Limits

- **Maxima**: $f(c)$ is local maximum if $f(c) \geq f(x)$ for x near c (note that inequality is non-strict)
- **Differentiability**: f is differentiable at a if $f'(a)$ exists
  f is differentiable (in its domain) if f is differentiable at every point in its domain
  f is differentiable at point $a \implies f$ is continuous at a (but not the converse)
- **Critical Point**: c is a critical point if c is an <u>interior</u> point in the domain and either $f'(c) = 0$ or $f'(c)$ does not exist (A critical point may not be a local extremum)
- **Inflection Point**: c is an inflection point if concavity changes at c (c is an inflection point $\implies f''(c) = 0$, but the converse is not always true)
- $\lim\limits_{x \to 0} \dfrac{\sin(x)}{x} = 1$
- **L'Hôpital's Rule**:
  $f(a) = g(a) = 0$ or $\infty \implies \lim\limits_{x \to a} \dfrac{f(x)}{g(x)} = \lim\limits_{x \to a} \dfrac{f'(x)}{g'(x)}$
  use $\ln(\dots)$ for other forms: $\infty - \infty$, 1^∞, ∞^0, 0^0

Differentiation & Integration

- **Common derivatives**:

$$y \longrightarrow \frac{dy}{dx}$$
$$\sin x \longrightarrow \cos x$$
$$\cos x \longrightarrow -\sin x$$
$$\tan x \longrightarrow \sec^2 x$$
$$\cot x \longrightarrow -\csc^2 x$$
$$\sec x \longrightarrow \sec x \tan x$$
$$\csc x \longrightarrow -\csc x \cot x$$
$$a^x \longrightarrow a^x \ln a$$
$$\log_a x \longrightarrow \tfrac{1}{x \ln a}$$
$$\sin^{-1} x \longrightarrow \tfrac{1}{\sqrt{1-x^2}}$$
$$\cos^{-1} x \longrightarrow -\tfrac{1}{\sqrt{1-x^2}}$$
$$\tan^{-1} x \longrightarrow \tfrac{1}{1+x^2}$$
$$\cot^{-1} x \longrightarrow -\tfrac{1}{1+x^2}$$
$$\sec^{-1} x \longrightarrow \tfrac{1}{|x|\sqrt{x^2-1}}$$
$$\csc^{-1} x \longrightarrow -\tfrac{1}{|x|\sqrt{x^2-1}}$$
$$x^x \longrightarrow x^x(\ln x + 1)$$

- **Common integrals**:

$$y \longrightarrow \int y\, dx$$
$$\sin x \longrightarrow -\cos x + C$$
$$\cos x \longrightarrow \sin x + C$$
$$\tan x \longrightarrow -\ln|\cos x| + C$$
$$\cot x \longrightarrow \ln|\sin x| + C$$
$$\sec x \longrightarrow \ln|\sec x + \tan x| + C$$
$$\csc x \longrightarrow -\ln|\csc x + \cot x| + C$$

- **Trigonometric formulae**: $1 + \cot^2 x = \csc^2 x$

- **Power rule**: $\dfrac{d}{dx} x^n = n x^{n-1}$
- **Product rule**: $(fg)'(x) = f'(x)g(x) + f(x)g'(x)$
- **Quotient rule**: $\left(\dfrac{f}{g}\right)'(x) = \dfrac{f'(x)g(x) - f(x)g'(x)}{[g(x)]^2}$
- **Chain rule**: $(f \circ g)'(x) = f'(g(x))g'(x)$
- **Parametric differentiation**: $\dfrac{dy}{dx} = \dfrac{dy}{dt} \div \dfrac{dx}{dt}$
- **Implicit differentiation**: Differentiate both sides of an equation containing x and y, then solve the resulting equation for the $\dfrac{dy}{dx}$ term.
- **Integration by parts**: $\displaystyle\int u\, dv = uv - \int v\, du$
  Try to differentiate in this order (highest to lowest priority): $\ln x$, x^n, e^x, e^{-x}, $\sin x$, $\cos x$
- $\dfrac{d}{dx} \displaystyle\int_a^x f(t)\, dt = f(x)$
- $\dfrac{d}{dx} \displaystyle\int_a^{g(x)} f(t)\, dt = f(g(x))\, g'(x)$
- Integral with $\sqrt{R^2 - x^2}$: Sub $x = R\sin\theta$ or $x = R\cos\theta$
- Integral with $\sqrt{R^2 + x^2}$: Sub $x = R\tan\theta$
- **Notable formula**:

