市场风险测量与管理

FRM Part II Program-强化班

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Market Risk

101% Contribution Breeds Professionalism

Framework
Market Risk Measurement and Management

① VaR and other Risk Measures
② Risk Measurement for the Trading Book
③ Modeling Dependence (相关性).
④ Empirical Approaches to Risk Metrics and Hedges
⑤ $\boxed{\text{Term Structure}}$ Models of Interest Rates
- Volatility Smiles

Reading 1

VaR and other Risk Measures

Estimating Market Risk Measures

VaR and other Risk Measures

;- (测量市场风险的方式)

计算VaR值:

Estimating Market Risk Measures

① Parametric approach (计算)
- Normal VaR
- Lognormal VaR

② Non-Parametric approach (不计算)
- Historical Simulation
- Non-parametric density estimation

③ Hybrid approach(Semi-Parametric approach) (非参数基础上加参数)
- Age-weighted(BRW)
- Volatility-weighted(HW)
- Correlation-weighted
- Filtered historical simulation

• The most popular semi-parameteric estimation method is the Hall estimator.
(consistent, asympototically).

Parametric approach

①假设 $R_t \sim$ normal. $R_t = \frac{P_t - P_{t-1}}{P_{t-1}}$

②公式:

➢ Normal VaR
- We assume that arithmetic returns are normally distributed with mean μ and standard deviation σ

单个资产
$$VaR_s = -(\mu - z_\alpha \sigma)$$ ③ VaR表示loss
$$VaR_s = -(\mu - z_\alpha \sigma)P_{t-1}$$ ④百分比/美元

Portfolio:
$$VaR_p = -(\mu_p - z_\alpha \sigma_p)P_{t-1} = \Delta \times VaR_s$$
$$VaR_p = \sqrt{VaR_1^2 + VaR_2^2 + 2 \times \rho \times VaR_1 \times VaR_2}$$ (保证 μ=0)

➢ Lognormal VaR
- Assume that geometric returns are normally distributed with mean μ and standard deviation σ. This assumption implies that the natural logarithm of *p* is normally distributed, or that *p* itself is lognormally distributed. Normally distributed geometric returns imply that the VaR is lognormally distributed.

① Assumption: Price ~ lognormal
$$VaR_s = 1 - e^{\mu - z_\alpha \sigma}$$
$$VaR_s = (1 - e^{\mu - z_\alpha \sigma})P_{t-1}$$

$r_t = \ln \frac{P_t}{P_{t-1}}$
$r_t \sim$ normal.

$VaR = loss = -P_t + P_{t-1}$
$= -P_{t-1} \cdot e^{r_t} + P_{t-1}$
$= P_{t-1}(1 - e^{r_t})$
$= P_{t-1}(1 - e^{\mu - z\sigma})$.

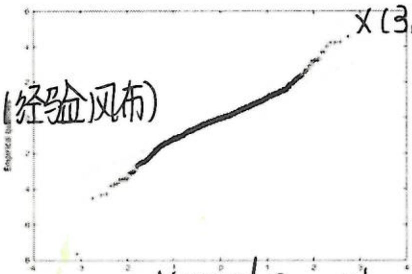
◆ Parametric approach

➢ **Quantile-Quantile Plots** ：判断未知分布是否为正态分布

- A plot of the quantiles of the empirical distribution against those of some specified distribution. The shape of the QQ plot tells us a lot about how the empirical distribution compares to the specified one.
- In particular, if the QQ plot is linear, then the specified distribution fits the data, and we have identified the distribution to which our data belong.

X(3,5)（累积概率一样）.

Empirical（经验分布）
quantiles



Normal Quantiles

Non-parametric 缺点 ① 数据要求高
②维度的诅咒 P.

◆ Non-parametric Methods

➢ **Introduction**

- All non-parametric approaches are based on the underlying assumption that the near future will be sufficiently like the recent past.（假设：历史会重演）.
- With non-parametric methods, there are no problems dealing with variance–covariance matrices, curses of dimensionality, etc. ①collect data → ②排序 → ③查数.

➢ **Drawbacks of HS** Historical simulation.

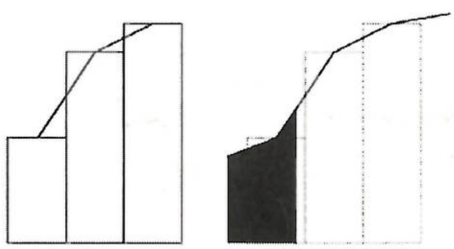
- Basic HS has the practical drawback that it only allows us to estimate VaRs at discrete confidence intervals determined by the size of our data set.
  - ✓ For instance, the VaR at the 95.1% confidence level is a problem because there is no corresponding loss observation to go with it.
  - ✓ With n observations, basic HS only allows us to estimate the VaRs associated with, at best, n different confidence levels.

◆ Non-parametric Methods

➢ **Non-parametric Methods** →（线性差值）.

- Non-parametric density estimation offers a potential solution.
  - ✓ Draw in straight lines connecting the mid-points at the top of each histogram bar(Polygon).
  - ✓ Treating the area under the lines as a pdf then enables us to estimate VaRs at any confidence level.

(a) Original histogram (b) Surrogate density function



◆ Non-parametric Methods

➢ **The Conditional VaR (expected shortfall)** (超过VaR的均值)

- The expected value of the loss when it exceeds VaR.
- Measures the average of the loss conditional on the fact that it is greater than VaR.
- CVaR indicates the potential loss if the portfolio is "hit" beyond VaR. Because CVaR is an average of the tail loss, one can show that it qualifies as a subadditive risk measure.

◆ Non-parametric Methods

➢ **Example:**

- Given the following 30 ordered percentage returns of an asset,

  −16, −14, −10, −7, −7, −5, −4, −4, −4, −3, −1, −1, 0, 0, 0, 1, 2, 2, 4, 6, 7, 8, 9, 11, 12, 12, 14, 18, 21, 23

  Calculate the VaR and expected shortfall at a 90% confidence level:
- Solution:

  VaR (90%) = 7, Expected Shortfall = 13.3

  置信

◆ Non-parametric Methods

➢ **A/D of Non-parametric Methods**

➢ **Advantages**

- Intuitive and conceptually simple;
- Do not depend on parametric assumptions; 不用满足特定分布
- Accommodate any type of position; {做多 做空.
- No need for covariance matrices, no curses of dimensionality;
- Use data that are (often) readily available;
- Are capable of considerable refinement and potential improvement if we combine them with parametric "add-ons" to make them semi-parametric.(主) (次).

Non-parametric Methods

- **A/D of Non-parametric Methods**
- **Disadvantages**
  - Very dependent on the historical data set;(假设历史会重演).
  - Subject to ghost effect;
  - If our data period was unusually quiet, non-parametric methods will often produce VaR or ES estimates that are too low, vice versa;
  - Have difficulty(act slowly) handling shifts(permanent risk change) that take place during our sample period; VaR的变动.很难⇒VaR对市场不敏感.
  - Have difficulty handling extreme value
    - ✓ If our data set incorporates extreme losses that are unlikely to recur, these losses can dominate non-parametric risk estimates even though we don't expect them to recur;
    - ✓ Make no allowance for plausible events that might occur, but did not actually occur, in our sample period.

Hybrid Approach

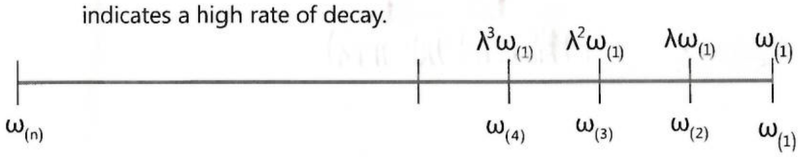
- **Hybrid approach(Semi-Parametric approach)**
  - Age-weighted(BRW)
  - Volatility-weighted(HW)
  - Correlation-weighted
  - Filtered historical simulation

$\omega_i = \frac{\lambda^{i-1}(1-\lambda)}{(1-\lambda^n)} = \lambda^{i-1}\cdot\frac{(1-\lambda)}{(1-\lambda^n)}$

$\frac{r_{adjust}}{r_{actual}} = \frac{\sigma_{today}, T}{\sigma_{current}, t}$

Hybrid Approach

- **Age-weighted Historical Simulation**
- **Boudoukh, Richardson and Whitelaw (BRW: 1998)**
  - $w_{(1)}$ is the probability weight given to an observation 1 day old.
  - A λ close to 1 indicates a slow rate of decay, and a λ far away from 1 indicates a high rate of decay.



①排序→②调整权重→③查数

$$\omega_{(1)} + \lambda\omega_{(1)} + \ldots + \lambda^{n-1}\omega_{(1)} = 1 \rightarrow$$

$$\omega_{(i)} = \frac{\lambda^{i-1}(1-\lambda)}{1-\lambda^n}$$

Hybrid Approach

- **Major attractions**
  - It provides a nice generalization of traditional HS, because we can regard traditional HS as a special case with zero decay, or $\lambda \to 1$.
  - Helps to reduce distortions caused by events that are unlikely to recur, and helps to reduce ghost effects.
    - ✓ As an observation ages, its probability weight gradually falls and its influence diminishes gradually over time, instead of from 1/n to zero. (这样可以反应当前市场状况)
  - Age-weighting allows us to let our sample period grow with each new observation, so we never throw potentially valuable information away. This would improve efficiency and eliminate ghost effects, because there would no longer be any "jumps" in our sample resulting from old observations being thrown away.

Hybrid Approach

- **Volatility-weighted Historical Simulation**
- **Hull and White (HW 1998)**

数据weight不动
数据大小改变

①调整数据
②排序.

