Assignment 70: Basis Set And Linear Equations

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Introduction

In this discussion, we will bridge the gap between coordinate geometry and linear algebra by exploring how geometric objects such as lines and planes can be represented algebraically. This journey begins with familiar notions from high school geometry, such as lines and planes in 3D, and leads us into the powerful abstractions provided by linear algebra, such as vector spaces, subspaces, and matrices.

We aim to answer the question: *Which is better for representing subspaces—basis sets or linear equations?* We will use examples from both geometry and algebra to build a unified understanding.

Coordinate Geometry: Lines, Planes, and Directional Cosines

In three-dimensional space, a line can be represented parametrically using a point $P_0(x_0, y_0, z_0)$ and a direction vector $\mathbf{d} = (a, b, c)$. The parametric equations for the line can be written as:

$$x = x_0 + at,$$
$$y = y_0 + bt,$$
$$z = z_0 + ct,$$

where t is a parameter.

Alternatively, we can express a line using directional cosines, which are the cosines of the angles that the line makes with the coordinate axes. If the direction cosines are denoted as l, m, n, where:

$$l = \cos(\alpha), \quad m = \cos(\beta), \quad n = \cos(\gamma),$$

the equations can be expressed as:

$$x = x_0 + k \cdot l,$$
$$y = y_0 + k \cdot m,$$
$$z = z_0 + k \cdot n,$$

where k is a scalar. This leads to the relationship:

$$\frac{x - x_0}{l} = \frac{y - y_0}{m} = \frac{z - z_0}{n},$$

which can be simplified into a set of two independent simultaneous linear equations, demonstrating a deeper connection between the geometric representation and linear algebraic formulation.

Planes in Three-Dimensional Space

The equation of a plane can be defined in multiple ways, but one common representation is through the normal vector and a point on the plane. If we have a point $P_0(x_0, y_0, z_0)$ on the plane and a normal vector $\mathbf{n} = (a, b, c)$, the equation of the plane can be expressed as:

$$a(x - x_0) + b(y - y_0) + c(z - z_0) = 0.$$

Interestingly, if we are given three non-collinear points $P_1(x_1, y_1, z_1)$, $P_2(x_2, y_2, z_2)$, and $P_3(x_3, y_3, z_3)$, we can also find the equation of the plane. The vectors $\mathbf{v_1} = P_2 - P_1$ and $\mathbf{v_2} = P_3 - P_1$ are formed, and their cross product $\mathbf{n} = \mathbf{v_1} \times \mathbf{v_2}$ gives us the normal vector to the plane. Using the normal vector and one of the points, we can derive the plane equation.

Linear Algebra: Vectors, Subspaces, and Basis

In linear algebra, we discuss concepts such as vector spaces, linear combinations, subspaces, and bases. A vector space is a collection of vectors that can be added together and multiplied by scalars. A subspace is a subset of a vector space that is itself a vector space under the same operations.

A basis of a vector space is a set of vectors that are linearly independent and span the space. For example, in $\mathbb{R}^3$, the standard basis consists of the vectors $\mathbf{e_1} = (1, 0, 0)$, $\mathbf{e_2} = (0, 1, 0)$, and $\mathbf{e_3} = (0, 0, 1)$.

The relationship between linear equations and subspaces becomes evident when considering the null space of a matrix. The null space consists of all vectors $\mathbf{x}$ such that $A\mathbf{x} = 0$, representing a subspace of the vector space defined by A.

Eigenvalues and Eigenvectors

Eigenvalues and eigenvectors play a significant role in linear algebra. Given a square matrix A, an eigenvalue λ is defined by the equation:

$$(A - \lambda I)\mathbf{x} = 0,$$

where I is the identity matrix and $\mathbf{x}$ is the corresponding eigenvector.