$$\begin{aligned}
&\int \frac{A\cos\theta + B\sin\theta}{\cos\theta + \sin\theta} d\theta \\
&= \int \frac{\frac{A+B}{2}(\cos\theta + \sin\theta) + \frac{A-B}{2}(\cos\theta - \sin\theta)}{\cos\theta + \sin\theta} d\theta \\
&= \int \frac{A+B}{2} d\theta + \int \frac{A-B}{2} \frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta} d\theta \\
&= \frac{A+B}{2}\theta + \frac{A-B}{2} \ln(\cos\theta + \sin\theta) + C
\end{aligned}$$

Series

The following is for $\sum a_n$ - a series of numbers.

- **Convergence**: A sequence $\{a_n\}$ is convergent if for some fixed $L \in \mathbb{R}$, we have $\lim\limits_{n \to \infty} a_n = L$
  Otherwise, the sequence is divergent
- **Partial sum of geometric series**: $s_n = a\dfrac{1 - r^n}{1 - r}$, $r \neq 1$
  s_n converges $\iff |r| < 1$
- **Ratio test for convergence of series**:

$$\lim_{n \to \infty} \left| \frac{a_{n+1}}{a_n} \right| = \rho,\ \{s_n\} = \left\{ \sum_{i=1}^{n} a_i \right\} \begin{cases} \text{converges} & \text{if } \rho < 1 \\ \text{diverges} & \text{if } \rho > 1 \\ \text{unknown} & \text{if } \rho = 1 \end{cases}$$

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- **Radius of convergence**: No concept of "radius". Series either converges if $\rho < 1$ or diverges when $\rho > 1$.
- ***p*-series**: $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^p} \begin{cases} \text{converges} & \text{if } p > 1 \\ \text{diverges} & \text{if } 0 \leq p < 1 \\ \text{no conclusion} & \text{if } p = 1 \end{cases}$

The following is for $\sum f_n(x)$ - a series of functions.

- **Power series about $x = a$**: $f(x) = \displaystyle\sum_{n=0}^{\infty} c_n (x-a)^n$
- **Radius of convergence**: The "allowed range" of value for x such that $\rho < 1$ — may be 0, some real no. h, or ∞
  For e.g., $\left|\frac{a_{n+1}}{a_n}\right| = \frac{n}{n+1}|x| \longrightarrow x$ as $n \longrightarrow \infty$, so it converges when $|x| < 1$ and diverges otherwise.
- **Differentiation & integration of power series**:
  $f'(x) = \displaystyle\sum_{n=1}^{\infty} n c_n (x-a)^{n-1}$
  radius of convergence R is unchanged by differentiation
  $\displaystyle\int f(x) dx = \sum_{n=0}^{\infty} c_n \frac{(x-a)^{n+1}}{n+1} + C$
  radius of convergence is unchanged by integration
- **Taylor series**: $f(x) = \displaystyle\sum_{k=0}^{\infty} \frac{f^{(k)}(a)}{k!}(x-a)^k$
- **Maclaurin series**: Taylor series with $a = 0$
- **<u>Common Maclaurin series</u>**:

$\boxed{\text{for } -1 < x < 1}$

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots$$
$$\frac{1}{1+x} = \sum_{n=0}^{\infty} (-1)^n x^n = 1 - x + x^2 - x^3 + \cdots$$
$$\frac{1}{1+x^2} = \sum_{n=0}^{\infty} (-1)^n x^{2n} = 1 - x^2 + x^4 - x^6 + \cdots$$
$$\ln(1+x) = \sum_{n=1}^{\infty} \frac{(-1)^{n-1} x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots$$
$$\tan^{-1} x = \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} x^{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots$$
$$\frac{1}{(1-x)^2} = \sum_{n=1}^{\infty} n x^{n-1} = 1 + 2x + 3x^2 + 4x^3 + \cdots$$
$$\frac{1}{(1-x)^3} = \frac{1}{2} \sum_{n=2}^{\infty} n(n-1) x^{n-2} = \frac{1}{2}(2 + 6x + 12x^2 + \cdots)$$
$$\frac{1}{(1+x)^2} = \sum_{n=0}^{\infty} (-1)^n (n+1) x^n = 1 - 2x + 3x^2 - 4x^3 + \cdots$$

