Nonlinear Dynamics coursework
Version 15

Siim Erik Pugal
214704YAFM

May 2022

1 Analysis of a 2-D system

Fixed points of a system

The governing equation of motion is a particle in a double well potential with linear damping

$$\ddot{x} + \gamma\dot{x} - \frac{1}{2}\left(1 - x^2\right)x = 0$$

where γ is the coefficient of damping and $\gamma = 0.1$.

Fixed points of the system are found by solving

$$\begin{cases} y^* = 0 \\ 0.5x^*(1 - x^{*2}) - 0.1y^{*2} = 0 \end{cases} \tag{1}$$

for x^* and y^*. The fixed points are: $(x^*, y^*) = (-1, 0), (0, 0), (+1, 0)$.

The Jacobian matrix of the system is

$$J = \begin{pmatrix} \frac{\partial \dot{x}}{\partial x} & \frac{\partial \dot{x}}{\partial y} \\ \frac{\partial \dot{y}}{\partial x} & \frac{\partial \dot{y}}{\partial y} \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 0.5 - 1.5x^2 & -0.1 \end{pmatrix}$$

The Jacobian matrix evaluated at fixed points $(x^*, y^*) = (-1, 0)$ and $(+1, 0)$ is equal to

$$J\mid_{(\pm 1, 0)} = \begin{pmatrix} 0 & 1 \\ -1. & -0.1 \end{pmatrix}.$$

And for the fixed point $(x^*, y^*) = (0, 0)$, the Jacobian is

$$J\,|_{(0,0)} = \begin{pmatrix} 0 & 1 \\ 0.5 & -0.1 \end{pmatrix}.$$

Linearization about the fixed point

The matrix form for $\vec{u} = (u, v)^T$ is the following:

$$\dot{\vec{u}} = \left.\begin{pmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{pmatrix}\right|_{(x^*,y^*)} \cdot \vec{u} \equiv J|_{(x^*,y^*)} \cdot \vec{u}$$

So the linearized form for fixed points $(x^*, y^*) = (\pm 1, 0)$ looks like

$$\begin{cases} \dot{x} = y \\ \dot{y} = -x - 0.1y \end{cases} \tag{2}$$

and for $(x^*, y^*) = (0, 0)$

$$\begin{cases} \dot{x} = y \\ \dot{y} = 0.5x - 0.1y \end{cases} \tag{3}$$

Plots of linearized phase portraits

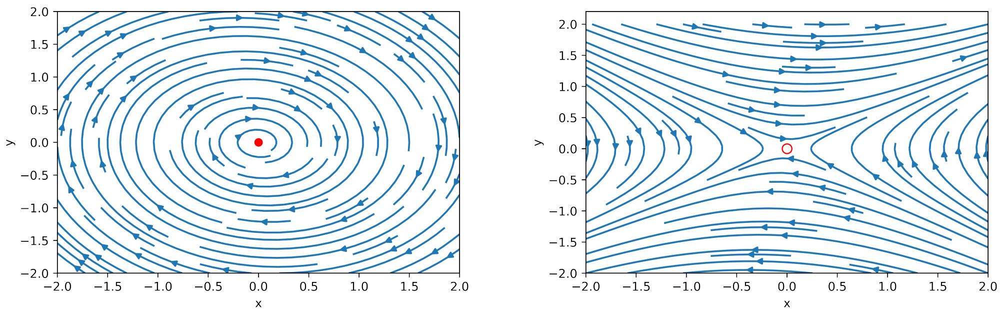

Figure 1: The linearized phase portraits of a stable spiral (left) and unstable saddle point (right).

Type of linearized fixed points

For fixed points $(x^*, y^*) = (\pm 1, 0)$ the determinant Δ of a jacobian is equal to

$$det(J)|_{(\pm 1,0)} = 1$$

and for $(x^*, y^*) = (0, 0)$

$$det(J)|_{(0,0)} = -0.5.$$

For all fixed points points, trace τ is equal to

$$tr(J) = -0.1.$$

Based on $\Delta = 1$ and $\tau = -0.1$, the fixed points $(x^*, y^*) = (\pm 1, 0)$ can be considered a **stable spiral**, because we have the condition

$$-\sqrt{4\Delta} < \tau < 0$$

or more specifically

$$-2 < -0.1 < 0.$$

For the unstable fixed point $(x^*, y^*) = (0, 0)$ we only need to consider the determinant $\Delta = -0.5$ and it can be classified as a **saddle point**, because we have the condition

$$\Delta < 0$$
$$-0.5 < 0.$$

Changes in the control parameter γ

If the γ parameter is increased enough, the 2-D phase portrait tends towards **stable non-isolated fixed points**.

