RESEARCH STATEMENT

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My scientific interests lie in mathematics, within the branch of mathematical logic called *set theory*. Set theory grew into a subject on its own during the 20th century, as the first truly scientific attempt to understand infinity. The difference of the finite and the infinite manifests itself in its clearest form in combinatorics. Indeed, a large part of set theoretic research concentrates on the combinatorial behavior of finite, countable and uncountable objects. My research contributes to understanding the combinatorics of the uncountable.

1. Universality problems

A *universality problem* is a problem of the form

Given a class $(\mathcal{K}, \hookrightarrow)$*, does it contain a universal object?*

Here $\mathcal{K}$ can be any class of structures (really anything: countable groups, graphs of size n, databases, or even all metric spaces) and $\hookrightarrow$ is any notion of embedding between them (for example homomorphism). A *universal object* in $\mathcal{K}$ is some object $A \in \mathcal{K}$ such that every $B \in \mathcal{K}$ embeds into A. Universality problems arise everywhere in mathematics and there are various methods for proving the existence of a universal object, whenever one exists. Two of the classical examples of universal objects are the rationals in the class of countable linear orders, equipped with order preserving maps, and the random graph in the class of countable graphs, equipped with graph homomorphism.

The picture changes radically when the objects are allowed to be uncountable. In this case, the existence of a universal object is often independent of the axioms of set theory. This means that its existence nor non-existence is unprovable. Equivalently, it means that there is one universe of set theory that satisfies the axioms of set theory in which $\mathcal{K}$ has a universal object and also an another one in which $\mathcal{K}$ does not. For instance, both the existence of a universal linear order of size $\aleph_1$ and the existence of a universal graph of size $\aleph_1$ are independent of set theory[1].

In general, the existence of a universal model in an *elementary class*, a class of the form $\mathsf{Mod}_\kappa(T) = \{M : M \vDash T \text{ and } |M| \leqslant \kappa\}$ for a complete first-order theory T equipped with elementary embeddings, is independent of ZFC whenever T is unstable and κ is uncountable. This follows from the fact that the existence of a saturated model of T, which is always universal, is equivalent to the statement $\kappa^{<\kappa} = \kappa$, whenever T is unstable in κ (See Theorem 4.7 on page 476 in [18]), and this combinatorial statement about cardinals is independent. No such combinatorial characterization is known for the existence of universal models, which is often strictly weaker than the existence of saturated models. Shelah showed in [19] that there can exist a universal graph of size $\aleph_1$, even without a saturated one (i.e. without the continuum hypothesis). This is an example of a scenario where a universal

Date: November 2025.

[1] Under the continuum hypothesis the theories of dense linear orders and random graph both have a saturated model, which are universal. See [22] for details. This shows the consistency of its existence. The consistency of non-existence follows by adding $\aleph_2$ many Cohen reals, see the appendix of [11].

model of size $\aleph_1$ does exist, despite the fact that a saturated model of size $\aleph_1$ cannot not exist by the failure of CH (in the case of graphs the relevant complete first-order theory is the theory of random graph).

In [14], Mekler provides a general method for producing a model of set theory with a universal object of size $\aleph_1$ in a certain type of an amalgamation class, together with $\neg\mathsf{CH}$. His approach suggests that a shift of attention away from elementary classes to classes with simpler morphisms could provide useful in solving universality problems at uncountable cardinalities. In my research, I have considered universality problems in some non-elementary classes consisting of uncountable objects.

A *tree* is a partial order such that the set of predecessors of each element is well-ordered. A *branch* in a tree is a linearly ordered subset. Denote

$$\mathcal{T}_\kappa := \text{ trees of size } \kappa \text{ with no branch of length } \kappa.$$

An embedding between two trees in $\mathcal{T}_\kappa$ is a map that preserves levels and the tree order. Together with Ben-neria, Magidor and Väänänen, we proved that consistently, there can be a universal object in $\mathcal{T}_{\aleph_2}$.

