfectly normal for a transport phenomenon with constant velocity o .

We will be mostly concerned about explicit time integration methods. This is because, in contrast to the case of parabolic equations, the solution of hyperbolic equations by explicit methods is not so costly, as we will see in Section 3.3. Moreover, for the same reason as in Section 3.1, we will study in detail only FD total discretisations. Nevertheless, in some situations it will be instructive to consider FE and FV counterparts of the space discretisation. We keep assuming in this section that we are given an equally spaced grid $\mathcal{G} = [0, h, 2h, \ldots, (n-1)h, nh = L]$, together with a time step $\tau = \Theta/l$. Then, starting from $u_j^0 = u_0(jh)$, we wish to determine approximations u_j^k of $u(jh, k\tau)$ for $k = 1, 2, \ldots, l$, and $j = 1, 2, \ldots, n$ setting $u_0^k = 0$ if $o > 0$, or $j = 0, 1, \ldots, n-1$ setting $u_n^k = 0$ if $o < 0$.

3.2.1 Natural Schemes

Like in the case of the Forward Euler scheme, it seems quite natural to approximate the time derivative at $(jh, [k-1]\tau)$ by $(u_j^k - u_j^{k-1})/\tau$. As for the space derivative at the same point, a simple Taylor expansion about $(jh, [k-1]\tau)$ indicates that the centred FD $(u_{j+1}^{k-1} - u_{j-1}^{k-1})/(2h)$ for $1 \leq j \leq n-1$ is very tempting, since it yields a second-order local truncation error. Notice that a modification of this approximation of the x -derivative would be necessary to treat the case $j = n$ if $o > 0$ and $j = 0$ if $o < 0$, but for the moment we disregard this issue by considering the case where only initial conditions are prescribed and the transport equation holds in the whole real axis Ox . In doing so, we can focus on the scheme's behavior for any integer j , namely,

$$\frac{u_j^k - u_j^{k-1}}{\tau} + o\frac{u_{j+1}^{k-1} - u_{j-1}^{k-1}}{2h} = 0. \tag{3.19}$$

Here, the pure trigonometric initial condition considered in Subsection 3.1.2 would not be conclusive regarding the stability of the above scheme. On the other hand, if we enlarge the range of the transport equation to the complex domain, we will be able to draw pertinent conclusions on this issue. As a matter of fact, we will apply the so-called **Von Neumann stability criterion**, by taking an initial datum $u_0 = e^{imx}$ with $i = \sqrt{-1}$. Let $o > 0$ and $a := o\tau/(2h)$. First, we observe that equation (3.19) leads to the following expression of u_j^k for every j and $k > 1$:

$$u_j^k = u_j^{k-1} + a(u_{j+1}^{k-1} - u_{j-1}^{k-1})$$

Applying this formula to the initial values $u_j^0 = e^{imjh}$, after straightforward calculations we obtain at the first time step,

$$u_j^1 = C(a,m,h)u_j^0 \text{ for every } j, \text{where } C(a,m,h) := 1 + 2ai\ \sin(mh).$$

It immediately follows that

$$u_j^k = [C(a,m,h)]^k u_j^0 \text{ for every } j.$$

We know that the exact solution for such an initial condition is given by $u(x,t) = e^{im(x-ot)}$, whose modulus equals one for all x and t. Therefore, the modulus of the numerical solution should grow indefinitely at no grid point. Unfortunately, undesirable growth is likely to happen to scheme (3.19), since $|C(a,m,h)| = \sqrt{1 + 4a^2\sin^2(mh)}$. In other words, whatever we do with h and τ, inevitably we will be far from a physically acceptable solution in one way or another.

