Chapter 6

Brownian Motion

6.1 Normal Distribution

Definition 6.1.1. *A r.v. X has a normal distribution with mean μ and variance σ^2, where $\mu \in \mathbb{R}$, and $\sigma > 0$, if its density is $f(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$.*

The previous definition makes sense because f is a nonnegative function and

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx = 1.$$

Note that by the changes of variables $\frac{x-\mu}{\sigma} = t$

$$\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx = \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\, dt.$$

By a change to polar coordinates,

$$\begin{aligned}&\left(\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\, dx\right)^2 = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}-\frac{y^2}{2}}\, dx\, dy = \int_0^{\infty}\int_0^{2\pi} e^{-\frac{r^2}{2}} r\, d\theta\, dr \\ &= 2\pi \int_0^{\infty} e^{-\frac{r^2}{2}} r\, dr = 2\pi(-e^{-\frac{r^2}{2}})\Big|_0^{\infty} = 2\pi.\end{aligned}$$

Theorem 6.1. *If X has a normal distribution with parameters μ and σ^2, then $E[X] = \mu$ and $\mathrm{Var}(X) = \sigma^2$.*

Proof. By the change of variables $\frac{x-\mu}{\sigma} = t$,

$$\begin{aligned}&E[X] = \int_{-\infty}^{\infty} x \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx = \int_{-\infty}^{\infty} (\mu + t\sigma) \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{t^2}{2}}\, dt \\ &= \mu \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{t^2}{2}}\, dt + \sigma \int_{-\infty}^{\infty} t \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{t^2}{2}}\, dt = \mu,\end{aligned}$$

because the function $t\frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{t^2}{2}}$ is odd.

By the change of variables $\frac{x-\mu}{\sigma} = t$ and integration by parts,

$$\begin{aligned}&E[(X-\mu)^2] = \int_{-\infty}^{\infty} (x-\mu)^2 \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(x-\mu)^2}{2\sigma^2}}\, dx = \int_{-\infty}^{\infty} \sigma^2 t^2 \frac{1}{\sqrt{2\pi}} e^{-\frac{t^2}{2}}\, dt \\ &= \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} t\, d\left(-e^{-\frac{t^2}{2}}\right) = \frac{\sigma^2}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{t^2}{2}}\, dt = \sigma^2.\end{aligned}$$

Q.E.D.

The following formulas could be useful:

$$\int_{-\infty}^{\infty} e^{-\frac{x^2}{2}}\,dx = \sqrt{2\pi},\ \int_{-\infty}^{\infty} xe^{-\frac{x^2}{2}}\,dx = 0,\ \int_{-\infty}^{\infty} x^2e^{-\frac{x^2}{2}}\,dx = \sqrt{2\pi}$$

A normal r.v. with mean zero and variance one is called a standard normal r.v. Probabilities for a standard normal r.v. can be found in table at the back of the book. Given a standard normal r.v. Z, to find $\mathrm{P}(Z \le -a)$, where $a > 0$, we do

$$\mathrm{P}[Z \le -a] = P[Z \ge a] = 1 - \mathrm{P}[Z \le a].$$

To find $\mathrm{P}(|Z| \le a)$, where $a > 0$, we do

$$\begin{aligned}&\mathrm{P}[|Z| \le a] = \mathrm{P}[-a \le Z \le a]\\ &= P[Z \le a] - \mathrm{P}[Z \le -a] = P[Z \le a] - (1 - P[Z \ge a]) = 2P[Z \ge a] - 1.\end{aligned}$$

We usually will denote the density of a standard normal r.v. by $\phi(x) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}$. We denote the cumulative distribution function of a standard normal r.v. by $\Phi(x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi}}e^{-\frac{t^2}{2}}\,dt$.

Theorem 6.2. *The moment generating function of a r.v. X with a normal distribution with parameters μ and σ^2, is*

$$M_X(t) = e^{t\mu+\frac{t^2\sigma^2}{2}}.$$

Proof. By the change of variables $\frac{x-\mu}{\sigma} = y$,

$$\begin{aligned}&M_X(t) = \int_{-\infty}^{\infty} e^{tx}\frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(x-\mu)^2}{2\sigma^2}}\,dx = \int_{-\infty}^{\infty} e^{t\mu+t\sigma y}\frac{1}{\sqrt{2\pi}}e^{-\frac{y^2}{2}}\,dy\\ &= e^{t\mu+\frac{t^2\sigma^2}{2}}\int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}}e^{-\frac{(y-\sigma t)^2}{2}}\,dy = e^{t\mu+\frac{t^2\sigma^2}{2}}.\end{aligned}$$

Q.E.D.

A standard normal r.v. Z has moment generating function: $M_Z(t) = e^{\frac{t^2}{2}}$.

Theorem 6.3. *If X has a normal distribution with parameters μ and σ^2, then $Z = \frac{X-\mu}{\sigma}$ has a standard normal r.v.*

Proof. The moment generating function of $Z = \frac{X-\mu}{\sigma}$ is

$$M_X(t) = E[e^{t\frac{X-\mu}{\sigma}}] = e^{-\frac{t\mu}{\sigma}}E[e^{\frac{tX}{\sigma}}] = e^{\frac{t^2}{2}},$$

which is the moment generating function of a standard normal r.v. Q.E.D.

Theorem 6.4. *Let $X_1, \dots, X_n$ be independent r.v.'s with a normal distribution. Then, $\sum_{i=1}^n X_i$ has a normal distribution.*

Proof. Let $\mu_i = E[e^{tX_i}]$ and let $\sigma_i^2 = \mathrm{Var}(X_i)$. Then, $E[e^{tX_i}] = \exp\left(t\mu_i + \frac{t^2\sigma_i^2}{2}\right)$. So, the moment generating function of $\sum_{i=1}^n X_i$ is

$$\begin{aligned}&M_{\sum_{i=1}^n X_i}(t) = \prod_{i=1}^n M_{X_i}(t) = \prod_{i=1}^n \exp\left(t\mu_i + \frac{t^2\sigma_i^2}{2}\right)\\ &= \exp\left(t\sum_{i=1}^n \mu_i + \frac{t^2\sum_{i=1}^n \sigma_i^2}{2}\right).\end{aligned}$$

Now, $\exp\left(t\sum_{i=1}^n \mu_i + \frac{t^2\sum_{i=1}^n \sigma_i^2}{2}\right)$ is the moment generating function of a normal distribution with mean $\sum_{i=1}^n \mu_i$ and variance $\sum_{i=1}^n \sigma_i^2$. Since the moment generating function determines the distribution, we conclude that $\sum_{i=1}^n X_i$ has a normal distribution with mean $\sum_{i=1}^n \mu_i$ and variance $\sum_{i=1}^n \sigma_i^2$. Q.E.D.

Example 6.1. *If X is a normal random variable with parameters $\mu = 1$ and $\sigma^2 = 4$, compute* $\mathrm{P}[5 \geq X > 3]$.

Solution: We standardize the variable: $Z = \frac{X-1}{\sqrt{4}}$ has a standard normal distribution. So,

$$\begin{aligned}&\mathrm{P}[5 \geq X > 3] = \mathrm{P}[3 < X \leq 5] = \mathrm{P}\left(\tfrac{3-1}{\sqrt{4}} \leq Z \leq \tfrac{5-1}{\sqrt{4}}\right)\\&= \mathrm{P}[1 < Z \leq 2] = \mathrm{P}[Z \leq 2] - \mathrm{P}[Z \leq 1] = 0.9772 - 0.8413 = 0.1359.\end{aligned}$$

Example 6.2. *A manufacturer produces an item which has a label weight of* 20.4 *grams. Let X denote the weight of a single mint selected at random from the production line. X has a normal distribution with mean* 21.37 *and variance* 0.16*. Find* $\mathrm{P}[X < 20.4]$.

Solution: $Z = \frac{X-\mu}{\sigma}$ has a standard normal distribution. So,

$$\mathrm{P}[X < 20.4] = \mathrm{P}\left[Z \leq \frac{20.4 - 21.37}{\sqrt{0.16}}\right] = \mathrm{P}[Z \leq -2.45] = .008.$$

Example 6.3. *Let Z be a standard normal r.v. Find c such that $P[|Z| \leq c] = .95$.*

Solution: Using the previous formula,

$$0.95 = P[|Z| \leq c] = 2P[Z \leq c] - 1.$$

So, $\mathrm{P}[Z \leq c] = \frac{1+0.95}{2} = 0.975$. From the normal table, $c = 1.96$.