Theorem 1 (Ben-Neria, K., Magidor, Väänänen [1])**.** *Assuming the consistency of a weakly compact cardinal*[2]*, consistently, there is a universal object in the class* $\mathcal{T}_{\aleph_2}$.

The result contrasts with Kurepa's result [12] according to which under the hypothesis $\kappa^{<\kappa} = \kappa$, there is no universal object in the class $\mathcal{T}_\kappa$, for any regular cardinal κ. I extended the result to $\aleph_1$:

Theorem 2 (K. [8])**.** *Assuming the consistency of a weakly compact cardinal, consistently, there is a universal object in the class* $\mathcal{T}_{\aleph_1}$.

In addition to Kurepa's result, the case of $\mathcal{T}_{\aleph_1}$ contrasts with a result by Džamonja and Shelah [3] by which $\mathcal{T}_{\aleph_1}$ does not have a universal object under the forcing-theoretic maximality principle Martin's Axiom. In fact, the proof of Theorem 1 works for every double successor cardinal κ that is below a weakly compact. Theorems 1 and 2 thus leave open whether $\mathcal{T}_\kappa$ can have a universal object when κ is a successor of a singular.

Question 3. *Can* $\mathcal{T}_\kappa$ *have a universal object when* κ *is a successor of a singular cardinal?*

Question 3 is potentially hard and would require combining different techniques. There is an easier question that states the existence of a universal object in the subclass of $\mathcal{T}_\kappa$ consisting of *special* trees (a tree T of size κ^+ is *special* if there is a function $f : T \to \kappa$ with the property that if $f(t) = f(s)$ for $t \neq s$, then t and s are incompatible in the tree order). It is a classical result by Laver and Shelah [13] that, assuming the consistency of a weakly compact cardinal, consistently all trees in $\mathcal{T}_{\aleph_2}$ are special.[3]

Question 4. *Is it consistent that* $\mathcal{T}_{\aleph_2}$ *has a universal object that is special?*

With Ben-Neria, Magidor and Väänänen (authors from [1]), we have started working towards answering this question. We hope to confirm that the answer is positive in the near future.

Next we consider universality question in linear orders. An embedding between two linear orders is an order-preserving map. If $2^\kappa = \kappa^+$, there is a saturated dense

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[2] The existence of a weakly compact cardinal is a mild strengthening of the axioms of set theory

[3] Strictly speaking, they show that consistently all $\aleph_2$-Aronszajn trees are special, but the proof adapts to the whole class $\mathcal{T}_{\aleph_2}$.

linear order of size κ, which is universal for linear orders of the same size. However, it is a longstanding open question whether there can be one without the assumption about cardinals.

Question 5. *Let $\kappa \geqslant \aleph_0$. Is is consistent that there is a universal linear order of size κ^+ and $2^\kappa > \kappa^+$?*

For $\aleph_1$, the question asks whether the existence of a universal linear order of size $\aleph_1$ together with the failure of continuum hypothesis is consistent. Consider the following subclass of linear orders:

$$\mathcal{L}_\kappa := \text{the class of linear orders of size } \kappa \text{ that do not order-embed } \kappa \text{ nor } \kappa^{-1}.$$

For instance, every subset of the real line $\mathbb{R}$ of size $\aleph_1$ belongs to $\mathcal{L}_{\aleph_1}$. Building on [1] and [8] and representing linear orders by trees, with H. Hammud, we managed to obtain an answer to a slightly modified version of Question 5:

Theorem 6 (Hammud, K. [4])**.** *Let $\kappa \in \{\aleph_1, \aleph_2\}$. Assuming the existence of a weakly compact cardinal, it is consistent that $\mathcal{L}_\kappa$ has a universal object.*

In [15], Moore proved that the forcing-theoretic maximality principle PFA, which strengthens the above-mentioned Martin's Axiom, implies the existence of a universal *Aronszajn line*. This is a linear order of size $\aleph_1$ that does not contain an order-isomorphic copy of ω_1 nor its inverse ω_1^{-1}, nor an uncountable subset of the real numbers. Aronszajn lines thus form a proper subclass of $\mathcal{L}_{\aleph_1}$. It should be noted that the consistency strength of PFA is much more than a weakly compact and it is not known what is the consistency stregth of the existence of a universal Aronszajn line. Our proof of Theorem 6 as such does not generalize to produce a universal Aronszajn line from a weakly compact cardinal.

