Computational Linear Algebra

Siju Swamy

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Preface

- - -

I was born in the darkness of ignorance, and my master opened my eyes with the torch of knowledge. I offer my respectful obeisances unto him

Welcome to the course on **Computational Linear Algebra**. This course is designed to provide a practical perspective on linear algebra, bridging the gap between mathematical theory and real-world applications. As we delve

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````
## Pseudocode: the new language for algorithm design

Pseudocode is a way to describe algorithms in a structured but plain language. It helps in pl

:::{.callout-caution}
There are varities of approaches in writing pseudocode. Students can adopt any of the standar
:::

### Matrix Sum

**Mathematical Procedure:**

To add two matrices $A$ and $B$, both matrices must have the same dimensions. The sum $C$ of

$$C[i][j] = A[i][j] + B[i][j]$$

**Example:**

Let $A$ and $B$ be two $2 \times 2$ matrices:

$$ A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}, \quad B = \begin{bmatrix} 5 & 6 \\ 7 & 8

The sum $C$ is:

$$ C = A + B = \begin{bmatrix} 1+5 & 2+6 \\ 3+7 & 4+8 \end{bmatrix} = \begin{bmatrix} 6 & 8 \

**Pseudocode:**

```python
FUNCTION matrix_sum(A, B):
    Get the number of rows and columns in matrix A
    Create an empty matrix C with the same dimensions
    FOR each row i:
````
````

```
        FOR each column j:
            Set C[i][j] to the sum of A[i][j] and B[i][j]
    RETURN the matrix C
END FUNCTION
```

Explanation:

1. Determine the number of rows and columns in matrix A.
2. Create a new matrix C with the same dimensions.
3. Loop through each element of the matrices and add corresponding elements.
4. Return the resulting matrix C.

Matrix Difference

Mathematical Procedure:

To subtract matrix B from matrix A, both matrices must have the same dimensions. The dif

$$ C[i][j] = A[i][j] - B[i][j] $$

Example:

Let A and B be two 2×2 matrices:

$$ A = \begin{bmatrix} 9 & 8 \\ 7 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 3 & 4

The difference C is:

$$ C = A - B = \begin{bmatrix} 9-1 & 8-2 \\ 7-3 & 6-4 \end{bmatrix} = \begin{bmatrix} 8 & 6 \

Pseudocode:

```python
FUNCTION matrix_difference(A, B):
    # Determine the number of rows and columns in matrix A
    rows = number_of_rows(A)
    cols = number_of_columns(A)

    # Create an empty matrix C with the same dimensions as A and B
    C = create_matrix(rows, cols)

    # Iterate through each row
```

```
    FOR i FROM 0 TO rows-1:
        # Iterate through each column
        FOR j FROM 0 TO cols-1:
            # Calculate the difference for each element and store it in C
            C[i][j] = A[i][j] - B[i][j]

    # Return the result matrix C
    RETURN C
END FUNCTION
```

In more human readable format the above pseudocode can be written as:

```python
FUNCTION matrix_difference(A, B):
    Get the number of rows and columns in matrix A
    Create an empty matrix C with the same dimensions
    FOR each row i:
        FOR each column j:
            Set C[i][j] to the difference of A[i][j] and B[i][j]
    RETURN the matrix C
END FUNCTION
```
Explanation:

1. Determine the number of rows and columns in matrix A.
2. Create a new matrix C with the same dimensions.
3. Loop through each element of the matrices and subtract corresponding elements.
4. Return the resulting matrix C.

Matrix Product

Mathematical Procedure:

To find the product of two matrices A and B, the number of columns in A must be equal t

$$C[i][j] = \sum_{k} A[i][k] \cdot B[k][j]$$

Example:

Let A be a 2×3 matrix and B be a 3×2 matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 7 & 8 \

````
The product $C$ is:

$$C = A \cdot B = \begin{bmatrix} 58 & 64 \\ 139 & 154 \end{bmatrix}$$

**Pseudocode:**

```python
FUNCTION matrix_product(A, B):
    # Get the dimensions of A and B
    rows_A = number_of_rows(A)
    cols_A = number_of_columns(A)
    rows_B = number_of_rows(B)
    cols_B = number_of_columns(B)

    # Check if multiplication is possible
    IF cols_A != rows_B:
        RAISE Error("Incompatible matrix dimensions")

    # Initialize result matrix C
    C = create_matrix(rows_A, cols_B)

    # Calculate matrix product
    FOR each row i FROM 0 TO rows_A-1:
        FOR each column j FROM 0 TO cols_B-1:
            # Compute the sum for C[i][j]
            sum = 0
            FOR each k FROM 0 TO cols_A-1:
                sum = sum + A[i][k] * B[k][j]
            C[i][j] = sum

    RETURN C
END FUNCTION
```

A more human readable version of the `pseudocode` is shown below:
```python
FUNCTION matrix_product(A, B):
    Get the number of rows and columns in matrix A
    Get the number of columns in matrix B
    Create an empty matrix C with dimensions rows_A x cols_B
    FOR each row i in A:
        FOR each column j in B:
````
````

```
            Initialize C[i][j] to 0
            FOR each element k in the common dimension:
                Add the product of A[i][k] and B[k][j] to C[i][j]
    RETURN the matrix C
END FUNCTION
```
Explanation:

1. Determine the number of rows and columns in matrices A and B.
2. Create a new matrix C with dimensions $\text{rows}(A)\times \text{columns}(B)$.
3. Loop through each element of the resulting matrix $C[i][j]$ and calculate the dot product
4. Return the resulting matrix C.

Determinant

Mathematical Procedure:

To find the determinant of a square matrix A, we can use the Laplace expansion, which invol

$$\text{det}(A) = A[0][0] \cdot A[1][1] - A[0][1] \cdot A[1][0]$$

For larger matrices, the determinant is calculated recursively.

Example:

Let A be a 2×2 matrix:

$$A = \begin{bmatrix} 4 & 3 \\ 6 & 3 \end{bmatrix}$$

The determinant of A is:

$$\text{det}(A) = (4 \cdot 3) - (3 \cdot 6) = 12 - 18 = -6$$

Pseudocode:

```python
FUNCTION determinant(A):
    # Step 1: Get the size of the matrix
    n = number_of_rows(A)

    # Base case for a 2x2 matrix
    IF n == 2:
        RETURN A[0][0] * A[1][1] - A[0][1] * A[1][0]
```

```
    # Step 2: Initialize determinant to 0
    det = 0

    # Step 3: Loop through each column of the first row
    FOR each column j FROM 0 TO n-1:
        # Get the submatrix excluding the first row and current column
        submatrix = create_submatrix(A, 0, j)
        # Recursive call to determinant
        sub_det = determinant(submatrix)
        # Alternating sign and adding to the determinant
        det = det + ((-1) ^ j) * A[0][j] * sub_det

    RETURN det
END FUNCTION

FUNCTION create_sub_matrix(A, row, col):
    sub_matrix = create_matrix(number_of_rows(A)-1, number_of_columns(A)-1)
    sub_i = 0
    FOR i FROM 0 TO number_of_rows(A)-1:
        IF i == row:
            CONTINUE
        sub_j = 0
        FOR j FROM 0 TO number_of_columns(A)-1:
            IF j == col:
                CONTINUE
            sub_matrix[sub_i][sub_j] = A[i][j]
            sub_j = sub_j + 1
        sub_i = sub_i + 1
    RETURN sub_matrix
END FUNCTION
```
A human readable version of the same pseudocode is shown below:

```python
FUNCTION determinant(A):
    IF the size of A is 2x2:
        RETURN the difference between the product of the diagonals
    END IF
    Initialize det to 0
    FOR each column c in the first row:
        Create a sub_matrix by removing the first row and column c
        Add to det: the product of (-1)^c, the element A[0][c], and the determinant of the su
```

```
    RETURN det
END FUNCTION

FUNCTION create_sub_matrix(A, row, col):
    Create an empty sub_matrix with dimensions one less than A
    Set sub_i to 0
    FOR each row i in A:
        IF i is the row to be removed:
            CONTINUE to the next row
        Set sub_j to 0
        FOR each column j in A:
            IF j is the column to be removed:
                CONTINUE to the next column
            Copy the element A[i][j] to sub_matrix[sub_i][sub_j]
            Increment sub_j
        Increment sub_i
    RETURN sub_matrix
END FUNCTION
```

Explanation:

1. If the matrix is $2×2$, calculate the determinant directly.
2. For larger matrices, use the Laplace expansion to recursively calculate the determinant.
3. Create submatrices by removing the current row and column.
4. Sum the determinants of the submatrices, adjusted for the sign and the current element.

Rank of a Matrix

Mathematical Procedure:

The rank of a matrix A is the maximum number of linearly independent rows or columns in A

Example:

Let A be a 3×3 matrix:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

After performing Gaussian elimination, we obtain:

$$\text{REF}(A) = \begin{bmatrix} 1 & 2 & 3 \\ 0 & -3 & -6 \\ 0 & 0 & 0 \end{bmatrix}$$

The rank of A is the number of non-zero rows, which is 2.

Pseudocode:

```python
FUNCTION matrix_rank(A):
    # Step 1: Get the dimensions of the matrix
    rows = number_of_rows(A)
    cols = number_of_columns(A)

    # Step 2: Transform the matrix to row echelon form
    row_echelon_form(A, rows, cols)

    # Step 3: Count non-zero rows
    rank = 0
    FOR each row i FROM 0 TO rows-1:
        non_zero = FALSE
        FOR each column j FROM 0 TO cols-1:
            IF A[i][j] != 0:
                non_zero = TRUE
                BREAK
        IF non_zero:
            rank = rank + 1

    RETURN rank
END FUNCTION

FUNCTION row_echelon_form(A, rows, cols):
    # Perform Gaussian elimination
    lead = 0
    FOR r FROM 0 TO rows-1:
        IF lead >= cols:
            RETURN
        i = r
        WHILE A[i][lead] == 0:
            i = i + 1
            IF i == rows:
                i = r
                lead = lead + 1
                IF lead == cols:
                    RETURN
        # Swap rows i and r
        swap_rows(A, i, r)
```

```
        # Make A[r][lead] = 1
        lv = A[r][lead]
        A[r] = [m / float(lv) for m in A[r]]
        # Make all rows below r have 0 in column lead
        FOR i FROM r + 1 TO rows-1:
            lv = A[i][lead]
            A[i] = [iv - lv * rv for rv, iv in zip(A[r], A[i])]
        lead = lead + 1
END FUNCTION

FUNCTION swap_rows(A, row1, row2):
    temp = A[row1]
    A[row1] = A[row2]
    A[row2] = temp
END FUNCTION
```
A more human readable version of the above pseudocode is shown below:

```python
FUNCTION rank(A):
    Get the number of rows and columns in matrix A
    Initialize the rank to 0
    FOR each row i in A:
        IF the element in the current row and column is non-zero:
            Increment the rank
            FOR each row below the current row:
                Calculate the multiplier to zero out the element below the diagonal
                Subtract the appropriate multiple of the current row from each row below
        ELSE:
            Initialize a variable to track if a swap is needed
            FOR each row below the current row:
                IF a non-zero element is found in the current column:
                    Swap the current row with the row having the non-zero element
                    Set the swap variable to True
                    BREAK the loop
            IF no swap was made:
                Decrement the rank
    RETURN the rank
END FUNCTION
```
Explanation:

1. Initialize the rank to 0.

```
2. Loop through each row of the matrix.
3. If the diagonal element is non-zero, increment the rank and perform row operations to zero
4. If the diagonal element is zero, try to swap with a lower row that has a non-zero element
5. If no such row is found, decrement the rank.
6. Return the resulting rank of the matrix.

### Practice Problems

Find the rank of the following matrices.

1. $\begin{pmatrix} 1&1&3\\ 2&2&6\\ 2&5&3\end{pmatrix}$.
2. $\begin{pmatrix} 2&0&2\\ 1&2&3\\ 3&2&7\end{pmatrix}$

### Solving a System of Equations

**Mathematical Procedure:**

To solve a system of linear equations represented as $A \mathbf{x} = \mathbf{b}$, where$A$ is

**Example:**

Consider the system of equations:

$$\begin{cases}
x + 2y + 3z &= 9 \\
4x + 5y + 6z& = 24 \\
7x + 8y + 9z& = 39
\end{cases}$$

The augmented matrix is:

$$[A | \mathbf{b}] = \begin{bmatrix} 1 & 2 & 3 & | & 9 \\ 4 & 5 & 6 & | & 24 \\ 7 & 8 & 9 &


After performing Gaussian elimination on the augmented matrix, we get:

$$\text{REF}(A) = \begin{bmatrix} 1 & 2 & 3 & | & 9 \\ 0 & -3 & -6 & | & -12 \\ 0 & 0 & 0 &

Performing back substitution, we solve for $z$,$y$, and $x$:

$$\begin{cases}
z = 1 \\
y = 0 \\
```

```
x = 3
\end{cases}$$

Therefore, the solution vector is $\mathbf{x} = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix}$.


**Pseudocode:**

```python
FUNCTION solve_system_of_equations(A, b):
    # Step 1: Get the dimensions of the matrix
    rows = number_of_rows(A)
    cols = number_of_columns(A)

    # Step 2: Create the augmented matrix
    augmented_matrix = create_augmented_matrix(A, b)

    # Step 3: Transform the augmented matrix to row echelon form
    row_echelon_form(augmented_matrix, rows, cols)

    # Step 4: Perform back substitution
    solution = back_substitution(augmented_matrix, rows, cols)

    RETURN solution
END FUNCTION

FUNCTION create_augmented_matrix(A, b):
    # Combine A and b into an augmented matrix
    augmented_matrix = []
    FOR i FROM 0 TO number_of_rows(A)-1:
        augmented_matrix.append(A[i] + [b[i]])
    RETURN augmented_matrix
END FUNCTION

FUNCTION row_echelon_form(augmented_matrix, rows, cols):
    # Perform Gaussian elimination
    lead = 0
    FOR r FROM 0 TO rows-1:
        IF lead >= cols:
            RETURN
        i = r
        WHILE augmented_matrix[i][lead] == 0:
            i = i + 1
```
```

```
            IF i == rows:
                i = r
                lead = lead + 1
                IF lead == cols:
                    RETURN
        # Swap rows i and r
        swap_rows(augmented_matrix, i, r)
        # Make augmented_matrix[r][lead] = 1
        lv = augmented_matrix[r][lead]
        augmented_matrix[r] = [m / float(lv) for m in augmented_matrix[r]]
        # Make all rows below r have 0 in column lead
        FOR i FROM r + 1 TO rows-1:
            lv = augmented_matrix[i][lead]
            augmented_matrix[i] = [iv - lv * rv for rv, iv in zip(augmented_matrix[r], augmer
        lead = lead + 1
END FUNCTION

FUNCTION back_substitution(augmented_matrix, rows, cols):
    # Initialize the solution vector
    solution = [0 for _ in range(rows)]
    # Perform back substitution
    FOR i FROM rows-1 DOWNTO 0:
        solution[i] = augmented_matrix[i][cols-1]
        FOR j FROM i+1 TO cols-2:
            solution[i] = solution[i] - augmented_matrix[i][j] * solution[j]
    RETURN solution
END FUNCTION

FUNCTION swap_rows(matrix, row1, row2):
    temp = matrix[row1]
    matrix[row1] = matrix[row2]
    matrix[row2] = temp
END FUNCTION
```
Explanation:

1. Augment the coefficient matrix A with the constant matrix B.
2. Perform Gaussian elimination to reduce the augmented matrix to row echelon form.
3. Back-substitute to find the solution vector X.
4. Return the solution vector X.

Review Problems

Q1: Fill in the missing parts of the pseudocode to yield a meaningful algebraic operation

Pseudocode:
```python
FUNCTION matrix_op1(A, B):
    rows = number_of_rows(A)
    cols = number_of_columns(A)
    result = create_matrix(rows, cols, 0)

    FOR i FROM 0 TO rows-1:
        FOR j FROM 0 TO cols-1:
            result[i][j] = A[i][j] + ---

    RETURN result
END FUNCTION
```

Q2: Write the pseudocode to get useful derivable from a given a matrix by fill in the mis

Pseudocode:
```python
FUNCTION matrix_op2(A):
    rows = number_of_rows(A)
    cols = number_of_columns(A)
    result = create_matrix(cols, rows, 0)

    FOR i FROM 0 TO rows-1:
        FOR j FROM 0 TO cols-1:
            result[j][i] = A[i][--]

    RETURN result
END FUNCTION
```

Transition from Pseudocode to Python Programming

In this course, our initial approach to understanding and solving linear algebra problems has

However, to fully leverage the power of computational tools and prepare for real-world applic

1. **Practical Implementation:** Python provides numerous libraries and tools, such as NumPy

2. **Hands-On Experience:** Moving to Python programming gives students hands-on experience

```
3. **Industry Relevance:** Python is extensively used in industry for data analysis, machine

4. **Integration with Other Tools:** Python's compatibility with various tools and platforms

5. **Enhanced Learning:** Implementing algorithms in Python helps reinforce theoretical conc

By transitioning to Python programming, we not only achieve our course objectives but also e

## Python Fundamentals

### Python Programming Overview

Python is a high-level, interpreted programming language that was created by Guido van Rossu


### Variables

In Python, variables are used to store data that can be used and manipulated throughout a pr

**Basic Input/Output Functions**

Python provides built-in functions for basic input and output operations. The `print()` funct

*Output with `print()` function*

>Example 1

```python
# Printing text
print("Hello, World!")

# Printing multiple values
x = 5
y = 10
print("The value of x is:", x, "and the value of y is:", y)
```

>Example 2

```python
# Assigning values to variables
a = 10
```
```

```
b = 20.5
name = "Alice"

# Printing the values
print("Values Stored in the Variables:")
print(a)
print(b)
print(name)
```
Input with `input()` Function:

```python
# Taking input from the user
name = input("Enter usr name: ")
print("Hello, " + name + "!")

# Taking numerical input
age = int(input("Enter usr age: "))
print("us are", age, "years old.")
```

:::{.callout-note}
The `print()` function in Python, defined in the built-in `__builtin__` module, is used to di
:::

Combining Variables and Input/Output

us can combine variables and input/output functions to create interactive programs.

>Example:

```python
# Program to calculate the sum of two numbers
num1 = float(input("Enter first number: "))
num2 = float(input("Enter second number: "))

# Calculate sum
sum = num1 + num2

# Display the result
print("The sum of", num1, "and", num2, "is", sum)
```

Python Programming Style

Indentation

Python uses indentation to define the blocks of code. Proper indentation is crucial as it af

```python
if a > b:
    print("a is greater than b")
else:
    print("b is greater than or equal to a")
```
Comments
Use comments to explain user code. Comments begin with the `#` symbol and extend to the end

```python
# This is a comment
a = 10  # This is an inline comment
```
Variable Naming

Use meaningful variable names to make usr code more understandable. Variable names should be

```python
student_name = "John"
total_score = 95
```

Consistent Style

Follow the `PEP 8` style guide for Python code to maintain consistency and readability. Use

```python

def calculate_sum(x, y):
    return x + y

result = calculate_sum(5, 3)
print(result)
```
Basic Datatypes in Python

In Python, a datatype is a classification that specifies which type of value a variable can

Numeric Types

Numeric types represent data that consists of numbers. Python has three distinct numeric type

1. **Integers (`int`)**:
   - Whole numbers, positive or negative, without decimals.
   - Example: `a = 10`, `b = -5`.

2. **Floating Point Numbers (`float`)**:
   - Numbers that contain a decimal point.
   - Example: `pi = 3.14`, `temperature = -7.5`.

3. **Complex Numbers (`complex`)**:
   - Numbers with a real and an imaginary part.
   - Example: `z = 3 + 4j`.

```python
# Examples of numeric types
a = 10         # Integer
pi = 3.14      # Float
z = 3 + 4j     # Complex
```

Sequence Types

Sequence types are used to store multiple items in a single variable. Python has several sequ

String Type

Strings in Python are sequences of characters enclosed in quotes. They are used to handle and

Characteristics of Strings

- *Ordered*: Characters in a string have a defined order.
- *Immutable*: Strings cannot be modified after they are created.
- *Heterogeneous*: Strings can include any combination of letters, numbers, and symbols.

Creating Strings

Strings can be created using single quotes, double quotes, or triple quotes for multiline str

>Example:

```python
# Creating strings with different types of quotes
single_quoted = 'Hello, World!'
double_quoted = "Hello, World!"
multiline_string = """This is a
multiline string"""
```
Accessing String Characters

Characters in a string are accessed using their index, with the first character having an in

>Example:

```python
# Accessing characters in a string
first_char = single_quoted[0]  # Output: 'H'
last_char = single_quoted[-1]  # Output: '!'
```

Common String Methods

Python provides various methods for string manipulation:

1. `upper()`: Converts all characters to uppercase.
2. `lower()`: Converts all characters to lowercase.
3. `strip()`: Removes leading and trailing whitespace.
4. `replace(old, new)`: Replaces occurrences of a substring with another substring.
5. `split(separator)`: Splits the string into a list based on a separator.

>Example:

```python
# Using string methods
text = "   hello, world!   "
uppercase_text = text.upper()       # Result: "   HELLO, WORLD!   "
stripped_text = text.strip()        # Result: "hello, world!"
replaced_text = text.replace("world", "Python")  # Result: "   hello, Python!   "
words = text.split(",")             # Result: ['hello', ' world!   ']
```

List Type

Lists are one of the most versatile and commonly used sequence types in Python. They allow fo

>**Characteristics of Lists**

- *Ordered*: The items in a list have a defined order, which will not change unless explicitl
- *Mutable*: The content of a list can be changed after its creation (i.e., items can be adde
- *Dynamic*: Lists can grow or shrink in size as items are added or removed.
- *Heterogeneous*: Items in a list can be of different data types (e.g., integers, strings, l

Creating Lists

Lists are created by placing comma-separated values inside square brackets.

>Example:

```python
# Creating a list of fruits
fruits = ["apple", "banana", "cherry"]

# Creating a mixed list
mixed_list = [1, "Hello", 3.14]
```

Accessing List Items

List items are accessed using their index, with the first item having an index of 0.

>Example:

```python

# Accessing the first item
first_fruit = fruits[0]  # Output: "apple"

# Accessing the last item
last_fruit = fruits[-1]  # Output: "cherry"
```

Modifying Lists

Lists can be modified by changing the value of specific items, adding new items, or removing

>Example:

```python
# Changing the value of an item
fruits[1] = "blueberry"  # fruits is now ["apple", "blueberry", "cherry"]

# Adding a new item
fruits.append("orange")  # fruits is now ["apple", "blueberry", "cherry", "orange"]

# Removing an item
fruits.remove("blueberry")  # fruits is now ["apple", "cherry", "orange"]
```

List Methods

`Python` provides several built-in methods to work with lists:

1. `append(item)`: Adds an item to the end of the list.
2. `insert(index, item)`: Inserts an item at a specified index.
3. `remove(item)`: Removes the first occurrence of an item.
4. `pop(index)`: Removes and returns the item at the specified index.
5. `sort()`: Sorts the list in ascending order.
6. `reverse()`: Reverses the order of the list.

>Example:

```python
# Using list methods
numbers = [5, 2, 9, 1]

numbers.append(4)     # numbers is now [5, 2, 9, 1, 4]
numbers.sort()        # numbers is now [1, 2, 4, 5, 9]
numbers.reverse()     # numbers is now [9, 5, 4, 2, 1]
first_number = numbers.pop(0)  # first_number is 9, numbers is now [5, 4, 2, 1]
```
Tuple Type

Tuples are a built-in sequence type in Python that is used to store an ordered collection of

Characteristics of Tuples

- *Ordered*: Tuples maintain the order of items, which is consistent throughout their lifeti
- *Immutable*: Once a tuple is created, its contents cannot be modified. This includes addin
- *Fixed Size*: The size of a tuple is fixed; it cannot grow or shrink after creation.

- *Heterogeneous*: Tuples can contain items of different data types, such as integers, string

Creating Tuples

Tuples are created by placing comma-separated values inside parentheses. Single-element tuple

>Example:

```python
# Creating a tuple with multiple items
coordinates = (10, 20, 30)

# Creating a single-element tuple
single_element_tuple = (5,)

# Creating a tuple with mixed data types
mixed_tuple = (1, "Hello", 3.14)
```

Accessing Tuple Items

Tuple items are accessed using their index, with the first item having an index of 0. Negativ

>Example:

```python
# Accessing the first item
x = coordinates[0]  # Output: 10

# Accessing the last item
z = coordinates[-1]  # Output: 30
```

Modifying Tuples

Since tuples are immutable, their contents cannot be modified. However, us can create new tu

>Example:

```python
# Combining tuples
new_coordinates = coordinates + (40, 50)  # Result: (10, 20, 30, 40, 50)
```

```
# Slicing tuples
sub_tuple = coordinates[1:3]  # Result: (20, 30)

```

Tuple Methods

Tuples have a limited set of built-in methods compared to lists:

1. `count(item)`: Returns the number of occurrences of the specified item.
2. `index(item)`: Returns the index of the first occurrence of the specified item.

>Example:

```python
# Using tuple methods
numbers = (1, 2, 3, 1, 2, 1)

# Counting occurrences of an item
count_1 = numbers.count(1)  # Result: 3

# Finding the index of an item
index_2 = numbers.index(2)  # Result: 1
```
Mapping Types

Mapping types in Python are used to store data in key-value pairs. Unlike sequences, mapping

Dictionary (`dict`)

The primary mapping type in Python is the `dict`. Dictionaries store data as key-value pairs

Characteristics of Dictionaries

- *Unordered*: The order of items is not guaranteed and may vary.
- *Mutable*: us can add, remove, and change items after creation.
- *Keys*: Must be unique and immutable (e.g., strings, numbers, tuples).
- *Values*: Can be of any data type and can be duplicated.

Creating Dictionaries

Dictionaries are created using curly braces `{}` with key-value pairs separated by colons `:

>Example:

```python
# Creating a dictionary
student = {
    "name": "Alice",
    "age": 21,
    "major": "Computer Science"
}
```
Accessing and Modifying Dictionary Items

Items in a dictionary are accessed using their keys. us can also modify, add, or remove items

>Example:

```python
# Accessing a value
name = student["name"]  # Output: "Alice"

# Modifying a value
student["age"] = 22  # Updates the age to 22

# Adding a new key-value pair
student["graduation_year"] = 2024

# Removing a key-value pair
del student["major"]
```

Dictionary Methods

Python provides several built-in methods to work with dictionaries:

1. `keys()`: Returns a view object of all keys.
2. `values()`: Returns a view object of all values.
3. `items()`: Returns a view object of all key-value pairs.
4. `get(key, default)`: Returns the value for the specified key, or a default value if the ke
5. `pop(key, default)`: Removes and returns the value for the specified key, or a default val

>Example:

```python
```

```
# Using dictionary methods
keys = student.keys()        # Result: dict_keys(['name', 'age', 'graduation_year'])
values = student.values()    # Result: dict_values(['Alice', 22, 2024])
items = student.items()      # Result: dict_items([('name', 'Alice'), ('age', 22), ('graduat
name = student.get("name")  # Result: "Alice"
age = student.pop("age")    # Result: 22
```

Set Types

Sets are a built-in data type in Python used to store unique, unordered collections of items

Characteristics of Sets

- *Unordered* : The items in a set do not have a specific order and may change.
- *Mutable* : us can add or remove items from a set after its creation.
- *Unique* : Sets do not allow duplicate items; all items must be unique.
- *Unindexed* : Sets do not support indexing or slicing.

Creating Sets

Sets are created using curly braces `{}` with comma-separated values, or using the `set()` fu

>Example:

```python
# Creating a set using curly braces
fruits = {"apple", "banana", "cherry"}

# Creating a set using the set() function
numbers = set([1, 2, 3, 4, 5])
```
Accessing and Modifying Set Items

While us cannot access individual items by index, us can check for membership and perform ope

>Example:

```python
# Checking membership
has_apple = "apple" in fruits  # Output: True

# Adding an item
```

```
fruits.add("orange")

# Removing an item
fruits.remove("banana")  # Raises KeyError if item is not present
```
Set Operations
Sets support various mathematical set operations, such as `union`, `intersection`, and `diffe

>Example:

```python
# Union of two sets
set1 = {1, 2, 3}
set2 = {3, 4, 5}
union = set1 | set2  # Result: {1, 2, 3, 4, 5}

# Intersection of two sets
intersection = set1 & set2  # Result: {3}

# Difference between two sets
difference = set1 - set2  # Result: {1, 2}

# Symmetric difference (items in either set, but not in both)
symmetric_difference = set1 ^ set2  # Result: {1, 2, 4, 5}
```

Set Methods

Python provides several built-in methods for set operations:

1. `add(item)`: Adds an item to the set.
2. `remove(item)`: Removes an item from the set; raises KeyError if item is not present.
3. `discard(item)`: Removes an item from the set if present; does not raise an error if item
4. `pop()`: Removes and returns an arbitrary item from the set.
5. `clear()`: Removes all items from the set.

>Example:

```python
# Using set methods
set1 = {1, 2, 3}

set1.add(4)        # set1 is now {1, 2, 3, 4}
```

```
set1.remove(2)     # set1 is now {1, 3, 4}
set1.discard(5)    # No error, set1 remains {1, 3, 4}
item = set1.pop()  # Removes and returns an arbitrary item, e.g., 1
set1.clear()      # set1 is now an empty set {}
```
Frozen Sets

Frozen sets are a built-in data type in Python that are similar to sets but are immutable. O

Characteristics of Frozen Sets

- *Unordered* : The items in a frozen set do not have a specific order and may change.
- *Immutable* : Unlike regular sets, frozen sets cannot be altered after creation. No items
- *Unique* : Like sets, frozen sets do not allow duplicate items; all items must be unique.
- *Unindexed* : Frozen sets do not support indexing or slicing.

Creating Frozen Sets

Frozen sets are created using the `frozenset()` function, which takes an iterable as an argu

>Example:

```python
# Creating a frozen set
numbers = frozenset([1, 2, 3, 4, 5])

# Creating a frozen set from a set
fruits = frozenset({"apple", "banana", "cherry"})
```
Accessing and Modifying Frozen Set Items

Frozen sets do not support modification operations such as adding or removing items. However

>Example:

```python
# Checking membership
has_apple = "apple" in fruits  # Output: True

# Since frozenset is immutable, us cannot use add() or remove() methods
```

Set Operations with Frozen Sets

Frozen sets support various mathematical set operations similar to regular sets, such as unio

>Example:

```python
# Union of two frozen sets
set1 = frozenset([1, 2, 3])
set2 = frozenset([3, 4, 5])
union = set1 | set2  # Result: frozenset({1, 2, 3, 4, 5})

# Intersection of two frozen sets
intersection = set1 & set2  # Result: frozenset({3})

# Difference between two frozen sets
difference = set1 - set2  # Result: frozenset({1, 2})

# Symmetric difference (items in either set, but not in both)
symmetric_difference = set1 ^ set2  # Result: frozenset({1, 2, 4, 5})
```

Frozen Set Methods

Frozen sets have a subset of the methods available to regular sets. The available methods inc

1. `copy()` : Returns a shallow copy of the frozen set.
2. `difference(other)` : Returns a new frozen set with elements in the original frozen set bu
3. `intersection(other)` : Returns a new frozen set with elements common to both frozen sets
4. `union(other)` : Returns a new frozen set with elements from both frozen sets.
5. `symmetric_difference(other)` : Returns a new frozen set with elements in either frozen se

>Example:

```python
# Using frozen set methods
set1 = frozenset([1, 2, 3])
set2 = frozenset([3, 4, 5])

# Getting the difference
difference = set1.difference(set2)  # Result: frozenset({1, 2})

# Getting the intersection
intersection = set1.intersection(set2)  # Result: frozenset({3})
```

```
# Getting the union
union = set1.union(set2)  # Result: frozenset({1, 2, 3, 4, 5})

# Getting the symmetric difference
symmetric_difference = set1.symmetric_difference(set2)  # Result: frozenset({1, 2, 4, 5})
```
Control Structures in Python

Control structures in Python allow us to control the flow of execution in our programs. They

Conditional Statements

Conditional statements are used to execute code based on certain conditions. The primary cond

>Syntax:

```python
if condition:
    # Code block to execute if condition is True
elif another_condition:
    # Code block to execute if another_condition is True
else:
    # Code block to execute if none of the above conditions are True
```

>Example: Program to classify a person based on his/her age.

```python
age = 20

if age < 18:
    print("us are a minor.")
elif age < 65:
    print("us are an adult.")
else:
    print("us are a senior citizen.")
```
Looping Statements
Looping statements are used to repeat a block of code multiple times. Python supports for loo

For Loop
The `for` loop iterates over a sequence (like a list, tuple, or string) and executes a block

>Syntax:

```python
for item in sequence:
    # Code block to execute for each item

```

>Example: Program to print names of fruits saved in a list.

```python
# Iterating over a list
fruits = ["apple", "banana", "cherry"]
for fruit in fruits:
    print(fruit)
```
While Loop

The `while` loop repeatedly executes a block of code as long as a specified condition is True

>Syntax:

```python
while condition:
    # Code block to execute while condition is True

```

>Example: Print all counting numbers less than 5.

