Aronszajn trees and maximality*

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Abstract

Assuming the consistency of a weakly compact cardinal above a regular uncountable cardinal μ, we prove the consistency of the existence of a wide μ^+-Aronszajn tree, i.e. a tree of height and cardinality μ^+ with no branches of length μ^+, into which every wide μ^+-Aronszajn tree can be embedded.

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1 Introduction

The topic of this paper is maximality among wide κ-Aronszajn trees, i.e. trees of cardinality and height κ without branches of length κ. Such a tree is called maximal if every such tree can be embedded into it. We show the consistency of maximal trees relative to the consistency of a weakly compact cardinal. This has been an open problem for 30 years.

Trees in this paper are partial orders in which the set of predecessors of every element are well-ordered by the partial order, and there is a unique smallest element. The order-type of the set of predecessors of an element of a tree is called the *height* of the element, and the supremum of all heights in a tree is called the *height* of the tree. The set of elements of a fixed height is called a

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level of the tree. There is a natural quasi-ordering of the class of all trees: a tree T is below a tree T' if T can be monomorphically embedded into T'. For any class $\mathcal{C}$ of trees it is natural to ask if $\mathcal{C}$ has a maximal element T under embeddability. Then, up to isomorphism, the class $\mathcal{C}$ consists just of subtrees of T in $\mathcal{C}$.

Our focus is on trees of cardinality and height $\kappa \geq \omega_1$ with no branches, i.e. linearly ordered subsets, of size κ. Following [4], we call such trees *wide* κ*-Aronszajn trees*. Our main result (Theorem 1 below) is that it is consistent, relative to the consistency of a weakly compact cardinal, to have a maximal wide κ-Aronszajn tree. Our proof works for any successor of a regular cardinal $> \aleph_0$. This result complements the fact that it is a consequence of the Generalized Continuum Hypothesis that there are no maximal wide κ^+-Aronszajn trees for any infinite regular κ [11]. Under the stronger assumption $V = L$, no wide κ^+-Aronszajn tree is maximal even just for κ^+-Souslin trees, for it can be proved, improving a result in [25], that, assuming $V = L$, for every κ^+-Aronszajn tree T, κ regular, there is a κ^+-Souslin tree which is not embeddable into T.

If the levels of a wide κ-Aronszajn tree are of cardinality $< \kappa$, we drop "wide" and call such trees just κ-Aronszajn, or, to be more specific, *narrow* κ-Aronszajn. Furthermore, if $\kappa = \omega_1$, we call the trees Aronszajn, or wide Aronszajn, respectively. For some κ there may be no κ-Aronszajn trees, and then κ is said to have the *tree property*. By Kőnig's Lemma, ω has the tree property. No singular cardinal has the tree property for trivial reasons. An inaccessible cardinal has the tree property if and only if it is weakly compact.

Examples of Aronszajn trees are so-called Souslin trees, which are instrumental in understanding, and proving the independence of, the so-called Souslin Hypothesis i.e. the hypothesis that the order-type of the real numbers is the unique, up to isomorphism, dense complete linear order without end-points in which all families of disjoint non-empty open sets are countable.

Wide κ-Aronszajn trees are important in the study of model theoretic properties of uncountable structures, namely, trees can be used as a weak substitute for ordinals when uncountable models are investigated by means of transfinite games [11] and, more generally, in the study of generalized Baire spaces [17]. For example, the existence of a particular kind of maximal tree, a so-called Canary Tree (see below), is equivalent, assuming CH, to the isomorphism class of the free Abelian group of cardinality $\aleph_1$ being Δ^1_1 in the generalized Baire Space $\omega_1^{\omega_1}$. This emphasises the importance of understanding better the global ordering of trees, especially the existence of maximal trees.

There is no maximal countable ordinal, but if we identify ordinals with trees without infinite branches, and consider generalized ordinals i.e. the class $\mathcal{T}_\alpha$ of trees of cardinality $\aleph_\alpha$ without branches of length $\aleph_\alpha$, the situation is more opaque. When $\alpha > 0$, the structure of the class of such trees is much more complicated than the structure of ordinals. For example, the structure of $\mathcal{T}_1$ is highly non-linear as it is easy to construct pairs of wide (or narrow) Aronszajn trees so that neither can be mapped even by a strict order preserving homomorphism to the other. Furthermore, the structure is highly non-absolute. Several partial results are known about $\mathcal{T}_\alpha$, $\alpha > 0$, [17, 18, 25, 24, 4].

The main result of this paper is:

Theorem 1. *Suppose that κ is a weakly compact cardinal and $\mu < \kappa$ is regular uncountable. There is a (set) forcing extension of the universe in which $\kappa = \mu^+$ and there is a maximal wide κ-Aronszajn tree i.e. a wide κ-Aronszajn tree T such that any other wide κ-Aronszajn tree can be monomorphically embedded into T.*

To simplify our presentation, we will prove the theorem for the case $\mu = \omega_1$ (i.e., $\kappa = \omega_2$). It will be apparent throughout this work that modification to an arbitrary regular uncountable cardinal μ is straightforward. The theorem holds also for $\mu = \omega$, see [13].

This theorem further emphasises the difference between the order of trees with no infinite branches and the class of trees with no branches of length $\kappa > \omega$.

We shall now define in detail the central concepts of this paper. We have already defined the concept of a wide κ-Aronszajn tree as well as its special case, the (narrow) κ-Aronszajn tree, agreeing to drop κ if $\kappa = \omega_1$. While Aronszajn trees always exist, the existence of an $\aleph_2$-Aronszajn tree is independent of ZFC, assuming the consistency of weakly compact cardinals: Specker proved the existence of an $\aleph_2$-Aronszajn tree from CH [23]. Mitchell and Silver proved the consistency of the non-existence of $\aleph_2$-Aronszajn trees, relative to the consistency of a weakly compact cardinal [19, Theorem 5.8]. They also showed that if there are no $\aleph_2$-Aronszajn trees, then $\aleph_2$ is weakly compact in L. As opposed to the case of (narrow) κ-Aronszajn trees, it is easy to construct a *wide* κ-Aronszajn tree in ZFC by bundling together isolated branches of all lengths $< \kappa$.

As discussed already, our main topic in this paper is the existence of trees that are maximal in some specific sense. There are several ways in which two trees T and T' can be compared to each other in order for the concept of maximality to make sense. Originally the question of maximality was raised [17] in connection with comparing trees by asking whether there is a *homomorphism* from one to the other i.e. a mapping from one tree T to another T', that preserves strict ordering:

$$t <_T t' \Rightarrow f(t) <_{T'} f(t').$$

Note that such a mapping need not be one-one because incomparable elements can be mapped to the same element. We follow [4] in calling such a mapping a *weak embedding*[1]. The reason for the emergence of weak embeddings as a way to order classes of trees is its close connection to certain games, introduced below. While proving Theorem 1 the authors realized that they can actually prove the consistency of the existence of a maximal tree under a stronger order, namely the order according to (monomorphic) embeddability. In the end, ordering trees by the existence of an embedding is very natural. From a general

[1]There was a claim in [17] that a maximal tree exists in the sense of weak embeddings in $\mathcal{T}_1$ if Martin's Axiom (MA) and $2^\omega > \omega_1$ were assumed. This claim was proved wrong in [4].

mathematical perspective it can be considered even more natural than ordering by weak embeddings.

Let us write $T \leq^* T'$ if there is an embedding (i.e. a monomorphism) $T \to T'$. If $T \leq^* T'$ and $T' \leq^* T$ we write $T \equiv^* T'$. Respectively, if $T \leq^* T'$ but $T' \not\leq^* T$, we write $T <^* T'$. If there is a weak embedding from T to T', we write $T \leq T'$. If $T \leq T'$ and $T' \leq T$ we write $T \equiv T'$. Finally, if $T \leq T'$ but $T' \not\leq T$, we write $T < T'$. Of course, $T \leq^* T'$ implies $T \leq T'$.

If B_α is the tree of descending chains of elements of α, ordered by end-extension, then $\alpha \leq \beta$ if and only if $B_\alpha \leq B_\beta$. Thus in the class of trees without infinite branches the weak embedding order reflects the received ordering of the class of all ordinal numbers. Again, we may ask, whether there is a maximal tree under the weaker ordering $\leq$ in the class of all Aronszajn trees. If we assume $\mathrm{MA}_{\aleph_1}$, then no wide Aronszajn tree is $\leq$-above all Aronszajn trees [4, 24]. Similarly, if $V = L$, then for every wide Aronszajn tree T there is a Souslin tree S such that $S \not\leq T$ [25].

We may now ask in two different senses whether there is a maximal tree in a given class of trees:

> **The Maximality Question:** Given a class $\mathcal{C}$ of trees, is there a tree T in $\mathcal{C}$ such that $S \leq^* T$ for every $S \in \mathcal{C}$?
>
> **The Weak Maximality Question:** Given a class $\mathcal{C}$ of trees, is there a tree T in $\mathcal{C}$ such that $S \leq T$ for every $S \in \mathcal{C}$?

Trivially, a positive solution to the Maximality Question gives a positive solution to the Weak Maximality Question.

Both the full and the Weak Maximality Questions are meaningful even if the maximal tree T is not in $\mathcal{C}$ but satisfies some weaker constraints. For example, it is consistent, relative to the consistency of ZF, that CH holds and every Aronszajn-tree is special (Jensen, [21, Theorem 8.5]). Thus in this model there is a wide Aronszajn-tree that is $\leq$-above all Aronszajn-trees. However, this tree T is (a priori) not Aronszajn, so we do not obtain a solution to the Weak Maximality Question for the class of Aronszajn trees. Let us call a wide $\aleph_2$-Aronszajn-tree T *special* if there is $f : T \to \omega_1$ such that $t < t'$ always implies $f(t) \neq f(t')$. Consistency of a weakly compact cardinal implies the consistency of $2^{\aleph_0} = \aleph_1 + 2^{\aleph_1} > \aleph_2 +$ every wide $\aleph_2$-Aronszajn-tree is special [16]. In this model there is a tree T $\leq$-above all $\aleph_2$-Aronszajn-trees such that T has no $\aleph_2$-branches. Here $|T| > \aleph_2$, so again T is not an answer to the Weak Maximality Question for wide $\aleph_2$-Aronszajn trees. Our Theorem 1 gives a positive solution to the (full) Maximality Question for wide κ-Aronszajn trees, κ a successor of a regular cardinal $> \aleph_0$. As we see below, it is impossible to combine this with GCH.

Both $T \leq^* T'$ and $T \leq T'$ measure in their own ways how big the trees T and T' are with respect to each other. If B_{ω^*} denotes the tree consisting of the single branch of length ω, then $B_\alpha \leq B_{\omega^*}$ holds for all α but of course $B_\alpha \not\leq^* B_{\omega^*}$ when $\alpha > 1$. Thus B_{ω^*} is $\leq$-above a proper class of non-$\equiv$-equivalent trees.

There can be only $2^{|T|}$ trees $\leq^*$-below a given tree T, up to $\equiv^*$. This illustrates the different senses in which $\leq$ and $\leq^*$ measure the bigness of trees.

A still further ordering of Aronszajn trees is the following: If T is an Aronszajn tree and $C \subseteq \omega_1$, then we use $T \restriction C$ to denote the suborder of T consisting of nodes in T the height of which is in C. Suppose T and T' are Aronszajn trees. We say that a partial map $\pi : T \to T'$ is an embedding (or an isomorphism) *on a club* if there is a club $C \subseteq \omega_1$ such that π is an embedding (or respectively an isomorphism) $T \restriction C \to T' \restriction C$. It follows from the Proper Forcing Axiom that any two Aronszajn trees are isomorphic on a club [1, 14].

The following useful operation on trees is due to Kurepa [15]: If T is a tree, let $\sigma(T)$ be the tree of ascending chains in T, ordered by end-extension. For well-founded trees this is like the successor function on ordinals in the sense that $\sigma(B_\alpha) \equiv B_{\alpha+1}$. It is easy to see that if T is any tree, then $T < \sigma(T)$. Moreover, if T has no branches of length κ, neither has $\sigma(T)$. So from the point of view of lengths of branches $\sigma(T)$ is similar to T. However, it is perfectly possible that $|T[< |\sigma(T)|$. For example, if every node in T splits, then $|\sigma(T)| \geq 2^\omega$.

The σ-operation shows that if $\aleph_\alpha^{<\aleph_\alpha} = \aleph_\alpha$, the class $\mathcal{T}_\alpha$ does not have a $\leq$-maximal element. So in that case even the Weak Maximality Question has a negative answer for the class $\mathcal{T}_\alpha$. In consequence, $\mu^{<\mu} > \mu$ holds in the final model of our Theorem 1.

If $A \subseteq \omega_1$ is co-stationary, let $T(A)$ be the tree of closed increasing sequences of elements of A. The class of such trees $T(A)$ is an interesting subclass of trees without uncountable branches. A tree without uncountable branches which is of cardinality $\leq 2^\omega$ and $\leq$-above all such $T(A)$ is known as a *Canary Tree.* The existence of Canary Trees is independent of $ZFC + GCH$ [18, 6]. Assuming CH, a Canary Tree is perhaps not maximal in the entire class of trees in $\mathcal{T}_1$ but it still $\leq$-majorises the large class of trees of the form $T(A)$.

Trees as game clocks

We already alluded to the fact that, assuming CH, Canary Trees can be used to show that the isomorphism class of a particular structure, in this case the free Abelian group of cardinality $\aleph_1$, is Δ_1^1 in the generalized Baire space $\omega_1^{\omega_1}$. This is an example of the use of trees as clocks in games in the way we now describe. A maximal tree would represent a kind of universal clock. To see what this means, suppose δ is an ordinal and G is a game of length δ between I and II in which I and II produce a δ-sequence of elements of a fixed set M, alternating moves, I starting each round. We fix a set $W \subseteq M^\delta$ and say that II wins if the sequence played is in W. Otherwise I wins. We assume the game is closed in the sense that if $s \notin W$ then there is an initial segment s' of s such that no extension of s' is in W. If T is any tree (of height δ), we can define a new game G_T, a kind of approximation of G, as follows. Every time I moves in G he also picks a node t in T in such a way that t is above all nodes he has picked during previous rounds of G. If he cannot pick such a t then he loses. Otherwise the game is played as G. Clearly, if I has a winning strategy in G_T, he has also in G. The role of T in G_T is to make it harder for I to win. If T is well-founded, player I can only win G_T if he can win G in finitely many moves but he does not have to tell in advance how many moves he needs in order to win. If T has no branches of length δ, player I can only win G_T if he can win G in $< \delta$ moves but, again, he does not have to tell in advance how long δ'-sequence, $\delta' < \delta$, of moves he needs in order to win. He can change his mind about this during the game.

The following implications are immediate:

1. If II has a winning strategy in $G_{T'}$ and $T \leq T'$, then II has a winning strategy in G_T.

2. If I has a winning strategy in G_T and $T \leq T'$, then I has a winning strategy in $G_{T'}$.

3. If II has a winning strategy in G_T and I has a winning strategy in $G_{T'}$, then $T < T'$.

These implications emphasise the role of maximal trees for the games G_T. Let us see how this works, first on a general level and then more specifically. Let $\mathcal{C}_G$ be the class of trees T such that II has a winning strategy in G_T. If II has a winning strategy even in the non-approximated game G, the class $\mathcal{C}_G$ is simply the class of all trees. The other extreme is that $\mathcal{C}_G = \emptyset$, which happens if $W = \emptyset$. Suppose S is $\leq$-above all trees in $\mathcal{C}_G$. Then $\sigma(S) \notin \mathcal{C}_G$. So, maximality of the tree gives us negative information about winning strategies of II. Let $\mathcal{C}'_G$ be the possibly bigger class of trees T such that I does not have a winning strategy in G_T. Again, $\mathcal{C}'_G$ may be the class of all trees and it is also possible that $\mathcal{C}'_G = \emptyset$. Suppose S' is $\leq$-above all trees in $\mathcal{C}'_G$. Then $\sigma(S') \notin \mathcal{C}'_G$ i.e. I has a winning strategy in $G_{\sigma(S')}$. So, maximality of the tree gives us positive information about winning strategies of I.

A particular closed game of interest in this connection is the transfinite EF-game. Suppose M and N are models of the same vocabulary, $|M| = |N| = \delta$ and $M \not\cong N$. Let τ be the canonical enumeration strategy (i.e. I enumerates $M \cup N$) of I in the EF-game $\mathrm{EF}^\delta(M,N)$ of length δ on M and N such that both players are allowed to play a sequence of $< |\delta|$ elements at a time. Because we assume $M \not\cong N$, τ is a winning strategy of I. The pairs (T, T'), $T \leq T'$, of trees such that Player II has a winning strategy in $\mathrm{EF}^\delta(M,N)_T$ but Player I has a winning strategy in $\mathrm{EF}^\delta(M,N)_{T'}$ give information about how far or close M and N are from being isomorphic. Such pairs outline a kind of boundary where advantage in the game $\mathrm{EF}^\delta(M,N)$ moves from Player II to Player I. Every tree with a branch of length δ is above the boundary. If $\delta = \omega$, the boundary is (up to $\equiv$) exactly one tree, namely B_α for some (unique) countable ordinal α. If $\delta > \omega$, the boundary may be quite wide. Let us assume $\delta = \omega_1$. If the first order theory of M is classifiable in the sense of stability theory, the boundary lies between well-founded trees and non-well-founded trees [20]. In the opposite case the boundary may be quite high in the class of trees without uncountable branches. For models of size $\aleph_1$ of unstable theories it is above any tree in $\mathcal{T}_1$, if CH is assumed [10].

Open Question: Are there, for every tree $T \in \mathcal{T}_1$ non-isomorphic models M and N of cardinality $\aleph_1$ such that Player II has a winning strategy in $\mathrm{EF}^{\omega_1}(M,N)_T$?

A positive answer is known only for the extremely simple trees which consist of countable branches bunched together at the root [22]. A positive answer follows also from CH [10]. If there is a weakly maximal tree T in $\mathcal{T}_1$, solving the above question for T gives automatically a positive answer for all trees in $\mathcal{T}_1$.

The analogue of the Scott height of a countable model in this context is the following, introduced in [10]: A tree T without branches of length ω_α is called a *universal non-equivalence tree* for a model M of cardinality $\aleph_\alpha$ if for all models N of cardinality $\aleph_\alpha$ in the same vocabulary, if $M \not\cong N$, then Player I has a winning strategy in $\mathrm{EF}^{\omega_\alpha}(M,N)_T$. For example, a Canary Tree is (if it exists) a universal non-equivalence tree for the free Abelian group of cardinality $\aleph_1$. A tree T without branches of length ω_α is called a *universal equivalence tree* for a model M of cardinality $\aleph_\alpha$ if for all models N of cardinality $\aleph_\alpha$ in the same vocabulary, if Player II has a winning strategy in $\mathrm{EF}^{\omega_\alpha}(M,N)_T$, then $M \cong N$. If $\alpha = 0$, every countable model has a universal non-equivalence tree $B_{\alpha+1}$ and a universal equivalence tree B_α, where α is the Scott height of the model. For uncountable models the existence of such universal trees depends on stability theoretic properties of the first order theory of the model [10, 7, 8, 9]. By and large, depnding on cardinal arithmetic, models whose first order theory is unstable have no universal equivalence tree [10]. Models whose first order theory is superstable, NOTOP and NDOP, have a universal equivalence tree [10].

Our main result leaves open the possibility of having a narrow κ-Aronszajn tree which is maximal with respect to wide κ-Aronszajn trees under strict order preserving homomorphisms.

An outline of the paper

After some preliminaries in Section 2, we use in Section 3 a weakly compact cardinal κ to force a wide $\aleph_2$-Aronszajn tree T by Levy-collapsing κ to $\aleph_2$. The tree T is the tree that will be the desired maximal tree in the final model. The levels of T are sufficiently collapse-generic to permit the wide $\aleph_2$-Aronszajn trees arising in the construction to be embedded into T. We then define in Section 4 a σ-closed countable support iteration of length $\aleph_3$ of forcing with side conditions designed by means of an appropriate book-keeping to force for every wide $\aleph_2$-Aronszajn tree S an embedding $S \to T$. Naturally, we have to make sure $\aleph_2$ is not collapsed during this forcing. Section 5 is devoted to showing that our iterated forcing has the right kind of strong properness to guarantee the κ^+-chain condition and thereby the preservation of $\aleph_2$. We have to also make sure that our tree T will not acquire a long branch during the iteration. This is shown in Section 6. Theorem 1 is then proved. We conclude this work with a short open problems section 7.

Our methodology

When we want to get the consistency of the existence of a maximal wide κ-Aronszajn tree, we face the challenges of preserving κ, and of showing that the intended universal tree does not obtain a cofinal branch by the iteration. We deal with the former challenge by maintaining that the forcing is strongly proper with respect to sufficiently many structures of cardinality $< \kappa$. Proper forcing methods involving specializing Aronszajn trees has been used in [3]. The transition to wide trees and tree embeddings requires the development of a new type of argument for maintaining strong properness, which is developed in Section 5 of this paper. In addition, having no cofinal branch is a typical example of a second order property of an object that is supposed to be preserved under the iteration.

To secure the preservation of κ and to make sure that no cofinal branches are introduced to the maximal tree, we introduce a forcing with certain special features:

- The original object we intend to become the maximal tree is highly generic. Specifically, a key requirement of each individual poset is that whenever it maps a node s at level $\alpha < \kappa$ of a given tree S to a node t in the intended universal tree then the local branch b_t below t is mutually generic from the generic information of the local branch b_s below s.
- We make use of substructures M as "side-conditions" to guide the generically constructed embeddings. It is crucial that the chosen structures reflect second order assertions about the objects involved in the forcing. The existence of such structures requires the involvement of large cardinals.

We adopt the following general schema for proving the consistency of the existence of a universal object of cardinality κ in a class of structures satisfying some second order sentence Φ. The scheme consists of:

1. Force an object A intended to be the universal object for the property Φ.
2. By dovetailing, iterate forcings which embed each individual structure satisfying Φ into A.
3. Show that the iteration is proper (or strongly proper) so that we do not collapse the relevant cardinals.
4. Show that A satisfies Φ after the iteration by using a "splitting argument" and the fact that the iteration satisfies a certain variation of strong properness.

2 Preliminaries

The rest of the paper is devoted to showing that consistently, there can be a maximal wide κ-Aronszajn tree, for any double successor cardinal κ. We review some facts needed for the proof. Our notation is standard and follows [12].

Trees: Preliminaries

A tree $(T, <_T)$ is a partially ordered set with a minimal element (root) and with the property that for every $t \in T$, the set of its $<_T$-predecessors $b_t = \{\bar{t} \in T \mid \bar{t} <_T t\}$ is well-ordered by $<_T$. We refer to b_t as the *branch below* t. For an ordinal α, the α-th level of T, denoted $\mathrm{Lev}_\alpha(T)$ is the set of all $t \in T$ so that b_t has ordertype α in $<_T$. The union $\bigcup_{\alpha'<\alpha} \mathrm{Lev}_{\alpha'}(T)$ is denoted by $\mathrm{Lev}_{<\alpha}(T)$. The height of the tree T is the minimal κ such that $\mathrm{Lev}_\kappa(T) = \emptyset$. Let T be a tree of height and size κ. We say that T is *narrow* if $|\mathrm{Lev}_\alpha(T)| < \kappa$ for every $\alpha < \kappa$. Otherwise, we say that T is *wide*. A subset $b \subseteq T$ is a cofinal branch if it is well ordered by $<_T$ and has order-type κ. We say that T is κ-Aronszajn if it has no cofinal branches, i.e. branches of ordertype κ. If M is a transitive set that is closed under taking predecessors in the tree order $<_T$, and $t \in T \setminus M$, we define the *exit node* $e_T(t, M)$ of t from M to be the $<_T$-minimal node $e \in b_t \cup \{t\}$ that does not belong to M.