$\boxed{\text{for } -\infty < x < +\infty}$

$$\sin x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots$$

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots$$

- **Deriving more Maclaurin/Taylor series**:
  Substitution:
  $f(x) = \sum g(x) \implies f(h(x)) = \sum g(h(x))$
  Multiplication: $f(x) = \sum g(x) \implies h(x)f(x) = h(x)\sum g(x) = \sum h(x)g(x)$
  Differentiation: $f(x) = \sum g(x) \implies f'(x) = \sum g'(x)$
  Integration:
  $f(x) = \sum g(x) \implies \int_0^x f(x)dx = \int_0^x \sum g(x)dx$
  ... and other usual operations on functions
  Example: Find Taylor series of $\frac{1}{2x+1}$ at $x = -2$ (Note: this means finding Taylor series $(x-a)$ where $a = -2$).
  One way is to **hardcore** differentiate $\frac{1}{2x+1}$ and calculate $\sum_{n=0}^{\infty} \frac{f^{(k)}(-2)}{k!}(x-(-2))^k$. Better soln: write $\frac{1}{2x+1}$ as $q\frac{1}{1-p(x+2)}$ and apply $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 \cdots$
- **Taylor polynomial**:
  n^{th} order Taylor polynomial: $P_n(x) :=$ terms until (and including) x^n, use for approximation
  remainder of order n: $R_n(x) :=$ remaining terms
  error $:=$ absolute value of remaining terms $= |R_n(x)|$
- **Taylor's Theorem**: $R_n(x) = \frac{f^{(n+1)}(c)}{(n+1)!}(x-a)^{n+1}$
  where $x \le c \le a$ (when $x \le a$) or $a \le c \le x$ (when $a \le x$)
  This provides an upper bound for the error term

Partial Differentiation

- Partial derivative wrt x – denoted by $f_x(a,b)$ or $\left.\frac{\partial f}{\partial x}\right|_{(a,b)}$
- $f_{xy} = (f_x)_y = \frac{\partial^2 f}{\partial y \partial x}$
- $f_{xy}(a,b) = f_{yx}(a,b)$ (if f_x, f_y, f_{xy}, f_{yx} are continuous in the neighbourhood around (a,b))
- To check if $f(x,y)$ has partial derivatives of all orders, check if $f_{xy} = f_{yx}$
- **Chain rule**: For $z = f(x,y)$ and $x = x(t)$, $y = y(t)$:

$$\frac{dz}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

  For $z = f(x,y)$ and $x = x(s,t)$, $y = y(s,t)$, $w = w(s,t)$:

$$\frac{\partial z}{\partial s} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial s} + \frac{\partial f}{\partial w}\frac{\partial w}{\partial s}$$

- **Directional derivative** $D_u f$ measures change δf when f moves a small distance δt **in direction of vector** u. For <u>unit vector</u> $u = u_1\mathbf{i} + u_2\mathbf{j}$:

$$D_u f(a,b) = f_x(a,b)u_1 + f_y(a,b)u_2$$

$$\Delta f(\text{actual change}) \approx \delta f = D_u f(a,b) \cdot \delta t$$

  Convert u if not a unit vector! $u' = \frac{1}{\sqrt{u_1^2+u_2^2}}(u_1\mathbf{i} + u_2\mathbf{j})$
- **Gradient vector** essentially $D_u f$ without direction:
  $\nabla f(a,b) = f_x(a,b)\mathbf{i} + f_y(a,b)\mathbf{j}$ or $\nabla f(a,b) \cdot u = D_u f(a,b)$

