1 **a)** Express $\cos^2 3x$ in terms of $\cos 6x$. [2]

$$\cos^2 x = \frac{1+\cos 2x}{2} \quad — 1$$

$$\therefore \cos^2(3x) = \boxed{\frac{1+\cos(6x)}{2}} \quad — 1$$

b) Hence find the exact value of $\int_0^{\frac{1}{18}\pi} \cos^2 3x \, dx$. [3]

$$\int_0^{\pi/18} \frac{1}{2} + \frac{\cos(6x)}{2} \, dx$$

$$\left[\frac{x}{2} + \frac{\sin(6x)}{6 \cdot 2}\right]_0^{\pi/18} \quad 1$$

$$\left(\frac{\pi}{18 \cdot 2} + \frac{\sin(\pi/3)}{12}\right) - \left(0 + \frac{\sin 0}{12}\right) \quad — 1$$

$$= \boxed{\frac{\pi}{36} + \frac{\sqrt{3}}{24}} \quad — 1$$

2) Using the substitution $u = \sqrt{x}$, find the exact value of

$$\int_3^\infty \frac{1}{(x+1)\sqrt{x}}\,dx.$$ [6]

$u = x^{1/2}$

$\frac{du}{dx} = \frac{1}{2}x^{-1/2}$

$\frac{du}{dx} = \frac{1}{2\sqrt{x}}$ 1

$\frac{du}{dx} = \frac{1}{2u}$

$dx = 2u\,du$

$\sqrt{x} = u$

$x = u^2$

when $x=3$

$u = \sqrt{3}$

when $x = \infty$

$u = \infty$

$$\int_3^\infty \frac{1}{(x+1)\sqrt{x}}\,dx$$

$$\int_{\sqrt{3}}^\infty \frac{1}{(u^2+1)\cancel{u}} \cdot 2\cancel{u}\,du$$ 1 subs

$$= \int_{\sqrt{3}}^a \frac{2}{1+u^2}\,du \to 1$$

$$= 2\left[\tan^{-1}(u)\right]_{\sqrt{3}}^a$$ 1

$$\Rightarrow 2\left[\tan^{-1}(a) - \tan^{-1}(\sqrt{3})\right]$$

As $a \to \infty$

$\tan^{-1}(a) \to \pi/2$

$$\Rightarrow 2\left[\frac{\pi}{2} - \frac{\pi}{3}\right]$$ 1 limits

$$= 2\left(\frac{\pi}{6}\right)$$

$$= \frac{\pi}{3}$$

$$\Rightarrow \int_3^\infty \frac{1}{(x+1)\sqrt{x}}\,dx = \frac{\pi}{3}$$ 1A.

3). Solve the equation $2\sin 2\theta + \cos 2\theta = 1$ for $0° \le \theta \le 360°$. [6]

$2\sin 2\theta + \cos 2\theta = 1 \qquad 0 \le \theta \le 360$

$2(2\sin\theta\cos\theta) + \cancel{\cos^2\theta} - \sin^2\theta = \sin^2\theta + \cancel{\cos^2\theta}$ 1 1

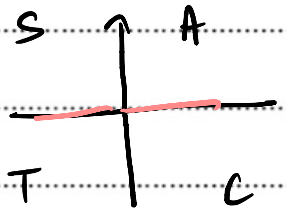
$\Rightarrow 4\sin\theta\cos\theta - 2\sin^2\theta = 0$

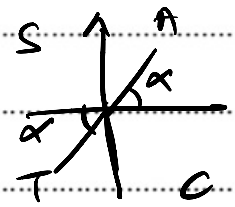
$\Rightarrow 2\sin\theta(2\cos\theta - \sin\theta) = 0$

$\sin\theta = 0 \qquad\qquad 2\cos\theta = \sin\theta$

1 $\alpha = 0° \qquad\qquad \tan\theta = 2$ 1

$\alpha = 63.435°$





$\theta = 0, 180°, 360°$ 1

$\theta = 63.4°, 243.4°$ 1

$\Rightarrow \theta = 0°, 63.4°, 180°, 243.4°, 360°$

4) The equation of a curve is $y = \sqrt{\tan x}$, for $0 \leqslant x < \frac{1}{2}\pi$.