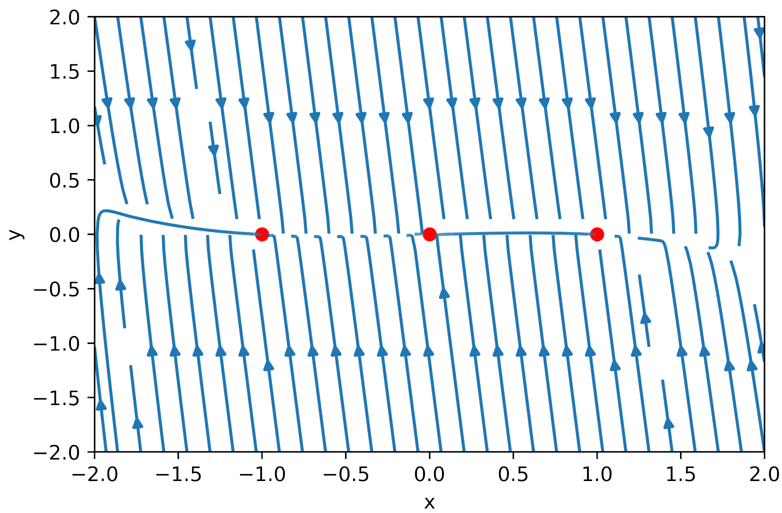

Figure 2: The phase portrait for the double well potential with increased value for γ.

On the other hand, if the opposite is done (parameter being decreased), the 2-D phase portrait will tend towards **unstable non-isolated fixed points**.

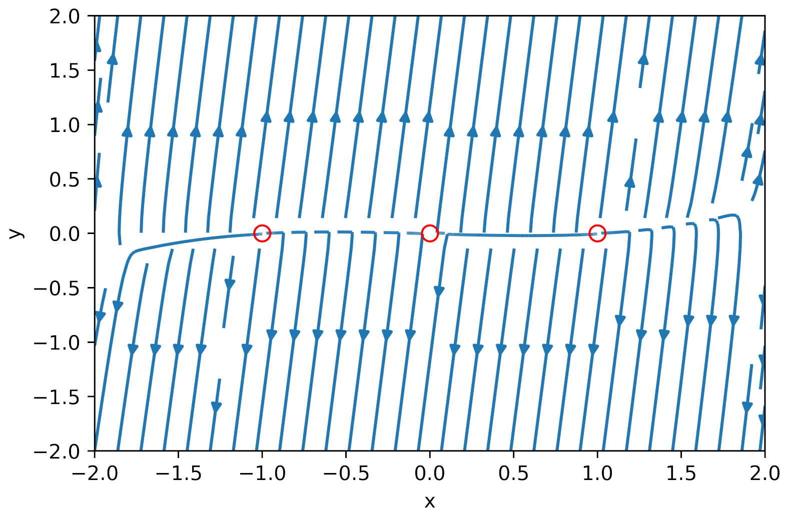

Figure 3: The phase portrait for the double well potential with decreased value for γ.

Comparing the type of nonlinear fixed points with corresponding linearized fixed points

Starting with the similarities. When we look at the fixed point $(x^*, y^*) = (0, 0)$ and compare the nonlinear system (1) to the linear one (3), the type of the fixed point stays the same (saddle point).

However, when we look at the alternative case and compare systems (1) and (2), this is not the case anymore. This time the linear one is a stable spiral like points $(x^*, y^*) = (\pm 1, 0)$ for the nonlinear one.

Other applications

The particle in a double well potential with linear damping has been a subject of many different papers as it is one of the simplest examples of a dynamical system exhibiting chaotic behavior, and thus represents a landmark chaotic system. Similar areas of interest, which are not limited to, include:

- Quantum mechanics - tunneling effect [1]
- The anomalous perihelion precession of Mercury [2]
- Applications of Melnikov theory [3]
- Stochastic analysis [4]
- Turbulence [5]