```python
# Counting from 0 to 4
count = 0
while count < 5:
    print(count)
    count += 1

```
Control Flow Statements

Control flow statements alter the flow of execution within loops and conditionals.

Break Statement

The `break` statement exits the current loop, regardless of the loop's condition.

>Example: Program to exit from the printing of whole numbers less than 10, while trigger 5.

```python
for i in range(10):
    if i == 5:
        break
    print(i)
# Output: 0 1 2 3 4
```

Continue Statement

The `continue` statement skips the rest of the code inside the current loop iteration and pro

>Example: Program to print all the whole numbers in the range 5 except 2.

```python
for i in range(5):
    if i == 2:
        continue
    print(i)
# Output: 0 1 3 4
```

Pass Statement

The `pass` statement is a placeholder that does nothing and is used when a statement is synta

>Example: Program to print all the whole numbers in the range 5 except 3.

```python
for i in range(5):
    if i == 3:
        pass  # Placeholder for future code
    else:
        print(i)
# Output: 0 1 2 4
```

::: {.callout-caution collapse="true"}

````
## Cautions When Using Control Flow Structures
Control flow structures are essential in Python programming for directing the flow of executi

**Infinite Loops**

- **Issue**: A `while` loop with a condition that never becomes `False` can lead to an infini
- **Caution**: Always ensure that the condition in a `while` loop will eventually become `Fal

**Example:**

```python
# Infinite loop example
count = 0
while count < 5:
    print(count)
    # Missing count increment, causing an infinite loop
```
:::

## Functions in Python Programming

Functions are a fundamental concept in Python programming that enable code reuse, modularity

**What is a Function?**

A function is a named block of code designed to perform a specific task. Functions can take i

**Defining a Function**

In Python, functions are defined using the `def` keyword, followed by the function name, pare

>Syntax:

```python
def function_name(parameters):
    # Code block
    return result
```
>Example:

```python
def greet(name):
    """
````
````

```
    Returns a greeting message for the given name.
    """
    return f"Hello, {name}!"
```

Relevance of functions in Programming
1. *Code Reusability* : Functions allow us to define a piece of code once and reuse it in mul

2. *Modularity* : Functions break down complex problems into smaller, manageable pieces. Eac

3. *Abstraction* : Functions enable us to abstract away the implementation details. We can us

4. *Testing and Debugging* : Functions allow us to test individual components of our code sep

5. *Library Creation* : Functions are the building blocks of libraries and modules. By organi

>Example: Creating a Simple Library

Stage 1: Define Functions in a Module

```python
# my_library.py

def add(a, b):
    """
    Returns the sum of two numbers.
    """
    return a + b

def multiply(a, b):
    """
    Returns the product of two numbers.
    """
    return a * b
```
Stage 2: Use the Library in Another Program

```python
# main.py

import my_library

result_sum = my_library.add(5, 3)
```

````
result_product = my_library.multiply(5, 3)

print(f"Sum: {result_sum}")
print(f"Product: {result_product}")
```
## Object-Oriented Programming (OOP) in Python

Object-Oriented Programming (OOP) is a programming paradigm that uses "objects" to design and

### Key Concepts of OOP

1. **Classes and Objects**

- **Class**: A class is a blueprint for creating objects. It defines a set of attributes and
- **Object**: An object is an instance of a class. It is a specific realization of the class

#### Example

```python
# Defining a class
class Dog:
    def __init__(self, name, age):
        self.name = name  # Attribute
        self.age = age    # Attribute

    def bark(self):
        return "Woof!"   # Method

# Creating an object of the class
my_dog = Dog(name="Buddy", age=3)

# Accessing attributes and methods
print(my_dog.name)  # Output: Buddy
print(my_dog.age)   # Output: 3
print(my_dog.bark())  # Output: Woof!
```
2. **Encapsulation**

Encapsulation is the concept of bundling data (attributes) and methods (functions) that opera

>Example: Controll the access to member variables using encapsulation.

```python
````

```
class Account:
    def __init__(self, balance):
        self.__balance = balance  # Private attribute

    def deposit(self, amount):
        if amount > 0:
            self.__balance += amount

    def get_balance(self):
        return self.__balance

# Creating an object of the class
my_account = Account(balance=1000)
my_account.deposit(500)

print(my_account.get_balance())  # Output: 1500
# print(my_account.__balance)  # This will raise an AttributeError
```

3. **Inheritance**

Inheritance is a mechanism in which a new class (child or derived class) inherits attributes

>Example: Demonstrating usage of attributes of base class in the derived classes.

```python
# Base class
class Animal:
    def __init__(self, name):
        self.name = name

    def speak(self):
        return "Some sound"

# Derived class
class Dog(Animal):
    def __init__(self, name, breed):
        super().__init__(name)  # Calling the constructor of the base class
        self.breed = breed

    def speak(self):
        return "Woof!"

# Another derived class
```

```
class Cat(Animal):
    def __init__(self, name, color):
        super().__init__(name)  # Calling the constructor of the base class
        self.color = color

    def speak(self):
        return "Meow!"

# Creating objects of the derived classes
dog = Dog(name="Buddy", breed="Golden Retriever")
cat = Cat(name="Whiskers", color="Gray")

print(f"{dog.name} is a {dog.breed} and says {dog.speak()}")  # Output: Buddy is a Golden Re
print(f"{cat.name} is a {cat.color} cat and says {cat.speak()}")  # Output: Whiskers is a Gra
```
4. **Polymorphism**

Polymorphism allows objects of different classes to be treated as objects of a common superc

>Example:

```python
class Bird:
    def fly(self):
        return "Flies in the sky"

class Penguin(Bird):
    def fly(self):
        return "Cannot fly, swims instead"

# Creating objects of different classes
bird = Bird()
penguin = Penguin()

print(bird.fly())      # Output: Flies in the sky
print(penguin.fly())  # Output: Cannot fly, swims instead
```
Working with Files in Python

File handling is an essential part of programming that allows us to work with data stored in

Opening a File

In Python, we use the `open()` function to open a file. This function returns a file object,

>Syntax:

```python
file_object = open(file_path, mode)
```

Where,

- `file_path` : Path to the file (can be a relative or absolute path).
- `mode` : Specifies the file access mode (e.g., 'r' for reading, 'w' for writing, 'a' for a

>Example:

```python
# Opening a file in read mode
file = open('example.txt', 'r')
```

Reading from a File

Once a file is opened, we can read its contents using various methods. Common methods include

- `read()` : Reads the entire file content.
- `readline()` : Reads a single line from the file.
- `readlines()` : Reads all the lines into a list.

>Example:

```python
# Reading the entire file
file_content = file.read()
print(file_content)

# Reading a single line
file.seek(0)  # Move cursor to the start of the file
line = file.readline()
print(line)

# Reading all lines
file.seek(0)
lines = file.readlines()
print(lines)
```

```

**Writing to a File**

To write data to a file, we need to open the file in write ('w') or append ('a') mode. When

>Example:

```python
# Opening a file in write mode
file = open('example.txt', 'w')

# Writing data to the file
file.write("Hello, World!\n")
file.write("Python file handling example.")

# Closing the file
file.close()
```

Closing a File

It is important to close a file after performing operations to ensure that all changes are sa

>Example:

```python
f_1 = open('example.txt', 'w') # open the file example.txt to f_1
f_1.close() # close the file with handler 'f_1'
```
Using Context Managers

Context managers provide a convenient way to handle file operations, automatically managing

>Example:

```python
# Using context manager to open and write to a file
with open('example.txt', 'w') as file:
    file.write("This is written using a context manager.")

```
## From Theory to Practice
```

In this section, we transition from theoretical concepts to practical applications by explori

Applications of Matrix Operations in Digital Image Processing

Matrix operations play a pivotal role in digital image processing, enabling a wide range of t

Matrix Addition in Image Blending

Matrix addition is a fundamental operation in image processing, particularly useful in the te

Concept

When working with grayscale images, each image can be represented as a matrix where each elem

>Example:

Consider two 2x2 grayscale images represented as matrices:

```python
image1= [[100, 150],[200, 250]]
image2=[[50, 100],[100, 150]]
```

To blend these images, we add the corresponding pixel values as:

```python
blended_image[i][j] = image1[i][j] + image2[i][j]
```

Ensure that the resulting pixel values do not exceed the maximum value allowed (255 for 8-bit

Python Implementation of image blending

Below is the Python code for blending two images using matrix addition:

::: {.cell execution_count=1}
``` {.python .cell-code}
def matrix_addition(image1, image2):
    rows = len(image1)
    cols = len(image1[0])
```

```
    blended_image = [[0] * cols for _ in range(rows)]

    for i in range(rows):
        for j in range(cols):
            blended_pixel = image1[i][j] + image2[i][j]
            blended_image[i][j] = min(blended_pixel, 255)  # Clip to 255

    return blended_image

# Example matrices (images)
image1 = [[100, 150], [200, 250]]
image2 = [[50, 100], [100, 150]]

blended_image = matrix_addition(image1, image2)
print("Blended Image:")
for row in blended_image:
    print(row)
```

::: {.cell-output .cell-output-stdout}
```
Blended Image:
[150, 250]
[255, 255]
```
:::
:::

Image blending is a powerful technique with numerous real-time applications. It is widely us

::: {.callout-tip title="Image Blending as Basic Arithmetic with Libraries"}
In upcoming chapters, we will explore how specific libraries for image handling simplify the
adding two objects. Using these libraries, such as PIL (Python Imaging Library) or OpenCV, a
:::

Let's summarize a few more matrix operations and its uses in digital image processing tasks

Matrix Subtraction in Image Sharpening

Matrix subtraction is a fundamental operation in image processing, essential for techniques

Concept

In grayscale images, each pixel value represents the intensity of light at that point. Image

>Example:

Consider a grayscale image represented as a matrix:
```python
original_image [[100, 150, 200],[150, 200, 250],[200, 250, 100]]
```

To sharpen the image, we subtract a blurred version (smoothed image) from the original. This

```python
sharpened_image[i][j] = original_image[i][j] - blurred_image[i][j]
```

Python Implementation

Below is a simplified Python example of image sharpening using matrix subtraction:

```python
# Original image matrix (grayscale values)
original_image = [
    [100, 150, 200],
    [150, 200, 250],
    [200, 250, 100]
]

# Function to apply Gaussian blur (for demonstration, simplified as average smoothing)
def apply_blur(image):
    blurred_image = []
    for i in range(len(image)):
        row = []
        for j in range(len(image[0])):
            neighbors = []
            for dx in [-1, 0, 1]:
                for dy in [-1, 0, 1]:
                    ni, nj = i + dx, j + dy
                    if 0 <= ni < len(image) and 0 <= nj < len(image[0]):
                        neighbors.append(image[ni][nj])
            blurred_value = sum(neighbors) // len(neighbors)
            row.append(blurred_value)
        blurred_image.append(row)
```

```
    return blurred_image

# Function for matrix subtraction (image sharpening)
def image_sharpening(original_image, blurred_image):
    sharpened_image = []
    for i in range(len(original_image)):
        row = []
        for j in range(len(original_image[0])):
            sharpened_value = original_image[i][j] - blurred_image[i][j]
            row.append(sharpened_value)
        sharpened_image.append(row)
    return sharpened_image

# Apply blur to simulate smoothed image
blurred_image = apply_blur(original_image)

# Perform matrix subtraction for image sharpening
sharpened_image = image_sharpening(original_image, blurred_image)

# Print the sharpened image
print("Sharpened Image:")
for row in sharpened_image:
    print(row)

```

Matrix Multiplication in Image Filtering (Convolution)

Matrix multiplication, specifically convolution in the context of image processing, is a func

Concept
In grayscale images, each pixel value represents the intensity of light at that point. Convol

>Example:

Consider a grayscale image represented as a matrix:

```python
original_image= [[100, 150, 200, 250],
 [150, 200, 250, 300],
 [200, 250, 300, 350],
 [250, 300, 350, 400]]
```

To perform smoothing (averaging) using a simple kernel:

```python
[[1/9, 1/9, 1/9],
 [1/9, 1/9, 1/9],
 [1/9, 1/9, 1/9]]
```
The kernel is applied over the image using convolution:

```python
smoothed_image[i][j] = sum(original_image[ii][jj] * kernel[k][l] for all (ii, jj) in neighbo
```
Python Implementation

Here's a simplified Python example demonstrating convolution for image smoothing without exte

```python
# Original image matrix (grayscale values)
original_image = [
    [100, 150, 200, 250],
    [150, 200, 250, 300],
    [200, 250, 300, 350],
    [250, 300, 350, 400]
]

# Define a simple kernel/filter for smoothing (averaging)
kernel = [
    [1/9, 1/9, 1/9],
    [1/9, 1/9, 1/9],
    [1/9, 1/9, 1/9]
]

# Function for applying convolution (image filtering)
def apply_convolution(image, kernel):
    height = len(image)
    width = len(image[0])
    ksize = len(kernel)
    kcenter = ksize // 2  # Center of the kernel

    # Initialize result image
    filtered_image = [[0]*width for _ in range(height)]
```

```
    # Perform convolution
    for i in range(height):
        for j in range(width):
            sum = 0.0
            for k in range(ksize):
                for l in range(ksize):
                    ii = i + k - kcenter
                    jj = j + l - kcenter
                    if ii >= 0 and ii < height and jj >= 0 and jj < width:
                        sum += image[ii][jj] * kernel[k][l]
            filtered_image[i][j] = int(sum)

    return filtered_image

# Apply convolution to simulate smoothed image (averaging filter)
smoothed_image = apply_convolution(original_image, kernel)

# Print the smoothed image
print("Smoothed Image:")
for row in smoothed_image:
    print(row)
```


Determinant: Image Transformation

Concept
The determinant of a transformation matrix helps understand how transformations like scaling

>Example:

Here, we compute the determinant of a scaling matrix to understand how the scaling affects tl

```python
def calculate_determinant(matrix):
    a, b = matrix[0]
    c, d = matrix[1]
    return a * d - b * c

# Example transformation matrix (scaling)
transformation_matrix = [[2, 0], [0, 2]]
determinant = calculate_determinant(transformation_matrix)
print(f"Determinant of the transformation matrix: {determinant}")
```

```
This value indicates how the transformation scales the image area.

#### Rank: Image Rank and Data Compression

**Concept**
The rank of a matrix indicates the number of linearly independent rows or columns. In image c

>Example:

Here, we compute the rank of a matrix representing an image. A lower rank might indicate that

```python
def matrix_rank(matrix):
    def is_zero_row(row):
        return all(value == 0 for value in row)

    def row_echelon_form(matrix):
        A = [row[:] for row in matrix]
        m = len(A)
        n = len(A[0])
        rank = 0
        for i in range(min(m, n)):
            if A[i][i] != 0:
                for j in range(i + 1, m):
                    factor = A[j][i] / A[i][i]
                    for k in range(i, n):
                        A[j][k] -= factor * A[i][k]
                rank += 1
        return rank

    return row_echelon_form(matrix)

# Example matrix (image)
image_matrix = [[1, 2], [3, 4]]
rank = matrix_rank(image_matrix)
print(f"Rank of the image matrix: {rank}")
```
Matrix Operations Using `Python` Libraries

Introduction

In this section, we will explore the computational aspects of basic matrix algebra using Pyth
```

````
### Introduction to SymPy

`SymPy` is a powerful Python library designed for symbolic mathematics. It provides tools fo

#### Key Matrix Functions in SymPy

- **Matrix Addition**: Adds two matrices element-wise.
- **Matrix Subtraction**: Subtracts one matrix from another element-wise.
- **Matrix Multiplication**: Multiplies two matrices using the dot product.
- **Matrix Power**: Raises a matrix to a given power using matrix multiplication.

>**Example 1: Matrix Addition**

**Pseudocode**

```python
FUNCTION matrix_add():
    # Define matrices A and B
    A = [[1, 2], [3, 4]]
    B = [[5, 6], [7, 8]]

    # Check if matrices A and B have the same dimensions
    if dimensions_of(A) != dimensions_of(B):
         raise ValueError("Matrices must have the same dimensions")

    # Initialize result matrix with zeros
    result = [[0 for _ in range(len(A[0]))] for _ in range(len(A))]

    # Add corresponding elements from A and B
    for i in range(len(A)):
        for j in range(len(A[0])):
            result[i][j] = A[i][j] + B[i][j]

# Return the result matrix
    return result
ENDFUNCTION
```
Python implementation of the above pseudocode is given below:

::: {.cell execution_count=2}
``` {.python .cell-code}
import sympy as sy
````
````

````
sy.init_printing()
# Define matrices A and B
A = sy.Matrix([[1, 2], [3, 4]])
B = sy.Matrix([[5, 6], [7, 8]])

# Add matrices
C = A + B

# Print the result in symbolic form
print("Matrix Addition Result:")
display(C)

# Convert to LaTeX code for documentation or presentation
#latex_code = sy.latex(C)
#print("LaTeX Code for Addition Result:")
#print(latex_code)
```

::: {.cell-output .cell-output-stdout}
```
Matrix Addition Result:
```
:::

::: {.cell-output .cell-output-display}
![](module_1_files/figure-pdf/cell-3-output-2.png){fig-pos='H'}
:::
:::


>*Example 2: Matrix Subtraction*

*Pseudocode:*

```python
# Define matrices A and B
A = [[5, 6], [7, 8]]
B = [[1, 2], [3, 4]]

# Check if matrices A and B have the same dimensions
if dimensions_of(A) != dimensions_of(B):
    raise ValueError("Matrices must have the same dimensions")
````

```
# Initialize result matrix with zeros
result = [[0 for _ in range(len(A[0]))] for _ in range(len(A))]

# Subtract corresponding elements from A and B
for i in range(len(A)):
    for j in range(len(A[0])):
        result[i][j] = A[i][j] - B[i][j]

# Return the result matrix
return result
```
Python implementation of the above pseudocode is given below:

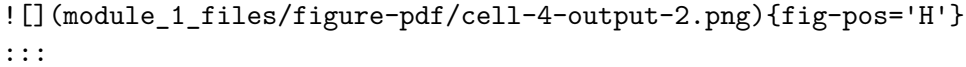
::: {.cell execution_count=3}
``` {.python .cell-code}
from sympy import Matrix
# Define matrices A and B
A = Matrix([[5, 6], [7, 8]])
B = Matrix([[1, 2], [3, 4]])

# Subtract matrices
C = A - B

# Print the result in symbolic form
print("Matrix Subtraction Result:")
display(C)

# Convert to LaTeX code for documentation or presentation
latex_code = sy.latex(C)
print("LaTeX Code for Subtraction Result:")
print(latex_code)
```

::: {.cell-output .cell-output-stdout}
```
Matrix Subtraction Result:
```
:::

::: {.cell-output .cell-output-display}

:::

```
::: {.cell-output .cell-output-stdout}
```
LaTeX Code for Subtraction Result:
\left[\begin{matrix}4 & 4\\4 & 4\end{matrix}\right]
```
:::
:::


>**Example 3: Matrix Multiplication**

*Pseudocode:*

```python
# Define matrices A and B
A = [[1, 2], [3, 4]]
B = [[5, 6], [7, 8]]

# Check if the number of columns in A equals the number of rows in B
if len(A[0]) != len(B):
    raise ValueError("Number of columns in A must equal number of rows in B")

# Initialize result matrix with zeros
result = [[0 for _ in range(len(B[0]))] for _ in range(len(A))]

# Multiply matrices A and B
for i in range(len(A)):
    for j in range(len(B[0])):
        for k in range(len(B)):
            result[i][j] += A[i][k] * B[k][j]

# Return the result matrix
return result

```

Python implementation of the above pseudocode is given below:

::: {.cell execution_count=4}
``` {.python .cell-code}
# Define matrices A and B
A = Matrix([[1, 2], [3, 4]])
B = Matrix([[5, 6], [7, 8]])
```
```

````
# Multiply matrices
M = A * B

# Print the result in symbolic form
print("Matrix Multiplication Result:")
display(M)

# Convert to LaTeX code for documentation or presentation
latex_code = sy.latex(M)
print("LaTeX Code for Multiplication Result:")
print(latex_code)
```

::: {.cell-output .cell-output-stdout}
```
Matrix Multiplication Result:
```
:::

::: {.cell-output .cell-output-display}
![](module_1_files/figure-pdf/cell-5-output-2.png){fig-pos='H'}
:::

::: {.cell-output .cell-output-stdout}
```
LaTeX Code for Multiplication Result:
\left[\begin{matrix}19 & 22\\43 & 50\end{matrix}\right]
```
:::
:::


>**Example 3: Matrix Multiplication**

*Pseudocode:*

```python
# Define matrices A and B
A = [[1, 2], [3, 4]]
B = [[5, 6], [7, 8]]

# Check if the number of columns in A equals the number of rows in B
````

```
if len(A[0]) != len(B):
    raise ValueError("Number of columns in A must equal number of rows in B")

# Initialize result matrix with zeros
result = [[0 for _ in range(len(B[0]))] for _ in range(len(A))]

# Multiply matrices A and B
for i in range(len(A)):
    for j in range(len(B[0])):
        for k in range(len(B)):
            result[i][j] += A[i][k] * B[k][j]

# Return the result matrix
return result
```
Python code for implementing the above pseudocode is shown below:

::: {.cell execution_count=5}
``` {.python .cell-code}
# Define matrices A and B
A = Matrix([[1, 2], [3, 4]])
B = Matrix([[5, 6], [7, 8]])

# Multiply matrices
M = A * B

# Print the result in symbolic form
print("Matrix Multiplication Result:")
display(M)

# Convert to LaTeX code for documentation or presentation
latex_code = sy.latex(M)
print("LaTeX Code for Multiplication Result:")
print(latex_code)
```

::: {.cell-output .cell-output-stdout}
```
Matrix Multiplication Result:
```
:::

::: {.cell-output .cell-output-display}

```
![](module_1_files/figure-pdf/cell-6-output-2.png){fig-pos='H'}
:::

::: {.cell-output .cell-output-stdout}
```
LaTeX Code for Multiplication Result:
\left[\begin{matrix}19 & 22\\43 & 50\end{matrix}\right]
```
:::
:::


>**Example 4: Matrix Power**

*Pseudocode:*

```python
# Define matrix A and power n
A = [[1, 2], [3, 4]]
n = 2

# Initialize result matrix as identity matrix
result = identity_matrix_of(len(A))

# Compute A raised to the power of n
for _ in range(n):
    result = matrix_multiply(result, A)

# Return the result matrix
return result

```
Python implementation of the above pseudocode is shown below:

::: {.cell execution_count=6}
``` {.python .cell-code}
# Define matrix A
A = Matrix([[1, 2], [3, 4]])

# Compute matrix A raised to the power of 2
n = 2
C = A**n
```
```

```
# Print the result in symbolic form
print("Matrix Power Result:")
display(C)

# Convert to LaTeX code for documentation or presentation
latex_code = sy.latex(C)
print("LaTeX Code for Power Result:")
print(latex_code)
```

::: {.cell-output .cell-output-stdout}
```
Matrix Power Result:
```
:::

::: {.cell-output .cell-output-display}

:::

::: {.cell-output .cell-output-stdout}
```
LaTeX Code for Power Result:
\left[\begin{matrix}7 & 10\\15 & 22\end{matrix}\right]
```
:::
:::

Introduction to PIL for Image Manipulation

The `PIL` (Python Imaging Library), now known as `Pillow`, provides essential tools for open

Matrix operations have significant applications in digital image processing. These operations

1. **Matrix Addition: Image Blending**

Matrix addition can be used to blend two images by adding their pixel values. This process ca

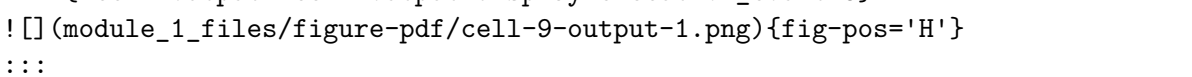
Example 1: Simple Image Blending

::: {.cell execution_count=7}
``` {.python .cell-code}
```

```
import numpy as np
from PIL import Image
import urllib.request
urllib.request.urlretrieve('http://lenna.org/len_top.jpg',"input.jpg")
img1 = Image.open("input.jpg") #loading first image

urllib.request.urlretrieve('https://www.keralatourism.org/images/destination/large/thekkekudi

img2 = Image.open("input2.jpg")# loading second image

# Resize second image to match the size of the first image
img2 = img2.resize(img1.size)
# Convert images to numpy arrays
arr1 = np.array(img1)
arr2 = np.array(img2)

# Add the images
blended_arr = arr1 + arr2

# Clip the values to be in the valid range [0, 255]
blended_arr = np.clip(blended_arr, 0, 255).astype(np.uint8)

# Convert back to image
blended_img = Image.fromarray(blended_arr)

# Save or display the blended image
#blended_img.save('blended_image.jpg')
#blended_img.show()
#blended_img #display the blended image
```
:::

The input and output images are shown below:

::: {.cell execution_count=8}
``` {.python .cell-code}
img1
```

::: {.cell-output .cell-output-display execution_count=8}

:::

````
:::


::: {.cell execution_count=9}
``` {.python .cell-code}
img2
```

::: {.cell-output .cell-output-display execution_count=9}
![](module_1_files/figure-pdf/cell-10-output-1.png){fig-pos='H'}
:::
:::


::: {.cell execution_count=10}
``` {.python .cell-code}
blended_img
```

::: {.cell-output .cell-output-display execution_count=10}
![](module_1_files/figure-pdf/cell-11-output-1.png){fig-pos='H'}
:::
:::


>**Example 2: Weighted Image Blending**

::: {.cell execution_count=11}
``` {.python .cell-code}
# Blend with weights
alpha = 0.7
blended_arr = alpha * arr1 + (1 - alpha) * arr2

# Clip the values to be in the valid range [0, 255]
blended_arr = np.clip(blended_arr, 0, 255).astype(np.uint8)

# Convert back to image
blended_img = Image.fromarray(blended_arr)

# Save or display the weighted blended image
#blended_img.save('weighted_blended_image.jpg')
#blended_img.show()
blended_img
````
````

```

::: {.cell-output .cell-output-display execution_count=11}

:::
:::


#### Matrix Subtraction: Image Sharpening

Matrix subtraction can be used to sharpen images by subtracting a blurred version of the ima

>**Example 1: Sharpening by Subtracting Blurred Image**

::: {.cell execution_count=12}
``` {.python .cell-code}
from PIL import Image, ImageFilter
# Convert image to grayscale for simplicity
img_gray = img1.convert('L')
arr = np.array(img_gray)

# Apply Gaussian blur
blurred_img = img_gray.filter(ImageFilter.GaussianBlur(radius=5))
blurred_arr = np.array(blurred_img)

# Sharpen the image by subtracting blurred image
sharpened_arr = arr - blurred_arr

# Clip the values to be in the valid range [0, 255]
sharpened_arr = np.clip(sharpened_arr, 0, 255).astype(np.uint8)

# Convert back to image
sharpened_img = Image.fromarray(sharpened_arr)

# Save or display the sharpened image
#sharpened_img.save('sharpened_image.jpg')
#sharpened_img.show()
sharpened_img
```

::: {.cell-output .cell-output-display execution_count=12}

:::
```

````
:::


#### Matrix Multiplication: Image Filtering (Convolution)

Matrix multiplication is used in image filtering to apply convolution kernels for various ef

>**Example 1: Applying a Convolution Filter**

::: {.cell execution_count=13}
``` {.python .cell-code}
# Define a simple convolution kernel (e.g., edge detection)
kernel = np.array([
    [1, 0, -1],
    [1, 0, -1],
    [1, 0, -1]
])

# Convert the image to grayscale for simplicity
img_gray = img1.convert('L')
arr = np.array(img_gray)

# Apply convolution
filtered_arr = np.zeros_like(arr)
for i in range(1, arr.shape[0] - 1):
    for j in range(1, arr.shape[1] - 1):
        region = arr[i-1:i+2, j-1:j+2]
        filtered_arr[i, j] = np.sum(region * kernel)

# Convert back to image
filtered_img = Image.fromarray(np.clip(filtered_arr, 0, 255).astype(np.uint8))

# Save or display the filtered image
#filtered_img.save('filtered_image.jpg')
#filtered_img.show()
filtered_img
```

::: {.cell-output .cell-output-display execution_count=13}
![](module_1_files/figure-pdf/cell-14-output-1.png){fig-pos='H'}
:::
:::
````

```
>**Example 2: Applying a Gaussian Blur Filter**

::: {.cell execution_count=14}
``` {.python .cell-code}
# Define a Gaussian blur filter
blurred_img = img1.filter(ImageFilter.GaussianBlur(radius=5))

# Save or display the blurred image
#blurred_img.save('blurred_image.jpg')
#blurred_img.show()
blurred_img
```

::: {.cell-output .cell-output-display execution_count=14}
{fig-pos='H'}
:::
:::

Solving Systems of Equations and Applications

Introduction

Solving systems of linear equations is crucial in various image processing tasks, such as ima

>**Example 1: Solving a System of Equations**

Consider the system:

$$\begin{cases}
2x + 3y& = 13 \\
4x - y &= 7
\end{cases}$$

>**Python Implementation**

::: {.cell execution_count=15}
``` {.python .cell-code}
from sympy import Matrix
# Define the coefficient matrix and constant matrix
A = Matrix([[2, 3], [4, -1]])
B = Matrix([13, 7])
```
```

````
# Solve the system of equations
solution = A.solve_least_squares(B)
# Print the solution
print("Solution to the System of Equations:")
print(solution)
```

::: {.cell-output .cell-output-stdout}
```
Solution to the System of Equations:
Matrix([[17/7], [19/7]])
```
:::
:::


## Conclusion

In this chapter, we transitioned from understanding fundamental matrix operations to applying

We further explored the application of these matrix operations to real-world image processing

By integrating theoretical understanding with practical implementation, this chapter reinfor



`<!-- quarto-file-metadata: eyJyZXNvdXJjZURpciI6Ii4ifQ== -->`{=html}

```{=html}
<!-- quarto-file-metadata: eyJyZXNvdXJjZURpciI6Ii4iLCJib29rSXRlbVR5cGUiOiJjaGFwdGVyIiwiYm9vaG
```

# Transforming Linear Algebra to Computational Language

```````{.quarto-title-block template='C:\Program Files\Quarto\share\projects\book\pandoc\titl
---
title: Transforming Linear Algebra to Computational Language
execute:
  enabled: true
jupyter: python3

---
````

Introduction

In the first module, we established a solid foundation in matrix algebra by exploring pseudocode and implementing fundamental matrix operations using `Python`. We practiced key concepts such as matrix addition, subtraction, multiplication, and determinants through practical examples in image processing, leveraging the `SymPy` library for symbolic computation.

As we begin the second module, **"Transforming Linear Algebra to Computational Language,"** our focus will shift towards applying these concepts with greater depth and actionable insight. This module is designed to bridge the theoretical knowledge from matrix algebra with practical computational applications. You will learn to interpret and utilize matrix operations, solve systems of equations, and analyze the rank of matrices within a variety of real-world contexts.

A new concept we will introduce is the **Rank-Nullity Theorem**, which provides a fundamental relationship between the rank of a matrix and the dimensions of its null space. This theorem is crucial for understanding the solution spaces of linear systems and the properties of linear transformations. By applying this theorem, you will be able to gain deeper insights into the structure of solutions and the behavior of matrix transformations.

This transition will not only reinforce your understanding of linear algebra but also enhance your ability to apply these concepts effectively in computational settings. Through engaging examples and practical exercises, you will gain valuable experience in transforming abstract mathematical principles into tangible solutions, setting a strong groundwork for advanced computational techniques.

Relearning of Terms and Operations in Linear Algebra

In this section, we will revisit fundamental matrix operations such as addition, subtraction, scaling, and more through practical examples. Our goal is to transform theoretical linear algebra into modern computational applications. We will demonstrate these concepts using `Python`, focusing on practical and industrial applications.

Matrix Addition and Subtraction in Data Analysis

Matrix addition and subtraction are fundamental operations that help in combining datasets and analyzing differences.

Simple Example: Combining Quarterly Sales Data

We begin with quarterly sales data from different regions and combine them to get the total sales. The sales data is given in Table 2. A ar plot of the total sales is shown in Fig 1.

Table 2: Quarterly Sales Data

Region	Q1	Q2	Q3	Q4
A	2500	2800	3100	2900
B	1500	1600	1700	1800

From Scratch `Python` Implementation:

```
import numpy as np
import matplotlib.pyplot as plt

# Quarterly sales data
sales_region_a = np.array([2500, 2800, 3100, 2900])
sales_region_b = np.array([1500, 1600, 1700, 1800])

# Combine sales data
total_sales = sales_region_a + sales_region_b

# Visualization
quarters = ['Q1', 'Q2', 'Q3', 'Q4']
plt.bar(quarters, total_sales, color='skyblue')
plt.xlabel('Quarter')
plt.ylabel('Total Sales')
plt.title('Combined Quarterly Sales Data for Regions A and B')
plt.show()
```

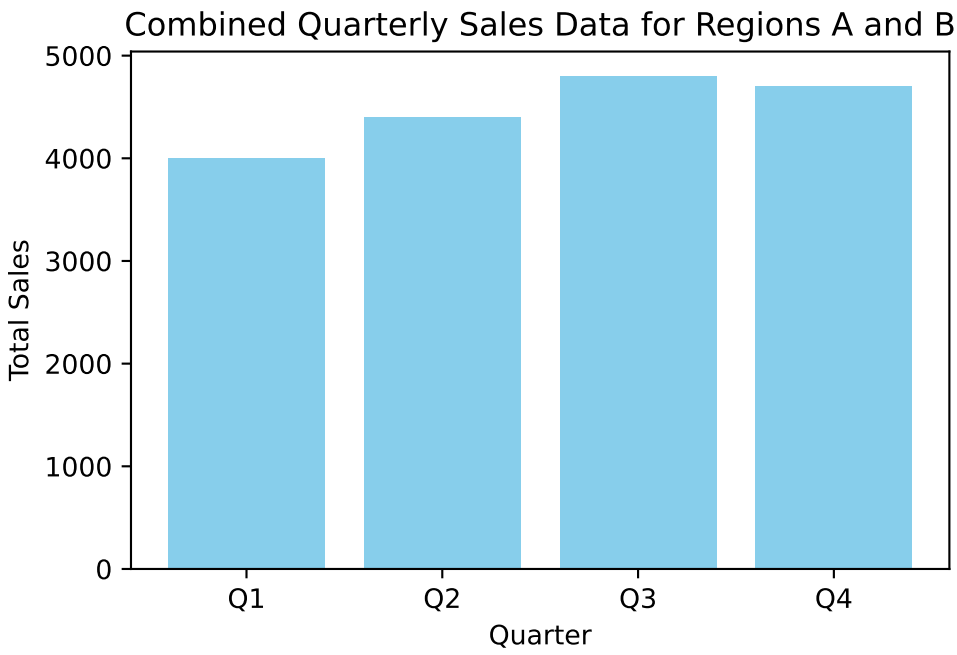


Figure 1: Computing Total Sales using `Numpy` aggregation method

In the above `Python` code, we have performed the aggregation operation with the `NumPy` method. Same can be done in a more data analysis style using `pandas` inorder to handle tabular data meaningfully. In this approach, quarterly sales data of each region is stored as `DataFrames`(like an excel sheet). The we combine these two `DataFrames` into one. After that create a new row with index 'Total' and populate this row with sum of quarterly sales in Region A and Region B. Finally a bar plot is created using this 'Total' sales. Advantage of this approach is that we don't need the `matplotlib` library to create visualizations!. The EDA using this approach is shown in Fig 2.

