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DEFINITIONS OF VENSAR REFERENCE FRAMES AND POINTING ANGLE ERRORS

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DISTRIBUTION

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1. APPLICABLE AND REFERENCE DOCUMENTS

1.1. Applicable Documents

[AD1] *Mission Requirements Document*. Tech. rep. ESA-ENVIS-EST-MIS-RS-001.

[AD2] *EnVision Payload Definition Document (VenSAR)*. Tech. rep. ESA-ENVIS-JPL-PL-RS-001.

[AD3] EnVisionTeam. *EnVision Acronyms Terms and Definition for Phase B1*. Tech. rep. TBW. 2022.

1.2. Reference Documents

[RD1] E. Boerner et al. "A new method for Total Zero Doppler Steering". In: *IGARSS 2004. 2004 IEEE International Geoscience and Remote Sensing Symposium*. Vol. 2. 2004, 1526–1529 vol.2. DOI: 10.1109/IGARSS.2004.1368712.

[RD2] H. Fiedler et al. "Total zero Doppler Steering-a new method for minimizing the Doppler centroid". In: *Geoscience and Remote Sensing Letters, IEEE* 2.2 (Apr. 2005), pp. 141–145. ISSN: 1545-598X. DOI: 10.1109/LGRS.2005.844591.

Acronyms

AAEU

Actual Azimuth, Elevation, Unit look-direction frame.

AKE

Absolute Knowledge Error.

AOC

Attitude and Orbit Control.

AOCS

Attitude and Orbit Control Subsystem.

APE

Absolute Pointing Error.

DAEU

Desired Azimuth, Elevation, Unit look-direction frame.

ESOC

European Space Operations Centre.

MAG

Mission Advisory Group.

MOC

Mission Operations Center.

MRD

Mission Requirements Document.

OBT

On-Board Timing.

OE

Orbit Ellipse frame.

PCI

Planet-Centered Inertial frame.

PCIP

Planet-Centered Inertial Plane frame.

PCR

Planet-Centered Rotating frame.

RMS

Root Mean-Square.

ROI

Region Of Interest.

RSS

Root Sum Squares.

S/C

Spacecraft.

SAE

Swath Angular Error.

SAR

Synthetic Aperture Radar.

SC

Spacecraft.

SNR

Signal to Noise Ratio.

SOP

Science Operations Planning.

SORS

Science Operations Reference Scenario.

TCN

Geocentric Track, Cross-Track, Nadir frame.

tcn

Tangent, Cross-track, Normal frame.

VCI

Venus-Centered Inertial frame.

VCIP

Venus-Centered Inertial Plane frame.

VCR

Venus-Centered Rotating frame.

YPR

Yaw, Pitch, Roll.

Definitions

Actual Azimuth, Elevation, Unit look-direction frame

The actual azimuth, elevation and look-direction reference frame.

Antenna boresight

The direction of maximum gain of radiated energy..

Azimuth

In radar, azimuth corresponds to the direction/dimension aligned with the radar platform track. Azimuth is often called the slow-time dimension as radar samples in this direction are collected at the pulse repetition frequency..

Azimuth pointing error

The azimuth pointing error is given by the angle α in the error diagram and in the specified equation showing the transformation from a desired AEU frame to the actual AEU frame..

Desired Azimuth, Elevation, Unit look-direction frame

The desired azimuth, elevation and look-direction reference frame.

Elevation

In radar, elevation corresponds to the complement of the incidence angle. It reflects the elevation of the radar above the horizon as seen from the target..

Elevation pointing error

The elevation pointing error is given by the angle ϵ in the error diagram and in the specified equation showing the transformation from a desired AEU frame to the actual AEU frame..

Envision Spacecraft frame

The Envision spacecraft fixed reference frame.

Geocentric Track, Cross-Track, Nadir frame

Geocentric track, cross-track and nadir reference frame.

Incidence angle

The angle between the normal to the local tangent plane on the planet's surface and the direction of the incident radiation..

Look angle

The angle between the nadir and the antenna boresight..

Look vector

Unit vector in the antenna boresight direction..

Mixed statistical interpretation

Variation across both time and ....

Orbit Ellipse frame

A reference frame centered on a satellite orbit ellipse.

Planet-Centered Inertial frame

Planet-centered inertial reference frame.

Planet-Centered Inertial Plane frame

Planet-centered inertial reference frame with two basis vectors in the inertial reference plane.

Planet-Centered Rotating frame

Planet-centered rotating reference frame.

Range

In radar, range corresponds to the direction/dimension perpendicular to the radar platform track. Range is often called the fast-time dimension as radar samples in this direction are collected at the pulse sampling frequency..

Swath angular error

Minimum angle (in absolute value) between the look vector and the vector between the spacecraft and target..

Tangent, Cross-track, Normal frame

A satellite tangent, cross-track normal reference frame.

Tilt pointing error

The error in the rotation of the azimuth axis of the DAEU about the look vector.

Venus-Centered Inertial frame

Venus-centered inertial reference frame.

Venus-Centered Inertial Plane frame

Planet-centered inertial reference frame with two basis vectors in the Envision orbit plane.

Venus-Centered Rotating frame

Venus-centered rotating reference frame.

Notation

$\dot{\mathbf{x}}(t)$

Derivative of matrix, vector or scalar with respect to time, t.

$\mathcal{E}\{W\}$

Expected value of a random variable.

$\hat{\mathbf{x}}$

Unit vector.

$\mathbf{x}'(s)$

Derivative of matrix, vector or scalar with respect to parameter, s.

$\mathbf{x} \times \mathbf{y}$

Cross-product of two vectors.

$\mathbf{x}^T$

Vector transpose.

$\mathbf{R}$

Matrix.

$\mathbf{x}$

Column vector.

2. INTRODUCTION

This document presents several ESA-defined coordinate reference frames for the Envision mission. Coordinate reference frames are first defined generically, then specific reference frames for Envision in orbit around Venus are defined. These reference frames as well as definitions for angular pointing errors specify a foundation for deriving and defining pointing requirements.

3. PLANET-CENTERED FRAMES

3.1. Planet-Centered Inertial (PCI) frame

Given orthonormal basis vectors $\hat{\mathbf{i}}_i$, $\hat{\mathbf{j}}_i$ and $\hat{\mathbf{k}}_i$, the rotational axis of the planet, an origin at the center of the planet and some inertial reference plane, the inertial PCI reference frame is defined in the following way (see Figure 1):

1. $\hat{\mathbf{k}}_i$ aligns with the planet rotational axis,
2. if the vector $\mathbf{n}$ denotes the normal to the inertial reference plane, on the side of the plane where the dot product with $\hat{\mathbf{k}}_i$ is positive, then the $\hat{\mathbf{j}}_i$ aligns with the cross product $\hat{\mathbf{k}}_i \times \mathbf{n}$,
3. $\hat{\mathbf{i}}_i$ completes the right-handed coordinate system.