A major observation in this context is that solving $(A - \lambda I)\mathbf{x} = 0$ results in finding a vector $\mathbf{x}$ that is orthogonal to the transformation defined by $A - \lambda I$. Furthermore, since $A - \lambda I$ contains only two independent rows, the eigenvector $\mathbf{x}$ can be interpreted geometrically as the cross product of the independent vectors of $A - \lambda I$ in 3D space.

Direction Cosines and Simultaneous Equations

Using the concept of directional cosines, we can represent a line in 3D. By simplifying this representation, we can observe that it reduces to a system of two independent simultaneous linear equations. This further emphasizes the connection between geometric concepts and algebraic formulations.

A Unified View of Coordinate Geometry and Linear Algebra

By synthesizing concepts from coordinate geometry and linear algebra, we find a rich interplay between algebraic equations and geometric interpretations. We observe how algebraic manipulations of vectors, represented in matrices, connect with geometric representations in $\mathbb{R}^3$.

This unification reveals deeper mathematical structures, particularly the concept of matrices as versatile tools for data representation, operations, and abstract computations. Matrices allow us to encapsulate geometric transformations, linear relationships, and eigenvalue problems in a single mathematical framework. Their compatibility with computation and analysis enhances our ability to model, analyze, and solve real-world problems in multi-dimensional spaces.

The abstract nature of matrices helps bridge various mathematical concepts, leading to innovative applications across disciplines.

Tasks

1. **Find the equation of the plane passing through three points:** $(1, 2, 0)$, $(3, 4, 5)$, $(6, 7, 8)$**.**

Solution:
To find the equation of the plane passing through the points $P_1(1, 2, 0)$, $P_2(3, 4, 5)$, and $P_3(6, 7, 8)$, we start by forming two vectors from these points:

$$\mathbf{v_1} = P_2 - P_1 = (3 - 1, 4 - 2, 5 - 0) = (2, 2, 5)$$

$$\mathbf{v_2} = P_3 - P_1 = (6 - 1, 7 - 2, 8 - 0) = (5, 5, 8)$$

Next, we find the normal vector $\mathbf{n}$ to the plane by taking the cross product of $\mathbf{v_1}$ and $\mathbf{v_2}$:

$$\mathbf{n} = \mathbf{v_1} \times \mathbf{v_2} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ 2 & 2 & 5 \\ 5 & 5 & 8 \end{vmatrix}$$

Calculating the determinant, we have:

$$\mathbf{n} = \hat{i}(2 \cdot 8 - 5 \cdot 5) - \hat{j}(2 \cdot 8 - 5 \cdot 5) + \hat{k}(2 \cdot 5 - 2 \cdot 5)$$

This simplifies to:

$$\mathbf{n} = \hat{i}(-9) + \hat{j}(9) + \hat{k}(0) = (-9, 9, 0)$$

The equation of the plane can be expressed as:

$$ax + by + cz = d$$

where $\mathbf{n} = (a, b, c) = (-9, 9, 0)$. Using the point $P_1(1, 2, 0)$ to find d:

$$-9(1) + 9(2) + 0(0) = d$$

This gives:

$$-9 + 18 = d \implies d = 9$$

Thus, the equation of the plane is:

$$-9x + 9y = 9 \quad \text{or} \quad x - y = -1.$$

2. **Find the directional cosines of the line passing through points** $(1, 1, 1)$ **and** $(4, 5, 6)$**.**

Solution:

The direction vector of the line is:

$$\mathbf{d} = (4 - 1, 5 - 1, 6 - 1) = (3, 4, 5).$$

The magnitude of the direction vector is:

$$|\mathbf{d}| = \sqrt{3^2 + 4^2 + 5^2} = \sqrt{9 + 16 + 25} = \sqrt{50}.$$

The directional cosines are:

$$l = \frac{3}{\sqrt{50}}, \quad m = \frac{4}{\sqrt{50}}, \quad n = \frac{5}{\sqrt{50}}.$$

Thus, the directional cosines are $l = \frac{3}{\sqrt{50}}$, $m = \frac{4}{\sqrt{50}}$, and $n = \frac{5}{\sqrt{50}}$.