Weakly compact cardinals: Preliminaries

A second-order formula ψ is Π^1_1 if it has at most one second-order quantifier, and that quantifier is universal.

A cardinal κ is *weakly compact* if for every $B \subseteq V_\kappa$ and every Π^1_1 statement ψ true in $(V_\kappa, \in, B)$ the set

$$A_\psi = \{\alpha < \kappa \mid (V_\alpha, \in, B \cap V_\alpha) \models \psi\}$$

is nonempty. It follows from the definition that the collection of sets

$$\{A_\psi \mid (V_\kappa, \in, B) \models \psi,\ \psi \text{ is } \Pi^1_1 \text{ and } B \subseteq V_\kappa\}$$

generates a κ-complete normal filter on κ, denoted by $\mathcal{F}_{WC}$.

Definition 2 (Reflecting structures)**.** Let $\theta \geq \kappa^{++}$ be a regular cardinal and $<_\theta$ a well-ordering of H_θ. For every $P \in H_\theta$ we define the reflecting sequence of P,

$$\vec{M}^P = \langle M^P_\alpha \mid \alpha \in \mathrm{dom}(\vec{M}^P)\rangle$$

to consist of all Skolem-hull substructures of H_θ of the form

$$M^P_\alpha = Hull^{(H_\theta, \in, <_\theta, P)}(\alpha)$$

with the following properties:

- $M^P_\alpha \cap V_\kappa = V_\alpha$,
- for every $Q \subseteq V_\kappa$ such that $Q \in M^P_\alpha$ and a Π^1_1 statement ψ true in $(V_\kappa, \in, Q)$, if $(V_\kappa, \in, Q) \models \psi$ then $(V_\alpha, \in, Q \cap V_\alpha) \models \psi$.

It follows from a standard argument that $\mathrm{dom}(\vec{M}^P)$ belongs to $\mathcal{F}_{WC}$ for every $P \in H_\theta$.

Levy Collapse: Preliminaries

Let $\mathbb{P} = \mathrm{Coll}(\omega_1, < \kappa)$ be the Levy-collapse poset. Conditions $p \in \mathbb{P}$ are countable partial functions $p : \omega_1 \times \kappa \to \kappa$ with the property that $p(\nu, \alpha) < \alpha$ for every $(\nu, \alpha) \in \mathrm{dom}(p)$. Let $G \subseteq \mathbb{P}$ be a generic filter. For each $\alpha < \kappa$ let $f^G_\alpha : \omega_1 \to \alpha$ be given by $f^G_\alpha(\nu) = \alpha'$ iff there is $p \in G$ and $p(\nu, \alpha) = \alpha'$. We refer to f^G_α as the collapse generic surjection from ω_1 onto α that is derived from G. By a well-known argument, $\mathbb{P}$ is isomorphic to any number $\tau \leq \kappa$ of copies of itself. Fix an isomorphism between $\mathbb{P}$ and $\kappa \times \kappa \times \kappa$ copies of itself,

$$\mathbb{P} \cong \prod_{(\eta,\beta,\alpha)\in\kappa^3} \mathbb{P}(\eta, \beta, \alpha)$$

i.e., $\mathbb{P}(\eta, \beta, \alpha) = \mathrm{Coll}(\omega_1, < \kappa)$ for all $\eta, \beta, \alpha \in \kappa$. The isomorphism breaks a generic filter $G \subseteq \mathrm{Coll}(\omega_1, < \kappa)$ to mutually generic filters

$$\langle G(\eta, \beta, \alpha) \mid (\eta, \beta, \alpha) \in \kappa^3 \rangle,$$

$G(\eta, \beta, \alpha) \subseteq \mathbb{P}(\eta, \beta, \alpha)$. For each $\tau < \kappa$, let

$$f^{G(\eta,\beta,\alpha)}_\tau : \omega_1 \to \tau$$

denote the collapse generic surjection from ω_1 to τ, derived from $G(\eta, \beta, \alpha)$.

3 Building the Wide Tree T

In this section, we construct a wide tree T in a generic extension $V[G]$, where G is a generic filter on $\mathrm{Coll}(\omega_1, < \kappa)$. This will be the maximal tree in the final model.

A key ingredient of the construction of the tree T is that its local branches are generically independent. More precisely, considering all $\alpha < \kappa$ of uncountable cofinality, and $t \in \mathrm{Lev}_\alpha(T)$, their associated branches $b_t = \{t' \in \mathrm{Lev}_{<\alpha}(T) \mid t' <_T t\}$ are generically independent of each other in the sense that the parts of the collapse generic filter G that is required to determine their identity are independent. To do this, we associate to each node $t \in \mathrm{Lev}_\alpha(T)$ four parameters $(\alpha, \beta, \gamma, \delta) \in \kappa^4$ called the *collapse index of* t and use them to determine a segment of G that will define b_t.

We can now give the construction of the wide tree $T \in V[G]$. The domain of T is the set $\kappa \times \kappa$, and each α-th level is the set

$$\mathrm{Lev}_\alpha(T) = \{\alpha\} \times \kappa.$$

The ordering $<_T$ is constructed level-by-level and makes use of a fixed isomorphism between $\mathrm{Coll}(\omega_1, < \kappa)$ and the poset obtained by taking κ^3 many copies of $\mathrm{Coll}(\omega_1, < \kappa)$,

$$\prod_{(\alpha,\beta,\gamma)\in\kappa^3} \mathbb{P}(\alpha, \beta, \gamma)$$

with the conventions given in Section 2 above: the conditions in $\mathbb{P}$

We maintain an inductive assumption that the restriction of $<_T$ to $\mathrm{Lev}_{<\alpha}(T)$ belongs to the intermediate extension $V[G \restriction \alpha \times \kappa \times \kappa]$, generic over V for the partial product poset

$$\mathbb{P} \restriction (\alpha \times \kappa \times \kappa) = \prod_{(\alpha',\beta,\gamma)\in\alpha\times\kappa^2} \mathbb{P}(\alpha', \beta, \gamma).$$

Suppose that $<_T\restriction \alpha \times \kappa$ has been defined for some $\alpha < \kappa$, i.e. the tree order of T has been defined on $\mathrm{Lev}_{<\alpha}(T)$.

If $\alpha = \alpha'+1$ is a successor ordinal then we extend the order to the new α-th level of T by adding κ many successive nodes above each $t' \in \{\alpha'\} \times \kappa = \mathrm{Lev}_{\alpha'}(T)$. We use a pairing bijection $\langle\cdot,\cdot\rangle : \kappa \times \kappa \to \kappa$ from V to do this.

Suppose that $\alpha < \kappa$ is a limit ordinal. To define the extension of $<_T$ to the α-th level, it suffices to assign each $t \in \{\alpha\} \times \kappa$ a cofinal branch $b_t \subseteq \mathrm{Lev}_{<\alpha}(T)$, as we can then define $t' <_T t$ if and only if $t' \in b_t$. We consider separately the cases when α is of countable cofinality and when α is of uncountable cofinality.

If $\mathrm{Cof}(\alpha) = \omega$ and $\langle \alpha_n \mid n < \omega\rangle$ is an increasing cofinal sequence in α, then each cofinal branch $b \subseteq \mathrm{Lev}_{<\alpha}(T)$ is determined by its ω-subseqence $\langle b(\alpha_n) \mid n < \omega\rangle \in (\alpha \times \kappa)^\omega$. As $|(\alpha \times \kappa)^\omega| = \kappa$, we may pair the cofinal branches through $\mathrm{Lev}_{<\alpha}(T)$ and the nodes at the α-th level. Indeed, we enumerate all cofinal branches in $\mathrm{Lev}_{<\alpha}(T)$ as $\langle b'_\nu \mid \nu < \kappa\rangle$, and for each node $t = (\alpha, \nu) \in \mathrm{Lev}_\alpha(T)$, define $b_t = b'_\nu$. So each cofinal branch has a supremum in $\mathrm{Lev}_\alpha(T)$, and each node in $\mathrm{Lev}_\alpha(T)$ has a unique cofinal branch below it. The tree T will be σ-closed.

Suppose that $\alpha < \kappa$ is a limit ordinal of uncountable cofinality. The following definition will be used to define the cofinal branches $b \subseteq \mathrm{Lev}_{<\alpha}(T)$ that will be extended to nodes at level α.

Definition 3. Suppose that T' is a σ-closed normal tree on $\alpha \times \beta$, $\delta \geq \alpha \cdot \beta$ is an ordinal and $f : \omega_1 \to \delta$ is a function such that for every $\mu < \delta$, $f^{-1}(\mu) \subseteq \omega_1$ is unbounded.

1. Define the branch $b^f \subseteq T'$ determined by f to be the sequence $\langle t_i \mid i < \omega_1\rangle$ defined as follows.

   To define t_0 we look at the minimal $j < \omega_1$ for which $f(j) < \delta$ is of the form $f(j) = \alpha \cdot \beta_0 + \alpha_0 < \alpha \cdot \beta$. We then set $t_0 = (\alpha_0, \beta_0)$.

   Suppose that $\langle t_i \mid i < i^*\rangle$ has been defined. If i^* is a limit ordinal then t_{i^*} is the limit of the countable sequence $\langle t_i \mid i < i^*\rangle$. Otherwise $i^* = i + 1$. Define $t_{i^*} = (\alpha^*, \beta^*)$ by looking at $j^* < \omega_1$ minimal such that $f(j^*) = \alpha \cdot \beta^* + \alpha^* < \alpha \cdot \beta$, and $t_i <_{T'} (\alpha^*, \beta^*)$ for all $i < i^*$. We then set $t_{i^*} = (\alpha^*, \beta^*)$.

2. If $q = f \restriction \nu : \nu \to \delta$ is an initial segment of f, then q naturally determines an initial sequence $\langle t_i \mid i \leq i_q \rangle$ of b^f which has a maximal element t_{i_q}. We denote the last node t_{i_q} by $\pi_q(b^f)$ and call it the *projection* of b_f determined by q.

The next lemma follows from a standard density argument and the definition of cofinal branches b_f (3).

Lemma 4. *Suppose that T' is a σ-closed normal tree on $\alpha \times \beta$. If $\delta \geq \alpha \cdot \beta$ and $f : \omega_1 \to \delta$ is a $\mathrm{Coll}(\omega_1, \delta)$-generic over a model V' that contains T', then $b^f \subseteq T'$ is a cofinal branch in T'.*

We shall use Definition 3 to determine the branches b_t for $t \in \mathrm{Lev}_\alpha(T)$ by choosing subtrees T' of $\mathrm{Lev}_{<\alpha}(T)$ and using collapse generic functions f to form branches b^f through T'. In this setup, T' belongs to the intermediate extension V' of V by $\mathbb{P}' = \mathbb{P} \restriction \alpha \times \kappa \times \kappa$. This means that the tree projection maps $\dot{\pi}_p(t)$ from definition 3 will be $\mathbb{P}'$-names of nodes in $\mathrm{Lev}_{<\alpha}(T)$. We make the following observation about the nature of the construction that follows from this setup.

Lemma 5. *Suppose that $\dot{T}'$ is $\mathbb{P}'$-name of a tree on $\alpha \times \beta$ for some poset $\mathbb{P}'$, $\delta \geq \alpha \cdot \beta$ is an ordinal, and $\dot{f}$ is a $\mathrm{Coll}(\omega_1, \delta)$-name for the generic collapse. Let $b^{\dot{f}}$ be the $\mathbb{P}' \times \mathrm{Coll}(\omega_1, \delta)$-name for the associated generic branch through T', and let $q \mapsto \dot{\pi}_q(b^{\dot{f}})$ denote the $\mathbb{P}'$-name for the projection assignment to conditions $q \in \mathrm{Coll}(\omega_1, \delta)$. Suppose that $(p', q) \in \mathbb{P}' \times \mathrm{Coll}(\omega_1, \delta)$ is such that for some $t^* \in T'$,*

$$p' \Vdash_{\mathbb{P}'} \dot{\pi}_q(b^{\dot{f}}) = \check{t}^*.$$

For every $t' \in T'$ there is an extension $q' \leq q$ such that

$$p' \Vdash_{\mathbb{P}'} \text{ if } t' >_{T'} \check{t}^* \text{ then } t' \leq_{T'} \dot{\pi}_{q'}(\dot{b}^f).$$

We use definition 3 above to define the branches b_t for $t \in \mathrm{Lev}_\alpha(T)$.

Definition 6. Call a quadruple $(\alpha, \beta, \gamma, \delta) \in \kappa^4$ *valid for* $\mathrm{Lev}_\alpha(T)$ if the following conditions hold:

- The subset $\alpha \times \beta$ of $\mathrm{Lev}_{<\alpha}(T)$ is closed under $<_T$,[2]
- $\alpha \cdot \beta \leq \gamma \leq \delta$.

For each valid quadruple $(\alpha, \beta, \gamma, \delta)$, the function $f = f_\delta^{G(\alpha,\beta,\gamma)} : \omega_1 \to \delta$ satisfies the assumption of Lemma 4 with respect to the tree $T' = \alpha \times \beta \subseteq \mathrm{Lev}_{<\alpha}(T)$. Since the tree ordering on $\mathrm{Lev}_{<\alpha}(T)$ is assumed to be defined in $V' = V[G \restriction \alpha \times \kappa \times \kappa]$ and $f_\delta^{G(\alpha,\beta,\gamma)}$ is $\mathrm{Coll}(\omega_1, \delta)$ generic over V', it follows from Lemma 4 that the branch $b^{f_\delta^{G(\alpha,\beta,\gamma)}}$ is cofinal in $\mathrm{Lev}_{<\alpha}(T)$.

[2] I.e., $T \restriction (\alpha \times \beta)$ is a subtree of $\mathrm{Lev}_{<\alpha}(T)$.

It is clear that the set of valid quadruples for $\mathrm{Lev}_\alpha(T)$ has size κ. Let

$$\langle(\alpha_\nu, \beta_\nu, \gamma_\nu, \delta_\nu) \mid \nu < \kappa\rangle$$

be an enumeration of all valid quadruples for $\mathrm{Lev}_\alpha(T)$.

Definition 7 (Collapse index)**.** Let $t = (\alpha, \nu) \in \mathrm{Lev}_\alpha(T)$. The *collapse-index* of t is the quadruple $(\alpha_\nu, \beta_\nu, \gamma_\nu, \delta_\nu)$.

Convention 8. *Let $t \in T$. Whenever the collapse index of a node t is decided by a condition p and the idensityt of the condition p is clear from context, then the collapse index of t is denoted by $(\alpha_t, \beta_t, \gamma_t, \delta_t)$.*

Collapse index will indicate the part of the generic used to define the branch below t. We want to define a wide tree T which will provide branches that are very generic over any small part of it. Collapse index will take care of this.

Remark 9. *The assignment of $t \in T$ to its collapse index $(\alpha_t, \beta_t, \gamma_t, \delta_t) \in \kappa^4$ is injective.*

In $V[G \restriction ((\alpha + 1) \times \kappa \times \kappa)]$, the function associated to each node $t = (\alpha, \nu) \in \mathrm{Lev}_\alpha(T)$ is the function $f_t = f_{\delta_\nu}^{G(\alpha_\nu, \beta_\nu, \gamma_\nu)}$. Set $b_t = b^{f_t}$. We have defined the map from $\mathrm{Lev}_\alpha(T)$ to cofinal branches through $\mathrm{Lev}_{<\alpha}(T)$. We let

$$s <_T t :\iff s \in b_t.$$

This concludes the construction of the tree T. We state a number of basic properties of T that follow from its construction. The first is an immediate consequence of our level-by-level definition of T.

Lemma 10. *Suppose that $M_\alpha \prec (H_{\kappa^{++}}, \in)$ is an elementary substructure with ${}^\omega M_\alpha \subseteq M_\alpha$ and $M_\alpha \cap \kappa = \alpha$ is a regular cardinal, then*

$$\mathbb{P} \cap M_\alpha = \prod_{(\alpha', \beta', \gamma') \in \alpha^3} \mathbb{P}(\alpha', \beta', \gamma') \cap V_\alpha$$

is a regular subforcing of $\mathbb{P}$, and $T \cap M_\alpha$ is forced to be equivalent to a $(\mathbb{P} \cap M_\alpha)$-name of a subtree of T.

Let $t \in \mathrm{Lev}_\alpha(T)$ and let $(\alpha, \beta, \gamma, \delta)$ be the collapse index of t, for some $\alpha < \kappa$ of uncountable cofinality. Then b_t is a cofinal branch in the subtree $T \restriction (\alpha \times \beta)$ whose order is defined in the intermediate generic extension $V' = V[G \restriction \alpha \times \kappa \times \kappa]$. The branch itself comes from the V'-generic filter $G(\alpha, \beta, \gamma, \delta)$ on $\mathrm{Coll}(\omega_1, \delta)$. Working in V', definition 3 allows us to assign each $q \in \mathrm{Coll}(\omega_1, \delta)$ a node $\pi_q(t) \in T \restriction (\alpha \times \beta)$ which is the maximal node forced by q to be in b_t. Back in V, $\dot{\pi}_q(t)$ is a $\mathbb{P} \restriction (\alpha \times \kappa \times \kappa)$-name for a node in $\mathrm{Lev}_{<\alpha}(T)$ that is decided on a dense open set.

Definition 11. For each node $t \in \mathrm{Lev}_\alpha(T)$, let $D(t) \subseteq \mathbb{P}$ be the set of conditions p that decide the collapse index of t and so that $p \restriction \alpha \times \kappa \times \kappa$ decides the $\mathbb{P} \restriction (\alpha \times \kappa \times \kappa)$-name of the node $\dot{\pi}_q(t) \in \mathrm{Lev}_{<\alpha}(T)$ where $q = p \restriction \{(\alpha_t, \beta_t, \gamma_t, \delta_t)\} \in \mathrm{Coll}(\omega_1, \delta_t)$ (See Convention 8). Being decided by p, we denote it by $\dot{\pi}_p(t)$.

It follows from the construction of T that T is σ-closed, i.e. that every countable increasing sequence of nodes has a supremum. It follows from the construction of the tree that for each $t \in T$, the projection $\dot{\pi}_p(t)$ is defined for densely many $p \in \mathbb{P}$.

Lemma 12. *$D(t)$ is dense and σ-closed for each $t \in T$.*

Proof. For density, fix $p \in \mathbb{P}$ and $t \in T$. Up to extending p, we assume that it decides the collapse index of t. Suppose that the collapse index of t is $(\alpha, \beta, \gamma, \delta)$. Then the branch below t is determined by the countable partial function $f := p(\alpha, \beta, \gamma, \delta\cdot) : \omega_1 \to \delta \in \mathrm{Coll}(\omega_1, \delta)$. We may extend $p \cap V_{\alpha\cdot\beta}$ to some p' in $\mathbb{P} \cap V_{\alpha\cdot\beta}$, such that p' decides the tree-order among nodes in the set

$$\mathrm{rng}(f) \cap \{\alpha \cdot \beta^* + \alpha^* < \alpha \cdot \beta \mid (\alpha^*, \beta^*) \in \alpha \times \beta\}$$

Then the pointwise union $p' \cup p$ belongs to $D(t)$. The fact that $D(t)$ is σ-closed follows similarly. $\square$

The following is a straightforward application of Lemma 5 to countably many mutually generic branches.

Lemma 13. *Let $\langle t_n \mid n < \omega\rangle$ be a sequence of distinct nodes in T. Suppose that $p \in \bigcap_n D(t_n)$ and $\langle t'_n \mid n < \omega\rangle$ is a sequence of nodes such that for each $n < \omega$, $t'_n \in T \restriction (\alpha_{t_n} \times \beta_{t_n})$. Then there is an extension $p' \leq p$ with the following properties:*

1. *p' and p are equal, except maybe at the collapse indices $(\alpha_{t_n}, \beta_{t_n}, \gamma_{t_n}, \delta_{t_n})$ of the points t_n.*

2. *for every $q \leq p'$, if $q \Vdash \dot{\pi}_p(t_n) \leq_T t'_n$ then $q \Vdash t'_n \leq_T \dot{\pi}_q(t_n)$.*

Note that $q \Vdash t'_n \leq_T \dot{\pi}_q(t_n)$ implies $q \Vdash t'_n \leq_T t_n$.

Proof of Lemma 13. For each $n < \omega$, denote by f_n the partial function

$$p(\alpha_{t_n}, \beta_{t_n}, \gamma_{t_n}, \delta_{t_n}, \cdot) : \omega_1 \to \delta_{t_n}$$

in $\mathrm{Coll}(\omega_1, \delta_{t_n})$. For each $n < \omega$, let $j_n = \min(\omega_1 - \mathrm{dom}(f_n))$. Suppose that $t'_n = (\alpha'_n, \beta'_n)$. Define p' from p by adding the pair

$$(\alpha_{t_n}, \beta_{t_n}, \gamma_{t_n}, \delta_{t_n}, j_n, \alpha \cdot \beta'_n + \alpha'_n),$$

for every $n < \omega$. Then p' is as wanted. $\square$

We summarize the section by a lemma that shows that the tree T is wide Aronszajn in $V[G]$ for any collapse generic filter $G \subseteq \mathrm{Coll}(\omega_1, <\kappa)$.

Lemma 14. *Let $G \subseteq \mathrm{Coll}(\omega_1, <\kappa)$ be a generic filter. The tree T is a wide κ-Aronszajn tree in $V[G]$.*

Proof. It is clear that T is a wide κ-tree. Suppose to the contrary that there is $p \in \mathrm{Coll}(\omega_1, < \kappa)$ and a name $\dot{b}$ such that

$$p \Vdash \dot{b} \text{ is a cofinal branch in } \dot{T}.$$

Let $M \prec H_{\kappa^+}$ be such that $V_\alpha = V_\kappa \cap M$ and α is an ordinal of uncountable cofinality, and such that $\dot{b} \in M$. Up to extending p, we may assume that it decides the α-th node on the branch $\dot{b}$, say $\dot{b}(\alpha) = t$. Also, without loss of generality $p \in D(t)$. Up to extending $p \cap V_\alpha$ in $\mathbb{P} \cap V_\alpha$, we may assume that it decides the node $\dot{\pi}_p(t)$, and that there are two dictinct nodes t^L and t^R at some level $\bar{\alpha} < \alpha$ such that $p \cap V_\alpha$ forces that they are above $\dot{\pi}_p(t)$. Now by Lemma 13 there are two extensions $p^L, p^R \leq p$ such that

$$r := p^L \cap V_\alpha = p^R \cap V_\alpha = p \cap V_\alpha,$$

and such that $p^L \Vdash t^L <_{\dot{T}} t$ and $p^R \Vdash t^R <_{\dot{T}} t$. Now $r, \dot{b} \in M$. By elementarity, there is an extension $w \leq r$ in $\mathbb{P} \cap M$ that decides the node $\dot{b}(\bar{\alpha})$, say, to be $\bar{t}$. The pointwise unions $w \cup t^L$ and $w \cup t^R$ are both conditions in $\mathbb{P}$. But this is a contradiction: if $\bar{t} \neq t^L$, then $w \cup p^L$ cannot be a condition since it would force both $\bar{t} <_{\dot{T}} t$ and $t^L <_{\dot{T}} t$, and similarly, if $\bar{t} \neq t^R$, then $w \cup p^R$ cannot be a condition. $\square$

Our forcing iteration for embedding trees on κ into T will have to deal with κ^+ many trees $\langle S_\eta \mid \eta < \kappa^+ \rangle$. Ideally, each S_η would have its own part of T into which it would be embedded. This cannot be accommodated because T is supposed to be of size κ and there are κ^+ many trees to embed. To deal with this, we assign to each node t an ordinal $\gamma_t < \kappa$. This assignment will take care that, in critical places, two nodes from two different trees cannot be assigned to a node with the same label.