This definition assumes one of the following: that $\hat{\mathbf{k}}_i$ remains inertially constant, that $\hat{\mathbf{k}}_i$ is defined at some specific point in time, or that $\hat{\mathbf{k}}_i$ is the time average over a specific period. Note that the origin of the coordinate system moves non-linearly through true inertial space through the orbit of the planet; thus this reference frame is not truly or fully inertial. Several Earth-Centered Inertial (ECI) coordinate frames use the ecliptic (the orbit plane of the earth) as the inertial reference plane (e.g. J2000, M50). Since $\hat{\mathbf{k}}_i$ is normal to the celestial equator, $\hat{\mathbf{j}}_i$ points in the direction of the earth-sun vector at the vernal equinox.

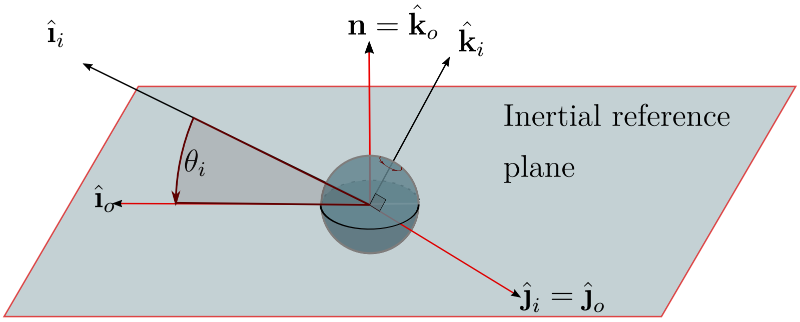

Figure 1: Planet-centered inertial reference frames. The PCIP reference frame has two basis vectors in the inertial reference plane. The PCIP is a rotation of the PCI reference frame, around $\hat{\mathbf{j}}_o = \hat{\mathbf{j}}_i$.

3.2. The Planet-Centered Inertial Plane (PCIP) frame

With orthonormal basis vectors $\hat{\mathbf{i}}_o$, $\hat{\mathbf{j}}_o$ and $\hat{\mathbf{k}}_o$, the PCIP reference frame is a rotation of the PCI frame around $\hat{\mathbf{j}}_o$ (which lies in the inertial reference plane) such that $\hat{\mathbf{i}}_o$ also lies in the inertial reference plane. In this case, $\hat{\mathbf{k}}_o$ aligns with the inertial reference plane normal and has a positive dot product with the PCI $\hat{\mathbf{k}}_i$ vector. For an illustration of the PCI frame relative to the PCIP frame, see Figure 1.

3.3. Planet-Centered Rotating (PCR) frame

With orthonormal basis vectors $\hat{\mathbf{i}}_p(t)$, $\hat{\mathbf{j}}_p(t)$ and $\hat{\mathbf{k}}_p(t)$, the PCR frame is a non-inertial reference frame with origin at the center of the planet, $\hat{\mathbf{k}}_p(t)$ aligned along the planet rotational axis, $\hat{\mathbf{i}}_p(t)$ passing through a specific feature on the planet equator (including the intersection of some special meridian with the equator) and $\hat{\mathbf{j}}_p(t)$ completing a right-handed coordinate system.

3.4. Transformation from PCI to PCIP

The transformation from the PCI system to the PCIP system is given by

$$\begin{bmatrix} \hat{\mathbf{i}}_o & \hat{\mathbf{j}}_o & \hat{\mathbf{k}}_o \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{i}}_i & \hat{\mathbf{j}}_i & \hat{\mathbf{k}}_i \end{bmatrix} \underbrace{\begin{bmatrix} \cos\theta_i & 0 & \sin\theta_i \\ 0 & 1 & 0 \\ -\sin\theta_i & 0 & \cos\theta_i \end{bmatrix}}_{\mathbf{M}_i^T(\theta_i)} \tag{1}$$

where θ_i is the angle between $\hat{\mathbf{k}}_i$ and $\mathbf{n}$, the normal to the inertial reference plane (this is illustrated in Figure 1). This leads to a transformation of coordinates via

$$\mathbf{x}_o(t) = \mathbf{M}_i(\theta_i)\mathbf{x}_i(t), \tag{2}$$

3.5. Transformation from PCI to PCR

The transformation from PCI to PCR is given by

$$\begin{bmatrix} \hat{\mathbf{i}}_p(t) & \hat{\mathbf{j}}_p(t) & \hat{\mathbf{k}}_p(t) \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{i}}_i & \hat{\mathbf{j}}_i & \hat{\mathbf{k}}_i \end{bmatrix} \underbrace{\begin{bmatrix} \cos\omega_p t & -\sin\omega_p t & 0 \\ \sin\omega_p t & \cos\omega_p t & 0 \\ 0 & 0 & 1 \end{bmatrix}}_{\mathbf{M}^T(t)} \tag{3}$$

where ω_p is the rotation rate of the planet in radians per second. In the above, the time origin is chosen so that the axes of both reference frames are aligned at $t = 0$.

A vector $\mathbf{x}_i(t)$ in the PCI system has coordinates in the PCR system given by

$$\mathbf{x}_p(t) = \mathbf{M}(t)\mathbf{x}_i(t), \tag{4}$$

By taking the derivative of the above with respect to t, one finds

$$\dot{\mathbf{x}}_p(t) = \dot{\mathbf{M}}(t)\mathbf{x}_i(t) + \mathbf{M}(t)\dot{\mathbf{x}}_i(t). \tag{5}$$

Note that

$$\dot{\mathbf{M}}(t) = \omega_p \begin{bmatrix} -\sin\omega_p t & \cos\omega_p t & 0 \\ -\cos\omega_p t & -\sin\omega_p t & 0 \\ 0 & 0 & 0 \end{bmatrix} \tag{6}$$

so that

$$\mathbf{M}^T(t)\dot{\mathbf{M}}(t) = \omega_p \underbrace{\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}_{\mathbf{Q}_2} \tag{7}$$

Note that a unit vector (or look vector) in the PCI frame, $\hat{\mathbf{u}}_i(t)$, transforms into the PCR frame via

$$\hat{\mathbf{u}}_p(t) = \mathbf{M}(t)\hat{\mathbf{u}}_i(t) \tag{8}$$

4. VENUS-CENTERED FRAMES

Venus-centered frames are special cases of planet-centered frames. The frames defined in this section inherit all transformations between planet-centered frames.