(a) Express $\frac{dy}{dx}$ in terms of $\tan x$, and verify that $\frac{dy}{dx} = 1$ when $x = \frac{1}{4}\pi$. [4]

$$\frac{dy}{dx} = \frac{1}{2} \cdot (\tan x)^{-1/2} \cdot \sec^2 x \quad 1$$

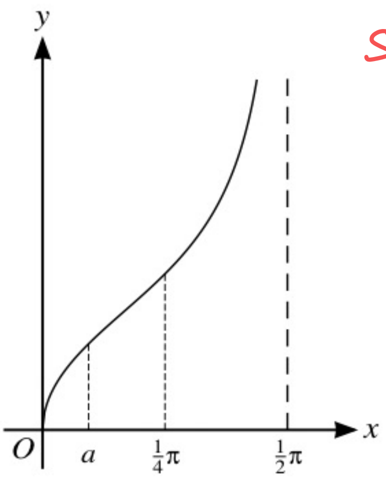
$$\frac{dy}{dx} = \frac{\sec^2 x}{2\sqrt{\tan x}} \quad 1$$

when $x = \pi/4$

$$\frac{dy}{dx} = \frac{2}{2\sqrt{1}} = \boxed{1} - 1$$

1

The value of $\frac{dy}{dx}$ is also 1 at another point on the curve where $x = a$, as shown in the diagram.



$\sec^2 x = 1 + \tan^2 x$

(b) Show that $t^3 + t^2 + 3t - 1 = 0$, where $t = \tan a$. [4]

$$\frac{\sec^2 a}{2\sqrt{\tan a}} = 1 \quad 1$$

$$1 + \tan^2 a = 2\sqrt{\tan a}$$

$$1 + t^2 = 2\sqrt{t}$$

$$(1+t^2)^2 = 4t$$

$$1 + 2t^2 + t^4 = 4t$$

$$t^4 + 2t^2 - 4t + 1 = 0 - 1$$

At $x = \frac{\pi}{4}$, $\frac{dy}{dx} = 1$ $\therefore$

$\tan(\pi/4) = t$

$t = 1 \quad \therefore \quad (t-1)$ is a factor

$$\begin{array}{r l}
 & t^3 + t^2 + 3t - 1 \\
t-1 \,\big) & t^4 + 2t^2 - 4t + 1 \\
 & -(t^4 - t^3) \\
 & t^3 + 2t^2 \\
 & -(t^3 - t^2) \\
 & 3t^2 - 4t + 1 \\
 & -(3t^2 - 3t) \\
 & -t + 1 \\
 & -(-t + 1) \\
 & 0
\end{array}$$

— 1 → proved.

1

(c) Use the iterative formula

$$a_{n+1} = \tan^{-1}\left(\tfrac{1}{3}(1 - \tan^2 a_n - \tan^3 a_n)\right)$$

to determine a correct to 2 decimal places, giving the result of each iteration to 4 decimal places. [3]

Use the iterative process correctly with the given formula at least once — 1

Obtain final answer $a = 0.29$ — 1

Show sufficient iterations to 4 d.p. to justify 0.29 to 2 d.p., or show there is a sign change in (0.285, 0.295)

* 1 for working

5) The variables x and y satisfy the differential equation

$$e^{2x}\frac{dy}{dx} = 4xy^2,$$

and it is given that $y = 1$ when $x = 0$.

Solve the differential equation, obtaining an expression for y in terms of x. [7]

$$e^{2x}\frac{dy}{dx} = 4xy^2$$

$$\frac{1}{y^2}\,dy = \frac{4x}{e^{2x}}\,dx \quad — 1$$

$$\int y^{-2}\,dy = \int 4xe^{-2x}\,dx \quad — 1$$

$u = x$, $\int dv = \int e^{-2x}$

$\frac{du}{dx} = 1$, $v = \frac{e^{-2x}}{-2}$

$dx = du$

$$\Rightarrow \frac{y^{-1}}{-1}\,dy = 4\left[uv - \int v\,du\right] \quad — 1$$

$$\Rightarrow -y = 4\left[-\frac{1}{2}xe^{-2x} + \frac{1}{2}\int e^{-2x}\,dx\right] \quad — 1$$

$$\Rightarrow -y = 4\left[-\frac{1}{2}xe^{-2x} + \frac{1}{2}\left[\frac{e^{-2x}}{-2}\right]\right] \quad — 1$$

$$\Rightarrow -y = -2xe^{-2x} - e^{-2x}$$

$$y = e^{-2x}(2x+1) \quad — 1$$

1