```
import pandas as pd
import matplotlib.pyplot as plt

# DataFrames for quarterly sales data
df_a = pd.DataFrame({'Q1': [2500], 'Q2': [2800], 'Q3': [3100], 'Q4': [2900]},
 ↪  index=['Region A'])
df_b = pd.DataFrame({'Q1': [1500], 'Q2': [1600], 'Q3': [1700], 'Q4': [1800]},
 ↪  index=['Region B'])

# Combine data
df_combined = df_a.add(df_b, fill_value=0)
```

```
df_combined.loc["Total"] = df_combined.sum(axis=0)
# Visualization
df_combined.loc["Total"].plot(kind='bar', color=['green'])
plt.xlabel('Quarter')
plt.ylabel('Total Sales')
plt.title('Combined Quarterly Sales Data for Regions A and B')
plt.show()
```

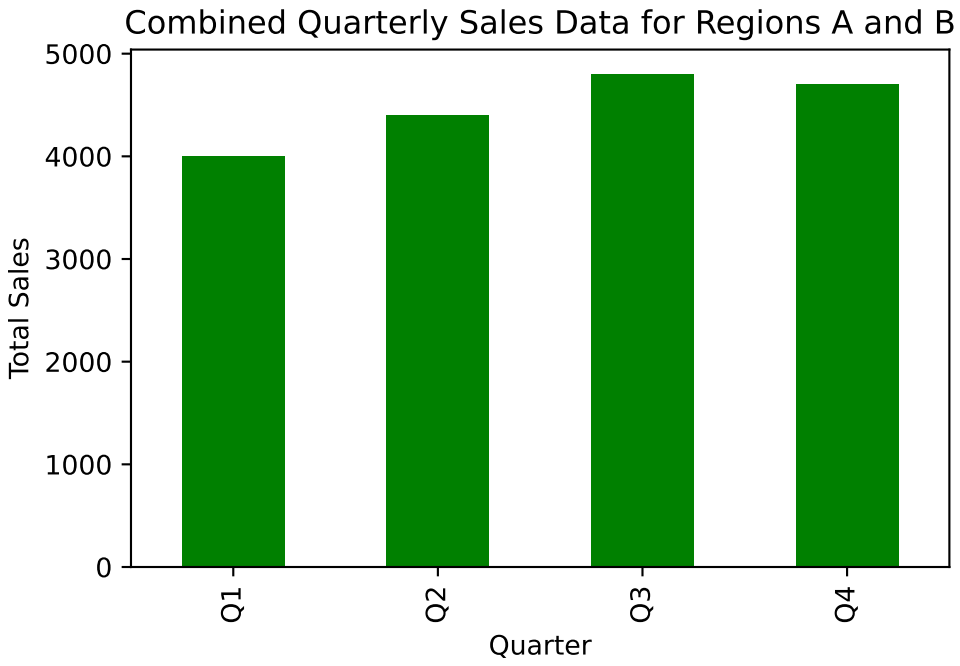


Figure 2: Computation of Total Sales using `Pandas` method

We can extend this in to more advanced examples. Irrespective to the size of the data, for representation and aggregation tasks matrix models are best options and are used in industry as a standard. Let us consider an advanced example to analyse difference in stock prices. For this example we are using a simulated data. The python code for this simulation process is shown in Fig 3.

```
import numpy as np
import matplotlib.pyplot as plt

# Simulated observed and predicted stock prices
observed_prices = np.random.uniform(100, 200, size=(100, 5))
```

```
predicted_prices = np.random.uniform(95, 210, size=(100, 5))

# Calculate the difference matrix
price_differences = observed_prices - predicted_prices

# Visualization
plt.imshow(price_differences, cmap='coolwarm', aspect='auto')
plt.colorbar()
plt.title('Stock Price Differences')
plt.xlabel('Stock Index')
plt.ylabel('Day Index')
plt.show()
```

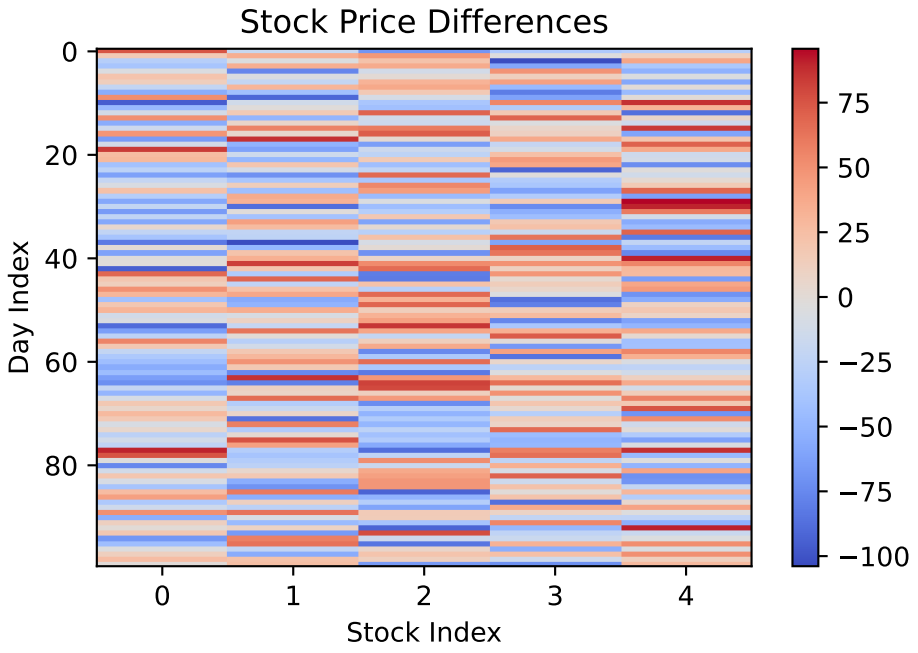


Figure 3: Demonstration of Stock Price simulated from a Uniform Distribution

Another important matrix operation relevant to data analytics and Machine Learning application is scaling. This is considered as a statistical tool to make various features (attributes) in to same scale so as to avoid unnecessary misleading impact in data analysis and its intepretation. In Machine Learning context, this pre-processing stage is inevitable so as to make the model relevant and usable.

Simple Example: Normalizing Employee Performance Data

Table 3: Employee Performance Data

Employee	Metric A	Metric B
X	80	700
Y	90	800
Z	100	900
A	110	1000
B	120	1100

Using simple python code we can simulate the model for `min-max` scaling. The formula for `min-max` scaling is:

$$min_max(X) = \frac{X - min(X)}{max(X) - min(X)}$$

For example, while applying the `min-max` scaling in the first value of Metric A, the scaled value is

$$min_max(80)\frac{80 - 80}{120 - 80} = 0$$

Similarly

$$min_max(100)\frac{100 - 80}{120 - 80} = 0.5$$

When we apply this formula to Metric A and Metric B, the scaled output from Table 3 will be as follows:

Table 4: Employee Performance Data

Employee	Metric A	Metric B
X	0.00	0.00
Y	0.25	0.25
Z	0.50	0.50
A	0.75	0.75
B	1.00	1.00

It is interesting to look into the scaled data! In the orginal table (Table 3) it is looked like Metric B is superior. But from the scaled table (Table 4), it is clear that both the Metrics are representing same relative information. This will help us to identify the redundency in measure and so skip any one of the Metric before analysis!.

The same can be achieved through a matrix operation. The `Python` implementation of this scaling process is shown in Fig 4.

```
import numpy as np
import matplotlib.pyplot as plt

# Employee performance data with varying scales
data = np.array([[80, 700], [90, 800], [100, 900], [110, 1000], [120, 1100]])

# Manual scaling
min_vals = np.min(data, axis=0)
max_vals = np.max(data, axis=0)
scaled_data = (data - min_vals) / (max_vals - min_vals)

# Visualization
plt.figure(figsize=(8, 5))
plt.subplot(1, 2, 1)
plt.imshow(data, cmap='viridis')
plt.title('Original Data')
plt.colorbar()

plt.subplot(1, 2, 2)
plt.imshow(scaled_data, cmap='viridis')
plt.title('Scaled Data')
plt.colorbar()

plt.show()
```

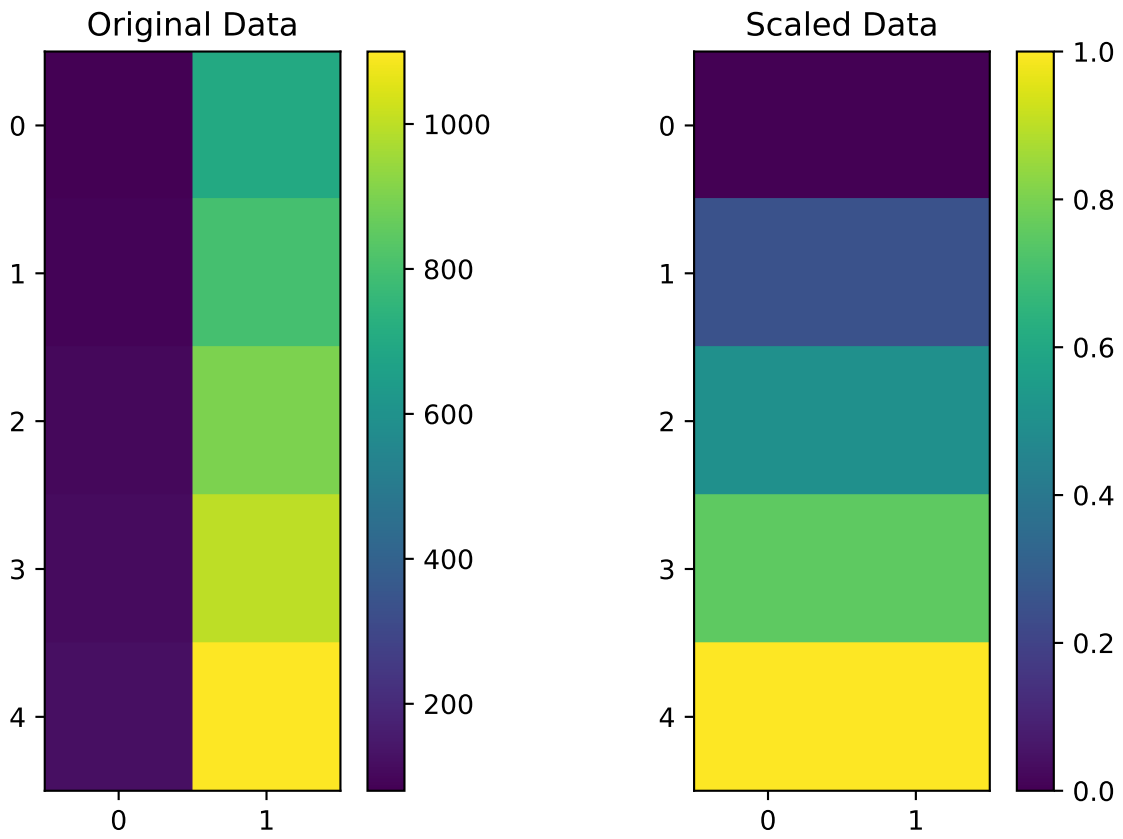


Figure 4: Total sales using `pandas` method

From the first sub plot, it is clear that there is a significant difference in the distributions (Metric A and Metric B values). But the second sub plot shows that both the distributions have same pattern and the values ranges between 0 and 1. In short the visualization is more appealing and self explanatory in this case.

> **Note**
>
> The `min-max` scaling method will confine the feature values (attributes) into the range $[0, 1]$. So in effect all the features are scaled proportionally to the data spectrum.

Similarly, we can use the `standard scaling` (transformation to normal distribution) using the transformation $\frac{x - \bar{x}}{\sigma}$. Scaling table is given as a practice task to the reader. The python code for this operation is shown in Fig 5.

```
# Standard scaling from scratch
def standard_scaling(data):
    mean = np.mean(data, axis=0)
    std = np.std(data, axis=0)
    scaled_data = (data - mean) / std
    return scaled_data

# Apply standard scaling
scaled_data_scratch = standard_scaling(data)

print("Standard Scaled Data (from scratch):\n", scaled_data_scratch)

# Visualization
plt.figure(figsize=(6, 5))
plt.subplot(1, 2, 1)
plt.imshow(data, cmap='viridis')
plt.title('Original Data')
plt.colorbar()

plt.subplot(1, 2, 2)
plt.imshow(scaled_data_scratch, cmap='viridis')
plt.title('Scaled Data')
plt.colorbar()

plt.show()
```

```
Standard Scaled Data (from scratch):
 [[-1.41421356 -1.41421356]
 [-0.70710678 -0.70710678]
 [ 0.          0.        ]
 [ 0.70710678  0.70710678]
 [ 1.41421356  1.41421356]]
```

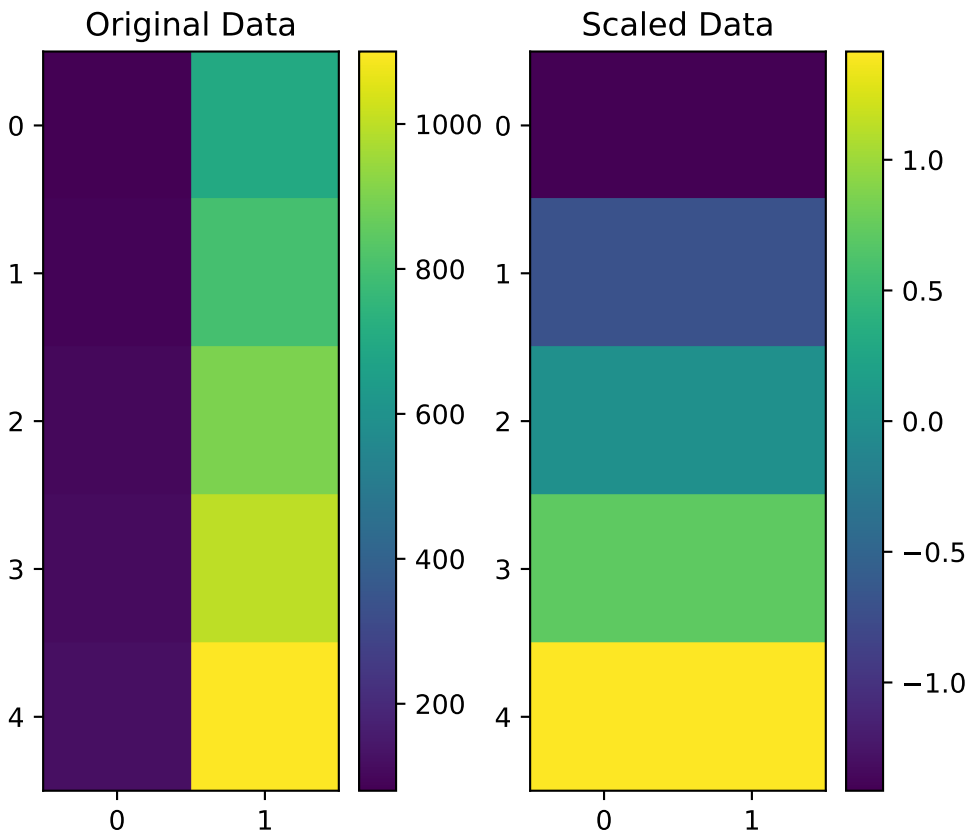


Figure 5: Min-max scaling using basic python

To understand the effect of standard scaling, let us consider Fig 6. This plot create the frequency distribution of the data as a histogram along with the density function. From the first sub-plot, it is clear that the distribution has multiple modes (peaks). When we apply the standard scaling, the distribution become un-modal(only one peek). This is demonstrated in the second sub-plot.

```
# Standard scaling from scratch
import seaborn as sns
# Create plots
plt.figure(figsize=(6, 5))

# Plot for original data
plt.subplot(1, 2, 1)
sns.histplot(data, kde=True, bins=10, palette="viridis")
plt.title('Original Data Distribution')
```

```
plt.xlabel('Value')
plt.ylabel('Frequency')

# Plot for standard scaled data
plt.subplot(1, 2, 2)
sns.histplot(scaled_data_scratch, kde=True, bins=10, palette="viridis")
plt.title('Standard Scaled Data Distribution')
plt.xlabel('Value')
plt.ylabel('Frequency')

plt.tight_layout()
plt.show()
```

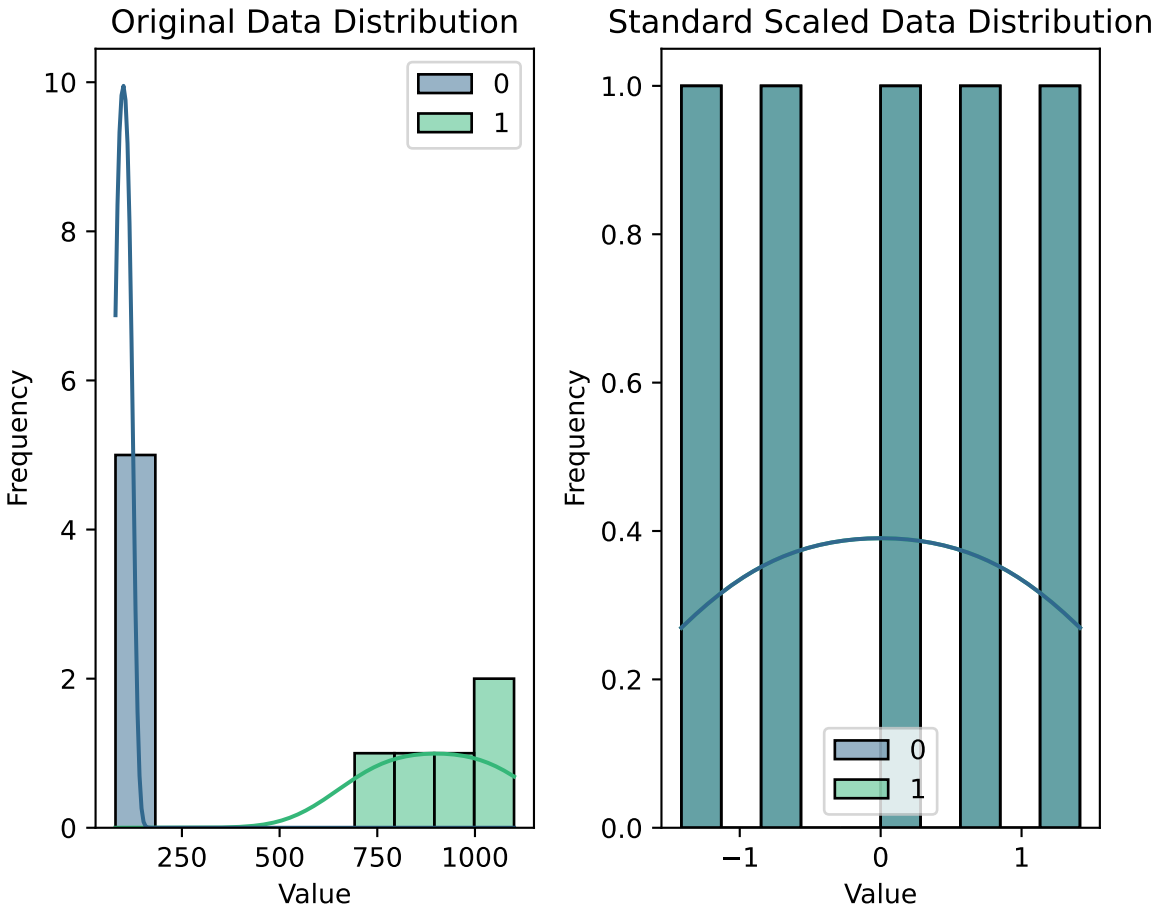


Figure 6: Impact of standard scaling on the distribution

A scatter plot showing the compare the impact of scaling on the given distribution is shown in Fig 7.

```
# Plot original and scaled data
plt.figure(figsize=(6, 5))

# Original Data
plt.subplot(1, 3, 1)
plt.scatter(data[:, 0], data[:, 1], color='blue')
plt.title('Original Data')
plt.xlabel('Metric A')
plt.ylabel('Metric B')

# Standard Scaled Data
plt.subplot(1, 3, 2)
plt.scatter(scaled_data_scratch[:, 0], scaled_data_scratch[:, 1],
 ↪  color='green')
plt.title('Standard Scaled Data')
plt.xlabel('Metric A (Standard Scaled)')
plt.ylabel('Metric B (Standard Scaled)')

# Min-Max Scaled Data
plt.subplot(1, 3, 3)
plt.scatter(scaled_data[:, 0], scaled_data[:, 1], color='red')
plt.title('Min-Max Scaled Data')
plt.xlabel('Metric A (Min-Max Scaled)')
plt.ylabel('Metric B (Min-Max Scaled)')

plt.tight_layout()
plt.show()
```

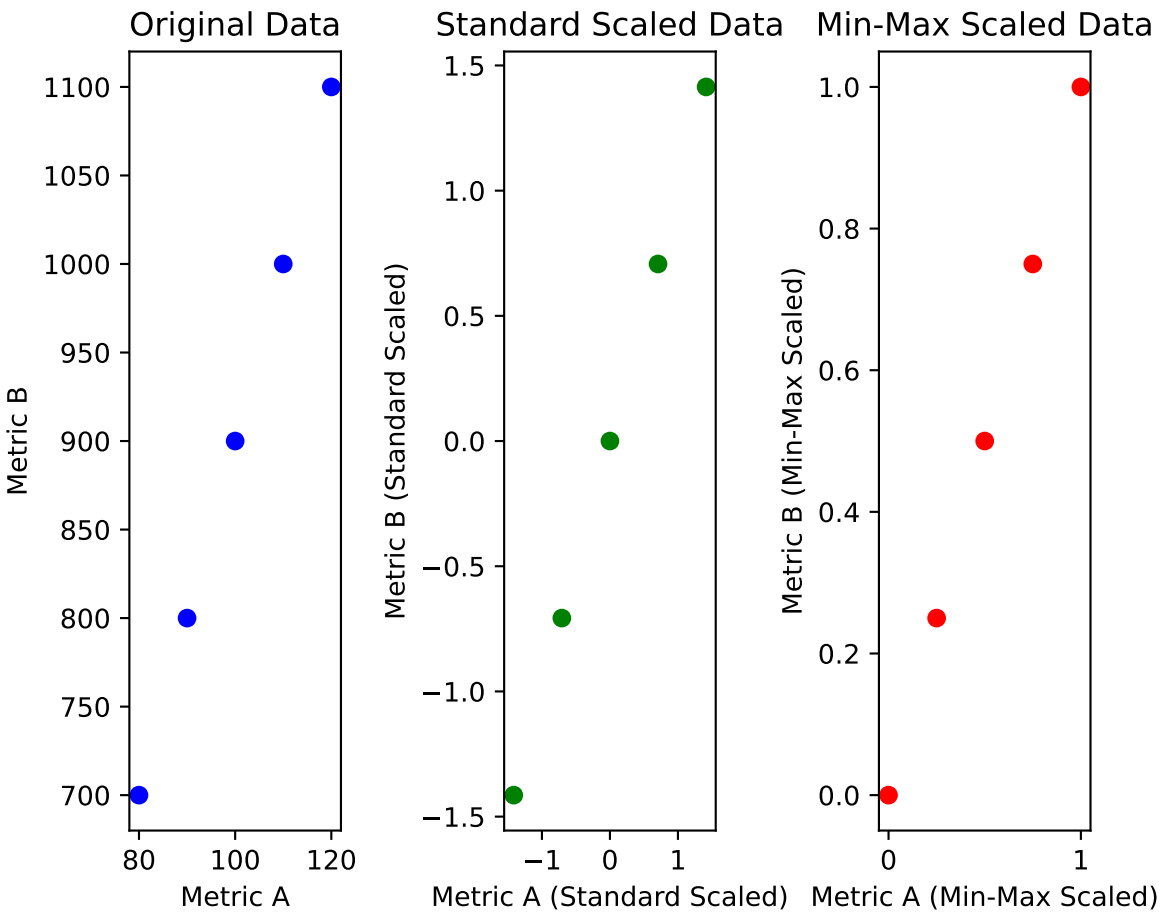


Figure 7: Comparison of impact of scaling on the distribution

From the Fig 7, it is clear that the scaling does not affect the pattern of the data, instead it just scale the distribution proportionally!

We can use the `scikit-learn` library for do the same thing in a very simple handy approach. The `python` code for this job is shown below.

```
from sklearn.preprocessing import MinMaxScaler

# Min-max scaling using sklearn
scaler = MinMaxScaler()
min_max_scaled_data_sklearn = scaler.fit_transform(data)

print("Min-Max Scaled Data (using sklearn):\n", min_max_scaled_data_sklearn)
```

```
Min-Max Scaled Data (using sklearn):
 [[0.   0.  ]
 [0.25 0.25]
 [0.5  0.5 ]
 [0.75 0.75]
 [1.   1.  ]]
```

```
from sklearn.preprocessing import StandardScaler

# Standard scaling using sklearn
scaler = StandardScaler()
scaled_data_sklearn = scaler.fit_transform(data)

print("Standard Scaled Data (using sklearn):\n", scaled_data_sklearn)
```

```
Standard Scaled Data (using sklearn):
 [[-1.41421356 -1.41421356]
 [-0.70710678 -0.70710678]
 [ 0.          0.        ]
 [ 0.70710678  0.70710678]
 [ 1.41421356  1.41421356]]
```

A scatter plot showing the impact on scaling is shown in Fig 8. This plot compare the mmin-max and standard-scaling.

```
# Plot original and scaled data
plt.figure(figsize=(6, 5))

# Original Data
plt.subplot(1, 3, 1)
plt.scatter(data[:, 0], data[:, 1], color='blue')
plt.title('Original Data')
plt.xlabel('Metric A')
plt.ylabel('Metric B')

# Standard Scaled Data
plt.subplot(1, 3, 2)
plt.scatter(scaled_data_sklearn[:, 0], scaled_data_sklearn[:, 1],
 ↪  color='green')
plt.title('Standard Scaled Data')
plt.xlabel('Metric A (Standard Scaled)')
```

```
plt.ylabel('Metric B (Standard Scaled)')

# Min-Max Scaled Data
plt.subplot(1, 3, 3)
plt.scatter(min_max_scaled_data_sklearn[:, 0], min_max_scaled_data_sklearn[:,
 ↪  1], color='red')
plt.title('Min-Max Scaled Data')
plt.xlabel('Metric A (Min-Max Scaled)')
plt.ylabel('Metric B (Min-Max Scaled)')

plt.tight_layout()
plt.show()
```

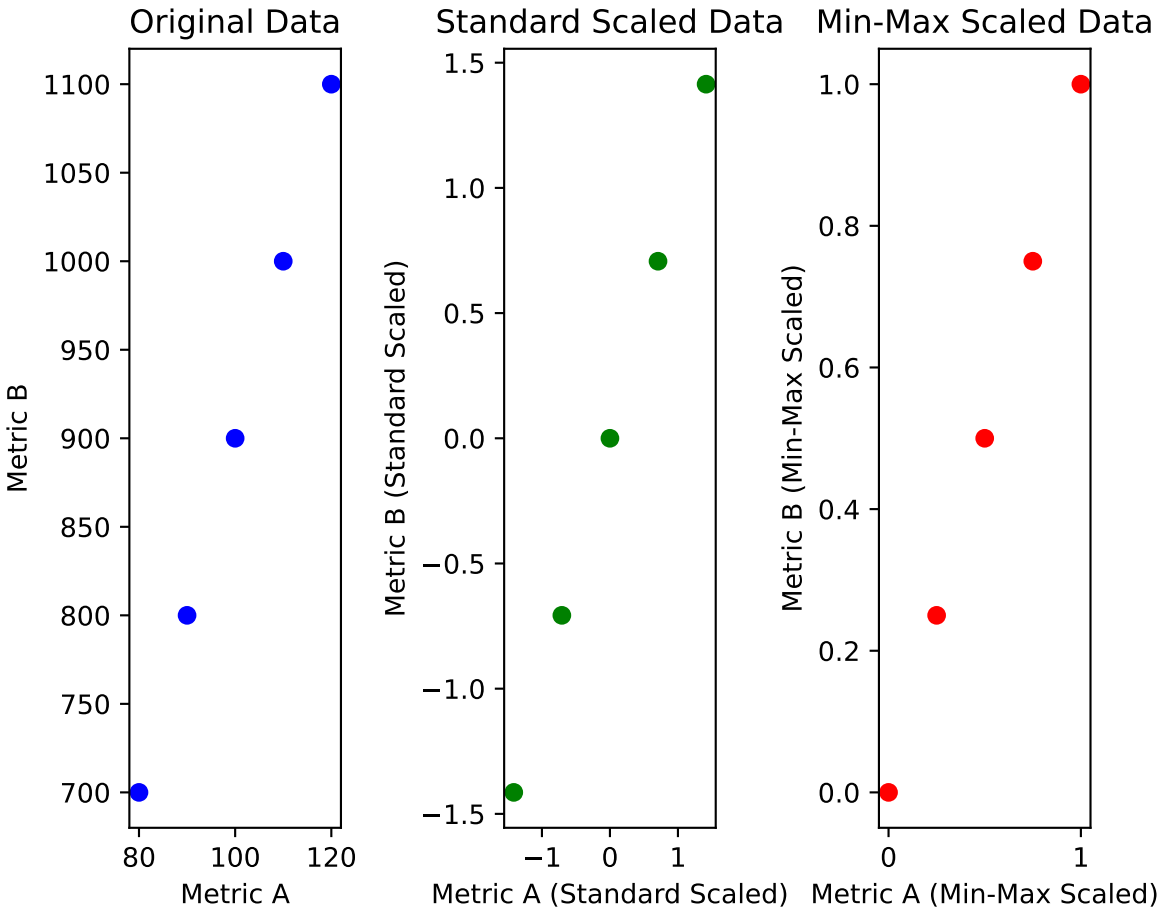


Figure 8: Camparison of Min-max and standard scalings with original data

More on Matrix Product and its Applications

In the first module of our course, we introduced matrix products as scalar projections, focusing on how matrices interact through basic operations. In this section, we will expand on this by exploring different types of matrix products that have practical importance in various fields. One such product is the *Hadamard product*, which is particularly useful in applications ranging from image processing to neural networks and statistical analysis. We will cover the definition, properties, and examples of the Hadamard product, and then delve into practical applications with simulated data.

Hadamard Product

The Hadamard product (or element-wise product) of two matrices is a binary operation that combines two matrices of the same dimensions to produce another matrix of the same dimensions, where each element is the product of corresponding elements in the original matrices.

! Definition (Hadamard Product):

For two matrices A and B of the same dimension $m \times n$, the Hadamard product $A \circ B$ is defined as:

$$(A \circ B)_{ij} = A_{ij} \cdot B_{ij}$$

where $\cdot$ denotes element-wise multiplication.

i Properties of Hadamard Product

1. **Commutativity**:
$$A \circ B = B \circ A$$
2. **Associativity**:
$$(A \circ B) \circ C = A \circ (B \circ C)$$
3. **Distributivity**:
$$A \circ (B + C) = (A \circ B) + (A \circ C)$$

Some simple examples to demonstrate the Hadamard product is given below.

Example 1: Basic Hadamard Product

Given matrices:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$$

The Hadamard product $A \circ B$ is:

$$A \circ B = \begin{pmatrix} 1 \cdot 5 & 2 \cdot 6 \\ 3 \cdot 7 & 4 \cdot 8 \end{pmatrix} = \begin{pmatrix} 5 & 12 \\ 21 & 32 \end{pmatrix}$$

Example 2: Hadamard Product with Larger Matrices

Given matrices:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}, \quad B = \begin{pmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{pmatrix}$$

The Hadamard product $A \circ B$ is:

$$A \circ B = \begin{pmatrix} 1 \cdot 9 & 2 \cdot 8 & 3 \cdot 7 \\ 4 \cdot 6 & 5 \cdot 5 & 6 \cdot 4 \\ 7 \cdot 3 & 8 \cdot & 9 \cdot 1 \end{pmatrix} = \begin{pmatrix} 9 & 16 & 21 \\ 24 & 25 & 24 \\ 21 & 16 & 9 \end{pmatrix}$$

In the following code chunks the computational process of Hadamard product is implemented in `Python`. Here both the from the scratch and use of external module versions are included.

1. Compute Hadamard Product from Scratch (without Libraries)

Here's how you can compute the Hadamard product manually:

```python
# Define matrices A and B
A = [[1, 2, 3], [4, 5, 6]]
B = [[7, 8, 9], [10, 11, 12]]

# Function to compute Hadamard product
def hadamard_product(A, B):
    # Get the number of rows and columns
    num_rows = len(A)
    num_cols = len(A[0])

    # Initialize the result matrix
    result = [[0]*num_cols for _ in range(num_rows)]

    # Compute the Hadamard product
    for i in range(num_rows):
        for j in range(num_cols):
            result[i][j] = A[i][j] * B[i][j]
```

```
    return result

# Compute Hadamard product
hadamard_product_result = hadamard_product(A, B)

# Display result
print("Hadamard Product (From Scratch):")
for row in hadamard_product_result:
    print(row)
```

```
Hadamard Product (From Scratch):
[7, 16, 27]
[40, 55, 72]
```

2. Compute Hadamard Product Using `SymPy`

Here's how to compute the Hadamard product using `SymPy`:

```
import sympy as sp

# Define matrices A and B
A = sp.Matrix([[1, 2, 3], [4, 5, 6]])
B = sp.Matrix([[7, 8, 9], [10, 11, 12]])

# Compute Hadamard product using SymPy
Hadamard_product_sympy = A.multiply_elementwise(B)

# Display result
print("Hadamard Product (Using SymPy):")
print(Hadamard_product_sympy)
```

```
Hadamard Product (Using SymPy):
Matrix([[7, 16, 27], [40, 55, 72]])
```

Practical Applications

Application 1: Image Masking

The Hadamard product can be used for image masking. Here's how you can apply a mask to an image and visualize it as shown in Fig 9.

