

LAVER-SHELAH ITERATION

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ABSTRACT. My personal notes from Laver and Shelah [3]: assuming a weakly compact cardinal, it is consistent that all \aleph_2 -Aronszajn trees are special and CH holds.

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The goal is to produce a model of set theory where there are \aleph_2 -Aronszajn trees and all of them are special. An \aleph_2 -tree T is **special** if there is a function $f : T \rightarrow \omega_1$ such that

$$f(s) = f(t) \implies s \sqsubset t$$

for all distinct nodes $s, t \in T$. We start by assuming GCH and the existence of a weakly compact cardinal κ and force over this model. The poset is an iteration $(\mathbb{P}_\delta : \delta \leq \kappa^+)$ such that the first poset collapses κ to \aleph_2 and the rest specialize all κ -Aronszajn trees. The existence of \aleph_2 -Aronszajn trees in the final model is guaranteed by preserving CH along the iteration.

Indeed, the first poset \mathbb{P}_0 will be the collapse $\text{Coll}(\omega_1, < \kappa)$, which consists of countable partial functions $p : \omega_1 \times \kappa \rightarrow \kappa$ such that $p(\alpha, \beta) \in \beta$ for all $\alpha \in \omega_1$ and $\beta \in \kappa$. This poset creates κ -Aronszajn trees, since κ becomes \aleph_2 and CH holds in $V^{\mathbb{P}_0}$. Each iterand $\dot{\mathbb{Q}}_\delta$ will be a \mathbb{P}_δ -name for the poset of countable approximations of a specialising function for a tree given by a bookkeeping function. The final poset \mathbb{P}_{κ^+} will be σ -closed and have κ -cc. The σ -closure guarantees preservation of CH and κ -cc guarantees preservation of κ .

Furthermore, each poset \mathbb{P}_δ for $\delta < \kappa^+$ will be **κ -strongly proper**: strongly proper with respect to stationarily many models of size $< \kappa$. By κ -strong properness, lots of new subsets of ω_1 are added, so the final model will satisfy $2^{\omega_1} = \omega_3$.

Note that it is *not possible* to specialize a κ -Aronszajn tree T which is in V with κ -strongly proper poset. Yet, the posets in this proof are κ -strongly proper. This is not a problem, since there are no κ -Aronszajn trees in V , since κ is weakly compact. All the trees appear along the iteration. This implies, in particular, that

the quotient posets $\mathbb{P}_\delta/\mathbb{P}_\gamma$ are not κ -strongly proper for any $\gamma < \delta < \kappa^+$, as they indeed do specialize κ -Aronszajn trees.

The fact that there are no κ -Aronszajn trees in V allows to perform a Mitchell-style splitting argument, which is described in detail in Lemma 4.8. The argument here does not generalize to wide \aleph_2 -Aronszajn trees, since this splitting lemma fails for them.

1. STRONG PROPERNESS

Definition 1.1. Let \mathbb{P} be a poset and let M be a set.

- (1) Let $p \in \mathbb{P}$. A condition $r \in \mathbb{P} \cap M$ is a **residue** for p into M if
$$\forall w \in \mathbb{P} \cap M (w \leq r \rightarrow w \parallel p).$$
- (2) A condition p is **strongly (\mathbb{P}, M) -generic** if every $q \leq p$ has a residue into M .
- (3) The poset \mathbb{P} is **strongly proper with respect to M** if for every $p \in \mathbb{P} \cap M$ there is $q \leq p$ which is strongly (\mathbb{P}, M) -generic.
- (4) The poset \mathbb{P} is **κ -strongly proper** if it is strongly proper with respect to stationarily many $M \in \mathcal{P}_\kappa(H_\theta)$ for any large enough regular θ .

A proof of the following lemma can be found in [1]:

Lemma 1.2. *A condition p is strongly (\mathbb{P}, M) -generic if and only if*

$$p \Vdash \check{G} \cap M \text{ is a } V\text{-generic filter on } \mathbb{P} \cap M.$$

Thus, strong properness is a generalisation of properness.

We define the notion of *common* or *dual residue* because we will do a following kind of *splitting argument*; we are in a situation where we have a poset \mathbb{P} , a name for a tree \dot{T} , a suitable model M , a condition $p \in \mathbb{P}$ and a node $t \notin M$. We will split the node t in the following sense: we find two distinct nodes s^L and s^R in M and two conditions q^L and q^R extending p such that they have a common residue r into M and such that $q^L \Vdash s^L < t$ and $q^R \Vdash s^R < t$.

