Least squares filling of DEM anomalies

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Abstract

This note describes a least-squares approach to filling anomalies in a Digital Elevation Model with substitute data.

1 Introduction

Digital Elevation Models (DEMs) are an important source of geometrical information in the process of orthorectifying both optical and radar satellite imagery. A DEM, often referenced to the WGS84 ellipsoid, is often used in conjuction with satellite state vectors in the radar case and with Rational Polynomial Coefficients (RPCs) in the optical case.

One important requirement of a DEM for thes purpose of orthorectification, is that the model reflects the terrain of the scene rather than the elevation of all features in the scene. A Digital Surface Model (DSM) which reports the height of all features in the scene including buildings, e.t.c. differs from a Digital Terrain Model (DTM) in this regard. Many users consider that a DEM can be either a DSM or a DTM. The use of a DSM for orthorectification can lead to significant distortions in the imagery, particularly for data collected at large oblique angles.

Unfortunately most freely available DEMs are DSMs. These include all of SRTM2, ASTER-GDEM, ALOS 3D DEM. There may be locations in the DEM that reflect terrain because no other features were present when the DEM was created, but there are significantly many signatures of tall buildings in all of these DEMs. This means that in many cases, to use these elevation models, unwanted surface features must be removed.

Features can be removed from the DSM through low-pass filtering or smoothing. Unwanted features are simply smoothed into larger resolution cells, thereby mitigating distortions. Unfortunately, this comes at the slight expense of geographic accuracy.

This document proposes another approach to removing unwanted features from a DEM. This approach requires initial human interaction to identify the features that are to be removed.

The next section provides detail on the approach.

2 Least-squares filling

The main idea of the approach is to select a polygon around an anomaly to be removed. The geographic polygon (a shapefile, geojson or other) spatially intersects with sample points in the DEM. This spatial intersection set can be partitioned into two subsets, the border, $\mathcal{B}$ and the interior $\mathcal{I}$. Assume that these sets are just sets of integers (labels) corresponding to DEM samples in each set. The idea is to use DEM samples in $\mathcal{B}$ to compute some appropriate values for DEM samples in $\mathcal{I}$.

A cost function for such an algorithm could be defined as follows:

$$J = \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{B}} n_{ij}(y_j - x_i)^2 + \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{I}} m_{ij}(x_j - x_i)^2, \tag{1}$$

where y_j are the DEM sample values on the boundary, x_i are DEM sample values in the interior $\mathcal{I}$,

$$n_{ij} = \begin{cases} 1 & \text{if sample } i \text{ in the interior is beside sample } j \text{ on the boundary} \\ 0 & \text{otherwise,} \end{cases} \tag{2}$$

and

$$m_{ij} = \begin{cases} 1 & \text{if interior samples } i \text{ and } j \text{ are beside each other} \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Minimization of J over x_i is equivalent to minimizing the changes in height between all samples that neighbour each other. Expansion of J leads to

$$J = \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{B}} n_{ij} y_j^2 - n_{ij} 2 y_j x_i + n_{ij} x_i^2 + \sum_{i \in \mathcal{I}} \sum_{j \in \mathcal{I}} m_{ij} x_j^2 - m_{ij} 2 x_j x_i + m_{ij} x_i^2, \tag{4}$$

which in matrix notation can be written as

$$\begin{aligned} J = {} & \mathbf{1}_M^T \mathbf{N}(\mathbf{y}^T \mathbf{I}_N \mathbf{y}) - 2\mathbf{x}^T \mathbf{N}\mathbf{y} + (\mathbf{x}^T \mathbf{I}_M \mathbf{x})\mathbf{N}\mathbf{1}_N \\ & + \mathbf{1}_M^T \mathbf{M}(\mathbf{x}^T \mathbf{I}_M \mathbf{x}) - 2\mathbf{x}^T \mathbf{M}\mathbf{x} + (\mathbf{x}^T \mathbf{I}_M \mathbf{x})\mathbf{M}\mathbf{1}_M \end{aligned}, \tag{5}$$

where $\mathbf{N}$ is the matrix defined with element n_{ij} and is therefore $|\mathcal{I}| \times |B|$, where $|\mathcal{X}|$ simply means the cardinality, or number of elements in the set $\mathcal{X}$. $\mathbf{M}$ is the matrix defined with element m_{ij} and is therefore square $|\mathcal{I}| \times |\mathcal{I}|$. $\mathbf{1}_N$ is a vector of ones of length $|B|$, $\mathbf{I}_N$ is the $|B| \times |B|$ identity matrix, $\mathbf{1}_M$ is a vector of ones of length $|\mathcal{I}|$, and $\mathbf{I}_M$ is the $|\mathcal{I}| \times |\mathcal{I}|$ identity matrix.

This can be re-written as

$$\begin{aligned} J = {} & \mathbf{y}^T \operatorname{diag}(\mathbf{1}_M^T \mathbf{N})\mathbf{y} - 2\mathbf{x}^T \mathbf{N}\mathbf{y} + \mathbf{x}^T \operatorname{diag}(\mathbf{N}\mathbf{1}_N)\mathbf{x} \\ & + \mathbf{x}^T \operatorname{diag}(\mathbf{1}_M^T \mathbf{M})\mathbf{x} - 2\mathbf{x}^T \mathbf{M}\mathbf{x} + \mathbf{x}^T \operatorname{diag}(\mathbf{M}\mathbf{1}_M)\mathbf{x} \end{aligned} \tag{6}$$

where $\operatorname{diag}(\mathbf{x})$ creates a diagonal matrix from a column or row vector $\mathbf{x}$.

The cost function has thus become

$$J = \mathbf{x}^T \mathbf{R}\mathbf{x} + -2\mathbf{x}^T \mathbf{v} + \mathbf{y}^T \operatorname{diag}(\mathbf{1}_M^T \mathbf{N})\mathbf{y}, \tag{7}$$

where

$$\mathbf{R} = \operatorname{diag}(\mathbf{1}_M^T \mathbf{M}) + \operatorname{diag}(\mathbf{M}\mathbf{1}_M) - 2\mathbf{M} + \operatorname{diag}(\mathbf{N}\mathbf{1}_N) \tag{8}$$

and

$$\mathbf{v} = \mathbf{N}\mathbf{y} \tag{9}$$

The cost function can be re-written as

$$J = [\mathbf{x} - \mathbf{R}^{-1}\mathbf{v}]^T \mathbf{R} [\mathbf{x} - \mathbf{R}^{-1}\mathbf{v}] - \mathbf{v}^T \mathbf{R}^{-1} \mathbf{v} \mathbf{y}^T \mathrm{diag}(\mathbf{1}_M^T \mathbf{N}) \mathbf{y} \tag{10}$$

and one sees that the solution to minimizing J over $\mathbf{x}$ is:

$$\boxed{\begin{array}{l} \mathbf{x} = \mathbf{R}^{-1}\mathbf{v}, \text{with} \\ \mathbf{R} = \mathrm{diag}(\mathbf{1}_M^T \mathbf{M}) + \mathrm{diag}(\mathbf{M}\mathbf{1}_M) - 2\mathbf{M} + \mathrm{diag}(\mathbf{N}\mathbf{1}_N) \\ \mathbf{v} = \mathbf{N}\mathbf{y} \end{array}}$$