```python
import matplotlib.pyplot as plt
import numpy as np

# Simulated large image (2D array) using NumPy
image = np.random.rand(100, 100)

# Simulated mask (binary matrix) using NumPy
mask = np.random.randint(0, 2, size=(100, 100))

# Compute Hadamard product
masked_image = image * mask

# Plot original image and masked image
fig, ax = plt.subplots(1, 2, figsize=(12, 5))
ax[0].imshow(image, cmap='gray')
ax[0].set_title('Original Image')
ax[1].imshow(masked_image, cmap='gray')
ax[1].set_title('Masked Image')
plt.show()
```

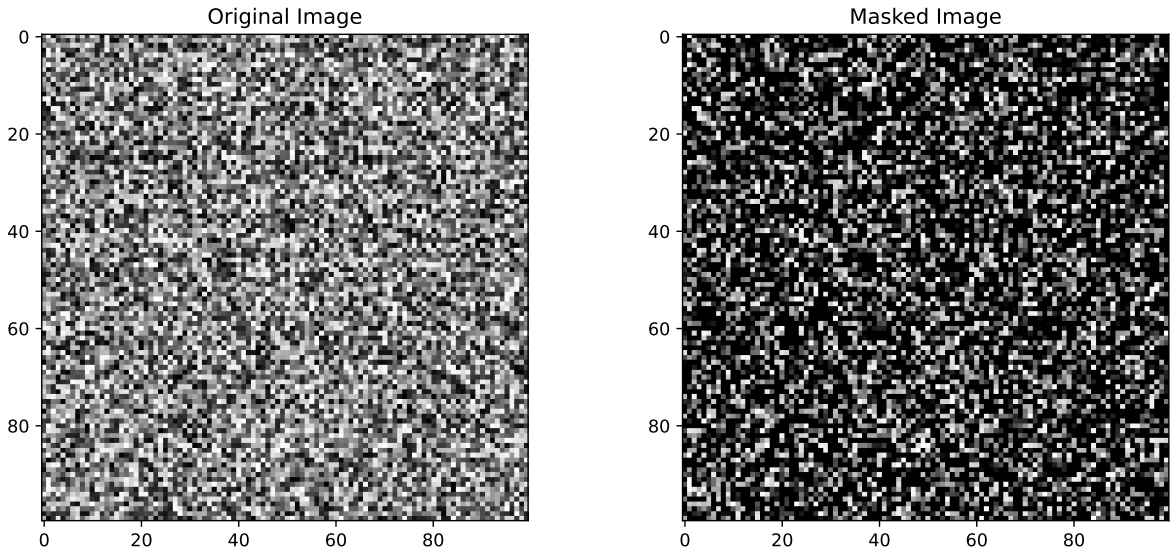


Figure 9: Demonstration of Masking in DIP using Hadamard Product

Application 2: Element-wise Scaling in Neural Networks

The Hadamard product can be used for dropout[1] in neural networks. A simple simulated example is given below.

```
# Simulated large activations (2D array) using NumPy
activations = np.random.rand(100, 100)

# Simulated dropout mask (binary matrix) using NumPy
dropout_mask = np.random.randint(0, 2, size=(100, 100))

# Apply dropout
dropped_activations = activations * dropout_mask

# Display results
print("Original Activations:")
print(activations)
print("\nDropout Mask:")
print(dropout_mask)
print("\nDropped Activations:")
print(dropped_activations)
```

```
Original Activations:
[[1.50241810e-01 4.33985567e-01 8.06799811e-01 ... 3.17691660e-02
  2.65296312e-01 1.95520882e-01]
 [3.95437889e-01 2.63829859e-01 5.16712381e-01 ... 4.27303912e-01
  8.20997228e-01 4.15547921e-01]
 [5.98635672e-01 5.22905270e-01 8.08365101e-01 ... 7.49319840e-01
  3.02072551e-01 4.00749514e-01]
 ...
 [5.51947232e-01 7.93479508e-01 3.29339876e-01 ... 6.07145901e-01
  7.96351298e-01 3.83801103e-01]
 [8.97921944e-01 5.98465485e-01 2.37418723e-02 ... 1.44208520e-01
  6.93063855e-01 3.02194139e-01]
 [1.10657022e-01 4.26475997e-01 2.83044675e-02 ... 5.35278273e-04
  5.20851155e-01 2.01807675e-01]]

Dropout Mask:
[[1 0 0 ... 0 1 0]
 [0 1 0 ... 0 0 1]
 [0 1 0 ... 0 0 0]
 ...
```

[1] A regularization techniques in Deep learning. This approach deactivate some selected neurons to control model over-fitting

```
 [1 1 0 ... 1 1 1]
 [1 0 0 ... 1 0 0]
 [0 0 1 ... 1 1 0]]

Dropped Activations:
[[1.50241810e-01 0.00000000e+00 0.00000000e+00 ... 0.00000000e+00
  2.65296312e-01 0.00000000e+00]
 [0.00000000e+00 2.63829859e-01 0.00000000e+00 ... 0.00000000e+00
  0.00000000e+00 4.15547921e-01]
 [0.00000000e+00 5.22905270e-01 0.00000000e+00 ... 0.00000000e+00
  0.00000000e+00 0.00000000e+00]
 ...
 [5.51947232e-01 7.93479508e-01 0.00000000e+00 ... 6.07145901e-01
  7.96351298e-01 3.83801103e-01]
 [8.97921944e-01 0.00000000e+00 0.00000000e+00 ... 1.44208520e-01
  0.00000000e+00 0.00000000e+00]
 [0.00000000e+00 0.00000000e+00 2.83044675e-02 ... 5.35278273e-04
  5.20851155e-01 0.00000000e+00]]
```

Application 3: Statistical Data Analysis

In statistics, the Hadamard product can be applied to scale covariance matrices. Here's how we can compute the covariance matrix using matrix operations and apply scaling. Following `Python` code demonstrate this.

```python
import sympy as sp
import numpy as np

# Simulated large dataset (2D array) using NumPy
data = np.random.rand(100, 10)

# Compute the mean of each column
mean = np.mean(data, axis=0)

# Center the data
centered_data = data - mean

# Compute the covariance matrix using matrix product operation
cov_matrix = (centered_data.T @ centered_data) / (centered_data.shape[0] - 1)
cov_matrix_sympy = sp.Matrix(cov_matrix)

# Simulated scaling factors (2D array) using SymPy Matrix
scaling_factors = sp.Matrix(np.random.rand(10, 10))
```

```
# Compute Hadamard product
scaled_cov_matrix = cov_matrix_sympy.multiply(scaling_factors)

# Display results
print("Covariance Matrix:")
print(cov_matrix_sympy)
print("\nScaling Factors:")
print(scaling_factors)
print("\nScaled Covariance Matrix:")
print(scaled_cov_matrix)
```

```
Covariance Matrix:
Matrix([[0.0829729792128583, -0.00130926119215640, -0.00449358582009456, -0.0085411236596643
Scaling Factors:
Matrix([[0.529376470070988, 0.563200130818408, 0.318460183719583, 0.797238283210820, 0.21628
Scaled Covariance Matrix:
Matrix([[0.0392621161636811, 0.0379768336785228, 0.0163507664815586, 0.0591680411074049, 0.0
```

Practice Problems

Problem 1: Basic Hadamard Product

Given matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

Find the Hadamard product $C = A \circ B$.

Solution:

$$C = \begin{bmatrix} 1 \cdot 5 & 2 \cdot 6 \\ 3 \cdot 7 & 4 \cdot 8 \end{bmatrix} = \begin{bmatrix} 5 & 12 \\ 21 & 32 \end{bmatrix}$$

Problem 2: Hadamard Product with Identity Matrix

Given matrices:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$$

Find the Hadamard product $C = A \circ I$.

Solution:

$$C = \begin{bmatrix} 1 \cdot 1 & 2 \cdot 0 & 3 \cdot 0 \\ 4 \cdot 0 & 5 \cdot 1 & 6 \cdot 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 5 & 0 \end{bmatrix}$$

Problem 3: Hadamard Product with Zero Matrix

Given matrices:

$$A = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$$

$$Z = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Find the Hadamard product $C = A \circ Z$.

Solution:

$$C = \begin{bmatrix} 3 \cdot 0 & 4 \cdot 0 \\ 5 \cdot 0 & 6 \cdot 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

Problem 4: Hadamard Product of Two Identity Matrices

Given identity matrices:

$$I_2 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$I_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Find the Hadamard product $C = I_2 \circ I_3$ (extend I_2 to match dimensions of I_3).

Solution:

Extend I_2 to I_3:

$$I_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 \cdot 1 & 0 \cdot 0 & 0 \cdot 0 \\ 0 \cdot 0 & 1 \cdot 1 & 0 \cdot 0 \\ 0 \cdot 0 & 0 \cdot 0 & 0 \cdot 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Problem 5: Hadamard Product with Random Matrices

Given random matrices:

$$A = \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 5 \\ 6 & 2 \end{bmatrix}$$

Find the Hadamard product $C = A \circ B$.

Solution:

$$C = \begin{bmatrix} 2 \cdot 0 & 3 \cdot 5 \\ 1 \cdot 6 & 4 \cdot 2 \end{bmatrix} = \begin{bmatrix} 0 & 15 \\ 6 & 8 \end{bmatrix}$$

Problem 6: Hadamard Product of 3x3 Matrices

Given matrices:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

$$B = \begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$$

Find the Hadamard product $C = A \circ B$.

Solution:

$$C = \begin{bmatrix} 1 \cdot 9 & 2 \cdot 8 & 3 \cdot 7 \\ 4 \cdot 6 & 5 \cdot 5 & 6 \cdot 4 \\ 7 \cdot 3 & 8 \cdot 2 & 9 \cdot 1 \end{bmatrix} = \begin{bmatrix} 9 & 16 & 21 \\ 24 & 25 & 24 \\ 21 & 16 & 9 \end{bmatrix}$$

Problem 7: Hadamard Product of Column Vectors

Given column vectors:

$$u = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$v = \begin{bmatrix} 5 \\ 6 \end{bmatrix}$$

Find the Hadamard product $w = u \circ v$.

Solution:

$$w = \begin{bmatrix} 2 \cdot 5 \\ 3 \cdot 6 \end{bmatrix} = \begin{bmatrix} 10 \\ 18 \end{bmatrix}$$

Problem 8: Hadamard Product with Non-Square Matrices

Given matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \\ 5 & 6 \end{bmatrix}$$

$$B = \begin{bmatrix} 7 & 8 \\ 9 & 10 \end{bmatrix}$$

Find the Hadamard product $C = A \circ B$ (extend B to match dimensions of A).

Solution:

Extend B to match dimensions of A:

$$B = \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 7 & 8 \end{bmatrix}$$

$$C = \begin{bmatrix} 1 \cdot 7 & 2 \cdot 8 \\ 3 \cdot 9 & 4 \cdot 10 \\ 5 \cdot 7 & 6 \cdot 8 \end{bmatrix} = \begin{bmatrix} 7 & 16 \\ 27 & 40 \\ 35 & 48 \end{bmatrix}$$

Problem 9: Hadamard Product in Image Processing

Given matrices representing image pixel values:

$$A = \begin{bmatrix} 10 & 20 \\ 30 & 40 \end{bmatrix}$$

$$B = \begin{bmatrix} 0.5 & 1.5 \\ 2.0 & 0.5 \end{bmatrix}$$

Find the Hadamard product $C = A \circ B$.

Solution:

$$C = \begin{bmatrix} 10 \cdot 0.5 & 20 \cdot 1.5 \\ 30 \cdot 2.0 & 40 \cdot 0.5 \end{bmatrix} = \begin{bmatrix} 5 & 30 \\ 60 & 20 \end{bmatrix}$$

Problem 10: Hadamard Product in Statistical Data

Given matrices representing two sets of statistical data:

$$A = \begin{bmatrix} 5 & 6 & 7 \\ 8 & 9 & 10 \end{bmatrix}$$
$$B = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

Find the Hadamard product $C = A \circ B$.

Solution:

$$C = \begin{bmatrix} 5 \cdot 1 & 6 \cdot 2 & 7 \cdot 3 \\ 8 \cdot 4 & 9 \cdot 5 & 10 \cdot 6 \end{bmatrix} = \begin{bmatrix} 5 & 12 & 21 \\ 32 & 45 & 60 \end{bmatrix}$$

Inner Product of Matrices

The inner product of two matrices is a generalized extension of the dot product, where each matrix is treated as a vector in a high-dimensional space. For two matrices A and B of the same dimension $m \times n$, the inner product is defined as the sum of the element-wise products of the matrices.

! Definition (Inner product)

For two matrices A and B of dimension $m \times n$, the inner product $\langle A, B \rangle$ is given by:

$$\langle A, B \rangle = \sum_{i=1}^{m} \sum_{j=1}^{n} A_{ij} \cdot B_{ij}$$

where $\cdot$ denotes element-wise multiplication.

! Properties

1. **Commutativity**:
$$\langle A, B \rangle = \langle B, A \rangle$$
2. **Linearity**:
$$\langle A + C, B \rangle = \langle A, B \rangle + \langle C, B \rangle$$
3. **Positive Definiteness**:
$$\langle A, A \rangle \geq 0$$
with equality if and only if A is a zero matrix.

Some simple examples showing the mathematical process of calculating the inner product is given bellow.

Example 1: Basic Inner Product

Given matrices:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$$

The inner product $\langle A, B \rangle$ is:

$$\langle A, B \rangle = 1 \cdot 5 + 2 \cdot 6 + 3 \cdot 7 + 4 \cdot 8 = 5 + 12 + 21 + 32 = 70$$

Example 2: Inner Product with Larger Matrices

Given matrices:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}, \quad B = \begin{pmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{pmatrix}$$

The inner product $\langle A, B \rangle$ is calculated as:

$$\begin{aligned} \langle A, B \rangle &= 1 \cdot 9 + 2 \cdot 8 + 3 \cdot 7 + 4 \cdot 6 + 5 \cdot 5 + 6 \cdot 4 + 7 \cdot 3 + 8 \cdot 2 + 9 \cdot 1 \\ &= 9 + 16 + 21 + 24 + 25 + 24 + 21 + 16 + 9 \\ &= 175 \end{aligned}$$

Practice Problems

Problem 1: Inner Product of 2x2 Matrices

Given matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} 5 & 6 \\ 7 & 8 \end{bmatrix}$$

Solution:

$$\begin{aligned}\langle A, B\rangle &= \sum_{i,j} A_{ij} B_{ij} \\ &= 1 \cdot 5 + 2 \cdot 6 + 3 \cdot 7 + 4 \cdot 8 \\ &= 5 + 12 + 21 + 32 \\ &= 70\end{aligned}$$

Problem 2: Inner Product of 3x3 Matrices

Given matrices:

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 3 & 4 & 5 \\ 6 & 7 & 8 \end{bmatrix}$$

$$B = \begin{bmatrix} 8 & 7 & 6 \\ 5 & 4 & 3 \\ 2 & 1 & 0 \end{bmatrix}$$

Solution:

$$\begin{aligned}\langle A, B\rangle &= \sum_{i,j} A_{ij} B_{ij} \\ &= 1 \cdot 8 + 0 \cdot 7 + 2 \cdot 6 + \\ &\quad 3 \cdot 5 + 4 \cdot 4 + 5 \cdot 3 + \\ &\quad 6 \cdot 2 + 7 \cdot 1 + 8 \cdot 0 \\ &= 8 + 0 + 12 + 15 + 16 + 15 + 12 + 7 + 0 \\ &= 85\end{aligned}$$

Problem 3: Inner Product of Diagonal Matrices

Given diagonal matrices:

$$A = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} 5 & 0 & 0 \\ 0 & 6 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

Solution:

$$\begin{aligned} \langle A, B \rangle &= \sum_{i,j} A_{ij} B_{ij} \\ &= 2 \cdot 5 + 0 \cdot 0 + 0 \cdot 0 + \\ &\quad 0 \cdot 0 + 3 \cdot 6 + 0 \cdot 0 + \\ &\quad 0 \cdot 0 + 0 \cdot 0 + 4 \cdot 7 \\ &= 10 + 0 + 0 + 0 + 18 + 0 + 0 + 0 + 28 \\ &= 56 \end{aligned}$$

Problem 4: Inner Product of Column Vectors

Given column vectors:

$$u = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$v = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$$

Solution:

$$\begin{aligned} \langle u, v \rangle &= \sum_{i} u_i v_i \\ &= 1 \cdot 4 + 2 \cdot 5 + 3 \cdot 6 \\ &= 4 + 10 + 18 \\ &= 32 \end{aligned}$$

Problem 5: Inner Product with Random Matrices

Given matrices:

$$A = \begin{bmatrix} 3 & 2 \\ 1 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} 5 & 7 \\ 8 & 6 \end{bmatrix}$$

Solution:

$$\begin{aligned} \langle A, B \rangle &= \sum_{i,j} A_{ij} B_{ij} \\ &= 3 \cdot 5 + 2 \cdot 7 + \\ &\quad 1 \cdot 8 + 4 \cdot 6 \\ &= 15 + 14 + 8 + 24 \\ &= 61 \end{aligned}$$

Problem 6: Inner Product of 2x3 and 3x2 Matrices

Given matrices:

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$$

$$B = \begin{bmatrix} 7 & 8 \\ 9 & 10 \\ 11 & 12 \end{bmatrix}$$

Solution:

$$\begin{aligned} \langle A, B \rangle &= \sum_{i,j} A_{ij} B_{ij} \\ &= 1 \cdot 7 + 2 \cdot 8 + 3 \cdot 11 + \\ &\quad 4 \cdot 9 + 5 \cdot 10 + 6 \cdot 12 \\ &= 7 + 16 + 33 + 36 + 50 + 72 \\ &= 214 \end{aligned}$$

Problem 7: Inner Product with Transpose Operation

Given matrices:

$$A = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$$

$$B = \begin{bmatrix} 6 & 7 \\ 8 & 9 \end{bmatrix}$$

Solution:

$$\begin{aligned} \langle A, B \rangle &= \sum_{i,j} A_{ij} B_{ij} \\ &= 2 \cdot 6 + 3 \cdot 7 + \\ &\quad 4 \cdot 8 + 5 \cdot 9 \\ &= 12 + 21 + 32 + 45 \\ &= 110 \end{aligned}$$

Problem 8: Inner Product of Symmetric Matrices

Given symmetric matrices:

$$A = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 4 & 5 \\ 5 & 6 \end{bmatrix}$$

Solution:

$$\begin{aligned} \langle A, B \rangle &= \sum_{i,j} A_{ij} B_{ij} \\ &= 1 \cdot 4 + 2 \cdot 5 + \\ &\quad 2 \cdot 5 + 3 \cdot 6 \\ &= 4 + 10 + 10 + 18 \\ &= 42 \end{aligned}$$

Problem 9: Inner Product with Complex Matrices

Given matrices:

$$A = \begin{bmatrix} 1+i & 2-i \\ 3+i & 4-i \end{bmatrix}$$

$$B = \begin{bmatrix} 5-i & 6+i \\ 7-i & 8+i \end{bmatrix}$$

Solution:

$$\begin{aligned} \langle A, B \rangle &= \sum_{i,j} \mathrm{Re}(A_{ij}\overline{B_{ij}}) \\ &= (1+i)\cdot(5+i) + (2-i)\cdot(6-i) + \\ &\quad (3+i)\cdot(7+i) + (4-i)\cdot(8+i) \\ &= (5+i+5i-i^2) + (12-i-6i+i^2) + \\ &\quad (21+i+7i-i^2) + (32+i-8i-i^2) \\ &= 5+5+12-6+21+32-2 \\ &= 62 \end{aligned}$$

Problem 10: Inner Product of 4x4 Matrices

Given matrices:

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 5 & 6 & 7 & 8 \\ 9 & 10 & 11 & 12 \\ 13 & 14 & 15 & 16 \end{bmatrix}$$

$$B = \begin{bmatrix} 16 & 15 & 14 & 13 \\ 12 & 11 & 10 & 9 \\ 8 & 7 & 6 & 5 \\ 4 & 3 & 2 & 1 \end{bmatrix}$$

Solution:

$$
\begin{aligned}
\langle A, B \rangle &= \sum_{i,j} A_{ij} B_{ij} \\
&= 1 \cdot 16 + 2 \cdot 15 + 3 \cdot 14 + 4 \cdot 13 + \\
&\quad 5 \cdot 12 + 6 \cdot 11 + 7 \cdot 10 + 8 \cdot 9 + \\
&\quad 9 \cdot 8 + 10 \cdot 7 + 11 \cdot 6 + 12 \cdot 5 + \\
&\quad 13 \cdot 4 + 14 \cdot 3 + 15 \cdot 2 + 16 \cdot 1 \\
&= 16 + 30 + 42 + 52 + 60 + 66 + 70 + 72 + \\
&\quad 72 + 70 + 66 + 60 + 52 + 42 + 30 + 16 \\
&= 696
\end{aligned}
$$

Now let's look into the computational part of *inner product*.

1. Compute Inner Product from Scratch (without Libraries)

Here's how you can compute the inner product from the scratch:

```
# Define matrices A and B
A = [[1, 2, 3], [4, 5, 6]]
B = [[7, 8, 9], [10, 11, 12]]

# Function to compute inner product
def inner_product(A, B):
    # Get the number of rows and columns
    num_rows = len(A)
    num_cols = len(A[0])

    # Initialize the result
    result = 0

    # Compute the inner product
    for i in range(num_rows):
        for j in range(num_cols):
            result += A[i][j] * B[i][j]

    return result

# Compute inner product
```

```
inner_product_result = inner_product(A, B)

# Display result
print("Inner Product (From Scratch):")
print(inner_product_result)
```

```
Inner Product (From Scratch):
217
```

2. Compute Inner Product Using `NumPy`

Here's how to compute the inner product using Numpy:

```
import numpy as np
# Define matrices A and B
A = np.array([[1, 2, 3], [4, 5, 6]])
B = np.array([[7, 8, 9], [10, 11, 12]])
# calculating innerproduct
inner_product = (A*B).sum() # calculate element-wise product, then column sum

print("Inner Product (Using numpy):")
print(inner_product)
```

```
Inner Product (Using numpy):
217
```

The same operation can be done using `SymPy` functions as follows.

```
import sympy as sp
import numpy as np
# Define matrices A and B
A = sp.Matrix([[1, 2, 3], [4, 5, 6]])
B = sp.Matrix([[7, 8, 9], [10, 11, 12]])

# Compute element-wise product
elementwise_product = A.multiply_elementwise(B)

# Calculate sum of each column
inner_product_sympy = np.sum(elementwise_product)
```

```
# Display result
print("Inner Product (Using SymPy):")
print(inner_product_sympy)
```

```
Inner Product (Using SymPy):
217
```

A vector dot product (in Physics) can be calculated using `SymPy .dot()` function as shown below.

Let $A = \begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$ and $B = \begin{pmatrix} 4 & 5 & 6 \end{pmatrix}$, then the dot product, $A \cdot B$ is computed as:

```
import sympy as sp
A=sp.Matrix([1,2,3])
B=sp.Matrix([4,5,6])
display(A.dot(B)) # calculate fot product of A and B
```

32

> **A word of caution**
>
> In `SymPy` , `sp.Matrix([1,2,3])` create a column vector. But `np.array([1,2,3])` creates a row vector. So be careful while applying matrix/ dot product operations on these objects.

The same dot product using `numpy` object can be done as follows:

```
import numpy as np
A=np.array([1,2,3])
B=np.array([4,5,6])
display(A.dot(B.T))# dot() stands for dot product B.T represents the
 ↪  transpose of B
```

```
np.int64(32)
```

Practical Applications

Application 1: Signal Processing

In signal processing, the inner product can be used to measure the similarity between two signals. Here the most popular measure of similarity is the `cosine` similarity. This measure is defined as:

$$\cos\theta = \frac{A \cdot B}{||A||||B||}$$

Now consider two digital signals are given. It's cosine similarity measure can be calculated with a simulated data as shown below.

```
import numpy as np

# Simulated large signals (1D array) using NumPy
signal1 = np.sin(np.random.rand(1000))
signal2 = np.cos(np.random.rand(1000))

# Compute inner product
inner_product_signal = np.dot(signal1, signal2)
#cosine_sim=np.dot(signal1,signal2)/(np.linalg.norm(signal1)*np.linalg.norm(
 ↪  signal2))
# Display result
cosine_sim=inner_product_signal/(np.sqrt(np.dot(signal1,signal1))*np.sqrt(np
 ↪  .dot(signal2,signal2)))
print("Inner Product (Using numpy):")
print(inner_product_signal)
print("Similarity of signals:")
print(cosine_sim)
```

```
Inner Product (Using numpy):
376.2416744359054
Similarity of signals:
0.8656766327701758
```

Application 2: Machine Learning - Feature Similarity

In machine learning, the inner product is used to calculate the similarity between feature vectors.

```
import numpy as np

# Simulated feature vectors (2D array) using NumPy
features1 = np.random.rand(100, 10)
features2 = np.random.rand(100, 10)

# Compute inner product for each feature vector
```

```
inner_products = np.einsum('ij,ij->i', features1, features2) # use Einstien's
 ↪  sum

# Display results
print("Inner Products of Feature Vectors:")
display(inner_products)
```

```
Inner Products of Feature Vectors:
```

```
array([1.86618745, 2.51274116, 3.22480512, 1.75063304, 2.71721823,
       1.98680212, 2.49529215, 2.71034517, 2.13720204, 2.11147807,
       1.96657719, 3.5413832 , 1.79540191, 3.70295428, 3.65000013,
       2.73129036, 2.73390532, 2.72873231, 2.16813032, 2.70008207,
       2.5691309 , 3.8267985 , 2.92428261, 3.70546606, 1.76740125,
       1.87750167, 2.6071815 , 2.84333663, 2.18774745, 1.12079554,
       1.57466902, 1.64387872, 2.90078842, 1.99736669, 2.89345103,
       3.27215634, 1.15500698, 3.4420285 , 2.66379682, 1.97189201,
       2.69580422, 3.66770217, 2.17603749, 2.64082386, 1.84436202,
       2.52202546, 3.69840123, 2.65554438, 2.56514562, 3.30404738,
       2.9149493 , 3.20316329, 1.67700272, 2.56251505, 2.13654397,
       2.58640145, 2.0852157 , 1.729585  , 2.00190409, 2.36285034,
       1.89257731, 2.76695102, 2.22617247, 3.38664877, 2.44943569,
       3.23633599, 2.78761947, 2.42658745, 1.9870498 , 2.14063901,
       1.8996319 , 1.78367225, 2.55715996, 2.31152679, 2.77771334,
       1.62328576, 1.7009172 , 3.26510025, 3.36515712, 2.73481074,
       2.34406789, 1.77924725, 3.06492737, 1.87634451, 2.40285279,
       2.26284099, 2.40946304, 1.95833024, 3.13335721, 3.12926964,
       2.5930494 , 1.01589811, 1.54176088, 1.8962135 , 1.9040875 ,
       2.16756144, 1.2903262 , 2.26065421, 4.28424505, 1.82733956])
```

Application 3: Covariance Matrix in Statistics

The inner product can be used to compute covariance matrices for statistical data analysis. If X is a given distribution and $x = X - \bar{X}$. Then the covariance of X can be calculated as $cov(X) = \frac{1}{n-1}(x \cdot x^T)$ [2]. The python code a simulated data is shown below.

[2]Remember that the covariance of X is defined as $Cov(X) = \frac{\sum(X-\bar{X})^2}{n-1}$

```python
import sympy as sp
import numpy as np

# Simulated large dataset (2D array) using NumPy
data = np.random.rand(100, 10)

# Compute the mean of each column
mean = np.mean(data, axis=0)

# Center the data
centered_data = data - mean

# Compute the covariance matrix using matrix product operation
cov_matrix = (centered_data.T @ centered_data) / (centered_data.shape[0] - 1)
cov_matrix_sympy = sp.Matrix(cov_matrix)