Definition 1.3. Let \mathbb{P} be a poset and let M be a suitable model. Let $p, q \in \mathbb{P}$. A **common residue** for p and q in M is a condition $r \in \mathbb{P} \cap M$ which is a residue for p and residue for q , i.e.

$$\forall w \in \mathbb{P} \cap M (w \leq r \rightarrow w \parallel p \wedge w \parallel q).$$

Remark 1.4. If two conditions p and q have a common residue it does not follow that p and q are compatible.

2. THE POSET

Throughout, let κ be a fixed weakly compact cardinal. We will define a countable support iteration $(\mathbb{P}_\delta : \delta \leq \kappa^+)$ that will collapse κ onto \aleph_2 and specialize all κ -Aronszajn trees.

Notation 2.1. We fix a bookkeeping function

$$\kappa^+ \rightarrow H_{\kappa^+}, \quad \delta \mapsto \dot{T}_\delta$$

such that \dot{T}_δ is a \mathbb{P}_δ -name for a κ -Aronszajn tree with domain κ , whenever the poset \mathbb{P}_δ is defined, for every $\delta < \kappa^+$.

For a tree T , a **partial specializing function** is a partial function f on T such that for any two distinct nodes $s, t \in T$, if $f(s) = f(t)$, then s and t are incompatible in the tree order of T .

Definition 2.2. For each $\delta \leq \kappa^+$, conditions of the poset \mathbb{P}_δ are functions

$$p : \delta \rightarrow V_\kappa$$

such that

- (1) $p(0) \in \text{Coll}(\omega_1, < \kappa)$,
- (2) $p(\gamma)$ is a countable partial function $f_\gamma^p : \kappa \rightarrow \omega_1$ such that

$$p \upharpoonright \gamma \Vdash "f_\gamma^p \text{ is a partial specializing function of } \dot{T}_\gamma",$$

for every $\gamma > 0$,

- (3) $p \upharpoonright \gamma$ decides the tree order of \dot{T}_γ on $\text{dom}(f_\gamma^p)$,
- (4) the support $\text{sp}(p) := \{\gamma < \delta : p(\gamma) \neq \emptyset\}$ is countable.

The order is defined by pointwise inverse inclusion:

$$q \leq p \iff \forall \gamma < \delta \ q(\gamma) \supseteq p(\gamma).$$

Remark 2.3. For $\gamma < \delta$, there is a canonical complete embedding $\mathbb{P}_\gamma \hookrightarrow \mathbb{P}_\delta$ that takes a condition p in \mathbb{P}_γ to $p^\frown (\emptyset, \emptyset, \dots)$ and each poset \mathbb{P}_δ is a dense subset of the iteration with iterands $(\dot{\mathbb{Q}}_\gamma : \gamma < \delta)$, where $\mathbb{Q}_0 = \text{Coll}(\omega_1, < \kappa)$ and each $\dot{\mathbb{Q}}_\gamma$ is a name for the poset of countable approximations of specialising functions for \dot{T}_γ .

3. TRACES

Notation 3.1. Let $\theta > \kappa^+$ be a regular cardinal and let $<_\theta$ be a well-ordering of H_θ . For each $\delta < \kappa^+$ and $\alpha \in \kappa$, let

$$M_\alpha^\delta := \text{Skolem hull of } \alpha \text{ in } (H_\theta, \epsilon, <_\theta, \kappa, \delta).$$

Remark 3.2. We have $\omega_1 \cap M_\alpha^\delta = \alpha$ for club many $\alpha < \omega_1$.

Remark 3.3. We may assume without loss of generality that each M_α^δ contains the bookkeeping function $\gamma \rightarrow \dot{T}_\gamma$ as element.

Remark 3.4. $\mathbb{P}_\delta \in M_\alpha^\delta$.

Definition 3.5. Let $\delta < \kappa^+$. For each $p \in \mathbb{P}_\delta$ and $\alpha \in \kappa$, the **trace** $p|_\alpha$ of p into M_α^δ is a function on δ defined by

$$p|_\alpha(\gamma) := \begin{cases} p(\gamma) \upharpoonright M_\alpha^\delta & \text{if } \gamma \in \delta \cap M_\alpha^\delta, \\ \emptyset & \text{otherwise.} \end{cases}$$

Remark 3.6. $p|_\alpha(0) \in \text{Coll}(\omega_1, < \alpha)$ whenever $\omega_1 \cap M_\alpha^\delta = \alpha$.