Exercise 6.1. *The height, X, that a college high jumper will clear each time she jumps is a normal random variable with mean 6 feet and variance 5.765 inches^2. What is the probability the jumper will clear 6 feet 4 inches in on a single jump?*

Exercise 6.2. *The length of the time required to complete a college achievement test is found to be normally distributed with mean 70 minutes and standard deviation 12 minutes. When should the test be terminated if we wish to allow sufficient time of 90 % of the students to complete the test?*

Exercise 6.3. *Suppose that X is a standard normal random variable with parameters $\mu = 1$ and $\sigma^2 = 4$. Find $P[X^2 > 1]$.*

Example 6.4. *If X and Y are independent identically distributed standard normal random variables, compute* $\Pr\{-1 \leq X + Y \leq 3\}$.

Solution: $X + Y$ has normal distribution with mean zero and variance 2. So,

$$\begin{aligned}&\mathrm{P}[-1 \leq X + Y \leq 3] = \mathrm{P}[\tfrac{-1}{\sqrt{2}} \leq \mathrm{N}(0,1) \leq \tfrac{3}{\sqrt{2}}]\\&= \mathrm{P}[-0.70 \leq \mathrm{N}(0,1) \leq 2.12] = 0.98 - (1 - 0.75) = 0.73.\end{aligned}$$

Example 6.5. *If X and Y are independent identically distributed standard normal random variables, compute the density of $U = 1 + X + Y$.*

Solution: By the previous theorem, U has a normal distribution. $E[U] = 1$ and $\text{Var}(U) = \text{Var}(X) + \text{Var}(Y) = 2$. So, the density of U is

$$f_U(u) = \frac{e^{-\frac{(u-1)^2}{2\cdot 2}}}{\sqrt{2\pi 2}} = \frac{e^{-\frac{(u-1)^2}{4}}}{2\sqrt{\pi}}.$$

6.2 Central Limit Theorem

If Y is a normal r.v. with mean μ and variance σ^2, then $\frac{Y-\mu}{\sigma}$ has a standard normal distribution. The central limit theorem says that we can do something like that for sums of i.i.d.r.v.'s. Let $X_1, \dots, X_n$ be i.i.d.r.v.'s with mean μ and variance σ^2. Then,

$$E\left[\sum_{i=1}^n X_i\right] = \sum_{i=1}^n E[X_i] = n\mu$$

and

$$\text{Var}\left(\sum_{i=1}^n X_i\right) = \sum_{i=1}^n \text{Var}(X_i) = n\sigma^2.$$

The central limit theorem says that for n large enough

$$\frac{(\sum_{i=1}^n X_i) - E[\sum_{i=1}^n X_i]}{\sqrt{\text{Var}(\sum_{i=1}^n X_i)}} = \frac{\sum_{i=1}^n (X_i - \mu)}{\sigma\sqrt{n}}$$

has approximately a normal distribution. Precisely, the central limit theorem says that:

Theorem 6.5. *Let $\{X_i\}_{i=1}^\infty$ be a sequence of i.i.d.r.v.'s with finite second moment. Then, for each $a \in \mathbb{R}$,*

$$\lim_{n\to\infty} \text{P}\left[\frac{\sum_{i=1}^n (X_i - E[X_i])}{\sigma\sqrt{n}} \le a\right] = \Phi(a),$$

where $\Phi(a) = \text{P}[\text{N}(0,1) \le a]$.

The central limit theorem is also true for open intervals and bounded intervals:

$$\lim_{n\to\infty} \text{P}\left[\frac{\sum_{i=1}^n (X_i - E[X_i])}{\sqrt{n}\sigma} < a\right] = P[Z < a]$$

and

$$\lim_{n\to\infty} \text{P}\left[a \le \frac{\sum_{i=1}^n (X_i - E[X_i])}{\sqrt{n}\sigma} \le b\right] = \text{P}[a \le Z \le b],$$

where Z denotes a r.v. with a standard normal distribution.

Instead of $\sum_{i=1}^n X_i$, we can use $\bar{x} = \frac{1}{n}\sum_{i=1}^n X_i$, and get that

$$\lim_{n\to\infty} \text{P}\left[\frac{\bar{x} - \mu}{\sigma/\sqrt{n}} \le a\right] = \text{P}[\text{N}(0,1) \le a].$$

For most distributions, the central limit theorem gives a good approximation to the probability if $n \geq 30$. For some distributions $n = 20$ is enough.

A common application of the central limit theorem is the estimation of probabilities of a binomial distribution with large n. In this case,

Theorem 6.6. *Let S_n be the a r.v. with a binomial distribution with parameters n and p, then*

$$\lim_{n\to\infty} \mathrm{P}\left[\frac{S_n - np}{\sqrt{np(1-p)}} \leq a\right] = \Phi(a).$$

When handling discrete distributions, a continuity correction can be made. If a is an integer, then for each $0 \leq t < 1$

$$\mathrm{P}\left[S_n \leq a\right] = \mathrm{P}\left[S_n \leq a + t\right] = \mathrm{P}\left[\tfrac{S_n - np}{\sqrt{np(1-p)}} \leq \tfrac{a+t-np}{\sqrt{np(1-p)}}\right] \approx \Phi\left(\tfrac{a+t-np}{\sqrt{np(1-p)}}\right).$$

Since each $0 \leq t < 1$ gives a different approximation, the average of these t's is taken. So,

$$\begin{aligned}
&\mathrm{P}\left[S_n \leq a\right] \approx \Phi\left(\tfrac{a+0.5-np}{\sqrt{np(1-p)}}\right)\\
&\mathrm{P}\left[S_n < a\right] \approx \Phi\left(\tfrac{a-0.5-np}{\sqrt{np(1-p)}}\right)\\
&\mathrm{P}\left[a \leq S_n \leq b\right] \approx \Phi\left(\tfrac{b+0.5-np}{\sqrt{np(1-p)}}\right) - \Phi\left(\tfrac{a-0.5-np}{\sqrt{np(1-p)}}\right)\\
&\mathrm{P}\left[a < S_n < b\right] \approx \Phi\left(\tfrac{b-0.5-np}{\sqrt{np(1-p)}}\right) - \Phi\left(\tfrac{a+0.5-np}{\sqrt{np(1-p)}}\right)
\end{aligned}$$

Example 6.6. *Let S be the number of heads in 1000 independent tosses of a fair coin. Find the probability that $480 \leq S \leq 515$.*

Solution: S has a binomial distribution with parameters $n = 1000$ and $p = \frac{1}{2}$. Then, $E[S] = np = 1000 \cdot \frac{1}{2} = 500$ and $\mathrm{Var}(S) = np(1-p) = 1000 \cdot \frac{1}{2} \cdot \frac{1}{2} = 250$. So,

$$\begin{aligned}
&\mathrm{P}[480 \leq S \leq 515] \approx \mathrm{P}[\tfrac{479.5-500}{\sqrt{250}} \leq Z \leq \tfrac{515.5-500}{\sqrt{250}}]\\
&= \Phi(0.9803) - \Phi(-1.2965) = 0.8365 - (1 - 0.9026) = 0.7391.
\end{aligned}$$

Exercise 6.4. *Let X be the number of heads in 10000 independent tosses of a fair coin. Find the probability that $4900 \leq X \leq 5150$.*

Exercise 6.5. *One thousand independent throws of a fair coin will be made. Let X be the number of heads in these thousand independent throws. Find the smallest value of a so that*

$$P[500 - a \leq X \leq 500 + a] \geq 0.70.$$

Exercise 6.6. *Let X be the number of heads in 10,000 independent tosses of a fair coin. Find c so that the probability that $5000 - c \leq X \leq 5000 + c$ is 95 %.*

Exercise 6.7. *One thousand independent rolls of a fair die are made. Let X be the number of times that the number 6 appears. Find the probability that $150 \leq X < 200$.*

Exercise 6.8. *One thousand independent rolls of a fair die will be made. Let X be the number of sixes which will appear. Find the probability that $160 \leq X \leq 190$.*