4 The embedding poset

We fix in V a sequence of functions $\vec{\psi} = \langle \psi_\tau \mid \tau < \kappa^+ \rangle$, such that for each τ, $\psi_\tau : \kappa \to \tau$ is a bijection whenever $\tau \geq \kappa$.

Definition 15. We say that a set $a \subseteq \kappa^+$ is *α-closed with respect to $\vec{\psi}$* for some $\alpha < \kappa$ if $\psi_\delta``\alpha \subseteq a$ for every $\delta \in a$.

For example, if $M \prec H_{\kappa^{++}}$ is such that $\alpha = \kappa \cap M \in \mathrm{Ord}$ and $\vec{\psi} \in M$, then $\kappa^+ \cap M$ is α-closed with respect to $\vec{\psi}$.

We are going to define by induction a sequence of forcing notions $\mathbb{P}_\tau$ for $\tau < \kappa^+$, together with $\mathbb{P}_\tau$-names of wide trees $S_\tau \subseteq \kappa \times \kappa$ and sequences of structures $\vec{M}^\tau = \langle M^\tau_\alpha : \alpha \in \mathrm{dom}(\vec{M}^\tau) \rangle$. This is our iteration for the proof of Theorem 1.

Remark 16. *Before giving the exact definitions, we give a brief informal description of $\mathbb{P}_\tau$. The first poset $\mathbb{P}_0$ will be equivalent to the Levy collapse forcing*

$\mathrm{Coll}(\omega_1, < \kappa)$. For $\delta \geq 1$, $\mathbb{P}_\delta$ will consists of pairs $p = \langle f^p, N^p \rangle$ of countable sets, so that $f^p = \langle f^p_\gamma : \gamma \in \mathrm{supp}(p) \rangle$ is a sequence of functions, such that $\mathrm{supp}(p) \subseteq \delta$ is a countable set, and for each $\gamma \in \mathrm{supp}(p)$, f^p_γ is a countable partial function, which will be forced to be order preserving from the tree S_γ to T. The tree (names) S_γ, $\gamma < \kappa^+$ will be chosen by a book-keeping function, which is planned to exhaust all wide trees S of a certain kind. The set N^p corresponds to the side condition part, which specifies a collection of structures M for which we would like to secure the existence of strong generic conditions. This is realized by having N^p consist of pairs (α, a_α) so that $\alpha < \kappa$ and $a_\alpha \subseteq \delta$ is a nonempty α-closed set with respect to $\vec{\psi}$ of size $|a_\alpha| \leq \alpha$. The role of (α, a_α) is to specify a set of $\gamma < \delta$ for which we would like $M^\gamma_\alpha \in \vec{M}^\gamma$ to have a strong generic condition (a.k.a. a master condition). For this, we add several natural requirements to the working parts f^p_γ. For example, we require that nodes $s \in \mathrm{dom}(f^p_\gamma) \cap M^\gamma_\alpha$ are mapped to $f^p_\gamma(s) \in T \cap M^\gamma_\alpha$.

We proceed with the complete recursive definition of $\mathbb{P}_\tau$, $\vec{M}^\tau$, $\tau < \kappa^+$. The definition will be given in steps, and require introducing a number of auxiliary definitions and notations. The auxiliary definitions will be extensively used to prove that $\mathbb{P}_\tau$ has the desired properties.

To start, we fix in advance a well-order $<_{H_{\kappa^{++}}}$ of $H_{\kappa^{++}}$, as well as a book-keeping function Ψ whose domain is the set of all posets $\mathbb{P} \in H_{\kappa^{++}}$ which preserve κ and satisfy κ^+.c.c, and $\Psi(\mathbb{P})$ is a $\mathbb{P}$-name of (wide) tree S on κ whose domain is $\kappa \times \kappa$.

Definition 17 ($\mathbb{P}_0$)**.** The first poset $\mathbb{P}_0$ is equivalent to the Levy collapse forcing $\mathrm{Coll}(\omega_1, < \kappa)$. Formally, it consists of pairs $p = \langle f^p, N^p \rangle$ where $f^p = \langle f^p_0 \rangle$ is a sequence with an element $f^p_0 \in \mathrm{Coll}(\omega_1, < \kappa)$, and $N^p = \emptyset$.

Let $\tau < \kappa^+$ and suppose that $\mathbb{P}_\delta$ has been defined for every $\delta < \tau$. Before defining $\mathbb{P}_\tau$ we list seven inductive assumptions for $\mathbb{P}_\delta$, $\delta < \tau$.

Inductive Assumption I: For every $\delta < \tau$, $\mathbb{P}_\delta$ is a σ-closed poset of size $|\mathbb{P}_\delta| = \kappa$, and $\mathbb{P}_\delta \in H_{\kappa^{++}}$.

For each $\gamma < \tau$ let $\dot{S}_\gamma = \Psi(\mathbb{P}_\gamma)$ denote the $\mathbb{P}_\gamma$-name for a wide tree with domain $\kappa \times \kappa$ chosen by a fixed book-keeping function Ψ. Let

$$\mathcal{A}^\gamma = \langle H_{\kappa^{++}}, \in, <_{H_{\kappa^{++}}}, \gamma, \Psi, \mathbb{P}_\gamma, \vec{\psi} \rangle.$$

Note that $S_\gamma = \Psi(\mathbb{P}_\gamma)$ is definable in this structure. Let

$$\vec{M}^\gamma = \langle M^\gamma_\alpha : \alpha \in \mathrm{dom}(\vec{M}^\gamma) \rangle$$

be the associated sequence of Π^1_1-reflecting elementary substructures of $\mathcal{A}^\gamma$ from Definition 2. As mentioned after the definition, $\mathrm{dom}(\vec{M}^\gamma)$ belongs to the weakly compact filter on κ, $\mathcal{F}_{WC}$.

Lemma 18. *If $M^\delta_\alpha \in \vec{M}^\delta$ and $\gamma \in \delta \cap M^\delta_\alpha$, then $M^\gamma_\alpha \in \vec{M}^\gamma$ and $M^\gamma_\alpha \subseteq M^\delta_\alpha$.*

Proof. The poset $\mathbb{P}_\gamma$ is definable from $\mathbb{P}_\delta$ and γ if $\gamma < \delta$, so if $\gamma \in \delta \in M^\delta_\alpha$, then $M^\gamma_\alpha \subseteq M^\delta_\alpha$. The fact that $M^\gamma_\alpha \cap V_\kappa = V_\alpha$ follows too, if $x \in V_\alpha - M^\gamma_\alpha$, then x is definable from the $<_{\kappa^{++}}$-minimal bijection $\psi : \kappa \to V_\kappa$ that is an element in both M^δ_α and M^γ_α and that must map α to V_α by reflectivity of M^δ_α. The reflectivity follows from the reflectivity of M^δ_α using the facts that $M^\gamma_\alpha \subseteq M^\delta_\alpha$ and $M^\gamma_\alpha \cap V_\kappa = V_\alpha$. □

Inductive Assumption II: For every $\delta < \tau$ and $p \in \mathbb{P}_\delta$, p is of the form $\langle f^p, N^p \rangle$ where

1. $f^p = \langle f^p_\gamma : \gamma \in \operatorname{supp}(p) \rangle$, a sequence whose domain $\operatorname{supp}(p) \subseteq \delta$ (called the *support of* p) is a countable set so that
   - $0 \in \operatorname{supp}(p)$ and $f^p_0 \in \operatorname{Coll}(\omega_1, < \kappa)$,
   - for each positive ordinal $\gamma \in \operatorname{supp}(p)$, $f^p_\gamma : \kappa \times \kappa \to T$ is a partial countable function.
2. N^p is a countable set of pairs of the form (α, a_α), where $\alpha < \kappa$ and $a_\alpha \subseteq \delta$ is a nonempty α-closed set (w.r.t $\vec{\psi}$) of size $|a_\alpha| \leq \alpha$. For each α there is at most one pair (α, a_α) in N^p.

There is a natural projection operator between the posets:

Definition 19. For every $\delta < \delta^*$ and $p \in \mathbb{P}_{\delta^*}$, define $p \restriction \delta = \langle f^{p\restriction\delta}, N^{p\restriction\delta} \rangle$ by

$$f^{p\restriction\delta} = \langle f^p_\gamma : \gamma \in \operatorname{supp}(p) \cap \delta \rangle,$$

and

$$N^{p\restriction\delta} = \{(\alpha, a_\alpha \cap \delta) : (\alpha, a_\alpha) \in N^p \text{ and } a_\alpha \cap \delta \neq \emptyset\}.$$

It will follow from Inductive Assumption III below that if $\delta' < \delta$ and $p \in \mathbb{P}_\delta$, then $p \restriction \delta' \in \mathbb{P}_{\delta'}$.

Definition 20.

1. Let $p \in \mathbb{P}_\delta$, $\delta' < \delta$ and $M^{\delta'}_\alpha \in \vec{M}^{\delta'}$. We say that $M^{\delta'}_\alpha$ *appears* in p (at coordinate δ') if $\delta' \in a_\alpha$ where $(\alpha, a_\alpha) \in N^p$.
2. We say that a condition $p \in \mathbb{P}_\delta$ is *amenable* to a structure $M^\delta_\alpha \in \vec{M}^\delta$ if for every $\gamma \in M^\delta_\alpha \cap \delta$, the model M^γ_α appears in p.
3. For $\delta < \kappa^+$, let $\vec{N}^\delta$ be the sequence that consists of Π^1_1-reflecting elementary substructures of the expanded structure $(\mathcal{A}^\delta, \vec{M}^\delta)$ from Definition 2. We say that M^δ_α is $\vec{M}^\delta$*-reflective* if α is in the domain of $\vec{N}^\delta$.

Remark 21. *A condition p is amenable to M^δ_α if and only if $\delta \cap M^\delta_\alpha \subseteq a_\alpha$ where $(\alpha, a_\alpha) \in N^p$.*

Remark 22.

1. *The domain of the sequence $\vec{N}^\delta$ of $\vec{M}^\delta$-reflective structures is a subset of the domain of $\vec{M}^\delta$. Furthermore, $\delta \cap M^\delta_\alpha = \delta \cap N^\delta_\alpha$ for each $\alpha \in \mathrm{dom}(\vec{N}^\delta)$. This follows from the fact that both structures are Skolem hulls: if $\gamma \in \delta \cap N^\delta_\alpha$, then γ is definable from the $<_{\kappa^{+}+}$-least bijection $\psi : \delta \to \kappa$ and an ordinal below α, which belong to both N^δ_α and M^δ_α.*

2. *Since the domain of the sequence $\vec{M}^\delta$ is unbounded in κ we get that if M^δ_α is $\vec{M}^\delta$-reflective then $|\vec{M}^\delta \restriction \alpha| = \alpha$. In particular, M^δ_α is a limit point of the sequence $\vec{M}^\delta$.*

For a condition $p \in \mathbb{P}_\delta$ and $\delta' < \delta$, there are two associated sequences of side conditions:

Definition 23. Let $p \in \mathbb{P}_\delta$ be a condition and $\delta' < \delta$.

1. $N^p_{\delta'} := \{\alpha < \kappa : M^{\delta'}_\alpha \text{ appears in } p\}$,

2. $E^p_{\delta'} := \{\alpha < \kappa : p \restriction \delta' \text{ is amenable to } M^{\delta'}_\alpha\}$.

If $M^{\delta'}_\alpha$ appears in p, then $p \restriction \delta'$ is amenable to $M^{\delta'}_\alpha$, so $N^p_{\delta'} \subseteq E^p_{\delta'}$. Note that E^p_δ makes sense for $p \in \mathbb{P}_\delta$. It will hold that the generic sequence $N^G_{\delta'} = \bigcup_{p \in G} N^p_{\delta'}$ consists of limit points of the generic sequence $E^G_\delta = \bigcup_{p \in G} E^p_{\delta'}$, for every generic $G \subseteq \mathbb{P}_\delta$ and $\delta' < \delta$. See Inductive Assumption III below.

Definition 24 ($\vec{M}^{\delta,\dot{G}(\mathbb{P}_\delta)}$)**.** For each $\delta < \tau$ let $\vec{M}^{\delta,\dot{G}(\mathbb{P}_\delta)}$ be the $\mathbb{P}_\delta$-name of the sub-sequence of $\vec{M}^\delta$ consisting of all $M^\delta_\alpha \in \vec{M}^\delta$ which are $\vec{M}^\delta$-reflective and for which there is an M^δ_α-amenable condition p' in the (canonical $\mathbb{P}_\delta$-name) generic filter $\dot{G}(\mathbb{P}_\delta)$ for $\mathbb{P}_\delta$.

So the sequence $\vec{M}^{\delta,\dot{G}(\mathbb{P}_\delta)}$ is a $\mathbb{P}_\delta$-name for a subsequence of E^G_δ from above.

Inductive Assumption III: For all $\delta < \delta^* < \tau$ and $p \in \mathbb{P}_{\delta^*}$:

1. $p \restriction \delta \in \mathbb{P}_\delta$.

2. If M^δ_α appears in p, then $p \restriction \delta$ forces that M^δ_α is a limit point of the sequence $\vec{M}^{\delta,\dot{G}(\mathbb{P}_\delta)}$.

In particular, if M^δ_α appears in $p \in \mathbb{P}_{\delta^*}$, then $p \restriction \delta$ is amenable to M^δ_α.

Inductive Assumption IV: For every $\delta < \tau$, $M^\delta_\alpha \in \vec{M}^\delta$ and $p \in \mathbb{P}_\delta \cap M^\delta_\alpha$, there is an extension $p' \leq p$ which is amenable to M^δ_α.

Remark 25. *Suppose $\gamma < \delta$. The sequence $\vec{M}^\gamma$ is definable from $\mathbb{P}_\delta$ and γ, as the set of models that appear in conditions in $\mathbb{P}_\delta$ at coordinate γ. Thus if $M \prec \mathcal{A}^\delta$ and $\gamma \in \delta \cap M$, then the sequence $\vec{M}^\gamma$ is an element in M. In particular, if M^δ_α belongs to the sequence $\vec{M}^\delta$ and $\gamma \in \delta \cap M^\delta_\alpha$, then by elementarity the domain of $\vec{M}^\gamma$ must be unbounded in $\alpha = \kappa \cap M^\delta_\alpha$. So α is a limit point of each sequence $\vec{M}^\gamma$, where $\gamma \in \delta \cap M^\delta_\alpha$. By the same reasoning, if M^δ_α is $\vec{M}^\delta$-reflective, then in addition it must be a limit point of the sequence $\vec{M}^\delta$ as well.*

Lemma 26. *Let $\delta < \tau$.*

1. *$\vec{M}^{\delta,\dot{G}(\mathbb{P}_\delta)}$ is (forced to be) of cardinality κ.*

2. *If p forces some M^δ_α to be in $\vec{M}^{\delta,\dot{G}(\mathbb{P}_\delta)}$ then it forces $M^{\delta'}_\alpha$ to be in $\vec{M}^{\delta',\dot{G}(\mathbb{P}'_\delta)}$ for every $\delta' \in \delta \cap M^\delta_\alpha$.*

Proof. The first part is an immediate consequence of Inductive assumption IV. For the second item, note that for every $M^\delta_\alpha \in \vec{M}^\delta$, $\psi_\delta \in M^\delta_\alpha$ and therefore $M^\delta_\alpha \cap \delta = \psi_\delta``\alpha$. It follows that for every a which is α-closed (w.r.t $\vec{\psi}$) and $\delta \in a$, one must have $\delta \cap M^\delta_\alpha \subseteq a$. The statement of this part now follows from Inductive Assumption III and Remark 22. $\square$

The goal is to show that each poset $\mathbb{P}_\delta$, $\delta < \kappa^+$, is strongly proper with respect to the models in the sequence $\vec{M}^\delta$. This is done by showing that if $p \in \mathbb{P}_\delta \cap M^\delta_\alpha$, then there is an extension of p which is amenable to M^δ_α, and whenever p is amenable to M^δ_α, then it is a strong master condition for M^δ_α. Hence the following inductive assumption is natural.

Inductive Assumption V: The following requirements hold for every $\delta' < \delta < \tau$ and $p \in \mathbb{P}_\delta$ with $\delta' \in \mathrm{supp}(p)$:

1. For any $s_1, s_2 \in \mathrm{dom}(f^p_{\delta'})$ there is some $s \in \mathrm{dom}(f^p_{\delta'})$ which is forced by $p \restriction \delta'$ to be the meet of s_1, s_2 in the tree order of $S_{\delta'}$, and $f^p_0 \in \mathrm{Coll}(\omega_1, <\kappa)$ forces that $f^p_{\delta'}(s)$ is the meet in T of $f^p_{\delta'}(s_1)$ and $f^p_{\delta'}(s_2)$.

2. $p \restriction \delta'$ forces that $f^p_{\delta'}$ is level-preserving.

Remark 27. *Inductive Assumption V implies $p \restriction \delta'$ forces that $f^p_{\delta'}$ is injective and order preserving. This follows from the simple observation that a partial function $f : S \to T$ between two trees S, T, whose domain is closed under meets in S, mapping those meets to the meets of the images, is order preserving.*

The next inductive assumptions describe the connection/restrictions between the "working" parts f^p_δ in conditions p and the "side condition" parts M^δ_α that appear in p. We start with a brief discussion before giving the precise details.

Our main goal is to secure a strong properness property for a structure M^δ_α that appears in a condition p. As usual, strong properness will imply that the restriction of the generic embedding f_δ to M^δ_α will be generic over M^δ_α. It is therefore natural to include a restriction saying that for every $s \in \mathrm{dom}(f^p_\delta)$, $s \in M^\delta_\alpha$ if and only if $f^p_\delta(s) \in M^\delta_\alpha$. Now, to secure this property while allowing every node $s' \in S_\delta$ to be added to the domain of an extension, we impose a similar requirement for branches b_s of nodes $s \in \mathrm{dom}(f^p_\delta)$. Namely, we require that for every $(s,t) \in f^p_\delta$, $b_s \subseteq M^\delta_\alpha$ if and only if $b_t \subseteq M^\delta_\alpha$. We recall that by the construction of our trees T, for every $t \in T$, the identity of $\beta < \kappa$ for which $b_t \subseteq M^\delta_\beta$ is determined from the collapse index $(\alpha', \beta', \gamma', \delta')$ of t (specifically, we need $\alpha', \beta' \leq \beta$).

We point out that the last restriction introduces the following additional complication: Say, $(s,t) \in f^p_\delta$ are outside of a structure M^δ_α which appears in p. By our strong properness aspirations for M^δ_α we would like our ability to extend the conditions $f^p_\delta \cap M^\delta_\alpha$ inside M^δ_α to be "independent" from considerations outside of M^δ_α. Because of this, we are at a risk of adding a new structure M^δ_β to the side condition part, when working inside M^δ_α that will violate the branch requirement coming from (s,t), which are outside of the structure. One approach to avoid such a problem will be to add a notion of excluded intervals to the side condition part (such as when adding a club to ω_1 with finite side conditions, as in Baumgartner, [2]) of the poset to exclude "problematic" structures M^δ_β below M^δ_α. We take here a similar and slightly more implicit approach to avoid this problem: We can use the function f^p_δ to produce excluded intervals, essentially by mapping a node $s' \in M^\delta_\beta$ to $f^p_\delta(s')$ outside of M^δ_β, but inside M^δ_α so that this restriction appears when we move to $f^p_\delta \cap M^\delta_\alpha$ and excludes adding M^δ_β to the side condition part. The last description can be seen as a motivation for the next definition, of a refined sub-sequence $\vec{M}^{\delta,G(\mathbb{P}_\delta),f_\delta} \subseteq \vec{M}^{\delta,G(\mathbb{P}_\delta)}$ given by a partial countable function f_δ.

Definition 28. $(\vec{M}^{\delta,\dot{G}(\mathbb{P}_\delta),f_\delta})$
Let $f_\delta : S_\delta \to T_\delta$ be a countable partial function. Define a $\mathbb{P}_\delta$-name for a sub-sequence $\vec{M}^{\delta,\dot{G}(\mathbb{P}_\delta),f_\delta}$ of $\vec{M}^{\delta,\dot{G}(\mathbb{P}_\delta)}$ by having $\vec{M}^{\delta,\dot{G}(\mathbb{P}_\delta),f_\delta}$ consists of all $M^\delta_\beta \in \vec{M}^{\delta,\dot{G}(\mathbb{P}_\delta)}$ such that for every $(s,t) \in f_\delta$

1. $s \in M^\delta_\beta$ if and only if $t \in M^\delta_\beta$, and
2. s is an exit node from M^δ_β if and only if t is an exit node from M^δ_β.

Definition 29 $(\beta_{f_\delta}(s))$**.** For every condition $p \in \mathbb{P}_\delta$, node $s \in \mathrm{Lev}_\alpha(S_\delta)$ where $\alpha < \kappa$, and partial countable function $f_\delta : S_\delta \to T_\delta$ there is an extension $q \leq p$ which either decides the minimal $\beta < \kappa$ such that $\beta \geq \alpha$, $b_s \subseteq \alpha \times \beta$, $s \notin M^\delta_\beta$, and $M^\delta_\beta \in \vec{M}^{\delta,\dot{G}(\mathbb{P}_\delta),f_\delta}$ or else forces that such β does not exist. If β exists, we denote it by $\beta_{f_\delta}(s)$.

Suppose that $p' \in \mathbb{P}_{\delta^*}$ is a condition for some $\delta^* > \delta$ and $p = p^* \restriction \delta$ and $f_\delta = f^{p'}_\delta$. The requirement $M^\delta_\beta \in \vec{M}^{\delta,\dot{G}(\mathbb{P}_\delta),f_\delta}$ implies that the model M^δ_β is in some sense "addable" to the side conditions of p', the fact that it belongs to $\vec{M}^{\delta,\dot{G}(\mathbb{P}_\delta)}$ implies that p is amenable to it and the fact that it belongs to $\vec{M}^{\delta,\dot{G}(\mathbb{P}_\delta),f_\delta}$ implies that it is closed under the function f_δ.

Remark 30. *Given a condition $p \in \mathbb{P}_\delta$, $\delta' < \delta$, and a node $s \in S_{\delta'}$, we can extend $p \restriction \delta'$ to determine the minimal $\beta < \kappa$ such that $b_s \subseteq \beta \times \beta$ and $s \notin \beta \times \beta$. This makes β a natural candidate for $\beta_{f_{\delta'}}(s)$ but not necessarily the correct one. It might happen that either $M^{\delta'}_\beta$ does not exist, or it does, but is not forced to be closed under f_δ, i.e. inside the sequence $\vec{M}^{\delta,G(\mathbb{P}_\delta),f^p_{\delta'}}$. We thus have to further extend $p \restriction \delta'$ to decide $\beta_{f_{\delta'}}(s)$ to be some $\beta' \geq \beta$.*

Definition 31. For every $\delta < \tau$ and $s \in S_\delta$, we define a $\mathbb{P}_\delta$-name of a structure M_1^s to be M_β^δ for the minimal β for which $M_\beta^\delta \in \vec{M}^{\delta,\dot{G}(\mathbb{P}_\delta)}$ and $s \in M_\beta^\delta$.

It follows that $p \in \mathbb{P}_\delta$ is amenable to M_s^1, whenever it decides it.