- We adjust the historical returns to reflect how volatility tomorrow is believed to have changed from its past values.

adjusted.

$$\frac{r_{t,i}^*}{r_{t,i}} = \frac{\sigma_{T,i}}{\sigma_{t,i}}$$ → 当前的volatility.

✓ $r_{t,i}$ = actual return for asset i on day t

✓ $\sigma_{t,i}$ = volatility forecast for asset i on day t

✓ $\sigma_{T,i}$ = current forecast of volatility for asset i

Hybrid Approach

- **Major attractions**
  - It takes account of volatility changes in a natural and direct way.
  - It produces risk estimates that are appropriately sensitive to current volatility estimates.
  - It allows us to obtain VaR and ES estimates that can exceed the maximum loss in our historical data set. (可能超过历史max)
    - ✓ In recent periods of high volatility, historical returns are scaled upwards, and the HS P/L series used in the HW procedure will have values that exceed actual historical losses.
  - Produces superior VaR estimates to the BRW one.

◆ Hybrid Approach → reflect "current volatilities" and "correlation"

③ **Correlation-weighted historical simulation** (相关系数考虑)

- Correlation-weighting is a little more involved than volatility-weighting.
- To see the principles involved, suppose for the sake of argument that we have already made any volatility-based adjustments to our HS returns along Hull-White lines, but also wish to adjust those returns to reflect changes in correlations.

④ **Filtered historical simulation** (考虑不对称性) GARCH model $\sigma_n^2 = \omega + \alpha \cdot u_{n-1}^2 + \beta \cdot \sigma_{n-1}^2$

- Combines the traditional historical simulation model with GARCH or AGARCH model. HS 收益与损失分析.

$\omega V_n = \omega + \alpha u_{x,n-1} \cdot u_{y,n-1} + \beta \omega V_{n-1}^2$

$\rho_n \sigma_{nx} \sigma_{ny} = \omega + \alpha \cdot u_{n-1,x} \cdot u_{n-1,y} + \ldots$

- Major attractions
  - ✓ Combine the non-parametric attractions of HS with a sophisticated (e.g., GARCH) treatment of volatility, and so take account of changing market volatility conditions. It maintains the correlation structure in our return.
  - ✓ As with the earlier HW approach, FHS allows us to get VaR and ES estimates that can exceed the maximum historical loss in our data set.
  - ✓ There is evidence that FHS works well. (+ volatility, + correlation

Extreme value

VaR and other Risk Measures

◆ Extreme Value

➢ **Generalized Extreme-Value Theory (Block Maxima)** Block Maxima (选择极值方式1)

- Consider a sample of size n drawn from F(x), and let the maximum of this sample be M_n. If n is large, we can regard M_n as an extreme value.
- Under relatively general conditions, as n gets large the distribution of extremes (i.e., M_n) converges to **generalized extreme-value (GEV) distribution.** 当n足够大, distribution会收敛到广义极值分布
- This distribution has **three parameters**. GEV中CDF参数
  - ✓ **μ**, the **location parameter** of the limiting distribution, which is a measure of the central tendency of M_n. (极值的μ)
  - ✓ **σ**, the **scale parameter** of the limiting distribution, which is a measure of the dispersion of M_n.
  - ✓ **ξ**, the **tail index**, gives an indication of the shape (or heaviness) of the tail of the limiting distribution. If $\xi > 0$, the GEV becomes the Frechet distribution. This case is particularly useful for financial returns because they are typically heavy-tailed.

$\xi > 0$, fat tail (Frechet)
$\xi = 0$, Gumbel
$\xi < 0$, thin tail / Weibull.

Extreme Value

- **Peaks-Over-Threshold (POT) Approach** (选择极值方式2)
  - If x is a random i, i, d. loss with distribution function F(x), and u is a threshold value of x, we can define the distribution of excess losses over threshold u. threshold 难以确定
  - The distribution of x itself can be any of the commonly used threshold. distributions and will usually be unknown to us. However, as $\boxed{u}$ gets large, the distribution $F_U(x)$ converges to a **generalized Pareto distribution**, given by:

$$G_{\xi,\beta}(x) = \begin{cases} 1-(1+\xi x/\beta)^{-1/\xi} & if\ \xi \neq 0 \\ 1-exp(-x/\beta) & if\ \xi = 0 \end{cases}$$

  - Two parameters: a positive ① **scale parameter, β**, and a ② **shape or tail index** parameter, **ξ.** This latter parameter is the same as the tail index encountered already with GEV theory

Extreme Value

- **The expression for VaR and expected shortfall using POT parameters is given as follows:**

confidence level (here)

$$VaR = \mu + \frac{\beta}{\xi}\left(\left[\frac{n}{N_\mu}(1-\alpha)\right]^{-\xi} - 1\right)$$

$$ES = \frac{VaR}{1-\xi} + \frac{\beta - \xi\mu}{1-\xi}$$

  - ξ = shape parameter
  - β = scale parameter
  - μ = threshold
  - n = number of observations
  - N_μ = number of observations that exceed threshold

Extreme Value (两个方式对比):

- **Peaks-Over-Threshold (POT) Approach (cont'd)**
  - **Tradeoffs Involved in Setting the Threshold Level**
    - ✓ We want a threshold u to be sufficiently high for the theorem to apply reasonably closely; but if u is too high, we won't have enough excess-threshold observations on which to make reliable estimates.
  - **Differences with GEV Theory** (表现，显示) (两个方法在极值理论本表现相同)
    - ✓ Both are different manifestations of the same underlying EV theory.
    - ✓ POT model exceedances over a high threshold while GEV theory model the maxima of a large sample.
    - ✓ POT require fewer parameters.
    - ✓ The block maxima approach can involve some loss of useful data, because some blocks might have more than one extreme in them.
    - ✓ POT requires us to grapple with the problem of choosing the threshold. (处理)

Backtesting VaR

VaR and other Risk Measures

计算出来的VaR值是否正确)⇒假设检验.

已知 αVaR = 10 million，若1年内(250)出现15天 loss超过10 million.
99%，1day

①两项分布.
②kupiec↓考虑 fat tail
③fat tail+数据集聚性.

◆ Model Verification Based on Failure Rates

- The failure rate: the proportion of times VaR is exceeded in a given sample.

$$\frac{\text{time VaR} > \text{VaR Bench}}{\text{given sample}}$$

- The number of exceptions x follows a binomial probability distribution:

$$f(x) = C_T^x p^x (1-p)^{T-x}$$

$$E(x) = pT \qquad V(x) = p(1-p)T$$

x (例外)# ⇒ binomial.

- Approximate the binomial distribution by the normal distribution:

$$z = \frac{x - pT}{\sqrt{p(1-p)T}} \approx N(0,1)$$

p: 产生 exception 概率

二项分布
↓ $np \geq 10$, $n(1-p) \geq 10$.
normal distribution

$$z = \frac{x-\mu}{\sigma} = \frac{x-pT}{\sqrt{p(1-p)T}}$$

◆ Example

- In 1998, daily revenue of JP Morgan fell short of the downside (95% VaR) band on 20 days, or more than 5% of the time. Nine of these 20 occurrences fell within the August to October period.

two tail 1.96

$x=20$，$p=5\%$

$$z = \frac{x - pT}{\sqrt{p(1-p)T}} = \frac{20 - 0.05 * 252}{\sqrt{0.05(1-0.05)252}} = \boxed{2.14} > 1.96$$

- We reject the hypothesis that the VaR model is unbiased.

- What happens to test 99% VaR at 5% significance level and 95% VaR at 1% significance level? (计算). (回测).

P(reject | H0 is true)

一.定义: 一类错误: 弃真.

type II error: 取伪

P(not reject | H0 is false)

Type 1 and Type 2 errors

- Users of VaR models need to balance type I errors against type II errors.
- Ideally, one would want to set a low type I error rate and then have a test that creates a very low type II error rate, in which case the test is said to be powerful.

confidence interval.

Decision	Correct	Incorrect
Accept	OK	Type II
Reject	Type I $=\alpha$	Power of test (检验力度)

① H_0: VaR is true, H_a: VaR is false.

② two-tail 95% → 1.96, 99% → 2.58.

③ $z = \dfrac{x - pT}{\sqrt{p(1-p)T}}$

p → 计算VaR值用的显著性水平.

一类错误就是显著性水平.

二.性质: ① type I, type II error 此消彼长

② n↑, type I, type II ↓.

④ 查找关键值: 99% → 2.58

假设检验置信水平

⑤ 对比 $|z| > |c|$

reject.

Kupiec VaR Backtest (考虑了fat tail的情况).

- **Backtesting Exceptions**
  - Using Failure Rates in Model Verification
    - ✓ H_0: accurate model
    - ✓ H_a: inaccurate model
    - ✓ Test statistic:

$$LR_{uc} = -2ln[(1-p)^{T-N}p^N] + 2ln\{[1-(N/T)]^{T-N}(N/T)^N\}$$

  - ◆ p: the probability of exception, p = 1-c
  - ◆ N: the number of exceptions
  - ◆ T: the number of samples

confidence level

significant level

  - We would reject the null hypothesis if LR > 3.841.

① H_0: VaR is true;

H_a: VaR is not true;

$$-2ln[p^N(1-p)^{T-N}] + 2ln\{[\tfrac{N}{T}]^N \cdot (1-\tfrac{N}{T})^{T-N}\}.$$

② $\Rightarrow ln\dfrac{[(1-\frac{N}{T})^{T-N}(\frac{N}{T})^N]^2}{[(1-p)^{T-N} \cdot p^N]^2}$.

③ critical value,

④ 对比.

Blame on Unconditional Coverage Models

- So far the framework focuses on unconditional coverage because it ignores conditioning, or time variation in the data. The observed exceptions, however, could **cluster** or "**bunch**" closely in time, which also should invalidate the model.
- In theory, these occurrences should be evenly spread over time. If, instead, we observed that 10 of these exceptions occurred over the last 2 weeks, this should raise a red flag.