Exercise 6.9. *One thousand independent rolls of a fair die will be made. Let X be the sum of these thousand independent rolls. Find the value of a so that*

$$P[350 - a \leq X \leq 350 + a] = 0.9.$$

Exercise 6.10. *It is assumed that the probability that a student in college has a GPA in high school of A- or above is 15%. Suppose that 500 students from that college are chosen at random, what is the probability that 10 students or less of these 500 students have a GPA in high school of A- or above.*

Exercise 6.11. *Let $Y_1, \dots, Y_{200}$ independent identically distributed random variables with exponential distribution of parameter $\lambda = 2$. Find (approximately)*

$$\Pr[Y_1 + \cdots + Y_{200} \geq 110].$$

Exercise 6.12. *One has 100 bulbs whose light times are independent exponentials with mean 5 hours. If the bulbs are used one at time, with a failed bulb being immediately replaced by a new one, what is the probability that there is still a working bulb after 525 hours.*

Exercise 6.13. *From past experience a professor knows that the test score of a student taking her final examination is a random variable with mean 70 and variance 25. Assuming that the distribution of grades is approximately bell–shaped, what can be said about the probability that student's score is between 60 and 80?*

Exercise 6.14. *A manufacturer of booklets packages them in boxes of 1000. It is known that, on the average, the booklets weigh 1 ounce, with a standard deviation of 0.05 oz. The manufacturer is interested in calculating*

$$P[100 \ \text{ booklets } \ \text{weigh more } 100.4 \text{ oz}],$$

a number hat would help detect whether too many booklets are being put in a box. Estimate last probability.

Exercise 6.15. *The scores of a reference population in the Wechsler Intelligence Scale top 5% of the population?*

Exercise 6.16. *Suppose that the weight (in ounces) of major league baseball is a random variable with mean $\mu = 5$ and standard deviation $\sigma = 2/5$. Find the probability that a carton of 144 baseballs has a total weight less than 725 ounces.*

Exercise 6.17. *A manufacturer packages some item in boxes of hundred. The weight of each item has mean 1 oz. and standard deviation 0.1 oz. Approximate the probability that a box weights more than 102 oz.*

6.3 Brownian Motion

Definition 6.3.1. *A stochastic process* $\{B(t) : t \geq 0\}$ *is said to be a Brownian motion process with variance parameter* $\sigma^2 > 0$ *if:*
(i) $B(0) = 0$.
(ii) (independent increments) For each $0 \leq t_1 < t_2 < \cdots < t_m$,

$$B(t_1), B(t_2) - B(t_1), \ldots, B(t_m) - B(t_{m-1})$$

are independent r.v.'s.
(iii) (stationary increments) For each $0 \leq s < t$, $B(t) - B(s)$ *has a normal distribution with mean zero and variance* $\sigma^2(t-s)$.

If $\sigma^2 = 1$, we said that $\{B(t) : t \geq 0\}$ is a standard Brownian motion. If $\{B(t) : t \geq 0\}$ is a Brownian motion process with variance parameter $\sigma^2 > 0$, then $\{\sigma^{-1}B(t) : t \geq 0\}$ is a standard Brownian motion. So, the study of Brownian motion reduces to the case of a standard Brownian motion. Unless, it is said otherwise, $\{B(t) : t \geq 0\}$ will denote a standard Brownian motion.

Theorem 6.7. *Let* $\{B(t) : t \geq 0\}$ *a standard Brownian motion. Then, the probability density function of* $B(t)$ *is*

$$f_{B(t)}(x) = \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}.$$

Example 6.7. *Let* $\{B(t) : t \geq 0\}$ *a standard Brownian motion. Let* $0 < s < t$. *Let* $a, b \in \mathbb{R}$. *Show that:*
(i) $\mathrm{Cov}(B(s), B(t)) = s$.
(ii) $\mathrm{Var}(B(t) - B(s)) = t - s$.
(iii) $\mathrm{Var}(aB(s) + bB(t)) = (a+b)^2 s + b^2(t-s)$.
(iv) The distribution of $aB(s) + bB(t)$ *is normal with mean zero.*

Solution: (i) Since $B(s)$ and $B(t) - B(s)$ are independent r.v.'s,

$$\begin{aligned} & \mathrm{Cov}(B(s), B(t)) = \mathrm{Cov}(B(s), B(s) + B(t) - B(s)) \\ = \; & \mathrm{Cov}(B(s), B(s)) + \mathrm{Cov}(B(s), B(t) - B(s)) = \mathrm{Var}(B(s)) = s. \end{aligned}$$

(ii) Since $B(t) - B(s)$ has a normal distribution with mean zero and variance $t - s$, $\mathrm{Var}(B(t) - B(s)) = t - s$.

(iii)

$$\begin{aligned} & \mathrm{Var}(aB(s) + bB(t)) = \mathrm{Var}(aB(s) + b(B(s) + B(t) - B(s)) \\ = \; & \mathrm{Var}((a+b)B(s) + b(B(t) - B(s))) = \mathrm{Var}((a+b)B(s)) + \mathrm{Var}(b(B(t) - B(s)) \\ = \; & (a+b)^2 s + b^2(t-s) \end{aligned}$$

(iv) Since $B(s)$ and $B(t) - B(s)$ are independent r.v.'s and they have a normal distribution, $(a+b)B(s) + b(B(t) - B(s)) = aB(s) + bB(t)$ has a normal distribution.

Example 6.8. *Suppose that* $\{X(t) : t \geq 0\}$ *is a Brownian motion process with variance parameter* $\sigma^2 = 9$. *Find:*
(i) $\mathrm{P}\{X(2) \leq 15\}$
(ii) $\mathrm{Var}(3X(2) - 2X(5))$.
(iii) $\mathrm{P}\{X(2) - 2X(3) \leq 4\}$

Solution: We have that $\{B(t) : t \geq 0\}$ is a standard Brownian motion, where $B(t) = \frac{X(t)}{3}$.
(i) We have that

$$\mathrm{P}\{X(2) \leq 12\} = \mathrm{P}\{B(2) \leq 4\} = \mathrm{P}\{N(0,1) \leq \tfrac{4}{\sqrt{2}}\} = \mathrm{P}\{Z \leq 2.8284\} = 0.9976609$$

(ii)

$$\begin{aligned} & \mathrm{Var}(3X(2) - 2X(5)) = \mathrm{Var}(9B(2) - 6B(5)) = \mathrm{Var}(3B(2) - 6(B(5) - B(2))) \\ = \; & (3)^2 \cdot (2) + (-6)^2 \cdot (3) = 117 \end{aligned}$$

(iii) $X(2) - 2X(3)$ has a normal distribution with mean zero and variance

$$\begin{aligned} & \mathrm{Var}(X(2) - 2X(3)) = \mathrm{Var}(3B(2) - 6B(3)) = \mathrm{Var}(-3B(2) - 6(B(3) - B(2))) \\ & = (-3)^2(2) + (-6)^2(3-2) = 54 \end{aligned}$$

So,

$$\mathrm{P}\{X(2) - 2X(3) \leq 4\} = \mathrm{P}\{Z \leq \tfrac{4}{\sqrt{54}}\} = \mathrm{P}\{Z \leq 0.54433\} = 0.7068928$$

Example 6.9. *Let* $\{B(t) : t \geq 0\}$ *a standard Brownian motion. Let* $0 < s < t < u$. *Show that* $E[B(s)B(t)B(u)] = 0$.

Solution: Let $X = B(s)$, let $Y = B(t) - B(s)$ and let let $Z = B(u) - B(t)$. Then, Z and Y and Z are independent r.v.'s with mean zero and $E[X^2] = s$, $E[Y^2] = t - s$ and let let $E[Z^2] = u - t$. So,

$$\begin{aligned} & E[B(s)B(t)B(u)] = E[X(X+Y)(X+Y+Z)] = E[X^3 + 2X^2YZ + XY^2 + X^2Z + XYZ] \\ = \; & E[X^3] + 2E[X^2]E[Y]E[Z] + E[X]E[Y^2] + E[X^2]E[Z] + E[X]E[Y]E[Z] = 0 \end{aligned}$$

Exercise 6.18. *Let* $\{B(t) : t \geq 0\}$ *a standard Brownian motion. Let* $0 < s < t < u$. *Let* $a, b, c \in \mathbb{R}$. *Show that:*
(i) $\mathrm{Var}(aB(s) + bB(t) + cB(u)) = (a+b+c)^2 s + (b+c)^2(t-s) + c^2(u-t)$.
(ii) The distribution of $aB(s) + bB(t) + cB(u)$ *is normal with mean zero.*

Exercise 6.19. *Find the density function of* $X = B(s) + B(t)$, *where* $0 \leq s < t$?