The situation will be completely different if, instead of the explicit approach in equation (3.19), we use its implicit version, namely,

$$\frac{u_j^k - u_j^{k-1}}{\tau} + o\frac{u_{j+1}^{k} - u_{j-1}^{k}}{2h} = 0. \tag{3.20}$$

for $j = 1, 2, \ldots, n-1$ assorted with suitable adaptations in order to approximate equation (3.6) at $x = L$ if $o > 0$ and at $x = 0$ if $o < 0$. In this aim, we set as naturally,

$$\begin{cases} \frac{u_j^k - u_j^{k-1}}{\tau} + o\frac{u_j^k - u_{j-1}^k}{h} = 0 \text{ for } j = n \text{ if } o > 0, \\ \frac{u_j^k - u_j^{k-1}}{\tau} + o\frac{u_{j+1}^k - u_j^k}{h} = 0 \text{ for } j = 0 \text{ if } o < 0. \end{cases} \tag{3.21}$$

The combination of equations (3.20) and (3.21) corresponds to the following implicit scheme to determine the u_j^k s for $k = 1, 2, \ldots, l$:

$$\begin{cases} u_j^k + o\tau u_{j+1}^k/(2h) - o\tau u_{j-1}^k/(2h) = u_j^{k-1} \text{ for } 1 \le j \le n-1, \\ (1 + o\tau/h)u_j^k - o\tau u_{j-1}^k/h = u_j^{k-1} \text{ for } j = n \text{ if } o > 0, \\ (1 - o\tau/h)u_j^k + o\tau u_{j+1}^k/h = u_j^{k-1} \text{ for } j = 0 \text{ if } o < 0. \end{cases} \tag{3.22}$$

It is interesting to check that this scheme generates no change of sign for a non-negative (resp. non-positive) u_0 , taking $\tau = 1/4$, $L = 1$, $n = 4$, $o = 1$ and $u_0 = 1$, for instance. Notice that in this case, it is necessary to solve a 4×4 linear system in order to determine the solution at every time step, whose matrix is

$$A_{1/4} = \begin{bmatrix} 1 & 1/2 & 0 & 0 \\ -1/2 & 1 & 1/2 & 0 \\ 0 & -1/2 & 1 & 1/2 \\ 0 & 0 & -1 & 2 \end{bmatrix}$$

and whose right side at the first step is $[1, 1, 1, 1]^T$. The solution at this step is $\frac{[17, 24, 27, 28]^T}{29}$. In order to push further the calculations, we suggest that the reader compute the inverse of $A_{1/4}$. In doing so, she or he would find out that the components of the solution always remain strictly positive (except for $u_0^k = 0$). Moreover, the solution absolute values never increase without control. Of course, those computations prove nothing, for this is only a particular case. Nevertheless, we can say that such a favourable behaviour does not happen by chance; it is due to the fact that equation (3.22) is unconditionally stable (see e.g. [67]). However, we refrain from going into details on this point, because there are many explicit schemes for solving hyperbolic equations that work well under very reasonable conditions, as seen further in this chapter.

3.2.2 The Lax Scheme

This scheme is obtained by modifying the way the time derivative is approximated in equation (3.19). More specifically, we replace u_j^{k-1} by $(u_{j+1}^{k-1} + u_{j-1}^{k-1})/2$, though only for $j = 1, 2, \ldots, n-1$. This gives rise to the following relations for $k = 1, 2, \ldots, l$:

The Lax scheme

$$\boxed{\begin{array}{l} \frac{u_j^k-(u_{j+1}^{k-1}+u_{j-1}^{k-1})/2}{\tau}+o\frac{u_{j+1}^{k-1}-u_{j-1}^{k-1}}{2h}=0 \text{ for } 1<j<n-1, \\ \frac{u_j^k-u_j^{k-1}}{\tau}+o\frac{u_j^{k-1}-u_{j-1}^{k-1}}{h}=0 \text{ for } j=n \text{ if } o>0, \\ \frac{u_j^k-u_j^{k-1}}{\tau}+o\frac{u_{j+1}^{k-1}-u_j^{k-1}}{h}=0 \text{ for } j=0 \text{ if } o<0. \end{array}} \tag{3.23}$$

Equation (3.23) allows one to determine in an explicit manner approximations u_j^k of u at $x = jh$ and at the k th time step by

$$\begin{cases} u_j^k = [(1-o\tau/h)u_{j+1}^{k-1}+(1+o\tau/h)u_{j-1}^{k-1}]/2 \text{ for } 1\le j\le n-1 \\ u_j^k = [(1-o\tau/h)u_j^{k-1}+o\tau u_{j-1}^{k-1}/h \text{ for } j=n \text{ if } o>0, \\ u_j^k = [(1+o\tau/h)u_j^{k-1}-o\tau u_{j+1}^{k-1}/h \text{ for } j=0 \text{ if } o<0. \end{cases} \tag{3.24}$$