Remark 32. *It follows from Inductive Assumption IV and the fact that the sequence $\vec{M}^{\delta,\dot{G}(\mathbb{P}_\delta)}$ is forced to be cofinal in κ, that the model M_s^1 always exists. The model M_s^1 is always (forced to be) a successor point of the sequence $\vec{M}^{\delta,\dot{G}(\mathbb{P}_\delta)}$, being the least structure in $M^{\delta,\dot{G}(\mathbb{P}_\delta)}$ that sees the node s. Since the sequence $\vec{M}^{\delta,\dot{G}(\mathbb{P}_\delta)}$ is included in the limit points of the sequence $\vec{M}^\delta$, M_s^1 is a limit point in the sequence $\vec{M}^\delta$ and limit point in the sequences $\vec{M}^{\gamma,\dot{G}(\mathbb{P}_\gamma)}$, for $\gamma \in \delta \cap M_s^1$.*

Definition 33 (s-knowledgeable conditions)**.** We say that a condition $q \in \mathbb{P}_\delta$ is *s-knowledgeable* (or knowledgeable about s) for some $s \in S_\delta$ with respect to a partial countable function $f_\delta : S_\delta \to T$, if

- q decides the identity of M_1^s,
- q decides if $\beta_{f_\delta}(s)$ exists, and if so, determines its value.

Inductive Assumption VI: For every $p \in \mathbb{P}_\delta$, $\delta' < \delta$, every $M_\alpha^{\delta'}$ that appears in p and every $s \in \mathrm{dom}(f_{\delta'}^p)$:

1. $s \in M_\alpha^{\delta'}$ if and only if $f_{\delta'}^p(s) \in M_\alpha^{\delta'}$,
2. s is an exit node from $M_\alpha^{\delta'}$ if and only if $f_{\delta'}^p(s)$ is,
3. if $s \in \mathrm{dom}(f_{\delta'}^p) - M_\alpha^{\delta'}$, then there is $s' \in \mathrm{dom}(f_{\delta'}^p)$ such that $p \restriction \delta'$ forces that $s' \leq_{\dot{S}_{\delta'}} s$ and s' is an exit node from $M_\alpha^{\delta'}$.
4. if $p \restriction \delta'$ forces that s is an exit node from $M_\alpha^{\delta'}$, then $p \restriction \delta'$ is s-knowledgeable with respect to $f_{\delta'}^p$, and if $p \restriction \delta'$ forces $\beta = \beta_{f_{\delta'}^p}(s)$ exists then $M_\beta^{\delta'}$ appears in p.

Inductive Assumption VII: For every $p \in \mathbb{P}_\delta$ and $s \in \mathrm{dom}(f_{\delta'}^p)$ for some $\delta' < \delta$ in the support of p, if s is forced by $p \restriction \delta'$ to be an exit node from a model $M_\alpha^{\delta'}$ that appears in p, then $f_{\delta'}^p(s)$ is also forced by $p \restriction \delta'$ to be an exit node from $M_\alpha^{\delta'}$ and the collapse-index $(\alpha, \beta, \gamma, \epsilon)$ of $f_{\delta'}^p(s)$ is forced by $p \restriction \delta'$ to satisfy the following requirements:

1. α is the level of s,
2. $\beta = \beta_{f_{\delta'}^p}(s)$ if exists, and
3. $\gamma = M_1^s \cap \kappa$.

Remark 34. *It follows form Inductive Assumption VII that whenever $s \in \mathrm{dom}(f_{\delta'}^p)$ is an exit node from a model that appears in p at coordinate δ', then $s \in M_s^1$ and $f_{\delta'}^p(s) \notin M_s^1$. This is because if it happened that $f_{\delta'}^p(s) \in M_s^1$,* *then by elementarity the collapse index of $t := f^p_{\delta'}(s)$ would also be an element in M^1_s, which is not possible because $\gamma_t = M^1_s \cap \kappa$. At first glance this may seem to be at odds with the first clause of item 2 in Inductive Assumption VI, which requires that for every $M^{\delta'}_\alpha$ which appears in p, if s belongs to $M^{\delta'}_\alpha$ then so does $f^p_{\delta'}(s)$. The two requirements are compatible as M^1_s does not appear in p (in other words, if $M^1_s = M^{\delta'}_\alpha$, then α does not belong to $N^p_{\delta'}$, i.e. δ' does not belong to a_α), as M^1_s is a successor structure in $\vec{M}^{\delta',G(\mathbb{P}_{\delta'})}$ (being the first to see s) while the structures $M^{\delta'}_\alpha$ that are allowed to appear in p are required to be limit points of the sequence $\vec{M}^{\delta',\dot{G}(\mathbb{P}_{\delta'})}$. Rather, again writing $M^1_s = M^{\delta'}_\alpha$, the structure M^γ_α appears in p for every $\gamma \in \delta' \cap M^1_s$. So in the notation of Definition 23, $\alpha \in E^p_{\delta'} - N^p_{\delta'}$.*

We are ready to define $\mathbb{P}_\tau$. We split the definition between the case τ is a limit ordinal, and $\tau = \delta + 1$ is a successor ordinal. Recall that $\mathbb{P}_0$ was defined in Definition 17.

Definition 35 ($\mathbb{P}_\tau$ for a limit ordinal τ)**.** Conditions $p \in \mathbb{P}_\tau$ are pairs $\langle f^p, N^p\rangle$ that satisfy the following requirements:

- $f^p = \langle f^p_\delta : \delta \in \mathrm{supp}(p)\rangle$ is a sequence with $\mathrm{supp}(p) \subseteq \tau$ countable and $0 \in \mathrm{supp}(p)$,
- N^p is a countable set of pairs (α, a) satisfying $\alpha < \kappa$ and $a \in [\tau]^{\leq\alpha}$ nonempty and α-closed with respect to $\vec{\psi}$,
- For every $\delta < \tau$, $p \restriction \delta$ belongs to $\mathbb{P}_\delta$.

A condition $p' \in \mathbb{P}_\tau$ extends p, denoted $p' \leq p$, iff $p' \restriction \delta \leq_{\mathbb{P}_\delta} p \restriction \delta$ for all $\delta < \tau$.

Definition 36 ($\mathbb{P}_\tau$ for a successor ordinal τ)**.** Suppose that $\tau = \delta + 1$. The poset $\mathbb{P}_\tau = \mathbb{P}_{\delta+1}$ consists of all pairs $p = \langle f^p, N^p\rangle$ which satisfy the following conditions:

- $f^p = \langle f^p_\delta : \delta \in \mathrm{supp}(p)\rangle$ is a sequence with $\mathrm{supp}(p) \subseteq \delta + 1$ countable and $0 \in \mathrm{supp}(p)$,
- N^p is a countable set of pairs (α, a) satisfying $\alpha < \kappa$ and $a \in [\tau]^{\leq\alpha}$ nonempty and α-closed with respect to $\vec{\psi}$, and whenever $\delta \in a$, $M^\delta_\alpha \in \vec{M}^\delta$ exists,
- $p \restriction \delta$ is a condition in $\mathbb{P}_\delta$,
- If $\delta \in \mathrm{supp}(p)$ then $p \restriction \delta$ forces the statements from Inductive Assumptions V,VI, and VII where δ' and δ are replaced with δ and τ.
- If M^δ_α appears in p, then M^δ_α is $\vec{M}^\delta$-reflective and $p \restriction \delta$ is amenable to M^δ_α.

A condition $p' \in \mathbb{P}_{\delta+1}$ extends p if it satisfies the following requirements:

1. $p' \restriction \delta \leq_{\mathbb{P}_\delta} p \restriction \delta$,
2. $f^p_\delta \subseteq f^{p'}_\delta$,
3. for every $(a, \alpha) \in N^p$ there is some $a' \supseteq a$ such that $(a', \alpha) \in N^{p'}$.[3]

Lemma 37. *$\mathbb{P}_\tau$ satisfies inductive assumptions I-VII.*

Proof.

Assumption I: It is clear that $|\mathbb{P}_\tau|$ has size κ and is σ-closed.

Assumption II: The structural assumptions of conditions $p \in \mathbb{P}_\tau$ are immediate consequences of the definition of $\mathbb{P}_\tau$.

Assumption III: Immediate from the definition of $\mathbb{P}_\tau$.

Assumption IV: We need to show that if $p \in \mathbb{P}_\tau \cap M^\tau_\alpha$, there is an extension $p' \leq p$ which is amenable to M^τ_α. Let $p' = \langle f^{p'}, N^{p'} \rangle$ where $f^{p'} = f^p$ and $N^{p'} = N^p \cup \{(\alpha, a_\alpha)\}$, where $a_\alpha = \tau \cap M^\tau_\alpha$. Note that $p \in M^\tau_\alpha$ implies that for every $\delta < \tau$, $f^p_\delta \in M^\delta_\alpha$. This is because f^p_δ is a countable subset of $V_\alpha \subseteq M^\delta_\alpha$ and thus element of M^δ_α. Thus there are no exit nodes from M^δ_α for any $\delta < \tau$ in the support of p and Inductive Assumption VI about the side conditions is satisfied. It follows that p' is a condition. It is clear that p' is amenable to M^τ_α.

Assumptions V,VI,VII: If τ is limit then IA V,VI,VII are immediate. Suppose $\tau = \delta + 1$ is a successor ordinal. The fact that V,VI, and VII hold at $\delta' < \delta$ follows from the fact $p \restriction \delta \in \mathbb{P}_\delta$. The fact that V,VI,VII hold at δ follows from the definition of $\mathbb{P}_{\delta+1}$. □

Remark 38. *Let $p \in \mathbb{P}_\tau$ be a condition. Suppose that $s' \in \mathrm{dom}(f^p_\delta)$ for some $\delta \in \mathrm{supp}(p)$, and $s \in S_\delta$ are such that*

- *$p \restriction \delta$ decides the meet of s, s' in S_δ, denoted by $m(s, s') \in S_\delta$,*
- *$m(s, s') \in \mathrm{Lev}_{\bar{\alpha}}(S_\delta)$ for some $\bar{\alpha} < \kappa$,*
- *the collapse part f^p_0 of p decides the identity of the unique node below $f^p_\delta(s')$ at level $\bar{\alpha}$ to be some $\bar{t} \in \mathrm{Lev}_{\bar{\alpha}}(T)$.*

The assumptions do not imply that $m(s, s')$ belongs to $\mathrm{dom}(f^{p'}_\delta)$ as we do not assume s does. However, the image $f^q_\delta(m(s, s'))$ of $m(s, s')$ in all possible extensions q of p with $m(s, s') \in \mathrm{dom}(f^q_\delta)$ is already decided by p to be $\bar{t}$. We call $\bar{t}$ the implicit image *of the meet $m(s, s')$ of s, s' as determined by p, and denote it by $t(s, s')$ or $t_p(s, s')$. Furthermore, if we add the pair $(m(s, s'), t(s, s'))$ to f^p_δ, it is easy to see that the extension of p obtained this way is a condition in $\mathbb{P}_\tau$.*

[3]Equivalently, every $M^{\delta'}_\alpha$ that appears in p appears in p'.

Traces and simple amalgamations

We end this section with definitions of operations that attempt to capture small pieces of conditions $p \in \mathbb{P}_\tau$, and join those pieces together (possibly pieces from different conditions). These are given by *traces* to structures, and *simple amalgamations*, respectively. The two operations need not produce conditions in general, but as we will show, they do under quite natural additional assumptions.

For a condition $p \in \mathbb{P}_\tau$ and a structure $M = M^\tau_\alpha \in \vec{M}^\tau$ the trace of p to M, denoted $[p]_M$, is meant to capture all the information p has which is relevant to M. A priori, the definition makes sense for any set M.

Definition 39. Let $p \in \mathbb{P}_\tau$ for some $\tau < \kappa^+$ and $M \prec H_{\kappa^{++}}$. The *trace* of $p = \langle f, N \rangle$ to M, is the pair $[p]_M = \langle \bar{f}, \bar{N} \rangle$ where

- $\mathrm{dom}(\bar{f}) = M \cap \mathrm{dom}(f)$,
- for every $\gamma \in \mathrm{dom}(\bar{f})$, $\bar{f}_\gamma = f_\gamma \cap M$,
- $\bar{N}$ consists of pairs $(\alpha, \bar{a})$ for which there is some a such that $(\alpha, a) \in N$, $\alpha \in M$, and $\bar{a} = a \cap M$ is nonempty.[4]

It is not true in general that $[p]_M$ is a condition in $\mathbb{P}_\tau$. For example, it is possible that the trace part $[p]_M \restriction \delta$ does not decide the meet of nodes $s_1 \neq s_2$ for $s_1, s_2 \in \mathrm{dom}(\bar{f}_\delta)$. Our main argument below shows that when p is amenable to M, then it has an extension $p' \leq p$ such that $[p']_M$ is a condition in $\mathbb{P}_\tau \cap M$. Indeed, in the next section we introduce a property of a condition p being *super-nice with respect to* M and prove that it implies that $[p]_M$ is a condition and a residue of p into M (see Definitions 43 and 45). We also show that whenever p is amenable to M, then it has an extension that it super-nice with respect to M.

Next, we define the simple amalgamation operation of two conditions p, p'. Given two conditions $p, p' \in \mathbb{P}_\tau$, the natural attempt to find a common extension q to p, p' involves taking coordinate-wise unions of the "working parts" and the "side condition" parts. The result, which need not be a condition, is called the *simple amalgamation* of p, p'.

Definition 40 (Simple Amalgamations)**.** The *simple amalgamation* of conditions $p = \langle f, N \rangle$ and $p' = \langle f', N' \rangle$ is the pair $\langle f^*, N^* \rangle$ defined by

- $\mathrm{dom}(f^*) = \mathrm{dom}(f) \cup \mathrm{dom}(f')$
- for each $\delta \in \mathrm{dom}(f^*)$, $f^*_\delta = f_\delta \cup f'_\delta$ (with f_δ or f'_δ taken to be empty in the case $\delta \notin \mathrm{dom}(f)$ or $\delta \notin \mathrm{dom}(f')$, respectively)
- $(\alpha^*, a^*) \in N^*$ if and only if there is either a so that $(\alpha^*, a) \in N$ or a' so that $(\alpha^*, a') \in N'$, and then $a^* = a \cup a'$ (with a or a' taken to be empty in the case $\alpha^* \notin \mathrm{dom}(N)$ or $\alpha^* \notin \mathrm{dom}(N')$, respectively)

[4] Since $\vec{\psi} \in M$ and a is α-closed, then so is $(\alpha, \bar{a})$.

The simple amalgamation (f^*, N^*) need not be a condition even when p, p' are compatible. A reason why the simple amalgamation might fail to be a condition even if p and p' were compatible is that it might form a condition in $\mathbb{P}_\delta$ up to some coordinate $\delta < \tau$ (this is clearly the case for $\delta = 0$) but it need not decide the meets in S_δ of nodes $s \in \mathrm{dom}(f^p_\delta)$ with nodes $s' \in \mathrm{dom}(f^{p'}_\delta)$, or the exit node of some $s' \in \mathrm{dom}(f^{p'}_\delta)$ with all side condition structures M^δ_α that appear in p, and vice versa. However, it is clear that if a condition q is a common extension of p, p' then it extends their simple amalgamation, i.e. with the above notation, $f^*_\delta \subseteq f^q_\delta$ for every $\delta \in \mathrm{dom}(f^*)$, and for every $(\alpha^*, a^*) \in N^*$ there is $b \supseteq a^*$ with $(\alpha^*, b) \in N^q$.

In a very simple case where $p' \in \mathbb{P}_\delta$ extends an initial segment $p \restriction \delta$ of p, it is an immediate consequence of the definition that the simple amalgamation is a condition.

Lemma 41. *Suppose that $p \in \mathbb{P}_\tau$, and $p' \in \mathbb{P}_\delta$ extends $p \restriction \delta$ for some $\delta < \tau$. Then the simple amalgamation of p, p' belongs to $\mathbb{P}_\tau$*

5 Strong Properness

In this section, we show that $\mathbb{P}_\tau$ is strongly proper with respect to the structures $\vec{M}^\tau$, for $\tau < \kappa^+$. We commence by recalling the usual definition of strong properness:

Definition 42. A condition $p \in \mathbb{P}$ is *strongly proper* with respect to a model $M \prec (H_\theta, \in, \mathbb{P})$ (for some sufficiently large θ) if p forces that $\dot{G} \cap M$ is a V-generic filter for $\mathbb{P} \cap M$. We say that $\mathbb{P}$ is *strongly proper* with respect to a collection $\mathcal{T}$ of models M, if for every $M \in \mathcal{T}$ and $q \in M \cap \mathbb{P}$, q has an extension $p \in \mathbb{P}$ which is strongly proper with respect to M.

This definition is equivalent to the definition of a *strong master condition* in Neeman and Gilton [5]. We will use the following notion of residue function to prove strong properness results for our poset. We denote $\mathbb{P}/p = \{q \in \mathbb{P} \mid q \leq p\}$.

Definition 43. Let $p \in \mathbb{P}$ and $M \prec (H_\theta, \in, \mathbb{P})$. A *residue function* for M over p is a function $r : D \to M \cap \mathbb{P}$, where $D \subseteq \mathbb{P}/p$ is dense below p and for every $q \in D$ and $w \in M \cap \mathbb{P}$ such that $w \leq r(q)$, w and q are compatible in $\mathbb{P}$.

Lemma 44. *If p and M are as above, and $r : D \to \mathbb{P} \cap M$ is a residue function for M over p, then p is strongly proper with respect to M.*

See Proposition 1.7 in [5] for details.

The goal is to show that if $p \in \mathbb{P}_\tau$ is amenable to M^τ_α, then p is a strong master condition for M^τ_α, for models M^δ_α that are $\vec{M}^\delta$-reflective. Our residue function to M will be given by the trace $[p]_M$ operator (see Definition 39). We introduce the notion of a super-nice conditions with respect to a structure M, and prove that super-nice conditions are dense below conditions that are amenable to M, and when p is super-nice with respect to M then $[p]_M$ is a residue of p in M.

Definition 45.

1. We say that $p \in \mathbb{P}_\tau$ *nicely projects* to a structure M^τ_β if it satisfies the following requirements:

   - p is amenable to M^τ_β
   - $f^p_0 \in D(t)$ for every $t \in \bigcup_{\delta \in \mathrm{supp}(p)} \mathrm{rng}(f^p_\delta) - M^\tau_\beta$ (see Definition 11 for the σ-closed dense set $D(t)$)
   - For every $\delta \in M^\tau_\beta \cap \mathrm{supp}(p)$ and $s \in \mathrm{dom}(f^p_\delta)$ which is forced by $p \restriction \delta$ to be an exit node from M^δ_β, then the node projection $\bar{t} = \pi_{f^p_0}(f^p_\delta(s))$ satisfies that either

     (a) ($\bar{t}$ has a preimage)
     there is some $\bar{s} \in \mathrm{dom}(f^p_\delta) \cap M^\tau_\beta$ such that $p \restriction \delta \Vdash \bar{s} <_{S_\delta} s$ and $f^p_\delta(\bar{s}) = \bar{t}$, or

     (b) ($b_{\bar{t}}$ has cofinal preimages)
     there is a sequence of nodes $\langle \bar{s}_n \rangle_n \subseteq \mathrm{dom}(f^p_\delta) \cap M^\tau_\beta$ such that $p \restriction \delta$ forces it is increasing in S_δ, bounded by s, and the images by f^p_δ are cofinal in $\bar{t}$.

     And moreover, if the height of s is a successor-ordinal, then the immediate predecessor of s belongs to $\mathrm{dom}(f^p_\delta)$.

2. Let $p \in \mathbb{P}_\tau$ be a condition which satisfies that for every $\delta \in \mathrm{supp}(p) \setminus 1$ and $s \in \mathrm{dom}(f^p_\delta)$, the condition $p \restriction \delta$ decides the identity of M^1_s. In particular, $p \restriction \delta$ is M^1_s-amenable. We define $E(p, M^\tau_\alpha)$ to be the countable set of pairs (M^δ_β, δ) obtained by starting with the singleton $\{(M^\tau_\alpha, \tau)\}$ and closing under the following operation: given (M^δ_β, δ) in our set and $\gamma \in \mathrm{supp}(p) \cap \delta \cap M^\delta_\beta$, if $s \in \mathrm{dom}(f^p_\gamma)$ is forced by $p \restriction \delta$ to be an exit node from M^γ_β, then we add (M^1_s, γ).

3. p is *super-nice with respect to* M^τ_α if it decides M^1_s for every $s \in \mathrm{dom}(f^p_\delta)$, $\delta \in \mathrm{supp}(p)$, and $p \restriction \delta$ nicely projects to M^δ_β for every $(M^\delta_\beta, \delta) \in E(p, M^\tau_\alpha)$. The set of super-nice conditions $p \in \mathbb{P}_\tau$ with respect to M^τ_α is denoted by $D_\tau(M^\tau_\alpha)$.

Remark 46. *It follows from the definition of $E(p, M^\tau_\alpha)$ that for every pair $(M^\delta_\beta, \delta) \in E(p, M^\tau_\alpha)$ that is not the initial one (τ, M^τ_α), the structure M^δ_β is forced by $p \restriction \delta$ to have form M^1_s for some $s \in \mathrm{dom}(f^p_\delta)$ and thus to be a successor structure in the sequence $\vec{M}^{\delta, \dot{G}(\mathbb{P}_\delta)}$. This observation together with the fact that if $(M^\delta_\beta, \delta) \in E(p, M^\tau_\alpha)$ and $\delta' \in \delta \cap M^\delta_\beta$, then $M^{\delta'}_\beta$ is a limit structure in the sequence $\vec{M}^{\delta', \dot{G}(\mathbb{P}_{\delta'})}$, imply that for all $(M^\delta_\beta, \delta), (M^{\delta'}_{\beta'}, \delta') \in E(p, M^\tau_\alpha)$, if $\delta' \in \delta \cap M^\delta_\beta$, then $\beta \neq \beta'$.*

Remark 47. *It also follows that if $(\delta, M^\delta_\alpha) \in E(p, M^\tau_\alpha)$, then $E(p \restriction \delta, M^\delta_\beta) \subseteq E(p, M^\tau_\alpha)$.*

The set $E(p, M^\tau_\alpha)$ can be understood through the concept of simple "paths":

Definition 48. Let $\gamma < \tau < \kappa^+$ and $\alpha < \beta < \kappa$. A *path from M^τ_α to M^γ_β* is a finite sequence $\langle (M^{\gamma_i}_{\beta_i}, \gamma_i) \mid i \leq n \rangle$ such that $(M^{\gamma_0}_{\beta_0}, \gamma_0) = (M^\tau_\alpha, \tau)$ and $(M^{\gamma_n}_{\beta_n}, \gamma_n) = (M^\gamma_\beta, \gamma)$ and for every $k < n$:

1. $\gamma_{k+1} \in \gamma_k \cap M^{\gamma_k}_{\beta_k}$,
2. $\beta_{k+1} > \beta_k$.

A path is said to be *in* p, for $p \in \mathbb{P}_\tau$, if $\gamma_k \in \mathrm{supp}(p) \cup \{\tau\}$ and $\beta_k \in E^p_{\gamma_k}$, using notation from 23.

The set $E(p, M^\tau_\alpha)$ is included in the closure of the singleton $\{(M^\tau_\alpha, \tau)\}$ under paths from M^τ_α.

Building on Lemma 13, we obtain a lemma according to which, given a condition p that is super-nice with respect to M^τ_α, it is possible to extend p to modify the information about the branches below certain nodes in T while preserving super-niceness of p.