Exercise 6.20. *Find the density function of* $X = 2B(2) - 3B(3) + 4B(5)$.

Exercise 6.21. *Suppose that* $\{X(t) : t \geq 0\}$ *is a Brownian motion process with variance parameter* $\sigma^2 = 8$. *Find:*
(i) $\mathrm{P}\{|X(4) - X(2)| > 10\}$.
(ii) $\mathrm{Var}(3 + X(4) - 2X(2) + X(3))$.
(iii) $\mathrm{Cov}(3 + X(4) - 2X(2), 5 - X(3))$.

Recall that if $(X_1,\dots,X_m)$ is a r.v. with joint density function $f_{X_1,\dots,X_m}(x_1,\dots,x_m)$, and h is a smooth transformation on the domain of $(X_1,\dots,X_m)$, then the joint density of $(Y_1,\dots,Y_m)=h(X_1,\dots,X_m)$, is

$$f_{Y_1,\dots,Y_m}(y_1,\dots,y_m)=f_{X_1,\dots,X_m}(h^{-1}(y_1,\dots,y_m))|Jh^{-1}(y_1,\dots,y_m)|,$$

where $Jh^{-1}(y_1,\dots,y_m)$ is the determinant of the partial derivatives of the function h^{-1} (the Jacobian of the h^{-1}).

Theorem 6.8. *Let $\{B(t):t\ge 0\}$ a standard Brownian motion. Let $0<t_1<\cdots,t_m$. Then, the joint probability density function of $(B(t_1),\dots,B(t_m))$ is*

$$f_{B(t_1),\dots,B(t_m)}(x_1,\dots,x_m)=\frac{\exp\left(-\frac{1}{2}\left(\frac{x_1^2}{t_1}+\frac{(x_2-x_1)^2}{t_2-t_1}+\cdots+\frac{(x_m-x_{m-1})^2}{t_m-t_{m-1}}\right)\right)}{(\sqrt{2\pi})^m\sqrt{t_1(t_2-t_1)\cdots(t_m-t_{m-1})}}.$$

Proof. Let $X_1=B(t_1),X_2=B(t_2)-B(t_1),\dots,X_m=B(t_m)-B(t_{m-1})$. $X_1,X_2,\dots,X_m$ are independent r.v.'s. X_i has normal distribution with mean 0 and variance t_i-t_{i-1}. So, the density of X_i is $f_{X_i}(x_i)=\frac{1}{\sqrt{2\pi(t_i-t_{i-1})}}e^{-\frac{x_i^2}{2(t_i-t_{i-1})}}$ and the joint density function of $(X_1,X_2,\dots,X_m)$ is

$$\begin{aligned}&f_{X_1,\dots,X_m}(x_1,\dots,x_m)=\textstyle\prod_{i=1}^m f_{X_i}(x_i)=\prod_{i=1}^m\frac{1}{\sqrt{2\pi(t_i-t_{i-1})}}e^{-\frac{x_i^2}{2(t_i-t_{i-1})}}\\&=\tfrac{1}{\left(\sqrt{2\pi}\right)^m\sqrt{t_1(t_2-t_1)\cdots(t_m-t_{m-1})}}\exp\left(-\tfrac{x_1^2}{2t_1}-\tfrac{x_2^2}{2(t_2-t_1)}-\cdots--\tfrac{x_m^2}{2(t_m-t_{m-1})}\right)\end{aligned}$$

Consider the transformation $(y_1,y_2,\dots,y_m)=h(x_1,x_2,\dots,x_m)=(x_1,x_1+x_2,\dots,x_1+\cdots+x_m)$. We have that $h^{-1}(y_1,y_2,\dots,y_m)=(y_1,y_2-y_1,\dots,y_m-y_{m-1})$ and $|Jh^{-1}(y_1,\dots,y_m)|=1$. Then,

$$(Y_1,Y_2,\dots,Y_m)=h(X_1,X_2,\dots,X_m)=(B(t_1),B(t_2),\dots,B(t_m)).$$

The joint density of $(Y_1,Y_2,\dots,Y_m)$ is

$$\begin{aligned}&f_{Y_1,\dots,Y_m}(y_1,\dots,y_m)=f_{X_1,\dots,X_m)}(h^{-1}(y_1,\dots,y_m))|Jh^{-1}(y_1,\dots,y_m)\\&=f_{X_1,\dots,X_m)}(y_1,y_2-y_1,\dots,y_m-y_{m-1})\\&=\tfrac{1}{\left(\sqrt{2\pi}\right)^m\sqrt{t_1(t_2-t_1)\cdots(t_m-t_{m-1})}}\exp\left(-\tfrac{y_1^2}{2t_1}-\tfrac{(y_2-y_1)^2}{2(t_2-t_1)}-\cdots--\tfrac{(y_m-y_{m-1})^2}{2(t_m-t_{m-1})}\right)\end{aligned}$$

Q.E.D.

Markov property of the Brownian motion: Given $0<s_1<\cdots<s_k<t_1<\cdots<t_m$, we have that the distribution of $B(t_1),\dots,B(t_m)$ given $B(s_1),\dots,B(s_k)$ agrees with the distribution of $B(t_1),\dots,B(t_m)$ given $B(s_k)$.

Theorem 6.9. *Let $\{B(t):t\ge 0\}$ a standard Brownian motion. Given $0<s_1<\cdots<s_k<t_1<\cdots<t_m$, we have that the conditional density of $B(t_1),\dots,B(t_m)$ given $B(s_1),\dots,B(s_k)$ agrees with the conditional density of $B(t_1),\dots,B(t_m)$ given $B(s_k)$.*

Proof. Let

$$X_1=B(s_1),\dots,X_k=B(s_k),Y_1=B(t_1),\dots,Y_m=B(t_m).$$

The joint density of $(X_1,\dots,X_k,Y_1,\dots,Y_m)$ is

$$\begin{aligned}&f_{X_1,\dots,X_k,Y_1,\dots,Y_m}(x_1,\dots,x_k,y_1,\dots,y_m)\\&=\frac{1}{\left(\sqrt{2\pi}\right)^{m+k}\sqrt{s_1(s_2-s_1)\cdots(s_k-s_{k-1})(t_1-s_k)(t_2-t_1)\cdots(t_m-t_{m-1})}}\\&\times\exp\left(-\frac{x_1^2}{2s_1}-\frac{(x_2-x_1)^2}{2(s_2-s_1)}-\cdots-\frac{(x_k-x_{k-1})^2}{2(s_k-s_{k-1})}-\frac{(y_1-x_k)^2}{2(t_1-s_k)}-\frac{(y_2-y_1)^2}{2(t_2-s_1)}-\cdots-\frac{(y_m-y_{m-1})^2}{2(t_m-t_{m-1})}\right).\end{aligned}$$

The joint density of $(X_1,\dots,X_k)$ is

$$\begin{aligned}&f_{X_1,\dots,X_k}(x_1,\dots,x_k)\\&=\frac{1}{\left(\sqrt{2\pi}\right)^{k}\sqrt{s_1(s_2-s_1)\cdots(s_k-s_{k-1})}}\exp\left(-\frac{x_1^2}{2s_1}-\frac{(x_2-x_1)^2}{2(s_2-s_1)}-\cdots-\frac{(x_k-x_{k-1})^2}{2(s_k-s_{k-1})}\right)\end{aligned}$$

So, the conditional density of $(Y_1,\dots,Y_m)$ given $(X_1,\dots,X_k)$ is

$$\begin{aligned}&f_{Y_1,\dots,Y_m|X_1,\dots,X_k}(y_1,\dots,y_m|x_1,\dots,x_k)\\&=\frac{1}{\left(\sqrt{2\pi}\right)^{m}\sqrt{(t_1-s_k)(t_2-t_1)\cdots(t_m-t_{m-1})}}\exp\left(-\frac{(y_1-x_k)^2}{2(t_1-s_k)}-\frac{(y_2-y_1)^2}{2(t_2-t_1)}-\cdots-\frac{(y_m-y_{m-1})^2}{2(t_m-t_{m-1})}\right).\end{aligned}$$