Remark 3.7. In general, $p|_\alpha$ might not be a condition, but it will for stationarily many $\alpha \in \kappa$. The goal is to show that for stationarily many α , for every $p \in \mathbb{P}_\delta$, the trace $p|_\alpha$ is a *residue* for p , in the sense defined below in 1.1.

4. PRESERVATION OF κ

We show by induction on $\delta < \kappa^+$ that \mathbb{P}_δ has κ -cc and is strongly proper with respect to M_α^δ for \mathcal{F}_{wc} -many $\alpha \in \kappa$. Here \mathcal{F}_{wc} is the **weakly compact filter** generated by sets

$$\{\alpha \in \kappa : V_\kappa \models \varphi(A) \Rightarrow V_\alpha \models \varphi(A \cap V_\alpha)\},$$

for a Π_1^1 -formula $\varphi(X)$ and a set $A \subseteq V_\kappa$. The filter \mathcal{F}_{wc} is a normal filter on κ that extends the club filter. See Proposition 6.11. from Kanamori [2]. An important fact is that the statement “ T is a κ -Aronszajn tree” is Π_1^1 for any $T \subseteq V_\kappa$, and furthermore, so is the statement

$$\text{“}\Vdash_{\mathbb{P}} \dot{T} \text{ is a } \kappa\text{-Aronszajn tree”},$$

whenever $\mathbb{P}, T \subseteq V_\kappa$.

The goal is to show the following proposition:

Proposition 4.1. *Let $\delta < \kappa^+$.*

- (1) \mathbb{P}_δ has κ -cc.
- (2) *It holds for \mathcal{F}_{wc} -many $\alpha < \kappa$ that if $p, q \in \mathbb{P}_\delta$ have the same trace to M_α^δ , then they have a common residue into M_α^δ .*

The rest of the section is devoted to the prove Proposition 4.1. The proof is by induction on δ . **From now onwards until the end of the paper, we fix one $\delta < \kappa^+$ and assume that the proposition holds for all $\gamma < \delta$.** We begin by proving a series of lemmas.

4.1. Preliminary lemmas.

Remark 4.2. The poset \mathbb{P}_δ has size κ . We tacitly assume that each \mathbb{P}_δ is coded as a subset of V_κ . Up to choosing the $<_\theta$ -least function, we may assume that each model M_α^δ knows about this coding. This will be important when using the Π_1^1 -reflection of κ .

We say that a subposet $\mathbb{Q} \subseteq \mathbb{P}_\gamma$ **determines** a subset $S \subseteq T_\gamma$ if for all nodes $s, t \in S$ the set of conditions deciding the tree-order of \dot{T}_γ between s and t is dense in \mathbb{Q} .

Lemma 4.3. *Assume that \mathbb{P}_γ has κ -cc for all $\gamma < \delta$. Then, it holds for club many $\alpha \in \kappa$ that for every $\gamma \in \delta \cap M_\alpha^\delta$, the poset $\mathbb{P}_\gamma \cap V_\alpha$ determines the set $\text{Lev}_{<\alpha}(\dot{T}_\gamma)$.*

Proof. Follows from strong inaccessibility of κ and κ -cc of each \mathbb{P}_γ . \square

Lemma 4.4. *For club many $\alpha \in \kappa$, it holds that for every $p \in \mathbb{P}_\delta$, the trace $p|_\alpha$ is a condition in $\mathbb{P}_\delta \cap M_\alpha^\delta$.*

Proof. Follows from the lemma above. \square

Lemma 4.5. *If $p, q \in \mathbb{P}_\delta$ have a common residue r into M_α^δ , then there are $\hat{p} \leq p$ and $\hat{q} \leq q$ with a common trace into M_α^δ extending r .*

