Strain - Displacement Relation from Cylindrical Spherical Coordinate System

The relation between cylindrical and spherical coordinate is:

$$r = \rho \sin\phi \quad , \quad z = \rho \cos\phi \quad , \quad \theta = \theta$$

Where $\rho = \sqrt{r^2 + z^2}$, $\theta = \tan^{-1}(y/x)$, $\phi = \arccos\left(\frac{z}{\rho}\right)$

The partial derivatives for the above equations are

$$\frac{\partial}{\partial r} = \frac{\partial \rho}{\partial r} \cdot \frac{\partial}{\partial \rho} + \frac{\partial \phi}{\partial r} \cdot \frac{\partial}{\partial \phi}$$

$$= \sin\phi \frac{\partial}{\partial \rho} + \frac{r^2}{\sqrt{r^2 - z^2} \cdot \rho^{3/2}} \cdot \frac{\partial}{\partial \phi}$$

$$\frac{\partial}{\partial z} = \frac{\partial \rho}{\partial z} \cdot \frac{\partial}{\partial \rho} + \frac{\partial \phi}{\partial z} \cdot \frac{\partial}{\partial \phi}$$

$$= \cos\phi \frac{\partial}{\partial \rho} + \frac{rz}{\sqrt{r^2 - z^2} \cdot \rho^{3/2}} \cdot \frac{\partial}{\partial \phi}$$

Now

$$u_r = u_\rho \sin\phi + u_\phi \frac{r^2}{\sqrt{r^2 - z^2} \cdot \rho^{3/2}} \quad , \quad u_z = u_\rho \cos\phi + u_\phi \frac{rz}{\sqrt{r^2 - z^2} \cdot \rho^{3/2}} \quad ; u_\theta = u_\theta$$

Calculating $e_\rho = \frac{\partial u_r}{\partial r}$

$$\hat{e}_\rho = \sin\phi\left[\frac{\partial}{\partial \rho}\left(u_\rho \sin\phi + u_\phi \frac{r^2}{\sqrt{r^2 - z^2} \cdot \rho^{3/2}}\right)\right] + \frac{r^2}{\sqrt{r^2 - z^2} \cdot \rho^{3/2}} \cdot \frac{\partial}{\partial \phi}\left[u_\rho \sin\phi + u_\phi \frac{r^2}{\sqrt{r^2 - z^2} \cdot \rho^{3/2}}\right]$$

$$= \left[\frac{\partial u_\rho}{\partial \rho} \sin^2\phi + \frac{\partial u_\phi}{\partial \rho} \cdot \frac{r^2 \sin\phi}{\rho^{3/2}\sqrt{r^2 - z^2}} + \frac{u_\phi r^2}{\sqrt{r^2 - z^2}} \cdot \frac{\sin\phi}{\rho^{-5/2}} + \frac{\partial u_\rho}{\partial \phi} \cdot \frac{\sin\phi\, r^2}{\sqrt{r^2 - z^2} \cdot \rho^{3/2}} + \frac{r^2 u_\rho \cos\phi}{\sqrt{r^2 - z^2}} + \frac{\partial u_\phi}{\partial \phi} \cdot \frac{r^4}{\rho^3 \cdot (r^3 - z^3)}\right]$$

$$\hat{e}_\rho = \frac{\partial u_\rho}{\partial \rho} \sin^2\phi + \left(\frac{\partial u_\phi}{\partial \rho} \cdot \frac{1}{\rho^{3/2}} + \frac{u_\phi}{\rho^{-5/2}} + \frac{\partial u_\rho}{\partial \phi} \cdot \frac{1}{\rho^{3/2}}\right) \frac{r^3 \cdot \sin\phi}{\sqrt{r^2 - z^2}}$$

$$+ \left(\left(u_\rho \cos\phi + \frac{\partial u_\phi}{\partial \phi} \cdot \frac{1}{\rho^3}\right) \cdot \frac{r^2}{\sqrt{r^2 - z^2}}\right) \frac{r^2}{\sqrt{r^2 - z^2}}$$

$$e_\rho = \frac{\partial u_z}{\partial z}$$

$$= \cos\phi \frac{\partial}{\partial \rho}\left[u_\rho \cos\phi + u_\phi \cdot \frac{rz}{\rho^{3/2}\cdot\sqrt{r^2-z^2}}\right] + \frac{rz}{\sqrt{r^2-z^2}}\rho^{1/2}\frac{\partial}{\partial \phi}\left[u_\rho \cos\phi + u_\phi \frac{rz}{\sqrt{r^2-z^2}\cdot\rho^{3/2}}\right]$$

$$= \frac{\partial u_\rho}{\partial \rho}\cos^2\phi + \frac{\partial u_\rho}{\partial \rho}\cdot\frac{rz\cos\phi}{\rho^{3/2}\cdot\sqrt{r^2-z^2}} + \frac{u_\phi\, rz}{\sqrt{r^2-z^2}}\cdot\frac{\cos\phi}{\rho^{-5/2}} + \frac{\partial u_\rho}{\partial \phi}\frac{\cos\phi\, rz}{\rho^{3/2}\cdot\sqrt{r^2-z^2}} - \frac{\sin\phi\cdot rz\, u_\rho}{\sqrt{r^2-z^2}\cdot\rho^{5/2}}$$

$$\partial\phi = \frac{\partial u_\rho}{\partial \rho}\cos^2\phi + \left(\frac{\partial u_\rho}{\partial \rho}\cdot\frac{1}{\rho^{3/2}} + \frac{u_\theta}{\rho^{5/2}} + \frac{\partial u_\rho}{\partial \phi}\cdot\frac{1}{\rho^{3/2}}\right)\frac{\cos\phi\cdot rz}{r^2-z^2} + \frac{\partial u_\phi}{\partial \phi}\cdot\frac{r^2z^2}{(r^2-z^2)\rho^3}$$

$$+\left(\frac{\partial u_\phi}{\partial \phi}\cdot\frac{rz}{\sqrt{r^2-z^2}}\rho^{1/2} - \frac{u_\rho \sin\phi}{\rho^{5/2}}\right)\frac{rz}{\sqrt{r^2-z^2}}$$

Therefore the strain-displacement relation becomes

$$e_\rho = \frac{\partial u_r}{\partial r}, \quad e_\phi = \frac{1}{\rho}\left(U_r + \frac{\partial U_z}{\partial \phi}\right)$$

$$e_\theta = \frac{1}{\rho \sin\phi}\left(\frac{\partial U_\theta}{\partial \theta} + \sin\phi\, U_r + \cos\phi\, U_z\right)$$

$$e_{\rho\phi} = \frac{1}{2}\left(\frac{1}{\rho}\frac{\partial U_r}{\partial \phi} + \frac{\partial U_z}{\partial \rho} - \frac{U_z}{\rho}\right)$$

$$e_{\phi\theta} = \frac{1}{2\rho}\left(\frac{1}{\sin\theta}\cdot\frac{\partial U_z}{\partial \theta} + \frac{\partial U_\theta}{\partial \theta} - \cot\phi\, U_\theta\right)$$

$$e_{\theta\rho} = \frac{1}{2}\left(\frac{1}{\rho\sin\phi}\cdot\frac{\partial U_r}{\partial \theta} + \frac{\partial U_\theta}{\partial \rho} - \frac{U_\theta}{\rho}\right)$$