方法3

- **Christoffersen test is applied.** (考虑了数据的聚集性).

(加入参数考虑时间).

Conclusions

可以体现假设检验的价值.

Power of test: P(reject | H_0: is false).

- Ideally, one would want a framework that has very high $\boxed{\text{power,}}$ or high probability of rejecting an incorrect model.
- The current framework could be improved by choosing a ① lower VaR confidence level or by ② increasing the number of data observations. n↑.
  - The horizon should be as short as possible in order to increase the number of observations and to mitigate the effect of changes in the portfolio composition.
    horizon / holding period: 生成一个数据所需时间 ⇒ 决定相同时间段,生成数据个数
  - The confidence level should not be too high because this decreases the effectiveness, or power, of the statistical tests.

提高假设检验方法: ① CL↓ ⇒ α↑ ⇒ typeI↑ ⇒ typeII↓ ⇒ Power of test↑

② n↑ ⇒ typeI, typeII↓ ⇒ Power of test↑

相加为1.

Basel Committee Rules for Backtesting

- The Basel Committee requires that market VaR be calculated at the 99% confidence level and back testing over the past year. That is at the 99% confidence level, we would expect to have 2.5 exceptions (250×0.01) each year.
- Economic capital=VaR×(3+k)

Zone	Number of exceptions	Increase in K
Green	0-4	0
Yellow	5	0.4
	6	0.5
	7	0.65
	8	0.75
	9	0.85
Red	10+	1

Basel Committee Rules for Backtesting

- **Four categories of causes for exceptions:** (产生exception的原因)
  - ① **Basic integrity of the model is lacking**. Exceptions occurred because of incorrect data or errors in the programming. The penalty should apply.
  - ② **Model accuracy needs improvement**. The exceptions occurred because the model does not describe risks precisely. The penalty should apply. (模型精准度不够
  - ③ **Intraday trading**. Positions changed during the day. The penalty should be considered. exception↑.
  - ④ **Bad luck**. Markets were particularly volatile or correlations changed. These exceptions should be expected to occur at least some of the time.

一. Principal mapping:

1.过程：①找到本次Portfolio平均到期时间(加权).

②找到Portfolio对应的VaR值;

$VaR_p = D^* \cdot VaR(\text{risk factor}) \cdot N\,Principal.$

2.例子：Portfolio：{1年, coupon=4%., bond FV=100.
5年, coupon=6%, bond FV=100

$1\times\frac{100}{200} + 5\times\frac{100}{200} = 3$(年)

(把平均到期时间的zero coupon bond就看成VaRp

VaR Mapping

VaR and other Risk Measures

本质：计算Portfolio的VaR.：mapping的是risk factor

Asset 1 2 3 4

risk factor1 risk factor2 risk factor3

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二. Duration mapping:

①过程：

1)找到Portfolio的duration.(考虑coupon, Principal)

2)找到duration=2.733年的zero-coupon bond的VaR视为VaRp.

$VaR_p = VaR_B =$ duration × VaR(2.733, interest) × NP.

(线性差值)

2.5% (1.3%) 3.8%

2yr (1yr) 3yr

$\frac{1yr}{1000} = \left(\frac{1.3\%}{1000}\right)$

{1) fixed-income mapping
2) derivatives mapping

◆ VaR Mapping

Fixed-Income的mapping：①②③

➢ Three approaches for mapping

① ● **Principal mapping**. Only the risk associated with the return of principal at the maturity of the bond is mapped. average portfolio maturity.

② ● **Duration mapping**. With duration mapping, one risk factor is chosen that corresponds to the portfolio duration.

③ ● **Cash flow mapping**. With cash-flow mapping, the portfolio cash flows are grouped into maturity buckets.

produce a lower diversified VaR.

and each payment is associated with a different risk factor

三. Cash-flow. mapping.

一.过程：找到对应各时点的现金流.

②每个时点进行映射.

100 106 6 6 106

0 1 2 3 4 5

$VaR_p = \frac{110}{(1+r)^1} \times VaR(1yr, interest) \times 1$

$+ \frac{6}{(1+r)^2} \times VaR(2yr, int) \times 2$

$+ \frac{6}{(1+r)^3} \times VaR(3yr, int) \times 3.$

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专业·创新·增值

◆ Cash Flow Mapping

➢ Each cash flow is represented by the present value of the cash payment, discounted at the appropriate zero-coupon rate.

D-I，没有考虑diversity好处.

● Undiversified VaR: considering interest rate VaRs with each maturities

● Diversified VaR: also considering interest rate correlations

➢ VaR(Principal)>VaR(duration)>VaR(Undiversified)>VaR(Diversified).

~~VaRp=VaRt~~

(最粗糙). 2.duration ①过程：多考虑3 coupon.

②公式：$VaR_p = VaR_B = D \cdot VaR(1\,factor)$

3. cash-flow ①过程. ②$VaR_p = \sum D^* \cdot VaR(t, yr) \cdot P_t$

(二)derivatives：①远期：$F=S \cdot e^{-qT} \cdot e^{rT}$

②FRA 6×12：long 6 bill + short 12 bill.

③Interest rate swap: call: $SN(d_1) - Ke^{-rT}N(d_2)$

④option：BSM model put: $Ke^{-rT}N(-d_2) - SN(-d_1)$

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专业·创新·增值

◆ Mapping Linear Derivatives

➢ **Currency forward contracts**

$\uparrow F_t = \uparrow(S_t e^{-\downarrow yt}) e^{\uparrow rt}$

1) S_t↑ (long foreign currency spot)
2) y↑ → convenience yield. (y→外币利率)看跌利率:与看涨价格.
3) r↑ → 本币利率!

- Long currency forward contract = long foreign currency spot + long foreign currency $\boxed{\text{bill}}$ + short U.S. dollar bill → (本国货币)

➢ **Forward Rate Agreements** 借钱:锁定利率.

- Long 6×12 FRA = long 6-month bill + short 12-month bill

long bill 收利率'

0 —— 6 —— 12.

short bill 𝓉利率'

➢ **Interest Rate Swaps** 收浮动支固定.(如果看涨:支固

- A payer swap is equivalent to buying a floating rate bond and simultaneously shorting a fixed rate bond.
- **Or a series of forward rate agreement with same forward rate.**

➢ **BSM Model:**

$$c = Se^{-yt}N(d_1) - Ke^{-rt}N(d_2) \rightarrow (\text{份})$$

- Long call option = long $N(d_1)$*asset + short $N(d_2)$ bill
- Long put option = long $N(-d_2)$*bill + short $N(-d_1)$*asset

$Put = Ke^{-rt}N(-d_2) - Se^{-yt}N(-d_1)$.

Summary:

一. Parametric法:

1. normal VaR: ①假设: $R = \frac{P_t - P_{t-1}}{P_{t-1}} \sim$ normal

② 公式 $VaR = -\mu + \sigma z$ (单/组)

$= -(\mu - \sigma z)$.

$VaR_p = \sqrt{VaR_1^2 + VaR_2^2 + 2\rho \cdot VaR_1 \cdot VaR_2}$ (只有μ=0).

③ VaR表示 loss

④ VaR (dollar vs %)

2. lognormal VaR: ①假设 $r = \ln\left(\frac{P_t}{P_{t-1}}\right) \sim$ normal

② 公式: $VaR = (1 - e^{\mu - z \cdot \sigma}) P_{t-1}$.

③

3. QQ-Plot: 是否 normal distribution.

①散点

②坐标

四 极值理论: ① Block maxima

→ 先有价值的数据.

→ GEV. (3个 para, μ, σ, ξ).

ξ >0 Frecht, =0, <0

② POT: 门槛 μ

→ threshold 难确定

→ GPD

→ 二个 Parameter:

Scale ← β, ξ.

③ POT vs. GEV.

④ POT: VaR. ES 计算

Reading 2

Risk Measurement for the Trading Book

五. Backtesting ① 二项分布. • 公式 $\frac{x - np}{\sqrt{p(1-p)n}} \sim$ normal $N(0,1)$

P → significant level.

② Kupiec VaR: 对二项分布修正: 考虑 fat tail;

LR > 3.841. reject H_0 (VaR is true).

③ Christoffersen: 加3两个参数 (考虑时间)

二. Non-parametric

1. 定义: ① collect → 排序 → 查数.
2. 假设: 历史可以重演.
3. 优/缺点:
4. 非参数密度估计: 线性插值
5. VaR vs ES →

1) 不能测量尾部损失

2) 不满足次可加性.

◆ VaR Implementation

④ Basel: Capital = VaR 日 + 大了 exception.

➢ The three categories of implementation issues reviewed are:

- (1) Time horizon over which VaR is estimated;(难确定)
- (2) The recognition of time-varying volatility in VaR risk factors;
- (3) VaR backtesting (C↑ → α↓)
  - ✓ Backtesting is not effective when the number of VaR exceptions is small. In addition, backtesting is less effective over longer time horizons due to portfolio instability.

三. semi Hybrid Approach: (非参数法 + 参数元素)

① BRW: 调整权重, 不变数据.

• $w_i = \lambda^{i-1} \frac{(1-\lambda)}{(1-\lambda^n)}$.

• 无 ghost effect.

② HW: ①调整数据, 不变权重. (0不一样).

②公式: $r_t^* = r_t \cdot \frac{\sigma_T}{\sigma_t}$.

③优点:

③ CWHS: 基于HW. 同时考虑 σ 与 ρ.

④ FHS: 基于HW, σ, ρ 及过积比. data.

