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Multi-criteria decision-making approach with incomplete certain information based on ternary AHP

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Abstract: It is not uncommon in multiple criteria decision-making that the numerical values of alternatives of some criteria are subject to imprecision, uncertainty and indetermination and the information on weights of criteria is incomplete certain. A new multiple criteria decision- making method with incomplete certain information based on ternary AHP is proposed. This improves on Takeda's method. In this method, the ternary comparison matrix of the alternatives under each pseudo-criteria is constructed, the eigenvector associated with the maximum eigenvalue of the ternary comparison matrix is attained as to normalize priority vector of the alternatives, then the order of alternatives is obtained by solving two kinds of linear programming problems. Finally, an example is given to show the feasibility and effectiveness of the method.

Key words: multi-criteria decision-making, incomplete certain information, pseudo-criteria, ternary AHP, linear programming.

1. INTRODUCTION

In economics and social life, there exist many multiple criteria decision-making (MCDM) problems, but in many of them, the numerical values of some criteria are subject to imprecision, uncertainty, and indetermination, just as Roy et al explained[1]. The concept of pseudo-criteria and its two thresholds can allow such phenomenon taken into account. So far, outranking relation methods, which are ELECTRE IS, III and TRI, PROMETHEE and others, in which an outranking relation between alternatives is constructed from pseudo-criteria that have been developed[2~12]. Among them, ELECTRE III is the most familiar and has been widely used. In ELECTRE III, however the weights assigned to the criteria are required to build an outranking relation and discrimination threshold function involving a certain amount of arbitrariness is used to rank alternatives in ascending and descending order.

Though, the weights of the criteria are not known precisely there is incomplete certain information on the weights that can be divided into five forms[12]:

Form 1 $\{\omega : A_1\omega \geqslant b, \omega > 0, b \geqslant 0\}$

Form 2 $\{\omega : A_1\omega \leqslant b, \omega > 0, b \geqslant 0\}$

Form 3 $\{\omega : A_1\omega = b, \omega > 0, b \geqslant 0\}$

where $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^{\mathrm{T}}$, A_1 is an $l \times n$ matrix.

The binary AHP is introduced to deal with the pairwise comparisons judgment with only binary information such as "good or bad"[13]. When "equal" or "tie" occurs, it is called the ternary AHP. The binary or ternary AHP can derive a consistent vector from the situations involving inconsistent pairwise comparisons.

On the basis of analyzing the present and improved ELECTRE methods, Takeda brought forward a procedure based on the ternary AHP for treating the pseudo-criteria[14]. The procedure differs from ELECTRE III and requires the incomplete information on the weights but not precise weights. However, the incomplete certain information on weights in the procedure is part of the incomplete certain information in[12,15]. It only is concluded in Form 1. If the incomplete information includes Form 2 or Form 3, then the procedure will fail.

The purpose of this paper is to propose a new procedure for dealing the pseudo-criteria with incomplete certain information on the weights based on the ternary AHP.

2. PSEUDO-CRITERIA AND THE MULTIPLE CRITERIA DECISION METHODS

In this section, we shall briefly review the pseudo-criterion, ELECTERE III and Takeda's approach.

2.1 Pseudo-Criteria

Let us consider m alternative, denoted as $A = \{a_1, a_2, \cdots, a_m\}$. And let $C_1, C_2, \cdots, C_n$ be n criteria, and the value of a_i is expressed by x_{ij} under criteria C_j.

In ELECTRE III, some thresholds were introduced to describe the ranges for indifference, weak preference and strict preference.

Let p_k and q_k be two thresholds for criteria C_k $(0 \leqslant q_k \leqslant p_k)$. The strict preference relation (P), weak preference relation (W), and indifferent relation (I) are defined as

$$P_k(a_i, a_j) \text{ iff } x_{ik} - x_{jk} > p_k$$

$$W_k(a_i, a_j) \text{ iff } q_k < x_{ik} - x_{jk} \leqslant p_k$$

$$I_k(a_i, a_j) \text{ iff } | x_{ik} - x_{jk} | \leqslant q_k$$

If $0 < q_k < p_k$, criteria C_k with the strict preference relation, weak preference relation and indifferent relation described above is called pseudo-criteria.

If $p_k = q_k$, criteria C_k with that strict preference relation and indifferent relation described above is called semi-criteria.