The joint density of $(X_k,Y_1,\dots,Y_m)$ is

$$\begin{aligned}&f_{X_m,Y_1,\dots,Y_m}(x_k,y_1,\dots,y_m)\\&=\frac{1}{\left(\sqrt{2\pi}\right)^{1+m}\sqrt{s_k(t_1-s_k)(s_2-s_1)\cdots(t_m-t_{m-1})}}\\&\times\exp\left(-\frac{x_k^2}{2s_k}-\frac{(y_1-x_k)^2}{2(t_1-s_k)}-\frac{(y_2-y_1)^2}{2(t_2-t_1)}-\cdots-\frac{(y_m-y_{m-1})^2}{2(t_m-t_{m-1})}\right).\end{aligned}$$

The joint density of X_k is

$$f_{X_k}(x_k)=\frac{1}{\sqrt{2\pi s_k}}\exp\left(-\frac{x_k^2}{2s_k}\right)$$

So, the conditional density of $(Y_1,\dots,Y_m)$ given X_k is

$$\begin{aligned}&f_{Y_1,\dots,Y_m|X_k}(y_1,\dots,y_m|x_k)\\&=\frac{1}{\left(\sqrt{2\pi}\right)^{m}\sqrt{(t_1-s_k)(t_2-t_1)\cdots(t_m-t_{m-1})}}\exp\left(-\frac{(y_1-x_k)^2}{2(t_1-s_k)}-\frac{(y_2-y_1)^2}{2(t_2-t_1)}-\cdots-\frac{(y_m-y_{m-1})^2}{2(t_m-t_{m-1})}\right).\end{aligned}$$

Therefore, the two conditional densities $f_{Y_1,\dots,Y_m|X_1,\dots,X_k}(y_1,\dots,y_m|x_1,\dots,x_k)$ and $f_{Y_1,\dots,Y_m|X_k}(y_1,\dots,y_m|x_k)$ agree. Q.E.D.

Theorem 6.10. *For $0\le s<t$, the distribution of $B(t)$ given $B(s)$ is normal with mean $B(s)$ and variance $t-s$. $B(t)|B(s)\overset{d}{\sim}N(B(s),t-s)$.*

Proof. By the computations in the previous theorem, the conditional density of $Y=B(t)$ given $X=B(s)$ is

$$f_{Y|X}(y|x)=\frac{1}{\sqrt{2\pi(t-s)}}\exp\left(-\frac{(y-x)^2}{2(t-s)}\right).$$

This is the density of a normal r.v. with mean x and variance $t-s$. Q.E.D.

Example 6.10. *Suppose that $\{X(t):t\ge 0\}$ is a Brownian motion process with variance parameter $\sigma^2=4$. Find:*
(i) $\mathrm{P}\{X(3)\le 1|X(2)=1/2\}$
(ii) $\mathrm{P}\{X(4)-X(2)>1|X(2)=1/2\}$.

Solution: We have that $\{B(t) : t \geq 0\}$ is a standard Brownian motion, where $B(t) = \frac{X(t)}{2}$.
(i) So,

$$\begin{aligned} & \mathrm{P}\{X(3) \leq 1 | X(2) = \tfrac{1}{2}\} = \mathrm{P}\{B(3) \leq \tfrac{1}{2} | B(2) = \tfrac{1}{4}\} = \mathrm{P}\{B(3) - B(2) \leq \tfrac{1}{2} - \tfrac{1}{4} | B(2) = \tfrac{1}{4}\} \\ = \; & \mathrm{P}\{B(3) - B(2) \leq \tfrac{1}{4} | B(2) = \tfrac{1}{4}\} = \mathrm{P}\{B(3) - B(2) \leq \tfrac{1}{4}\} = \mathrm{P}\{Z \leq 0.25\} = 0.5987063 \end{aligned}$$

(ii) Since $X(2)$ and $X(4) - X(2)$ are independent r.v.'s

$$\begin{aligned} & \mathrm{P}\{X(4) - X(2) > 1 | X(2) = 1/2\} = \mathrm{P}\{X(4) - X(2) > 1\} = \mathrm{P}\{B(4) - B(2) > \tfrac{1}{2}\} \\ = \; & \mathrm{P}\{Z > \tfrac{1}{2 \cdot \sqrt{2}}\} = \mathrm{P}\{Z > 0.353553\} = 0.361837 \end{aligned}$$

Theorem 6.11. *For $0 \leq s < t$, the distribution of $B(s)$ given $B(t)$ is normal with mean $\frac{s}{t} B(t)$ and variance $\frac{s(t-s)}{t}$. $B(s) | B(t) \overset{d}{\sim} N(\frac{s}{t} B(t), \frac{s(t-s)}{t})$.*

Proof. Let $X = B(s)$ and let $Y = B(t)$. By the computations in two theorems above, the joint density of (X, Y)

$$f_{X,Y}(x, y) = \frac{1}{2\pi \sqrt{s(t-s)}} \exp\left(-\frac{x^2}{2s} - \frac{(y-x)^2}{2(t-s)}\right).$$

The marginal density of Y is

$$f_Y(y) = \frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{y^2}{2t}\right).$$

The conditional density of X given $Y = y$ is

$$\begin{aligned} f_{X|Y}(x|y) &= \frac{\frac{1}{2\pi\sqrt{s(t-s)}} \exp\left(-\frac{x^2}{2s} - \frac{(y-x)^2}{2(t-s)}\right)}{\frac{1}{\sqrt{2\pi t}} \exp\left(-\frac{y^2}{2t}\right)} = \frac{1}{\sqrt{2\pi \frac{s(t-s)}{t}}} \exp\left(\frac{y^2}{2t} - \frac{x^2}{2s} - \frac{(y-x)^2}{2(t-s)}\right) \\ &= \frac{1}{\sqrt{2\pi \frac{s(t-s)}{t}}} \exp\left(\frac{y^2}{2t} - \frac{x^2}{2s} - \frac{y^2}{2(t-s)} + \frac{xy}{t-s} - \frac{x^2}{2(t-s)}\right) \\ &= \frac{1}{\sqrt{2\pi \frac{s(t-s)}{t}}} \exp\left(-\frac{tx^2}{2s(t-s)} - \frac{sy^2}{2t(t-s)} + \frac{xy}{t-s}\right) \\ &= \frac{1}{\sqrt{2\pi \frac{s(t-s)}{t}}} \exp\left(-\frac{(x - \frac{sy}{t})^2}{2\frac{s(t-s)}{t}}\right), \end{aligned}$$

which is the density of a normal r.v. with mean $\frac{sy}{t}$ and variance $\frac{s(t-s)}{t}$. Q.E.D.

Example 6.11. *Suppose that $\{X(t) : t \geq 0\}$ is a Brownian motion process with variance parameter $\sigma^2 = 3$. Find:* $\mathrm{P}\{X(1) \leq 1 | X(2) = \frac{5}{4}\}$

Solution: We have that

$$\mathrm{P}\left\{X(1) \leq 1 | X(2) = \frac{5}{4}\right\} = \mathrm{P}\left\{B(1) \leq \frac{1}{\sqrt{3}} | B(2) = \frac{5}{4 \cdot \sqrt{3}}\right\} = \mathrm{P}\left\{B(1) \leq \frac{\sqrt{3}}{3} | B(2) = \frac{5 \cdot \sqrt{3}}{12}\right\}.$$

We have that

$$E\left[B(1) | B(2) = \frac{5 \cdot \sqrt{3}}{12}\right] = \frac{5 \cdot \sqrt{3}}{24}.$$

and

$$\operatorname{Var}\left(B(1)|B(2)=\frac{5\cdot\sqrt{3}}{12}\right)=\frac{1}{2}.$$

So,

$$\mathrm{P}\left\{B(1)\le\frac{\sqrt{3}}{3}|B(2)=\frac{5\cdot\sqrt{3}}{12}\right\}=\mathrm{P}\{Z\le\frac{\frac{\sqrt{3}}{3}-\frac{5\cdot\sqrt{3}}{24}}{\sqrt{\frac{1}{2}}}\}=\mathrm{P}\{Z\le 0.3061862\}=0.6202686.$$