4.1. Venus-Centered Inertial (VCI) frame

The VCI frame is a Venus-centered PCI frame where the inertial reference plane is defined by the orbit plane of Envision around Venus. Because of the very small J2 term for the Venus gravitational potential, the Envision orbit plane can be considered inertial. This reference frame is illustrated in Figure 2

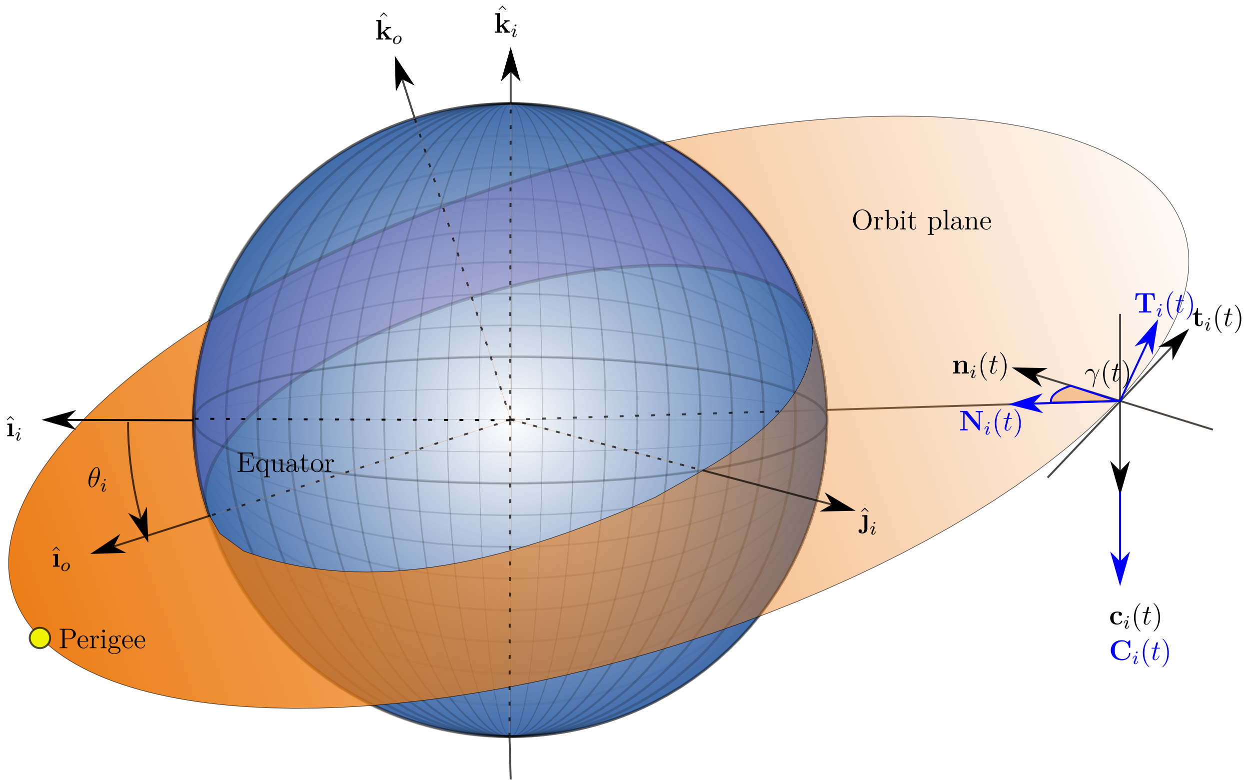

Figure 2: Satellite-centered reference frames. The VCI frame is represented by the orthonormal unit vectors $\{\hat{\mathbf{i}}_i, \hat{\mathbf{j}}_i, \hat{\mathbf{k}}_i\}$, the VCIP frame is represented by the orthonormal unit vectors $\{\hat{\mathbf{i}}_o, \hat{\mathbf{j}}_o = \hat{\mathbf{j}}_i, \hat{\mathbf{k}}_o\}$. The TCN frame is represented by the orthonormal basis $\{\mathbf{T}_i(t), \mathbf{N}_i(t), \mathbf{C}_i(t)\}$. The tcn frame is represented by the orthonormal basis $\{\mathbf{t}_i(t), \mathbf{n}_i(t), \mathbf{c}_i(t)\}$ which is a rotation of the TCN axis around $\mathbf{C}_i(t)$ by the angle $\gamma(t)$. Note that $\mathbf{C}_i(t) = \mathbf{c}_i(t)$ is normal to the orbit plane.

For an inclined orbit, ($|\theta_i| > 0$), the ascending node is the point on the orbit arc (in the inertial orbit plane) where the $\hat{\mathbf{k}}_i$-axis is perpendicular to the vector from the centre of the planet to the satellite when the satellie tracks northward. For $\theta_i = 0$, the ascending node can be chosen such that the $\hat{\mathbf{i}}_i$ axis points toward the star around which the planet orbits.

4.2. Venus-Centered Inertial Plane (VCIP) frame

The VCIP frame is a PCIP frame where the inertial reference plane is defined by the orbit plane of Envision around Venus. Because of the very small J2 term for the Venus gravitational potential, the orbit plane can be considered inertial. This frame is illustrated in Figure 2.

4.3. Venus-Centered Rotating (VCR) frame

With orthonormal basis vectors $\hat{\mathbf{i}}_p(t)$, $\hat{\mathbf{j}}_p(t)$ and $\hat{\mathbf{k}}_p(t)$, the VCR frame is a Venus-centered PCR frame. The special/prime meridian for Venus is defined as the meridian passing through a small impact crater named Ariadne, located in Sedna Planita.

5. SATELLITE-CENTERED REFERENCE FRAMES

5.1. Spacecraft (SC) reference frame

With orthonormal basis vectors $\hat{\mathbf{i}}_s(t)$, $\hat{\mathbf{j}}_s(t)$ and $\hat{\mathbf{k}}_s(t)$, the SC frame is a non-inertial satellite-centered reference frame. This reference frame is illustrated in Figure 3 and remains fixed with respect to the spacecraft. Note that $\hat{\mathbf{i}}_s(t)$, $\hat{\mathbf{j}}_s(t)$ and $\hat{\mathbf{k}}_s(t)$ are defined with respect to the VCIF reference frame.