# Display results
print("Covariance Matrix:")
display(cov_matrix_sympy)
```

```
Covariance Matrix:
```

$$\begin{bmatrix} 0.0688323944026016 & 0.0153740605980509 & -0.000140933353214011 & 0.00920489541395922 & -0. \\ 0.0153740605980509 & 0.0905182214346242 & 0.000712744651254393 & -0.00436319787429453 & -0. \\ -0.000140933353214011 & 0.000712744651254393 & 0.0585555577321913 & -0.00452162693037121 & 0. \\ 0.00920489541395922 & -0.00436319787429453 & -0.00452162693037121 & 0.0862201676624409 & 0. \\ -0.00387734773004558 & -0.000825052380333573 & 0.00126191421665622 & 0.0141142358813661 & 0. \\ -0.00571190548041544 & 0.00786460769477729 & 0.00217942019679709 & -0.0044274235853324 & -0. \\ -0.0049428689590279 & -7.22532226030962 \cdot 10^{-5} & 0.000750926068297655 & 0.0032882833965958 7 & -0. \\ -0.00463812193721968 & -0.00028578801387221 & -0.00674883183942774 & -0.00724495179921262 & -0 \\ 0.0105897102719774 & 0.00456567668280999 & 0.00612207233520953 & 0.00795687838697261 & 0.0 \\ 0.000992217456020462 & -0.000925756164661477 & -0.0137231192102392 & -0.00242695079854651 & 0. \end{bmatrix}$$

These examples demonstrate the use of inner product and dot product in various applications.

Outer Product

The outer product of two vectors results in a matrix, and it is a way to combine these vectors into a higher-dimensional representation.

Definition (Outer Product)

For two vectors $\mathbf{u}$ and $\mathbf{v}$ of dimensions m and n respectively, the outer product $\mathbf{u} \otimes \mathbf{v}$ is an $m \times n$ matrix defined as:

$$(\mathbf{u} \otimes \mathbf{v})_{ij} = u_i \cdot v_j$$

where $\cdot$ denotes the outer product operation. In matrix notation, for two column vectors u, v,

$$u \otimes v = uv^T$$

Properties

1. **Linearity**:
$$(\mathbf{u} + \mathbf{w}) \otimes \mathbf{v} = (\mathbf{u} \otimes \mathbf{v}) + (\mathbf{w} \otimes \mathbf{v})$$
2. **Distributivity**:
$$\mathbf{u} \otimes (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \otimes \mathbf{v}) + (\mathbf{u} \otimes \mathbf{w})$$
3. **Associativity**:
$$(\mathbf{u} \otimes \mathbf{v}) \otimes \mathbf{w} = \mathbf{u} \otimes (\mathbf{v} \otimes \mathbf{w})$$

Some simple examples of outer product is given below.

Example 1: Basic Outer Product

Given vectors:

$$\mathbf{u} = \begin{pmatrix} 1 \\ 2 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 3 \\ 4 \\ 5 \end{pmatrix}$$

The outer product $\mathbf{u} \otimes \mathbf{v}$ is:

$$\mathbf{u} \otimes \mathbf{v} = \begin{pmatrix} 1 \cdot 3 & 1 \cdot 4 & 1 \cdot 5 \\ 2 \cdot 3 & 2 \cdot 4 & 2 \cdot 5 \end{pmatrix} = \begin{pmatrix} 3 & 4 & 5 \\ 6 & 8 & 10 \end{pmatrix}$$

Example 2: Outer Product with Larger Vectors

Given vectors:

$$\mathbf{u} = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \quad \mathbf{v} = \begin{pmatrix} 4 \\ 5 \end{pmatrix}$$

The outer product $\mathbf{u} \otimes \mathbf{v}$ is:

$$\mathbf{u} \otimes \mathbf{v} = \begin{pmatrix} 1 \cdot 4 & 1 \cdot 5 \\ 2 \cdot 4 & 2 \cdot 5 \\ 3 \cdot 4 & 3 \cdot 5 \end{pmatrix} = \begin{pmatrix} 4 & 5 \\ 8 & 10 \\ 12 & 15 \end{pmatrix}$$

Practice Problems

Find the outer product of A and B where A and B are given as follows:

Problem 1:

Find the outer product of:

$$A = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 & 4 \end{bmatrix}$$

Solution:

$$\begin{aligned} A \otimes B &= \begin{bmatrix} 1 \\ 2 \end{bmatrix} \otimes \begin{bmatrix} 3 & 4 \end{bmatrix} \\ &= \begin{bmatrix} 1 \cdot 3 & 1 \cdot 4 \\ 2 \cdot 3 & 2 \cdot 4 \end{bmatrix} \\ &= \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix} \end{aligned}$$

Problem 2:

Find the outer product of:

$$A = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 4 & 5 & 6 \end{bmatrix}$$

Solution:

$$
\begin{aligned}
A \otimes B &= \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \otimes \begin{bmatrix} 4 & 5 & 6 \end{bmatrix} \\
&= \begin{bmatrix} 1 \cdot 4 & 1 \cdot 5 & 1 \cdot 6 \\ 2 \cdot 4 & 2 \cdot 5 & 2 \cdot 6 \\ 3 \cdot 4 & 3 \cdot 5 & 3 \cdot 6 \end{bmatrix} \\
&= \begin{bmatrix} 4 & 5 & 6 \\ 8 & 10 & 12 \\ 12 & 15 & 18 \end{bmatrix}
\end{aligned}
$$

Problem 3:

Find the outer product of:

$$A = \begin{bmatrix} 1 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Solution:

$$
\begin{aligned}
A \otimes B &= \begin{bmatrix} 1 & 2 \end{bmatrix} \otimes \begin{bmatrix} 3 \\ 4 \end{bmatrix} \\
&= \begin{bmatrix} 1 \cdot 3 & 1 \cdot 4 \\ 2 \cdot 3 & 2 \cdot 4 \end{bmatrix} \\
&= \begin{bmatrix} 3 & 4 \\ 6 & 8 \end{bmatrix}
\end{aligned}
$$

Problem 4:

Find the outer product of:

$$A = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & -1 \end{bmatrix}$$

Solution:

$$
\begin{aligned}
A \otimes B &= \begin{bmatrix} 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 0 \cdot 1 & 0 \cdot -1 \\ 1 \cdot 1 & 1 \cdot -1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 1 & -1 \end{bmatrix}
\end{aligned}
$$

Problem 5:

Find the outer product of:

$$A = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$

$$B = \begin{bmatrix} 5 & -2 \end{bmatrix}$$

Solution:

$$
\begin{aligned}
A \otimes B &= \begin{bmatrix} 2 \\ 3 \end{bmatrix} \otimes \begin{bmatrix} 5 & -2 \end{bmatrix} \\
&= \begin{bmatrix} 2 \cdot 5 & 2 \cdot -2 \\ 3 \cdot 5 & 3 \cdot -2 \end{bmatrix} \\
&= \begin{bmatrix} 10 & -4 \\ 15 & -6 \end{bmatrix}
\end{aligned}
$$

Problem 6:

Find the outer product of:

$$A = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & -1 & 0 \end{bmatrix}$$

Solution:

$$
\begin{aligned}
A \otimes B &= \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} \otimes \begin{bmatrix} 2 & -1 & 0 \end{bmatrix} \\
&= \begin{bmatrix} 1 \cdot 2 & 1 \cdot -1 & 1 \cdot 0 \\ 0 \cdot 2 & 0 \cdot -1 & 0 \cdot 0 \\ 1 \cdot 2 & 1 \cdot -1 & 1 \cdot 0 \end{bmatrix} \\
&= \begin{bmatrix} 2 & -1 & 0 \\ 0 & 0 & 0 \\ 2 & -1 & 0 \end{bmatrix}
\end{aligned}
$$

Problem 7:

Find the outer product of:

$$
A = \begin{bmatrix} 1 \\ -1 \end{bmatrix}
$$

$$
B = \begin{bmatrix} 2 & 0 \\ 3 & -1 \end{bmatrix}
$$

Solution:

$$
\begin{aligned}
A \otimes B &= \begin{bmatrix} 1 \\ -1 \end{bmatrix} \otimes \begin{bmatrix} 2 & 0 \\ 3 & -1 \end{bmatrix} \\
&= \begin{bmatrix} 2 & 3 & 0 & -1 \\ -2 & -3 & 0 & 1 \end{bmatrix}
\end{aligned}
$$

Problem 8:

Find the outer product of:

$$
A = \begin{bmatrix} 3 \\ 4 \end{bmatrix}
$$

$$
B = \begin{bmatrix} 1 & -2 & 3 \end{bmatrix}
$$

Solution:

$$
\begin{aligned}
A \otimes B &= \begin{bmatrix} 3 \\ 4 \end{bmatrix} \otimes \begin{bmatrix} 1 & -2 & 3 \end{bmatrix} \\
&= \begin{bmatrix} 3 \cdot 1 & 3 \cdot -2 & 3 \cdot 3 \\ 4 \cdot 1 & 4 \cdot -2 & 4 \cdot 3 \end{bmatrix} \\
&= \begin{bmatrix} 3 & -6 & 9 \\ 4 & -8 & 12 \end{bmatrix}
\end{aligned}
$$

Problem 9:

Find the outer product of:

$$
A = \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix}
$$

$$
B = \begin{bmatrix} 4 & -2 \end{bmatrix}
$$

Solution:

$$
\begin{aligned}
A \otimes B &= \begin{bmatrix} 2 \\ 3 \\ -1 \end{bmatrix} \otimes \begin{bmatrix} 4 & -2 \end{bmatrix} \\
&= \begin{bmatrix} 2 \cdot 4 & 2 \cdot -2 \\ 3 \cdot 4 & 3 \cdot -2 \\ -1 \cdot 4 & -1 \cdot -2 \end{bmatrix} \\
&= \begin{bmatrix} 8 & -4 \\ 12 & -6 \\ -4 & 2 \end{bmatrix}
\end{aligned}
$$

Problem 10:

Find the outer product of:

$$
A = \begin{bmatrix} 0 \\ 5 \end{bmatrix}
$$

$$
B = \begin{bmatrix} 3 & 1 \end{bmatrix}
$$

Solution:

$$
\begin{aligned}
A \otimes B &= \begin{bmatrix} 0 \\ 5 \end{bmatrix} \otimes \begin{bmatrix} 3 & 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 \cdot 3 & 0 \cdot 1 \\ 5 \cdot 3 & 5 \cdot 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 & 0 \\ 15 & 5 \end{bmatrix}
\end{aligned}
$$

---

1. Compute Outer Product of Vectors from Scratch (without Libraries)

Here's how you can compute the outer product manually:

```
# Define vectors u and v
u = [1, 2]
v = [3, 4, 5]

# Function to compute outer product
def outer_product(u, v):
    # Initialize the result
    result = [[a * b for b in v] for a in u]
    return result

# Compute outer product
outer_product_result = outer_product(u, v)

# Display result
print("Outer Product of Vectors (From Scratch):")
for row in outer_product_result:
    print(row)
```

```
Outer Product of Vectors (From Scratch):
[3, 4, 5]
[6, 8, 10]
```

2. Compute Outer Product of Vectors Using SymPy

Here's how to compute the outer product using `SymPy`:

```python
import sympy as sp

# Define vectors u and v
u = sp.Matrix([1, 2])
v = sp.Matrix([3, 4, 5])

# Compute outer product using SymPy
outer_product_sympy = u * v.T

# Display result
print("Outer Product of Vectors (Using SymPy):")
display(outer_product_sympy)
```

```
Outer Product of Vectors (Using SymPy):
```

$$\begin{bmatrix} 3 & 4 & 5 \\ 6 & 8 & 10 \end{bmatrix}$$

Outer Product of Matrices

The outer product of two matrices extends the concept from vectors to higher-dimensional tensors. For two matrices A and B, the outer product results in a higher-dimensional tensor and is generally expressed as block matrices.

Definition (Outer Product of Matrices)

For two matrices A of dimension $m \times p$ and B of dimension $q \times n$, the outer product $A \otimes B$ results in a tensor of dimension $m \times q \times p \times n$. The entries of the tensor are given by:

$$(A \otimes B)_{ijkl} = A_{ij} \cdot B_{kl}$$

where $\cdot$ denotes the outer product operation.

Properties

1. **Linearity**:
$$(A + C) \otimes B = (A \otimes B) + (C \otimes B)$$
2. **Distributivity**:
$$A \otimes (B + D) = (A \otimes B) + (A \otimes D)$$
3. **Associativity**:

$$(A \otimes B) \otimes C = A \otimes (B \otimes C)$$

Here are some simple examples to demonstrate the mathematical procedure to find outer product of matrices.

Example 1: Basic Outer Product of Matrices

Given matrices:

$$A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}, \quad B = \begin{pmatrix} 5 \\ 6 \end{pmatrix}$$

The outer product $A \otimes B$ is:

$$A \otimes B = \begin{pmatrix} 1 \cdot 5 & 1 \cdot 6 \\ 2 \cdot 5 & 2 \cdot 6 \\ 3 \cdot 5 & 3 \cdot 6 \\ 4 \cdot 5 & 4 \cdot 6 \end{pmatrix} = \begin{pmatrix} 5 & 6 \\ 10 & 12 \\ 15 & 18 \\ 20 & 24 \end{pmatrix}$$

Example 2: Outer Product with Larger Matrices

Given matrices:

$$A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} 7 \\ 8 \end{pmatrix}$$

The outer product $A \otimes B$ is:

$$A \otimes B = \begin{pmatrix} 1 \cdot 7 & 1 \cdot 8 \\ 2 \cdot 7 & 2 \cdot 8 \\ 3 \cdot 7 & 3 \cdot 8 \\ 4 \cdot 7 & 4 \cdot 8 \\ 5 \cdot 7 & 5 \cdot 8 \\ 6 \cdot 7 & 6 \cdot 8 \end{pmatrix} = \begin{pmatrix} 7 & 8 \\ 14 & 16 \\ 21 & 24 \\ 28 & 32 \\ 35 & 40 \\ 42 & 48 \end{pmatrix}$$

Example 3: Compute the outer product of the following vectors $\mathbf{u} = [0, 1, 2]$ and $\mathbf{v} = [2, 3, 4]$.

To find the outer product, we calculate each element (i, j) as the product of the (i)-th element of $\mathbf{u}$ and the (j)-th element of $\mathbf{v}$. Mathematically:

$$\mathbf{u} \otimes \mathbf{v} = \begin{bmatrix} 0 \cdot 2 & 0 \cdot 3 & 0 \cdot 4 \\ 1 \cdot 2 & 1 \cdot 3 & 1 \cdot 4 \\ 2 \cdot 2 & 2 \cdot 3 & 2 \cdot 4 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 2 & 3 & 4 \\ 4 & 6 & 8 \end{bmatrix}$$

1. Compute Outer Product of Matrices from Scratch (without Libraries)

Here's how you can compute the outer product manually:

```
# Define matrices A and B
A = [[1, 2], [3, 4]]
B = [[5], [6]]

# Function to compute outer product
def outer_product_matrices(A, B):
    m = len(A)
    p = len(A[0])
    q = len(B)
    n = len(B[0])
    result = [[0] * (n * p) for _ in range(m * q)]

    for i in range(m):
        for j in range(p):
            for k in range(q):
                for l in range(n):
                    result[i*q + k][j*n + l] = A[i][j] * B[k][l]

    return result

# Compute outer product
outer_product_result_matrices = outer_product_matrices(A, B)

# Display result
print("Outer Product of Matrices (From Scratch):")
for row in outer_product_result_matrices:
    print(row)
```

```
Outer Product of Matrices (From Scratch):
[5, 10]
[6, 12]
[15, 20]
[18, 24]
```

Here is the `Python` code to compute the outer product of these vectors using the `NumPy` function `.outer()`:

```
import numpy as np

# Define vectors
u = np.array([[1,2],[3,4]])
v = np.array([[5],[4]])

# Compute outer product
outer_product = np.outer(u, v)

print("Outer Product of u and v:")
display(outer_product)
```

```
Outer Product of u and v:

array([[ 5,  4],
       [10,  8],
       [15, 12],
       [20, 16]])
```

Example 3: Real-world Application in Recommendation Systems

In recommendation systems, the outer product can represent user-item interactions. A simple context is here. Let the user preferences of items is given as $u = [4, 3, 5]$ and the item scores is given by $v = [2, 5, 4]$. Now the recommendation score can be calculated as the outer product of these two vectors. Calculation of this score is shown below. The outer product $\mathbf{u} \otimes \mathbf{v}$ is calculated as follows:

$$\mathbf{u} \otimes \mathbf{v} = \begin{bmatrix} 4 \cdot 2 & 4 \cdot 5 & 4 \cdot 4 \\ 3 \cdot 2 & 3 \cdot 5 & 3 \cdot 4 \\ 5 \cdot 2 & 5 \cdot 5 & 5 \cdot 4 \end{bmatrix} = \begin{bmatrix} 8 & 20 & 16 \\ 6 & 15 & 12 \\ 10 & 25 & 20 \end{bmatrix}$$

The python code for this task is given below.

```
import numpy as np
import matplotlib.pyplot as plt

# Define the user and product ratings vectors
user_ratings = np.array([4, 3, 5])
product_ratings = np.array([2, 5, 4])

# Compute the outer product
```

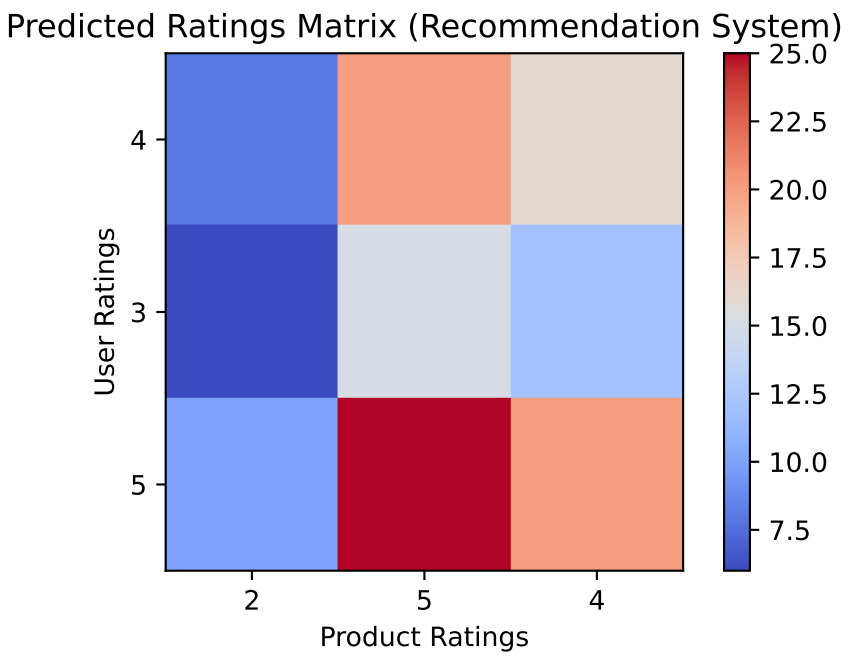
```
predicted_ratings = np.outer(user_ratings, product_ratings)

# Print the predicted ratings matrix
print("Predicted Ratings Matrix:")
display(predicted_ratings)

# Plot the result
plt.imshow(predicted_ratings, cmap='coolwarm', interpolation='nearest')
plt.colorbar()
plt.title('Predicted Ratings Matrix (Recommendation System)')
plt.xlabel('Product Ratings')
plt.ylabel('User Ratings')
plt.xticks(ticks=np.arange(len(product_ratings)), labels=product_ratings)
plt.yticks(ticks=np.arange(len(user_ratings)), labels=user_ratings)
plt.show()
```

```
Predicted Ratings Matrix:

array([[ 8, 20, 16],
       [ 6, 15, 12],
       [10, 25, 20]])
```



Additional Properties & Definitions

1. **Definition and Properties**

   Given two vectors:

   - $\mathbf{u} \in \mathbb{R}^m$
   - $\mathbf{v} \in \mathbb{R}^n$

   The outer product $\mathbf{u} \otimes \mathbf{v}$ results in an $m \times n$ matrix where each element (i, j) of the matrix is calculated as:

   $$(\mathbf{u} \otimes \mathbf{v})_{ij} = u_i \cdot v_j$$

2. **Non-Symmetry**

   The outer product is generally not symmetric. For vectors $\mathbf{u}$ and $\mathbf{v}$, the matrix $\mathbf{u} \otimes \mathbf{v}$ is not necessarily equal to $\mathbf{v} \otimes \mathbf{u}$:

   $$\mathbf{u} \otimes \mathbf{v} \neq \mathbf{v} \otimes \mathbf{u}$$

3. **Rank of the Outer Product**

   The rank of the outer product matrix $\mathbf{u} \otimes \mathbf{v}$ is always 1, provided neither $\mathbf{u}$ nor $\mathbf{v}$ is a zero vector. This is because the matrix can be expressed as a single rank-1 matrix.

4. **Distributive Property**

   The outer product is distributive over vector addition. For vectors $\mathbf{u}_1, \mathbf{u}_2 \in \mathbb{R}^m$ and $\mathbf{v} \in \mathbb{R}^n$:

   $$(\mathbf{u}_1 + \mathbf{u}_2) \otimes \mathbf{v} = (\mathbf{u}_1 \otimes \mathbf{v}) + (\mathbf{u}_2 \otimes \mathbf{v})$$

5. **Associativity with Scalar Multiplication**

   The outer product is associative with scalar multiplication. For a scalar α and vectors $\mathbf{u} \in \mathbb{R}^m$ and $\mathbf{v} \in \mathbb{R}^n$:

   $$\alpha(\mathbf{u} \otimes \mathbf{v}) = (\alpha\mathbf{u}) \otimes \mathbf{v} = \mathbf{u} \otimes (\alpha\mathbf{v})$$

6. **Matrix Trace**

   The trace of the outer product of two vectors is given by:

   $$\text{tr}(\mathbf{u} \otimes \mathbf{v}) = (\mathbf{u}^T\mathbf{v}) = (\mathbf{v}^T\mathbf{u})$$

   Here, tr denotes the trace of a matrix, which is the sum of its diagonal elements.

7. **Matrix Norm**

   The Frobenius norm of the outer product matrix can be expressed in terms of the norms of the original vectors:

$$\|\mathbf{u} \otimes \mathbf{v}\|_F = \|\mathbf{u}\|_2 \cdot \|\mathbf{v}\|_2$$

   where $\|\cdot\|_2$ denotes the Euclidean norm.

Example Calculation in `Python`

Here's how to compute and visualize the outer product properties using `Python`:

```
import numpy as np
import matplotlib.pyplot as plt

# Define vectors
u = np.array([1, 2, 3])
v = np.array([4, 5])

# Compute outer product
outer_product = np.outer(u, v)

# Display results
print("Outer Product Matrix:")
print(outer_product)

# Compute and display rank
rank = np.linalg.matrix_rank(outer_product)
print(f"Rank of Outer Product Matrix: {rank}")

# Compute Frobenius norm
frobenius_norm = np.linalg.norm(outer_product, 'fro')
print(f"Frobenius Norm: {frobenius_norm}")

# Plot the result
plt.imshow(outer_product, cmap='viridis', interpolation='nearest')
plt.colorbar()
plt.title('Outer Product Matrix')
plt.xlabel('Vector v')
plt.ylabel('Vector u')
plt.xticks(ticks=np.arange(len(v)), labels=v)
plt.yticks(ticks=np.arange(len(u)), labels=u)
plt.show()
```

```
Outer Product Matrix:
[[ 4  5]
 [ 8 10]
 [12 15]]
Rank of Outer Product Matrix: 1
Frobenius Norm: 23.958297101421877
```

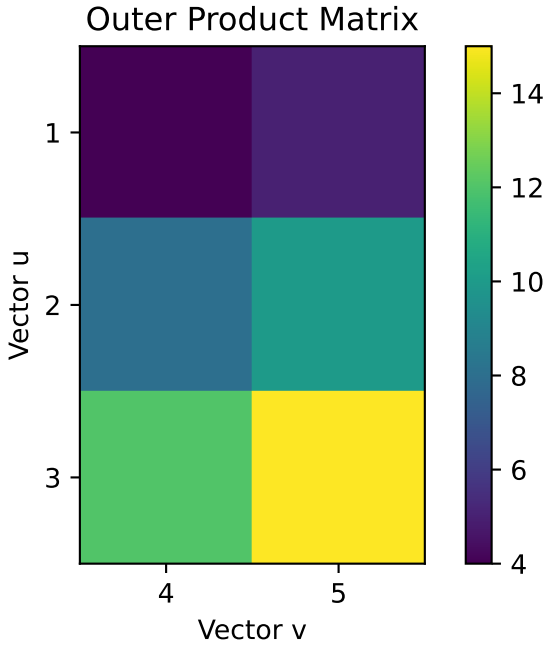


Figure 10: Demonstration of Outer Product and its Properties

Kronecker Product

In mathematics, the Kronecker product, sometimes denoted by $\otimes$, is an operation on two matrices of arbitrary size resulting in a *block matrix*. It is a specialization of the tensor product (which is denoted by the same symbol) from vectors to matrices and gives the matrix of the tensor product linear map with respect to a standard choice of basis. The Kronecker product is to be distinguished from the usual matrix multiplication, which is an entirely different operation. The Kronecker product is also sometimes called *matrix direct product*.

Note

If A is an $m \times n$ matrix and B is a $p \times q$ matrix, then the Kronecker product $A \otimes B$ is the $pm \times qn$ block matrix defined as: Each a_{ij} of A is replaced by the matrix $a_{ij}B$. Symbolically this will result in a block matrix defined by:

$$A \otimes B = A \otimes B = \begin{bmatrix} a_{11}B & a_{12}B & \cdots & a_{1n}B \\ a_{21}B & a_{22}B & \cdots & a_{2n}B \\ \vdots & \vdots & \ddots & vdots \\ a_{m1}B & a_{m2}B & \cdots & a_{mn}B \end{bmatrix}$$

Properties of the Kronecker Product

1. **Associativity**

   The Kronecker product is associative. For matrices $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{p \times q}$, and $C \in \mathbb{R}^{r \times s}$:

   $$(A \otimes B) \otimes C = A \otimes (B \otimes C)$$

2. **Distributivity Over Addition**

   The Kronecker product distributes over matrix addition. For matrices $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{p \times q}$, and $C \in \mathbb{R}^{p \times q}$:

   $$A \otimes (B + C) = (A \otimes B) + (A \otimes C)$$

3. **Mixed Product Property**

   The Kronecker product satisfies the mixed product property with the matrix product. For matrices $A \in \mathbb{R}^{m \times n}$, $B \in \mathbb{R}^{p \times q}$, $C \in \mathbb{R}^{r \times s}$, and $D \in \mathbb{R}^{r \times s}$:

   $$(A \otimes B)(C \otimes D) = (AC) \otimes (BD)$$

4. **Transpose**

   The transpose of the Kronecker product is given by:

   $$(A \otimes B)^T = A^T \otimes B^T$$

5. **Norm**

   The Frobenius norm of the Kronecker product can be computed as:

   $$\|A \otimes B\|_F = \|A\|_F \cdot \|B\|_F$$

   where $\|\cdot\|_F$ denotes the Frobenius norm.

Frobenius Norm

The Frobenius norm, also known as the Euclidean norm for matrices, is a measure of a matrix's magnitude. It is defined as the square root of the sum of the absolute squares of its elements. Mathematically, for a matrix A with elements a_{ij}, the Frobenius norm is given by:

$$\|A\|_F = \sqrt{\sum_{i,j} |a_{ij}|^2}$$

Example 1: Calculation of Frobenius Norm

Consider the matrix A:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

To compute the Frobenius norm:

$$\|A\|_F = \sqrt{1^2 + 2^2 + 3^2 + 4^2} = \sqrt{1 + 4 + 9 + 16} = \sqrt{30} \approx 5.48$$

Example 2: Frobenius Norm of a Sparse Matrix

Consider the sparse matrix B:

$$B = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 5 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

To compute the Frobenius norm:

$$\|B\|_F = \sqrt{0^2 + 0^2 + 0^2 + 5^2 + 0^2 + 0^2} = \sqrt{25} = 5$$

Example 3: Frobenius Norm in a Large Matrix

Consider the matrix C of size \$3 $\times$3 \$:

$$C = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{bmatrix}$$

To compute the Frobenius norm:

$$\begin{aligned}\|C\|_F &= \sqrt{1^2 + 2^2 + 3^2 + 4^2 + 5^2 + 6^2 + 7^2 + 8^2 + 9^2} \\ &= \sqrt{1 + 4 + 9 + 16 + 25 + 36 + 49 + 64 + 81} \\ &= \sqrt{285} \\ &\approx 16.88\end{aligned}$$

Applications of the Frobenius Norm

- *Application 1: Image Compression:* In image processing, the Frobenius norm can measure the difference between the original and compressed images, indicating how well the compression has preserved the original image quality.
- *Application 2: Matrix Factorization:* In numerical analysis, Frobenius norm is used to evaluate the error in matrix approximations, such as in Singular Value Decomposition (SVD). A lower Frobenius norm of the error indicates a better approximation.
- *Application 3: Error Measurement in Numerical Solutions:* In solving systems of linear equations, the Frobenius norm can be used to measure the error between the true solution and the computed solution, providing insight into the accuracy of numerical methods.

The `linalg` sub module of `NumPy` library can be used to calculate various norms. Basically norm is the generalized form of Euclidean distance.

```
import numpy as np

# Example 1: Simple Matrix
A = np.array([[1, 2], [3, 4]])
frobenius_norm_A = np.linalg.norm(A, 'fro')
print(f"Frobenius Norm of A: {frobenius_norm_A:.2f}")

# Example 2: Sparse Matrix
B = np.array([[0, 0, 0], [0, 5, 0], [0, 0, 0]])
frobenius_norm_B = np.linalg.norm(B, 'fro')
print(f"Frobenius Norm of B: {frobenius_norm_B:.2f}")

# Example 3: Large Matrix
C = np.array([[1, 2, 3], [4, 5, 6], [7, 8, 9]])
frobenius_norm_C = np.linalg.norm(C, 'fro')
print(f"Frobenius Norm of C: {frobenius_norm_C:.2f}")
```