It follows that, for the same test case, no value u_j^k can be negative if $|o|\tau \le h$. Of course, this does not allow us to conclude that this scheme is reliable whatever the datum u_0. Nevertheless, we will see in Section 3.3 that it is always stable if the above condition holds, and in Section 3.4 that it is convergent under the same condition. Scheme (3.23), or yet (3.24), is known as the **Lax scheme**. In principle, this scheme prevents the numerical solution from growing in absolute value as k increases if $|o|\tau \le h$. This assertion is corroborated by an argument exploited in Subsection 3.2.3, where a scheme with the very same property will be studied. For the moment, the reader might check this no-growth property by applying the Lax scheme to the data u_0, L, τ and h tested for the explicit scheme (3.19).

Equation (3.23) is only a first-order scheme in time as seen in Section 3.4. It can be improved in several manners, among which lies a modification known as the **Lax–Wendroff scheme**. For further details about this second-order scheme in both space and time, among several other possibilities to solve efficiently first-order hyperbolic PDEs, we refer to the vast literature on the subject such as reference [67].

3.2.3 Upwind Schemes

We next present another first-order scheme, this time in space too at every grid point. Akin to equation (3.23), the scheme is designed in order to avoid solution growth, among other shortcomings. Now the space derivative at point $(jh, k\tau)$ is approximated by $(u_j^k - u_{j-1}^k)/h$ if $o > 0$ and by $(u_{j+1}^k - u_j^k)/h$ if $o < 0$, for $k = 1, 2, \dots, l$. In more compact form, this approximation can be rewritten as

The Upwind scheme

(3.25)

Setting $D_o(k\tau, jh) := o^+(u_j^k - u_{j-1}^k)/h + o^-(u_{j+1}^k - u_j^k)/h$ with $o^+ := \max[0, o]$ and $o^- := \min[0, o]$, determine u_j^k for $k = 1, 2, \ldots, l$ by $\frac{u_j^k - u_j^{k-1}}{\tau} + D_o([k-1]\tau, jh) = 0$ for $j = 1, \ldots, n$ with $u_0^{k-1} = 0$ if $o > 0$, and $j = 0, \ldots, n-1$ with $u_n^{k-1} = 0$ if $o < 0$.

This means that, at the k th time step, u_j^k is given by

$$u_j^k = [1 - (o^+ - o^-)\tau/h)]u_j^{k-1} + [o^+ u_{j-1}^{k-1} - o^- u_{j+1}^{k-1}]\tau/h \quad (3.26)$$

for $1 \leq j \leq n$ with $u_0^k = 0$ if $o > 0$, and $0 \leq j \leq n-1$ with $u_n^k = 0$ if $o < 0$, for all k . Notice that the values u_{n+1}^{k-1} and u_{-1}^{k-1} are never used, but we keep them in equation (3.26) in order to unify the treatment of all grid points, setting them to zero for all k , for the sake of clarity.

The no-growth property of scheme (3.26) holds if $|o|\tau/h \leq 1$. Indeed, in this case the three coefficients of the approximations of u at the $(k-1)$ th step are non-negative and form a **partition of unity**, since their sum equals one. This property implies that

$$u_j^k \leq \max[u_{j-1}^{k-1}, u_j^{k-1}, u_{j+1}^{k-1}] \text{ and } u_j^k \geq \min[u_{j-1}^{k-1}, u_j^{k-1}, u_{j+1}^{k-1}].$$

for every j and k . Both inequalities can be established by arguments similar to those exploited in Chapter 2.

The denomination *upwind* for scheme (3.25) is due to the fact that the second point used to approximate the space derivative at a given point $x_j = jh$ is always located in the upwind direction with respect to the velocity o . In the next two sections, we will see that both equations (3.23) and (3.25) are stable and convergent provided $|o|\tau/h \leq 1$.

3.2.4 Extensions to the FVM and the FEM

To complete this presentation, we briefly address possible FV and FE counterparts of the explicit schemes (3.23) and (3.25). An Upwind scheme can be easily designed and implemented in the context of both types of FVM. It suffices to consider the **upwind CV** in the approximation of the space derivative. More specifically, recalling the definitions of Section 1.4, for $k = 1, 2, \ldots, l$, and sticking to the academic distinction between the two types of CV we considered there, the following schemes can be proposed.