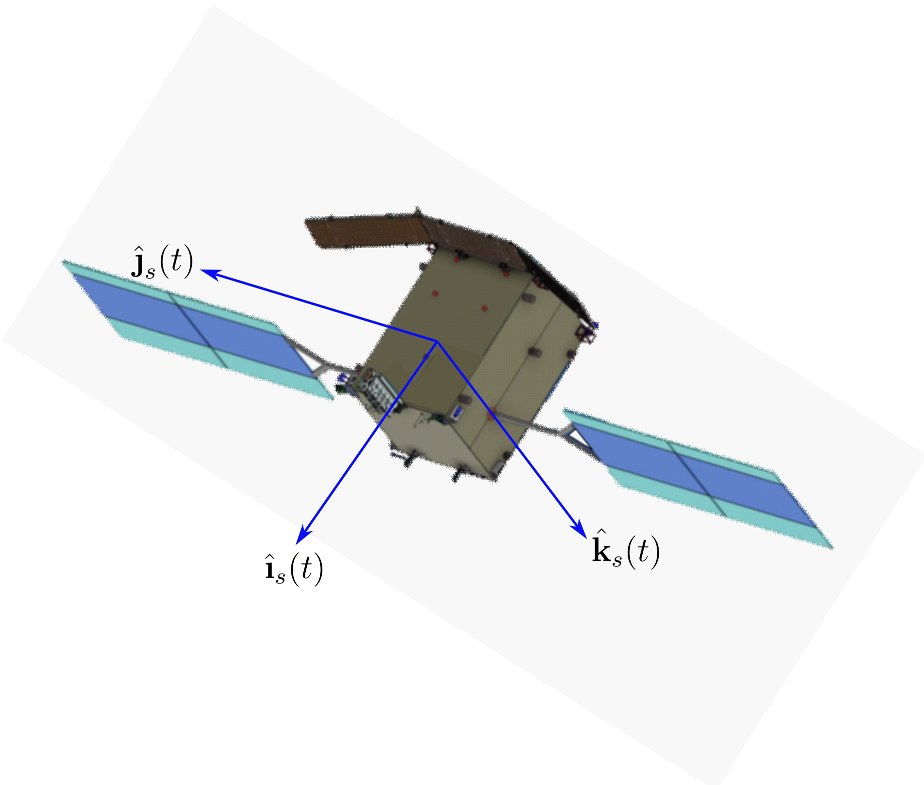

Figure 3: SC reference frame. The non-inertial SC reference frame is represented by the orthonormal unit vectors $\{\hat{\mathbf{i}}_s(t), \hat{\mathbf{j}}_s(t), \hat{\mathbf{k}}_s(t)\}$. These all remain fixed with respect to the satellite.

5.2. Roll, Pitch and Yaw

This section defines the roll, pitch and yaw angles.

These are, by definition, the angles that satisfy the following relation

$$\begin{bmatrix}\hat{\mathbf{j}}_s(t) & \hat{\mathbf{k}}_s(t) & \hat{\mathbf{i}}_s(t)\end{bmatrix} = \begin{bmatrix}\mathbf{t}_i(t) & \mathbf{c}_i(t) & \mathbf{n}_i(t)\end{bmatrix} \mathbf{M}_1^T[\theta_r(t)]\mathbf{M}_2^T[\theta_p(t)]\mathbf{M}_3^T[\theta_y(t)] \tag{9}$$

where

$$\mathbf{M}_1^T[\theta_r(t)] = \begin{bmatrix}1 & 0 & 0\\ 0 & \cos\theta_r(t) & -\sin\theta_r(t)\\ 0 & \sin\theta_r(t) & \cos\theta_r(t)\end{bmatrix}, \tag{10}$$

$$\mathbf{M}_2^T[\theta_p(t)] = \begin{bmatrix}\cos\theta_p(t) & 0 & \sin\theta_p(t)\\ 0 & 1 & 0\\ -\sin\theta_p(t) & 0 & \cos\theta_p(t)\end{bmatrix}, \tag{11}$$

$$\mathbf{M}_3^T[\theta_y(t)] = \begin{bmatrix}\cos\theta_y(t) & -\sin\theta_y(t) & 0\\ \sin\theta_y(t) & \cos\theta_y(t) & 0\\ 0 & 0 & 1\end{bmatrix}, \tag{12}$$

and where the pitch, roll and yaw angles are given by $\theta_p(t)$, $\theta_r(t)$ and $\theta_y(t)$, respectively.

5.2.1. Computation of Roll, Pitch and Yaw angles

Given $\begin{bmatrix}\hat{\mathbf{j}}_s(t) & \hat{\mathbf{k}}_s(t) & \hat{\mathbf{i}}_s(t)\end{bmatrix}$ and $\begin{bmatrix}\mathbf{t}_i(t) & \mathbf{c}_i(t) & \mathbf{n}_i(t)\end{bmatrix}$, $\theta_r(t)$, $\theta_p(t)$ and $\theta_y(t)$ in (9) can be determined with the algorithm described in this section.

Pre-multiply both sides of (9) by $\begin{bmatrix}\mathbf{t}_i(t) & \mathbf{c}_i(t) & \mathbf{n}_i(t)\end{bmatrix}^T$, to obtain

$$\begin{bmatrix}\mathbf{t}_i(t) & \mathbf{c}_i(t) & \mathbf{n}_i(t)\end{bmatrix}^T \begin{bmatrix}\hat{\mathbf{j}}_s(t) & \hat{\mathbf{k}}_s(t) & \hat{\mathbf{i}}_s(t)\end{bmatrix} = \mathbf{M}_1^T[\theta_r(t)]\mathbf{M}_2^T[\theta_p(t)]\mathbf{M}_3^T[\theta_y(t)]. \tag{13}$$

Now, suppose that

$$\begin{bmatrix}\mathbf{t}_i(t) & \mathbf{c}_i(t) & \mathbf{n}_i(t)\end{bmatrix}^T \begin{bmatrix}\hat{\mathbf{j}}_s(t) & \hat{\mathbf{k}}_s(t) & \hat{\mathbf{i}}_s(t)\end{bmatrix} = \begin{bmatrix}a & b & c\\ d & e & f\\ g & h & i\end{bmatrix}. \tag{14}$$

Then,

$$\begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} \begin{bmatrix} \cos\theta_y(t) & \sin\theta_y(t) & 0 \\ -\sin\theta_y(t) & \cos\theta_y(t) & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} a' & 0 & c' \\ d' & e' & f' \\ g' & h' & i' \end{bmatrix} \tag{15}$$

if $\theta_y(t)$ is chosen so that $a\sin\theta_y(t) + b\cos\theta_y(t) = 0$.

Further,

$$\begin{bmatrix} a' & 0 & c' \\ d' & e' & f' \\ g' & h' & i' \end{bmatrix} \begin{bmatrix} \cos\theta_p(t) & 0 & -\sin\theta_p(t) \\ 0 & 1 & 0 \\ \sin\theta_p(t) & 0 & \cos\theta_p(t) \end{bmatrix} = \begin{bmatrix} a'' & 0 & 0 \\ d'' & e'' & f'' \\ g'' & h'' & i'' \end{bmatrix} \tag{16}$$

by choosing $\theta_p(t)$ such that $c'\cos\theta_p(t) - a'\sin\theta_p(t) = 0$.

Finally,

$$\begin{bmatrix} a'' & 0 & 0 \\ d'' & e'' & f'' \\ g'' & h'' & i'' \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta_r(t) & \sin\theta_r(t) \\ 0 & -\sin\theta_r(t) & \cos\theta_r(t) \end{bmatrix} = \begin{bmatrix} a''' & 0 & 0 \\ d''' & e''' & 0 \\ g''' & h''' & i''' \end{bmatrix} \tag{17}$$

by choosing $\theta_r(t)$ such that $e''\sin\theta_r(t) + f''\cos\theta_r(t) = 0$.

This last expression is equivalent to the identity matrix, $\mathbf{I}$; thus, $d''' = g''' = h''' = 0$ and $a''' = e''' = i''' = 1$.