```
Frobenius Norm of A: 5.48
Frobenius Norm of B: 5.00
Frobenius Norm of C: 16.88
```

Frobenius norm of Kronecker product

Let us consider two matrices,

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

and

$$B = \begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix}$$

The Kronecker product $C = A \otimes B$ is:

$$C = \begin{bmatrix} 1 \cdot B & 2 \cdot B \\ 3 \cdot B & 4 \cdot B \end{bmatrix} = \begin{bmatrix} \begin{bmatrix} 0 & 5 \\ 6 & 7 \end{bmatrix} & \begin{bmatrix} 0 \cdot 2 & 5 \cdot 2 \\ 6 \cdot 2 & 7 \cdot 2 \end{bmatrix} \\ \begin{bmatrix} 0 \cdot 3 & 5 \cdot 3 \\ 6 \cdot 3 & 7 \cdot 3 \end{bmatrix} & \begin{bmatrix} 0 \cdot 4 & 5 \cdot 4 \\ 6 \cdot 4 & 7 \cdot 4 \end{bmatrix} \end{bmatrix}$$

This expands to:

$$C = \begin{bmatrix} 0 & 5 & 0 & 10 \\ 6 & 7 & 12 & 14 \\ 0 & 15 & 0 & 20 \\ 18 & 21 & 24 & 28 \end{bmatrix}$$

Computing the Frobenius Norm

To compute the Frobenius norm of C:

$$\|C\|_F = \sqrt{\sum_{i=1}^{4} \sum_{j=1}^{4} |c_{ij}|^2}$$

$$\|C\|_F = \sqrt{0^2 + 5^2 + 0^2 + 10^2 + 6^2 + 7^2 + 12^2 + 14^2 + 0^2 + 15^2 + 0^2 + 20^2 + 18^2 + 21^2 + 24^2 + 28^2}$$

$$\|C\|_F = \sqrt{0+25+0+100+36+49+144+196+0+225+0+400+324+441+576+784}$$

$$\|C\|_F = \sqrt{2896}$$
$$\|C\|_F \approx 53.87$$

Practice Problems

Find the Kronecker product of A and B where A and B are given as follows:

Problem 1:

Find the Kronecker product of:

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$
$$B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Solution:

$$\begin{aligned} A \otimes B &= \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & 2 \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ 3 \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & 4 \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 & 2 \\ 1 & 0 & 2 & 0 \\ 0 & 3 & 0 & 4 \\ 3 & 0 & 4 & 0 \end{bmatrix} \end{aligned}$$

Problem 2:

Find the Kronecker product of:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix}$$

Solution:

$$\begin{aligned} A \otimes B &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \\ &= \begin{bmatrix} 1 \cdot \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} & 0 \cdot \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \\ 0 \cdot \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} & 1 \cdot \begin{bmatrix} 2 & 3 \\ 4 & 5 \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} 2 & 3 & 0 & 0 \\ 4 & 5 & 0 & 0 \\ 0 & 0 & 2 & 3 \\ 0 & 0 & 4 & 5 \end{bmatrix} \end{aligned}$$

Problem 3:

Find the Kronecker product of:

$$A = \begin{bmatrix} 1 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

Solution:

$$\begin{aligned} A \otimes B &= \begin{bmatrix} 1 & 2 \end{bmatrix} \otimes \begin{bmatrix} 3 \\ 4 \end{bmatrix} \\ &= \begin{bmatrix} 1 \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix} & 2 \cdot \begin{bmatrix} 3 \\ 4 \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} 3 & 6 \\ 4 & 8 \end{bmatrix} \end{aligned}$$

Problem 4:

Find the Kronecker product of:

$$A = \begin{bmatrix} 0 & 1 \end{bmatrix}$$
$$B = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}$$

Solution:

$$\begin{aligned} A \otimes B &= \begin{bmatrix} 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 0 \cdot \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} & 1 \cdot \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & 2 & 0 \end{bmatrix} \end{aligned}$$

Problem 5:

Find the Kronecker product of:

$$A = \begin{bmatrix} 2 \\ 3 \end{bmatrix}$$
$$B = \begin{bmatrix} 4 & -2 \end{bmatrix}$$

Solution:

$$\begin{aligned} A \otimes B &= \begin{bmatrix} 2 \\ 3 \end{bmatrix} \otimes \begin{bmatrix} 4 & -2 \end{bmatrix} \\ &= \begin{bmatrix} 2 \cdot \begin{bmatrix} 4 & -2 \end{bmatrix} \\ 3 \cdot \begin{bmatrix} 4 & -2 \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} 8 & -4 \\ 12 & -6 \end{bmatrix} \end{aligned}$$

Problem 6:

Find the Kronecker product of:

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

Solution:

$$\begin{aligned} A \otimes B &= \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \otimes \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1 \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & -1 \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \\ 0 \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} & 2 \cdot \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 1 & 0 & -1 \\ 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix} \end{aligned}$$

Problem 7:

Find the Kronecker product of:

$$A = \begin{bmatrix} 2 \end{bmatrix}$$

$$B = \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix}$$

Solution:

$$\begin{aligned} A \otimes B &= \begin{bmatrix} 2 \end{bmatrix} \otimes \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} \\ &= 2 \cdot \begin{bmatrix} 3 & 4 \\ 5 & 6 \end{bmatrix} \\ &= \begin{bmatrix} 6 & 8 \\ 10 & 12 \end{bmatrix} \end{aligned}$$

Problem 8:

Find the Kronecker product of:

$$A = \begin{bmatrix} 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Solution:

$$\begin{aligned} A \otimes B &= \begin{bmatrix} 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} & 1 \cdot \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} \end{aligned}$$

Problem 9:

Find the Kronecker product of:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$$

Solution:

$$\begin{aligned} A \otimes B &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \otimes \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 \cdot \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} & 0 \cdot \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \\ 0 \cdot \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} & 1 \cdot \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \end{aligned}$$

Problem 10:

Find the Kronecker product of:

$$A = \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & 5 \\ -2 & 3 \end{bmatrix}$$

Solution:

$$\begin{aligned} A \otimes B &= \begin{bmatrix} 2 & -1 \\ 3 & 4 \end{bmatrix} \otimes \begin{bmatrix} 0 & 5 \\ -2 & 3 \end{bmatrix} \\ &= \begin{bmatrix} 2 \cdot \begin{bmatrix} 0 & 5 \\ -2 & 3 \end{bmatrix} & -1 \cdot \begin{bmatrix} 0 & 5 \\ -2 & 3 \end{bmatrix} \\ 3 \cdot \begin{bmatrix} 0 & 5 \\ -2 & 3 \end{bmatrix} & 4 \cdot \begin{bmatrix} 0 & 5 \\ -2 & 3 \end{bmatrix} \end{bmatrix} \\ &= \begin{bmatrix} 0 & 10 & 0 & -5 \\ -4 & 6 & 2 & -3 \\ 0 & 15 & 0 & 20 \\ -6 & 9 & -8 & 12 \end{bmatrix} \end{aligned}$$

Connection Between Outer Product and Kronecker Product

1. **Conceptual Connection:**
   - The **outer product** is a special case of the **Kronecker product**. Specifically, if $\mathbf{A}$ is a column vector and $\mathbf{B}$ is a row vector, then $\mathbf{A}$ is a $m \times 1$ matrix and $\mathbf{B}$ is a $1 \times n$ matrix. The Kronecker product of these two matrices will yield the same result as the outer product of these vectors.
   - For matrices $\mathbf{A}$ and $\mathbf{B}$, the Kronecker product involves taking the outer product of each element of $\mathbf{A}$ with the entire matrix $\mathbf{B}$.
2. **Mathematical Formulation:**
   - Let $\mathbf{A} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $\mathbf{B} = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}$. Then:

$$\mathbf{A} \otimes \mathbf{B} = \begin{bmatrix} a_{11}\mathbf{B} & a_{12}\mathbf{B} \\ a_{21}\mathbf{B} & a_{22}\mathbf{B} \end{bmatrix}$$

- If $\mathbf{A} = \mathbf{u}\mathbf{v}^T$ where $\mathbf{u}$ is a column vector and $\mathbf{v}^T$ is a row vector, then the Kronecker product of $\mathbf{u}$ and $\mathbf{v}^T$ yields the same result as the outer product $\mathbf{u} \otimes \mathbf{v}$.

> **Note**
>
> **Summary**
>
> - The **outer product** is a specific case of the **Kronecker product** where one of the matrices is a vector (either row or column).
> - The **Kronecker product** generalizes the outer product to matrices and is more versatile in applications involving tensor products and higher-dimensional constructs.

Matrix Multiplication as Kronecker Product

Given matrices $\mathbf{A}$ and $\mathbf{B}$, where: - $\mathbf{A}$ is an $m \times n$ matrix - $\mathbf{B}$ is an $n \times p$ matrix

The product $\mathbf{C} = \mathbf{AB}$ can be expressed using Kronecker products as:

$$\mathbf{C} = \sum_{k=1}^{n} (\mathbf{A}_{:,k} \otimes \mathbf{B}_{k,:})$$

where: - $\mathbf{A}_{:,k}$ denotes the k-th column of matrix $\mathbf{A}$ - $\mathbf{B}_{k,:}$ denotes the k-th row of matrix $\mathbf{B}$

Example:

Let:

$$\mathbf{A} = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}$$

and:

$$\mathbf{B} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

To find $\mathbf{C} = \mathbf{AB}$ using Kronecker products:

1. **Compute the Kronecker Product of Columns of A and Rows of B:**
   - For column $\mathbf{A}_{:,1} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and row $\mathbf{B}_{1,:} = \begin{bmatrix} 0 & 1 \end{bmatrix}$:

$$\mathbf{A}_{:,1} \otimes \mathbf{B}_{1,:} = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix}$$

- For column $\mathbf{A}_{:,2} = \begin{bmatrix} 2 \\ 4 \end{bmatrix}$ and row $\mathbf{B}_{2,:} = \begin{bmatrix} 1 & 0 \end{bmatrix}$:

$$\mathbf{A}_{:,2} \otimes \mathbf{B}_{2,:} = \begin{bmatrix} 2 & 0 \\ 4 & 0 \end{bmatrix}$$

2. **Sum the Kronecker Products:**

$$\mathbf{C} = \begin{bmatrix} 0 & 1 \\ 0 & 3 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 4 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$$

In the previous block we have discussed the Frobenius norm and its applications. Now came back to the discussions on the Kronecker product. The Kronecker product is particularly useful in scenarios where interactions between different types of data need to be modeled comprehensively. In recommendation systems, it allows us to integrate user preferences with item relationships to improve recommendation accuracy.

In addition to recommendation systems, Kronecker products are used in various fields such as:

- Signal Processing: For modeling multi-dimensional signals.
- Machine Learning: In building features for complex models.
- Communication Systems: For modeling network interactions.

By understanding the Kronecker product and its applications, we can extend it to solve complex problems and enhance systems across different domains. To understand the practical use of Kronecker product in a Machine Learning scenario let us consider the following problem statement and its solution.

Problem statement

In the realm of recommendation systems, predicting user preferences for various product categories based on past interactions is a common challenge. Suppose we have data on user preferences for different products and categories. We can use this data to recommend the best products for each user by employing mathematical tools such as the Kronecker product. The User Preference and Category relationships are given in Table 5 and Table 6 .

Table 5: User Preference

User/Item	Electronics	Clothing	Books
User 1	5	3	4
User 2	2	4	5
User 3	3	4	4

Table 6: Category Relationships

Category/Feature	Feature 1	Feature 2	Feature 3
Electronics	1	0	0
Clothing	0	1	1
Books	0	1	1

Predict user preferences for different product categories using the Kronecker product matrix.

Solution Procedure

1. *Compute the Kronecker Product:* Calculate the Kronecker product of matrices U and C to obtain matrix K.

   To model the problem, we use the Kronecker product of the user preference matrix U and the category relationships matrix C. This product allows us to predict the user's rating for each category by combining their preferences with the category features.

Formulating Matrices

User Preference Matrix (U): - Dimension: 3×3 (3 users, 3 items) - from the User preference data, we can create the User Preference Matrix as follows:

$$U = \begin{pmatrix} 5 & 3 & 4 \\ 2 & 4 & 5 \\ 3 & 4 & 4 \end{pmatrix}$$

Category Relationships Matrix (C): - Dimension: 3×3 (3 categories) - from the Category Relationships data, we can create the Category Relationship Matrix as follows:

$$C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 1 & 1 \end{pmatrix}$$

Kronecker Product Calculation

The Kronecker product K of U and C is calculated as follows:

1. **Matrix Dimensions:**

- U is 3×3 (3 users, 3 items).
- C is 3×3 (3 categories, 3 features).

2. **Calculate Kronecker Product:**

- For each element u_{ij} in U, multiply by the entire matrix C.

The Kronecker product K is computed as:

$$K = U \otimes C$$

Explicitly, the Kronecker product K is:

$$K = \begin{pmatrix} 5 \cdot C & 3 \cdot C & 4 \cdot C \\ 2 \cdot C & 4 \cdot C & 5 \cdot C \\ 3 \cdot C & 4 \cdot C & 4 \cdot C \end{pmatrix}$$

As an example the blocks in first row are:

$$5 \cdot C = \begin{pmatrix} 5 & 0 & 0 \\ 0 & 5 & 5 \\ 0 & 5 & 5 \end{pmatrix}, \quad 3 \cdot C = \begin{pmatrix} 3 & 0 & 0 \\ 0 & 3 & 3 \\ 0 & 3 & 3 \end{pmatrix}, \quad 4 \cdot C = \begin{pmatrix} 4 & 0 & 0 \\ 0 & 4 & 4 \\ 0 & 4 & 4 \end{pmatrix}$$

Combining these blocks:

$$K = \begin{pmatrix} 5 & 0 & 0 & 3 & 0 & 0 & 4 & 0 & 0 \\ 0 & 5 & 5 & 0 & 3 & 3 & 0 & 4 & 4 \\ 0 & 5 & 5 & 0 & 3 & 3 & 0 & 4 & 4 \\ 2 & 0 & 0 & 4 & 0 & 0 & 5 & 0 & 0 \\ 0 & 2 & 2 & 0 & 4 & 4 & 0 & 5 & 5 \\ 0 & 2 & 2 & 0 & 4 & 4 & 0 & 5 & 5 \\ 3 & 0 & 0 & 4 & 0 & 0 & 4 & 0 & 0 \\ 0 & 3 & 3 & 0 & 4 & 4 & 0 & 4 & 4 \\ 0 & 3 & 3 & 0 & 4 & 4 & 0 & 4 & 4 \end{pmatrix}$$

2. **Interpret the Kronecker Product Matrix:** The resulting matrix K represents all possible combinations of user preferences and category features.

3. **Predict Ratings:** For each user, use matrix K to predict the rating for each category by summing up the values in the corresponding rows.
4. **Generate Recommendations:** Identify the top categories with the highest predicted ratings for each user.

The `python` code to solve this problem computationally is given below.

```
import numpy as np
import pandas as pd
import matplotlib.pyplot as plt

# Define the matrices
U = np.array([[5, 3, 4],
              [2, 4, 5],
              [3, 4, 4]])

C = np.array([[1, 0, 0],
              [0, 1, 1],
              [0, 1, 1]])

# Compute the Kronecker product
K = np.kron(U, C)

# Create a DataFrame to visualize the Kronecker product matrix
df_K = pd.DataFrame(K,
                    columns=['Electronics_F1', 'Electronics_F2',
 ↪  'Electronics_F3',
                             'Clothing_F1', 'Clothing_F2', 'Clothing_F3',
                             'Books_F1', 'Books_F2', 'Books_F3'],
                    index=['User 1 Electronics', 'User 1 Clothing', 'User 1
 ↪  Books',
                           'User 2 Electronics', 'User 2 Clothing', 'User 2
                            ↪  Books',
                           'User 3 Electronics', 'User 3 Clothing', 'User 3
                            ↪  Books'])

# Print the Kronecker product matrix
print("Kronecker Product Matrix (K):\n", df_K)

# Predict ratings and create recommendations
def recommend(user_index, top_n=3):
    """ Recommend top_n categories for a given user based on Kronecker
     ↪  product matrix. """
```

```
    user_ratings = K[user_index * len(C):(user_index + 1) * len(C), :]
    predicted_ratings = np.sum(user_ratings, axis=0)
    recommendations = np.argsort(predicted_ratings)[::-1][:top_n]
    return recommendations

# Recommendations for User 1
user_index = 0  # User 1
top_n = 3
recommendations = recommend(user_index, top_n)

print(f"\nTop {top_n} recommendations for User {user_index + 1}:")
for rec in recommendations:
    print(df_K.columns[rec])
```

```
Kronecker Product Matrix (K):
                     Electronics_F1  Electronics_F2  Electronics_F3  \
User 1 Electronics                5               0               0
User 1 Clothing                   0               5               5
User 1 Books                      0               5               5
User 2 Electronics                2               0               0
User 2 Clothing                   0               2               2
User 2 Books                      0               2               2
User 3 Electronics                3               0               0
User 3 Clothing                   0               3               3
User 3 Books                      0               3               3

                     Clothing_F1  Clothing_F2  Clothing_F3  Books_F1  Books_F2  \
User 1 Electronics             3            0            0         4         0
User 1 Clothing                0            3            3         0         4
User 1 Books                   0            3            3         0         4
User 2 Electronics             4            0            0         5         0
User 2 Clothing                0            4            4         0         5
User 2 Books                   0            4            4         0         5
User 3 Electronics             4            0            0         4         0
User 3 Clothing                0            4            4         0         4
User 3 Books                   0            4            4         0         4

                     Books_F3
User 1 Electronics          0
User 1 Clothing             4
User 1 Books                4
```

```
User 2 Electronics         0
User 2 Clothing            5
User 2 Books               5
User 3 Electronics         0
User 3 Clothing            4
User 3 Books               4

Top 3 recommendations for User 1:
Electronics_F2
Electronics_F3
Books_F3
```

A simple visualization of this recomendation system is shown in Fig 11.

```python
# Visualization
def plot_recommendations(user_index):
    """ Plot the predicted ratings for each category for a given user. """
    user_ratings = K[user_index * len(C):(user_index + 1) * len(C), :]
    predicted_ratings = np.sum(user_ratings, axis=0)
    categories = df_K.columns
    plt.figure(figsize=(6, 5))
    plt.bar(categories, predicted_ratings)
    plt.xlabel('Categories')
    plt.ylabel('Predicted Ratings')
    plt.title(f'Predicted Ratings for User {user_index + 1}')
    plt.xticks(rotation=45)
    plt.show()

# Plot recommendations for User 1
plot_recommendations(user_index)
```

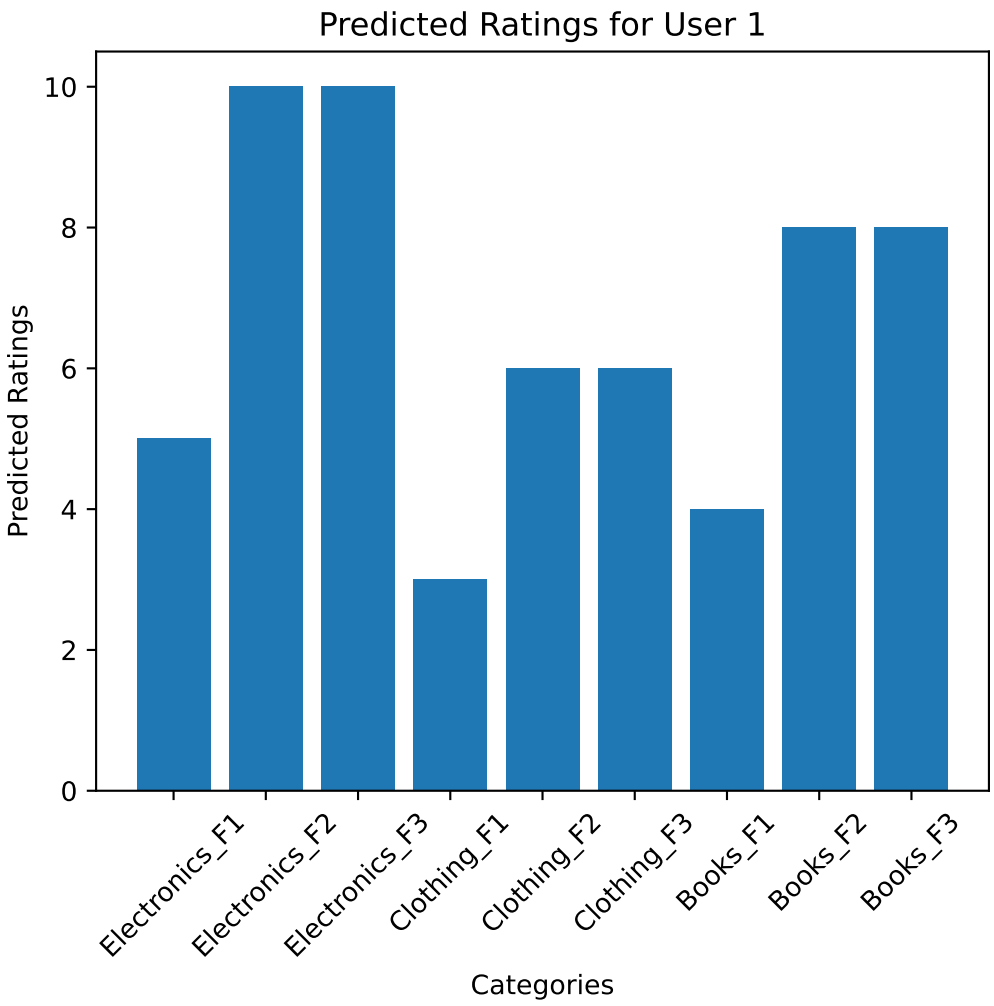


Figure 11: EDA for the Recommendation System

This micro project illustrate one of the popular use of Kronecker product on ML application.

Matrix Measures of Practical Importance

Matrix measures, such as rank and determinant, play crucial roles in linear algebra. While both rank and determinant provide valuable insights into the properties of a matrix, they serve different purposes. Understanding their roles and applications is essential for solving complex problems in computer science, engineering, and applied mathematics.

Determinant

Determinant of a 2×2 matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is defined as $|A| = ad - bc$. Determinant of higher order square matrices can be found using the Laplace method or Sarrus method.

The determinant of a matrix provides information about the matrix's invertibility and scaling factor for volume transformation. Specifically:

1. *Invertibility:* A matrix is invertible if and only if its determinant is non-zero.
2. *Volume Scaling:* The absolute value of the determinant gives the scaling factor by which the matrix transforms volume.
3. *Parallelism:* If the determinant of a matrix composed of vectors is zero, the vectors are linearly dependent, meaning they are parallel or redundant.
4. *Redundancy:* A zero determinant indicates that the vectors span a space of lower dimension than the number of vectors, showing redundancy.

! Least Possible Values of Determinant

1. *Least Positive Determinant:* For a 1×1 matrix, the smallest non-zero determinant is any positive value, typically ϵ, where ϵ is a small positive number.
2. Least Non-Zero Determinant: For higher-dimensional matrices, the smallest non-zero determinant is a non-zero value that represents the smallest area or volume spanned by the matrix's rows or columns. For example a 2×2 matrix with determinant ϵ could be:

$$B = \begin{pmatrix} \epsilon & 0 \\ 0 & \epsilon \end{pmatrix}$$

Here, ϵ is a small positive number, indicating a very small but *non-zero* area.

Now let's look into the most important matrix measure for advanced application in Linear Algebra.

As we know the matrix is basically a representation tool that make things abstract- remove unnecessary details. Then the matrix itself can be represented in many ways. This is the real story telling with this most promising mathematical structure. Consider a context of collecting feedback about a product in three aspects- cost, quality and practicality. For simplicity in calculation, we consider responses from 3 customers only. The data is shown in Table 7.

Table 7: User rating of a consumer product

User	Cost	Quality	Practicality
User-1	1	4	5

User	Cost	Quality	Practicality
User-2	3	2	5
User-3	2	1	3

It's perfect and nice looking. But both mathematics and a computer can't handle this table as it is. So we create an abstract representation of this data- the rating matrix. Using the traditional approach, let's represent this rating data as:

$$A = \begin{bmatrix} 1 & 4 & 5 \\ 3 & 2 & 5 \\ 2 & 1 & 3 \end{bmatrix}$$

Now both the column names and row indices were removed and the data is transformed into the abstract form. This representation has both advantages and disadvantages. Be positive! So we are focused only in the advantages.

Just consider the product. Its sales fully based on its features. So the product sales perspective will be represented in terms of the features- cost, quality and practicality. These features are columns of our rating matrix. Definitly people will have different rating for these features. Keeping all these in mind let's introduce the concept of *linear combination*. This leads to a new matrix product as shown below.

$$\begin{aligned} Ax &= \begin{bmatrix} 1 & 4 & 5 \\ 3 & 2 & 5 \\ 2 & 1 & 3 \end{bmatrix} x \\ &= \begin{bmatrix} 1 & 4 & 5 \\ 3 & 2 & 5 \\ 2 & 1 & 3 \end{bmatrix} \cdot \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \\ &= \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix} x_1 + \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix} x_2 + \begin{bmatrix} 5 \\ 5 \\ 3 \end{bmatrix} x_3 \end{aligned}$$

As the number of users increases, the product sales perspective become more informative. In short the span of the features define the feature space of the product. In real cases, a manufacture wants to know what are the features really inflence the customers. This new matrix product will help the manufactures to identify that features!

So we are going to define this new matrix product as the feature space, that will provide more insights to this context as:

$$A = CR$$

Where C is the column space and R is the row reduced Echelon form of A. But the product is not the usual scalar projection, Instead the weight of linear combination of elements in the column space.

Let's formally illustrate this in our example. From the first observation itself, it is clear that last column is just the sum of first and second columns (That is in our context the feature 'practicality' is just depends on 'cost' and 'quality'. meaningful?). So only first columns are independent and so spans the column space.

$$C = \begin{bmatrix} 1 & 4 \\ 3 & 2 \\ 2 & 1 \end{bmatrix}$$

Now look into the matrix R. Applying elementary row tansformations, A will transformed into:

$$R = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

Hence we can form a decomposition for the given rating matrix, A as:

$$\begin{aligned} A &= CR \\ &= \begin{bmatrix} 1 & 4 \\ 3 & 2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 1 \\ & & \end{bmatrix} \end{aligned}$$

This decomposition says that there are only two independent features (columns) and the third feature (column) is the sum of first two features (columns).

> **!** Interpretation of the R matrix
>
> Each column in the R matrix represents the weights for linear combination of vectors in the column space to get that column in A. In this example, third column of R is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$. This means that third column of A will be $1 \times C_1 + 1 \times C_2$ of the column space, C!

This first matrix decompostion donate a new type of matrix product (outer product) and a new measure- the number of independent columns and number of independent rows. This count is called the *rank* of the matrix A. In the case of features, if the rank of the column space

is less than the number of features then definitly a less number of feature set will perfectly represent the data. This will help us to reduce the dimension of the dataset and there by reducing computational complexities in data analysis and machine Learning jobs.

In the above discussion, we consider only the columns of A. Now we will mention the row space. It is the set of all linearly independent rows of A. For any matrix A, both the row space and column space are of same rank. This correspondance is a helpful result in many practical applications.

Now we consider a stable equation, $Ax = 0$. With the usual notation of dot product, it implies that x is orthogonal to A. Set of all those independent vectors which are orthogonal to A constitute a new space of interest. It is called the *null space* of A. If A represents a linear transformation, then the null space will be populated by those non-zero vectors which are *nullified* by the transformation A. As a summary of this discussion, the row space and null space of a matrix A creates an orthogonal system. Considering the relationship between A and A^T, it is clear that row space of A is same as the column space of A^T and vice verse are. So we can restate the orthogonality as: 'the null space of A is orthogonal to the column space of A^T' and 'the null space of A^T is orthogonal to the column space of A'. Mathematically this property can be represents as follows.

> **Note**
>
> $$\mathcal{N}(A) \perp \mathcal{C}(A^T)$$
> $$\mathcal{N}(A^T) \perp \mathcal{C}(A)$$

In the given example, solving $Ax = 0$ we get $x = \begin{bmatrix} 1 & 1 & -1 \end{bmatrix}^T$.

So the rank of $\mathcal{N}(A) = 1$. Already we have rank of $A = 2$. This leads to an interesting result:

$$\text{Rank}(A) + \text{Rank}(\mathcal{N}(A)) = 3$$

This observation can be framed as a theorem.

Rank Nullity Theorem

The rank-nullity theorem is a fundamental theorem in linear algebra that is important for understanding the connections between mathematical operations in engineering, physics, and computer science. It states that the sum of the rank and nullity of a matrix equals the number of columns in the matrix. The rank is the maximum number of linearly independent columns, and the nullity is the dimension of the nullspace.

Theorem 0.1 (Rank Nullitty Theorem)**.** *The Rank-Nullity Theorem states that for any $m \times n$ matrix A, the following relationship holds:*

$$Rank(A) + Nullity(A) = n$$

*where: - **Rank** of A is the dimension of the column space of A, which is also equal to the dimension of the row space of A. - **Nullity** of A is the dimension of the null space of A, which is the solution space to the homogeneous system $A\mathbf{x} = \mathbf{0}$.*

Steps to Formulate for Matrix A

1. **Find the Rank of** A: The rank of a matrix is the maximum number of linearly independent columns (or rows). It can be determined by transforming A into its row echelon form or reduced row echelon form (RREF).
2. **Find the Nullity of** A: The nullity is the dimension of the solution space of $A\mathbf{x} = \mathbf{0}$. This can be found by solving the homogeneous system and counting the number of free variables.
3. **Apply the Rank-Nullity Theorem**: Use the rank-nullity theorem to verify the relationship.

Example 1: Calculate the rank and nullity of $A = \begin{bmatrix} 1 & 4 & 5 \\ 3 & 2 & 5 \\ 2 & 1 & 3 \end{bmatrix}$ and verify the rank nullity theorem.

1. **Row Echelon Form**:

   Perform Gaussian elimination on A:

   $$A = \begin{bmatrix} 1 & 4 & 5 \\ 3 & 2 & 5 \\ 2 & 1 & 3 \end{bmatrix}$$

   Perform row operations to get it to row echelon form:

   - Subtract 3 times row 1 from row 2:

     $$\begin{bmatrix} 1 & 4 & 5 \\ 0 & -10 & -10 \\ 2 & 1 & 3 \end{bmatrix}$$

- Subtract 2 times row 1 from row 3:

$$\begin{bmatrix} 1 & 4 & 5 \\ 0 & -10 & -10 \\ 0 & -7 & -7 \end{bmatrix}$$

- Add $\frac{7}{10}$ times row 2 to row 3:

$$\begin{bmatrix} 1 & 4 & 5 \\ 0 & -10 & -10 \\ 0 & 0 & 0 \end{bmatrix}$$

The matrix is now in row echelon form.

Rank is the number of non-zero rows, which is 2.

2. **Find the Nullity**: The matrix A has 3 columns. The number of free variables in the solution of $A\mathbf{x} = \mathbf{0}$ is $3 - \text{Rank}$.

   So,

$$\text{Nullity}(A) = 3 - 2 = 1$$

3. **Apply the Rank-Nullity Theorem**:

$$\text{Rank}(A) + \text{Nullity}(A) = 2 + 1 = 3$$

   This matches the number of columns of A, confirming the theorem.

Fundamental Subspaces

In section (**note-ortho?**), we have seen that for any matrix A, there is two pairs of inter-related orthogonal spaces. This leads to the concept of Fundamental sup spaces.

Matrices are not just arrays of numbers; they can represent linear transformations too. A linear transformation maps vectors from one vector space to another while preserving vector addition and scalar multiplication. The matrix A can be viewed as a representation of a linear transformation T from $\mathbb{R}^n$ to $\mathbb{R}^m$ where:

$$T(\mathbf{x}) = A\mathbf{x}$$

In this context:

- The column space of A represents the range of T, which is the set of all possible outputs.

- The null space of A represents the kernel of T, which is the set of vectors that are mapped to the zero vector.

The Four Fundamental Subspaces

Understanding the four fundamental subspaces helps in analyzing the properties of a linear transformation. These subspaces are:

Definition 0.1 (Four Fundamental Subspaces)**.** Let $T : \mathbb{R}^n \longrightarrow \mathbb{R}^m$ be a linear transformation and A represents the matrix of transformation. The four fundamental subspaces are defined as:

1. **Column Space (Range)**: The set of all possible outputs of the transformation. For matrix A, this is the span of its columns. It represents the image of $\mathbb{R}^n$ under T.
2. **Null Space (Kernel)**: The set of all vectors that are mapped to the zero vector by the transformation. For matrix A, this is the solution space of $A\mathbf{x} = \mathbf{0}$.
3. **Row Space**: The span of the rows of A. This space is crucial because it helps in understanding the rank of A. The dimension of the row space is equal to the rank of A, which represents the maximum number of linearly independent rows.
4. **Left Null Space**: The set of all vectors $\mathbf{y}$ such that $A^T\mathbf{y} = \mathbf{0}$. It provides insight into the orthogonal complement of the row space.

This idea is depicted as a 'Big picture of the four sub spaces of a matrix' in the Strang's text book on Linear algebra for every one (Strang 2020). This 'Big Picture' is shown in Fig- 12.

A video session from Strang's session is here:

https://youtu.be/rwLOfdfc4dw?si=DsJb8KJTF05hHc76

Practice Problems

Problem 1: Express the vector $(1, -2, 5)$ as a linear combination of the vectors $(1, 1, 1)$, $(1, 2, 3)$ and $(2, -1, 1)$.

Problem 2: Show that the feature vector $(2, -5, 3)$ is not linearly associated with the features $(1, -3, 2)$, $(2, -4, -1)$ and $(1, -5, 7)$.

Problem 3: Show that the feature vectors $(1, 1, 1)$, $(1, 2, 3)$ and $(2, -1, 1)$ are non-redundant.

Problem 4: Prove that the features $(1, -1, 1)$, $(0, 1, 2)$ and $(3, 0, -1)$ form basis for the feature space.

Problem 5: Check whether the vectors $(1, 2, 1)$, $(2, 1, 4)$ and $(4, 5, 6)$ form a basis for $\mathbb{R}^3$.

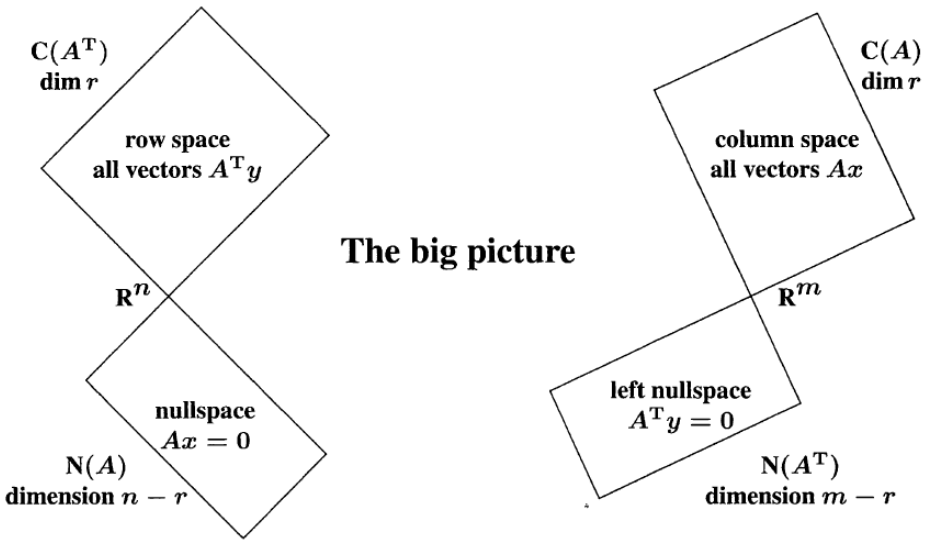


Figure 12: The Big Pictue of Fundamental Subspaces

Problem 6: Find the four fundamental subspaces of the feature space created by $(1, 2, 1)$, $(2, 1, 4)$ and $(4, 5, 6)$.

Problem 7: Find the four fundamental subspaces and its dimensions of the matrix $\begin{bmatrix} 1 & 2 & 4 \\ 2 & 1 & 5 \\ 1 & 4 & 6 \end{bmatrix}$.

Problem 8: Express $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 1 & -1 \\ 2 & -1 & 0 \end{bmatrix}$ as the Kronecker product of the column space and the row space in the form $A = C \otimes R$.

Problem 9: Find the four fundamental subspaces of $A = \begin{bmatrix} 1 & 2 & 0 & 2 & 5 \\ -2 & -5 & 1 & -1 & -8 \\ 0 & -3 & 3 & 4 & 1 \\ 3 & 6 & 0 & -7 & 2 \end{bmatrix}$.

Problem 10: Find the four fundamental subspaces of $A = \begin{bmatrix} -1 & 2 & -1 & 5 & 6 \\ 4 & -4 & -4 & -12 & -8 \\ 2 & 0 & -6 & -2 & 4 \\ -3 & 1 & 7 & -2 & 12 \end{bmatrix}$.

Problem 11: Express $A = \begin{bmatrix} 2 & 3 & -1 & -1 \\ 1 & -1 & -2 & -4 \\ 3 & 1 & 3 & -2 \\ 6 & 3 & 0 & -7 \end{bmatrix}$ in $A = C \otimes R$, where C is the column space and R is the row space of A.

Problem 12: Express $A = \begin{bmatrix} 0 & 1 & -3 & -1 \\ 1 & 0 & 1 & 1 \\ 3 & 1 & 0 & 2 \\ 1 & 1 & -2 & 0 \end{bmatrix}$ in $A = C \otimes R$, where C is the column space and R is the row space of A.

Problem 13: Show that the feature vectors $(2, 3, 0)$, $(1, 2, 0)$ and $(8, 13, 0)$ are redundant and hence find the relationship between them.

Problem 14: Show that the feature vectors $(1, 2, 1)$, $(4, 1, 2)$, $(-3, 8, 1)$ and $(6, 5, 4)$ are redundant and hence find the relationship between them.

Problem 15: Show that the feature vectors $(1, 2, -1, 0)$, $(1, 3, 1, 2)$, $(4, 2, 1, 0)$ and $(6, 1, 0, 1)$ are redundant and hence find the relationship between them.

! Important

Three Parts of the *Fundamental theorem* The fundamental theorem of linear algebra relates all four of the fundamental subspaces in a number of different ways. There are main parts to the theorem:
Part 1:(Rank nullity theorem) The column and row spaces of an $m \times n$ matrix A both have dimension r, the rank of the matrix. The nullspace has dimension $n - r$, and the left nullspace has dimension $m - r$.
Part 2:(Orthogonal subspaces) The nullspace and row space are orthogonal. The left nullspace and the column space are also orthogonal.
Part 3:(Matrix decomposition) The final part of the fundamental theorem of linear algebra constructs an orthonormal basis, and demonstrates a singular value decomposition: any matrix M can be written in the form $M = U\Sigma V^T$, where $U_{m\times m}$ and $V_{n\times n}$ are unitary matrices, $\Sigma_{m\times n}$ matrix with nonnegative values on the diagonal.
This part of the fundamental theorem allows one to immediately find a basis of the subspace in question. This can be summarized in the following table.

Subspace	Subspace of	Symbol	Dimension	Basis
Column space	$\mathbb{R}^m$	$\text{im}(A)$	r	First r columns of U
Nullspace (kernel)	$\mathbb{R}^n$	$\ker(A)$	$n - r$	Last $n - r$ columns of V
Row space	$\mathbb{R}^n$	$\text{im}(A^T)$	r	First r columns of V

Left nullspace (kernel)	$\mathbb{R}^m$	$\ker(A^T)$	$m-r$	Last $m-r$ columns of U

Computational methods to find all the four fundamental subspaces of a matrix

There are different approaches to find the four fundamental subspaces of a matrix using `Python`. Simplest method is just convert our mathematical procedure into `Python` functions and call them to find respective spaces. This method is illustrated below.