Aronszajn lines are tightly connected with *Aronszajn trees*: those trees in $\mathcal{T}_{\aleph_1}$ whose levels are countable. Each Aronszajn line is a linear extension of an Aronszajn tree order. Therefore, it is surprising that Todorcevic [23] showed that under Martin's Axiom, there cannot exist a universal Aronszajn tree. Thus, PFA, which implies Martin's Axiom, implies both the existence of a universal Aronszajn line and non-existence of a universal Aronszajn tree. It is a longstanding open problem whether a universal Aronszajn tree can exist.

Question 7. *Is the existence of a universal $\aleph_1$-Aronszajn tree consistent?*

There is a possibility that the new methods from [1] could provide a way to attack this problem, but certainly a new idea is required too.

For the class of graphs, as mentioned above, the existence of a universal graph of size $\aleph_1$ alongside with the failure of the continuum hypothesis is consistent ([19]). I managed to generalize this result to countably chromatic graphs of size $\aleph_1$. The proof is different than Shelah's proof from [19].

Theorem 8 (K. [7])**.** *The existence of a universal countably chromatic graph of size $\aleph_1$ along with the failure of continuum hypothesis is consistent.*

In Theorem 8, there is no need for large cardinal assumptions.

I am very interested in extending the result on multiple cardinals simultaneously:

Question 9. *Is it consistent to have a universal graph / wide tree / linear order on two successive cardinals simultaneously?*

This question seems like the most natural next step in the study of universality problems. I find the question interesting since it requires adaptation of known methods to new context - the method of side conditions of two types, developed by Neeman [16].

The proofs of Theorems 1, 2, 6 and 8 are forcing constructions, and in each of them, the preservation of cardinals is handled through a special use of side conditions - small models of set theory - as part of forcing conditions. In the case of the three first ones, the side conditions are models that are strong enough to reflect the second-order properties of the class in question (no cofinal branch). The existence of these is provided by the weakly compact cardinal.

Question 10. *Is the weakly compact needed in Theorems 1, 2 and 6?*

I conjectured in [8] that the results can be obtained without any large cardinal strength. Instead of forcing using the reflective properties of the weakly compact cardinal and the weakly compact filter, one could force over L and use Devlin's principle $\diamond^{\#}$ from [2] and the filter derived from it. Thus, there is a path to solve Question 10.

We end the section on universality problems with a difficult unprecise question.

Question 11. *Is there a way to render the methodology of Theorems 1, 2, 6 and 8 into a black box?*

As explained above, the proofs of Theorems 1, 2, 6 and 8 have the same structure, but differ in details. The far-reaching goal is to find, in the spirit of Mekler's result [14], a combinatorial condition P and a map $\mathcal{K} \mapsto \mathbb{P}_{\mathcal{K}}$ such that for any class $\mathcal{K}$ of objects of size κ satisfying the condition P, the poset $\mathbb{P}_{\mathcal{K}}$ is a poset that preserves κ and forces that the class $\mathcal{K}$ has a universal object together with $\kappa^{<\kappa} > \kappa$. This is what is meant by "black box". The combinatorial condition P should be general enough to cover the classes $\mathcal{T}_{\aleph_1}$, $\mathcal{L}_{\aleph_1}$ and the class of countably chromatic graphs and other natural non-elementary classes, and simple enough to be easily verifiable. It is possible that something even better could be around - these theorems could be glimpses of a new forcing-theoretic maximality principle.

2. Classifying uncountable models

In mathematical logic, the objects of study are *models*: tuples of the form $(M, R_0, R_1, \dots)$, where M is a set and $R_0, R_1, \dots$ are n-ary relations or functions on M. For instance, graphs, groups, fields, orders, trees, vector spaces, R-modules for a fixed ring R, etc. are all instances of models. The notion is thus very general. Two most important questions to understand are the following:

(1) How to measure the similarity of two models?
(2) How to classify all the models?