$$D_u f(a,b) = ||\nabla f(a,b)|| \cos\theta$$

  Direction of $\nabla f(a,b)$ is steepest, maximum value of $D_u f(x,y)$ is $\|\nabla f(x,y)\| = \sqrt{f_x(a,b)^2 + f_y(a,b)^2}$
- **Maxima**: $f(a,b)$ is local maximum if $f(a,b) \ge f(x_1,y_1)$ $\forall$ points (x_1,y_1) near (a,b) (note $\ge$)
- **Critical point**: (a,b) is a critical point if $(f_x(a,b) = 0$ <u>and</u> $f_y(a,b) = 0)$ or $(f_x(a,b)$ or $f_y(a,b)$ does not exist).
  Not all critical pts are min/max points
- **Discriminant**: $D = f_{xx}(a,b)f_{yy}(a,b) - (f_{xy}(a,b))^2$
  If $D > 0$ and $f_{xx}(a,b) > 0$ then (a,b) is a local minimum
  If $D > 0$ and $f_{xx}(a,b) < 0$ then (a,b) is a local maximum
  If $D < 0$ then (a,b) is a saddle point
  If $D = 0$ then no conclusion can be drawn
  For finding max/min, f_{yy} may be used in place of f_{xx}

Ordinary Differential Equations

- **Definition**: $\sum_{i=0}^{n} a_i(x)y^{(i)}(x) = F(x)$
  where $a_i(x)$ and $F(x)$ are functions of x
  and $y^{(i)}(x)$ is the i^{th} derivative of y w.r.t. x
- **Separable equations**: Separate and integrate both sides
- **Exponential decay**: $\frac{dx}{dt} = kx \implies x(t) = x(0)e^{kt}$
  $x(0)$ is the initial val., $k = -\frac{\ln 2}{\tau}$ where τ is the half-life
- **Exponential cooling/heating**:
  $\frac{dx}{dt} = k(x - x_0) \implies x(t) - x_0 = (x(0) - x_0)\, e^{kt}$
  $x(0)$ is the initial value, x_0 is the target value
- **Hyperbolic functions**: $\sinh(x) = \frac{e^x - e^{-x}}{2}$
  $\cosh(x) = \frac{e^x + e^{-x}}{2}$ $\quad \tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$
- **Linear 1st order ODEs**: $\frac{dy}{dx} + P(x)y = Q(x)$
  $\implies ye^{\int P(x)dx} = \int Q(x)e^{\int P(x)dx}dx$
- **Bernoulli equation**: $\frac{dy}{dx} + P(x)y = Q(x)y^n$
  $\implies \frac{dz}{dx} + (1-n)P(x)z = (1-n)Q(x)$ where $z = y^{1-n}$
  ... which is a linear 1st order ODE

