Functional analysis homeworks

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4. Prove Hölder inequality for L^p spaces.

Hölder's inequality shows that if $f \in L^p$, $g \in L^q$ and $\frac{1}{p} + \frac{1}{q} = 1$ then $fg \in L^1$.

Proof. The proof relies on Young's inequality: if $x, y \geq 0$ then $xy \leq \frac{x^p}{p} + \frac{y^q}{q}$. This inequality is obvious if x or y is zero, so assume $x, y > 0$. For any $t \in [0, 1]$, we have

$$\log(tx + (1-t)y) \geq t \log(x) + (1-t) \log(y).$$

Setting $t = \frac{1}{p}$ (which also means that $\frac{1}{q} = 1 - t$), we get

$$\log\left(\frac{x^p}{p} + \frac{y^q}{q}\right) \geq \frac{1}{p}\log(x^p) + \frac{1}{q}\log(y^q) = \log(xy)$$

from which Young's inequality follows.

Now we prove Hölder's inequality. If f or g are zero the it is trivial, so assume

$$\|f\|_p, \|g\|_q > 0$$

Defining

$$F = \frac{f}{\|f\|_p}, G = \frac{g}{\|g\|_q}$$

Then $F \in L^p$, $G \in L^p$ with $\|F\|_p = \|G\|_q = 1$. Using Young's inequality, we have

$$\int_X |FG| \, d\mu \leq \int_X \left(\frac{|F|^p}{p} + \frac{|G|^q}{q}\right) \, d\mu = \frac{1}{p}\|F\|_p^p + \frac{1}{q}\|G\|_q^q = 1.$$

We have shown

$$\frac{1}{\|f\|_p \|g\|_q} \int_X |fg| \, d\mu = \int_X \left| \frac{f}{\|f\|_p} \frac{g}{\|g\|_q} \right| \, d\mu \leq 1.$$

By carefully keeping track of where equality occurs in Young's inequality, one can show that equality occurs if and only if $|f|^p$ and $|g|^q$ are linearly dependent on L^1.

5. Prove Minkowski inequality for L^p spaces.

The Minkowski inequality establishes that the L^p spaces are normed vector spaces.

Proof. First, we prove that $f + g$ has a finite p-form if f and g both do, which follows by

$$|f+g|^p \leq 2^{p-1}(|f|^p + |g|^p).$$

Here we use the fact that $h(x) = |x|^p$ is convex (meaning the line segment between any two points on the graph of the function lies above the graph between the two points) over R^+ (for $p > 1$) and so, by definition of convexity,

$$\left|\tfrac{1}{2}f + \tfrac{1}{2}g\right|^p \leq \left|\tfrac{1}{2}|f| + \tfrac{1}{2}|g|\right|^p \leq \tfrac{1}{2}|f|^p + \tfrac{1}{2}|g|^p.$$

This means that

$$|f+g|^p \leq \tfrac{1}{2}|2f|^p + \tfrac{1}{2}|2g|^p = 2^{p-1}|f|^p + 2^{p-1}|g|^p.$$

Now, we can focus on $\|f+g\|_p$. If it is zero, then Minkowski's inequality holds, which is trivial. We now assume that $\|f+g\|_p$ is not zero. Using the triangle inequality and then Hölder's inequality, we find that

$$\begin{aligned}
\|f+g\|_p^p &= \int |f+g|^p \,\mathrm{d}\mu \\
&= \int |f+g| \cdot |f+g|^{p-1} \,\mathrm{d}\mu \\
&\leq \int (|f| + |g|)|f+g|^{p-1} \,\mathrm{d}\mu \\
&= \int |f||f+g|^{p-1} \,\mathrm{d}\mu + \int |g||f+g|^{p-1} \,\mathrm{d}\mu \\
&\leq \left(\left(\int |f|^p \,\mathrm{d}\mu \right)^{\frac{1}{p}} + \left(\int |g|^p \,\mathrm{d}\mu \right)^{\frac{1}{p}} \right) \left(\int |f+g|^{(p-1)\left(\frac{p}{p-1}\right)} \,\mathrm{d}\mu \right)^{1-\frac{1}{p}} && \text{Hölder's inequality} \\
&= (\|f\|_p + \|g\|_p) \frac{\|f+g\|_p^p}{\|f+g\|_p}.
\end{aligned}$$

We obtain Minkowski's inequality by multiplying both sides by

$$\frac{\|f+g\|_p}{\|f+g\|_p^p},$$

after which we are finally left with

$$\|f+g\|_p \leq \|f\|_p + \|g\|_p.$$

This shows that $f + g \in L^p(\mu)$ and that $\|f+g\|_p \leq \|f\|_p + \|g\|_p$.