六. VaR mapping:

(一) Bond 1) Principal: ①过程

②公式: $VaR_p = VaR_B = T \cdot VaR(\text{factor})$

Risk Measures

(迷补)

- **Value at Risk (VaR)** has become a standard risk measure in finance.
  - VaR measures only quantiles of losses, and thus disregards any loss beyond.
  - It has been criticized for its lack of subadditivity.
- **Expected Shortfall (ES)** is the most well-known risk measure following VaR.
  - **First**, ES does account for the severity of losses beyond the confidence threshold.
  - **Second**, it is always subadditive and coherent. 一致性
  - **Third**, it mitigates the impact that the particular choice of a single confidence level. ES比VaR更平滑.(ES是平均数).
  - However, expected shortfall is more complex and computationally intensive.
- **Spectral risk measures** (谱风险度量) → ES配权重
  - Are a promising generalization of expected shortfall.
    - ✓ Favorable **smoothness properties.**
    - ✓ Adapting the risk measure directly to the risk aversion of investors.
    - ✓ It requires little effort if the underlying risk model is simulations-based.

风险测量方法(1)(2)

Integrated Risk Measurement

- We survey the academic literature on the implications of modelling the aggregate risks present across a bank's trading and banking books using either a
  (分区化) $VaR_{总} = VaR_m + VaR_c + VaR_o$ $\rho=1$.

  Basel
  - (1) **Compartmentalised approach**: the sum of risks measured separately
  - (2) **Unified approach**: considers the interaction between these risks explicitly. (统一化).
- **The Basel framework** is based on a "building block" approach such that a bank's regulatory capital requirement is
  - The sum of the capital requirements for each of the defined risk categories (i.e., market, credit and operational risk), which are calculated separately within the formulas and rules that make up pillar 1.
  - Capital requirements for other risk categories are determined by the supervisory process that fits within Pillar 2;
  - This approach is therefore often referred to as a non-integrated approach to risk measurement.

Fundamental Review of the Trading Book:

FRTB对Basel修正:

- **Expected Shortfall (ES)**
  - The FRTB is proposing a change to the measure used for determining market risk capital.
  - Under the FRTB proposal, banks would forgo combining a 99% VaR with 99% Stress VaR, and instead, calculate expected shortfall (ES) with a 97.5% confidence level. 放弃.
  - Capital is based solely on the calculation of the expected shortfall using a 12-month stressed period(250 days).
    - ✓ Analogously to the way stressed VaR is determined for Basel II.5, banks are required to search back through time and choose a period that would be particularly difficult for the bank's current portfolio.

• Basel I and Basel II.5 required calculation of VaR with a 99% confidence interval.

◆ FRTB

- **Two approaches to calculate ES**
  - Internal models-based approach(**IMA**) (基础用)
  - Revised standardized approach
- **Liquidity Horizons**
  - FRTB requires the changes to market variables (referred to as shocks) to be the changes that would take place (in stressed market conditions) over periods of time that reflect the differing liquidities of market variables. The periods of time are referred to as liquidity horizons.
  - Five different liquidity horizons are used: 10 days, 20 days, $\boxed{40}$ days, 60 days, and $\boxed{120}$ days. VaR用10天.

◆ FRTB

- **Treatment of Credit Spread** 有些产品无法剥离 credit risk; Prepayment risk: interest rate↓, risk↑.
  - The FRTB provides **a modification of the IRC**. It recognizes that for instruments dependent on the credit risk of a particular company, two types of risk can be identified: incremental risk charge.
    - ✓ **Credit spread risk.** This is the risk that the company's credit spread will change, causing the mark-to-market value of the instrument to change. Still Use $\boxed{\text{ten-day}}$, $\boxed{\text{99\% VaR}}$ (信用质量一直在变化).
    - ✓ **Jump-to-default risk.** This is the risk that there will be a default by the company. Typically this leads to an immediate loss or gain to the bank. Change to $\boxed{\text{one-year}}$, $\boxed{\text{99.9\% VaR}}$.

直接转到 default

Reading 3

Modeling Dependence: Correlations And Copulas

Some Correlation Basics

Modeling Dependence: Correlations And Copulas

用来map marignal distribution

joint distribution.

Correlation Swap

看涨相关系数.

- Paying a fixed rate in a **correlation swap** is also called buying correlation. The payoff = NP × (realized ρ − fixed ρ). 了规定.

Principal

n: number of asset in a portfolio;

$$\rho_{realized} = \frac{2}{n^2 - n}\sum_{i>j}\rho_{i,j}$$

真实 float).

- This is because the present value of the correlation swap will increase for the correlation buyer if the realized correlation increases.
- The fixed rate receiver is selling correlation.

Correlation fixed rate payer	Fixed percentage (e.g. ρ = 10%) → / ← Realized ρ	Correlation fixed rate receiver

Correlation Swap

$\rho_{fix} = 0.1$.

- What is the payoff of a correlation swap with three assets, a fixed rate of 10%, a notional amount of $1,000,000, and a 1-year maturity? Let's assume the realized pairwise correlations of the log returns at maturity are as displayed.

Pairwise Pearson Correlation Coefficient at Swap Maturity			
	$S_{j=1}$	$S_{j=2}$	$S_{j=3}$
$S_{i=1}$	1	0.5	0.1
$S_{i=2}$	0.5	1	0.3
$S_{i=3}$	0.1	0.3	1

$\frac{2}{n^2 - n}$,

$$\rho_{realized} = \frac{2}{3^2 - 3}\sum(0.5 + 0.3 + 0.1) = 0.3$$

- So, the payoff for the correlation fixed rate payer at swap maturity is $1,000,000 × (0.3 -0.1) = $200,000

◆ Correlation Swap

risk

- There is a positive relationship between correlation and volatility.
  - Therefore, if correlation between the stocks of the Dow increases, so will the implied volatility of the call on the Dow. 指数. ρ↑→σ↑.
  - Another way of buying correlation (i.e., benefiting from an increase in correlation) is to **buy put options on an index such as the Dow Jones Industrial Average and sell put options on individual stocks of the Dow.**

on index spread.

on individual.

(买Put卖Put). 低买高卖

低strike price.

• 认为index变动比个股变动多

index↑↑, stock↑.

• ρ↑→市场变差.(赌风险↑)→index↓↓, stock↓

→long put on index + short put on stock

(赚多) + (亏少) ⟹对冲

(赚).

◆ Credit Crisis Resulting from CDOs

callateralized debt obligation (CDO): Pools of loan

CDS: long mezzanine, short equity

- **Failure 1: Loss from the strategy that long the equity tranche of the CDO and short the mezzanine tranche with CDO.** (赌错金融危机发生时间)
  - Going long the equity tranche means receiving the (high) equity tranche contract spread.(sell CDS on equity tranche)
  - Shorting the mezzanine tranche means paying the (fairly low) mezzanine tranche contract spread.(buy CDS on mezzanine tranche)
- **When the correlations of the assets in the CDO decreased, the hedge funds lost on both positions.**

leg

normal: Equity price↓ risk↑ Premium↑

senior price↑ risk↓ premium↓.

stress: ρ=1

Price↑, → Premium↓ (risk↓)

Price↓ → Premium↑ (risk↓).

ρ↓

ρ上升时,赚钱

ρ↓时, lost in both position.

◆ Credit Crisis Resulting from CDOs

(赚保费) Short equity CDS, (亏保费) long mezzanie CDS

- **Failure 2: Loss from the default on equity tranche** (同样的策略) (低估了financial crisis)
  - From 2007 to 2009, default correlations of the mortgages in the CDOs increased. **This actually helped equity tranche investors.**
  - **However, this increase was overcompensated by a strong increase in default probability of the mortgages.**
- As a consequence, tranche spreads increased sharply, resulting in huge losses for the equity tranche investors as well as investors in the other tranches.

• 盈亏在保费上.

Credit Crisis Resulting from CDOs

最后senior, equity 没什么区别).

> **Failure 3: Loss from senior tranche** (只long senior)

- Correlations between the tranches of the CDOs also increased during the crisis. **This had a devastating effect on the super-senior tranches.**
- In normal times, these tranches were considered extremely safe since:
  - ✓ They were AAA rated.
  - ✓ They were protected by the lower tranches.
- But with the increased tranche correlation and the generally deteriorating credit market, **these super-senior tranches were suddenly considered risky and lost up to 20% of their value.** (亏 3 80%)

Credit Default Swaps

④ > **Failure 4: Loss from greed** (贪心) (CDS的投机)

- CDSs can also be used as **speculative instruments.** (买保险,但不持有资产).
  - ✓ For example, the CDS seller (i.e., the insurance seller) hopes that the insured event (已投保的) (e.g., default of a company or credit deterioration of the company) will not occur.
  - ✓ In this case the CDS seller keeps the CDS spread (i.e., the insurance premium) as income.
- A CDS buyer who does not own the underlying asset is speculating on the credit deterioration of the underlying asset.

实证结论:

Correlation Risk and Other Risks

① > **The default correlation within sectors is higher than between sectors.** (ρ个 ρ行业内 > ρ行业间.)

- a lender is advised to have a sector-diversified loan portfolio to reduce default correlation risk.

② > **Time dependency of credit risk**

1) ● For most investment grade bonds, the longer the time horizon, the higher the probability of adverse events. (时间越长,违约概率越大).

2) ● For a distressed company, the longer the time horizon, the probability of default decreases.

③ > **Systemic risk and correlation risk are highly dependent, but not significant.** (β ρ.) (统计学上不显著).

Empirical Properties of Correlation

Modeling Dependence: Correlations And Copulas

◆ Equity Correlation Behaviors

- **Correlation levels are lowest in strong economic growth times.**
  - In recessions, correlation levels typically increase.
- **Correlation volatility is lowest in an economic expansion and highest in normal economic states.**
- **Positive correlation between correlation and correlation volatility** $\rho(\rho,\sigma)>0$.
- **Before every recession a downturn in correlation volatility occurred.** However, the relationship between a decline in volatility and the severity of **the recession is statistically non-significant**.