These two questions are, obviously, ridiculously too general. However, there are partial answers to them. In the field of logic, these questions are understood though their logical aspects and attacked using *formal language* as the principal weapon.

The first successful attempt to answer the two questions in case of countable models is due to Scott [17] and Karp [5], among others. If A and B are *countable* models in the same signature[4], then two countable models are the same, i.e. isomorphic, if and only if they are elementarily equivalent in the infinitary logic $\mathcal{L}_{\omega_1\omega}$ (Scott [17]). This gives a "grading" for the isomorphism relation in the class of countable models - two models are said to be *β-isomorphic* if they satisfy the same $\mathcal{L}_{\infty\omega}$-sentences up to quantifier rank β. Thus, the similarity of two models can now be given a grade $\beta \in \mathsf{Ord}$, answering the first question. Moreover, Scott [17] answered the second question. He proved that for every countable model A

[4]signature of a model specifies the number of the relations listed for the model and their arity.

there is a sentence $\psi_A \in \mathcal{L}_{\omega_1\omega}$ that works as an invariant; if B is a countable model, then A and B are isomorphic if and only if B satisfies the sentence ψ_A.

The notion of β-isomorphism is characterized via an Ehrenfeucht-Fraïssé game - two models are β-isomorphic if and only if player II has a winning strategy in the EF^β-game between them (Karp [5]). Such a characterization of a logic in terms of a game is called a *Karp characterization*. It is an empirical fact that every logic comes with a corresponding game-theoretic Karp characterization.

To analyse the two above questions in the case of uncountable models, one needs to study the extensions of first-order logic, and even the extensions of the "standard" infinitary logics $\mathcal{L}_{\kappa\lambda}$. There are various directions. In addition to the method of forcing and solving universality questions, I am also interested in the theory of abstract logics. What I find particularly appealing in this field are the proofs; they combine methods from set theory and all around logic in neat ways.

In [20], Shelah described a new kind of infinitary logic, that he called $\mathcal{L}^1_\kappa$. It was a solution to an open problem posed by Feferman in the 70s, whether there exists a logic strictly stronger than first-order logic, that satisfies a Lindström characterization. The classical Lindström theorem states that first-order logic is maximal[5] among logics satisfying Löwenheim-Skolem theorem and being countably compact. In the same spirit, a *Lindström characterization* is a characterization of a logic as maximal among logics that satisfy some model-theoretic property. In the case of $\mathcal{L}^1_\kappa$, this model-theoretic property is an amalgame of Löwenheim-Skolem type of property and undefinability of well-order. A Lindström characterization indicates that the logic has model-theoretic interest and could be useful in tackling the above two questions about studying the structure of uncountable models. There is indeed some evidence towards this, for instance this logic $\mathcal{L}^1_\kappa$ admits a Keisler-Shelah style algebraic characterization of elementary equivalence in terms of isomorphic iterated ultrapowers. This was proved by Shelah in [21] and an alternative proof can be found in our note with Veličković [9].

The problem of $\mathcal{L}^1_\kappa$, however, is that it is not known whether it admits a (generative) syntax. In [10], with J. Väänänen and A. Villaveces, we define a logic that we call *Cartagena logic* and denote by $\mathcal{L}^c_\kappa$. We show that it is a good approximation[6] of $\mathcal{L}^1_\kappa$ and has a generative syntax. We give a game-theoretic description of $\mathcal{L}^c_\kappa$ via a Karp-theorem:

Theorem 12 (K., Villaveces, Väänänen [10])**.** *The following are equivalent for any models A and B:*

(1) A and B satisfy the same $\mathcal{L}^c_\kappa$-sentences up to quantifier rank β,
(2) player II has a winning strategy in the game $\mathsf{CG}^\beta(A,B)$.