Lemma 49. *Let $\tau < \kappa^+$. Suppose that $p \in \mathbb{P}_\tau$ is super-nice with respect to M^τ_α and t_n, $n < \omega$, are countably nodes in T with $\gamma_{t_n} \leq \alpha$ for each $n < \omega$, and $\bar{t}_n$, $n < \omega$, are countably many nodes in $T \cap V_\alpha$ such that $\bar{t}_n \in \alpha_n \times \beta_n$ (See Convention 8 for α_{t_n}, β_{t_n}). Then for any $\sigma : \omega \to 2$ there is a condition $p' \leq p$ such that p' is super-nice with respect to M^τ_α, $[p']_{M^\tau_\alpha} = [p]_{M^\tau_\alpha}$ and any extension of p' that forces $\dot{\pi}_p(t_n) \leq \bar{t}_n$ must also force*

$$\bar{t}_n <^{\sigma(i)}_{\dot{T}} t_n,$$

where $<^0_{\dot{T}}$ is $\not<_{\dot{T}}$ and $<^1_{\dot{T}}$ is $<_{\dot{T}}$, for every $n < \omega$. Furthermore, the extension $p' \leq p$ is minimal in the sense that only the coordinate f^p_0 was extended, and for any $t \in T - (V_\alpha \cup \{t_n : n < \omega\})$,

$$\dot{\pi}^{p'}(t) = \dot{\pi}^p(t).$$

Proof. By Lemma 13 there is a collapse condition $g \in \mathrm{Coll}(\omega_1, < \kappa)$ that extends f^p_0, forces the wanted conclusion, and satisfies the minimality condition that $g \cap V_\alpha = f^p_0 \cap V_\alpha$ and for any $t \in T - (V_\alpha \cup \{t_n : n < \omega\})$,

$$\dot{\pi}^g(t) = \dot{\pi}^{f^p_0}(t).$$

It exists by the definition of the tree $\dot{T}$. Let p' be obtained from p by replacing f^p_0 by g.

Now clearly we still have $[p]_{M^\tau_\alpha} = [p']_{M^\tau_\alpha}$. We claim that p' is super-nice with respect to M^τ_α.

To show super-niceness, suppose towards contradiction that p' is not super-nice with respect to M^τ_α. Since p is super-nice with respect to M^τ_α, this implies that we modified the projection below a node that is relevant for super-niceness. Specifically, there is a path

$$(M^{\delta_0}_{\beta_0}, \delta_0), \dots, (M^{\delta_n}_{\beta_n}, \delta_n)$$

in p' such that $\delta_0 = \tau$ and $\beta_0 = \alpha$, and such that for some $\xi \in \delta_n \cap M^{\delta_n}_{\beta_n}$ and pair of exit nodes $(s,t) \in f^p_\xi$ from $M^\xi_{\beta_n}$, we have

$$\dot\pi^{p'}(t) \neq \dot\pi^p(t).$$

Note that the path is also in p since only the collapse condition was extended. By the minimality consition, we have $t = t_n$ for some n. In particular, $\gamma_t = \gamma_{t_n} \leq \alpha$. By definition of the poset, we have $s \in V_{\gamma_t}$. Then: $s \in M^1_s \cap V_\kappa = V_{\gamma_t} \subseteq V_\alpha = V_{\beta_0} \subseteq V_{\beta_n}$. This contradicts the assumption that s was exit node from $M^\xi_{\beta_n}$. Hence p' must be super-nice with respect to M^τ_α.

□

Lemma 50. *Let $\tau < \kappa^+$. Suppose that for every $\delta < \tau$ and for every M^δ_α that is $\vec{M}^\delta$-reflective, the following holds:*

- *The set $D_\delta(M^\delta_\alpha)$ is σ-closed and dense below every condition amenable to M^δ_α.*
- *For every $p \in D_\delta(M^\delta_\alpha)$, the trace $[p]_{M^\delta_\alpha}$ is a condition in $\mathbb{P}_\delta \cap M^\delta_\alpha$ and a residue of p into M^δ_α.*

Then:

1. (Node density) *For every $p \in \mathbb{P}_\tau$, $\delta < \tau$, and $s \in S_\delta$, there is an extension $p' \leq p$ with $s \in \mathrm{dom}(f^p_\delta)$.*
2. (Super-nice density) *For every $M^\tau_\alpha \in \vec{M}^\tau$, the set $D_\tau(M^\tau_\alpha)$ is σ-closed and dense below every condition that is amenable to M^τ_α.*
3. *For every condition $p \in D_\tau(M^\tau_\alpha)$, the trace $[p]_{M^\tau_\alpha}$ is a condition in $\mathbb{P}_\tau \cap M^\tau_\alpha$.*

Proof. Starting with the node density assertion, let $p \in \mathbb{P}_\tau$ and $s \in S_\delta$, $\delta < \tau$. We would like to extend p and add s to $\mathrm{dom}(f^p_\delta)$. Up to extending $p \restriction \delta$, we may assume that it decides the following information:

- Meets in the set $\mathrm{dom}(f^p_\delta) \cup \{s\}$ and their implicit images (see Remark 38).
- For every $\alpha \in N^p_\delta$ with $s \notin M^\delta_\alpha$, the node $\bar s_\alpha$ that is below s and an exit node from M^δ_α as well as the identity of $M^1_{\bar s_\alpha} \in \vec{M}^{\delta, \dot G(\mathbb{P}_\delta)}$.

By Remark 38, we may assume that the meet $s' \wedge s$ is already in the domain of f^p_δ for every $s' \in \mathrm{dom}(f^p_\delta)$. We need to find images t and $\bar t_\alpha$ for the nodes s and $\bar s_\alpha$, respectively, where $\alpha \in N^p_\delta$. We need to make sure that for every $\alpha \in N^p_\delta$ and $\beta \in N^p_\delta$ with $\beta > \alpha$,

$$\bar s_\alpha \in M^\delta_\beta \iff \bar t_\alpha \in M^\delta_\beta,$$

and $s \in M^\delta_\beta$ iff $t \in M^\delta_\beta$.

Claim 51. For all $\beta \in N^p_\delta$ and $\alpha \in N^p_\delta$ with $\alpha < \beta$, if $\bar{s}_\alpha \in M^\delta_\beta$, then $M^1_{\bar{s}_\alpha} \in M^\delta_\beta$.

Proof of Claim 51. Let $\bar{s}_\alpha \in X$ and let $\beta \in N^p_\delta$ be such that $\bar{s}_\alpha \in M^\delta_\beta$. Say $p \restriction \delta$ decides $M^1_{\bar{s}_\alpha}$ to be the model $M \in \vec{M}^{\delta,\dot{G}(\mathbb{P}_\delta)}$. Up to extending $p \restriction \delta$ further, we assume by induction that it is super-nice with respect to M^δ_β. Therefore, by assumption its trace $[p \restriction \delta]_{M^\delta_\beta}$ is a residue of $p \restriction \delta$ into M^δ_β. Since $\bar{s}_\alpha \in M^\delta_\beta$, by elementarity and by the fact that M^δ_β is reflective, we find $w \in \mathbb{P}_\delta \cap M^\delta_\beta$ extending the trace $[p \restriction \delta]_{M^\delta_\beta}$ that decides $M^1_{\bar{s}_\alpha}$, say to be the model $M' \in \vec{M}^{\delta,\dot{G}(\mathbb{P}_\delta)}$. But then w and $p \restriction \delta$ are compatible, so we must have $M = M' \in M^\delta_\beta$. $\square$

Next, let $\bar{t}_\alpha$, $\alpha \in N^p_\delta$, be nodes of collapse index $(\alpha_{\bar{t}_\alpha}, \beta_{\bar{t}_\alpha}, \gamma_{\bar{t}_\alpha}, \delta_{\bar{t}_\alpha})$ that satisfy:

- the collapse part f^p_0 does not decide anything about them, i.e.
$$(\alpha_{\bar{t}_\alpha}, \beta_{\bar{t}_\alpha}, \gamma_{\bar{t}_\alpha}, \delta_{\bar{t}_\alpha}) \notin \mathrm{dom}(f^p_0),$$
- $\alpha_t = \mathrm{ht}(\bar{s}_\alpha)$,
- and $\beta_t = \beta_p(\bar{s}_\alpha)$ if $\beta_p(\bar{s}_\alpha)$ exists,
- $\gamma_{\bar{t}_\alpha} = \kappa \cap M^1_{\bar{s}_\alpha}$,
- $\delta_{\bar{t}_\alpha} < \beta$ for every β such that $\bar{s}_\alpha \in M^\delta_\beta$,
- if $\bar{t}_\alpha \neq \bar{t}_{\alpha'}$ and $\alpha < \alpha'$, then $\bar{t}_\alpha \in \alpha_{\bar{t}_{\alpha'}} \times \beta_{\bar{t}_{\alpha'}}$.

Furthermore, let t be a node of collapse index $(\alpha_t, \beta_t, \gamma_t, \delta_t)$ which the collapse part f^p_0 does not decide anything and such that $\bar{t}_\alpha \in \alpha_t \times \beta_t$ for every $\alpha \in N^p_\delta$, and if $s \in M^\delta_\alpha$, then $t \in M^\delta_\alpha$. There is an extension of the collapse part $f \leq f^p_0$ that decides the following:

- $\bar{t}_\alpha <_{\dot{T}} t$,
- for every $s' \in \mathrm{dom}(f^p_\delta)$, $f^p_\delta(s') \wedge t = f^p_\delta(s \wedge s')$.

Let p' be the extension of p obtained by replacing the collapse part f^p_0 by f and by letting
$$f^{p'}_\delta := f^p_\delta \cup \{(\bar{s}_\alpha, \bar{t}_\alpha) : \alpha \in N^p_\delta, s \notin M^\delta_\alpha\} \cup \{(s,t)\}.$$
Then, by construction, p' is a condition in $\mathbb{P}_\tau$, it extends p, and satisfies $s \in \mathrm{dom}(f^{p'}_\delta)$. This ends the Node density assertion.

Next, we prove the second assertion of $D_\tau(M^\tau_\alpha)$ being dense below every condition p which is amenable to M^τ_α. Since $\mathbb{P}_\tau$ is σ-closed one can iterate ω many times the Node density assertion, by using a standard bookkeeping argument, to construct an extension $p' \leq p$ such that for every $\delta < \tau$:

- $f^p_0 \in D(t)$ for all $t \in \mathrm{rng}(f^{p'}_\delta)$,

- for every structure M^δ_β that appears in p', and pair $(s,t) \in f^{p'}_\delta$ that are forced by $p' \restriction \delta$ to be exit nodes from M^δ_β, $p' \restriction \delta$ decides a countable sequence of predecessors of s, $\langle \bar{s}_n \mid n < \omega \rangle$, whose height are cofinal in the height of $\dot{\pi}_{p'}(t)$, and that are in the domain of f^p_δ. They then satisfy that the sequence $\langle f^p_\delta(s_n) \mid n < \omega \rangle$ is cofinal in $\dot{\pi}_{p'}(t)$.

These two items take care of the super-niceness of p' with respect to M^τ_α, 45. (In fact, they take care of p' being super-nice with respect to all structures with respect to which p' is amenable.)

To achieve this, consider a single step of the construction. Let $(s,t) \in f^p_\delta$ be exit nodes from some M^δ_β that appears in p. We define a decreasing sequence of conditions $\langle q^n \mid n < \omega \rangle$ below $q^0 = p \restriction \delta$, whose collapse parts are all in $D(t)$. Extend first q^0 to q' to decide the node $\bar{t}_0 = \pi_{q^0}(t)$ and denote its level by $\bar{\alpha}_0$. Extend then $q^1 \leq q'$ to decide the unique predecessor of s at the height of $\bar{t}_0$, call it $\bar{s}_0$. This $\bar{s}_0$ is the only possible candidate for the pre-image of $\bar{t}_0$. Indeed, any extension of q^1 and p in the poset $\mathbb{P}_\tau$ that has $\bar{s}_0$ in the domain of the δ-th embedding must map $\bar{s}_0$ to $\bar{t}_0$. Continue like this: Extend q^1 to some q'' that decides the node $\bar{t}_1 := \pi_{q_1}(t)$ and find $q^2 \leq q''$ that decides the predecessor of s at the height of $\bar{t}_1$, call it $\bar{s}_1$. Continue ω many times. Finally, let q^ω extend each n. Then let p_1 be the simple amalgamation of q^ω and p. It is a condition in $\mathbb{P}_\tau$ by Lemma 41. By enumerating all relevant pairs of nodes (s,t) in the supports of the p_n, using a bookkeeping function, we finally obtain a condition in $\mathbb{P}_\tau$ that satisfies the above two bulletpoints and is thus super-nice with respect to M^τ_α.

We move to the third assertion of the lemma. Given $p \in D_\tau(M^\tau_\alpha)$ we would like to show that $[p]_{M^\tau_\alpha}$ is a condition in $\mathbb{P}_\tau$. Having that $[p \restriction \delta]_{M^\delta_\alpha} \in \mathbb{P}_\delta$ for every $\delta \in M^\tau_\alpha \cap \tau$, it suffices to verify the assertion for the case $\tau = \delta + 1$ is a successor ordinal. In this case, one has to verify that $[p]_{M^{\delta+1}_\alpha}$ satisfies the requirements in assumptions V,VI, and VII regarding decisions for pairs $(s,t) \in f^p_\delta \cap M^{\delta+1}_\alpha$ and structures M^δ_β, $\beta < \alpha$ that appear in p. Having $p \in D_\delta(M^\delta_\alpha)$ implies $p \restriction \delta \in D_\delta(M^\delta_\alpha)$, which by our assumption, implies that $\mathbb{P}_\delta \cap M^{\delta+1}_\alpha$ is a regular sub-forcing of $\mathbb{P}_\delta/(p \restriction \delta)$. It follows that all decisions about nodes $s \in \mathrm{dom}(f^\delta_p) \cap M^\delta_\alpha$, including (i) the meets, (ii) their exit nodes from every M^δ_β that appears in p, (iii) the identity of M^s_1, and (iv) $\beta_p(s)$, which are determined by $p \restriction \delta$, must already be determined by the trace $[p \restriction \delta]_{M^\delta_\alpha}$. Otherwise, there will be an extension $w \in \mathbb{P}_\delta \cap M^{\delta+1}_\alpha$ which gives incompatible information. But such a condition w could not be compatible with $p \restriction \delta$, contradicting the inductive assumption of the lemma for δ. $\square$

Our goal is to show the following:

Proposition 52. *Suppose $M^\tau_\alpha \in \vec{M}^\tau$ and $q \in \mathbb{P}_\tau$ is amenable to M^τ_α. The function which takes $p \in D_\tau(M^\tau_\alpha)$ to $[p]_{M^\tau_\alpha}$ is a residue function to M^τ_α over q.*

Before proving Proposition 52 we give its immediate corollary and make some preparations.

Corollary 53. *Let $\tau < \kappa^+$ and $\alpha < \kappa$.*

1. *Every condition $q \in \mathbb{P}_\tau$ which is amenable to M^τ_α is strongly proper with respect to M^τ_α. In particular, $\mathbb{P}_\tau \cap M^\tau_\alpha$ is a regular subforcing of $\mathbb{P}_\tau/q$.*

2. *For every condition $q \in D_\tau(M^\tau_\alpha)$, $[q]_{M^\tau_\alpha}$ forces that q belongs to the quotient forcing $\mathbb{P}_\tau/(\mathbb{P}_\tau \cap M^\tau_\alpha)$.*

Given a condition $p \in D_\tau(M^\tau_\alpha)$ as in the statement of Proposition 52, and an extension $w \leq [p]_{M^\tau_\alpha}$ in $\mathbb{P}_\tau \cap M^\tau_\alpha$, the goal of the proof is to find a common extension of w and p. Taking the simple amalgamation of w and p will not work in general. The main part of the argument is based on an inductive construction that results in a common extension p' of p and w.

Definition 54.

1. Let $\tau < \kappa^+$. A *τ-sequence* is a sequence of the form $\langle (M^{\tau_i}_{\alpha_i}, \tau_i) \mid i < \nu\rangle$ where ν is a countable ordinal and:

   (a) if $i < j$, then $\alpha_i \leq \alpha_j$,

   (b) if $i < \nu$, then $\tau_i \leq \tau$,

   (c) for every $i > 0$ there is $j < i$ such that $\tau_i \leq \tau_j$ and $\tau_i \in M^{\tau_j}_{\alpha_j}$.

2. The *root* of a τ-sequence $\langle (M^{\tau_i}_{\alpha_i}, \tau_i) \mid i < \nu\rangle$ is the model $M^{\tau_0}_{\alpha_0}$.

3. Suppose that $\langle (M^{\tau_i}_{\alpha_i}, \tau_i) \mid i < \nu\rangle$ is a τ-sequence and let $p \in \mathbb{P}_\tau$. We say that the sequence is *closed to p* if $M^{\tau_i}_{\alpha_i}$ appears in p, $p \restriction \tau_i \in D_{\tau_i}(M^{\tau_i}_{\alpha_i})$, and $E(p \restriction \tau_i, M^{\tau_i}_{\alpha_i})$ is contained in $\langle (M^{\tau_i}_{\alpha_i}, \tau_i) \mid i < \nu\rangle$, for all $i < \nu$.

4. Suppose that $\langle (M^{\tau_i}_{\alpha_i}, \tau_i) \mid i < \nu\rangle$ is a τ-sequence and let $p \in \mathbb{P}_\tau$. The *p-closure* of the sequence is the natural extension obtained by adding the pairs from $E(p \restriction \tau_i, M^{\tau_i}_{\alpha_i})$ for each $i < \nu$.

Remark 55. *It is clear from the definitions that the p-closure of a τ-sequence remains a τ-sequence (with the same root).*

The main combinatorial feature we will need from τ-sequence is given in the following Lemma.

Lemma 56. *Let $\langle (M^{\tau_i}_{\alpha_i}, \tau_i) : i < \nu\rangle$ be a τ-sequence. For every $i < j$, if $\tau_i \leq \tau_j$ then $\tau_i \in M^{\tau_j}_{\alpha_j}$.*

To prove the Lemma we make use of the idea that a τ-sequence can be described in terms of the simple paths from Definition 48.
We shall prove a claim :

Claim 57. Let $\langle (M^{\tau_i}_{\alpha_i}, \tau_i) : i < \nu\rangle$ be a τ-sequence of models. For all $i < j < \nu$,

$$M^{\tau_i}_{\alpha_i} \cap (\min(\tau_i, \tau_j) + 1) \subseteq M^{\tau_j}_{\alpha_j}.$$

Proof of Claim 57. For $\delta < \kappa^+$ let $f_\delta : \delta \to \kappa$ be an injection of δ into κ which is minimal in the well ordering of H_{κ^+}. Obviously $f_\delta \in M^\rho_\mu$ if $\delta \in M^\rho_\mu$.

We use the fact that the rooted sequence of models is a union of paths that start from (M^τ_α, τ).

We show first that the claim holds for any two nodes of our sequence that lie on the same path. Let

$$(M^{\gamma_i}_{\beta_i}, \gamma_i)_{i \le n}$$

be a path. We show by induction on n that for all $i < j$ along the path, $M^{\gamma_i}_{\beta_i} \cap \gamma_j \subseteq M^{\gamma_j}_{\beta_j}$. Suppose that claim holds for n and prove for $n+1$. It suffices to check that for every $i \le n$, $M^{\gamma_i}_{\beta_i} \cap \gamma_{n+1} \subseteq M^{\gamma_{n+1}}_{\beta_{n+1}}$. Since $\gamma_{n+1} < \gamma_n$ we can use the inductive assumption from $i < n$ to get $M^{\gamma_i}_{\beta_i} \cap \gamma_{n+1} \subseteq M^{\gamma_n}_{\beta_n} \cap \gamma_{n+1}$. As $\gamma_{n+1} \in M^{\gamma_n}_{\beta_n}$ then for every $\rho \in M^{\gamma_n}_{\beta_n} \cap \gamma_{n+1}$ the index $\nu < \kappa$ of ρ in a fixed well-order of the κ-sized structure $M^{\gamma_{n+1}}$ belongs to $M^{\gamma_n}_{\beta_n}$. Hence $\nu < \beta_n < \beta_{n+1}$ and therefore $\rho \in M^{\gamma_{n+1}}_{\beta_{n+1}}$.

So we now have to verify the claim for members of two different paths. Let

$$(M^{\gamma_i}_{\beta_i}, \gamma_i)_{i \le m}$$

and

$$(M^{\delta_j}_{\alpha_j}, \delta_j)_{j \le n}$$

be paths. The claim now is that if $i \le m, j \le n$ then if $\beta_i < \alpha_j$, then

$$M^{\gamma_j}_{\beta_j} \cap (\min(\gamma_j, \delta_j) + 1) \subseteq M^{\delta_i}_{\alpha_i}.$$

We prove this by induction on $n + m$. Note that if $n = 0$ or $m = 0$ then the models lie on the same path and therefore the claim follows from the previous argument. We therefore assume that $m, n > 0$ and without loss of generality that $\alpha_n \le \beta_m$. Using the induction assumption, we have the claim for the pair of paths

$$(M^{\gamma_j}_{\beta_j}, \gamma_j)_{j \le m-1}$$

and

$$(M^{\delta_i}_{\alpha_i}, \delta_i)_{i \le n}$$

So the only additional cases of the claim we have to consider are between $M^{\delta_i}_{\alpha_i}$ and $M^{\gamma_m}_{\beta_m}$ for all $i \le n$. By our assumption, $\alpha_i \le \beta_m$ for all $i \le n$.

We distinguish several cases :

Case 1: $\alpha_i \le \beta_{m-1}$**.** By the induction assumption for the two paths

$$(M^{\gamma_j}_{\beta_j}, \gamma_j)_{j \le m-1}$$

and

$$(M^{\delta_i}_{\alpha_i}, \delta_i)_{i \le n}$$

and the two members of these pairs $M^{\delta_i}_{\alpha_i}$ and $M^{\gamma_{m-1}}_{\beta_{m-1}}$ we have

$$M^{\delta_i}_{\alpha_i} \cap (\min(\delta_i, \gamma_{m-1}) + 1) \subseteq M^{\gamma_{m-1}}_{\beta_{m-1}}.$$

Since we proved the claim for the path

$$(M^{\delta_i}_{\alpha_i}, \delta_i)_{i \leq n}$$

$$M^{\gamma_{m-1}}_{\beta_{m-1}} \cap (\min(\gamma_{m-1}, \gamma_m) + 1) \subseteq M^{\gamma_m}_{\beta_m}.$$

Since $\gamma_m < \gamma_{m-1}$, $\min(\gamma_m, \gamma_{m-1}) = \gamma_{m-1}$ and $\min(\delta_i, \gamma_m) \leq \min(\delta_i, \gamma_{m-1})$ we get

$$M^{\delta_i}_{\alpha_i} \cap (\min(\delta_i, \gamma_m) + 1) \subseteq M^{\gamma_{m-1}}_{\beta_{m-1}} \cap (\min(\delta_i, \gamma_m) + 1) \subseteq M^{\gamma_m}_{\beta_m}.$$

Case 2: $\beta_{m-1} < \alpha_i \leq \beta_m$ and $\delta_i \geq \gamma_m$.
Recall that $\gamma_m \in M^{\gamma_{m-1}}_{\beta_{m-1}}$. By the induction assumption on the pair of paths:

$$(M^{\gamma_j}_{\beta_j}, \gamma_j)_{j \leq m-1}$$

and

$$(M^{\delta_i}_{\alpha_i}, \delta_i)_{i \leq n}$$

$M^{\gamma_{m-1}}_{\beta_{m-1}} \cap (\min(\gamma_{m-1}, \delta_i) + 1) \subseteq M^{\delta_i}_{\alpha_i}$. Since $\gamma_m \leq \delta_i$ and $\gamma_m < \gamma_{m-1}$, we get that $\gamma_m \in M^{\delta_i}_{\alpha_i}$. Hence $f_{\gamma_m} \in M^{\delta_i}_{\alpha_i}$.

Let $\rho \in M^{\delta_i}_{\alpha_i} \cap (\min(\delta_i, \gamma_m)+1)$. To see that $\rho \in M^{\gamma_m}_{\beta_m}$, consider the ordinal $\eta = f_{\gamma_m}(\rho)$. We have that $\eta \in \kappa \cap M^{\delta_i}_{\alpha_i} = \alpha_i$ and as $\alpha_i \leq \beta_m = \kappa \cap M^{\gamma_m}_{\beta_m}$, $\eta \in M^{\gamma_m}_{\beta_m}$. It follows that $\rho = f_{\gamma_m}(\eta) \in M^{\gamma_m}_{\beta_m}$.