Lemma 4.6. *The set*

- $$\{\alpha \in \kappa : \begin{aligned} (1) \quad & V_\kappa \cap M_\alpha^\delta = V_\alpha, \\ (2) \quad & \alpha \text{ is an inaccessible cardinal,} \\ (3) \quad & \forall \gamma \in \delta \cap M_\alpha^\delta \quad \mathbb{P}_\gamma \cap M_\alpha^\gamma \text{ has } \alpha\text{-cc,} \\ (4) \quad & \forall \gamma \in \delta \cap M_\alpha^\delta \quad \mathbb{P}_\gamma \cap M_\alpha^\gamma \subseteq_c \mathbb{P}_\gamma, \\ (5) \quad & \forall \gamma \in \delta \cap M_\alpha^\delta \quad \Vdash_{\mathbb{P}_\gamma \cap M_\alpha^\gamma} \text{"}\alpha = \aleph_2\text{"}, \\ (6) \quad & \forall \gamma \in \delta \cap M_\alpha^\delta \quad \Vdash_{\mathbb{P}_\gamma \cap M_\alpha^\gamma} \text{"}\dot{T}_\gamma \cap V_\alpha \text{ is an Aronszajn tree on }\alpha\text{"}, \\ (7) \quad & \forall \gamma \in \delta \cap M_\alpha^\delta \quad \forall p \in \mathbb{P}_\gamma \quad p|_\alpha \in \mathbb{P}_\gamma \cap M_\alpha^\gamma \end{aligned}\}$$

is in \mathcal{F}_{wc} .

Proof. Points (1), (2) follow directly from the fact that κ is weakly compact, in particular regular and Mahlo. Point (3) follows by Π_1^1 -reflection from the hypothesis that \mathbb{P}_γ has κ -cc. Point (4) follows from (3) and normality of \mathcal{F}_{wc} by a pressing down argument. Points (5), (6) follow by Π_1^1 -reflection. Point (7) is previous lemma. \square

4.2. Splitting.

Definition 4.7. Let $p, q \in \mathbb{P}_\delta$, $\gamma < \delta$ and let $s, t \in T_\gamma$. We say that the conditions p and q split the pair (s, t) below α if there are two distinct nodes $\hat{s}, \hat{t} \in T_\gamma \cap \alpha$ of the same height such that

$$\begin{aligned} p \upharpoonright \gamma &\Vdash \hat{s} < s, \\ q \upharpoonright \gamma &\Vdash \hat{t} < t. \end{aligned}$$

The case $s = t$ is not excluded; we say that p and q split the node s below α if p and q split the pair (s, s) below α .

The use of the weakly compact cardinal is substantial in the following lemma.

Lemma 4.8 (Splitting). *The following holds for \mathcal{F}_{wc} -many $\alpha < \kappa$: for any two conditions $p, q \in \mathbb{P}_\delta$ with same trace to M_α^δ there are two conditions $\hat{p} \leq p$ and $\hat{q} \leq q$ that have the same trace to M_α^δ and that split below α every pair of nodes $(s, t) \in (\text{dom}(f_\gamma^{\hat{p}}) \times \text{dom}(f_\gamma^{\hat{q}})) - \alpha$ where $\gamma \in \delta \cap M_\alpha^\delta$.*

Proof. We begin with a claim that we will then iterate to finish the proof of the lemma.

Claim 4.9. *The following holds for \mathcal{F}_{wc} -many $\alpha < \kappa$: for any $\gamma \in \delta \cap M_\alpha^\delta$, any two conditions $p, q \in \mathbb{P}_\gamma$ that have a common residue r into M_α^γ , and any nodes $s, t \in T_\gamma - \alpha$ there are two conditions $\hat{p} \leq p$ and $\hat{q} \leq q$ that have a common residue $\hat{r} \leq r$ into M_α^γ and that split the pair (s, t) below α .*

Proof of Claim 4.9. We show that the claim holds for every $\alpha < \kappa$ from the set in Lemma 4.6. Let $\gamma \in \delta \cap M_\alpha^\delta$ and suppose that $p, q \in \mathbb{P}_\gamma$ are conditions with same trace to M_α^γ . Let $s, t \in T_\gamma - \alpha$. The reflection properties of the weakly compact are essential in the following step. Let $G \subseteq \mathbb{P}_\gamma \cap M_\alpha^\gamma$ be a generic filter containing the common residue of p and q . It follows that p and q are in the quotient forcing \mathbb{P}_γ/G . By the choice of α , it holds (among other things) that

$$(*) \quad \Vdash_{\mathbb{P}_\gamma \cap M_\alpha^\gamma} \text{"}\dot{T}_\gamma \cap \alpha \text{ is an } \alpha\text{-Aronszajn tree".}$$