```
# importing numpy library for numerical computation
import numpy as np
# define the function create the row-reduced Echelon form of given matrix
def row_echelon_form(A):
    """Convert matrix A to its row echelon form."""
    A = A.astype(float)
    rows, cols = A.shape
    for i in range(min(rows, cols)):
        # Pivot: find the maximum element in the current column
        max_row = np.argmax(np.abs(A[i:, i])) + i
        if A[max_row, i] == 0:
            continue  # Skip if the column is zero
        # Swap the current row with the max_row
        A[[i, max_row]] = A[[max_row, i]]
        # Eliminate entries below the pivot
        for j in range(i + 1, rows):
            factor = A[j, i] / A[i, i]
            A[j, i:] -= factor * A[i, i:]
    return A

# define function to generate null space from the row-reduced echelon form
def null_space_of_matrix(A, rtol=1e-5):
    """Compute the null space of a matrix A using row reduction."""
    A_reduced = row_echelon_form(A)
    rows, cols = A_reduced.shape
    # Identify pivot columns
    pivots = []
    for i in range(rows):
        for j in range(cols):
            if np.abs(A_reduced[i, j]) > rtol:
                pivots.append(j)
```

```
                break
    free_vars = set(range(cols)) - set(pivots)

    null_space = []
    for free_var in free_vars:
        null_vector = np.zeros(cols)
        null_vector[free_var] = 1
        for pivot, row in zip(pivots, A_reduced[:len(pivots)]):
            null_vector[pivot] = -row[free_var]
        null_space.append(null_vector)

    return np.array(null_space).T

# define the function to generate the row-space of A

def row_space_of_matrix(A):
    """Compute the row space of a matrix A using row reduction."""
    A_reduced = row_echelon_form(A)
    # The non-zero rows of the reduced matrix form the row space
    non_zero_rows = A_reduced[~np.all(A_reduced == 0, axis=1)]
    return non_zero_rows

# define the function to generate the column space of A

def column_space_of_matrix(A):
    """Compute the column space of a matrix A using row reduction."""
    A_reduced = row_echelon_form(A)
    rows, cols = A_reduced.shape
    # Identify pivot columns
    pivots = []
    for i in range(rows):
        for j in range(cols):
            if np.abs(A_reduced[i, j]) > 1e-5:
                pivots.append(j)
                break
    column_space = A[:, pivots]
    return column_space
```

Examples:

1. Find all the fundamental subspaces of $A = \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{pmatrix}$.

```
A = np.array([[1, 2, 3],
              [4, 5, 6],
              [7, 8, 9]])

print("Matrix A:")
print(A)

# Null Space
null_space_A = null_space_of_matrix(A)
print("\nNull Space of A:")
print(null_space_A)

# Row Space
row_space_A = row_space_of_matrix(A)
print("\nRow Space of A:")
print(row_space_A)

# Column Space
column_space_A = column_space_of_matrix(A)
print("\nColumn Space of A:")
print(column_space_A)
```

```
Matrix A:
[[1 2 3]
 [4 5 6]
 [7 8 9]]

Null Space of A:
[[-9.        ]
 [-1.71428571]
 [ 1.        ]]

Row Space of A:
[[7.00000000e+00 8.00000000e+00 9.00000000e+00]
 [0.00000000e+00 8.57142857e-01 1.71428571e+00]
 [0.00000000e+00 5.55111512e-17 1.11022302e-16]]
```

```
Column Space of A:
[[1 2]
 [4 5]
 [7 8]]
```

Rank and Solution of System of Linear Equations

In linear algebra, the rank of a matrix is a crucial concept for understanding the structure of a system of linear equations. It provides insight into the solutions of these systems, helping us determine the number of independent equations and the nature of the solution space.

Definition 0.2 (Rank and System Consistency)**.** The rank of a matrix A is defined as the maximum number of linearly independent rows or columns. When solving a system of linear equations represented by $A\mathbf{x} = \mathbf{b}$, where A is an $m \times n$ matrix and $\mathbf{b}$ is a vector, the rank of A plays a crucial role in determining the solution's existence and uniqueness.

Consistency of the System

1. **Consistent System:** A system of linear equations is consistent if there exists at least one solution. This occurs if the rank of the coefficient matrix A is equal to the rank of the augmented matrix $[A|\mathbf{b}]$. Mathematically, this can be expressed as:

   $$\text{rank}(A) = \text{rank}([A|\mathbf{b}])$$

   If this condition is met, the system has solutions. The solutions can be:

   - **Unique** if the rank equals the number of variables.
   - **Infinitely many** if the rank is less than the number of variables.

2. **Inconsistent System:** A system is inconsistent if there are no solutions. This occurs when:

   $$\text{rank}(A) \neq \text{rank}([A|\mathbf{b}])$$

   In this case, the equations represent parallel or conflicting constraints that cannot be satisfied simultaneously.

> **i** Use of Null space in creation of general solution from particular solution
>
> If the system $AX = b$ has many solutions, then the general solution of the system can be found using a particular solution and the elements in the null space of the coefficient matrix A as
>
> $$X = x_p + tX_N$$

where X is the general solution and t is a free variable (parameter) and $X_N \in N(A)$.

Computational method to solve system of linear equations.

If for a system $AX = b$, $det(A) \neq 0$, then the system has a unique solution and can be found by `solve()` function from `NumPy`. If the system is consistant and many solutions, then computationally we will generate the general solution using the $N(A)$. A detailed `Python` code is given below.

```
import numpy as np

def check_consistency(A, b):
    """
    Check the consistency of a linear system Ax = b and return the solution
 ↪  if consistent.

    Parameters:
    A (numpy.ndarray): Coefficient matrix.
    b (numpy.ndarray): Right-hand side vector.

    Returns:
    tuple: A tuple with consistency status, particular solution (if
 ↪  consistent), and null space (if infinite solutions).
    """
    A = np.array(A)
    b = np.array(b)

    # Augment the matrix A with vector b
    augmented_matrix = np.column_stack((A, b))

    # Compute ranks
    rank_A = np.linalg.matrix_rank(A)
    rank_augmented = np.linalg.matrix_rank(augmented_matrix)

    # Check for consistency
    if rank_A == rank_augmented:
        if rank_A == A.shape[1]:
            # Unique solution
            solution = np.linalg.solve(A, b)
            return "Consistent and has a unique solution", solution, None
        else:
```

```
            # Infinitely many solutions
            particular_solution = np.linalg.lstsq(A, b, rcond=None)[0]
            null_space = null_space_of_matrix(A)
            return "Consistent but has infinitely many solutions",
             ↪  particular_solution, null_space
    else:
        return "Inconsistent system (no solution)", None, None

def null_space_of_matrix(A):
    """
    Compute the null space of matrix A, which gives the set of solutions to
 ↪  Ax = 0.

    Parameters:
    A (numpy.ndarray): Coefficient matrix.

    Returns:
    numpy.ndarray: Basis for the null space of A.
    """
    u, s, vh = np.linalg.svd(A)
    null_mask = (s <= 1e-10)  # Singular values near zero
    null_space = np.compress(null_mask, vh, axis=0)
    return null_space.T
```

Example 1: Solve

$$\begin{aligned} 2x - y + z &= 1 \\ x + 2y &= 3 \\ 3x + 2y + z &= 4 \end{aligned}$$

```
# Example usage 1: System with a unique solution
A1 = np.array([[2, -1, 1], [1, 0, 2], [3, 2, 1]])
b1 = np.array([1, 3, 4])

status1, solution1, null_space1 = check_consistency(A1, b1)
print("Example 1 - Status:", status1)

if solution1 is not None:
    print("Solution:", solution1)
if null_space1 is not None:
    print("Null Space:", null_space1)
```

```
Example 1 - Status: Consistent and has a unique solution
Solution: [0.27272727 0.90909091 1.36363636]
```

Example 2: Solve the system of equations,

$$\begin{aligned} x + 2y + z &= 3 \\ 2x + 4y + 2z &= 6 \\ x + y + z &= 2 \end{aligned}$$

```python
# Example usage 2: System with infinitely many solutions
A2 = np.array([[1, 2, 1], [2, 4, 2], [1, 1, 1]])
b2 = np.array([3, 6, 2])

status2, solution2, null_space2 = check_consistency(A2, b2)
print("\nExample 2 - Status:", status2)

if solution2 is not None:
    print("Particular Solution:", solution2)
if null_space2 is not None:
    print("Null Space (Basis for infinite solutions):", null_space2)
```

```
Example 2 - Status: Consistent but has infinitely many solutions
Particular Solution: [0.5 1.  0.5]
Null Space (Basis for infinite solutions): [[ 7.07106781e-01]
 [ 1.11022302e-16]
 [-7.07106781e-01]]
```

1 Python Libraries for Computational Linear Algebra

In the first two modules, we gained a foundational understanding of `Python` programming and the basics of linear algebra, including fundamental subspaces such as row space, column space, and null space, both theoretically and through Python implementations. These essential concepts provided the groundwork for solving linear algebra problems manually and computationally. Now, as we move into Module 3, the focus shifts toward leveraging advanced `Python` libraries to handle more complex and large-scale computations in linear algebra efficiently.

This module introduces the powerful computational tools available in `Python`, such as `NumPy`, `SymPy`, `SciPy`, and `Matplotlib`. These libraries are designed to enhance the ability to perform both numerical and symbolic operations on matrices, vectors, and systems of equations. With `NumPy`'s high-performance array operations, `SymPy`'s symbolic computation abilities, and `SciPy`'s extensive collection of scientific routines, students will be able to compute solutions for real-world problems with ease. The module also incorporates visualization techniques through `Matplotlib`, allowing students to graphically represent mathematical solutions, interpret data, and communicate their findings effectively. This module empowers students to move beyond manual calculations and explore advanced problem-solving strategies computationally.

1.1 Introduction to NumPy

In this section, we will introduce **NumPy**, the core library for scientific computing in Python. NumPy provides support for arrays, matrices, and a host of mathematical functions to operate on these structures. This is particularly useful for linear algebra computations, making it an essential tool in computational mathematics. The library also serves as the foundation for many other Python libraries like SciPy, Pandas, and Matplotlib.

1.1.1 Purpose of Using NumPy

The primary purpose of NumPy is to enable efficient numerical computations involving large datasets, vectors, and matrices. With NumPy, one can perform mathematical operations on arrays and matrices in a way that is highly optimized for performance, both in terms of memory and computational efficiency (Harris et al. 2020).

Some key advantages of using NumPy include:

- **Efficient handling of large datasets**: Arrays in NumPy are optimized for performance and consume less memory compared to native Python lists.
- **Matrix operations**: NumPy provides built-in functions for basic matrix operations, allowing one to perform tasks like matrix multiplication, transpose, and inversion easily.
- **Linear algebra**: It includes functions for solving systems of equations, finding eigenvalues and eigenvectors, computing matrix factorizations, and more.

1.2 Basic Operations in NumPy

This section will present several examples of using NumPy array manipulation to access data and subarrays, and to split, reshape, and join the arrays. While the types of operations shown here may seem a bit dry and pedantic, they comprise the building blocks of many other examples used throughout the book. Get to know them well!

We'll cover a few categories of basic array manipulations here:

- *Attributes of arrays*: Determining the size, shape, memory consumption, and data types of arrays
- *Indexing of arrays*: Getting and setting the value of individual array elements
- *Slicing of arrays*: Getting and setting smaller subarrays within a larger array
- *Reshaping of arrays*: Changing the shape of a given array
- *Joining and splitting of arrays*: Combining multiple arrays into one, and splitting one array into many

Loading `numpy` to a python programme

Syntax

```
import numpy as "name of instance"
```

eg: `import numpy as np`

1.2.0.1 Array Creation

At the core of NumPy is the **ndarray** object, which represents arrays and matrices. Here's how to create arrays using NumPy:

```
import numpy as np

# Creating a 1D array
arr = np.array([1, 2, 3, 4, 5])

# Creating a 2D matrix
matrix = np.array([[1, 2, 3], [4, 5, 6], [7, 8, 9]])

print("1D Array: \n", arr)
print("2D Matrix: \n", matrix)
```

1.2.1 Define different types of `numpy` arrays

As the first step to understand different types of arrays in `NumPy` let us consider the following examples.

1.2.1.1 1D Array (Vector)

In NumPy, a one-dimensional (1D) array is similar to a list or vector in mathematics. It consists of a single row or column of numbers, making it an ideal structure for storing sequences of values.

```
import numpy as np

# Creating a 1D array
arr = np.array([1, 2, 3, 4])
print(arr)
```

```
[1 2 3 4]
```

Here, `np.array()` is used to create a 1D array (or vector) containing the values `[1, 2, 3, 4]`. The array represents a single sequence of numbers, and it is the basic structure of `NumPy`.

> **i** Use:
>
> A 1D array can represent many things, such as a vector in linear algebra, a list of numbers, or a single dimension of data in a machine learning model.

1.2.1.2 2D Array (Matrix)

A two-dimensional (2D) array is equivalent to a matrix in mathematics. It consists of rows and columns and is often used to store tabular data or perform matrix operations.

```
from IPython.display import display, HTML
# Creating a 2D array (Matrix)
matrix = np.array([[1, 2, 3], [4, 5, 6]])
display(matrix)
```

```
array([[1, 2, 3],
       [4, 5, 6]])
```

In this example, the 2D array (or matrix) is created using `np.array()` by providing a list of lists, where each list represents a row in the matrix. The result is a matrix with two rows and three columns.

> **i** Use:
>
> Matrices are fundamental structures in linear algebra. They can represent anything from transformation matrices in graphics to coefficients in systems of linear equations.

1.2.1.3 Zero Arrays

Zero arrays are used to initialize matrices or arrays with all elements set to zero. This can be useful when creating placeholder arrays where the values will be computed or updated later.

```
# Creating an array of zeros
zero_matrix = np.zeros((3, 3))
print(zero_matrix)
```

```
[[0. 0. 0.]
 [0. 0. 0.]
 [0. 0. 0.]]
```

The `np.zeros()` function creates an array filled with zeros. In this example, we create a 3x3 matrix with all elements set to zero.

Use:

Zero arrays are commonly used in algorithms that require the allocation of memory for arrays that will be updated later.

1.2.1.4 Identity Matrix

An identity matrix is a square matrix with ones on the diagonal and zeros elsewhere. It plays a crucial role in linear algebra, especially in solving systems of linear equations and matrix factorizations.

```
# Creating an identity matrix
identity_matrix = np.eye(3)
print(identity_matrix)
```

```
[[1. 0. 0.]
 [0. 1. 0.]
 [0. 0. 1.]]
```

The `np.eye(n)` function creates an identity matrix with the specified size. In this case, we create a 3x3 identity matrix, where all diagonal elements are 1, and off-diagonal elements are 0.

1.2.1.5 Arange Function

The `np.arange()` function is used to create an array with evenly spaced values within a given range. It's similar to Python's built-in "range() function but returns a NumPy array instead of a list.

```
# Creating an array using arange
arr = np.arange(1, 10, 2)
print(arr)
```

```
[1 3 5 7 9]
```

Here, `np.arange(1, 10, 2)` generates an array of numbers starting at 1, ending before 10, with a step size of 2. The result is `[1, 3, 5, 7, 9]`.

> Use:
>
> This function is useful when creating arrays for loops, data generation, or defining sequences for analysis.

1.2.1.6 Linspace Function

The `np.linspace()` function generates an array of evenly spaced values between a specified start and end, with the number of intervals defined by the user.

```
# Creating an array using linspace
arr = np.linspace(0, 1, 5)
print(arr)
```

```
[0.   0.25 0.5  0.75 1.  ]
```

`np.linspace(0, 1, 5)` creates an array with 5 evenly spaced values between 0 and 1, including both endpoints. The result is `[0. , 0.25, 0.5 , 0.75, 1. ]`. :::{.callout-note} ### Use: linspace() is often used when you need a specific number of evenly spaced points within a range, such as for plotting functions or simulating data. :::

1.2.1.7 Reshaping Arrays

The `reshape()` function changes the shape of an existing array without changing its data. It's useful when you need to convert an array to a different shape for computations or visualizations.

```
# Reshaping an array
arr = np.arange(1, 10)
reshaped_arr = arr.reshape(3, 3)
print(reshaped_arr)
```

```
[[1 2 3]
 [4 5 6]
 [7 8 9]]
```

In this example, a 1D array with 9 elements is reshaped into a 3x3 matrix using the `reshape()` method. The data remains the same but is now structured in a 2D form.

Use:

Reshaping is critical in linear algebra and machine learning when working with input data of different dimensions.

1.2.1.8 Random Arrays

NumPy's random module is used to generate arrays with random values. These arrays are useful in simulations, testing algorithms, and initializing variables in machine learning.

```
# Creating a random array
random_arr = np.random.rand(3, 3)
print(random_arr)
```

```
[[0.63701895 0.39281355 0.68542753]
 [0.92000307 0.88941171 0.76594411]
 [0.39385117 0.87924491 0.65657706]]
```

`np.random.rand(3, 3)` creates a 3x3 matrix with random values between 0 and 1. The `rand()` function generates random floats in the range $[0, 1)$.

Use:

Random arrays are commonly used for initializing weights in machine learning algorithms, simulating stochastic processes, or for testing purposes.

Syntax

- **One-Dimensional Array:** `np.array([list of values])`
- **Two-Dimensional Array:** `np.array([[list of values], [list of values]])`
- **Zero Array:** `np.zeros(shape)`
  - `shape` is a tuple representing the dimensions (e.g., `(3, 3)` for a 3x3 matrix).
- **Identity Matrix:** `np.eye(n)`
  - `n` is the size of the matrix.
- **Arrange Function:** `np.arange(start, stop, step)`

  - `start` is the starting value, `stop` is the end value (exclusive), and `step` is the increment.

- **Linspace Function:** `np.linspace(start, stop, num)`
  - `start` and `stop` define the range, and `num` is the number of evenly spaced values.
- **Reshaping Arrays:** `np.reshape(array, new_shape)`
  - `array` is the existing array, and `new_shape` is the desired shape (e.g., `(3, 4)`).
- **Random Arrays without Using rand:** `np.random.randint(low, high, size)`
  - `low` and `high` define the range of values, and `size` defines the shape of the array.

1.2.2 Review Questions

Q1: What is the purpose of using `np.array()` in NumPy?
Ans: `np.array()` is used to create arrays in NumPy, which can be 1D, 2D, or multi-dimensional arrays.

Q2: How do you create a 2D array in NumPy?
Ans: A 2D array can be created using `np.array([[list of values], [list of values]])`.

Q3: What is the difference between `np.zeros()` and `np.eye()`?
Ans: `np.zeros()` creates an array filled with zeros of a specified shape, while `np.eye()` creates an identity matrix of size `n`.

Q4: What is the syntax to create an evenly spaced array using `np.linspace()`?
Ans: The syntax is `np.linspace(start, stop, num)`, where `num` specifies the number of evenly spaced points between `start` and `stop`.

Q5: How can you reshape an array in NumPy?
Ans: Arrays can be reshaped using `np.reshape(array, new_shape)`, where `new_shape` is the desired shape for the array.

Q6: How do you create a random integer array in a specific range using NumPy?
Ans: You can use `np.random.randint(low, high, size)` to generate a random array with integers between `low` and `high`, and `size` defines the shape of the array.

Q7: What does the function `np.arange(start, stop, step)` do?
Ans: It generates an array of values from `start` to `stop` (exclusive) with a step size of `step`.

Q8: What is array broadcasting in NumPy?
Ans: Array broadcasting allows NumPy to perform element-wise operations on arrays of different shapes by automatically expanding the smaller array to match the shape of the larger array.

Q9: How do you generate a zero matrix of size 4x4 in NumPy?
Ans: A zero matrix of size 4x4 can be generated using `np.zeros((4, 4))`.

Q10: What is the difference between `np.arange()` and `np.linspace()`?
Ans: `np.arange()` generates values with a specified step size, while `np.linspace()` generates evenly spaced values over a specified range and includes the endpoint.

1.2.3 Tensors in NumPy

A tensor is a generalized concept of matrices and vectors. In mathematical terms, tensors are multi-dimensional arrays, and their dimensionality (or rank) is what differentiates them from simpler structures like scalars (rank 0), vectors (rank 1), and matrices (rank 2). A tensor with three dimensions or more is often referred to as a higher-order tensor.

In practical terms, tensors can be seen as multi-dimensional arrays where each element is addressed by multiple indices. Tensors play a significant role in machine learning and deep learning frameworks, where operations on these multi-dimensional data structures are common.

1.2.3.1 Types of Tensors:

1. *Scalar (0-D Tensor)*: A single number.

Example: 5 Rank: 0 Shape: ()

2. *Vector (1-D Tensor)*: An array of numbers.

Example: [1, 2, 3] Rank: 1 Shape: (3)

3. *Matrix (2-D Tensor)*: A 2D array (rows and columns).

Example: [[1, 2, 3], [4, 5, 6]] Rank: 2 Shape: (2, 3)

4. *3-D Tensor*: An array of matrices.

Example: [[[1, 2, 3], [4, 5, 6]], [[7, 8, 9], [10, 11, 12]]] Rank: 3 Shape: (2, 2, 3)

5. *N-D Tensor*: A tensor with N dimensions, where $N > 3$.

Example: A 4-D tensor could represent data with shape `(n_samples, n_channels, height, width)` in image processing.

1.2.3.2 Creating Tensors Using NumPy

In `NumPy`, tensors are represented as multi-dimensional arrays. You can create tensors in a way similar to how you create arrays, but you extend the dimensions to represent higher-order tensors.

Creating a 1D Tensor (Vector)

A 1D tensor is simply a vector. You can create one using `np.array()`:

```
import numpy as np
vector = np.array([1, 2, 3])
print(vector)
```

```
[1 2 3]
```

Creating a 2D Tensor (Matrix)

A 2D tensor is a matrix:

```
matrix = np.array([[1, 2, 3], [4, 5, 6]])
print(matrix)
```

```
[[1 2 3]
 [4 5 6]]
```

Creating a 3D Tensor

To create a 3D tensor (a stack of matrices):

```
tensor_3d = np.array([[[1, 2, 3], [4, 5, 6]], [[7, 8, 9], [10, 11, 12]]])
print(tensor_3d)
```

```
[[[ 1  2  3]
  [ 4  5  6]]

 [[ 7  8  9]
  [10 11 12]]]
```

Creating a 4D Tensor

In applications like deep learning, a 4D tensor is often used to represent a batch of images, where the dimensions could be (batch_size, channels, height, width):

```
tensor_4d = np.random.randint(10, size=(2, 3, 4, 5))  # 2 batches, 3
 ↪  channels, 4x5 images
print(tensor_4d)
```

```
[[[[6 3 6 1 9]
   [7 3 9 4 1]
   [2 7 1 3 8]
   [1 6 8 9 4]]

  [[2 2 9 2 7]
   [5 4 7 2 2]
   [1 1 3 0 7]
   [9 5 6 2 1]]

  [[3 9 6 2 8]
   [0 3 5 1 5]
   [0 4 5 5 7]
   [0 3 6 0 3]]]


 [[[2 4 6 0 0]
   [3 6 8 9 5]
   [2 4 9 9 7]
   [0 7 1 3 7]]

  [[9 5 2 5 8]
   [5 0 8 6 8]
   [4 6 4 7 8]
   [0 5 0 3 4]]

  [[7 5 5 9 2]
   [7 2 7 3 1]
   [4 9 2 4 3]
   [3 6 8 0 9]]]]
```

General Syntax for Creating Tensors Using NumPy

```
np.array(object, dtype=None, copy=True, order='K', subok=False, ndmin=0)
```

- object: An array-like object (nested lists) that you want to convert to a tensor.
- dtype: The desired data type for the tensor elements.
- copy: Whether to copy the data (default True).
- order: Row-major (C) or column-major (F) order.
- ndmin: Specifies the minimum number of dimensions for the tensor.

In the next section we will discuss the various attributes of the `NumPy` array.

1.2.3.3 Attributes of arrays

Each array has attributes `ndim` (the number of dimensions), `shape` (the size of each dimension), and `size` (the total size of the array):

To illustrate this attributes, consider the following arrays:

```
#np.random.seed(0)  # seed for reproducibility

x1 = np.random.randint(10, size=6)  # One-dimensional array
x2 = np.random.randint(10, size=(3, 4))  # Two-dimensional array
x3 = np.random.randint(10, size=(3, 4, 5))  # Three-dimensional array
```

The array attributes of x_3 is shown below.

```
print("x3 ndim: ", x3.ndim)
print("x3 shape:", x3.shape)
print("x3 size: ", x3.size)
```

```
x3 ndim:  3
x3 shape: (3, 4, 5)
x3 size:  60
```

Another useful attribute are the `dtype` which return the data type of the array , `itemsize`, which lists the size (in bytes) of each array element, and `nbytes`, which lists the total size (in bytes) of the array:

```
print("dtype:", x3.dtype)
print("itemsize:", x3.itemsize, "bytes")
print("nbytes:", x3.nbytes, "bytes")
```

```
dtype: int32
itemsize: 4 bytes
nbytes: 240 bytes
```

1.2.4 Array Indexing: Accessing Single Elements

If you are familiar with Python's standard list indexing, indexing in `NumPy` will feel quite familiar. In a one-dimensional array, the i^{th} value (counting from zero) can be accessed by specifying the desired index in square brackets, just as with Python lists:

To demonstrate indexing, let us consider the one dimensional array:

```
x1=np.array([8, 5, 4, 7,4,1])
```

The fourth element of `x1` can be accessed as

```
print(x1[3])
```

```
7
```

Now the second element from the end of the the arrray `x1` can be accessed as:

```
print(x1[-2])
```

```
4
```

1.2.4.1 Acessing elements in multi-dimensional arrays

In a multi-dimensional array, items can be accessed using a comma-separated tuple of indices. An example is shown below.

```
x2=np.array([[3, 3, 9, 2],
       [5, 2, 3, 5],
       [7, 2, 7, 1]])
print(x2)# list the 2-D array
```

```
[[3 3 9 2]
 [5 2 3 5]
 [7 2 7 1]]
```

Now print the third element in the first row, we will use the following code.

```
x2[0, 2] ## access the element in first row and thrid column
```

```
np.int64(9)
```

```
x2[2, -1] ## access the element in the 3rd row and last column
```

```
np.int64(1)
```

1.2.4.2 Modification of array elements

Values can also be modified using any of the above index notation. An example is shown below.

```
x2[2, -1]=20 ## replace the 3rd row last column element of x2 by 20
print(x2)
```

```
[[ 3  3  9  2]
 [ 5  2  3  5]
 [ 7  2  7 20]]
```

> **Homogenity of data in `NumPy` arrays**
>
> Keep in mind that, unlike Python lists, NumPy arrays have a fixed type. This means, for example, that if you attempt to insert a floating-point value to an integer array, the value will be silently truncated. Don't be caught unaware by this behavior!

1.2.4.3 Array Slicing: Accessing Subarrays

Just as we can use square brackets to access individual array elements, we can also use them to access subarrays with the *slice* notation, marked by the colon (`:`) character. The NumPy slicing syntax follows that of the standard Python list; to access a slice of an array `x`, use this:

```
x[start:stop:step]
```

If any of these are unspecified, they default to the values `start=0`, `stop=`*`size of dimension`*, `step=1`.

We'll take a look at accessing sub-arrays in one dimension and in multiple dimensions.

1. One-dimensional subarrays

```
x = np.arange(0,10)
x
```

```
array([0, 1, 2, 3, 4, 5, 6, 7, 8, 9])
```

```
x[1:6]  # first five elements
```

```
array([1, 2, 3, 4, 5])
```

```
x[5:]  # elements after index 5
```

```
array([5, 6, 7, 8, 9])
```

```
x[4:7]  # middle sub-array
```

```
array([4, 5, 6])
```

```
x[::2]  # every other element with step 2 (alternate elements)
```

```
array([0, 2, 4, 6, 8])
```

2. Multi-dimensional subarrays (slicing)

Multi-dimensional slices work in the same way, with multiple slices separated by commas. For example:

```
# creating a two dimensional array
x2=np.array([[1,2,3],[3,4,5],[5,6,7]])
print(x2)
```

```
[[1 2 3]
 [3 4 5]
 [5 6 7]]
```

```
# selecting first 3 rows and first two columns from x2
print(x2[:3,:2])
```

```
[[1 2]
 [3 4]
 [5 6]]
```

```
print(x2[:3:2,:3:2]) # slice alternate elements in first three rows and first
 ↪  three columns
```

```
[[1 3]
 [5 7]]
```

> **Accessing array rows and columns**
>
> One commonly needed routine is accessing of single rows or columns of an array. This can be done by combining indexing and slicing, using an empty slice marked by a single colon (`:`)

For example *all the elements* in first column can be accessed as:

```
print(x2[:, 0])  # first column of x2
```

```
[1 3 5]
```

1.2.4.4 Creating copies of arrays

Despite the nice features of array views, it is sometimes useful to instead explicitly copy the data within an array or a subarray. This can be most easily done with the `copy()` method.

This concept can be illustrated through an example. Consider the array `x2` previously defined:

```
print(x2)
```

```
[[1 2 3]
 [3 4 5]
 [5 6 7]]
```

Now take a copy of a slice of `x2` as follows.

```
# create a copy of subarray and store it with the new name
x2_sub_copy = x2[:2, :2].copy()
print(x2_sub_copy)
```

```
[[1 2]
 [3 4]]
```

Now the changes happend in the copy will not affect the orginal array. For example, replace one element in the copy slice and check how it is refelected in both arrays.

```
x2_sub_copy[0, 0] = 42
print(x2_sub_copy)
```

```
[[42  2]
 [ 3  4]]
```

```
print(x2)
```

```
[[1 2 3]
 [3 4 5]
 [5 6 7]]
```

1.2.4.5 More on reshaping

Another useful type of operation is reshaping of arrays. The most flexible way of doing this is with the `reshape` method. There are various approaches in reshaping of arrays. For example, if you want to put the numbers 1 through 9 in a 3×3 grid, you can do the following:

```
np.arange(1, 10)
```

```
array([1, 2, 3, 4, 5, 6, 7, 8, 9])
```

```
grid = np.arange(1, 10).reshape((9, 1))
print(grid)
```

```
[[1]
 [2]
 [3]
 [4]
 [5]
 [6]
 [7]
 [8]
 [9]]
```

> **Note**
>
> Note that for this to work, the size of the initial array must match the size of the reshaped array. Where possible, the reshape method will use a no-copy view of the initial array, but with non-contiguous memory buffers this is not always the case.
> Another common reshaping pattern is the conversion of a one-dimensional array into a two-dimensional row or column matrix. This can be done with the reshape method, or more easily done by making use of the newaxis keyword within a slice operation:

More Examples

```
x = np.array([1, 2, 3])
print(x)
```

```
[1 2 3]
```

Now check the dimension of the array created.

```
x.shape
```

```
(3,)
```

Reshaping the array as a matrix.

```
# row vector via reshape
x1=x.reshape((1, 3))
x1.shape
```

```
(1, 3)
```

We can achieve the same using the `newaxis` function as shown below.

```
# row vector via newaxis
print(x[np.newaxis, :])
```

```
[[1 2 3]]
```

Some other similar operations are here.