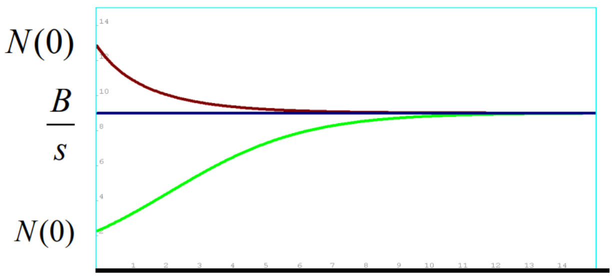
Homogeneous linear 2nd order ODEs

$$\frac{d^2y}{dx^2} + p(x)\frac{dy}{dx} + q(x)y = 0$$

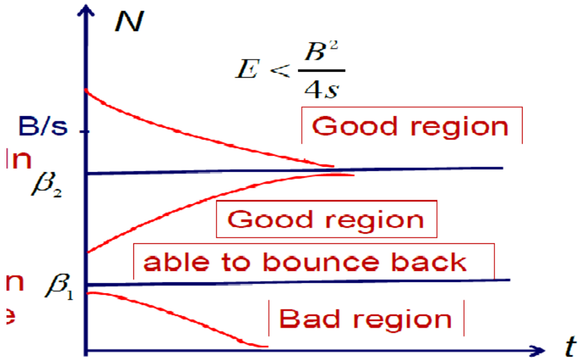
- **Superposition principle**: If y_1 and y_2 are **linearly independent** solns then $c_1y_1 + c_2y_2$ is also a solution.
- **Dimension**: The solution space has dimension 2, so finding 2 **linearly independent** solutions is sufficient to obtain the general solution
- **Guessing**: Obtain 2 lin. indep. solutions by guessing
- **Constant** $p(x)$ **and** $q(x)$:
  Let $\frac{d^2y}{dx^2} + A\frac{dy}{dx} + By = 0$, then solutions have form $y = e^{\lambda x}$ for some value λ
  By substitution, $\lambda^2 + A\lambda + B = 0$, solve for $\lambda = \lambda_1, \lambda_2$
  Two distinct real roots: $y = c_1e^{\lambda_1 x} + c_2e^{\lambda_2 x}$
  Two repeated (real) roots: $y = c_1e^{\lambda x} + c_2xe^{\lambda x}$
  Two distinct complex roots: If λ_1 or $\lambda_2 = a + b\sqrt{-1}$ then $y = e^{ax}(c_1\cos bx + c_2\sin bx)$
  Given λ_1, λ_2 we can recover $A = -(\lambda_1 + \lambda_2), B = \lambda_1\lambda_2$

Mathematical Modelling

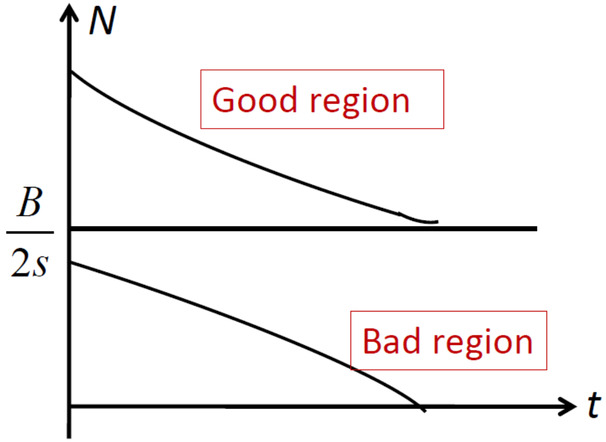
- **Malthusian growth model**:
  $B :=$ per capita birth rate (constant, +ve)
  $D :=$ per capita death rate (constant, +ve)
  $\frac{dN}{dt} = (B - D)N \implies N(t) = N(0)e^{(B-D)t}$
  $B > D \implies$ population explosion
  $B = D \implies$ stable
  $B < D \implies$ extinction
- **Logistic growth model**:
  $B :=$ per capita birth rate (constant)
  $sN :=$ per capita death rate (linear to population)
  $\frac{dN}{dt} = (B - sN)N \implies$
  $N(t) = \dfrac{B}{s + (\frac{B}{N(0)} - s)e^{-Bt}} = \dfrac{\frac{B}{s}}{1 + \left(\frac{B}{s} \cdot \frac{1}{N(0)} - 1\right) e^{-Bt}}$
  $B - sN(t) > 0 \quad \forall t \implies \frac{B}{s} > N(t)$
  $\implies$ smaller than sustainable population
  $B - sN(t) = 0 \quad \forall t \implies \frac{B}{s} = N(t)$
  $\implies$ sustainable population (equilibrium)
  $B - sN(t) < 0 \quad \forall t \implies \frac{B}{s} < N(t)$
  $\implies$ larger than sustainable population
  Population always tends to $\boxed{\frac{B}{s}}$ — "carrying capacity"



- **Harvesting growth model**:
  $B :=$ per capita birth rate (constant)
  $sN :=$ per capita death rate (linear to population)
  $E :=$ harvest rate
  $\frac{dN}{dt} = (B - sN)N - E = BN - sN^2 - E$
  Solve $-sN^2 + BN - E = 0$ for equilibrium solutions
  $\implies \beta_1 < \beta_2 < \frac{B}{s}$, $\beta_1 + \beta_2 = \frac{B}{s}$ and $\beta_1\beta_2 = \frac{E}{s}$
  $B^2 - 4sE > 0 \implies \frac{B^2}{4s} > E \implies$ two equilibriums



β_2 is stable but not β_1
$B^2 - 4sE = 0 \implies \frac{B^2}{4s} = E \implies$ one equilibrium



$\implies \beta = \frac{B}{2s}$
$B^2 - 4sE < 0 \implies \frac{B^2}{4s} < E \implies$ no equilibrium