Thus $(\dot{T}_\gamma)_G$ is an α -Aronszajn tree, and thus the branch below the node s must be introduced by the quotient forcing \mathbb{P}_γ/G , for otherwise it would be a cofinal branch in $(\dot{T}_\gamma \cap \alpha)_G$. Thus, there must be two conditions $p^L, p^R \leq p$ in \mathbb{P}_γ/G that split s at some level $\beta < \alpha$ with some distinct nodes s^L and s^R . Since p^L and p^R are in the quotient forcing \mathbb{P}_γ/G , there is some $r_0 \in G$ that forces $p, q \in \mathbb{P}_\gamma/\check{G}$. This r_0 is a common residue for p and q . Then we find an extension $\hat{q} \leq q$, also in the quotient \mathbb{P}_δ/G , such that \hat{q} decides the node below t at level β , call it \bar{t} . There is $r \in G$ that extends r_0 and forces $\hat{q} \in \mathbb{P}_\delta/\check{G}$. Then r is a common residue for p^L, p^R and \hat{q} . If $\bar{t} \neq s^L$, then p^L and \hat{q} are as wanted, and if $\bar{t} \neq s^R$, then p^R and \hat{q} are as wanted. This ends the proof of the claim. \square

We now complete the proof of Lemma 4.8 by iterating Claim 4.9 countably many times. Using a suitable enumeration, we find conditions $(p_n)_{n < \omega}$ and $(q_n)_{n < \omega}$ such that conditions p_n and q_n have a common residue r_n into M_α^δ and split a pair of nodes (s_n, t_n) from $T_{\gamma_n} - \alpha$, where $\gamma_n \in \delta \cap M_\alpha^\delta$, $s_n \in \text{dom}(f_{\gamma_n}^{p_n})$ and $t_n \in \text{dom}(f_{\gamma_n}^{q_n})$.

This is accomplished by first fixing a bijection $\langle \cdot \rangle : \omega \times \omega \times \omega \rightarrow \omega$ that is suitable in the sense that whenever $n = \langle m, k, l \rangle$, then $m \leq n$. We begin by letting $p_0 := p$ and $q_0 := q$. At step $n+1$, we assume that p_n and q_n have been defined and have the same trace to M_α^δ . We first enumerate the union of the supports $(\text{sp}(p_n) \cup \text{sp}(q_n)) \cap M_\alpha^\delta$ as $(\gamma_k^n : k < \omega)$, and the set of all nodes in $(\text{dom}(f_{\gamma_k^n}^{p_n}) - \alpha) \times (\text{dom}(f_{\gamma_k^n}^{q_n}) - \alpha)$, for each $k < \omega$, as $((s_{(n,k,l)}, t_{(n,k,l)} : l < \omega)$. Then we pick the unique m, k, l such that $n = \langle m, k, l \rangle$ and look at the conditions $p_n \upharpoonright \gamma_m^k$ and $q_n \upharpoonright \gamma_m^k$. We apply the induction hypothesis of Proposition 4.1(2): since $p_n \upharpoonright \gamma_m^k$ and $q_n \upharpoonright \gamma_m^k$ have the same trace to $M_\alpha^{\gamma_m^k}$, there is $r \in \mathbb{P}_{\gamma_m^k}^k \cap M_\alpha^{\gamma_m^k}$ that is a common residue for them. Then we apply Claim 4.9 to the nodes $s_{(m,k,l)}$ and $t_{(m,k,l)}$: we find two conditions $p' \leq p_n \upharpoonright \gamma_m^k$ and $q' \leq q_n \upharpoonright \gamma_m^k$ in $\mathbb{P}_{\gamma_m^k}^k$ that split $s_{(m,k,l)}$ and $t_{(m,k,l)}$ and have a common residue $r' \leq r$ into $M_\alpha^{\gamma_m^k}$. By Lemma 4.5 we may extend further to assume that p' and q' have the same trace into $M_\alpha^{\gamma_m^k}$, and the same trace extends r' . We define p_{n+1} and q_{n+1} by taking the pointwise union:

$$\begin{aligned} p_{n+1} &:= p' \text{ } p_n \upharpoonright [\gamma_m^k, \delta), \\ q_{n+1} &:= q' \text{ } q_n \upharpoonright [\gamma_m^k, \delta). \end{aligned}$$

Then p_{n+1} and q_{n+1} have the same trace to M_α^δ and split the nodes $s_{(m,k,l)}$ and $t_{(m,k,l)}$.

Finally, we let $\hat{p} := \bigcup_n p_n$ and $\hat{q} := \bigcup_n q_n$. They are as wanted. \square

4.3. Proof of κ -cc and existence of common residues.

We finally are ready to prove Proposition 4.1.

