### 4.2.1 Model

This mathematical model builds upon the model presented in [] by incorporating an inter-colony interaction. Specifically, it focuses on modeling two colonies, designated as Colony 1 and Colony 2, to illustrate how different behaviors impact disease transmission between colonies. The mathematical model for Colony 1 is given by the following system:

$$\frac{dB_{1s}}{dt} = \alpha_1 P_1 B_1 + \frac{C_1}{v} M_1 - \varphi_1 \varphi_2 B_{1s} \tag{1}$$

$$\frac{dB_1}{dt} = M_1 - \alpha_1 P_1 B_1 - \frac{C_1}{v} M_1 - \beta_1 B_1 \tag{2}$$

$$\frac{dB_{1a}}{dt} = \beta_1 B_1 - \beta_2 B_{1a} \tag{3}$$

$$\frac{dC_1}{dt} = \varphi_1 \varphi_2 B_{1s}(\sigma(v - C_1) \tag{4}$$

$$\frac{dA_{1s}}{dt} = \beta_2 a_1 P_1 B_{1a} + a_2 \frac{C_1}{v} A_1 + a_3 P_1 A_1 - \mu A_{1s} - d_1 A_{1s} + r_1 \gamma_1 d_1 A_{1s} \tag{5}$$

$$\frac{dA_1}{dt} = \beta_2 (1 - a_1 P_1) B_{1a} - a_2 \frac{C_1}{v} A_1 - a_3 P_1 A_1 - \mu A_1 - d_1 A_1 + r_1 \gamma_1 d_1 A_1 \tag{6}$$

The dynamics of Colony 2 exhibit similarities to Colony 1, albeit with distinct subscripts of the state variables and parameters $C_n$, $P_n$, $M_n$, $d_n$, $s_n$, and $r_n$. The dynamics of Colony 2 are governed by the following system:

$$\frac{dB_{2s}}{dt} = \alpha_1 P_2 B_2 + \frac{C_2}{v} M_2 - \varphi_1 \varphi_2 B_{2s} \tag{7}$$

$$\frac{dB_2}{dt} = M_2 - \alpha_1 P_2 B_2 - \frac{C_2}{v} M_2 - \beta_1 B_2 \tag{8}$$

$$\frac{dB_{2a}}{dt} = \beta_1 B_2 - \beta_2 B_{2a} \tag{9}$$

$$\frac{dC_2}{dt} = \varphi_1 \varphi_2 B_{2s}(\sigma(v - C_2) \tag{10}$$

$$\frac{dA_{2s}}{dt} = \beta_2 a_1 P_2 B_{2a} + a_2 \frac{C_2}{v} A_2 + a_3 P_2 A_2 - \mu A_{2s} - d_2 A_{2s} + r_2 \gamma_2 d_2 A_{2s} \tag{11}$$

$$\frac{dA_2}{dt} = \beta_2 (1 - a_1 P_2) B_{2a} - a_2 \frac{C_2}{v} A_2 - a_3 P_2 A_2 - \mu A_2 - d_2 A_2 + r_2 \gamma_2 d_2 A_2 \tag{12}$$

The drifting component's dynamics are governed through the following system:

$$\frac{dA_{21s}}{dt} = d_2A_{2s} - r_2\gamma_2d_2A_{2s} - (1-r_2)\gamma_2d_2A_{2s} - uA_{21s} + \alpha_1P_1B_{1a} + a_2\frac{C_1}{v}A_{21} + \alpha_3P_1A_{21} \tag{13}$$

$$\frac{dA_{21}}{dt} = d_2A_2 - r_2\gamma_2d_2A_2 - (1-r_2)\gamma_2d_2A_2 - uA_{21} - a_2\frac{C_1}{v}A_{21} - \alpha_3P_1A_{21} \tag{14}$$

$$\frac{dA_{12s}}{dt} = d_1A_{1s} - r_1\gamma_1d_1A_{1s} - (1-r_1)\gamma_1d_1A_{1s} - uA_{12s} + \alpha_1P_2B_{2a} + a_2\frac{C_2}{v}A_{12} + \alpha_3P_2A_{12} \tag{15}$$

$$\frac{dA_{12}}{dt} = d_1A_1 - r_1\gamma_1d_1A_1 - (1-r_1)\gamma_1d_1A_1 - uA_{12} - a_2\frac{C_2}{v}A_{12} - \alpha_3P_2A_{12} \tag{16}$$

From the equations, the following terms $M_1 = L_1\frac{A_1+A_{1s}+A_{21}+A_{21s}}{w_1+A_1+A_{1s}+A_{21}+A_{21s}}$, $M_2 = L_2\frac{A_2+A_{2s}+A_{12}+A_{12s}}{w_2+A_2+A_{2s}+A_{12}+A_{12s}}$, $P_1 = \frac{A_{1s}+A_{21s}}{1+A_1+A_{1s}+A_{21}+A_{21s}}$, and $P_2 = \frac{A_{2s}+A_{12s}}{1+A_2+A_{2s}+A_{12}+A_{12s}}$. The parameters that have been introduced include $d_1$ and $d_2$, representing the drifting rate of forager honey bees; $\gamma_1$ and $\gamma_2$, indicating the rejection rate of drifting honey bees not accepted by the other colony; and finally $r_1$ and $r_2$, signifying the return rate of drifting adult bees to their original colony upon failure to integrate into the other colony.