Case 3: $\beta_{m-1} < \alpha_i \leq \beta_m$ and $\delta_i < \gamma_m$.
Let $k \leq n$ be maximal such that $\alpha_k \leq \beta_{m-1}$ or $\delta_k \geq \gamma_m$. Note that $k < i$. We prove by induction on l, where $k < l \leq i$, that $\delta_l \in M^{\gamma_m}_{\beta_m}$. For each such l we have that $\alpha_l > \beta_{m-1}$ and $\delta_l < \gamma_m$.

The base case of $l = k + 1$ requires a separate argument. We distinguish two cases.

($l = k + 1$, first subcase) Assume first that $\alpha_k \leq \beta_{m-1}$.
Then $\delta_{k+1} < \gamma_m < \gamma_{m-1}$ and $\delta_{k+1} \in M^{\delta_k}_{\alpha_k}$. By the induction assumption $M^{\delta_k}_{\alpha_k} \cap (\min(\delta_k, \gamma_m) + 1) \subseteq M^{\gamma_{m-1}}_{\beta_{m-1}}$. Hence $\delta_{k+1} \in M^{\gamma_{m-1}}_{\beta_{m-1}}$. It follows from our proof for the case of a single path that $\delta_{k+1} \in M^{\gamma_m} \beta_m$.

($l = k + 1$, second subcase) Assume now $\beta_{m-1} < \alpha_k$.
Hence $\beta_{m-1} < \alpha_k \leq \beta_m$ and $\delta_k \geq \gamma_m$. By case 2 above, $M^{\delta_k}_{\alpha_k} \cap \gamma_m \subseteq M^{\gamma_m}_{\beta_m}$. $\delta_{k+1} < \delta_m$ and $\delta_{k+1} \in M^{\delta_k}_{\alpha_k}$ we get $\delta_{k+1} \in M^{\gamma_m}_{\beta_m}$.

Establishing the assertion for $l = k + 1$ we proceed by induction to show the same holds for other $l \leq i$.

(Inductive step $l \to l + 1$) Assume the claim for l and prove it for $l + 1$. By assumption $\delta_l \in M^{\gamma_m}_{\beta_m}$. We have $\delta_{l+1} < \delta_l$ and $\delta_{l+1} \in M^{\delta_l}_{\alpha_l}$. By assumption $f_{\delta_l} \in M^{\gamma_m}_{\beta_m} \cap M^{\delta_l}_{\alpha_l}$. So $\rho = f^{-1}_{\delta_l}(\delta_{l+1}) \in \alpha_i$. Hence $\rho \in \beta_m$. So $f^{-1}_{\delta_l}(\rho) \in M^{\gamma_m}_{\beta_m}$.

Using this for $l = i$ we get that $\delta_i \in M^{\gamma_m}_{\beta_m}$. As in the previous cases we get that $M^{\delta_i}_{\alpha_i} \cap (\min(\delta_i, \gamma_m) + 1) \subseteq M^{\gamma_m}_{\beta_m}$.

This concludes the proof of the Claim. □

Proof of Lemma 56. The lemma follows easily from the claim: if $\tau_i < \tau_j$, then by the claim we have $\tau_i \in M^{\tau_i}_{\alpha_i} \cap (\tau_i + 1) \subseteq M^{\tau_j}_{\alpha_j}$.

□

Claim 57 gives us another lemma that will be useful:

Lemma 58. *Suppose that $\langle (M^{\tau_i}_{\alpha_i}, \tau_i) \mid i < \nu \rangle$ is a τ-sequence. If $k < j < \nu$ are such that $\tau_j \leq \tau_k$ and $\tau_j \in M^{\tau_k}_{\alpha_k}$, then the sequence*

$$\langle (M^{\min\{\tau_i,\tau_j\}}_{\alpha_i}, \min\{\tau_i, \tau_j\}) \mid k \leq i < j \rangle$$

is a τ_j-sequence and an element of $M^{\tau_j}_{\alpha_j}$.

Proof. The fact that the sequence is an element of $M^{\tau_j}_{\alpha_j}$ follows from the fact that $M^{\tau_j}_{\alpha_j}$ is closed for countable sequences and using Claim 57.

We first show that if i satisfies $k \leq i < j$, then $M^{\min\{\tau_i,\tau_j\}}_{\alpha_i}$ is well-defined, i.e. $\vec{M}^{\min\{\tau_i,\tau_j\}}$-reflective. For this, it suffices to show that if $\tau_j < \tau_i$, then

$$\tau_j \in M^{\tau_i}_{\alpha_i},$$

since this implies that α_i belongs to the sequence relevant to the coordinate τ_j and thus $M^{\tau_j}_{\alpha_i}$ is defined. To this end, suppose that i satisfies $k \leq i < j$ and $\tau_j < \tau_i$. If $i = k$, we are done. Otherwise, applying Claim 57 to $k < i$, we obtain the conclusion:

$$\tau_j \in M^{\tau_k}_{\alpha_k} \cap (\min\{\tau_k, \tau_i\} + 1) \subseteq M^{\tau_i}_{\alpha_i}.$$

This is enough to conclude that each model $M^{\min\{\tau_i,\tau_j\}}_{\alpha_i}$ for $k \leq i < j$ is well-defined.

We then show that the sequence is a τ_j-sequence. For $i < j$, we denote

$$\tau^j_i := \min\{\tau_i, \tau_j\}.$$

It suffices to verify that for any $i \in (k, j)$ there is $l \in [k, i)$ such that $\tau^j_i \leq \tau^j_l$ and $\tau^j_i \in M^{\tau^j_l}_{\alpha_l}$. Fix some such i. If $\tau_j \leq \tau_i$, then $l := k$ is as wanted, since then $\tau^j_i = \tau_j$, so in particular $\tau^j_i \leq \tau^j_k$ and $\tau^j_i \in M^{\tau^j_k}_{\alpha_k}$. Suppose then $\tau_i < \tau_j$. Note that $\tau_j \leq \tau_k$ by assumption. Since we started with a τ-sequence, there is $i' < i$ such that $\tau_i \leq \tau_{i'}$ and $\tau_i \in M^{\tau_{i'}}_{\alpha_{i'}}$.

- If $i' \leq k$, then Claim 57 gives

$$\tau_i \in M^{\tau_{i'}}_{\alpha_{i'}} \cap (\min\{\tau_{i'}, \tau_k\} + 1) \subseteq M^{\tau_k}_{\alpha_k},$$

and since $\tau_i, \tau_j \in M^{\tau_k}_{\alpha_k}$ and $\tau_i \leq \tau_j$, we have $\tau_i \in M^{\tau_j}_{\alpha_k}$, so $l := k$ is as wanted.

- Suppose that $k < i'$.
  - If $\tau_{i'} \leq \tau_j$, then $l := i'$ is as wanted, since then $M^{\tau^j_{i'}}_{\alpha_{i'}} = M^{\tau_{i'}}_{\alpha_{i'}}$.
  - If $\tau_j < \tau_{i'}$, then by Claim 57 applied to $k < i'$, we have

  $$\tau_j \in M^{\tau_k}_{\alpha_k} \cap (\min\{\tau_k, \tau_{i'}\} + 1) \subseteq M^{\tau_{i'}}_{\alpha_{i'}}.$$

  So $\tau_i, \tau_j \in M^{\tau_{i'}}_{\alpha_{i'}}$, and since $\tau_i < \tau_j$, we have $\tau_i \in M^{\tau_j}_{\alpha_{i'}} = M^{\tau^j_{i'}}_{\alpha_{i'}}$. Thus $l := i'$ is as wanted.

We have now seen that $\langle (M^{\tau^j_i}_{\alpha_i}, \tau^j_i) \mid k \leq i < j \rangle$ is a τ_j-sequence in $M^{\tau_j}_{\alpha_j}$. $\square$

Definition 59.

1. Let $p \in \mathbb{P}_\tau$. A pair of countable sequences $\langle (M^{\tau_i}_{\alpha_i}, \tau_i) \mid i < \nu \rangle$ and $\langle q^i \mid i < \nu \rangle$ is a *p-Multi-Extension*, p-M.E. in short, if it satisfies
   (a) $p \restriction \tau_i \in D_{\tau_i}(M^{\tau_i}_{\alpha_i})$ for every $i < \nu$.
   (b) $q^i \in \mathbb{P}_{\tau_i} \cap M^{\tau_i}_{\alpha_i}$ and $q^i \leq [p \restriction \tau_i]_{M^{\tau_i}_{\alpha_i}}$ for every $i < \nu$.
   (c) $\langle (M^{\tau_i}_{\alpha_i}, \tau_i) \mid i < \nu \rangle$ is a τ-sequence and is an initial closure of its p-closure.
   (d) For each non-zero $j < \nu$, there is $k < j$ such that denoting $\tau^j_i = \min\{\tau_i, \tau_j\}$ then the sequence $\langle (M^{\tau^j_i}_{\alpha_i}, \tau^j_i) \mid k \leq i < j \rangle$ is a τ_j-sequence.
   (e) For every $i < j < \nu$, $q^j \restriction \tau^j_i \leq q^i \restriction \tau^j_i$.
   (f) For every $i < \nu$ and $s \in (\mathrm{dom}(f^p_{\tau_i}) \cap M^{\tau_i}_{\alpha_i}) - \bigcup_{j<i} V_{\alpha_j}$, q^i decides the following information about b_s:
      - the meet $m(s, s')$ in S_{τ_i} for every $s' \in \mathrm{dom}(f^{q^j}_{\tau_i})$ for any $j < i$ with $\tau_i \in \mathrm{supp}(q^i)$.
      - the implicit image $t(s, s')$ of $m(s, s')$ (as described in Remark 38).
      - the exit node $\bar{s}_\beta := e(s, M^{\tau_i}_\beta)$ for every structure $M^{\tau_i}_\beta$ which appears in q^j at coordinate τ_i and its $M^1_{\bar{s}_\beta}$, for every $j < i$.
   (g) For all $i < i' < \nu$ and $s \in \mathrm{dom}(f^p_{\tau_i})$ such that $t := f^p_{\tau_i}(s) \in M^{\tau_{i'}}_{\alpha_{i'}}$, the collapse coordinate of $q^{i'}$ forces the following information about b_t:
      - the implicit image $t(s, s')$ is the meet of $f^p_{\tau_i}(s)$ and $f^{q^j}_{\tau_i}(s')$ in $\dot{T}$, for any $s' \in \mathrm{dom}(f^{q^j}_{\tau_i})$ and $j < i$ with $\tau_i \in \mathrm{supp}(q^j)$,

- for every structure $M^{\tau_i}_{\beta}$ which appears in q^j at coordinate τ_i for any $j < i$, there is an exit node $\bar{t}_\beta$ from $M^{\tau_i}_{\beta}$ at the height of $\bar{s}_\beta$ that is below t and satisfies $\gamma_{\bar{t}_\beta}) = \kappa \cap M^1_{\bar{s}_\beta}$, and furthermore for every $s' \in \mathrm{dom}(f^{q^j}_{\tau_i})$,

$$t \wedge f^{q^j}_{\tau_i}(s') = \bar{t}_\beta \wedge f^{q^j}_{\tau_i}(s').$$

2. The *simple amalgamation* of a sequence of conditions $\langle q^i \mid i < \nu\rangle$ is the pair (f, N) defined by

   - $\mathrm{dom}(f) = \bigcup_{i<\nu} \mathrm{supp}(q^i)$.
   - for each $\delta \in \mathrm{dom}(f)$, $f_\delta = \bigcup\{f^{q^i}_\delta \mid i < \nu \text{ and } \delta \in \mathrm{supp}(q^i)\}$.
   - for each $\alpha < \kappa$, the set N includes a pair (α, a_α) if and only if there is some pair (α, a') in $\bigcup_i N^{q^i}$, in which case a_α is taken to be the union of all such a'.

Lemma 60.
For every $p \in \mathbb{P}_\tau$ and p-M.E. pair of sequences $\langle (M^{\tau_i}_{\alpha_i}, \tau_i) \mid i < \nu\rangle$ and $\langle q^i \mid i < \nu\rangle$ there is a condition $q \in \mathbb{P}_\tau$ such that $q \leq p$ and for each $i < \nu$, $q \restriction \tau_i \leq q^i$.

Proof. The proof is by induction on pairs (τ, α) (with the usual lexicographic ordering) where $\alpha < \kappa$ is minimal so that $\alpha \geq \cup_{i<\nu}\alpha_i$, and $p \in M^\tau_\alpha$.

Let $\langle (\tau_j, M^{\tau_j}_{\alpha_j}) \mid j < \nu^*\rangle$ be the p-closure of $\langle (M^{\tau_i}_{\alpha_i}, \tau_i) \mid i < \nu\rangle$. Denote

$$\tau_{\nu^*} := \tau \text{ and } \alpha_{\nu^*} := \kappa,$$

with the convention that $M^\tau_\kappa = Hull^{\mathcal{A}^\tau}(\kappa)$. By the assumption, the p-closure is an end extension of $\langle (M^{\tau_i}_{\alpha_i}, \tau_i) \mid i < \nu\rangle$. Moreover, it follows from the definition of super-nice conditions and the fact $p \restriction \tau_i \in D_{\tau_i}(M^{\tau_i}_{\alpha_i})$ for all $i < \nu$ that $p \restriction \tau_j \in D_{\tau_j}(M^{\tau_j}_{\alpha_j})$ for all $j < \nu^*$.
Our next step is to extend the corresponding sequence of conditions $\langle q^i \mid i < \nu\rangle$ to a sequence $\langle q^j \mid j \leq \nu^*\rangle$, so that $\langle (\tau_j, M^{\tau_j}_{\alpha_j}) \mid j \leq \nu^*\rangle$ and $\langle q^j \mid j \leq \nu^*\rangle$ form a p-Multi-Extension. Suppose that $\langle q^j \mid j < \eta\rangle$ has been defined for some $\nu \leq \eta \leq \nu^*$, such that $\langle (\tau_j, M^{\tau_j}_{\alpha_j}) \mid j < \eta\rangle$ and $\langle q^j \mid j < \eta\rangle$ are p-M.E. We define q^η in four steps that go through auxiliary conditions $q^{\eta,0}$ and $q^{\eta,1}$. The entire construction happens inside $M^{\tau_\eta}_{\alpha_\eta}$.

($q^{\eta,0}$) Let $q^{\eta,0} = [p \restriction \tau_\eta]_{M^{\tau_\eta}_{\alpha_\eta}}$. Also, denote for each $j < \eta$, $\tau^\eta_j = \min(\tau_j, \tau_\eta)$. It follows from the definition of a p-M.E. pair and the fact $p \restriction \tau_j \in D_{\tau_j}(M^{\tau_j}_{\alpha_j})$ for all $j < \eta$, that there is $k < \eta$ such that the pair of sequences $\langle (M^{\tau^\eta_j}_{\alpha_j}, \tau^\eta_j) \mid k \leq j < \eta\rangle$ and $\langle q^j \restriction \tau^\eta_j \mid j < \eta\rangle$ forms a $q^{\eta,0}$-M.E.

($q^{\eta,1}$) To obtain $q^{\eta,1}$, we extend the collapse part of $q^{\eta,0}$ to make decisions about nodes in T that fit relevant decisions about meets and exit nodes in various trees S_δ, which were made by previous conditions q^j, $j < \eta$, according to property (1g) in the definition of $M.E.$ sequences. More precisely, let $\langle (s_n, t_n) \mid n < M \rangle_{n<M}$, $M \leq \omega$, enumerate all the pairs (s,t) for which there is $j < \eta$ such that

– $(s,t) \in f^p_{\tau_j}$,

– $t \in M^{\tau_\eta}_{\alpha_\eta} - \bigcup_{j<\eta} M^{\tau_j}_{\alpha_j}$,

Note that it follows that $\gamma_t \leq \sup_{j<\eta} \alpha_j$, since the model M^1_s occurs in the τ-sequence before $M^{\tau_j}_{\alpha_j}$ and $\gamma_t = \kappa \cap M^1_s$.

By Lemma 49, there is an extension $q^{\eta,1} \leq q^{\eta,0}$ such that for every $j < \eta$, it is super-nice with respect to every $M^{\tau^\eta_j}_{\alpha_j}$, satisfies $[q^{\eta,1}]_{M^{\tau^\eta_j}_{\alpha_j}} = [q^{\eta,0}]_{M^{\tau^\eta_j}_{\alpha_j}}$ and forces the information satisfying (1g) for t_n, $n < M$.

(q^η) This step breaks into two cases: either $\tau_\eta = \tau$, or $\tau_\eta < \tau$.

Case $\boldsymbol{\tau_\eta < \tau}$: We want to apply the inductive assumption of the lemma inside the structure $M^{\tau_\eta}_{\alpha_\eta}$. To this end, we claim that the sequences $\langle q^\eta \restriction \tau^\eta_j : j < \eta \rangle$ and $\langle (M^{\tau^\eta_j}_{\alpha_j}, \tau^\eta_j) : k \leq j < \eta \rangle$ form a $q^{\eta,1}$-Multi-Extension. But this follows from Lemma 58 and the fact that, by assumption, they form a $q^{\eta,0}$-Multi-Extension, which implies together with the fact that the condition $q^{\eta,1}$ is super-nice with respect to each $M^{\tau^\eta_j}_{\alpha_j}$, and was obtained from $q^{\eta,0}$ by extending only the collapse coordinate, that they also form a $q^{\eta,1}$-Multi-Extension.

Thus, by induction hypothesis applied inside the model $M^{\tau_\eta}_{\alpha_\eta}$, there is a condition $q^{\eta,2} \in \mathbb{P}_{\tau_\eta} \cap M^{\tau_\eta}_{\alpha_\eta}$ that extends each $q^j \restriction \tau^\eta_j$ and $q^{\eta,1}$.

Finally, working inside $M^{\tau_\eta}_{\alpha_\eta}$, we find the final extension $q^\eta \leq q^{\eta,2}$ which determines the following information for every $s \in (\mathrm{dom}(f^p_{\tau_\eta}) \cap M^{\tau_\eta}_{\alpha_\eta}) - \bigcup_{j<\eta} V_{\alpha_j}$:

– The meets of all pairs of nodes $m(s,s')$ where $s' \in \bigcup_{i<j} \mathrm{dom}(f^{q^i}_{\tau_\eta})$

– The implicit images $t(s,s')$ of $m(s,s')$ (see Remark 38).

– For every $j < \eta$ with $\tau_\eta \in \mathrm{supp}(q^j)$ and every structure $M^{\tau_\eta}_\beta$ that appears in q^j, the exit node below s from $M^{\tau_\eta}_\beta$.

Case $\boldsymbol{\tau_\eta = \tau}$: First, let q' be the simple amalgamation of the conditions $\langle q^j : j < \eta \rangle$. It follows from item (1e) the definition of Multi-Extension that q' is a condition in $\mathbb{P}_{\sup_{j<\eta} \tau_j}$. Then, for every $\gamma \in \tau \cap M^\tau_{\alpha_\eta}$, let f^1_γ be the extension of the function $f^{q'}_\gamma$ by the pairs

$$(s,t), (m(s,s'), t(s,s'))$$

where $(s,t) \in f^{q^{\eta,0}}_\gamma$ and $s' \in \mathrm{dom}(f^{q^j}_\gamma)$, $j < \eta$. By construction, $q' \restriction \gamma$ decides meets in the domain of f^1_γ and forces that f^1_γ is a level- and meet-preserving injective tree-embedding from $\dot{S}_\gamma \cap M^\tau_{\alpha_\eta}$ to $\dot{T}$. Furthermore, still by construction, using item 1g, for every $s \in \mathrm{dom}(f_\gamma^{q^{\eta,1}})$ and model M_β^γ that appears in q^j at coordinate γ for any $j < \eta$, the condition $q' \restriction \gamma$ decides the exit node $\bar{s}_\beta$ from M_β^γ below s, and there is a "image" $\bar{t}_\beta$ such that if we let f_γ^2 to be the extension of f_γ^1 by the pairs

$$(\bar{s}_\beta, \bar{t}_\beta),$$

where $s \in \mathrm{dom}(f_\gamma^{q^{\eta,1}})$ and M_β^γ appears in q^j at coordinate γ for some $j < \eta$.

Finally, let q^η be the condition obtained by extending $q^{\eta,1}$ at each coordinate $\gamma \in \tau \cap M_{\alpha_\eta}^\tau$ by the function f_γ^2. By construction, q^η is a condition in $\mathbb{P}_\tau \cap M_{\alpha_\eta}^\tau$.

It is clear from the construction that the sequences $\langle (M_{\alpha_j}^{\tau_j}, \tau_j) \mid j \leq \eta \rangle$ and $\langle q^j \mid j \leq \eta \rangle$ constitute a $[p \restriction \tau_\eta]_{M_{\alpha_\eta}^{\tau_\eta}}$-Multi-Extension.

This concludes the recursion. Now, by construction, the condition q^{ν^*} is a condition in $\mathbb{P}_\tau$ that extends p and each q^j.

$\square$

We can now prove Proposition 52.

Proof of Proposition 52. Let $p \in D_\tau(M_\alpha^\tau)$. To show that $[p]_{M_\alpha^\tau}$ is a residue for p, we need to verify that it is compatible with every condition $w \in \mathbb{P}_\tau \cap M_\alpha^\tau$ which extends $[p]_{M_\alpha^\tau}$. This is an immediate consequence of the previous Lemma with the p-M.E. pair of sequences of length $\nu = 1$ with $(M_{\alpha_0}^{\tau_0}, \tau_0) = (M_\alpha^\tau, \tau)$, and $q^0 = w$. $\square$

Phrased using notation from 23, Proposition 52 says that if $p \in \mathbb{P}_\tau$ and $\alpha \in E_\tau^p$, then p has a residue into M_α^τ.

Theorem 61. *Let $\mathbb{P}_{\kappa^+} = \bigcup_{\tau<\kappa^+} \mathbb{P}_\tau$.*

1. *$\mathbb{P}_{\kappa^+}$ has κ^+-c.c.*
2. *$\mathbb{P}_{\kappa^+}$ is σ-closed and thus, does not collapse ω_1.*
3. *$\mathbb{P}_{\kappa^+}$ collapses all cardinals between ω_1 and κ.*
4. *$\mathbb{P}_{\kappa^+}$ does not collapse κ*
5. *$2^{\aleph_1} = \kappa^+$ in $V^{\mathbb{P}_{\kappa^+}}$.*
6. *For each $\tau < \kappa^+$, the τ-th wide-tree S^τ chosen by the book-keeping function Ψ embeds into T. In particular, if Ψ covers all $\mathbb{P}_\tau$ names of wide trees on κ for all $\tau < \kappa^+$, then T is a maximal wide tree in the generic extension by $\mathbb{P}_{\kappa^+}$.*

Proof.

1. This is a standard consequence of the fact that each $\mathbb{P}_\tau$, $\tau < \kappa^+$ has size κ, and that there is a stationary set of $\tau < \kappa^+$ for which $\mathbb{P}_\tau$ is a direct limit of $\mathbb{P}_\delta$, $\delta < \tau$ (the set of all limit $\tau < \kappa^+$ of uncountable cofinality).

2. Immediate from the fact each $\mathbb{P}_\tau$, $\tau < \kappa^+$ is σ-closed (inductive assumption I).

3. Immediate. As $\mathbb{P}_{\kappa^+}$ embeds $\mathbb{P}_0 = \mathrm{Coll}(\omega_1, < \kappa)$.

4. This follows from the fact that $\mathbb{P}_{\kappa^+}$ is κ^+.c.c, and from the strong properness of all $\mathbb{P}_\tau$, $\tau < \kappa^+$, for structures of size $\aleph_1$.