If $p_k = q_k = 0$, criteria C_k with the strict preference relation and indifferent relation described above is called true criteria.

2.2 ELECTRE III

In ELECTRE III, for each pseudo criterion, an outranking relation $c_k(a_i, a_j)$ is defined as follows

$$P_k(a_i, a_j): c_k(a_i, a_j) = 1 \quad \text{and } c_k(a_j, a_i) = 0$$

$$W_k(a_i, a_j): c_k(a_i, a_j) = 1 \quad \text{and } c_k(a_j, a_i) = \frac{p_k + x_{jk} - x_{ik}}{p_k - q_k}$$

$$I_k(a_i, a_j): c_k(a_i, a_j) = 1 \quad \text{and } c_k(a_j, a_i) = 1$$

Using the weights ω (the value is certain), the concordance index $C(a_i, a_j)$ is defined as follows

$$C(a_i, a_j) = \sum_{k=1}^{n} \omega_k c_k(a_i, a_j)$$

On the other hand, by introducing a veto threshold v_k for each pseudo criterion, a discordance index $d_k(a_i, a_j)$, which rejects the assertion "a_i outranks a_j", is defined as follows

If $x_{ik} - x_{jk} \leqslant p_k$, then $d_k(a_i, a_j) = 0$

If $p_k < x_{ik} - x_{jk} \leqslant v_k$, then $d_k(a_i, a_j) = \dfrac{x_{ik} - x_{jk} - p_k}{v_k - p_k}$

If $x_{ik} - x_{jk} > v_k$, then $d_k(a_i, a_j) = 1$

The reliability index of the outranking relation is defined as follows

$$\sigma(a_i, a_j) = C(a_i, a_j) \prod_{k \in F} \frac{1 - d_k(a_i, a_j)}{1 - C(a_i, a_j)}$$

where $F = \{j : d_j(a_i, a_j) > C(a_i, a_j)\}$

In the final stage, the distillation approach using a discrimination threshold is used to rank the alternatives.

2.3 Pseudo-Criteria Based on the Ternary AHP

Takeda defined the ternary comparison matrix $T_k = (t_k(a_i, a_j))$ of alternatives for pseudo-criteria C_k according to the preference relation of the pseudo-criteria[14].

$$P_k(a_i, a_j): t_k(a_i, a_j) = \theta, t_k(a_j, a_i) = 1/\theta$$

$$W_k(a_i, a_j): t_k(a_i, a_j) = \theta, t_k(a_j, a_i) = 1$$

$$I_k(a_i, a_j): t_k(a_i, a_j) = 1, t_k(a_j, a_i) = 1$$

We generally adopt $\theta = 2$[14], and compute the maximum eigenvector associated with the maximum eigenvalue of the ternary comparison matrix T_k, and normalize the eigenvector so as to maximum its elements to be 1.

Let $(v_{1k}, v_{2k}, \cdots, v_{mk})$ be the maximum eigenvector (whose elements are normalized) of T_k.

Let us assume that the criteria are additively independent. Thus, the overall evaluation of alternatives is denoted by additive weighting rule.

$$V_k = \sum_{j=1}^{n} \omega_j v_{jk}$$

where $\omega = (\omega_1, \omega_2, \cdots, \omega_n)^{\mathrm{T}}$ is a weight vector whose elements represent the relative importance of the criteria.

Consider the incomplete information on weights. Let the set of weights be

$$\overline{W} = \{\omega : A_1 \omega \geqslant 0, \omega > 0\}$$

where A_1 is an $l \times n$ matrix.

Let A_p be a subset of A, and consider the following linear programming problems for each alternative a_k in A_p,

$$\max V_k = \sum_{j=1}^{n} \omega_j v_{kj}$$

$$\text{s.t.}\begin{cases} V_i = \sum_{j=1}^{n} \omega_j v_{ij} \leqslant 1 \quad i = 1,2,\cdots,v_p \\ \omega \in \overline{W} \end{cases} \tag{1}$$

$$\min V_k = \sum_{j=1}^{n} \omega_j v_{kj}$$

$$\text{s.t.}\begin{cases} V_i = \sum_{j=1}^{n} \omega_j v_{ij} \geqslant 1 \quad i = 1,2,\cdots,v_p \\ \omega \in \overline{W} \end{cases} \tag{2}$$

If there is the optimal solution (V_k^*) in (1) and $V_k^* = 1$, then a_k becomes a top ranking alternative.