5.3. The Track, Cross-Track, Nadir (TCN) frame

With orthonormal basis vectors $\mathbf{T}_i(t)$, $\mathbf{C}_i(t)$ and $\mathbf{N}_i(t)$, illustrated in Figure 2, the TCN frame is a non-inertial satellite-centered reference frame with origin at the center of the satellite, $\mathbf{N}_i(t)$ pointing from the instantaneous satellite position vector to the center of the orbited planet, $\mathbf{C}_i(t)$ pointing in the direction of the cross product of $\mathbf{N}_i(t)$ and the instantaneous inertial satellite velocity vector and $\mathbf{T}_i(t)$ aligned in the direction of $\mathbf{C}_i(t) \times \mathbf{N}_i(t)$. The unit vectors $\mathbf{T}_i(t)$, $\mathbf{C}_i(t)$ and $\mathbf{N}_i(t)$ are defined with respect to a PCI frame and change over time as the satellite proceeds along its orbit. With this definition, both $\mathbf{T}_i(t)$ and $\mathbf{N}_i(t)$ lie in the satellite orbit plane and $\mathbf{T}_i(t)$ need not align with the instantaneous inertial satellite velocity vector. Note that a common variation of this reference frame applies a rotation around the $\mathbf{C}_i(t)$ vector so that the $\mathbf{N}_i(t)$ vector is perpendicular to the sub-satellite surface tangent plane. The amount of rotation depends on both surface topography and the degree to which the planet is spherical. The variation may be distinguished from the original by using the terms geodetic TCN versus geocentric TCN, respectively.

5.4. The tangent, cross-track, normal (tcn) frame

With orthonormal basis vectors $\mathbf{t}_i(t)$, $\mathbf{c}_i(t)$ and $\mathbf{n}_i(t)$, the tcn frame is a non-inertial satellite-centered reference frame with origin at the center of the satellite, $\mathbf{t}_i(t)$ in the direction of the tangent vector (same direction as the instantaneous inertial satellite velocity vector), $\mathbf{n}_i(t)$ perpendicular to $\mathbf{t}_i(t)$ and in the orbit plane pointing inwards, and $\mathbf{c}_i(t)$ completing the right-handed coordinate basis. The unit vectors $\mathbf{t}_i(t)$, $\mathbf{c}_i(t)$ and $\mathbf{n}_i(t)$ are defined with respect to a PCI frame and change over time as the satellite proceeds along its orbit. With this definition, $\mathbf{t}_i(t)$ and $\mathbf{n}_i(t)$ lie in the satellite orbit plane so tcn is a rotation of the TCN frame around the common $\mathbf{c}_i(t)$ vector - this is illustrated in Figure 2. The tcn and TCN frames share the same three letter acronym; thus, to distinguish them, lower case is used for the tangent, cross-track normal frame.

Mathematical model

Imagine that in inertial space, the satellite traces a curve parameterized by either arclength or time with arclength, s, related to time via

$$s = \int_0^t |\dot{\mathbf{x}}_i(\tau)| \mathrm{d}\tau. \tag{18}$$

With the satellite position given as $\mathbf{x}_i(s)$, this formalism admits the following vectors: $\mathbf{t}_i(s) = \mathbf{x}_i'(s)$ which is a unit vector in the direction of motion, and, if we define the curvature as $\kappa(s) = |\mathbf{t}_i'(s)|$, $\mathbf{n}_i(s) = \mathbf{t}_i'(s)/\kappa(s)$ which is perpendicular to $\mathbf{t}_i(s)$. To complete a right-handed orthonormal basis, we introduce also $\mathbf{c}_i(s) = \mathbf{n}_i(s) \times \mathbf{t}_i(s)$. This basis has an origin at $\mathbf{x}_i(s)$.

This arclength parameterization of a curve can be developed further, but what we have suffices for the discussion in this document. Although we have written $\mathbf{t}_i(s)$, $\mathbf{n}_i(s)$ and $\mathbf{c}_i(s)$ as vectors as functions of s, we see from (18) that $s = s(t)$, thus we write from here onwards $\mathbf{t}_i(t) = \mathbf{t}_i(s[t])$, $\mathbf{n}_i(t) = \mathbf{n}_i(s[t])$ and $\mathbf{c}_i(t) = \mathbf{c}_i(s[t])$.

Note that

$$\frac{\mathrm{d}\mathbf{x}_i(t)}{\mathrm{d}t} = \frac{\mathrm{d}\mathbf{x}_i(s[t])}{\mathrm{d}t} = \mathbf{x}_i'(s[t])\frac{\mathrm{d}s[t]}{\mathrm{d}t} = \mathbf{x}_i'(s[t])|\dot{\mathbf{x}}_i(t)| = v_s(t)\mathbf{t}_i(t), \tag{19}$$

where $v_s(t) = |\dot{\mathbf{x}}_i(t)|$ (the inertial satellite speed) and the fourth equality follows from the fundamental theorem of calculus and (18).

5.5. Transformation from VCIP to TCN frame

As illustrated in Figure 4, the transformation from the VCIP frame to the TCN frame may be represented by

$$\begin{bmatrix}\mathbf{T}_o(t) & \mathbf{C}_o(t) & \mathbf{N}_o(t)\end{bmatrix} = \begin{bmatrix}\hat{\mathbf{i}}_o & \hat{\mathbf{j}}_o & \hat{\mathbf{k}}_o\end{bmatrix}\begin{bmatrix}-\cos\beta(t) & 0 & -\sin\beta(t)\\ \sin\beta(t) & 0 & -\cos\beta(t)\\ 0 & -1 & 0\end{bmatrix}, \tag{20}$$

where the orbit angle, $\beta(t)$ is defined and illustrated in Figure 4 and

$$\begin{bmatrix}\mathbf{T}_o(t) & \mathbf{C}_o(t) & \mathbf{N}_o(t)\end{bmatrix} = \begin{bmatrix}\mathbf{T}_i(t) & \mathbf{C}_i(t) & \mathbf{N}_i(t)\end{bmatrix}\mathbf{M}_i^T(\theta_i). \tag{21}$$

5.6. Transformation from tcn to TCN

As illustrated in Figures 2 and 4, a rotation of the tcn reference frame around $\mathbf{c}_o(t)$ yields the geo-centric TCN frame. With the cross-track vector remaining unchanged, the following yields the TCN basis vectors

$$\begin{bmatrix}\mathbf{T}_o(t) & \mathbf{C}_o(t) & \mathbf{N}_o(t)\end{bmatrix} = \begin{bmatrix}\mathbf{t}_o(t) & \mathbf{c}_o(t) & \mathbf{n}_o(t)\end{bmatrix}\begin{bmatrix}\cos\gamma(t) & 0 & -\sin\gamma(t)\\ 0 & 1 & 0\\ \sin\gamma(t) & 0 & \cos\gamma(t)\end{bmatrix}. \tag{22}$$

The angle $\gamma(t)$ relates to the orbit angle, $\beta(t)$, the argument of perigee, ω and the orbit eccentricity, e (all defined and illustrated in Figure 4) through

$$\cos\gamma(t) = \frac{1 + e\cos[\beta(t) - \omega]}{\sqrt{1 + 2e\cos[\beta(t) - \omega] + e^2}}, \tag{23}$$

$$\sin\gamma(t) = \frac{e\sin[\beta(t) - \omega]}{\sqrt{1 + 2e\cos[\beta(t) - \omega] + e^2}}. \tag{24}$$

Interesting fact: the expression for $\cos\gamma(t)$ is the same expression as used for pitch control law in [RD1, RD2]. Note that the left side of the above, and the first term on the right, are matrices composed of the indicated column vectors.