5. For every $\tau < \kappa^+$, the poset $\mathbb{P}_{\tau+1}$ introduces a tree embedding $f_\tau : S_\tau \to T$, so that for every structure $M_\alpha^{\tau+1} \in \vec{M}^{\tau+1, G(\mathbb{P}_{\tau+1})}$, $f_\tau \restriction M_\alpha^{\tau+1}$ is generic over $V[G(\mathbb{P}_\tau)]$ and introduces an embedding of $S_\tau \cap M_\alpha^{\tau+1}$ to $T \cap M_\alpha^{\tau+1}$, which is a new subset of $M_\alpha^{\tau+1}$ of size $\aleph_1$.

6. Immediate by the construction of the posets $\mathbb{P}_\tau$, $\tau < \kappa^+$.

$\square$

5.1 Strong properness of quotients

To prove that the forcing does not add new branches to T given the chosen trees S_τ, $\tau < \kappa^+$, are all wide Aronszajn, we need a slightly stronger version of strong properness that can be applied in quotients. The setup to have in mind is having some $\tau < \kappa^+$, $\alpha < \kappa$ and a condition $p \in D_\tau(M_\alpha^\tau)$.

Lemma 62. *Let $M_\alpha^\tau, M_\beta^\tau \in \vec{M}^\tau$, $\alpha < \beta$. Suppose that $p \in D_\tau(M_\beta^\tau) \cap D_\tau(M_\alpha^\tau)$. Let $G_\alpha \subseteq \mathbb{P}_\tau \cap M_\alpha^\tau$ be generic over V with $[p]_{M_\alpha^\tau} \in G_\alpha$.*

1. *The quotient $\mathbb{P}_\tau / G_\alpha$ has a σ-closed dense subset.*

2. *For every $w \in (\mathbb{P}_\tau/G_\alpha) \cap M_\beta^\tau[G_\alpha]$ with $w \leq [p]_{M_\beta^\tau}$, there is a common extension $q \leq_{\mathbb{P}_\tau/G_\alpha} w, p$, such that $q \in D_\tau(M_\beta^\tau)$.*

Proof.

1. The quotient is σ-closed with respect to conditions in $D_\tau(M_\alpha^\tau)$. This is an immediate consequence of the fact that $D_\tau(M_\alpha^\tau)$ is σ-closed dense below p, of Proposition 52, and the fact that the trace map $p \mapsto [p]_{M_\alpha^\tau}$ respects countable joins.

2. Let $p \in D_\tau(M_\alpha^\tau) \cap D_\tau(M_\beta^\tau)$, $p \in \mathbb{P}_\tau/G_\alpha$ and $w \leq [p]_{M_\beta^\tau}$, $w \in \mathbb{P}_\tau/G_\alpha$, $w \in M_\beta^\tau[G_\alpha]$. By extending w we may assume that it belongs to $D_\tau(M_\alpha^\tau)$, and therefore that $[w]_{M_\alpha^\tau}$ forces that w belongs to the quotient $\mathbb{P}_\tau/G_\alpha$.

   Suppose towards contradiction that there exists an extension $r \in M_\beta^\tau$ of w which is incompatible with p in the quotient forcing $\mathbb{P}_\tau/G_\alpha$.

   This means that in V, there is $w' \leq [w]_{M_\alpha^\tau}$, $w' \in \mathbb{P}_\tau \cap M_\alpha^\tau$, which forces that r and p are incompatible as conditions of the quotient forcing, over an extension by $\mathbb{P}_\tau \cap M^\tau_\alpha$. By strong properness (Proposition 52) with respect to M^τ_α, r and w' are compatible by some $r' \in M^\tau_\beta$. Since $r' \leq r \leq w \leq [p]_{M^\tau_\beta}$ we can apply strong properness again with respect to M^τ_β and conclude that r' and p are compatible by some $q \in D^\tau(M^\tau_\beta) \cap D^\tau(M^\tau_\alpha)$. But now $[q]_{M^\tau_\alpha}$ extends w' and forces contradictory information.

□

We also need an extrapolation of item (2) of the previous lemma. In what follows, we will have a generic filter $G \subseteq \mathbb{P}_\tau \cap M^\tau_\alpha$ and a path from M^τ_α to some $M^{\tau_i}_{\alpha_i}$, where $\tau_i \leq \tau$. By Claim 57 it follows that

$$\tau_i \cap M^\tau_\alpha \subseteq M^{\tau_i}_{\alpha_i}.$$

Thus, if $\alpha_i > \alpha$, then the poset $\mathbb{P}_{\tau_i} \cap M^\tau_\alpha$ belongs to $M^{\tau_i}_{\alpha_i}$ and thus filter $G \restriction \tau_i = \{p \restriction \tau_i : p \in G\}$ is also an element in $M^{\tau_i}_{\alpha_i}$ and $M^{\tau_i}_{\alpha_i}$-generic on $\mathbb{P}_{\tau_i} \cap M^\tau_\alpha$.

Lemma 63. *Let $\tau < \kappa^+$. If $p \in \mathbb{P}_\tau$ is super-nice with respect to M^τ_α, $G \subseteq \mathbb{P}_\tau \cap M^\tau_\alpha$ is a generic that contains the trace of p and $\langle((M^{\tau_i}_{\alpha_i}, \tau_i) : i < \nu\rangle$ and $\langle q^i : i < \nu\rangle$ is a p-M.E. pair of sequences such that $\alpha_i > \alpha$ and $q^i \in \mathbb{P}_{\tau_i}/G \restriction \tau_i$ for every $i < \nu$, then there is $q \in \mathbb{P}_\tau/G$ such that $q \leq p$ and $q \restriction \tau_i \leq q^i$ for every $i < \nu$.*

Proof. As the proof of Lemma 60, but using the above observation along with the stronger inductive assumption that a common extension can be found the in the quotient. □

6 No new branches

In this section, we prove that if all chosen names $\dot{S}^\tau$ are names for wide Aronszajn trees, then T does not get a cofinal branch, and so can become a universal wide Aronszajn tree on $\kappa = \aleph_2$.

If p is a condition in $\mathbb{P}_\delta$ and $\alpha \in N^p_\gamma$ for every $\gamma \in \delta \cap M^\tau_\alpha$, then automatically p' obtained from p by letting $N^{p'} := N^p \cup \{(\gamma, \alpha) : \gamma \in \tau \cap M^\tau_\alpha\}$ is a condition in $\mathbb{P}_\tau$ that satisfies $p' \restriction \delta = p$.

Definition 64. Let $\delta \leq \tau$ and let α be such that M^τ_α belongs to the τ-th sequence of side conditions. A condition $q \in \mathbb{P}_\delta$ is *super-nice with respect to* M^τ_α if $\alpha \in N^q_\gamma$ for every $\gamma \in \delta \cap M^\tau_\alpha$ and the condition q' obtained from q by extending the side conditions by replacing every $(\alpha, a_\alpha) \in N^q$ by

$$\begin{cases} (\alpha, a_\alpha \cup \{\gamma\}) & \text{if } \gamma \in \tau \cap M^\tau_\alpha, \\ (\alpha, a_\alpha) & \text{otherwise,} \end{cases}$$

is a condition in $\mathbb{P}_\tau$ that is super-nice with respect to M^τ_α.

Also the trace operator from Definition 39 is defined for conditions in $\mathbb{P}_\delta$ and models M^τ_α, where possibly $\delta < \tau$. It works as expected in the case of this generalised version of super-niceness:

Lemma 65. *Let $\delta \leq \tau$ and let $\alpha < \kappa$. If $p \in \mathbb{P}_\delta$ is super-nice with respect to M^τ_α, then:*

1. *the conditions that are super-nice with respect to M^τ_α are dense below p in $\mathbb{P}_\delta$,*

2. *the trace $[p]_{M^\tau_\alpha}$ is a condition in $\mathbb{P}_\delta \cap M^\tau_\alpha$ and a residue of p into M^τ_α.*

It follows that if $p \in \mathbb{P}_\delta$ is super-nice with respect to M^τ_α, then every generic $G \subseteq \mathbb{P}_\delta \cap M^\tau_\alpha$ that contains $[p]_{M^\tau_\alpha}$ extends to a generic on $\mathbb{P}_\delta$ that contains p.

Lemma 66. *If there is a path from M^τ_α to M^δ_β (see Definition 48), then the trace map from $\mathbb{P}_\delta \cap M^\delta_\beta$ to $\mathbb{P}_\delta \cap M^\tau_\alpha$ is a residue map and element in M^δ_β.*

Proof. Note that if there is a path from M^τ_α to M^δ_β, then $\delta \cap M^\tau_\alpha \subseteq M^\delta_\beta$. Thus, if $p \in \mathbb{P}_\delta \cap M^\delta_\beta$ is super-nice with respect to M^τ_α and $w \in \mathbb{P}_\delta \cap M^\tau_\alpha$ extends $[p]_{M^\tau_\alpha}$, then $w \in M^\delta_\beta$, and thus by elementarity w and p have a common extension in $\mathbb{P}_\delta \cap M^\delta_\beta$. □

Definition 67. Two conditions $p, q \in \mathbb{P}_\tau$ are said to *split a pair of nodes* (s, s') in a tree $\dot{S}$ if there are distinct nodes $\bar{s} \neq \bar{s}'$ of the same height $\bar{\alpha}$ such that

1. $p \Vdash \bar{s} <_{\dot{S}} s$
2. $q \Vdash \bar{s}' <_{\dot{S}} s'$

We say that p and q split the node s if they split the pair (s, s).

Lemma 68. *Let $\tau < \kappa^+$ and suppose that α belongs to the τ-th sequence of side conditions. If $G \subseteq \mathbb{P}_\tau \cap M^\tau_\alpha$ is a generic filter and if $s \in \dot{S}_\tau$ is an exit node from M^τ_α at a limit level, then the branch below s is not introduced by G.*

Proof. Suppose towards contradiction that $b \subseteq (\dot{S}_\tau \cap V_\alpha)^G$ is forced to be the branch below s. By the Π^1_1-reflection, the tree $(\dot{S}_\tau \cap V_\alpha)^G$ is a wide α-Aronszajn tree. If s has height α, then the branch below it cannot be introduced by G, for it would be a cofinal branch in $(\dot{S}_\tau \cap V_\alpha)^G$. Thus the height $\bar{\alpha}$ of s must satisfy $\bar{\alpha} < \alpha$. Look at $\bar{\beta} :=$ the minimal ordinal such that $b \subseteq \bar{\beta} \times \bar{\alpha}$. We have $\bar{\beta} \leq \alpha$ because s is an exit node from M^τ_α.

If $\bar{\beta} = \alpha$, then b induces a cofinal function from $\bar{\alpha}$ to α, which is absurd since $\alpha = \aleph_2^{V[G]}$.

Thus $\bar{\beta} < \alpha$. Let G' be a generic on $\mathbb{P}_\tau$ that extends G. By a standard argument the set $M^\tau_\alpha[G'] = \{\dot{a}^{G'} : \dot{a} \in V^{\mathbb{P}_\tau} \cap M^\tau_\alpha\}$ is an elementary submodel of $H_{\kappa^{++}}[G']$ and closed under $< \alpha$-sequences. Thus $b \in M^\tau_\alpha[G']$. And by elementarity, b must have a supremum in $M^\tau_\alpha[G']$. Since s is at a limit level, it is the unique supremum of b. This is absurd since s is an exit node from $M^\tau_\alpha[G']$. □

Lemma 69. *Suppose that there is a path from M^τ_α to M^δ_β. Suppose that $p \in \mathbb{P}_\delta$ and $q \in \mathbb{P}_\delta$ are conditions that are super-nice with respect to M^τ_α and $[p]_{M^\tau_\alpha} =$*

$[q]_{M^\tau_\alpha}$. *Suppose also that* p *is super-nice with respect to* M^δ_β. *For any countable sets* $A, B \subseteq \dot{S}_\delta$ *of exit nodes from* M^δ_β, *there are two conditions* $\hat{p} \leq p$ *and* $\hat{q} \leq q$ *that split every pair in* $A \times B$, *are super-nice with respect to* M^τ_α *and* $[p]_{M^\tau_\alpha} = [q]_{M^\tau_\alpha}$. *Furthermore, for any* $\beta' > \beta$ *such that* $A \subseteq M^\delta_{\beta'}$, *if* $p \in M^\delta_{\beta'}$, *then we may assume that* $\hat{p} \in M^\delta_{\beta'}$.

Proof. Fix a generic $G \subseteq \mathbb{P}_\delta \cap M^\tau_\alpha$ that contains the common trace of p and q. Enumerate $A \times B$ as $(a_0, b_0), (a_1, b_1), \ldots$. By recursion on n, define conditions p_n and q_n in $\mathbb{P}_\delta$. Let $p_0 := p$ and $q_0 := q$. At step $n+1$, assume that we have defined p_n and q_n and they satify:

- p_n and q_n are super-nice with respect to M^τ_α and $[p_n]_{M^\tau_\alpha} = [q_n]_{M^\tau_\alpha}$,
- p_n and q_n split every pair (a_i, b_i), $i < n$,
- p_n is super-nice with respect to M^δ_β and $p_n \in M^\delta_{\beta'}$.

Look at the node a_n. The branch below a_n is not introduced by $\mathbb{P}_\delta \cap M^\delta_\beta$ by Lemma 68, so it is not introduced by $\mathbb{P}_\delta \cap M^\tau_\alpha$ either. Thus there are $p^L, p^R \leq p_n$ in $\mathbb{P}_\delta / G$ that split a_n, at some level $\bar{\beta} < \beta$ with some distinct nodes a^L and a^R. Since $A \subseteq M^\delta_{\beta'}$, by elementarity we may choose $p^L, p^R \in M^\delta_{\beta'}$. Find $q_{n+1} \leq q_n$ in $\mathbb{P}_\delta / G$ that decides the predecessor $\bar{b}$ of b_n at height $\bar{\beta}$. If $\bar{b} \neq a^L$, let $p_{n+1} := p^L$ and otherwise let $p_{n+1} := p^R$. Finally, let $\hat{p}$ be the pointwise union of the p_n and let $\hat{q}$ be the pointwise union of the q_n. They are both in $\mathbb{P}_\delta / G$ so they can be assumed to be super-nice with respect to M^τ_α and $[\hat{p}]_{M^\tau_\alpha} = [\hat{q}]_{M^\tau_\alpha}$. They split every pair in $A \times B$, as desired. Furthermore, $\hat{p} \in M^\delta_{\beta'}$, since $M^\delta_{\beta'}$ is closed for countable sequences. □

Lemma 70. *Let* $\tau < \kappa^+$. *For every* $p \in \mathbb{P}_\tau$ *that is super-nice with respect to* M^τ_α *there is* $q \leq p$ *that is super-nice with respect to* M^τ_α *and satisfies that if* $\langle (M^{\tau_i}_{\alpha_i}, \tau_i) : i < \nu \rangle$ *is the* p'*-closure of* $\langle (M^\tau_\alpha, \tau) \rangle$, *then*

$$\sup\{\alpha_i : i < \nu\} = \sup\{\alpha_i : \tau_i \in \tau \cap M^\tau_\alpha, i < \nu\}.$$

Moreover, this supremum can be chosen as high in κ *as wanted.*

Proof. By adding nodes to coordinates in M^τ_α using Node density Lemma. □

Definition 71. Let $\delta \leq \tau$. A pair of conditions (p, q) from $\mathbb{P}_\delta$ *nicely splits with respect to* M^τ_α if

1. $\alpha \in N^p_\gamma \cap N^q_\gamma$ for every $\gamma \in \delta \cap M^\tau_\alpha$,
2. for every $\gamma \in \tau \cap M^\tau_\alpha$ and every pair (s, s') of exit nodes from M^γ_α at limit levels where $s \in \operatorname{dom}(f^p_\gamma)$ and $s' \in \operatorname{dom}(f^q_\gamma)$, the conditions $p \restriction \gamma$ and $q \restriction \gamma$ split the pair (s, s') with a pair $(\bar{s}, \bar{s}')$ where $\bar{s} \in \operatorname{dom}(f^p_\gamma)$ and $\bar{s}' \in \operatorname{dom}(f^q_\gamma)$.

3. for every $\gamma \in \tau \cap M^\tau_\alpha$ and every $s \in \mathrm{dom}(f^p_\gamma)$ (resp. $s \in \mathrm{dom}(f^q_\gamma)$) exit node from M^γ_α at a successor level, the condition $p \restriction \gamma$ (resp. $q \restriction \gamma$) decides the immediate predecessor $\bar{s}$ of s and $\bar{s} \in \mathrm{dom}(f^p_\gamma)$ (resp. $\bar{s} \in \mathrm{dom}(f^q_\gamma)$).

Lemma 72. *Let $\tau < \kappa^+$, $\alpha < \kappa$ and $p, q \in \mathbb{P}_\tau$.*

1. *If p and q nicely split with respect to M^τ_α and $[p]_{M^\tau_\alpha} = [q]_{M^\tau_\alpha}$ then there are $\hat{p} \leq p$ and $\hat{q} \leq q$ that are super-nice with respect to M^τ_α and $[\hat{p}]_{M^\tau_\alpha} = [\hat{q}]_{M^\tau_\alpha}$.*

2. *If p and q are super-nice with respect to M^τ_α and $[p]_{M^\tau_\alpha} = [q]_{M^\tau_\alpha}$, then there are $\hat{p} \leq p$ and $\hat{q} \leq q$ that nicely split with respect to M^τ_α and $[\hat{p}]_{M^\tau_\alpha} = [\hat{q}]_{M^\tau_\alpha}$. Moreover, for any $t \in T$ that is an exit node from M^τ_α at a limit level, we may choose $\hat{p}$ and $\hat{q}$ such that they also split t.*

3. *$\dot{T}$ is a wide κ-Aronszajn tree in $V^{\mathbb{P}_\tau}$.*

Proof. The proof is by induction on τ.

Remark 73. *If Lemma 72 holds for τ, then any pair of conditions $p, q \in \mathbb{P}_\tau$ having the same trace to M^τ_α satisfies that if either they nicely split or are super-nice with respect to M^τ_α, then there are extensions that still have the same trace to M^τ_α and both nicely split and are super-nice with respect to M^τ_α. This can be proved by iterating ω many times items (1) and (2).*

Proof of item 1.

We look at the successor case $\tau + 1$ first. Let $p, q \in \mathbb{P}_{\tau+1}$ be such that they have a common trace to $M^{\tau+1}_\alpha$ and nicely split with respect to it. By induction hypothesis and the above remark we may assume that $p \restriction \tau$ and $q \restriction \tau$ nicely split and are super-nice with respect to M^τ_α.

Let $G \subseteq \mathbb{P}_\tau \cap M^\tau_\alpha$ be a generic that contains the common trace of $p \restriction \tau$ and $q \restriction \tau$. Then $p \restriction \tau, q \restriction \tau \in \mathbb{P}_\tau / G$ and the common trace of f^p_τ and f^q_τ to M^τ_α is a meet- and level-preserving tree-embedding from $(\dot{S}_\tau \cap V_\alpha)^G$ to $(\dot{T} \cap V_\alpha)^G$. In the quotient, find $p_0, q_0 \in \mathbb{P}_\tau / G$ such that

- $p_0 \leq p \restriction \tau$ and $q_0 \leq q \restriction \tau$,
- p_0 is super-nice with respect to every M^τ_β where $\beta \in E^p_\tau$ and q_0 is super-nice with respect to every M^τ_β where $\beta \in E^q_\tau$,
- p_0 decides, for every pair $(s, t) \in f^p_\tau$ of exit nodes from V_α, cofinal implicit preimages for the node projection $\dot{\pi}^{p_0}(t)$, i.e. there are nodes $(\bar{t}^p_n : n < \omega)$ cofinal in $\dot{\pi}^{p_0}(t)$ and p_0 decides, for each n, the node $\bar{s}^p_n$ below s at the height of $\bar{t}^p_n$.
- same for q_0: q_0 decides, for every pair $(s, t) \in f^p_\tau$ of exit nodes from V_α, cofinal implicit preimages for the node projection $\dot{\pi}^{q_0}(t)$, i.e. there are nodes $(\bar{t}^q_n : n < \omega)$ cofinal in $\dot{\pi}^{q_0}(t)$ and q_0 decides, for each n, the node $\bar{s}^q_n$ below s at the height of $\bar{t}^q_n$.

- the conditions p_0 and q_0 nicely split and are super-nice with respect to M^τ_α.

For every $s \in \mathrm{dom}(f^p_\tau)$ and $s' \in \mathrm{dom}(f^q_\tau)$, look at $\bar{s}$ and $\bar{s}'$ in V_α that witness the splitting, i.e. $p_0 \Vdash \bar{s} < s$ and $q_0 \Vdash \bar{s}' < s'$. Then the meet of the nodes $\bar{s}$ and $\bar{s}'$ is decided by G, so in particular p_0 and q_0 decide it the same way. Furthermore, for any n and m we have $\bar{s}^p_n \wedge \bar{s}'^q_m = \bar{s} \wedge \bar{s}'$ and moreover,

$$\begin{aligned} p_0 &\Vdash s \wedge \bar{s}'^q_m = \bar{s} \wedge \bar{s}', \\ q_0 &\Vdash \bar{s}^p_n \wedge s' = \bar{s} \wedge \bar{s}'. \end{aligned}$$

We also have that the height of the node

$$f^p_\tau(\bar{s}) \wedge f^q_\tau(\bar{s}')$$

in the tree $(\dot{T} \cap V_\alpha)^G$ is exactly the height of the node $\bar{s} \wedge \bar{s}'$ in the tree $(\dot{S}_\tau \cap V_\alpha)^G$. This follows from the fact that the trace are the same,

$$[p_0{}^\frown(f^p_\tau, N^p_\tau)]_{M^{\tau+1}_\alpha} = [p_0{}^\frown(f^p_\tau, N^p_\tau)]_{M^{\tau+1}_\alpha}$$

so in particular the nodes $\bar{s}$ and $\bar{s}'$ belong to the τth embedding of the common trace, call it $f := f^p_\tau \cap V_\alpha = f^q_\tau \cap V_\alpha$. Thus, since f is meet-preserving, we have for any $(s,t) \in f^p_\tau$ and $(s',t') \in f^q_\tau$, that

$$f(\bar{s} \wedge \bar{s}') = \bar{t}^p_m \wedge \bar{t}'^q_m.$$

Hence, the condition p_0 decides meets in the set

$$\begin{aligned} \mathrm{dom}(f^p_\tau) \cup \{\bar{s}^p_n : n < \omega, s \in \mathrm{dom}(f^p_\tau) \text{ exit node from } V_\alpha\} \\ \cup \{\bar{s}'^q_n : n < \omega, s' \in \mathrm{dom}(f^q_\tau) \text{ exit node from } V_\alpha\} \end{aligned}$$

and the condition q_0 decides meets in the set

$$\begin{aligned} \mathrm{dom}(f^q_\tau) \cup \{\bar{s}^p_n : n < \omega, s \in \mathrm{dom}(f^p_\tau) \text{ exit node from } V_\alpha\} \\ \cup \{\bar{s}'^q_n : n < \omega, s' \in \mathrm{dom}(f^q_\tau) \text{ exit node from } V_\alpha\}. \end{aligned}$$

and the meets are already in the domain of the function f. Hence we may let

$$\begin{aligned} \hat{f}^p &:= f^p_\tau \cup \{(\bar{s}^p_n, \bar{t}^p_n) : (s,t) \in f^p_\tau\} \\ &\qquad \cup \{(\bar{s}'^q_m, \bar{t}'^q_m) : (s',t') \in f^q_\tau\} \\ \hat{f}^q &:= f^q_\tau \cup \{(\bar{s}^p_n, \bar{t}^p_n) : (s,t) \in f^p_\tau\} \\ &\qquad \cup \{(\bar{s}'^q_m, \bar{t}'^q_m) : (s',t') \in f^q_\tau\}. \end{aligned}$$

It follows from above that both $\hat{f}^p$ and $\hat{f}^q$ are injective functions, p_0 forces that the domain of the function $\hat{f}^p$ is closed under meets and is level- and meet-preserving, and q_0 forces that the function $\hat{f}^q$ is closed under meets and is level- and meet-preserving. It also follows that

$$\begin{aligned} \hat{p} &:= p_0{}^\frown(\hat{f}^p, N^p_\tau), \\ \hat{q} &:= q_0{}^\frown(\hat{f}^q, N^q_\tau) \end{aligned}$$

are conditions in $\mathbb{P}_{\tau+1}$ that are super-nice with respect to $M_\alpha^{\tau+1}$ and have the same trace to it.