If there is the optimal solution (V_k^*) in (2) and $V_k^* = 1$, then a_k becomes a low ranking alternative.

The Takeda's procedure for constructing a descending order (ascending order) is as follows.

Step 1 To begin with, let $p=1, A_p=A$.

Step 2 For each $a_k \in A_p$, solve the linear programming (1) and (2).

Step 3 Let $C_1(D_1)$ be the set of all alternatives with $V_k^* = 1$ in $A_1 = A$. Removing $C_1(D_1)$ from the set of alternative, let $p=2$ and the remaining set is $A_p = A_1 \setminus C_1 (A_p = A_1 \setminus D_1)$. For each $a_k \in A_p$, again solve the linear programming (1) and (2). Let $C_2(D_2)$ be the set consisting of all alternatives with $V_k^* = 1$ in A_p. Continue the above process until the remaining set A_p is empty.

So, the descending order obtained is

$$C_1 \rightarrow C_2 \rightarrow \cdots \rightarrow C_{q-1} \rightarrow C_q$$

The ascending order obtained is

$$D_p \leftarrow D_{p-1} \leftarrow \cdots \leftarrow D_2 \leftarrow D_1$$

The final order is obtained by taking the intersection of the descending order and the ascending order.

But in above procedure, the incomplete certain information on weights only includes in Form 1. If it includes Form 2 or Form 3, then the procedure will fail. If the incomplete certain information on weights includes the Form 2 or Form 3, then it is possible that there will be no optimal solution in (1) or (2).

Example Let the eigenvectors: $v_1 = (1, 0.5)$, $v_2 = (0.3, 1)$ from two pseudo-criteria in two alternatives, and the weights accord with the term as follows

$$0.1 \leqslant \omega_1 \leqslant 0.8, \quad 0.3 \leqslant \omega_2 \leqslant 0.5$$

And according to (2), there is a linear programming as follows

$$\min V_1 = \omega_1 + 0.5\omega_2$$

$$\text{s.t.}\begin{cases} \omega_1 + 0.5\omega_2 \geqslant 1 \\ 0.3\omega_1 + \omega_2 \geqslant 1 \\ 0.1 \leqslant \omega_1 \leqslant 0.8 \\ 0.3 \leqslant \omega_2 \leqslant 0.5 \\ \omega_1 \geqslant 0, \quad \omega_2 \geqslant 0 \end{cases} \tag{3}$$

Consider the second restriction in (3): $0.3\omega_1 + \omega_2 \leqslant 0.3 * 0.8 + 0.5 = 0.74 < 1$, and $0.3\omega_1 + \omega_2 \geqslant 1$ in (3), it is contradictory, and there is no optimal solution to (3).

3. A NEW DECISION METHOD WITH INCOMPLETE CERTAIN INFORMATION BASED ON TERNARY AHP

The incomplete certain information types we are considering here are linear inequalities and linear equalities on criteria weights[12]. Let H be the incomplete certain information set on weights.

We make sure the ternary comparison matrix T_k of the alternatives on the condition of criteria C_k according to section 2.3.

Let ($v_{1k}, v_{2k}, \cdots, v_{mk}$) be maximum eigenvector (whose elements are normalized so as to the maximum value to be 1) of the ternary comparison matrix T_k.

Let $V_k = \sum_{j=1}^{n} \omega_j v_{jk}$, and A_p be a subset of alternatives. For each alternative a_k in A_p

$$\max V_k = \sum_{j=1}^{n} \omega_j v_{kj}$$

$$\text{s.t.}\begin{cases} V_i - V_k = \sum_{j=1}^{n} \omega_j (v_{ij} - v_{kj}) \geqslant 0, i = 1,2,\cdots,v_p \\ \omega \in H \\ \sum_{j=1}^{n} \omega_j = 1 \end{cases} \tag{4}$$

If there exist the optimal solution ω^* to (4), $V_i \geqslant V_k$ for $\omega^* \in H$.

On the other hand, if there is no optimal solution to (4), there do no exist weights such that $V_i \geqslant V_k$. That is to say that for every weights $V_i \leqslant V_k$, there exists at least one i such that $V_i \leqslant V_k$.