$S_t - S_{t-1} = a\cdot(\mu_s - S_{t-1})\cdot\Delta t + \sigma_s\cdot\sqrt{\Delta t}$.

◆ Mean reversion & Autocorrelation in correlations

令$\Delta t=1$，不考虑波动因素，$\sigma_s\sqrt{\Delta t}=0$.

均值复归.

- **Mean reversion** is present if there is a negative relationship between the change of a variable, $S_t - S_{t-1}$, and the variable S_{t-1}.
  - $S_t = a(\mu_s - S_{t-1}) + S_{t-1}$ a=1时，瞬间回归均值.
- **In that case, a is called the mean reversion coefficient, or simply, mean reversion.** $S_t - S_{t-1} = a(\mu_s - S_{t-1})$.

$\Rightarrow S_t - S_{t-1} = a(\mu_s - S_{t-1})$.

$\Delta S_t = a(\mu_s - S_{t-1})\cdot\Delta t + \sigma\sqrt{\Delta t}$

$dr = k(\theta - r_0)\cdot dt + \sigma\cdot dw$.

ΔS_t $\quad S_t = a\mu_s + (1-a)S_{t-1}$

$S_t - S_{t-1} = a\cdot(\mu_s - S_{t-1})$.

$$S_t = a(\mu_s - S_{t-1}) + S_{t-1} \quad \text{Or} \quad S_t - S_{t-1} = a\mu_s - aS_{t-1}$$

a: mean reversion coefficient.
(a=1，瞬间回归mean)

(1-a) 自相关 autocorrelation.

- To find the mean reversion rate a, we can run a standard regression analysis of the form

$$Y = \alpha + \beta X$$

- We are regressing $S_t - S_{t-1}$ with respect to S_{t-1}:

$$\underbrace{S_t - S_{t-1}}_{Y} = \underbrace{a\,\mu_s}_{\alpha} - \underbrace{a\,S_{t-1}}_{\beta X} \quad X = S_{t-1}.$$

$\Delta S_t \qquad \beta = -a$.

◆ Mean reversion & Autocorrelation in correlations

- Autocorrelation is the degree to which a variable is correlated to its past values. (昨天的我对今天的我们影响).
  - In finance, positive autocorrelation is also termed **persistence**. (持续性) In mutual fund or hedge fund performance analysis, an investor typically wants to know if an above-market performance of a fund has persisted for some time.
- **Autocorrelation is the opposite property of mean reversion.**

mean reversion + autocorrelation = 1.

$a + (1-a) = 1.$

• Copulas: 关联结构，为处理统计中随机变量相关性问题的一种方法，由一组随机变量的边际分布来确定它们的联合分布.

Financial Correlation Modeling

Modeling Dependence: Correlations And Copulas

A 未知分布.	(Prob) CDF $G(U_i)$		normal
X_1	10%		Y_1
X_2	20%	$\xrightarrow{\text{mapping}}$	Y_2
X_3	30%		Y_3.

◆ Copulas (纯统计学模型) → 联合概率

- **Copula functions are designed to simplify statistical problems.** They allow the joining of multiple univariate distributions to a single multivariate distribution.
- **There exists a copula function C such that:**

univariate distribution. → Probability

$$C[G_1(u_1), \ldots, G_n(u_n)] = F_n[F_1^{-1}(G_1(u_1)), \ldots, F_n^{-1}(G_n(u_n)); \rho_F]$$

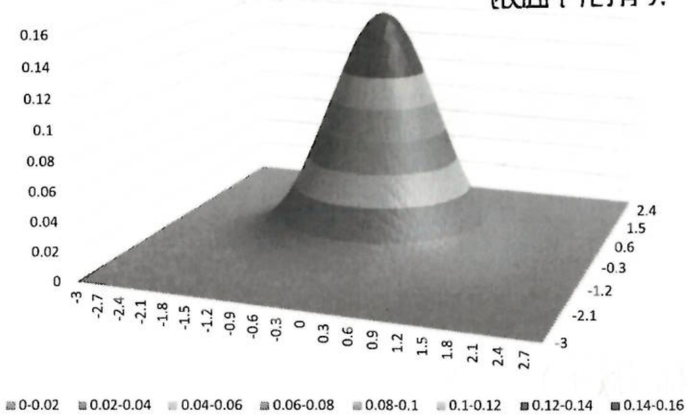
Copula. 未知分布的CDF.

- $G_i(u_i)$: marginal uniform distributions
- F_n: the joint cumulative distribution function
- $F_i^{-1}(G_i(u_i))$: the inverse of F_i, standard normal distribution →随机变量
- ρ_F: the correlation structure of F_n

◆ David Li's Copula

Bivariate Normal Distribution M_2

表面不光滑).



Reading 4

Empirical Approaches to Risk Metrics and Hedges

◆ Regression Hedge (线性对冲).

➢ **Let Δy_t^N and Δy_t^R be the changes in the yields of the nominal and bonds.**

$$\Delta y_t^N = \alpha + \beta \Delta y_t^R + \varepsilon_t$$

$\beta = \dfrac{Cov(X,Y)}{Var(X)}$

- Δy_t^N = changes in the nominal yield
- Δy_t^R = changes in the real yield

$\alpha = \bar{Y} - \beta \bar{X}$.

➢ Least-squares estimation of α and β finds the estimates $\hat{\alpha}$ and $\hat{\beta}$ that minimize the sum of the squares of the realized error terms over the observation period.

$$\sum_t \hat{e}_t^2 = \sum_t (\Delta y_t^N - \hat{\alpha} - \widehat{\beta \Delta y_t^R})^2$$

Regression Hedge

- The trader plans to short \$100 million of the (nominal) 35/8 s of August 15, 2019, and, against that, to buy some amount of the TIPS 17/8 s of July 15, 2019. The nominal yield in the data changes by 1.0189 basis points per basis-point change in the real yield. How to hedge the position?

$\Rightarrow F^R \cdot DV01 \cdot \Delta y^R = F^{hedge} \cdot DV01_H \cdot \Delta y^{nominal}$

Bond	Yield (%)	DV01
TIPS 1 7/8	1.237	0.081
T-Bond 3 5/8	3.275	0.067

$$F^R \times \frac{0.081}{100} = 100mm \times \frac{0.067}{100} \times 1.0189$$

$$F^R = \$100mm \times \frac{0.067}{0.081} \times 1.0189 = \$84.3mm$$

$\Delta P = -D^* \cdot \Delta y \cdot P$ (标的资产).

$\parallel$

$\Delta P = -D^* \cdot \Delta y \cdot P$ (对冲资产)

$\Rightarrow D_S^* \cdot P_S = D_F \cdot P_F$.

$\Rightarrow \Delta y_S \times D_S^* \times P_S = \Delta y_F \cdot D_F \cdot P_F$

1.0189   1

Regression Hedge

- The market maker in question has bought or received fixed in relatively illiquid 20-year swaps from a customer and needs to hedge the resulting interest rate exposure.
- Immediately paying fixed or **selling 20-year swaps** would sacrifice too much if not all of spread paid by the customer, **so the market maker chooses instead to sell a combination of 10- and 30-year swaps.**
- Two variable regression model to **describe the relationship between changes in 20-year swap rates and changes in 10-and 30-year swap rates:**

$$\Delta y_t^{20} = \alpha + \beta^{10}\Delta y_t^{10} + \beta^{30}\Delta y_t^{30} + \varepsilon_t$$

(解决 curse of dimensionality)

Principal Components Analysis

- Principal component analysis is a powerful statistical tool that can help solve the curse of dimensionality.
- PCs of rates are particularly useful because of an empirical regularity:
  - The sum of the variances of the first three PCs is usually quite close to the sum of variances of all the rates.
  - Hence, rather than describing movements in the term structure by describing the variance of each rate and all pairs of correlation, one can simply describe the structure and volatility of each of only three PCs.

$PC_1 \geqslant PC_2 \geqslant PC_3$.

Summary: model 1: $dr = \sigma \cdot dw$ (无drift).

model 2: $dr = \lambda dt + \sigma \cdot dw$ (一般式,均衡).

Holee: $dr = \lambda(t)dt + \sigma \cdot dw$.

$\underline{dr} = k(\theta - r)dt + \sigma \cdot dw$. (mean reversion). } 无套利.

↳ 只对drift项调整

Reading 5

Term Structure Models of Interest Rates

model 3: $dr = \lambda(t)dt + \sigma(t)dw$. 一般式.

$dr = \lambda(t)dt + \sigma \cdot e^{-at}dw$

CIR: $dr = k(\theta - r)dt + \sigma\sqrt{r}dw$ (mean revers

总结: 会对drift and volatility 做调整.

model 4: $\underbrace{\frac{dr}{r}}_{return} = a\,dt + \sigma \cdot dw$. (一般式.

$d(\ln r) = a(t)dt + \sigma \cdot dw$ (类似Ho-lee)

$d(\ln r) = k(t)[\ln\theta(t) - \ln r]dt + \sigma(t)d$

(类似CIR模型).

mean reversion ↗

The Science of Term Structure Models

Term Structure Models of Interest Rates

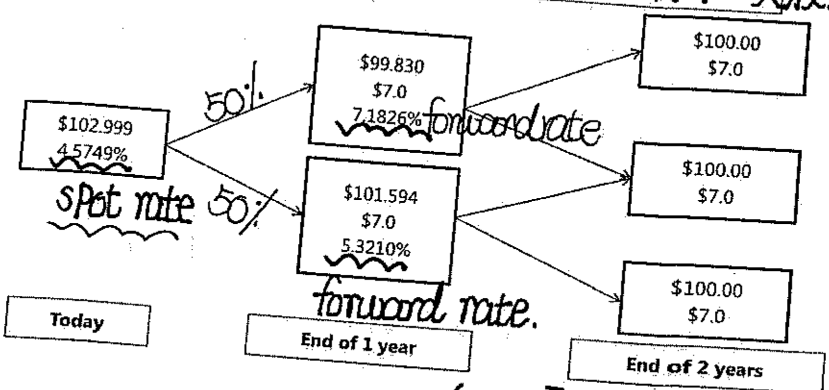
利率小(平) 不知

107.