Nonlinear phase portrait and comparison to linearized ones

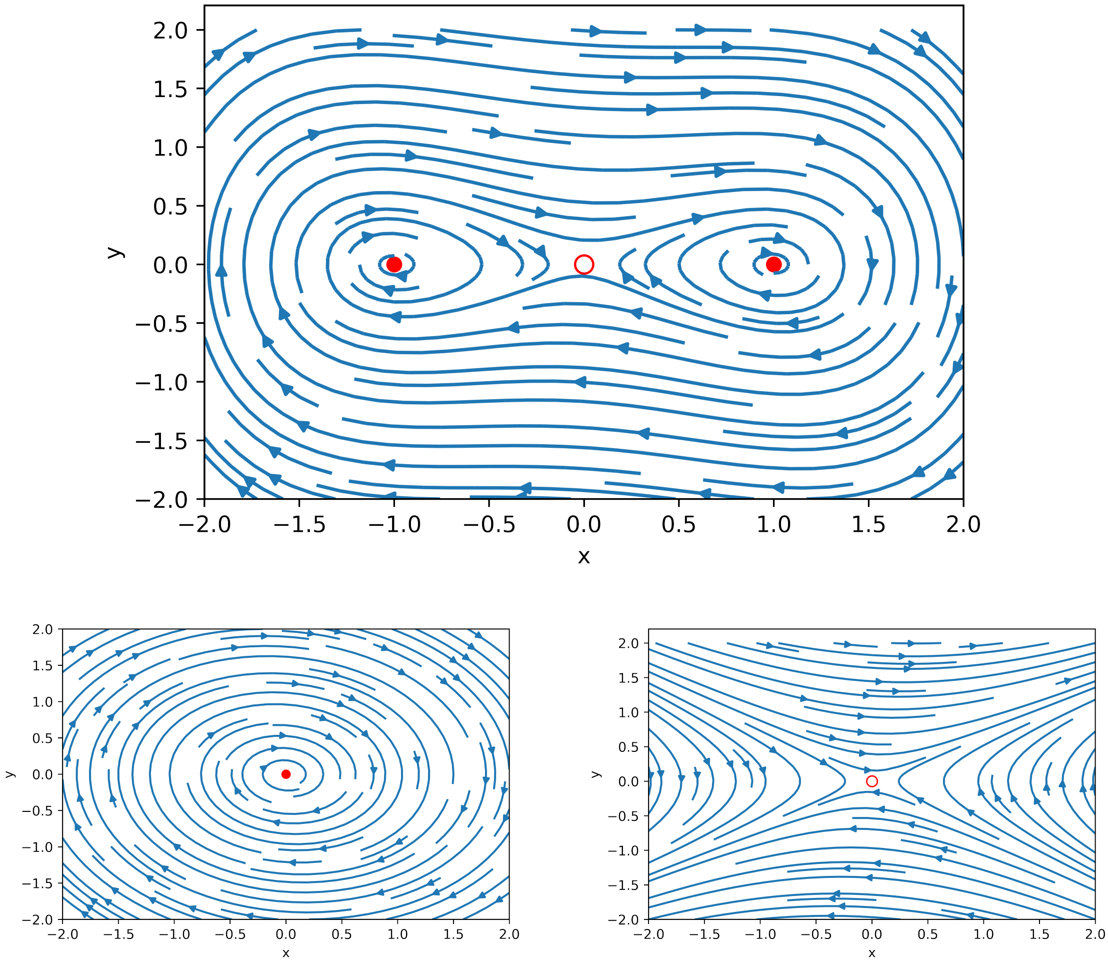

Figure 4: All three phase portraits together. The top one shows the nonlinear (1) and bottom ones the linearized cases (right - (2); left - (3))

2 Analysis of a 3-D system

My 3-D system is governed by these equations:

$$\begin{cases} \dot{x} = a(y - x) \\ \dot{y} = (c - a)x - xz + cy + m, \\ \dot{z} = xy - bz, \end{cases}$$

where the constants have values $a = 35, b = 3, c = 28, m = 23.1$.

The 3-D plot for these conditions looks like this:

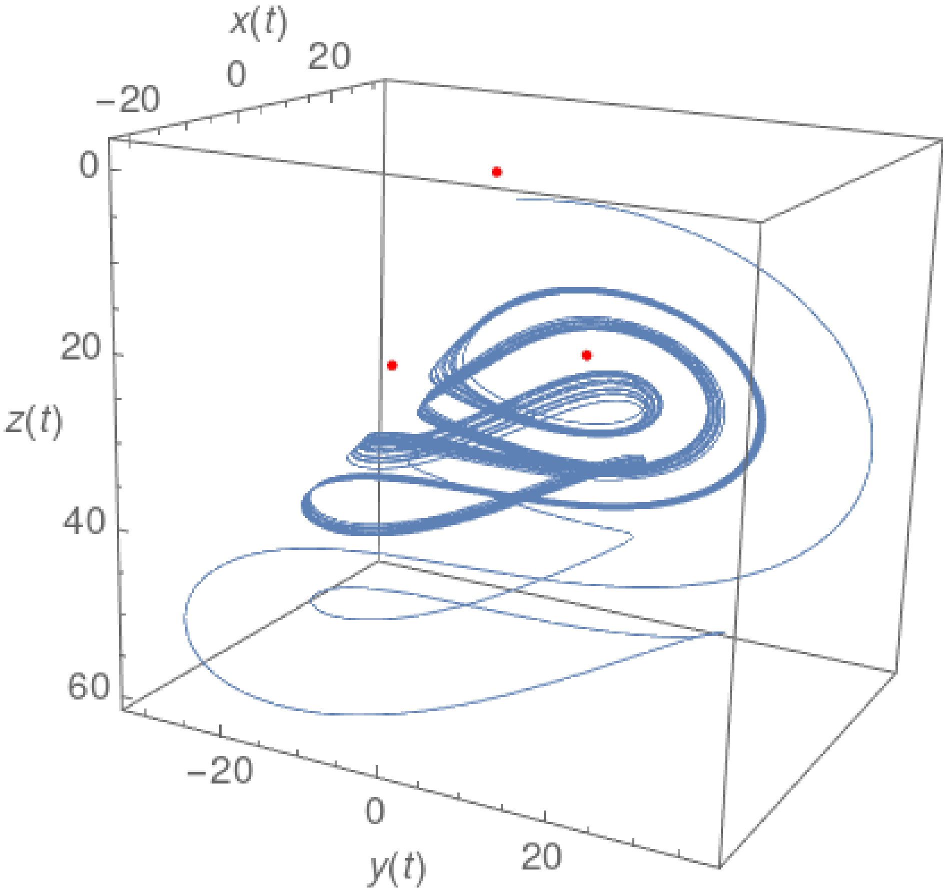

Figure 5: The modified Chen attractor.

One of the key properties of strange attractors is their sensitivity to initial conditions. A slight alterations can introduce new discrepancies in the system and is a characteristics of chaotic system.

Additionally, from the `Mathematica` code we can see that the divergence $\nabla \cdot \vec{x} = -10$ is negative, so it is contracting.

Here is a graph similar to the first one, but the initial conditions that were before $\vec{u} = (1, 2, 3)$ are now changed to $\vec{u} = (0, 1, 2)$:

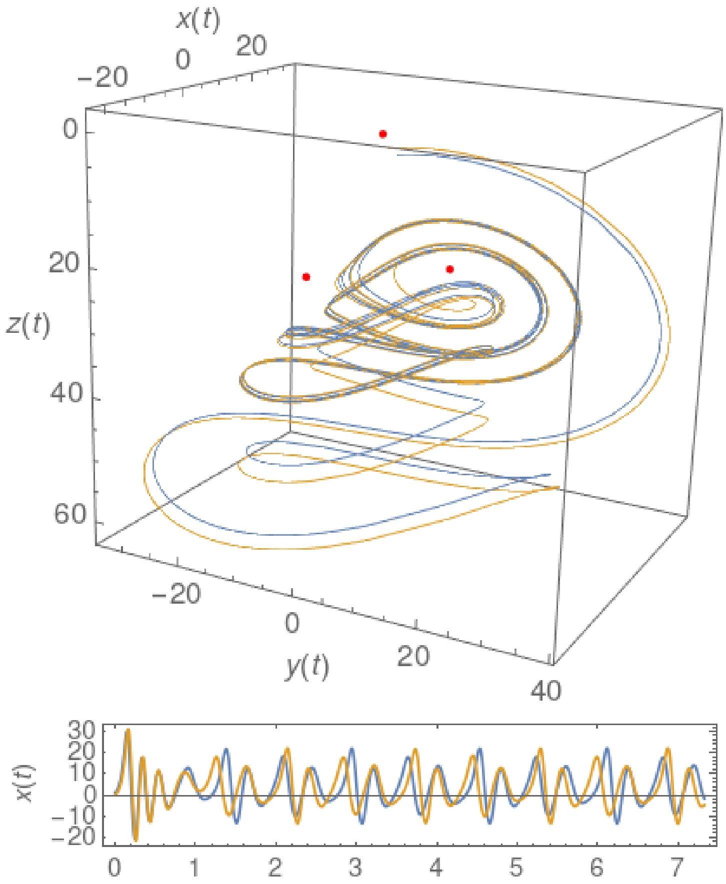

Figure 6: 3-D and 1-D map of two different trajectories for the modified Chen attractor.

As it can be seen from the 1-D graph down below, already at $t = 1$ small variations start developing and from the 3-D graph above it is clear to see the difference between the two trajectories and how they deviate from one another.

Another important characteristic of strange attractors is that the system will come very close to the desired unstable periodic orbit in a relatively short time [6]. This is visible for both figures 5 and 6, as the lines where the object spends a small amount of time has few wide sparse lines in the area. As it gets more closer to the periodic orbit, the density of the lines thickens referring to the fact that the object spends a lot more time in that area.