Referring to Figure 1.5, for the Vertex-centred FVM the unknowns are constant approximations u_j^k of u in the CVs V_j at time $k\tau$, for $j = 0, 1, \ldots, n$, where $u_j^0 = u_0(x_j)$. We set $u_0^k = 0$ if $o > 0$ and $u_n^k = 0$ if $o < 0$ for every k. In this case, the upwind CV with respect to $V_j = (x_{j-1/2}, x_{j+1/2})$ is V_{j-1} if $o > 0$ and V_{j+1} if $o < 0$. Recalling that the length of V_j is $h_{j+1/2} = (h_j + h_{j+1})/2$ for $j = 1, 2, \ldots, n-1$ and those of V_0 and V_n are $h_{1/2} = h_1/2$ and $h_{n+1/2} = h_n/2$, respectively, the term $o\partial_x u$ at time $k\tau$ is approximated in V_j by $[o^+(u_j^k - u_{j-1}^k) + o^-(u_{j+1}^k - u_j^k)]/h_{j+1/2}$ for $j = 1, 2, \ldots, n$ if $o > 0$ and $j = 0, 1, \ldots, n-1$ if $o < 0$. Then, integrating in CV V_j, we obtain for $k = 1, 2, \ldots$:

The Upwind Vertex-centred FV scheme

$$\boxed{\begin{array}{l}\text{Determine } u_j^k \text{ satisfying,}\\ h_{j+1/2}\frac{u_j^k - u_j^{k-1}}{\tau} + o^+[u_j^{k-1} - u_{j-1}^{k-1}] + o^-[u_{j+1}^{k-1} - u_j^k] = 0,\\ \text{for } j = 1, 2, \ldots, n \text{ with } u_0^k = u_0^{k-1} = u_{n+1}^{k-1} = 0 \text{ if } o > 0,\\ \text{and } j = 0, 1, \ldots, n-1 \text{ with } u_{-1}^{k-1} = u_n^{k-1} = u_n^k = 0 \text{ if } o < 0.\end{array}} \tag{3.27}$$

At the k th time step, the unknowns are explicitly determined by

$$\begin{cases} u_j^k = [1 - \tau(o^+/h_{j+1/2} - o^-/h_{j+1/2})]u_j^{k-1} + \tau[o^+u_{j-1}^{k-1}/h_{j+1/2} - o^-u_{j+1}^{k-1}/h_{j+1/2}],\\ \text{for } j = 1, 2, \ldots, n \text{ with } u_0^k = u_0^{k-1} = u_{n+1}^{k-1} = 0 \text{ if } o > 0,\\ \text{and } j = 0, 1, \ldots, n-1 \text{ with } u_{-1}^{k-1} = u_n^{k-1} = u_n^k = 0 \text{ if } o < 0. \end{cases} \tag{3.28}$$

Like before in equations (3.27) and (3.28), $u_{-1}^{k-1}, u_{n+1}^{k-1}$, though never used, were set to zero for all k.

We refer to Figure 1.6 for the notation we adopted to formulate the Cell-centred FVM with a slight difference in the notation of the approximations of u, namely, subscript $-1/2$ is used instead of 0 and $n + 1/2$ replaces n. The unknowns are constant approximations $u_{j-1/2}^k$ of u in the CVs $T_j = (x_{j-1}, x_j)$ at time $k\tau$, for every k, and in principle for $j = 0, 1, \ldots, n$, except for $j = 0$ if $o > 0$ and for $j = n$ if $o < 0$. We also set $u_{j-1/2}^0 = u_0(x_{j-1/2})$, for $j = 0, 1, \ldots, n$, with $x_{j-1/2} = (x_j + x_{j-1})/2$ for $j = 1, 2, \ldots, n$, $x_{-1/2} = x_0 = 0$ and $x_{n+1/2} = x_n = L$. In this case, the upwind CV with respect to T_j is T_{j-1} if $o > 0$ and T_{j+1} if $o < 0$. Recalling that the measure of T_j is $h_j = x_j - x_{j-1}$ for $j = 1, 2, \ldots, n$, the term $o\partial_x u$ at time $k\tau$ is approximated in T_j by $[o^+(u_{j-1/2}^k - u_{j-3/2}^k) + o^-(u_{j+1/2}^k - u_{j-1/2}^k)]/h_j$ for $j = 1, 2, \ldots, n$ if $o > 0$ and $j = 0, 1, \ldots, n-1$ if $o < 0$. Then, integrating in CV T_j, we obtain for $k = 1, 2, \ldots$:

Ruas, Vitoriano. Numerical Methods for Partial Differential Equations : An Introduction, John Wiley & Sons, Incorporated, 2016. ProQuest Ebook Central, http://ebookcentral.proquest.com/lib/tuee/detail.action?docID=4517661.
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The Upwind -centred FV scheme

$$\boxed{\begin{array}{l}\text{Determine } u^k_{j-1/2} \text{ satisfying,}\\ h_j\frac{u^k_{j-1/2}-u^{k-1}_{j-1/2}}{\tau}+o^+[u^{k-1}_{j-1/2}-u^{k-1}_{j-3/2}]+o^-[u^{k-1}_{j+1/2}-u^{k-1}_{j-1/2}]=0,\\ \text{for } j=1,2,\ldots,n \text{ with } u^k_{-1/2}=u^{k-1}_{-1/2}=u^{k-1}_{n+1/2}=0 \text{ if } o>0,\\ \text{and } j=0,1,\ldots,n-1 \text{ with } u^{k-1}_{-3/2}=u^{k-1}_{n-1/2}=u^k_{n-1/2}=0 \text{ if } o<0.\end{array}} \tag{3.29}$$

At the k th time step, the unknowns are explicitly determined by

$$\begin{cases} u^k_{j-1/2}=[1-\tau(o^+-o^-)/h_j]u^{k-1}_{j-1/2}+\tau[o^+u^{k-1}_{j-3/2}/h_j-o^-u^{k-1}_{j+1/2}/h_j],\\ \text{for } j=1,2,\ldots,n \text{ with } u^k_{-1/2}=u^{k-1}_{-1/2}=u^{k-1}_{n+1/2}=0 \text{ if } o>0,\\ \text{and } j=0,1,\ldots,n-1 \text{ with } u^{k-1}_{-3/2}=u^{k-1}_{n-1/2}=u^k_{n-1/2}=0 \text{ if } o<0.\end{cases} \tag{3.30}$$

Akin to u^{k-1}_{-1} and u^{k-1}_{n+1} in equation (3.28), the values $u^{k-1}_{-3/2}$ and $u^{k-1}_{n+1/2}$ are never used, but were set to zero for all k in equations (3.29) and (3.30).

Like equation (3.26), both schemes above avoid solution unphysical growth or oscillations provided $\tau|o| \leq ch$, where c is the constant of the quasi-uniform family of meshes in use, and h is the maximum CV length. This result can be established as Exercise 3.2. On the other hand, in contrast to equation (3.26), except for particular situations such as uniform meshes, none of the above schemes attain a first-order of consistency in both space and time in the pointwise sense. This fact rules out any simple proof of convergence in the maximum norm, even though it does not prevent the schemes from being convergent. In this respect, we refer to reference [68].

An interesting FE analog of the Lax scheme, strictly avoiding uncontrolled growth of solution absolute values, results from a combination of the **mass lumping** technique and the exact integration of coefficients attached to the time derivative. This scheme has been mainly exploited by Kawahara and co-workers (cf. [114]) in intensive fluid flow simulations. In order to see how this works, let us assume again that the mesh is uniform with n elements and mesh size $h = L/n$. Assuming that $o > 0$, for instance, and recalling the notations of Sections 1.3 and 2.2, a natural semi-discretisation in space by the linear FEM of equations (3.6)–(3.7) is

$$\text{Find } u_h \in V_h \text{ such that } (\partial_t u_h|v)_0 + o(\partial_x u_h|v)_0 = 0 \ \forall v \in V_h.$$

Like in the case of the FDM, if we want to solve the above problem, in principle we have to further proceed to a time discretisation. In this aim, we first expand $u_h(x,t)$ as the sum $\sum_{j=1}^n u_j(t)\varphi_j(x)$. Then, by a natural and standard procedure for an explicit solution, we consider that at time $(k-1)\tau$ the time derivative of u_h is approximated by $(u_h^k - u_h^{k-1})/\tau$, where $u_h^k(x) = \sum_{j=1}^{n} u_j^k \varphi_j(x)$ for every $x \in [0, L]$, the u_j^k s being the nodal unknowns at the k th step.

