Two-Point Boundary Value Problems

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In Chapter 1 above we encountered the wave equation in Section 1.4.3 and the heat equation in Section 1.4.4. These equations occur rather frequently in applications, and are therefore often referred to as fundamental equations. We will return to these equations in later chapters. Another fundamental equation is Poisson's equation, given by

$$-\sum_{j=1}^{n} \frac{\partial^2 u}{\partial x_j^2} = f,$$

where the unknown function u is a function of n spatial variables $x_1, \dots, x_n$.

The main purpose of this chapter is to study Poisson's equation in one space dimension with Dirichlet boundary conditions, i.e. we consider the two-point boundary value problem given by

$$-u''(x) = f(x), \quad x \in (0,1), \quad u(0) = u(1) = 0. \tag{2.1}$$

Although the emphasis of this text is on partial differential equations, we must first pay attention to a simple ordinary differential equation of second order, since the properties of such equations are important building blocks in the analysis of certain partial differential equations. Moreover, the techniques introduced for this problem also apply, to some extent, to the case of partial differential equations.

We will start the analysis of (2.1) by investigating the analytical properties of this problem. Existence and uniqueness of a solution will be demonstrated, and some qualitative properties will be derived. Then we will turn our attention to numerical methods for solving this simple problem, and we will carefully study how well the numerical solutions mimic the properties of the exact solutions. Finally, we will study eigenvalue problems associated with the boundary value problem (2.1). The results of this analysis will be a fundamental tool in later chapters.

Although the equations investigated in this chapter are very simple and allow analytical solution formulas, we find it appropriate to start our study of numerical methods by considering these problems. Clearly, numerical values of the solutions of these problems could have been generated without the brute force of finite difference schemes. However, as we will encounter more complicated equations later on, it will be useful to have a feeling for how finite difference methods handle the very simplest equations.

2.1 Poisson's Equation in One Dimension

In this section we will show that the problem (2.1) has a unique solution. Moreover, we will find a representation formula for this solution.

We start by recalling a fundamental theorem of calculus: There is a constant c_1 such that

$$u(x) = c_1 + \int_0^x u'(y)\,dy, \tag{2.2}$$

and similarly, there is a constant c_2 such that

$$u'(y) = c_2 + \int_0^y u''(z)\,dz. \tag{2.3}$$

This is true for any twice continuously differentiable function u. Suppose now that u satisfies the differential equation (2.1). Then (2.3) implies that

$$u'(y) = c_2 - \int_0^y f(z)\,dz. \tag{2.4}$$

Then, inserting this equation into (2.2), we obtain

$$u(x) = c_1 + c_2 x - \int_0^x \left(\int_0^y f(z)\,dz \right) dy. \tag{2.5}$$

In order to rewrite this expression in a more convenient form, we define

$$F(y) = \int_0^y f(z)\,dz,$$