```
# column vector via reshape
x.reshape((3, 1))
```

```
array([[1],
       [2],
       [3]])
```

```
# column vector via newaxis
x[:, np.newaxis]
```

```
array([[1],
       [2],
       [3]])
```

1.2.5 Array Concatenation and Splitting

All of the preceding routines worked on single arrays. It's also possible to combine multiple arrays into one, and to conversely split a single array into multiple arrays. We'll take a look at those operations here.

1.2.5.1 Concatenation of arrays

Concatenation, or joining of two arrays in NumPy, is primarily accomplished using the routines `np.concatenate`, `np.vstack`, and `np.hstack`. `np.concatenate` takes a tuple or list of arrays as its first argument, as we can see here:

```
x = np.array([1, 2, 3])
y = np.array([3, 2, 1])
np.concatenate([x, y])
```

```
array([1, 2, 3, 3, 2, 1])
```

Another example is shown here:

```
np.concatenate([y, y, y])
```

```
array([3, 2, 1, 3, 2, 1, 3, 2, 1])
```

It can also be used for two-dimensional arrays:

```
grid1 = np.array([[1, 2, 3],
                  [4, 5, 6]])
grid2=np.array([[5,5,5],[7,7,7]])
# concatenate along the first axis
nm=np.concatenate([grid1, grid2],axis=0)
nm.shape
print(nm)
```

```
[[1 2 3]
 [4 5 6]
 [5 5 5]
 [7 7 7]]
```

Row-wise concatenation is showm below.

```
# concatenate along the second axis (horrizontal) (zero-indexed)
np.concatenate([grid1, grid2], axis=1)
```

```
array([[1, 2, 3, 5, 5, 5],
       [4, 5, 6, 7, 7, 7]])
```

For working with arrays of mixed dimensions, it can be clearer to use the `np.vstack` (vertical stack) and `np.hstack` (horizontal stack) functions:

```
x = np.array([1, 2, 3])
grid = np.array([[9, 8, 7],
                 [6, 5, 4]])

# vertically stack the arrays
grid
```

```
array([[9, 8, 7],
       [6, 5, 4]])
```

Now the new vector `x` has the same number of columns of `grid`. So we can only vertically stack it `grid`. For this the numpy function `vstack` will be used as follows.

```
grid2=np.vstack([grid,x])
print(grid2)
```

```
[[9 8 7]
 [6 5 4]
 [1 2 3]]
```

Similarly the horrizontal stacking can be shown as follows.

```
# horizontally stack the arrays
y = np.array([[99],
              [99],[3]])
np.hstack([grid2, y])
```

```
array([[ 9,  8,  7, 99],
       [ 6,  5,  4, 99],
       [ 1,  2,  3,  3]])
```

1.2.5.2 Splitting of arrays

The opposite of concatenation is splitting, which is implemented by the functions `np.split`, `np.hsplit`, and `np.vsplit`. For each of these, we can pass a list of indices giving the split points:

Let's begin with one dimensional arrays. First we split this array at specified locations and save it into sub arrays.

```
x = [1, 2, 3, 99, 99, 3, 2, 1]
```

Now split the list into two sub lists at index 2

```
x1,x2=np.split(x,[2])
```

Now see the sub-arrays:

```
print("the first array is:", x1)
print("the second array is:", x2)
```

```
the first array is: [1 2]
the second array is: [ 3 99 99  3  2  1]
```

More sub arrays can be created by passing the splitting locations as a list as follows.

```
x1,x2,x3=np.split(x,[2,4])
print(x1,"\n",x2,'\n',x3)
```

```
[1 2]
 [ 3 99]
 [99  3  2  1]
```

> **Note**
>
> Notice that N split-points, leads to $N+1$ subarrays. The related functions `np.hsplit` and `np.vsplit` are similar:

Now use the `vsplit` and `hsplit` functions on multi dimensional arrays.

```
grid = np.arange(16).reshape((4, 4))
grid
```

```
array([[ 0,  1,  2,  3],
       [ 4,  5,  6,  7],
       [ 8,  9, 10, 11],
       [12, 13, 14, 15]])
```

```
# vsplit
upper, lower = np.vsplit(grid, [2])
print(upper)
print(lower)
```

```
[[0 1 2 3]
 [4 5 6 7]]
[[ 8  9 10 11]
 [12 13 14 15]]
```

```
#hsplit
left, right = np.hsplit(grid, [2])
print("Left array:\n",left,"\n Right array:\n",right)
```

```
Left array:
 [[ 0  1]
 [ 4  5]
 [ 8  9]
 [12 13]]
 Right array:
 [[ 2  3]
 [ 6  7]
 [10 11]
 [14 15]]
```

1.2.6 Review Questions

Short Answer Questions (SAQ)

Q1: What is the main purpose of the NumPy library in Python?
Ans: The main purpose of NumPy is to provide support for large, multi-dimensional arrays

and matrices, along with a collection of mathematical functions to perform operations on these arrays efficiently.

Q2: How can a 1D array be created in NumPy?
Ans: A 1D array can be created using `np.array()` function, like:

```
np.array([1, 2, 3])
```

Q3: How do you access the shape of a NumPy array?

Ans: You can access the shape of a NumPy array using the .shape attribute. For example, array.shape gives the dimensions of the array.

Q4: What does the np.reshape() function do?

Ans: The np.reshape() function reshapes an array to a new shape without changing its data.

Q5: Explain the difference between vstack() and hstack() in NumPy.

Ans: vstack() vertically stacks arrays (along rows), while hstack() horizontally stacks arrays (along columns).

Q6: How does NumPy handle array slicing?

Ans: Array slicing in NumPy is done by specifying the start, stop, and step index like array[start:stop:step], which returns a portion of the array.

Q7: What is the difference between the np.zeros() and np.ones() functions?

Ans: np.zeros() creates an array filled with zeros, while np.ones() creates an array filled with ones.

Q8: What is array broadcasting in NumPy?

Ans: Broadcasting in NumPy allows arrays of different shapes to be used in arithmetic operations by stretching the smaller array to match the shape of the larger array.

Q9: How can you stack arrays along a new axis in NumPy? Ans: You can use np.stack() to join arrays along a new axis.

Q10: How do you generate a random integer array using NumPy?

Ans: You can generate a random integer array using np.random.randint(low, high, size).

Long Answer Questions (LAQ)

Q1: Explain how array slicing works in NumPy.

Ans: Array slicing in NumPy is a method to access or modify a subset of elements from a larger array. The syntax for slicing is array[start:stop:step], where:

start is the index from which slicing begins (inclusive), stop is the index where slicing ends (exclusive), step is the interval between indices to include in the slice. For example, in a 1D array, arr[1:5:2] will return every second element between the indices 1 and 4.

Q2: Discuss the difference between the .reshape() function and the .ravel() function in NumPy.

Ans: The .reshape() function changes the shape of an array without modifying its data, allowing a multi-dimensional array to be flattened or reshaped into any compatible shape. On the other hand, .ravel() returns a flattened 1D version of an array, but it tries to avoid copying the data by returning a flattened view where possible. If modifying the flattened array is necessary, ravel() returns a copy instead.

Q3: Explain how NumPy handles broadcasting during array operations.

Ans: Broadcasting in NumPy is a method to perform element-wise operations on arrays of different shapes. Smaller arrays are "broadcast" across the larger array by repeating their elements to match the shape of the larger array. For example, when adding a scalar to a 2D array, the scalar is added to each element of the array by broadcasting the scalar to match the array's shape. Similarly, operations between arrays of different shapes follow the broadcasting rules to make them compatible.

Q4: Describe how you would split an array in NumPy using the np.split() function. Provide an example.

Ans: The np.split() function in NumPy divides an array into multiple sub-arrays based on the indices provided. The syntax is: np.split(array, indices) Here, array is the array to be split, and indices is a list of indices where the split will occur. For example:

```
arr = np.array([1, 2, 3, 4, 5, 6])
np.split(arr, [2, 4])
```

This splits the array at indices 2 and 4, resulting in three sub-arrays: [1, 2], [3, 4], and [5, 6].

Q5: What are the key differences between np.hsplit() and np.vsplit()? Provide examples.

Ans: np.hsplit() horizontally splits an array along its columns, while np.vsplit() vertically splits an array along its rows. For example, if we have a 2D array:

```
arr = np.array([[1, 2, 3], [4, 5, 6]])
```

np.hsplit(arr, 3) splits the array into three columns, each with two rows: [[1], [4]], [[2], [5]], [[3], [6]]. np.vsplit(arr, 2) splits the array into two sub-arrays along rows: [[1, 2, 3]] and [[4, 5, 6]].

Q6: How can you create a 2D array with random integers between 1 and 10 using NumPy? Provide an example.

Ans: A 2D array with random integers between 1 and 10 can be created using np.random.randint(low, high, size). Example:

```
np.random.randint(1, 10, size=(3, 3))
```

This generates a 3x3 array with random integers between 1 and 9.

Q7: Describe how you would reshape an array from 1D to 2D in NumPy.

Ans: Reshaping an array from 1D to 2D in NumPy can be done using the .reshape() function. For example, given a 1D array:

```
arr = np.array([1, 2, 3, 4, 5, 6])
```

To reshape it into a 2D array with 2 rows and 3 columns:

```
arr.reshape(2, 3)
```

This results in `[[1, 2, 3], [4, 5, 6]]`.

Q8: Explain the concept of stacking arrays in NumPy using np.stack(). Provide an example.

Ans: np.stack() joins arrays along a new axis, unlike hstack() and vstack(), which concatenate along existing axes. For example:

```
arr1 = np.array([1, 2, 3])
arr2 = np.array([4, 5, 6])
np.stack((arr1, arr2), axis=0)
```

This stacks the arrays along a new axis, resulting in [[1, 2, 3], [4, 5, 6]].

Q9: How does NumPy's array_split() differ from split()? Provide an example.

Ans: The array_split() function allows unequal splitting of an array, whereas split() requires the splits to result in equal-sized sub-arrays. For example:

```
arr = np.array([1, 2, 3, 4, 5])
np.array_split(arr, 3)
```

This will split the array into three parts: [1, 2], [3, 4], and [5].

Q10: How would you flatten a multi-dimensional array into a 1D array in NumPy?

Ans: You can flatten a multi-dimensional array using the .ravel() or .flatten() methods. Example using ravel():

```
arr = np.array([[1, 2], [3, 4]])
arr.ravel()
```

This flattens the array into [1, 2, 3, 4].

Q11: Discuss the importance of NumPy in scientific computing and how it handles large datasets efficiently.

Ans: NumPy is crucial in scientific computing because it provides efficient storage and operations for large datasets through its n-dimensional array objects. It uses continuous memory blocks, making array operations faster than traditional Python lists, and supports a variety of mathematical functions and broadcasting, which simplifies computation.

NumPy operates efficiently by:

- Avoiding type checking at each operation due to its homogeneous data type constraint.
- Leveraging vectorization to reduce the need for explicit loops in operations.
- Providing optimized C and Fortran libraries for core computations.

Example of large dataset handling:

```
large_array = np.random.rand(1000000)
sum_large_array = np.sum(large_array)  # Efficient summation
```

This efficiency makes NumPy a foundation for data-driven scientific applications like machine learning, signal processing, and simulations.

Q12: What is the difference between a view and a copy in NumPy? Why does this matter in array operations?

Ans: A view is a reference to the original array, meaning changes in the view will affect the original array. A copy creates a new, independent array.

Example:

```
arr = np.array([1, 2, 3])
view = arr[:2]  # Creates a view
copy = arr[:2].copy()  # Creates a copy
view[0] = 99  # This will change arr
```

Views are more memory-efficient, but changes to them affect the original data, whereas copies do not.

Q13: How are higher-dimensional arrays handled in NumPy, and how can they be reshaped and indexed? Provide a practical example.

Ans: Higher-dimensional arrays (tensors) in NumPy can be created and manipulated like 1D and 2D arrays. You can reshape tensors using reshape() and index them similarly, using one index for each dimension.

```
tensor = np.arange(24).reshape(2, 3, 4)  # 3D tensor with shape (2, 3, 4)
element = tensor[1, 2, 3]  # Access element at specified indices
```

we can reshape tensors:

```
reshaped_tensor = tensor.reshape(4, 6)
```

1.3 Some important `NumPy` function for Linear Algrbra

Let's start with some basic matrix operations. Suppose we have two matrices, A and B, and we want to add them together. In NumPy, we can do this with the simple command, `A+B`. Let's look into the detailed computational steps.

```
# Addition and Subtraction
matrix1 = np.array([[1, 2], [3, 4]])
matrix2 = np.array([[5, 6], [7, 8]])
print(matrix1 + matrix2) # prints [[6, 8], [10, 12]]
```

```
[[ 6  8]
 [10 12]]
```

Similary, matrix difference and other matrix operations can be illustrated as follows.

```
print(matrix1 - matrix2) # prints [[-4, -4], [-4, -4]]
```

```
[[-4 -4]
 [-4 -4]]
```

```
# Scalar Multiplication
matrix = np.array([[1, 2], [3, 4]])
print(2 * matrix) # prints [[2, 4], [6, 8]]
```

```
[[2 4]
 [6 8]]
```

```
# Matrix Multiplication
matrix1 = np.array([[1, 2], [3, 4]])
matrix2 = np.array([[5, 6], [7, 8]])
print(np.dot(matrix1, matrix2)) # prints [[19, 22], [43, 50]]
```

```
[[19 22]
 [43 50]]
```

```
# Matrix Hadamards product
matrix1 = np.array([[1, 2], [3, 4]])
matrix2 = np.array([[5, 6], [7, 8]])
print(matrix1*matrix2)
```

```
[[ 5 12]
 [21 32]]
```

```
# Transpose
matrix = np.array([[1, 2, 3], [4, 5, 6]])
print(np.transpose(matrix)) # prints [[1, 4], [2, 5], [3, 6]]
```

```
[[1 4]
 [2 5]
 [3 6]]
```

```
# inverse of a matrix
a = np.array([[1, 2], [3, 4]])
a_inv = np.linalg.inv(a)
```

Next, let's talk about vectors. A vector is simply a matrix with one column. They're often used to represent things like forces or velocities in physics. In NumPy, we can represent vectors as arrays with one dimension.

```
vector = np.array([1, 2, 3])
print(vector) # prints [1, 2, 3]
```

```
[1 2 3]
```

Let's say we have two vectors, $\vec{u}$ and $\vec{v}$, and we want to compute their dot product (i.e., the sum of the products of their corresponding entries). We can do this with the command:

```
vector1 = np.array([1, 2, 3])
vector2 = np.array([4, 5, 6])
print(np.dot(vector1, vector2)) # prints 32
```

```
32
```

Like that there are some other operations too.

```
# Cross Product
vector1 = np.array([1, 2, 3])
vector2 = np.array([4, 5, 6])
print(np.cross(vector1, vector2)) # prints [-3, 6, -3]
```

```
[-3  6 -3]
```

> **Note**
>
> *Norm:* The norm of a vector is a scalar that represents the "length" of the vector. In NumPy, we can compute the norm using the `numpy.linalg.norm` function. The inner product of two vectors is a matrix that is computed by multiplying the first vector by the transpose of the second vector. In NumPy, we can compute the inner product using the numpy.inner function.

```
# finding norm
vector = np.array([1, 2, 3])
print(np.linalg.norm(vector)) # prints 3.74165738677
```

```
3.7416573867739413
```

```
#finding inner product
vector1 = np.array([1, 2, 3])
vector2 = np.array([4, 5, 6])
print(np.inner(vector1, vector2)) # prints 32
```

```
32
```

To handle higher dimensional mutrix multiplication, one can use `matmul()` function. The `np.matmul()` function is another way to perform matrix multiplication. Unlike `np.dot()`, it handles higher-dimensional arrays correctly by broadcasting. The syntax for this operation is `np.matmul(a, b)`.

```
A = np.array([[1, 0], [0, 1]])
B = np.array([[4, 1], [2, 2]])

# Matrix multiplication using matmul
result = np.matmul(A, B)
print(result)
```

```
[[4 1]
 [2 2]]
```

The function `np.linalg.inv()` computes the inverse of a square matrix. Syntax for this function is `np.linalg.inv(a)`. An example is shown below.

```
A = np.array([[1, 2], [3, 4]])

# Compute inverse
inv_A = np.linalg.inv(A)
print(inv_A)
```

```
[[-2.   1. ]
 [ 1.5 -0.5]]
```

The `np.linalg.det()` function computes the determinant of a square matrix. The determinant is useful for solving linear systems and understanding matrix properties. The syntax is `np.linalg.det(a)`.

```
A = np.array([[1, 2], [3, 4]])

# Compute determinant
det_A = np.linalg.det(A)
print(det_A)
```

```
-2.0000000000000004
```

The `np.linalg.solve()` function solves a linear matrix equation or system of linear scalar equations. It finds the vector x that satisfies $Ax = b$. Syntax for this function is `np.linalg.solve(A, b)`.

```
A = np.array([[3, 1], [1, 2]])
b = np.array([9, 8])

# Solve system of equations
x = np.linalg.solve(A, b)
print(x)
```

```
[2. 3.]
```

This function computes the QR decomposition of a matrix. QR decomposition is used to solve linear systems, least squares problems, and compute eigenvalues. Syntax for this function is `np.linalg.qr()`.

```
A = np.array([[1, 2, 3], [4, 5, 6], [7, 8, 9]])

# QR decomposition
Q, R = np.linalg.qr(A)
print("Q:", Q)
print("R:", R)
```

```
Q: [[-0.12309149  0.90453403  0.40824829]
 [-0.49236596  0.30151134 -0.81649658]
 [-0.86164044 -0.30151134  0.40824829]]
R: [[-8.12403840e+00 -9.60113630e+00 -1.10782342e+01]
 [ 0.00000000e+00  9.04534034e-01  1.80906807e+00]
 [ 0.00000000e+00  0.00000000e+00 -8.88178420e-16]]
```

The `np.linalg.lstsq()` function solves a linear least-squares problem, which is useful in regression tasks. The syntax for the function is `np.linalg.lstsq(a, b, rcond=None)`.

```
A = np.array([[1, 1], [1, 2], [2, 2], [2, 3]])
b = np.array([6, 8, 9, 11])

# Least-squares solution
x, residuals, rank, s = np.linalg.lstsq(A, b, rcond=None)
print("Solution:", x)
```

```
Solution: [2.09090909 2.54545455]
```

The Kronecker product is a matrix operation used in various applications like tensor products and matrix calculus. Syntax for this function is `np.kron(a, b)`.

```
A = np.array([[1, 2], [3, 4]])
B = np.array([[0, 5], [6, 7]])

kronecker = np.kron(A, B)
print(kronecker)
```

```
[[ 0  5  0 10]
 [ 6  7 12 14]
 [ 0 15  0 20]
 [18 21 24 28]]
```

Cosine similarity is used to find the cosine of the angle between two vectors. The mathematical formula for this operation is $\cos\theta = \frac{\vec{a}\cdot\vec{b}}{|\vec{a}\cdot\vec{b}|}$. The python function to calculate the cosine similarity is shown below.

```
a = np.array([1, 2, 3])
b = np.array([4, 5, 6])

cosine_sim = np.dot(a, b) / (np.linalg.norm(a) * np.linalg.norm(b))
print(cosine_sim)
```

```
0.9746318461970762
```

1.3.1 Linear Regression using `NumPy`

The `np.polyfit()` function fits a polynomial of a specified degree to the data, making it useful for regression. Syntax for this function is `np.polyfit(x, y, deg)`. Where x Independent variable (input), y is the dependent variable (output) and `deg` degree of the fitting polynomial. A simple example is given below.

```
x = np.array([0, 1, 2, 3, 4])
y = np.array([1, 3, 5, 7, 9])

# Linear fit (degree 1)
coefficients = np.polyfit(x, y, 1)
print("Coefficients:", coefficients)
```

```
Coefficients: [2. 1.]
```

1.3.2 Some interesting handy matrix operations using numpy arrays

In matrix decomposition, we need the matrix representation, $A - \lambda I$. For any matrix, we can do this by just `A-lambda np.eye(3)`. This can be deomonstarted here.

```
A=np.array([[1,2,3],[3,4,5],[7,6,7]])
lamda=3
A-lamda*np.eye(3)
```

```
array([[-2.,  2.,  3.],
       [ 3.,  1.,  5.],
       [ 7.,  6.,  4.]])
```

Task: Create a matrix, A using `numpy` and find the covariance , $cov(A)$ using matrix operation.

```
# creating a random matrix
import numpy as np
A=np.arange(16).reshape(4,4)
A
```

```
array([[ 0,  1,  2,  3],
       [ 4,  5,  6,  7],
       [ 8,  9, 10, 11],
       [12, 13, 14, 15]])
```

Now find $A - \bar{A}$.

```
A_bar=np.mean(A,axis=0) # calculating column-wise sum
A_bar
```

```
array([6., 7., 8., 9.])
```

```
# calculating A-A bar with outer product opertation
print(A-np.outer(A_bar,np.ones(4)).T)
```

```
[[-6. -6. -6. -6.]
 [-2. -2. -2. -2.]
 [ 2.  2.  2.  2.]
 [ 6.  6.  6.  6.]]
```

> **Note**
>
> The same can be done using reshaping method
>
> ```
> #calculating A-A_bar using broadcasting
> X=A-np.mean(A,axis=0).reshape(1,4)
> print(X)
> ```

```
[[-6. -6. -6. -6.]
 [-2. -2. -2. -2.]
 [ 2.  2.  2.  2.]
 [ 6.  6.  6.  6.]]
```

Calculating the covariance.

```
#mannualy calculating covariance
CoV=(1/3)*np.dot(X.T,X)
CoV
```

```
array([[26.66666667, 26.66666667, 26.66666667, 26.66666667],
       [26.66666667, 26.66666667, 26.66666667, 26.66666667],
       [26.66666667, 26.66666667, 26.66666667, 26.66666667],
       [26.66666667, 26.66666667, 26.66666667, 26.66666667]])
```

We can verify the same using default function as follows.

```
#calculating covariance using numpy function
np.cov(A, rowvar=False)
```

```
array([[26.66666667, 26.66666667, 26.66666667, 26.66666667],
       [26.66666667, 26.66666667, 26.66666667, 26.66666667],
       [26.66666667, 26.66666667, 26.66666667, 26.66666667],
       [26.66666667, 26.66666667, 26.66666667, 26.66666667]])
```

Note

It is interesting to compare the two ways of flattening an array using `reshape()`.

```
#comparing two ways of flattening a matrix using numpy
A.reshape(-1)==A.reshape(16,)
```

```
array([ True,  True,  True,  True,  True,  True,  True,  True,  True,
        True,  True,  True,  True,  True,  True,  True])
```

1.4 Basics of `SciPy` Library for Computational Linear Algebra

Following the comprehensive exploration of the NumPy library, which forms the foundation of array operations and basic linear algebra computations, it is essential to expand into more advanced tools for scientific computing. The **SciPy** library builds upon NumPy, offering a vast collection of functions and utilities specifically designed for higher-level operations in scientific and technical computing. While NumPy provides efficient array handling and basic matrix operations, SciPy extends these capabilities by incorporating advanced functions for optimization, integration, interpolation, and linear algebra, among other tasks (Virtanen et al. 2020).

For computational linear algebra, SciPy provides specialized modules like `scipy.linalg`, which can handle everything from solving linear systems to eigenvalue decompositions and matrix factorizations. This transition from NumPy to SciPy enables the handling of more complex problems efficiently and allows users to leverage optimized algorithms for large-scale numerical computations. By integrating SciPy into the workflow, computations can be carried out more robustly, expanding on the basic linear algebra concepts introduced through NumPy with advanced techniques necessary for practical applications.

1.4.1 Basic Matrix operations

`SciPy` builds on the functionality of NumPy, offering more sophisticated and optimized algorithms, particularly suited for numerical computing tasks. While `NumPy` provides essential operations for linear algebra, `SciPy`'s `scipy.linalg` module extends these capabilities with more advanced functions. `SciPy` functions are often better optimized for large-scale systems, making them highly efficient for computational linear algebra applications.

As the first step to use `SciPy`, we need to import the (only) necessary submodules for our specific tasks. In our discussion, we consider only linear algebra. So we import the `linalg` submodule as follows

```
#import scipy
import numpy as np # for matrix definition
from scipy import linalg
#print(scipy.__version__) # check version
```

Now let's discuss various `SciPy` functions for linear algebra with examples.

1.4.1.1 Computing the Determinant

The determinant is a scalar value that can be computed from the elements of a square matrix and is often used to determine whether a system of linear equations has a unique solution.

Syntax: `scipy.linalg.det(A)`

```
# example
A = np.array([[1, 2], [3, 4]])
det_A = linalg.det(A)
print(det_A)
```

```
-2.0
```

> **Note**
>
> Similar functionality is provided by `np.linalg.det()`. However, `scipy.linalg.det()` is often preferred when working with very large matrices due to the efficiency of SciPy's backend implementations.

1.4.1.2 Solving Linear Systems of Equations

One of the fundamental tasks in linear algebra is solving a system of linear equations of the form $AX = b$, where A is a matrix and b is a vector or matrix of known values. >*Syntax:* `scipy.linalg.solve(A, b)`

```
#example
A = np.array([[3, 1], [1, 2]])
b = np.array([9, 8])
x = linalg.solve(A, b)
print(x)
```

```
[2. 3.]
```

> **Note**
>
> NumPy's `np.linalg.solve()` also provides this functionality, but SciPy's version is better suited for larger and more complex matrices because it uses more efficient algorithms for decomposing the matrix.

1.4.1.3 Matrix Inversion

Matrix inversion is a critical operation in many linear algebra problems, particularly in solving systems of linear equations. >*Syntax:* `scipy.linalg.inv(A)`

```
A = np.array([[1, 2], [3, 4]])
inv_A = linalg.inv(A)
print(inv_A)
```

```
[[-2.   1. ]
 [ 1.5 -0.5]]
```

1.4.1.4 Kronecker Product

The Kronecker product is used in various applications, including constructing block matrices and expanding the dimensionality of matrices.

Syntax: `scipy.linalg.kron(A, B)`

```
A = np.array([[1, 2], [3, 4]])
B = np.array([[0, 5], [6, 7]])
kron_product = linalg.kron(A, B)
print(kron_product)
```

```
[[ 0  5  0 10]
 [ 6  7 12 14]
 [ 0 15  0 20]
 [18 21 24 28]]
```

1.4.1.5 Eigenvalues and Eigenvectors

Eigenvalues and eigenvectors are fundamental in many areas of linear algebra, including solving systems of differential equations and performing dimensionality reduction in machine learning.

Syntax: `scipy.linalg.eig(A)`

```
A = np.array([[3, 2], [4, 1]])
eigenvalues, eigenvectors = linalg.eig(A)
print("Eigenvalues:", eigenvalues)
print("Eigenvectors:", eigenvectors)
```

```
Eigenvalues: [ 5.+0.j -1.+0.j]
Eigenvectors: [[ 0.70710678 -0.4472136 ]
 [ 0.70710678  0.89442719]]
```

1.5 Sparse Matrices

Sparse matrices are often useful in numerical simulations dealing with large systems, if the problem can be described in matrix form where the matrices or vectors mostly contains zeros. Scipy has a good support for sparse matrices, with basic linear algebra operations (such as equation solving, eigenvalue calculations, etc.).

There are many possible strategies for storing sparse matrices in an efficient way. Some of the most common are the so-called coordinate form (COO), list of list (LIL) form, and compressed-sparse column CSC (and row, CSR). Each format has some advantages and disadvantages. Most computational algorithms (equation solving, matrix-matrix multiplication, etc.) can be efficiently implemented using CSR or CSC formats, but they are not so intuitive and not so easy to initialize. So often a sparse matrix is initially created in COO or LIL format (where we can efficiently add elements to the sparse matrix data), and then converted to CSC or CSR before used in real calculations.

For more information about these sparse formats, see e.g. http://en.wikipedia.org/wiki/Sparse_matrix

1.5.1 Sparse Matrix operations in `SciPy`

Sparse matrices are a key feature of `SciPy`, providing an efficient way to store and manipulate large matrices with a significant number of zero elements. `SciPy` offers a variety of sparse matrix formats and supports operations like matrix multiplication, addition, transposition, and solving systems of equations.

Here is a guide to working with sparse matrices in `SciPy`.

Types of Sparse Matrices in SciPy

`SciPy` provides different types of sparse matrices depending on the use case:

1. **CSR (Compressed Sparse Row) Matrix**: Efficient for row slicing and matrix-vector products.
2. **CSC (Compressed Sparse Column) Matrix**: Efficient for column slicing and fast arithmetic operations.
3. **COO (Coordinate) Matrix**: Suitable for constructing sparse matrices by specifying individual entries.
4. **DIA (Diagonal) Matrix**: For matrices where non-zero elements are primarily on the diagonals.
5. **LIL (List of Lists) Matrix**: Good for constructing matrices incrementally.

1.5.1.1 Importing Sparse Matrices

```
from scipy.sparse import csr_matrix, csc_matrix, coo_matrix
import numpy as np
```

Creating Sparse Matrices

```
# Create a dense matrix
dense_matrix = np.array([[0, 0, 3], [4, 0, 0], [0, 5, 6]])

# Convert dense matrix to CSR format
csr = csr_matrix(dense_matrix)

# Display CSR matrix
print(csr)
```

```
<Compressed Sparse Row sparse matrix of dtype 'int64'
    with 4 stored elements and shape (3, 3)>
  Coords    Values
  (0, 2)    3
  (1, 0)    4
  (2, 1)    5
  (2, 2)    6
```

```
# creating sparse matrix in COO format
# Define row indices, column indices, and values
row = np.array([0, 1, 2, 2])
col = np.array([2, 0, 1, 2])
data = np.array([3, 4, 5, 6])
```

```
# Create COO sparse matrix
coo = coo_matrix((data, (row, col)), shape=(3, 3))

print(coo)
```

```
<COOrdinate sparse matrix of dtype 'int64'
    with 4 stored elements and shape (3, 3)>
  Coords    Values
  (0, 2)    3
  (1, 0)    4
  (2, 1)    5
  (2, 2)    6
```

Basic Operations with Sparse Matrices

The basic matrix operations can be performed on the sparse matrix too. The difference is that in the case of sparse matices, the respective operations will be done only on non-zero enties. Now look into the basic matrix operations thorugh following examples.

Matrix Multiplication:

```
A = csr_matrix([[1, 0, 0], [0, 0, 1], [0, 2, 0]])
B = csr_matrix([[4, 5], [0, 0], [7, 8]])

# Matrix multiplication (dot product)
result = A.dot(B)
print(result.toarray())  # Convert to dense array for display
```

```
[[4 5]
 [7 8]
 [0 0]]
```

Transposition:

```
# Transpose the matrix
transposed = A.transpose()

print(transposed.toarray())
```

```
[[1 0 0]
 [0 0 2]
 [0 1 0]]
```

Addition:

```
# Adding two sparse matrices
C = csr_matrix([[0, 1, 2], [3, 0, 0], [0, 0, 5]])
D = csr_matrix([[0, 1, 0], [0, 0, 0], [2, 0, 5]])

sum_matrix = C + D

print(sum_matrix.toarray())
```

```
[[ 0  2  2]
 [ 3  0  0]
 [ 2  0 10]]
```

Solving Sparse Linear Systems

We can solve systems of linear equations using sparse matrices with the `spsolve()` function:

```
from scipy.sparse.linalg import spsolve

# Create a sparse matrix (A) and a dense vector (b)
A = csr_matrix([[3, 1, 0], [1, 2, 0], [0, 0, 1]])
b = np.array([5, 5, 1])

# Solve the system Ax = b
x = spsolve(A, b)

print("Solution x:", x)
```

```
Solution x: [1. 2. 1.]
```

1.5.1.2 Conversion from one sparse matrix system to another

We can convert between different sparse matrix formats using the `.tocsc()`,`.tocoo()`, `.todia()`, and similar methods:

```
# Convert CSR to COO format
coo = A.tocoo()
print(coo)
print("The matrix is :\n",coo.toarray())
```

```
<COOrdinate sparse matrix of dtype 'int64'
    with 5 stored elements and shape (3, 3)>
  Coords    Values
  (0, 0)    3
  (0, 1)    1
  (1, 0)    1
  (1, 1)    2
  (2, 2)    1
The matrix is :
 [[3 1 0]
 [1 2 0]
 [0 0 1]]
```

Summary of sparse matrix operations

Function	Description
`csr_matrix()`	Compressed Sparse Row matrix.
`csc_matrix()`	Compressed Sparse Column matrix.
`coo_matrix()`	Coordinate format matrix.
`spsolve()`	Solves sparse linear systems.
`spdiags()`	Extracts or constructs diagonal sparse matrices.
`lil_matrix()`	List of lists sparse matrix.

1.6 Visualization Libraries

References

Harris, Charles R., K. Jarrod Millman, Stéfan J. van der Walt, Ralf Gommers, Pauli Virtanen, David Cournapeau, Eric Wieser, et al. 2020. "Array Programming with NumPy." *Nature* 585 (7825): 357–62. https://doi.org/10.1038/s41586-020-2649-2.

Strang, Gilbert. 2020. *Linear Algebra for Everyone*. SIAM.

Virtanen, Pauli, Ralf Gommers, Travis E. Oliphant, Matt Haberland, Tyler Reddy, David Cournapeau, Evgeni Burovski, et al. 2020. "SciPy 1.0: Fundamental Algorithms for Scientific Computing in Python." *Nature Methods* 17: 261–72. https://doi.org/10.1038/s41592-019-0686-2.