6. SATELLITE-ORBIT-CENTERED FRAMES

6.1. Orbit Ellipse (OE) frame

With orthonormal basis vectors $\hat{\mathbf{i}}_e$ and $\hat{\mathbf{j}}_e$, the OE frame is a two-dimensional reference frame with origin at the center of the satellite orbit ellipse, unit vector $\hat{\mathbf{i}}_e$ pointing from the origin towards periapsis and unit vector $\hat{\mathbf{j}}_e$ as a rotation of 90 degrees around the normal to the orbit plane. Figure 4 illustrates the OE reference frame.

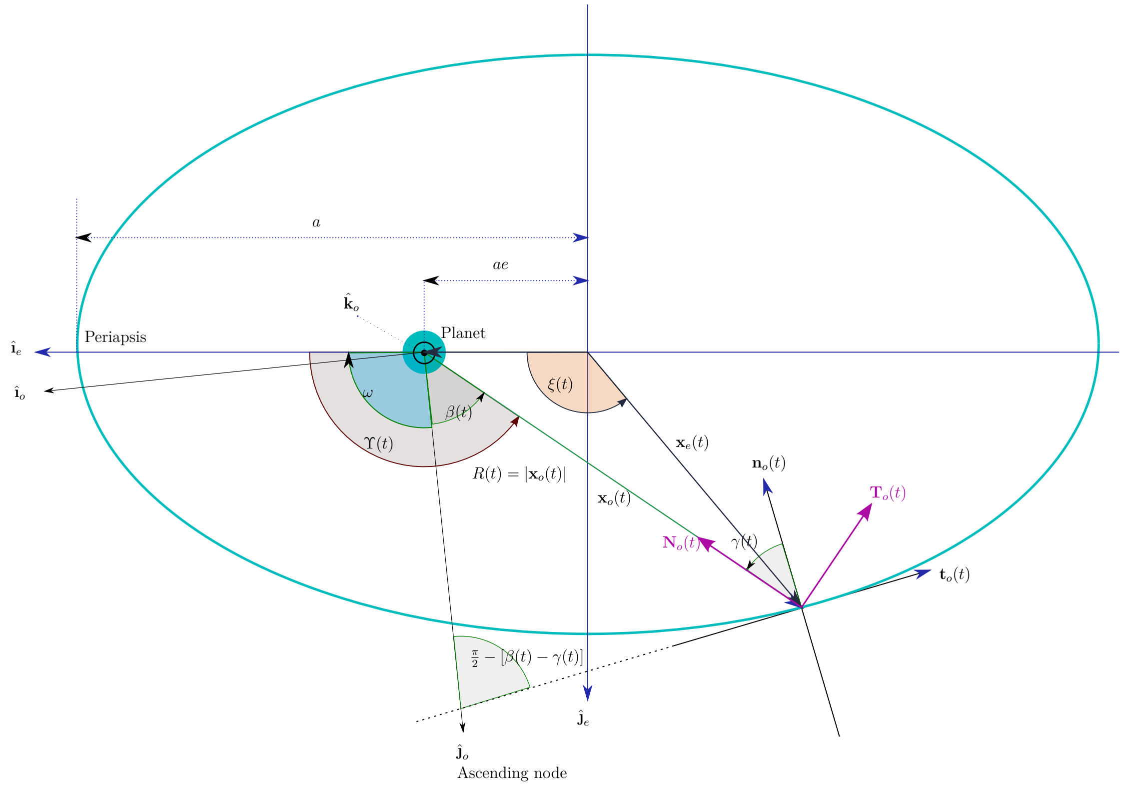

Figure 4: [Satellite-orbit-centered reference frames. Relative to $\hat{\mathbf{j}}_e$, ω denotes the argument of perigee, $\Upsilon(t) = \beta(t) - \omega$ represents the true anomaly, e denotes the orbit eccentricity, a the semi-major axis length, $\mathbf{x}_e(t)$ and $\mathbf{x}_o(t)$ denote the satellite position vector in the OE and VCIP frames, respectively, $\gamma(t)$ represents the angle between tcn and TCN frames.

7. ANTENNA-CENTERED FRAMES

7.1. Desired Azimuth, Elevation, Unit look direction (DAEU) frame

With orthonormal basis vectors $\hat{\mathbf{a}}_i(t)$, $\hat{\mathbf{e}}_i(t)$ and $\hat{\mathbf{u}}_i(t)$, the DAUE frame is a non-inertial radar-antenna-centered reference frame with $\hat{\mathbf{a}}_i(t)$ in the direction of the theoretically desired long axis of the antenna (or reflector), $\hat{\mathbf{u}}_i(t)$ in the theoretically desired boresight direction and $\hat{\mathbf{e}}_i(t)$ completing the right-handed coordinate system. Note that it is expected that $\hat{\mathbf{u}}_i(t)$ and $\hat{\mathbf{a}}_i(t)$ are perpendicular for Envision. Further $\hat{\mathbf{e}}_i(t)$ will not be perpendicular the the face of the reflector due to the fact that the reflector is rotated/tilted around the $\hat{\mathbf{a}}_i(t)$ axis with respect to the feed array. The unit vectors $\hat{\mathbf{a}}_i(t)$, $\hat{\mathbf{e}}_i(t)$ and $\hat{\mathbf{u}}_i(t)$ are defined with respect to a PCI frame and change over time as the satellite attitude changes over time. This reference frame stays fixed relative to the satellite platform. Figure 5 presents an illustration of this reference frame