In addition, we show that $\mathcal{L}^c_\kappa$ satisfies multiple model-theoretic properties, such as a Löwenheim-Skolem theorem, closure under countable limits (Theorems 3.3, 3.5, 3.10 from [10]). In addition, we explicitly exhibit multiple $\mathcal{L}^c_\kappa$-sentences that define interesting and natural model classes:

Theorem 13 (K., Villaveces, Väänänen [10])**.** *The following classes of structures are definable in the Cartagena logic $\mathcal{L}^c_\kappa$, for each cardinal $\theta < \kappa$:*

(1) Models of cardinality θ, as well as models with a predicate or a definable subset of size θ.
(2) Graphs with a clique of size θ.

[5] with respect to expressive power

[6] It is only a Δ-closure away: $\Delta(\mathcal{L}^c_\kappa) = \mathcal{L}^1_\kappa$. The Δ-closure of a logic is a closure operator that takes a logic $\mathcal{L}$ and gives a possibly stronger logic $\Delta(\mathcal{L}) \geqslant \mathcal{L}$. The Δ-operator preserves many properties such as compactness and Löwenheim-Skolem theorem. It is the least logic above $\mathcal{L}$ in which those models classes that are projective and co-projective are in fact definable.

(3) Graphs of size θ that admit an ω-coloring.
(4) For each cardinal $\theta < \kappa$ of uncountable cofinality: θ-Aronszajn trees.
(5) Partially ordered models with an uncountable descending chain.

Despite being strong enough to describe these classes, $\mathcal{L}^1_\kappa$ is quite weak since it is not strong enough to define the class of well orderings. There is a natural further question.

Question 14. *Is there a logic that is strong enough to define well-order which admits a Lindström characterization?*

The natural candidate for a positive answer to Question 14 is a certain tree logic $\mathcal{L}_\mathcal{T}$ from [6]. The difficulty in the question is that the satisfaction relation is defined in terms of a game that is not always determined. This logic is particularly interesting, since it is the natural approach to answer both questions from the beginning of the section in the case of models of size $\aleph_1$ - how to measure the difference of non-isomorphic models, and how to find invariants for them. Its Karp characterization is in terms of an Ehrenfeucht-Fraïssé game that is forced to have an arbitrarily large but countable length using a tree without uncountable branches as a "game clock". Such trees are called *clock trees*. The class $\mathcal{T}_{\aleph_1}$ is a subclass of the class of clock trees and in fact, the study of the class of trees $\mathcal{T}_\kappa$ from the first section initiated in the study of uncountable models. See [24]. The universality question in the class of clock trees is of particular interest due to the following observation: if there is a tree-embedding $S \to T$ and player II has a winning strategy in a game against the clock tree T, then she has one against the clock tree S. Thus a universal tree provides a sort of a "master clock". We say that a tree T is *very wide* if it has size less than continuum (provided that continuum has size at least $\aleph_2$). In the context of classification of uncountable models, a natural further question from Theorem 2 is the following:

Question 15. *Is the existence of a universal very wide clock tree consistent with large continuum?*

3. Conclusion

This research statement outlines a research program which is far from its end. It divides into two parts: universality problems and the study of infinitary logics (with the goal to understand the structure of uncountable models). Both directions derive motivation from model theory, and from combinatorial questions of classes of structures, but the techniques and tools used in the proofs are purely set-theoretic. The methods involve heavy forcing machinery and the use of large cardinals as a combinatorial booster. These two seemingly different directions meet each other in the universality problems in the classes consisting of trees without cofinal branches. My contributions in the universality problems can be found in papers [1], [8], [4] and [7], and in the study of infinitary logics can be found in [10].

In the future years, I wish to continue the study of these questions and develop new and better methods that can be used to attack even deeper problems. In particular I wish to attempt answering the natural questions 4, 15, 5, 7, 9 and 3 (ordered in an increasing order of expected difficulty) and gain a better understanding of Question 11. I see Questions 9 and 3 as an excellent excuse to widen my mathematical toolpack and gain confidence in adjacent branches of set theory.

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