So all the alternative in A_p are divided into two subsets. One is the alternative set with an optimal solution in (4), and the other is the alternative set without an optimal solution to (4).

Thus, if a_k is an alternative which has an optimal solution to (4), and a_i is an alternative which does not has an optimal solution to (4), then a_k has a lower rank than a_i.

Conversely, let us consider the following linear programming

$$\min V_k = \sum_{j=1}^{n} \omega_j v_{kj}$$

$$\text{s.t.} \begin{cases} V_i - V_k = \sum_{j=1}^{n} \omega_j (v_{ij} - v_{kj}) \leqslant 0, \ i = 1,2,\cdots, v_p \\ \omega \in H \\ \sum_{j=1}^{n} \omega_j = 1 \end{cases} \tag{5}$$

If there exists the optimal solution ω_* to (5), then a_k ranks highly for $\omega_* \in H$, since $V_i \leqslant V_k$ for $\omega_* \in H$.

If there is no optimal solution to (5), there not exist weights such that $V_i \leqslant V_k$. That is to say that for every criteria weight $\omega \in H$, there exists at least one i such that $V_i \geqslant V_k$.

So all the alternative in A_p are divided into two subsets. One is the alternative with an optimal solution to (5), and the other is the alternative without an optimal solution to (5).

Thus, if a_k is an alternative which has an optimal solution to linear programming (5), and a_i is an alternative which does not have an optimal solution to (5), then a_k has a higher rank than a_i.

The decision-making procedure for constructing a descending order (ascending order) in our method is as follows.

Step 1 Let $p=1, A_p = A$;

Step 2 For each a_k in A_p, solve the linear programms (4) and (5);

Step 3 Let C_p be the set of alternatives that has the optimal solution to linear programm (4), and let $p=2, A_p = A_{p-1} \setminus C_{p-1}$. Solve linear program (4) for each alternative a_k. Continue the above process until set A_p is empty;

Step 4 Let D_p be the set of alternatives that has the optimal solution to linear programming (5), and let $p=2, A_p = A_{p-1} \setminus D_{p-1}$. Solve linear programing (5) for each alternative a_k. Continue the above process until set A_p is empty.

The ascending order of C_p is

$$C_q \leftarrow C_{q-1} \leftarrow \cdots \leftarrow C_2 \leftarrow C_1$$

The descending order of D_p is

$$D_1 \rightarrow D_2 \rightarrow \cdots \rightarrow C_{p-1} \rightarrow D_p$$

The final order of alternatives is obtained by taking the intersection of the descending order and the ascending order.

4. AN ILLUSTRATIVE EXAMPLE

In this section, we shall compare the result of our method with Takeda's method.

There are 5 criteria and 7 alternatives in the decision-making example from Ref. [14] and the following Table 1 summarizes the relevant data.

Table 1 The relevant data

Criteria	C_1	C_2	C_3	C_4	C_5
a_1	-120	-284	5	3.5	18
a_2	-150	-269	2	4.5	24
a_3	-100	-413	4	5.5	17
a_4	-60	-596	6	8.0	20
a_5	-30	-1 321	8	7.5	16
a_6	-80	-734	5	4.0	21
a_7	-45	-982	7	8.5	13
p_k	40	350	3	3.5	5
q_k	15	80	1	0.5	1
v_k	100	850	5	4.5	8

The criteria weights are restricted by the terms as follows

$$\omega_2 > \omega_1 > \omega_4 > \omega_3 > \omega_5 > 0$$
$$2\omega_1 > \omega_2, \quad \omega_1 > 3\omega_5, \quad \omega_2 > 4\omega_3$$

Now, we apply our method in this example to get an order of alternatives.

By applying computing programming in Matlab 6.5, we obtain the results as follows.