$102.999 = \frac{\cancel{98}\ 99.83+7}{1+4.5749\%} \times 50\% + \frac{101.594+7}{1+4.5749\%} \times 50\%$

Valuing an Option Bond with the Tree

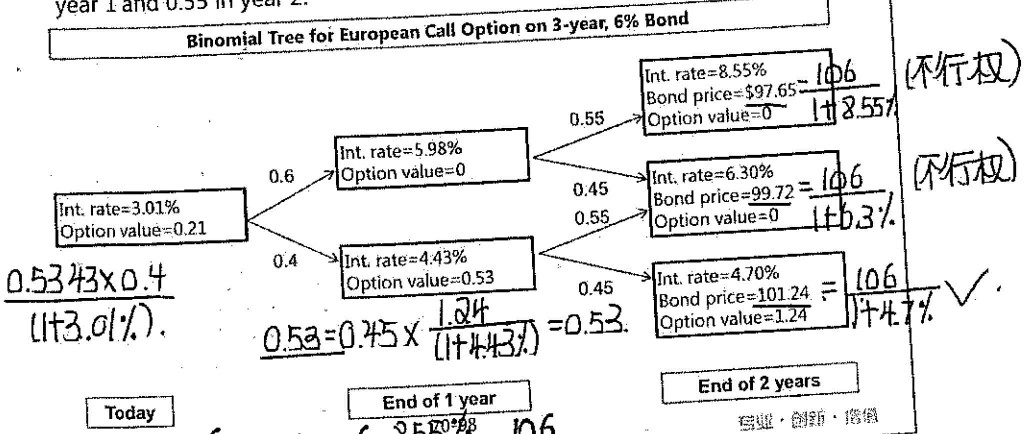
Valuing a 2-year, 7.0% Coupon, Option-free Bond 利率二叉树. (一步利率二叉树).

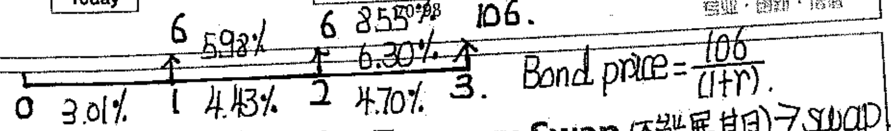


$\left(\frac{107}{1+7.1826\%} \times 50\% + \frac{107}{1+5.3210\%} \times 50\% + 7\right) \times \frac{1}{1+4.5749\%}$

◆ Valuing an Option on Bond with the Tree

- Value a European call option with two years to expiration and a strike price of $100.00. The underlying is a 6%, annual-coupon bond with three years to maturity. Assume that the risk-neutral probability of an up move is 0.6 in year 1 and 0.55 in year 2.



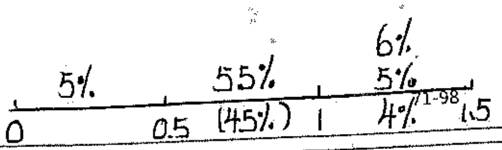


◆ Constant Maturity Treasury Swap (不能展期)→swap (期限保持不变)

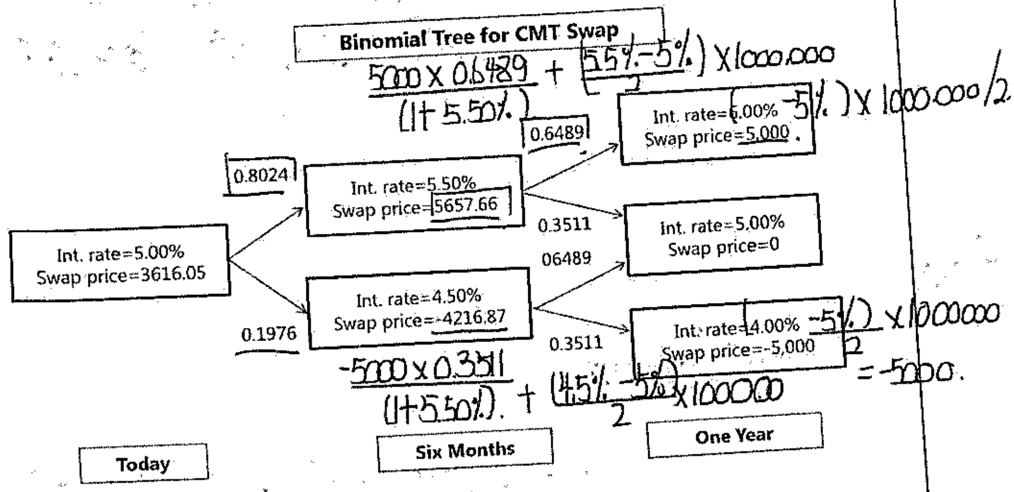
- **A CMT swap is an agreement to swap a floating rate for a Treasury rate such as the 10-year rate.**
- **Example: CMT swap**
  - The rate tree can be used to price a constant-maturity Treasury (CMT) swap. In the example, the strike (fixed rate) is 5.0% such that every six months until maturity the swap pays:

$$\left(\frac{\$1{,}000{,}000}{2}\right)\times(y_{CMT}-5\%)$$ 每6个月换1次.

  - y_{CMT}: a semiannually compounded yield, of a predetermined maturity, on the payment date. The text prices a one-year CMT swap on the six month yield. In practice, CMT swaps trade most commonly on the yields of the most liquid maturities, i.e., on 2-, 5- and 10-year yields.



◆ Constant Maturity Treasury Swap



Fixed Income Securities & Black-Scholes-Merton

- Why the Black-Scholes-Merton model to value equity derivatives is not appropriate to value derivatives on fixed-income securities? Price ~ lognormal, 因为股票的价格可以随机 bond价格不可以随机波动.

① The price of a bond must converge to its face value at maturity while the random process describing the stock price need not be constrained in any similar way.

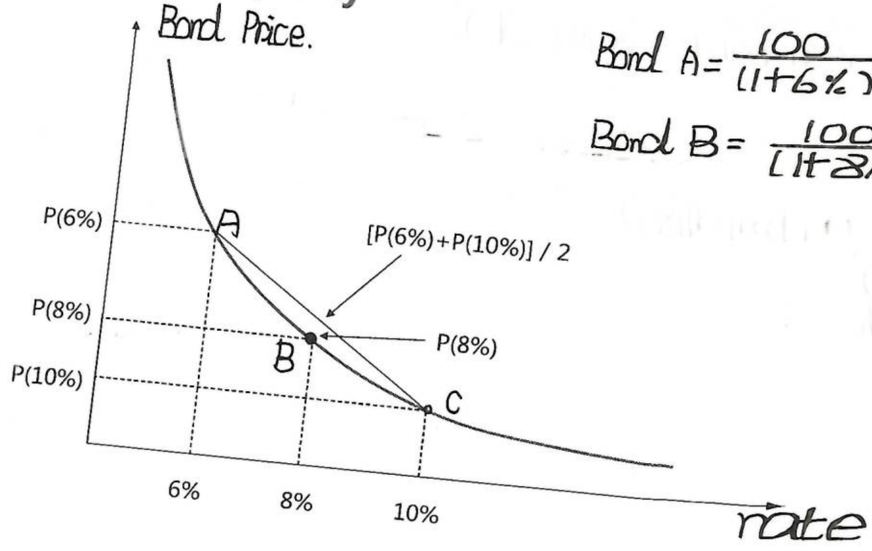
② Because of the maturity constraint, the volatility of a bond's price must eventually get smaller as the bond approaches maturity. The simpler assumption that the volatility of a stock is constant is not so appropriate for bonds.

③ Since stock volatility is very large relative to short-term rate volatility, it may be relatively harmless to assume that the short-term rate is constant. By contrast, it can be difficult to defend the assumption that a bond price 相矛盾 follows some random process while the short-term interest rate is constant. (矛盾.)

The Evolution of Short Rates and the Shape of the Term Structure

Term Structure Models of Interest Rates

Jensen's Inequality



$$\text{Bond } A = \frac{100}{(1+6\%)} \times 50\% + \frac{100}{(1+10\%)} \times 50\% \quad (\text{价高})$$

$$\text{Bond } B = \frac{100}{(1+8\%)}$$

◆ Jensen's Inequality

➢ **Convexity Effect** 两张债券的价格差异.

- The convexity effect arises from a special case of Jensen's Inequality:

expectation. (0 时刻)

$$E\left[\frac{1}{(1+r)}\right] > \frac{1}{E(1+r)}$$

- All else held equal, the value of convexity increases with maturity and volatility.

$$E\left(\frac{1}{1+r}\right) = \frac{1}{1+6\%} \times 0.5 + \frac{1}{1+8\%10\%} \times 0.5$$

$$\frac{1}{E(1+r)} = \frac{1}{(1+6\%)\times 0.5 + (1+10\%)\times 0.5} = \frac{1}{1+8\%}$$

Summary: ① model 1: $dr = \sigma \cdot dw$ ($dw = \varepsilon\sqrt{dt}$). 无drift.

② model 2: $dr = \lambda dt + \sigma \cdot dw$ (一般式) 均衡模型.