We consider the case when τ is a limit ordinal of countable cofinality. This is straightforward. Let $p, q \in \mathbb{P}_\tau$ nicely split with respect to M_α^τ satisfying $[p]_{M_\alpha^\tau} = [q]_{M_\alpha^\tau}$. Fix a cofinal sequence $(\tau_j)_{j<\omega}$ converging to τ. Proceed by recursion on $j < \omega$. Let $p_0 := p \restriction \tau_0$ and $q_0 := q \restriction \tau_0$. At step $j+1$, look at p_j and q_j. Suppose that they nicely split and are super-nice with respect to M_α^τ. Then $p' := p_j {}^\frown p \restriction [\tau_j, \tau_{j+1})$ and $q' := q_j {}^\frown q \restriction [\tau_j, \tau_{j+1})$ nicely split and satisfy $[p']_{M_\alpha^\tau} = [q']_{M_\alpha^\tau}$. By induction hypothesis there are $p_{j+1} \leq p'$ and $q_{j+1} \leq q'$ that nicely split and are super-nice with respect to M_α^τ and satisfy $[p_{j+1}]_{M_\alpha^\tau} = [q_{j+1}]_{M_\alpha^\tau}$. Finally the pointwise unions of the p_j and q_j, respectively, are as wanted.

The limit case of uncountable cofinality is a straightforward implication of induction hypothesis.

Proof of item 2.

We prove by induction on τ: *If p and q are super-nice with respect to M_α^τ and have the same trace to it, and $t \in T$ is an exit node from V_α at a limit level, then there are $\hat{p} \leq p$ and $\hat{q} \leq q$ that nicely split with respect to M_α^τ, have the same trace to it, and split t, and furthermore, for any β it holds that if $p \in M_\beta^\tau$, then $\hat{p} \in M_\beta^\tau$.*

Suppose that p and q are super-nice with respect to M_α^τ and have the same trace to it. Let $t \in T$ be an exit node from M_α^τ at a limit level. We find $\hat{p} \leq p$ and $\hat{q} \leq q$ that split t and nicely split with respect to M_α^τ, and satisfy $[\hat{p}]_{M_\alpha^\tau} = [\hat{q}]_{M_\alpha^\tau}$, and moreover such that $\hat{p} \in M_\beta^\tau$ for any large enough β such that $p \in M_\beta^\tau$.

We may assume without loss of generality that for every $\gamma \in \tau \cap M_\alpha^\tau$ and $s \in \mathrm{dom}(f_\gamma^p)$ (resp. $s \in \mathrm{dom}(f_\gamma^q)$) that is an exit node from M_α^τ at a successor level, the condition $p \restriction \gamma$ (resp. $q \restriction \gamma$) decides the immediate predecessor $\bar{s}$ of s and $\bar{s} \in \mathrm{dom}(f_\gamma^p)$ (resp. $\bar{s} \in \mathrm{dom}(f_\gamma^q)$). This assumption can be made by fixing a generic $G \subseteq \mathbb{P}_\tau \cap M_\alpha^\tau$ containing the common trace of p and q and extending p and q in the quotient $\mathbb{P}_\tau / G$ using Node density. Similarly, if $\langle (M_{\alpha_i}^{\tau_i}, \tau_i) : i < \nu \rangle$ is the p-closure of M_α^τ, up to extending p in the quotient we may assume that

$$\sup\{\alpha_i : i < \nu\} = \sup\{\alpha_i : i < \nu \text{ and } \tau_i \in M_\alpha^\tau\},$$

and moreover, that $t \in V_{\alpha_i}$ for some $i < \nu$. See Lemma 70.

Fix thus such p and q and let $\langle (M_{\alpha_i}^{\tau_i}, \tau_i) : i < \nu \rangle$ be the p-closure of M_α^τ. Denote $\tau_\nu := \tau$ and $\alpha_\nu := \kappa$. By recursion on $j \leq \nu$, we define conditions $p_j \in \mathbb{P}_{\tau_j} \cap M_{\alpha_j}^{\tau_j}$ and conditions $q_j \in \mathbb{P}_\tau$. The final conditions p_ν and q_ν will nicely split with repect to M_α^τ, split t, and satisfy $[p_\nu]_{M_\alpha^\tau} = [q_\nu]_{M_\alpha^\tau}$. Let $p_0 := [p]_{M_\alpha^\tau}$ and $q_0 := q$.

At step *j*: Assume that we have defined p_i and q_i for $i < j$ and they satisfy:

1. $(p_i \restriction \min\{\tau_i, \tau_j\} : i < j)$ forms a $[p \restriction \tau_j]_{M_{\alpha_j}^{\tau_j}}$-M.E.,

2. $(q_i : i < j)$ is a decreasing sequence of conditions in $\mathbb{P}_\tau$,

3. p_i and $q_i \restriction \tau_i$ nicely split and are super nice with respect to M^τ_α and satisfy $[p_i]_{M^\tau_\alpha} = [q_i \restriction \tau_i]_{M^\tau_\alpha}$,

4. If $\tau_i \in \tau \cap M^\tau_\alpha$ and $t \in V_{\alpha_i}$, then p_i and $q_i \restriction \tau_i$ split t,

5. If $\tau_i \in \tau \cap M^\tau_\alpha$, then for every pair $(s, s') \in A_i \times B_i$, where

$$A_i = \{s \in \mathrm{dom}(f^p_{\tau_i}) : s \text{ is an exit node from } V_\alpha\} - (V_{\alpha_i} - \bigcup_{i'<i} V_{\alpha_{i'}}),$$

$$B_i = \{s \in \mathrm{dom}(f^{q_i}_{\tau_i}) : s \text{ is an exit node from } V_\alpha\},$$

   p_i and $q_i \restriction \tau_i$ split the pair (s, s') with a pair $(\bar{s}, \bar{s}')$. Also, the common trace $[p_i]_{M^\tau_\alpha} = [q_i \restriction \tau_i]_{M^\tau_\alpha}$ decides the meet $\bar{s} \wedge \bar{s}'$ and p_i forces

$$s \wedge \bar{s}' = \bar{s} \wedge \bar{s}'.$$

   Furthermore, $\bar{s}, \bar{s}' \in \mathrm{dom}(f^{q_i}_{\tau_i})$, and if $i' \in [i+1, j)$ is such that $f^p_{\tau_i}(s) \in V_{\alpha_{i'}}$, then $p_{i'}$ forces that

$$f^{q_i}_{\tau_i}(\bar{s}) <_{\dot{T}} f^p_{\tau_i}(s).$$

6. The extension $q_i \leq \bigcup_{i'<i} q_{i'}$ is minimal in the sense that we only extended the part of q_i up to τ_i and the function $f^{q_i}_{\tau_i}$ below α by adding witnesses for splitting. Specifically:

   - $q_i \restriction [\tau_i + 1, \tau) = \bigcup_{i'<i} q_{i'} \restriction [\tau_i + 1, \tau)$,
   - $N^{q_i}_{\tau_i} = \bigcup_{i'<i} N^{q_{i'}}_{\tau_i}$,
   - $f^{q_i}_{\tau_i} - V_\alpha = \bigcup_{i'<i} f^{q_{i'}}_{\tau_i} - V_\alpha$,
   - $q_i \restriction \tau_i \Vdash \mathrm{dom}(f^{q_i}_{\tau_i})$ is the least set closed under meets and exit nodes from models in $\bigcup_{i'<i} N^{q_{i'}}_{\tau_{i'}}$ that contains $\bigcup_{i'<i} f^{q_{i'}}_{\tau_{i'}}$ and $\{\bar{s}, \bar{s}' : (s, s') \in A_i \times B_i\}$.

We define p_j and q_j.

Claim 74. There is $\tilde{p} \leq [p \restriction \tau_j]_{M^{\tau_j}_{\alpha_j}}$ in $\mathbb{P}_{\tau_j} \cap M^{\tau_j}_{\alpha_j}$ such that $\tilde{p}$ is super-nice with respect to $M^{\tau_i}_{\alpha_i}$ and $[\tilde{p} \restriction \min\{\tau_j, \tau_i\}]_{M^{\tau_i}_{\alpha_i}} = [p \restriction \min\{\tau_j, \tau_i\}]_{M^{\tau_i}_{\alpha_i}}$ for cofinally many $i < j$, and such that for any $i < j$ and any $(s, s') \in A_i \times B_i$, if $f^p_{\tau_i}(s) \in V_{\alpha_j}$, then

$$\tilde{p} \Vdash f^{q_i}_{\tau_i}(\bar{s}) <_{\dot{T}} f^p_{\tau_i}(s).$$

Proof. By Lemma 49. $\square$

Let $\tilde{p}$ be as in Claim 74.

Claim 75. There are conditions $p_j \in \mathbb{P}_{\tau_j} \cap M^{\tau_j}_{\alpha_j}$ and $q_j \in \mathbb{P}_\tau$ such that:

1. $p_j \leq \tilde{p}$ and $p_j \leq p_i \restriction \tau_j$ for every $i < j$,

2. q_j extends every q_i, $i < j$, and agrees with their pointwise union after τ_j, i.e. $q_j \upharpoonright [\tau_j, \tau) = \bigcup_{i<j} q_i \upharpoonright [\tau_j, \tau)$,

3. p_j and $q_j \upharpoonright \tau_j$ nicely split with respect to M^τ_α and $[p_j]_{M^\tau_\alpha} = [q_j \upharpoonright \tau_j]_{M^\tau_\alpha}$.

Proof of Claim 75. Suppose first that $\tau_j < \limsup_{i<j} \tau_i$. Let q_j be the pointwise union of the q_i, $i < j$. Let $G \subseteq \mathbb{P}_{\tau_j} \cap M^\tau_\alpha$ be a generic filter that contains the common trace $[p_i \upharpoonright \tau_j]_{M^\tau_\alpha} = [q_i \upharpoonright \tau_j]_{M^\tau_\alpha}$ for every $i < j$. Then q_j is in the quotient $\mathbb{P}_{\tau_j}/G$, and so is every $p_i \upharpoonright \tau_j$. By strong properness for τ_j-sequences in the quotient, Lemma 63, there is a condition $p_j \in (\mathbb{P}_{\tau_j} \cap M^{\tau_j}_{\alpha_j})/G$ that extends $\tilde{p}$ and each $p_i \upharpoonright \tau_j$, $i < j$. Up to extending $q_j \upharpoonright \tau_j$ and p_j in the quotient $\mathbb{P}_{\tau_j}/G$, we may assume that they are super-nice with respect to M^τ_α and have the same trace to it, $[p_j]_{M^\tau_\alpha} = [q_j \upharpoonright \tau_j]_{M^\tau_\alpha}$. By induction hypothesis for τ_j we may assume further that they nicely split with respect to M^τ_α. Then p_j and q_j are as wanted.

Suppose then that $\tau_j \geq \limsup_{i<j} \tau_i$. Again, let q_j be the pointwise union of the q_i, $i < j$. Let p_j be the condition obtained by taking the pointwise union of the conditions $p_i \upharpoonright \tau_j$, $i < j$ and $[p \upharpoonright \tau_j]_{M^{\tau_j}_{\alpha_j}}$, and furthermore, whenever $\gamma \in \tau_j \cap M^\tau_\alpha \cap M^{\tau_j}_{\alpha_j}$, then the γ-th coordinate is

$$f_\gamma := (f^{q_i}_\gamma \cap V_\alpha) \cup \bigcup_{i<j} \bigcup_{\gamma \in \operatorname{supp}(p_i)} f^{p_i}_\gamma \cup (f^p_\gamma \cap V_{\alpha_j}).$$

It follows by construction, as in the proof of strong properness (at step q^η of proof of Lemma 60), that p_j is a condition in $\mathbb{P}_{\tau_j} \cap M^{\tau_j}_{\alpha_j}$, and it follows by construction that p_j and $q_j \upharpoonright \tau_j$ nicely split and $[p_j]_{M^\tau_\alpha} = [q_j]_{M^\tau_\alpha}$.

$\square$

Let p_j and q_j be as in Claim 75. Up to extending p_j inside $\mathbb{P}_{\tau_j} \cap M^{\tau_j}_{\alpha_j}$, we may assume that $(p_i \upharpoonright \min\{\tau_i, \tau_{j+1}\} : i < j+1)$ forms a $[p \upharpoonright \tau_{j+1}]_{M^{\tau_{j+1}}_{\alpha_{j+1}}}$-Multi-Extension.

If $j = \nu$, we are done.
If $\tau_j \notin M^\tau_\alpha$, we keep p_j and q_j as they are.
If $j < \nu$ and $\tau_j \in M^\tau_\alpha$, we make two more extensions to p_j and q_j.

Since $j < \nu$, we have $\tau_j < \tau$. Thus by induction hypothesis, we may assume that p_j and q_j are super-nice with respect to M^τ_α, in addition to nicely splitting and having the same trace to M^τ_α.

Let $G \subseteq \mathbb{P}_{\tau_j} \cap M^\tau_\alpha$ be a generic filter that contains the common trace of p_j and $[q_j \upharpoonright \tau_j]_{M^\tau_\alpha}$. Then $p_j, q_j \upharpoonright \tau_j \in \mathbb{P}_{\tau_j}/G$.

Claim 76. If $t \in V_{\alpha_j}$, then there are extensions of p_j and $q_j \upharpoonright \tau_j$ that split t in $\mathbb{P}_{\tau_j}/G$ such that $p_j \in M^{\tau_j}_{\alpha_j}$.

Proof of Claim 76. By assumption $\tau_j < \tau$, so by induction hypothesis the tree $\dot{T}$ is a wide κ-Aronszajn tree in $V^{\mathbb{P}_{\tau_j}}$. As in Lemma 68, we argue that the branch below t cannot be introduced by $\mathbb{P}_{\tau_j} \cap M^\tau_\alpha$: it follows from the case assumption that $\mathbb{P}_{\tau_j} \in M^\tau_\alpha$, so in particular M^τ_α reflects the fact that $\Vdash_{\mathbb{P}_{\tau_j}} \dot{T}$

is wide κ-Aronszajn tree. Hence the branch below t cannot be introduced by $\mathbb{P}_{\tau_j} \cap M^\tau_\alpha$. There are two extensions $p^L, p^R \leq p_j$ in $\mathbb{P}_{\tau_j}/G$ that split t at some level $\bar{\alpha} < \alpha$ with some nodes $t^L \neq t^R$. Extend $q' \leq q_j \restriction \tau_j$ in $\mathbb{P}_{\tau_j}/G$ to decide the predecessor of t at level $\bar{\alpha}$, call it $\bar{t}$. If $\bar{t} \neq t^L$, then p^L and $q'^\frown q_j \restriction [\tau_j, \tau)$ are as wanted and otherwise $\bar{t} \neq t^R$ and p^R and $q'^\frown q_j \restriction [\tau_j, \tau)$ are as wanted. $\square$

By Claim 76, we assume that p_j and $q_j \restriction \tau_j$ split t in case $t \in V_{\alpha_j}$. We make a last extension inside the quotient $\mathbb{P}_{\tau_j}/G$. Let

$$A_j = \{s \in \mathrm{dom}(f^p_{\tau_j}) : s \text{ is an exit node from } V_\alpha\} - (V_{\alpha_j} - \bigcup_{i<j} V_{\alpha_i}),$$
$$B_j = \{s \in \mathrm{dom}(f^{q_j}_{\tau_j}) : s \text{ is an exit node from } V_\alpha\}.$$

Up to extending p_j and $q_j \restriction \tau_j$ in $(\mathbb{P}_{\tau_j} \cap M^{\tau_j}_{\alpha_j})/G$ and $\mathbb{P}_{\tau_j}/G$, respectively, we may assume that they split every pair $(s, s') \in A_j \times B_j$ with some pair $(\bar{s}, \bar{s}')$ of distinct nodes of same height. For every $i < j$ such that $\tau_j \in \mathrm{supp}(p_i)$, consider the function $f^{p_i}_{\tau_j}$. We have

$$f^{p_i}_{\tau_j} \cap V_\alpha = f^{q_i}_{\tau_j} \cap V_\alpha.$$

Let

$$f := \bigcup_{i<j, \tau_j \in \mathrm{supp}(p_i)} f^{p_i}_{\tau_j} \cap V_\alpha.$$

It follows that $f = f^{q_j}_{\tau_j} \cap V_\alpha$, and further, that f is a level- and meet-preserving tree-embedding

$$f : (\dot{S}_{\tau_j} \cap V_\alpha)^G \to (\dot{T} \cap V_\alpha)^G.$$

Note that $\tau_j + 1 \leq \tau$ and $\tau_{j+1} \in M^\tau_\alpha$. By Node density Lemma applied in the poset $(\mathbb{P}_{\tau_j+1} \cap M^\tau_\alpha)/G$, there is a function $f' \supseteq f$ whose domain is the closure under meets of the set

$$\mathrm{dom}(f) \cup \{\bar{s}, \bar{s}' : (s, s') \in A_j \times B_j\},$$

and which is level- and meet-preserving tree-embedding from $(\dot{S}_{\tau_j} \cap V_\alpha)^G$ to $(\dot{T} \cap V_\alpha)^G$.

Note also that if $(s, s') \in A_j \times B_j$, then , the meet $\bar{s} \wedge \bar{s}'$ is decided by the generic G, since $\bar{s}, \bar{s}' \in (\dot{S}_{\tau_j} \cap V_\alpha)^G$, and furthermore

$$p_j \Vdash s \wedge \bar{s}' = \bar{s} \wedge \bar{s}',$$
$$q_j \restriction \tau_j \Vdash \bar{s} \wedge s' = \bar{s} \wedge \bar{s}'.$$

This implies in particular, that if $s \neq s'$, then p_j forces that $\bar{s}'$ is not below s. And further, that p_j decides the meets in the set

$$\mathrm{dom}(f') \cup (\mathrm{dom}(f^p_{\tau_j}) \cap V_{\alpha_j}) \cup \bigcup_{i<j, \tau_j \in \mathrm{supp}(p_i)} f^{p_i}_{\tau_j}.$$

Recall that if $s \in A_j$, then $f^p_{\tau_j}(s) \notin M^{\tau_j}_{\alpha_j}$, so in particular we are free to extend the collapse coordinate of p to force $f'(\bar{s} \wedge \bar{s}') < f^p_{\tau_j}(s)$ and $f'(\bar{s}') \not< f^p_{\tau_j}(s)$ at the least step $j' > j$ when it happens that $f^p_{\tau_j}(s) \in V_{\alpha_{j'}}$.

Up to extending p_j and $q_j \upharpoonright \tau_j$ in $(\mathbb{P}_{\tau_j}/G) \cap M^{\tau_j}_{\alpha_j}$ and $\mathbb{P}_{\tau_j}/G$, respectively, we may assume that the common trace $[p_j]_{M^\tau_\alpha} = [q_j \upharpoonright \tau_j]_{M^\tau_\alpha}$ decides meets in $\mathrm{dom}(f')$ and forces that f' is level- and meet-preserving tree-embedding from $\dot{S}_{\tau_j}$ to $\dot{T}$. Then, up to extending q_j at the coordinate τ_j, we may assume that

$$f^{q_j}_{\tau_j} = f' \cup \bigcup_{i<j} f^{q_i}_{\tau_j}.$$

Then p_j and q_j are as wanted. This ends item (2).

Proof of item 3.

Assume that $\mathbb{P}_\tau$ introduces a cofinal branch to $\dot{T}$. Fix a $\mathbb{P}_\tau$-name $\dot{b}$ and a condition p such that $p \Vdash \dot{b}$ is a cofinal branch in $\dot{T}$. Find α such that $p, \dot{b} \in M^\tau_\alpha$ and $q \leq p$ which is super-nice with respect to M^τ_α. Up to extending q, assume that it forces $\dot{b}(\alpha) = t$ for some node $t \in T$. By item 2 applied to the pair $q_1 := q_2 := q$, there are $\hat{q}_1 \leq q_1$ and $\hat{q}_2 \leq q_2$ that are super-nice with respect to M^τ_α, have the same trace $[\hat{q}_1]_{M^\tau_\alpha} = [\hat{q}_2]_{M^\tau_\alpha}$, and split t with some distinct nodes t^L and t^R at some level $\bar{\alpha} < \alpha$. The the common trace $[\hat{q}_1]_{M^\tau_\alpha} = [\hat{q}_2]_{M^\tau_\alpha}$ is a common residue for $\hat{q}_1$ and $\hat{q}_2$. Let $w \in \mathbb{P}_\tau \cap M^\tau_\alpha$ extend the common trace to decide the $\bar{\alpha}$-th node on the branch, say it forces $\dot{b}(\bar{\alpha}) = \bar{t}$. If $\bar{t} \neq t^L$, then w cannot be compatible with $\hat{q}_1$ and if $\bar{t} \neq t^R$, then w cannot be compatible with $\hat{q}_2$. This is in contradiction with the fact that the common trace was a common residue for $\hat{q}_1$ and $\hat{q}_2$.

$\square$

Theorem 77. *Suppose that the book-keeping function Ψ is such that for each $\tau < \kappa^+$, the name S_τ is a $\mathbb{P}_\tau$-name for q (wide) κ-Aronszajn tree. Then no cofinal branches are added to the wide tree T by $\mathbb{P}_{\kappa^+}$.*

Proof. By Lemma 72 item (3) and κ^+-cc of $\mathbb{P}_{\kappa^+}$. $\square$

Theorem 78. *Suppose that the book-keeping function Ψ which picks the trees S_τ, picks only names for wide κ-Aronszajn trees and covers all $\mathbb{P}_\tau$-names for wide κ-Aronszajn trees, for all $\tau < \kappa^+$. Let $G \subseteq \mathbb{P}_{\kappa^+}$ be a generic filter. In $V[G]$, $\kappa = \aleph_2$, T is a wide $\aleph_2$-Aronszajn tree on κ which embeds all wide $\aleph_2$-Aronszajn trees.*

Proof. It follows from the proof 61 that T embeds all wide $\aleph_2$-Aronszajn trees on κ. By the last theorem T does not get a cofinal branch, and hence remains $\aleph_2$-Aronszajn. $\square$

7 Open Problems

The following problems are left open by this work:

1. Is the weakly compact cardinal needed for Theorem 1?
   We conjecture that the answer is yes.

2. Is it consistent to have a universal (narrow) Aronszajn tree?
   For $\aleph_2$-Aronszajn trees, we expect this to be possible by incorporating ideas from Mitchell's construction of a model without Aronszajn trees on ω_2 and work towards verifying the details. The case of $\aleph_1$-Aronszajn trees remains unclear.

3. Can the universality result for wide Aronszajn trees hold at successors of singular cardinals?
   A positive answer would likely require developing new methods for iteration at successors of singular cardinals.

4. Can one consistently have a maximal wide $\aleph_2$-Aronszajn and, at the same time, a maximal wide $\aleph_3$-Aronszajn tree?

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