Firstly, for each criteria, the maximum eigenvector of the ternary comparison matrix are obtained as follows

Criteria 1 $v=(0.489\,659,\ 0.361\,486,\ 0.605\,673,\ 0.889\,558,\ 1.000\,000,\ 0.736\,091,\ 0.889\,558)$
Criteria 2 $v=(1.000\,000,\ 1.000\,000,\ 0.890\,925,\ 0.793\,748,\ 0.360\,262,\ 0.598\,095,\ 0.440\,980)$
Criteria 3 $v=(0.671\,495,\ 0.454\,620,\ 0.610\,649,\ 0.745\,807,\ 1.000\,000,\ 0.671\,495,\ 0.909\,240)$
Criteria 4 $v=(0.446\,714,\ 0.606\,573,\ 0.798\,213,\ 0.893\,427,\ 0.893\,427,\ 0.500\,000,\ 1.000\,000)$
Criteria 5 $v=(0.612\,549,\ 1.000\,000,\ 0.547\,800,\ 0.808\,801,\ 0.547\,800,\ 0.808\,801,\ 0.404\,400)$

Then, we get the ascending order and descending order as follows

$$\{a_4\}\to\left\{\begin{matrix}a_1\\a_3\end{matrix}\right\}\to\{a_2\}\to\left\{\begin{matrix}a_5\\a_6\\a_7\end{matrix}\right\}$$

and $$\{a_4\}\to\left\{\begin{matrix}a_1\\a_3\end{matrix}\right\}\to\left\{\begin{matrix}a_2\\a_5\\a_7\end{matrix}\right\}\to\{a_6\}$$

Finally, we get the final order as follows

$$\{a_4\}\to\left\{\begin{matrix}a_1\\a_3\end{matrix}\right\}\to\{a_2\}\to\left\{\begin{matrix}a_5\\a_7\end{matrix}\right\}\to\{a_6\}$$

When using the method of Takeda[17], we get the order as follows

$$\{a_4\}\to\left\{\begin{matrix}a_1\\a_2\\a_3\end{matrix}\right\}\to\left\{\begin{matrix}a_5\\a_7\end{matrix}\right\}\to\{a_6\}$$

In the final order of our method, the alternative a_2 is separated out as an independent style. This is the only difference with the order according to Takeda's method[14]. The difference is that there is the constrained condition $\sum_{j=1}^{n}\omega_j=1$ in our method. If we delete the constrained condition $\sum_{j=1}^{n}\omega_j=1$ in (4) and (5), the final order of our method is the same as the order of Takeda's method.

If the criteria weights are restricted by the terms as follows

$$\omega_2>\omega_1>\omega_4>\omega_3>\omega_5>0$$
$$0.1\leqslant\omega_1\leqslant0.35,\ 0.15\leqslant\omega_2\leqslant0.45$$
$$0.05\leqslant\omega_3\leqslant0.2,\ 0.1\leqslant\omega_4\leqslant0.3$$
$$0.08\leqslant\omega_5\leqslant0.15$$

When we use Takeda's method, we won't get the order of the set of alternatives. Additionally if the alternative a_7 is restricted by the terms as follows

$$0.736\,091\omega_1+0.598\,095\omega_2+0.671\,495\omega_3+0.5\omega_4+0.808\,801\omega_5\leqslant0.736\,091*0.35+0.598\,095*0.45+0.671\,495*0.2+0.5*0.3+0.808\,801*0.15=0.932\,58<1$$

Then, the optimal solution to (3) will not exist.

According to the method of this paper, the ascending order and descending order is

$$\{a_4\}\to\{a_3\}\to\left\{\begin{matrix}a_1\\a_2\\a_5\\a_7\end{matrix}\right\}\to\{a_6\}\ \text{and}\ \{a_4\}\to\left\{\begin{matrix}a_2\\a_3\\a_5\\a_7\end{matrix}\right\}\to\left\{\begin{matrix}a_1\\a_6\end{matrix}\right\}$$

The final order is

$$\{a_4\}\to\{a_3\}\to\left\{\begin{matrix}a_2\\a_5\\a_7\end{matrix}\right\}\to\{a_1\}\to\{a_6\}$$

The different information of the criteria weights plays an important rôle in the difference between the two results. And the maximum eigenvectors for each criteria by obtained applying computing programm in Matlab 6.5 differ a little from the results of the Ref. [14], but it doesn't matter.

5. CONCLUSIONS

We have proposed a new multi-criteria decision making method for dealing with pseudo-criteria and incomplete certain information on weights, which can improve Takeda's method. In this method, we adopt the ternary comparison on the criteria, employ the ternary AHP to obtain a normalized priority vector, construct linear programming by using the incomplete certain information on weights, obtain a descending order and ascending order of alternatives by solving these linear programming, and obtain the overall order by taking the intersection of the descending order and the ascending order. We also use an example to show the feasibility and effectiveness of this method.

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