→ Holee: $dr = \lambda(t)dt + \sigma dw$.

$dr = k(\theta - r)\cdot dt + \sigma \cdot dw$. (mean reverse).

· 模型 2 只会对 drift 项做调整.

③ model 3: $dr = \lambda(t)dt + \sigma(t)\,dw$ 一般式.

· $dr = \lambda(t)dt + \sigma \cdot e^{-\alpha t}dw$

CIR: $dr = k(\theta - r)dt + \sigma\sqrt{r}\,dw$.

(mean reversion)

会对 drift and volatility 做调整

model 4: $\frac{dr}{r} = a\,dt + \sigma dw$. (一般式)

$d\ln(r) = a(t)dt + \sigma\,dw$. (类似 Ho-Lee)

$d\ln(r) = k(t)\,[\ln\theta(t) - \ln r]\,dt + \sigma(t)dw$.

→ (类似 CIR)

mean reversion.

The Art of Term Structure Models: Drift & Volatility and Distribution

Term Structure Models of Interest Rates

$dr = \sigma \cdot \varepsilon \cdot \sqrt{dt}$. (人为规定 $\varepsilon = +1$ or -1).

$$dr = \begin{cases} \sigma\cdot\sqrt{dt} \\ -\sigma\sqrt{dt} \end{cases}$$

σ: basis point volatility (利率变动的来源). constant 常数.

dr: 利率变动: $\Delta r = dr = r_1 - r_0$

◆ Term Structure Model with No Drift (Model 1)

dw: random variable. $\sim N(0, dt)$.

$$dr = \sigma dw \qquad dw = \varepsilon\sqrt{dt}$$

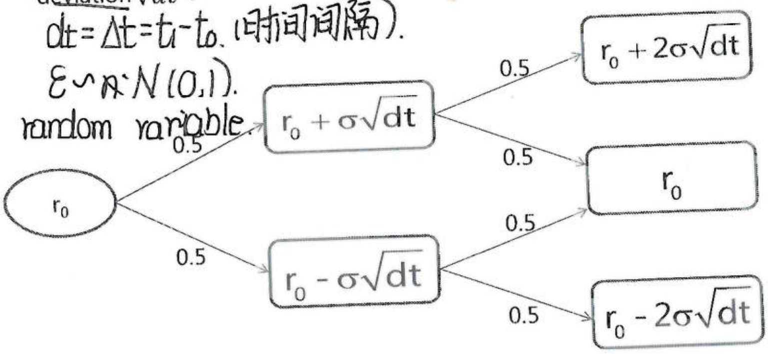
$dw = \varepsilon\sqrt{dt}$.

- dw = normally distributed random variable with mean 0 and standard deviation $\sqrt{dt}$.

$dt = \Delta t = t_1 - t_0$. (时间间隔).

$\varepsilon \sim N(0,1)$.

random variable



◆ Term Structure Model with No Drift (Model 1)

- A problem with Gaussian models is that the short-term rate can become negative.
- Solutions for negative interest rates
  - **A non-normal distribution can be assumed**: For example, if we assume interest rates are lognormally distributed, then the short-term rate cannot become negative . However, building a model around a probability distribution that rules out negative rates or makes them less likely may result in volatilities that are unacceptable. 不服从正态分布).
  - **Use shadow rates** (force the "adjusted" tree rates to be non-negative): Another popular method of ruling out negative rates is to construct rate trees with whatever distribution is desired, as done in this section, and then simply set all negative rates to zero. In this methodology, rates in the original tree are called the shadow rates of interest while the rates in the adjusted tree could be called the observed rates of interest.

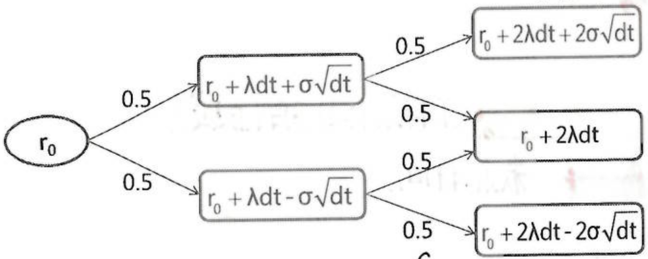
◆ Term Structure Model with Drift (Model 2) 均衡模型:

λ: constant. (>0)正数).

$$dr = \lambda dt + \sigma dw$$

$$dr = \begin{cases} \lambda dt + \sigma\sqrt{dt} \\ \lambda dt - \sigma\sqrt{dt} \end{cases}$$

·model 2 is more capable of producing an upward-sloping term structure



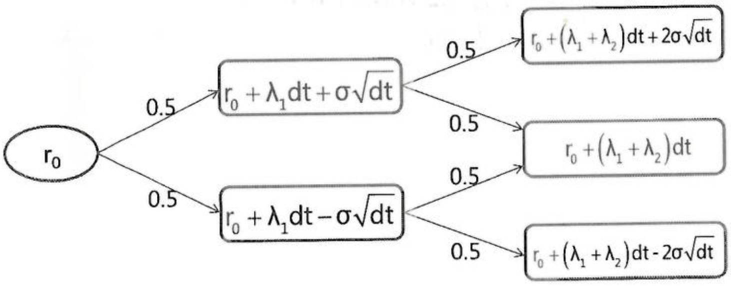
优点: 解决model 1中负利率问题
缺点: λ>0, 但利率不应该一直上升).
(λ根据市场上流动性好的bond, 反推出来的).

allows for a risk premium to be applied to interest rates that changes over time.

◆ Ho-lee Model with Time-dependent Drift 变形. (无套利模型): 根据市场状况对模型进行反馈.

- It is clear that if $\lambda_1 = \lambda_2$ then Ho-Lee model reduces to Model 2.

$$dr = \lambda(t)\,dt + \sigma \cdot dw.$$



▲计算. $S_t - S_{t-1} = a(\mu s - S_{t-1})$. a: mean reversion

◆ Vasicek Model (变形a).

- The Vasicek Model introduces mean reversion into the rate model, which is a common assumption for the level of interest rates. The Vasicek Model is given by:

coefficient

mean reversion coeffec

$$dr = \kappa(\theta - r)dt + \sigma dw$$

均值复归速度.

- k = a parameter that measures the speed of reversion adjustment
- θ = long-run value of the short-term rate assuming risk neutrality
- r = current interest rate level

- The greater the difference between r and θ, the greater the expected change in the short-term rate toward θ.
- **Basis point volatility** refers to the volatility of dr, which is positively correlated to the level of current interest rate.

◆ Vasicek Model 新增

- For completeness, the expectation of the rate in the Vasicek model after T years is :

$$r_T = r_0 e^{-kT} + \theta(1 - e^{-kT})$$

- The half-life of the interest rate, τ, is given by the following equation:

半衰期

$$\theta - [r_0 e^{-k\tau} + \theta(1 - e^{-k\tau})] = \frac{1}{2}(\theta - r_0)$$

$$(\theta - r_0)e^{-k\tau} = \frac{1}{2}(\theta - r_0)$$ (T年后, r_0与θ的距离).

$$\tau = \frac{\ln 2}{k}$$ 表示时间.

- In words, the interest rate factor takes τ years to cover half the distance between its starting value and its goal. Larger mean-reverting parameters produce shorter half lives.

$\frac{1}{2}(\theta - r_0) = \theta - r_T$.

$r_T = r_0 e^{-rT} + \theta(1-e^{-rT})$

$\Rightarrow \frac{1}{2}(\theta - r_0) = \theta - [r_0 e^{-kT} + \theta(1-e^{-kT})]$

$\Rightarrow r_T = \theta - (\theta - r_0)\cdot e^{-rT} = \theta(1-e^{-rT}) + r_0 \cdot e^{-rT}$.

$(\theta - r_0)\cdot e^{-rT} = (\theta - r_T)$.

$\frac{1}{2}(\theta - r_0) = \theta - [\theta(1-e^{-rT}) + r_0 \cdot e^{-kT}]$

$= \theta - \theta + \theta \cdot e^{-kT} - r_0 \cdot e^{-kT}$.

$\frac{1}{2}(\theta - r_0) = e^{-kT}(\theta - r_0)$. $\Rightarrow -\ln 2 = -kT \Rightarrow T = \frac{\ln 2}{k}$.

◆ Term Structure Model with Time-dependent Volatility (Model 3)

model 2: $dr = \lambda dt + \sigma(t) dw$.

$$dr = \lambda(t)dt + \sigma(t)dw$$

$$dr = \lambda(t)dt + \sigma e^{-at}dw$$ $\rightarrow 0$ (整体趋向为0).

$a > 0, t \rightarrow \infty$,

- σ: volatility at t = ~~0~~ 0, which decreases exponentially to 0.

◆ Other Models

➢ **Cox-Ingersoll-Ross (CIR) Model** (model 3 的变形) mean reversion.

$$dr = \kappa(\theta - r)dt + \sigma\sqrt{r}dw$$

r → 当期利率水平.

$\sigma\sqrt{r}$: basis point volatility.

➢ **Lognormal model (Model 4)** (利率变动百分比).

$$dr = ardt + \sigma rdw$$

$$\frac{dr}{r} = a\,dr + \sigma dw.$$

$$\frac{r_1 - r_0}{r_0} = \text{Return}.$$

➢ **Lognormal model with deterministic drift**

$$d[ln(r)] = a(t)dt + \sigma dw$$

➢ **Lognormal model with mean reversion**

$$d[ln(r)] = k(t)[ln\theta(t) - ln(r)]dt + \sigma(t)dw$$

Reading 6

Volatility Smiles

◆ Volatility Smiles is the Same for Calls and Puts

① $P + S = C + Ke^{-rT}$: BSM.