6.4 Hitting Time

Let $\{B(t):t\ge 0\}$ be a standard Brownian motion. Let $a>0$. Let T_a denote the first time the Brownian motion process hits a. Then for $t>0$,

$$\begin{aligned}&\mathrm{P}\{B(t)\ge a\}=\Pr\{T_a\le t\}\Pr\{B(t)\ge a|T_a\le t\}+\Pr\{T_a>t\}\Pr\{X(t)\ge a|T_a>t\}\\&=\Pr\{T_a\le t\}\tfrac{1}{2}+\Pr\{T_a>t\}0=\Pr\{T_a\le t\}\tfrac{1}{2}.\end{aligned}$$

So,

$$\Pr\{T_a\le t\}=2\mathrm{P}\{B(t)\ge a\}=2\mathrm{P}\{N(0,1)\ge\tfrac{a}{\sqrt{t}}\}=2\left(1-\Phi\left(\tfrac{a}{\sqrt{t}}\right)\right)=\tfrac{2}{\sqrt{2\pi}}\int_{\frac{a}{\sqrt{t}}}^{\infty}e^{-\frac{y^2}{2}}\,dy.$$

The density of T_a is

$$f_{T_a}(t)=\frac{2}{\sqrt{2\pi}}e^{-\frac{a^2}{2t}}\frac{a}{2t^{3/2}}=\frac{ae^{-\frac{a^2}{2t}}}{\sqrt{2\pi}t^{3/2}}.$$

We also have that

$$\Pr\{T_a\le t\}=2\mathrm{P}\{B(t)\ge a\}=\mathrm{P}\{|B(t)|\ge a\}.$$

Example 6.12. *Let T_a be the time until a standard Brownian motion process hits a. Show that $E[T_a]=\infty$.*

Solution:

$$E[T_a]=\int_0^\infty tf_{T_a}(t)\,dt=\int_0^\infty t\frac{ae^{-\frac{a^2}{2t}}}{\sqrt{2\pi}t^{3/2}}\,dt=\int_0^\infty\frac{ae^{-\frac{a^2}{2t}}}{\sqrt{2\pi}t^{1/2}}\,dt=\infty,$$

because $\lim_{t\to\infty}\frac{\frac{ae^{-\frac{a^2}{2t}}}{\sqrt{2\pi}t^{1/2}}}{\frac{a}{\sqrt{2\pi}t^{1/2}}}=1$ and $\int_1^\infty\frac{a}{\sqrt{2\pi}t^{1/2}}\,dt=\infty$.

Exercise 6.22. *Suppose that you own one share of a stock whose price changes according to a standard Brownian motion process. Suppose that you purchased the stock at a price a and the present time price is b, where $b<a$. You decide to sell the stock when it reaches the price a. What is the average time it takes the stock to recover to the original purchase price?*

Example 6.13. *Let T_a be the time until a standard Brownian motion process hits a. Calculate the following:*
(i) $\mathrm{P}\{T_2\le 8\}$.
(ii) The median of T_2.

Solution: (i)

$$\mathrm{P}\{T_2 \le 8\} = 2\mathrm{P}\{B(8) \ge 2\} = 2\mathrm{P}\left\{Z \ge \frac{2}{\sqrt{8}}\right\} = 2\mathrm{P}\{Z \ge 0.7071068\} = 0.4795001$$

(ii) Let m be the median of T_2. Then,

$$1/2 = \mathrm{P}\{T_2 \le m\} = 2\mathrm{P}\{B(m) \ge 2\}.$$

So, $0.75 = \mathrm{P}\{B(m) < 2\} = \mathrm{P}\{Z < \frac{2}{\sqrt{m}}\}$, $\frac{2}{\sqrt{m}} = 0.6744898$ and $m = 8.792437$.

Exercise 6.23. *Let T_a be the time until a standard Brownian motion process hits a. Calculate the following probabilities and percentiles:*
(i) $\mathrm{P}\{T_3 \le 8\}$.
(ii) The 99-th percentile of T_3.

Problem 6.1. *(# 15, November 2000). You own one share of a stock. The price is 18. The stock price changes according to a standard Brownian motion process, with time measured in months. Calculate the probability that the stock reaches a price of 21 at some time within the next 4 months.*
(A) 0.0668 (B) 0.0735 (C) 0.0885 (D) 0.1096 (E) 0.1336

Solution: Let T_3 be the hitting time of 3. We need to find

$$\Pr\{T_3 \le 4\} = 2(1 - \Phi(3/2)) = 0.1336$$

Theorem 6.12. *Given $a > 0 > b$, then*

$$\mathrm{P}\{B(t) \text{ hits } a \text{ before } b\} = \frac{|b|}{a + |b|}.$$

Example 6.14. *You own one share of a stock. The price is 18. The stock price changes according to a standard Brownian motion process, with time measured in months. Calculate the probability that the stock reaches a price of 21 before it reaches 12.*

Solution: $\frac{6}{9}$.

Let $\{B(t) : t \ge 0\}$ be a standard Brownian motion. The maximum process over $[0, t]$ is $M_t = \max_{0 \le s \le t} B(s)$. For $a > 0$,

$$\Pr\{M_t \ge a\} = \Pr\{T_a \le t\} = \mathrm{P}\{|B(t)| \ge a\}$$

and

$$\Pr\{M_t \le a\} = \mathrm{P}\{|B(t)| \ge a\} = 2\int_0^a \frac{1}{\sqrt{2\pi t}} e^{-\frac{x^2}{2t}}\, dx.$$

So, the density of M_t is

$$f_{M_t}(a) = \begin{cases} \frac{\sqrt{2}}{\sqrt{\pi t}} e^{-\frac{a^2}{2t}} & \text{if } a \ge 0 \\ 0 & \text{if } a < 0 \end{cases}$$

M_t has the density of $|B(t)|$.

Example 6.15. $E[M_t] = \sqrt{\frac{2t}{\pi}}$ *and* $\mathrm{Var}(M_t) = \frac{t(\pi-2)}{\pi}$

Solution:

$$E[M_t] = \int_0^\infty a\frac{\sqrt{2}}{\sqrt{\pi t}}e^{-\frac{a^2}{2t}}\,da = -\frac{\sqrt{2t}}{\sqrt{\pi}}e^{-\frac{a^2}{2t}}\Big|_{a=-0}^{\infty} = \frac{\sqrt{2t}}{\sqrt{\pi}}$$

and

$$E[(M_t)^2] = E[(B(t))^2] = t.$$

Example 6.16. *Suppose that* $\{X(t)|0 \le t < \infty\}$ *is a Brownian motion process with variance parameter* 9*. Calculate the following:*
(i) $\mathrm{P}\{\max_{t\in[0,2]} X(t) \le 4\}$*.*
(ii) The median value of $\max_{t\in[0,2]} X(t)$*.*
(iii) The mean value of $\max_{t\in[0,2]} X(t)$*.*

Solution: (i)

$$\begin{aligned}&\mathrm{P}\{\max_{t\in[0,2]} X(t) \le 4\} = \mathrm{P}\{\max_{t\in[0,2]} B(t) \le \tfrac{4}{3}\} = \mathrm{P}\{|B(2)| \le \tfrac{4}{3}\}\\ &= \mathrm{P}\{-\tfrac{4}{3\cdot\sqrt{2}} \le Z \le \tfrac{4}{3\cdot\sqrt{2}}\} = 0.6542214\end{aligned}$$

(ii)

$$E[\max_{t\in[0,2]} X(t)] = 3E[\max_{t\in[0,2]} B(t)] = 3\sqrt{\frac{4}{\pi}} = \frac{6}{\sqrt{\pi}}.$$

(iii) Let m be mean value of $\max_{t\in[0,2]} X(t)$. Then,

$$\begin{aligned}&1/2 = \mathrm{P}\{\max_{t\in[0,2]} X(t) \le m\} = \mathrm{P}\{\max_{t\in[0,2]} B(t) \le m/3\} = \mathrm{P}\{|B(2)| \le m/3\}\\ &= 2\mathrm{P}\{B(2) \le m/3\} - 1 = 2\mathrm{P}\{Z \le \tfrac{m}{3\cdot\sqrt{2}}\} - 1.\end{aligned}$$

So, $\mathrm{P}\{Z \le \frac{m}{3\cdot\sqrt{2}}\} = 3/4$, $\frac{m}{3\cdot\sqrt{2}} = 0.6744898$ and $m = 2.861618$

Continuity of the Brownian motion: In some sense, a function of t, the Brownian motion is a continuous function on t. A Brownian motion can be used to give probabilities to continuous function. The price of a stock is a continuous function. A Brownian motion can be used to estimate probabilities related with the price of a stock over time.