Proof of Proposition 4.1. The proof is by induction on δ . We assume that the proposition holds for $\gamma \in \delta$ and show that it holds for δ .

- (1) We show that \mathbb{P}_δ has κ -cc. Let $\{p_\alpha : \alpha \in \kappa\} \subseteq \mathbb{P}_\delta$. We find distinct α and β such that p_α and p_β are compatible.

The general idea is as follows. By applying Splitting Lemma 4.8, for \mathcal{F}_{wc} many α , find conditions $p_\alpha^L, p_\alpha^R \leq p_\alpha$ which split relevant pairs of nodes and satisfy $p_\alpha^L|_\alpha = p_\alpha^R|_\alpha$. Then, by pressing down with Fodor's lemma, it is possible to show that for \mathcal{F}_{wc} many $\alpha < \beta$, the left extension p_α^L is compatible with the right extension p_β^R . This allows to finish.

Now in more detail. Look at the set $\mathcal{S}_\delta \in \mathcal{F}_{\text{wc}}$ from Lemma 4.8. For every $\alpha \in \mathcal{S}_\delta$, find

$$p_\alpha^L, p_\alpha^R \leq p_\alpha$$

such that $p_\alpha^L|_\alpha = p_\alpha^R|_\alpha$ and for all $\gamma \in \delta \cap M_\alpha^\delta$, the pair (p_α^L, p_α^R) splits each pair of nodes (s, t) in $(\text{dom}(f_\gamma^{p_\alpha^L}) \times \text{dom}(f_\gamma^{p_\alpha^R})) - V_\alpha$ with some pair of nodes (\bar{s}, \bar{t}) from V_α . By pigeonhole principle there is an \mathcal{F}_{wc} -positive set $U \subseteq \mathcal{S}_\delta$ such that for all $\alpha, \beta \in U$ there are isomorphisms

$$\begin{aligned} p_\alpha^L &\cong p_\beta^L, \\ p_\alpha^R &\cong p_\beta^R, \end{aligned}$$

which fix the traces

$$p_\alpha^L|_\alpha = p_\beta^L|_\beta = p_\beta^R|_\beta = p_\alpha^R|_\alpha.$$

Up to further refining U , we may assume that for all $\alpha < \beta$ from U and $\gamma \in \delta$,

$$\begin{aligned} &\sup\{ht(t) : t \in (\text{dom}(f_\gamma^{p_\alpha^L}) \cup \text{dom}(f_\gamma^{p_\alpha^R})) - V_\alpha\} \\ &< \min\{ht(t) : t \in (\text{dom}(f_\gamma^{p_\beta^L}) \cup \text{dom}(f_\gamma^{p_\beta^R})) - V_\beta\}, \end{aligned}$$

and that

$$(\text{sp}(p_\alpha^L) - \alpha) \cap (\text{sp}(p_\beta^R) - \beta) = \emptyset.$$

Choose $\alpha < \beta$ from U . We claim that $p_\alpha^L \parallel p_\beta^R$. Define \hat{p} to be the pointwise union, by letting

$$\begin{aligned} \hat{p}(0) &:= p_\alpha^L(0) \cup p_\beta^R(0), \\ \hat{p}(\gamma) &:= f_\gamma^{p_\alpha^L} \cup f_\gamma^{p_\beta^R}. \end{aligned}$$

We claim that \hat{p} extends to a condition. We need to make sure that $\hat{p} \upharpoonright \gamma$ decides the tree-order of \dot{T}_γ relativised to $\text{dom}(f_\gamma^{\hat{p}})$ and that show that $f_\gamma^{\hat{p}}$ is forced to be a specializing function. If to the contrary this is not the case, then there are two distinct nodes $s, t \in \text{dom}(f_\gamma^{\hat{p}})$ such that

$$f_\gamma^{\hat{p}}(s) = f_\gamma^{\hat{p}}(t)$$

and such that $\hat{p} \upharpoonright \gamma \Vdash s \leq t$. If $s, t \in \text{dom}(f_\gamma^{p_\alpha^L})$ or $s, t \in \text{dom}(f_\gamma^{p_\beta^R})$, we are done. Assume thus that $s \in \text{dom}(f_\gamma^{p_\alpha^L})$ and $t \in \text{dom}(f_\gamma^{p_\beta^R})$. Now, by construction,

$$\begin{aligned} p_\alpha^L \upharpoonright \gamma &\Vdash \bar{s} < s, \\ p_\beta^R \upharpoonright \gamma &\Vdash \bar{t} < t. \end{aligned}$$