② $P_{mkt} + S = C_{mkt} + Ke^{-rT}$: market.

①-②: $P_{mkt} - P_{BSM} = C_{mkt} - C_{BSM}$

➢ **Based on the put-call parity** 隐含波动率

- The implied volatility of a **European call option** is always the same as the implied volatility of a **European put option** when the two have the same strike price and maturity date.

BSM: $C = SN(d_1) - Ke^{-rT}N(d_2)$. 得到 call 市场报价

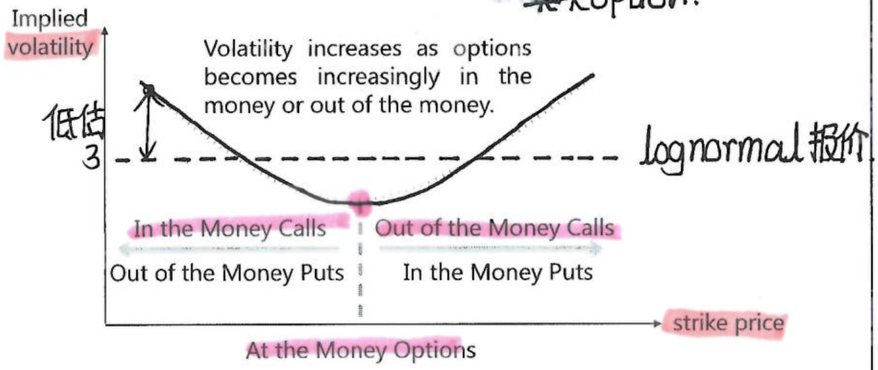
$C_{mkt} = SN(d_1) - Ke^{-rT}N(d_2)$.

$d_{1,2} = \frac{\ln(\frac{S}{K})e^{-rT}}{\sigma\sqrt{T}} \pm \frac{1}{2}\sigma\sqrt{T}$ 反推出 σ: (implied volatility). 反推出来.

Volatility Smile for Foreign Currency Options

> The implied volatility is relatively low for **at-the-money** options. It becomes progressively higher as an option moves either **into the money or out of the money.**

真实option.



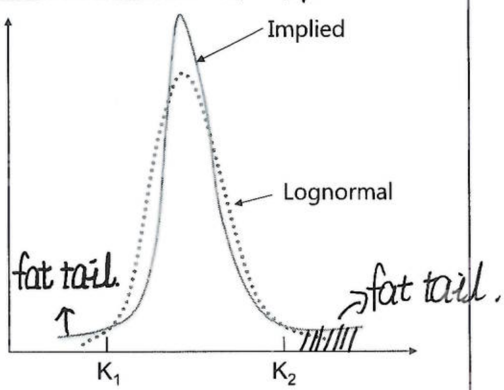
Reasons for Smile in Foreign Currency Options

为什么 exchange rate

> Why are exchange rate **not lognormally distributed?** Two of the contidions for an asset price to have a lognormal distribution are:

1. **The volatility of the asset is constant.**
2. **The price of the asset changes smoothly with no jumps.** 图二：PDF

In practice, neither of these conditions is satisfied for an exchange rate. The volatility of an exchange rate is far from constant, and exchange rates frequently exhibit jumps. (sometimes the jumps are in response to the actions of **central banks**).



Reasons for Smiles in Foreign Currency Options

1. > **Alternative ways of characterizing the volatility smile**

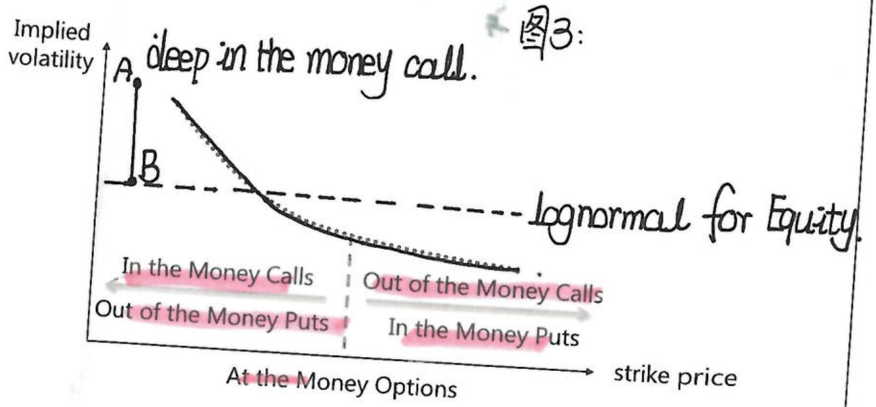
- The volatility smile is often calculated as the relationship between the **implied volatility** and **K/S_0** rather than as the relationship between the implied volatility and K.
  - ✓ A refinement of this is to calculate the volatility smile as the relationship between the **implied volatility** and **K/F_0**, where F_0 is the forward pirce of the asset for a contract maturing at the same time as the options that are considered.

2. - Another approach to defining the volatility smile is as the relationship between **the implied volatility** and the **delta of the option**.

◆ Volatility Smiles (skew) for Equity Options

➢ The volatility used to price a low-strike-price option (i.e., a deep out of the money put or a deep in the money call) is significantly higher than that used to price a high-strike-price option (i.e., a deep in the money put or a deep out of the money call).

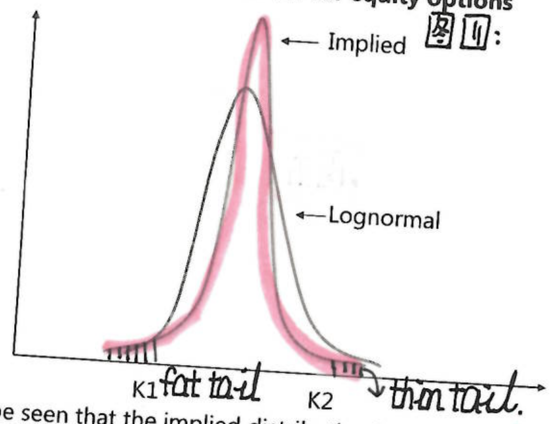
图3:



◆ Reasons for the Smile in Equity Options

➢ **Implied and lognormal distribution for equity options**

图II:



● It can be seen that the implied distribution has a heavier left tail and a less heavy right tail than the lognormal distribution.

◆ Reasons for the Smile in Equity Options

(原因). $\frac{\text{Equity}}{\text{Asset}}\uparrow \Rightarrow$ liability↑ ⇒ risk↑ ⇒ premium↑ ⇒ price↓ ⇒ leverage↑ (惰性循环).

➢ **Leverage**

● As a company's equity declines in value, the company's leverage increases. This means that the equity becomes more risky and its volatility increases. ⇒ Increasing leverage at lower equity prices suggests increasing volatility.

➢ **Volatility Feedback Effect** σ↑ ⇒ Premium↑ ⇒ Price↓ ⇒ risk↑

● As volatility increases (decreases) because of external factors, investors require a higher (lower) return and as a result the stock price declines (increases).

➢ **Crashophobia** (崩盘恐惧症) Put option (on index)

● 1987 stock market crash: higher premiums for $\boxed{\text{put}}$ prices when the strike prices lower.

(指↓23% ⇒ large price declined are more likely than assumed in BSM

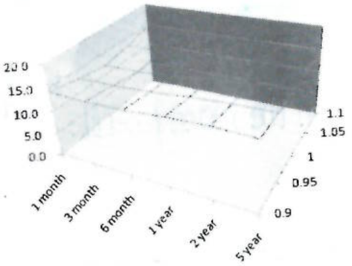
◆ Volatility Term Structure and Volatility Surface

波动率微笑.

到期时间

Volatility surface

	K/S_0				
	0.90	0.95	1.00	1.05	1.10
1 month	14.2	13.0	12.0	13.1	14.5
3 month	14.0	13.0	12.0	13.1	14.2
6 month	14.1	13.3	12.5	13.4	14.3
1 year	14.7	14.0	13.5	14.0	14.8
2 year	15.0	14.4	14.0	14.5	15.1
5 year	14.8	14.6	14.4	14.7	15.0

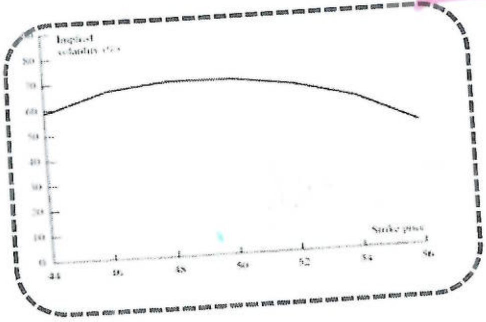


For example, when valuing a 9-month option with a K/S_0 ratio of 1.05, a financial engineer would interpolate between 13.4 and 14.0 in table to obtain a volatility of 13.7%. When valuing a 1.5-year option with a K/S_0 ratio of 0.925, a two-dimensional(bilinear) interpolation would be used to give an implied volatility of 14.525%.

stock price

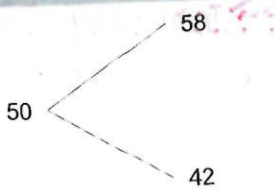
◆ Asset Price Jumps on Volatility Smiles 波动率与曲面.

- Suppose that a stock price is currently \$50 and an important news announcement due in a few days is expected either to increase the stock price by \$8 or to reduce it by \$8.
- The probability distribution of the stock price in, say, 1 month might then consist of a mixture of two lognormal distribution, the first corresponding to favorable news, the second to unfavorable news. 图五.



◆ The Impact of Large Asset Price Jumps on Volatility Smiles

Change in stock price in 1 month



Effect of a single large jump. The solid line is the true distribution; the dashed line is the lognormal distribution.

PDF(双峰) implied.