6.5 Modeling Stock Prices

Definition 6.5.1. *X is said to have a lognormal distribution with parameters μ and σ, if* $\ln X$ *has a normal distribution with mean μ and variance σ^2.*

If X is said to have a lognormal distribution with parameters μ and σ, then X has the distribution of $e^{\mu+\sigma Z}$, where Z is a standard normal r.v.

Example 6.17. *Let X be a r.v. with lognormal distribution with parameters $\mu = 2$ and $\sigma^2 = 5$. Find* $\mathrm{Pr}\{X \le 3\}$*.*

Solution:

$$\Pr\{X \le 3\} = \Pr\{\ln X \le \ln 3\} = \Pr\{Z \le \frac{\ln 3 - 2}{\sqrt{5}}\} = \Pr\{Z \le -0.4031128\} = 0.3434326$$

Example 6.18. *Let X be a r.v. with lognormal distribution with parameters μ and σ. Show that:*
(i) $E[X] = e^{\mu+\frac{\sigma^2}{2}}$.
(ii) $E[X^2] = e^{2\mu+2\sigma^2}$.
(iii) $\mathrm{Var}(X) = e^{2\mu+\sigma^2}(e^{\sigma^2} - 1)$.
(iv) For $t > 0$, $\Pr\{X \le t\} = \Phi\left(\frac{\ln t - \mu}{\sigma}\right)$, *where* $\Phi(t) = \Pr\{Z \le t\}$.

Solution: We have that X has the distribution of $e^{\mu+\sigma Z}$. So,

$$E[X] = E[e^{\mu+\sigma Z}] = e^{\mu+\frac{\sigma^2}{2}}$$

and

$$E[X^2] = E[e^{2\mu+2\sigma Z}] = e^{2\mu+2\sigma^2}.$$

Example 6.19. *Let X be a lognormal r.v. with $E[X] = e^5$ and $E[X^2] = e^{14}$. Find:*
(i) $\Pr(X \le e^7)$.
(ii) The third quartile of X.

Solution: We have that

$$e^5 = E[X] = e^{\mu+\frac{\sigma^2}{2}},\ e^{14} = E[X^2] = e^{2\mu+2\sigma^2}.$$

So,

$$5 = \mu + \frac{\sigma^2}{2},\ 14 = 2\mu + 2\sigma^2$$

Hence, $\mu = 3$ and $\sigma^2 = 4$. Thus,

$$\Pr(X \le e^7) = \Pr(\ln X \le m) = \mathrm{P}(Z \le \frac{7-3}{\sqrt{4}}) = \mathrm{P}(Z \le 2) = 0.9772499$$

Let q be the third quartile of X. Then,

$$0.75 = \Pr(X \le m) = \Pr(\ln X \le 7) = \mathrm{P}(Z \le \frac{\ln m - 3}{\sqrt{4}})$$

and $0.6744898 = \frac{\ln m - 3}{2}$ and $m = 77.39944$.

Definition 6.5.2. *We say that a stochastic process $\{X(t) : t \ge 0\}$ is a Brownian motion with drift coefficient μ and variance parameter σ^2 if $X(t) = \sigma B(t) + \mu t$, where $\{B(t) : t \ge 0\}$ is a standard Brownian motion.*

If $\{X(t) : t \ge 0\}$ is a Brownian motion with drift coefficient μ and variance parameter σ^2, then:
(i) $X(0) = 0$.
(ii) $\{X(t) : t \ge 0\}$ has stationary and independent increments
(iii) $X(t)$ is normally distributed with mean μt and variance $\sigma^2 t$.

Definition 6.5.3. *We say that a stochastic process $\{X(t) : t \geq 0\}$ is a geometric Brownian motion with drift coefficient μ and variance parameter σ^2 if $X(t) = e^{\sigma B(t)+\mu t}$, where $\{B(t) : t \geq 0\}$ is a standard Brownian motion.*

Suppose that a share of a certain stock is currently (i.e. $t = 0$) selling for P_0. Set $P(t) = P_0 Z(t) = P_0 e^{\sigma B(t)+\mu t}$, where $Z(t)$ is a geometric Brownian motion with drift coefficient μ and variance parameter σ^2 and $B(t)$ is a standard Brownian motion. $\{P(t) : t \geq 0\}$ is called a price process model.

The geometric Brownian motion is used to model prices of stock, because it is assumed that the rate of interest earned over disjoint interval of times are independent r.v.'s.

The difference $P(t) - P(s)$ is the interest $I[s,t]$ earned over the period $[s,t]$. The effective rate of interest earned in the period $[s,t]$ is $\frac{P(t)-P(s)}{P(s)}$, i.e. this is the interest earned per unit of investment in that period of time. $I[s,t] = \frac{P(t)-P(s)}{P(s)}$.

For the price process and $0 < t_1 < \cdots < t_m$,

$$
\begin{aligned}
I[t_1,t_2] &= \tfrac{P(t_2)-P(t_1)}{P(t_1)} = e^{\sigma(B(t_2)-B(t_1))+\mu(t_2-t_1)} - 1, \ldots, \\
I[t_{m-1},t_m] &= \tfrac{P(t_m)-P(t_{m-1})}{P(t_{m-1})} = e^{\sigma(B(t_m)-B(t_{m-1}))+\mu(t_m-t_{m-1})} - 1.
\end{aligned}
$$

So, $I[t_1,t_2], \ldots, I[t_{m-1},t_m]$ are independent r.v.'s.

The average accumulation of $P(t)$ is $A(t) = E[P(t)] = P_0 e^{\mu t + \frac{\sigma^2 t}{2}}$. Given an accumulation function $A(t)$, the force of interest at time t is

$$\delta_t = \frac{A'(t)}{A(t)} = \frac{d}{dt} \ln A(t).$$

The force of interest of $A(t) = E[P(t)]$ is

$$\delta_t = \frac{d}{dt} \ln A(t) = \mu + \frac{\sigma^2}{2}.$$

Exercise 6.24. *To model the price of a share of stock with current price of 20, we use the model $P(t) = 20X(t)$, where $X(t)$ is a geometric Brownian motion model with $\mu = 0$ and $\sigma^2 = 0.09$.*
(i) Calculate the probability that the price hits $20e^{0.15}$ by time 2.
(ii) Calculate the median time that it is required for the price to hits $20e^{0.15}$.
(iii) Calculate the mean time that it is required for the price to hits $20e^{0.15}$.
(iv) Calculate the probability that the maximum price of the stock over the time interval $[0,2]$ exceeds $20e^{0.30}$.
(v) Calculate the median value of the maximum price of the stock over the time interval $[0,2]$.
(vi) Calculate the median value of the maximum price of the stock over the time interval $[0,2]$.
(vii) Calculate the expected value and the variance of the price of the stock at the time 2.
(viii) Calculate the probability that the stock increases by more than 20% in a two year period.

Exercise 6.25. *To model the price of a share of stock with current price of 20, we use the model $P(t) = 12X(t)$, were $X(t)$is a geometric Brownian motion model with $\mu = 1$ and*

$\sigma^2 = 0.09$.
(i) Calculate the probability that the price increases by more than 20% before it decreases by more than 10%.
(ii) Calculate the probability that the price increases by more than 15% before it decreases by more than 10%.

Exercise 6.26. *Suppose that stock currently sells for 100 and that the price process* $\{P(t) : t \geq 0\}$*is modeled by a multiple of a geometric Brownian motion, where the drift parameter* $\mu = 0.02$ *and the variance parameter is* $\sigma^2 = 0.02$*. At a force of interest of* $\delta = 0.02$ *the present value of the price at time* t *is* $e^{-\delta t}P(t)$*. Calculate the following:*
(i) $E[P(1)]$, $\mathrm{Var}(P(1))$ *and* $\Pr\{P(1) \geq 100e^{0.03}\}$.
(ii) The probability the maximum present value of the price over $[0,1]$ *exceeds* $100e^{0.03}$.
(iii) The probability the present value of the price hits $100e^{0.4}$ *before time 5.*
(iv) The probability the present value of the price hits $100e^{0.02}$ *before it hits* $100e^{-0.1}$.