After setting $v = \varphi_i$ for $i = 1, 2, \ldots, n$, the u_j^k s are easily found to be given by

$$\sum_{j=1}^{n} u_j^k (\varphi_j|\varphi_i)_0 - \sum_{j=1}^{n} u_j^{k-1}[(\varphi_j|\varphi_i)_0 + o\tau(\varphi_j'|\varphi_i)_0] = 0. \tag{3.31}$$

Now, if the mass lumping technique is used to compute the integrals $(\varphi_j|\varphi_i)_0$ on both sides of the above relation, it is not difficult to conclude that the resulting explicit scheme for computing u_j^k will be the same as equation (3.19) (cf. Exercise 3.3). Therefore, we have to discard this possibility. However, if the resolution is to remain explicit, the simplest choice is to use **mass lumping** on the left side. On the other hand, we note that exact integration of $(\varphi_j|\varphi_i)_0$ on the right side does not prevent the resolution from being explicit. That is exactly what we will do, in order to recover our FE analog of the Lax scheme. However, here a word of caution is in order: in principle, this recipe introduces an inconsistency in the space–time discretisation, due to the unbalance between both sides of the scheme. Thus, it has to be duly validated by means of a careful consistency analysis. Anyway, recalling the coefficients determined in Section 1.3 for the integrals of the product $\varphi_j \varphi_i$, and setting $u_0^k = 0$ if $o > 0$ and $u_n^k = 0$ if $o < 0$ for all $k > 0$, after straightforward calculations, which the reader is encouraged to carry out as Exercise 3.4, we come up with

An FE analog of the Lax scheme for uniform meshes

$$\boxed{\begin{array}{l} \text{For } j = 1, 2, \ldots, n-1, \\ u_j^k - 2u_j^{k-1}/3 - [1/6 + o\tau/(2h)]u_{j-1}^{k-1} - [1/6 - o\tau/(2h)]u_{j+1}^{k-1}) = 0, \\ \text{and} \\ u_j^k - [2/3 - o\tau/h]u_j^{k-1} - [1/3 + o\tau/h]u_{j-1}^{k-1} = 0 \text{ for } j = n \text{ if } o > 0; \\ u_j^k - [2/3 + o\tau/h]u_j^{k-1} - [1/3 - o\tau/h]u_{j+1}^{k-1} = 0 \text{ for } j = 0 \text{ if } o < 0. \end{array}} \tag{3.32}$$

After expressing the u_j^k s in terms of the u_j^{k-1} s in equation (3.32), by inspection we can state that, provided $|o|\tau/h \leq 1/3$, all the coefficients on the right side of the resulting relations are non-negative. Moreover, they form a partition of unity for all j . Hence, using the same argument as before, we conclude that the FE scheme (equation (3.32)), though being different from the Lax scheme, behaves very much like it. In particular, neither changes of sign can be observed for a positive (resp. negative) u_0 , nor can the numerical solution grow in absolute value as k increases. Although both properties still hold whatever the case, unfortunately this FE scheme is no longer consistent for non-uniform meshes. Therefore, some modifications are necessary in order to recover consistency and hence convergence in this case. We refer to reference [172] for further details, and also to Example 7.3.

3.3 Stability of the Numerical Models

As we saw in Chapter 2, the stability of a numerical method is not a purely mathematical property. In some sense, it must lead to the conclusion that the numerical solution is able to conform to a physical behaviour, close to the one the mathematical model represents.

In the case of the heat equation, the basic property to be searched for states that, as it evolves, the absolute value of the temperature always remains controlled by the maximum in absolute value of the heat source f , its value at the bar ends and its value u_0 at the initial time. In particular, this means that in the absence of heat sources, if the temperature is kept fixed at both ends, say $u(0) = 0$ and $u(L) = 0$, the maximum of $|u|$ at every time is bounded by the maximum of $|u_0|$. Finally, if under the same conditions the initial temperature is non-negative (resp. non-positive), then the temperature will gradually decrease (resp. increase) as time goes by.

All these properties are a consequence of the following **maximum principle**