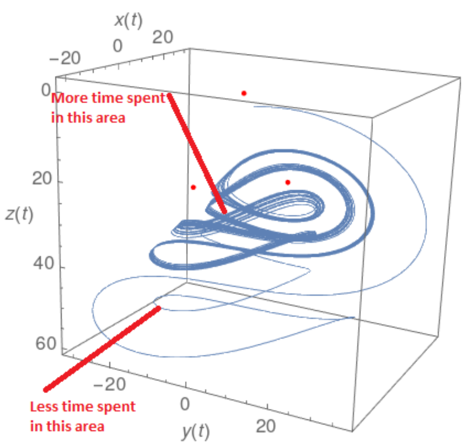

Figure 7: Specified areas, where the object spends its time traveling along the trajectories

When looking at the overall shape of the trajectories, it has this sort of constant stretching, folding and reinjection into itself, which is a property of chaos. Furthermore, the disks in these denser parts of the trajectories would continue to develop this sort of characteristic shape for this sort of attractor. Informally speaking, this shape would be referred to as a fractal [7].

Lastly, why the modified Chen attractor can be considered chaotic is by giving a proof of an existence of horseshoes in these systems. The horseshoe is a mathematical object, a transformation combining a scaling and a contraction, but also a folding. It transforms a square into the shape of a horseshoe. Its dynamics are continuous forever towards the future as well as the past. This referenced paper [8] gives a rigorous verification that horseshoes exist for this system.

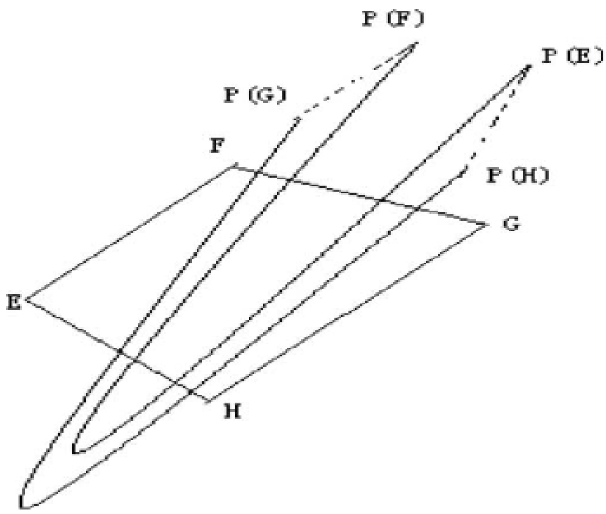

Figure 8: Sketch of the horshoe for the modified Chen attractor.

References

[1] Cho Cho Win, *Symmetrical Double-Well Potential and its Application*, 2018. Available at `https://meral.edu.mm/record/678/files/Symmetrical%20Double-Well%20Potential%20and%20its%20Application.pdf`

[2] K. Johannessen, *The Solution to the Differential Equation with Linear Damping describing a Physical Systems governed by a Cubic Energy Potential.* Available at `https://arxiv.org/ftp/arxiv/papers/1810/1810.10336.pdf`

[3] G. Litak,M. Borowiec, A. Syta, *Vibration of Generalized Double Well Oscillators*, 2013. Available at `https://www.academia.edu/10815425/Vibration_of_generalized_double_well_oscillators`.

[4] W. Q. Zhu, G. Q. Cai, R. C. Hu, *Stochastic analysis of dynamical system with double-well potential*, 2013. Available at `https://link.springer.com/content/pdf/10.1007%2Fs40435-013-0002-4.pdf`.

[5] Jin-Han Xie, *Quantifying the linear damping in two-dimensional turbulence*, 2020.Available at `https://www.researchgate.net/profile/Jin-Han-Xie/publication/346398516_Quantifying_the_linear_damping_in_two-dimensional_turbulence/links/5fbfbe8e299bf104cf7ae08f/Quantifying-the-linear-damping-in-two-dimensional-turbulence.pdf?origin=publication_detail`.

[6] Joshua Socolar, *Encyclopedia of Physical Science and Technology*, 3^{rd} Edition, 2003.

[7] Glenn Elert, *The Chaos Hypertextbook: 2.1 Strange Attractors*, 2016-2022, Available at `https://hypertextbook.com/chaos/strange/`.

[8] Yan Huang, Xiao-Song Yang, *Horseshoes in modified Chen's attractors*, 2004, Available at `https://www.sciencedirect.com/science/article/abs/pii/S0960077905000342`