Problem 6.2. *(# 27, Sample Test). You are given:*

- *The logarithm of the price of a stock can be modeled by Brownian motion with drift coefficient* $\mu = 0$ *and variance parameter* $\sigma^2 = 0.04$.
- *The price of the stock at time* $t = 0$ *is* 10.

Calculate the probability that the price of the stock will be 12 *or greater at some time between* $t = 0$ *and* $t = 1$.
(A) 0.023 (B) 0.181 (C) 0.273 (D) 0.362 (E) 0.543

Solution: Let $P(t)$ be the price of the stock at time t. $P(t) = 10e^{\sqrt{0.04}B(t)}$, where $B(t)$ is a standard Brownian motion. We need to find

$$\begin{aligned}&\Pr\{\sup_{0\leq t\leq 1} P(t) \geq 12\} = \Pr\{\sup_{0\leq t\leq 1} 10e^{\sqrt{0.04}B(t)} \geq 12\}\\ &= \Pr\{\sup_{0\leq t\leq 1} B(t) \geq (5)\ln(12/10)\} = \Pr\{|N(0,1)| \geq (5)\ln(12/10)\} = 0.362.\end{aligned}$$

Problem 6.3. *(# 8, November 2002). The value of currency in country M is currently the same as in country N (i.e., 1 unit in country M can be exchanged for 1 unit in country N). Let* $C(t)$ *denote the difference between the currency values in country M and N at any point in time (i.e., 1 unit in country M will exchange for* $1 + C(t)$ *at time* t*).* $C(t)$ *is modeled as a Brownian motion process with drift 0 and variance parameter 0.01. An investor in country M currently invests 1 in a risk free investment in country N that matures at 1.5 units in the currency of country N in 5 years. After the first year, 1 unit in country M is worth 1.05 in country N. Calculate the conditional probability after the first year that when the investment matures and the funds are exchanged back to country M, the investor will receive at least 1.5 in the currency of country M.*
(A) 0.3 (B) 0.4 (C) 0.5 (D) 0.6 (E) 0.7

Solution: We have that $C(t) = \sqrt{0.01}B(t) = 0.1B(t)$, where $B(t0$ is a standard Brownian motion. We have to find

$$\begin{aligned}
& \mathrm{P}\{\tfrac{1.5}{C(5)} > 1.5 | C(1) = 1.05\} = \mathrm{P}\{1 > 1 + 0.1B(5) | 1 + 0.1B(1) = 1.05\} \\
= \ & \mathrm{P}\{0 > B(5) | 1 + 0.1B(1) = 1.05\} = \mathrm{P}\{0 > B(5) | B(1) = .5\} \\
= \ & \mathrm{P}\{B(5) - B(1) < -0.5 | B(1) = .5\} = \mathrm{P}\{B(5) - B(1) < -0.5\} \\
= \ & \mathrm{P}\{N(0,1) \le -0.25\} = 0.4.
\end{aligned}$$

Problems

1. Let $\{B(t) : t \ge 0\}$ be a standard Brownian motion.
   (i) Find the mean and the variance of $B(2) + B(5) - 3B(3)$.
   (ii) Find $\Pr(-2 \le B(2) + B(5) - 3B(3) \le 3)$.

2. Let $\{B(t) : t \ge 0\}$ be a standard Brownian motion. Show that for $0 < s < t < u < v$, the variance of
$$aB(s) + bB(t) + cB(u) + dB(v)$$
is
$$(a + b + c + d)^2 s + (b + c + d)^2 (t - s) + (c + d)^2 (u - t) + d^2 (v - u).$$

3. Let $\{X(t) : t \ge 0\}$ be a Brownian motion process with variance parameter $\sigma^2 = 3$. Find the following:
   (i) $\Pr(-1 \le X(2) \le 5)$.
   (ii) $\Pr(-1 \le X(3) - X(4) \le 5)$.
   (iii) $\Pr(-1 \le X(1) - 2X(2) + 3X(4) \le 5)$.

4. Let $\{B(t) : t \ge 0\}$ be a standard Brownian motion process. Find the following:
   (i) $\Pr(-1 \le B(2) \le 3 | B(1) = 1)$.
   (ii) $\Pr(-2 \le B(3) - B(4) \le 1 | B(1) = 3)$.
   (iii) $\Pr(0 \le B(3) \le 4 | B(5) = 3)$.

5. Let $\{X(t) : t \ge 0\}$ be a Brownian motion process with variance parameter $\sigma^2 = 9$. Find the following:
   (i) $\Pr(3 \le X(2) \le 9 | X(1) = 6)$.
   (ii) $\Pr(-2 \le X(2) \le 1 | X(5) - X(3) = 3)$

6. Let $\{B(t) : t \ge 0\}$ be a standard Brownian motion. Let $t > 0$ and let $M_t = \sup_{0 \le s \le t} B(s)$ be the maximum process over t. Find:
   (i) $\Pr(1 \le M_9 \le 5)$.
   (ii) The median of M_9.

7. Let $\{B(t) : t \ge 0\}$ be a standard Brownian motion. Let $t > 0$ and let $M_t = \sup_{0 \le s \le t} B(s)$ be the maximum process over t. Find:
   (i) $\Pr(M_4 \le 5)$.
   (ii) The density of M_4.
   (iii) The median of M_4.
   (iv) The first and the third quartile of M_4.
   (v) The mean and the variance of M_4.

8. Let $\{B(t) : t \geq 0\}$ be a standard Brownian motion. Let T_a be the first time that the Brownian motion hits a. Find the following:
(i) $\Pr(T_2 \leq 4)$.
(ii) The density of T_2.
(iii) The median of T_2.
(iv) $\Pr(2 \leq T_{-3} \leq 5)$.
(v) The first and the third quartile (the 25 % and 75 % percentile) of T_{-3}.

9. Use a Geometric Brownian Motion model with $\mu = 0$ and $\sigma^2 = 0.09$ to model the ratio $P(t)/20$, where $P(t)$ is the price at time t for a share of a stock currently selling at 20. What is the probability that the stock increases by more than 20% in a two-year period?

10. You own one share of a stock. The price is 12. The stock price $P(t)$ changes according to the model $P(t) = 12e^{0.5B(t)}$, where $B(t)$, $t \geq 0$, is a standard Brownian motion and the time is measured in months.
(i) Calculate the probability that the stock reaches a price of 30 at some time within the next 4 months.
(ii) Calculate the median of the time that it takes the price to reach 30.

11. You own one share of a stock. The current price of the stock is 10. Let $P(t)$ be the stock price at time t. $\frac{P(t)}{P(0)}$ changes according to a Geometric Brownian Motion model with $\mu = 0.1$ and $\sigma^2 = 0.4$, with time measured in months. Find:
(i) The mean and the variance of the price of the stock one year from now.
(ii) The probability that the price of the stock 6 months from now is more than 30.

12. You own 1 share of stock currently worth 102. Assume that the change in value of this share over time follows a standard Brownian motion process where time is measured in months. What is the probability that the price three months from now is greater than 105?

13. You own one share of a stock. The price is 50. The stock price $P(t)$ changes according to a geometric Brownian Motion model with $\mu = 0$ and $\sigma^2 = 0.36$, with time measured in months. Let S be the time, $P(t)$ hits 70. Show that the distribution of S is that of T_a, where T_a is the time that a standard Brownian motion hits $a > 0$, for some a. Find:
(i) a.
(ii) $\Pr(2 \leq S \leq 4)$.
(iii) The median of S.
(iv) The first and the third quartile (the 25 % and 75 % percentile) of S.

14. Using a Geometric Brownian Motion model with $\mu = 0$, $\sigma^2 = 0.09$ to model the ratio $P(t)/20$, where $P(t)$ is the price at time for a share of a stock currently selling at 20, calculate the probability that the price hits 25 by time 2.

15. You own one share of a stock. The price is 28. Let $P(t)$ be the stock price at time t. $\frac{P(t)}{P(0)}$ changes according to a Geometric Brownian Motion model with $\mu = 0.1$ and $\sigma^2 = 0.49$,

with time measured in months.
(i) Find the mean and the variance of the price of the stock in 6 months.
(ii) Find the probability that the price of the stock will be higher 20% or more in 6 months.
(iii) The mean and the variance of $P(6)$.
(iv) The median of $P(6)$.
(v) The first and the third quartile of $P(6)$.