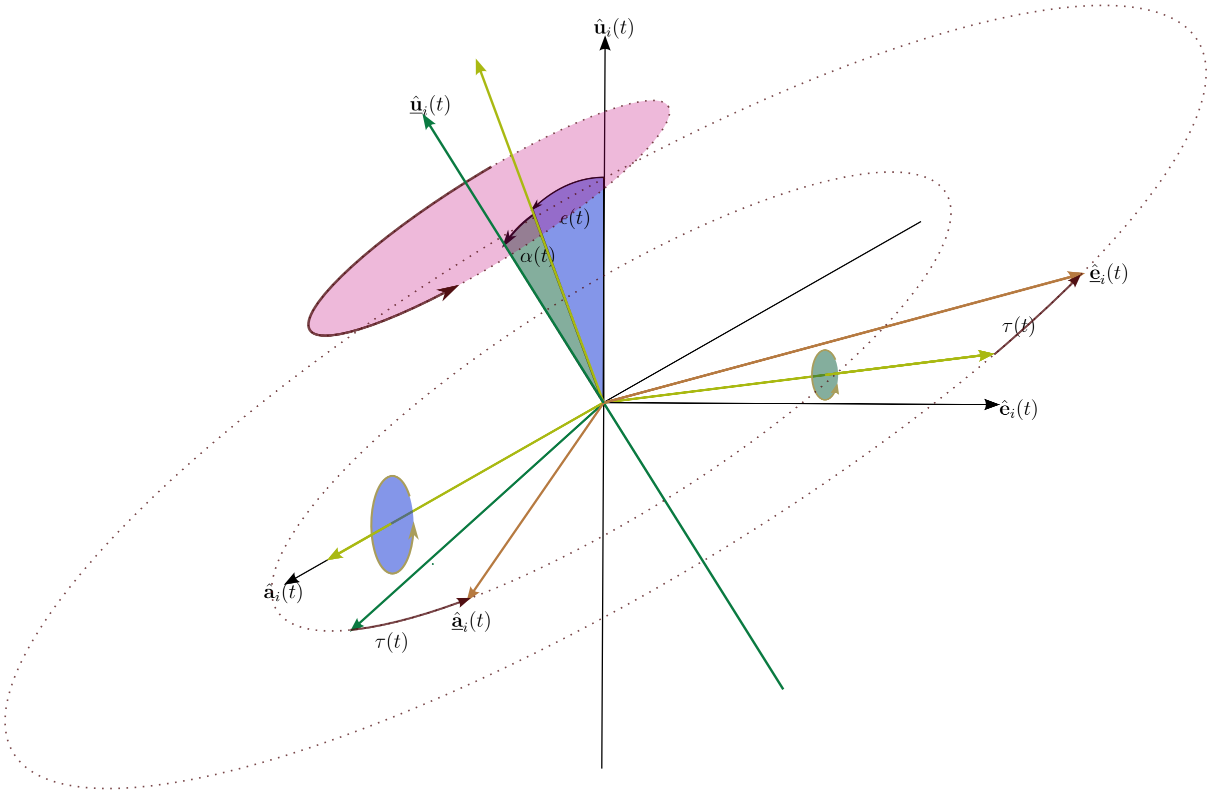

Figure 5: Antenna-centered reference frames and angular pointing errors. The elevation, azimuth and tilt pointing errors are represented by $\epsilon(t)$, $\alpha(t)$ and $\tau(t)$ respectively. The figure illustrates the DAEU frame with the unit basis vectors, $\hat{\mathbf{a}}_i(t), \hat{\mathbf{e}}_i(t), \hat{\mathbf{u}}_i(t)$ compared to the AAEU frame defined with the unit basis vectors $\underline{\hat{\mathbf{a}}}_i(t), \underline{\hat{\mathbf{e}}}_i(t), \underline{\hat{\mathbf{u}}}_i(t)$. If the two frames are initially aligned, the sequence of intrinsic rotations of the actual pointing frame by, $\epsilon(t)$ around the $\underline{\hat{\mathbf{a}}}_i(t)(t)$, followed by $\alpha(t)$ around $\underline{\hat{\mathbf{e}}}_i(t)(t)$, and then $\tau(t)$ around the $\underline{\hat{\mathbf{u}}}_i(t)$ yields the final perturbed AAUE frame.

7.2. Actual Azimuth, Elevation, Unit look direction (AAEU) frame

With orthonormal basis vectors $\underline{\hat{\mathbf{a}}}_i(t)$, $\underline{\hat{\mathbf{e}}}_i(t)$ and $\underline{\hat{\mathbf{u}}}_i(t)$, the AAUE frame is a non-inertial radar-antenna-centered reference frame with $\underline{\hat{\mathbf{a}}}_i(t)$ in the direction of the actual (not necessarily the desired) long axis of the antenna (or reflector), $\underline{\hat{\mathbf{u}}}_i(t)$ in the actual (not necessarily the desired) boresight direction and $\underline{\hat{\mathbf{e}}}_i(t)$ completing the right-handed coordinate system. Note that it is expected that $\underline{\hat{\mathbf{u}}}_i(t)$ and $\underline{\hat{\mathbf{a}}}_i(t)$ are perpendicular for Envision. Further $\underline{\hat{\mathbf{e}}}_i(t)$ will not be perpendicular the the face of the reflector due to the fact that the reflector is rotated/tilted around the $\underline{\hat{\mathbf{a}}}_i(t)$ axis with respect to the feed array. The unit vectors $\underline{\hat{\mathbf{a}}}_i(t)$, $\underline{\hat{\mathbf{e}}}_i(t)$ and $\underline{\hat{\mathbf{u}}}_i(t)$ are defined with respect to a PCI frame and change over time as the satellite attitude changes over time. This reference frame stays fixed relative to the satellite platform. Figure 5 presents an illustration of this reference frame.

7.3. Transformation from DAEU to AAEU

The transformation from DAEU to AAEU frames is given by the elevation pointing error, the azimuth pointing error and the tilt pointing error (see figure 5). These angles and order of operations yield the following transformation

$$\underline{\mathbf{B}}_i(t) = \mathbf{B}_i(t) \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\epsilon(t) & -\sin\epsilon(t) \\ 0 & \sin\epsilon(t) & \cos\epsilon(t) \end{bmatrix} \begin{bmatrix} \cos\alpha(t) & 0 & \sin\alpha(t) \\ 0 & 1 & 0 \\ -\sin\alpha(t) & 0 & \cos\alpha(t) \end{bmatrix} \begin{bmatrix} \cos\tau(t) & -\sin\tau(t) & 0 \\ \sin\tau(t) & \cos\tau(t) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \tag{25}$$

where

$$\underline{\mathbf{B}}_i(t) = \left[\underline{\hat{\mathbf{a}}}_i(t) \quad \underline{\hat{\mathbf{e}}}_i(t) \quad \underline{\hat{\mathbf{u}}}_i(t)\right], \tag{26}$$

and

$$\mathbf{B}_i(t) = \left[\hat{\mathbf{a}}_i(t) \quad \hat{\mathbf{e}}_i(t) \quad \hat{\mathbf{u}}_i(t)\right]. \tag{27}$$

The expression in (25) implies an order of operations given by elevation followed by azimuth followed by tilt.

Note that the naming convention can lead to some confusion. In order of application, the $\epsilon(t)$ angle is a right-handed rotation around the $\hat{\mathbf{a}}_i(t)$, corresponding to a roll. The $\alpha(t)$ is a right-handed rotation around the rotated (from previous operation) $\hat{\mathbf{e}}_i(t)$, and the $\tau(t)$ is a right-handed rotation around the rotated $\hat{\mathbf{u}}_i(t)$. In contrast to the RPY (roll, pitch, yaw) euler angles, where the order of operations is implicit in the acronym, the DAEU and AAEU reference frames have acronyms that correspond to a right-handed coordinate system of the basis vectors, but do not correspond to the order of operations of the error angles, $\alpha(t)$, $\epsilon(t)$ and $\tau(t)$.