This is because the pair $(p_\alpha^L \upharpoonright \gamma, p_\alpha^R \upharpoonright \gamma)$ splits some pair (s, t') with (\bar{s}, \bar{t}') and the pair $(p_\beta^L \upharpoonright \gamma, p_\beta^R \upharpoonright \gamma)$ splits a pair (s', t) with (\bar{s}', \bar{t}) . This implies

$$\hat{p} \Vdash s \perp t.$$

- (2) We show that for \mathcal{F}_{wc} many $\alpha \in \kappa$, whenever p and q are two conditions in \mathbb{P}_δ with the same trace $p|_\alpha = q|_\alpha$ to M_α^δ , then they have a common residue to M_α^δ .

- Suppose to the contrary that there is an \mathcal{F}_{wc} -positive set S such that for every $\alpha \in S$, there are two conditions p_α and q_α with the same trace $p_\alpha|_\alpha = q_\alpha|_\alpha$ and such that no condition in $\mathbb{P}_\delta \cap M_\alpha^\delta$ is a common residue for them. We may choose the set S such that every $\alpha \in S$ satisfies all properties listed in Lemmas 4.3, 4.4 and 4.6. Furthermore, we may assume by the splitting lemma 4.8 that for every $\alpha \in S$, there are $\hat{p}_\alpha \leq p_\alpha$ and $\hat{q}_\alpha \leq q_\alpha$ with same traces $\hat{p}_\alpha|_\alpha = \hat{q}_\alpha|_\alpha$ and such that all relevant pairs of nodes are split.
- By stratifying by levels we may arrange that for $\alpha < \beta$ in S' , the conditions \hat{p}_α and \hat{q}_β are compatible, as in the proof of κ -cc.
- For every $\alpha \in S$ there is a maximal antichain

$$W_\alpha \subseteq \{r \in \mathbb{P}_\delta \cap M_\alpha^\delta : r \leq \hat{p}_\alpha|_\alpha = \hat{q}_\alpha|_\alpha \text{ and } (r \perp \hat{p}_\alpha \text{ or } r \perp \hat{q}_\alpha)\}.$$

- Since we already proved κ -cc of \mathbb{P}_δ , we may assume that $\mathbb{P}_\delta \cap M_\alpha^\delta \subseteq_c \mathbb{P}_\delta/\hat{p}_\alpha$ for every $\alpha \in S$. This implies that W_α is a maximal antichain in $\mathbb{P}_\delta/\hat{p}_\alpha|_\alpha$.
- Each antichain W_α has size $< \kappa$. There is a subset $S' \subseteq S$ such that $W_\alpha = W_\beta$ for $\alpha, \beta \in S'$, and such that for $r \in W$, $\hat{p}_\alpha \perp r$ iff $\hat{p}_\beta \perp r$ and $\hat{q}_\alpha \perp r$ iff $\hat{q}_\beta \perp r$. This follows by Fodor's lemma applied to the fact that wlog we may fix an enumeration of all antichains \mathbb{P}_δ of length κ and since there are stationarily many inaccesibles below κ , there are many $\alpha < \kappa$ such that the enumeration givies an enumeration of antichains of $\mathbb{P}_\delta \cap M_\alpha^\delta$ of length α ; then the function mapping α to the index of W_α is regressive.
- Choose some $v \leq \hat{p}_\alpha, \hat{q}_\beta$. Then $v \leq \hat{p}_\alpha|_\alpha$ and without loss of generality v does not belong to $\mathbb{P}_\delta \cap M_\alpha^\delta$. Thus $v \notin W_\alpha = W$ either.
- We claim that $W \cup \{v\}$ is an antichain. This will contradict the maximality of W . Indeed, if $w \in W$, then either $w \perp \hat{p}_\alpha$ or $w \perp \hat{q}_\beta$. The first case implies that $w \perp v$ and the second case implies that $w \perp v$. Thus $W \cup \{v\}$ is indeed an antichain. This is a contradiction.

□

Corollary 4.10. *Each poset \mathbb{P}_δ is κ -strongly proper and have κ -cc, for $\delta < \kappa^+$. The poset \mathbb{P}_{κ^+} has κ -cc.*

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