7.4. Transformation from DAEU to desired SC frame

Given basis vectors for the SC reference frame that relate to the desired pointing direction, $\hat{\mathbf{i}}_s(t)$, $\hat{\mathbf{j}}_s(t)$ and $\hat{\mathbf{k}}_s(t)$, the following transformation relates the SC frame to the DAEU frame via

$$\begin{bmatrix}\hat{\mathbf{a}}_i(t) & \hat{\mathbf{e}}_i(t) & \hat{\mathbf{u}}_i(t)\end{bmatrix} = \begin{bmatrix}\hat{\mathbf{i}}_s(t) & \hat{\mathbf{j}}_s(t) & \hat{\mathbf{k}}_s(t)\end{bmatrix} \underbrace{\begin{bmatrix}0 & \sin\epsilon_0 & \cos\epsilon_0 \\ 1 & 0 & 0 \\ 0 & \cos\epsilon_0 & -\sin\epsilon_0\end{bmatrix}}_{\mathbf{M}_{\epsilon_0}^T} \tag{28}$$

7.5. Transformation from AAEU to actual SC frame

Given basis vectors for the SC reference frame that relate the actual pointing direction, $\hat{\underline{\mathbf{i}}}_s(t)$, $\hat{\underline{\mathbf{j}}}_s(t)$ and $\hat{\underline{\mathbf{k}}}_s(t)$, the following transformation relates these actual SC frame basis vectors to the AAEU frame via

$$\begin{bmatrix}\hat{\underline{\mathbf{a}}}_i(t) & \hat{\underline{\mathbf{e}}}_i(t) & \hat{\underline{\mathbf{u}}}_i(t)\end{bmatrix} = \begin{bmatrix}\hat{\underline{\mathbf{i}}}_s(t) & \hat{\underline{\mathbf{j}}}_s(t) & \hat{\underline{\mathbf{k}}}_s(t)\end{bmatrix} \underbrace{\begin{bmatrix}0 & \sin\epsilon_0 & \cos\epsilon_0 \\ 1 & 0 & 0 \\ 0 & \cos\epsilon_0 & -\sin\epsilon_0\end{bmatrix}}_{\mathbf{M}_{\epsilon_0}^T} \tag{29}$$

Figure 6 illustrates the geometry of the relation.

7.6. Pitch, Roll and Yaw Errors from Elevation, Azimuth and Tilt Errors

With an SC frame related to the desired pointing direction given by (28), and the SC frame relating to the actual pointing direction given by (29), one finds that

$$\begin{bmatrix}\hat{\mathbf{i}}_s(t) & \hat{\mathbf{j}}_s(t) & \hat{\mathbf{k}}_s(t)\end{bmatrix}^T \begin{bmatrix}\hat{\underline{\mathbf{i}}}_s(t) & \hat{\underline{\mathbf{j}}}_s(t) & \hat{\underline{\mathbf{k}}}_s(t)\end{bmatrix} = \mathbf{M}_{\epsilon_0}^T \mathbf{B}_i^T(t) \underline{\mathbf{B}}_i(t) \mathbf{M}_{\epsilon_0}. \tag{30}$$

which means

$$\begin{bmatrix}\hat{\mathbf{j}}_s(t) & \hat{\mathbf{k}}_s(t) & \hat{\mathbf{i}}_s(t)\end{bmatrix}^T \begin{bmatrix}\hat{\underline{\mathbf{j}}}_s(t) & \hat{\underline{\mathbf{k}}}_s(t) & \hat{\underline{\mathbf{i}}}_s(t)\end{bmatrix} = \mathbf{I}_p^T \mathbf{M}_{\epsilon_0}^T \mathbf{B}_i^T(t) \underline{\mathbf{B}}_i(t) \mathbf{M}_{\epsilon_0} \mathbf{I}_p, \tag{31}$$

where

$$\mathbf{I}_p = \begin{bmatrix}0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0\end{bmatrix}. \tag{32}$$

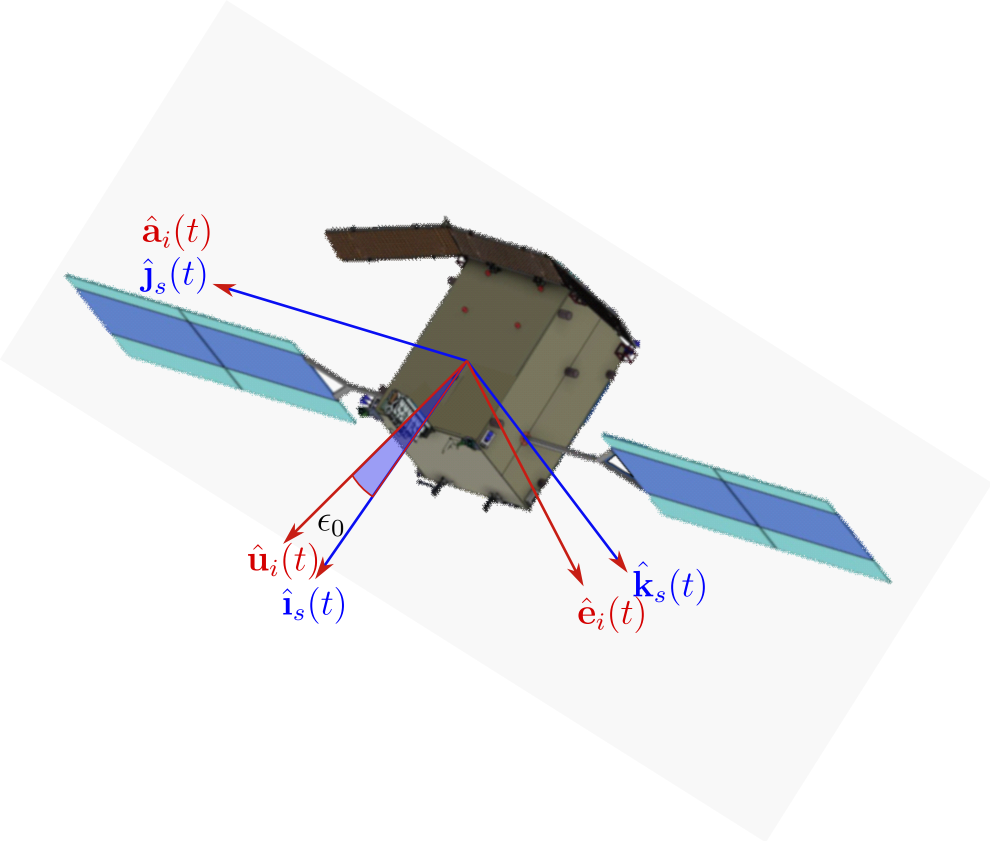

Figure 6: Relation between the AC frame and the DAEU frame with fixed angle ϵ_0.

This expression can then be used with the algorithm in Section 5.2.1 to compute the pitch, roll and yaw angles that would be needed to transform the SC to point the DAEU frame into the AAEU frame.