4.2.1 Model

This mathematical model builds upon the model presented in [] by incorporating an inter-colony interaction. Specifically, it focuses on modeling two colonies, designated as Colony 1 and Colony 2, to illustrate how different behaviors impact disease transmission between colonies. The mathematical model for Colony 1 is given by the following system:

$$\frac{dB_{1s}}{dt} = \alpha_1 P_1 B_1 + \frac{C_1}{v} M_1 - \varphi_1 \varphi_2 B_{1s} \tag{1}$$

$$\frac{dB_1}{dt} = M_1 - \alpha_1 P_1 B_1 - \frac{C_1}{v} M_1 - \beta_1 B_1 \tag{2}$$

$$\frac{dB_{1a}}{dt} = \beta_1 B_1 - \beta_2 B_{1a} \tag{3}$$

$$\frac{dC_1}{dt} = \varphi_1 \varphi_2 B_{1s}(\sigma(v - C_1) \tag{4}$$

$$\frac{dA_{1s}}{dt} = \beta_2 a_1 P_1 B_{1a} + a_2 \frac{C_1}{v} A_1 + a_3 P_1 A_1 - \mu A_{1s} - d_1 s_1 A_{1s} + r_1(1 - s_1) A_{12s} \tag{5}$$

$$\frac{dA_1}{dt} = \beta_2 (1 - a_1 P_1) B_{1a} - a_2 \frac{C_1}{v} A_1 - a_3 P_1 A_1 - \mu A_{1s} - d_1 s_1 A_1 + r_1 (1 - s_1) A_{12} \tag{6}$$

The dynamics of Colony 2 exhibit similarities to Colony 1, albeit with distinct subscripts of the state variables and parameters C_n, P_n, M_n, d_n, s_n, and r_n. The dynamics of Colony 2 are governed by the following system:

$$\frac{dB_{2s}}{dt} = \alpha_1 P_2 B_2 + \frac{C_2}{v} M_2 - \varphi_1 \varphi_2 B_{2s} \tag{7}$$

$$\frac{dB_2}{dt} = M_2 - \alpha_1 P_2 B_2 - \frac{C_2}{v} M_2 - \beta_1 B_2 \tag{8}$$

$$\frac{dB_{2a}}{dt} = \beta_1 B_2 - \beta_2 B_{2a} \tag{9}$$

$$\frac{dC_2}{dt} = \varphi_1 \varphi_2 B_{2s}(\sigma(v - C_2) \tag{10}$$

$$\frac{dA_{2s}}{dt} = \beta_2 a_1 P_2 B_{2a} + a_2 \frac{C_2}{v} A_2 + a_3 P_2 A_2 - \mu A_{2s} - d_2 s_2 A_{2s} + r_2(1 - s_2) A_{21s} \tag{11}$$

$$\frac{dA_2}{dt} = \beta_2 (1 - a_1 P_2) B_{2a} - a_2 \frac{C_2}{v} A_2 - a_3 P_2 A_2 - \mu A_{2s} - d_2 s_2 A_2 + r_2 (1 - s_2) A_{21} \tag{12}$$

The drifting component's dynamics are governed through the following system:

$$\frac{dA_{21s}}{dt} = d_2 s_2 A_{2s} - r_2(1 - s_2)A_{21s} - (1 - r_2)(1 - s_2)A_{21s} - uA_{21s} + \alpha_1 P_1 B_{1a} + a_2 \frac{C_1}{v} A_{21} + \alpha_3 P_1 A_{21} \tag{13}$$

$$\frac{dA_{21}}{dt} = d_2 s_2 A_2 - r_2(1 - s_2)A_{21} - (1 - r_2)(1 - s_2)A_{21} - uA_{21} - a_2 \frac{C_1}{v} A_{21} - \alpha_3 P_1 A_{21} \tag{14}$$

$$\frac{dA_{12s}}{dt} = d_1 s_1 A_{1s} - r_1(1 - s_1)A_{12s} - (1 - r_1)(1 - s_1)A_{12s} - uA_{12s} + \alpha_1 P_2 B_{2a} + a_2 \frac{C_2}{v} A_{12} + \alpha_3 P_2 A_{12} \tag{15}$$

$$\frac{dA_{12}}{dt} = d_1 s_1 A_1 - r_1(1 - s_1)A_{12} - (1 - r_1)(1 - s_1)A_{12} - uA_{12} - a_2 \frac{C_2}{v} A_{12} - \alpha_3 P_2 A_{12} \tag{16}$$

From the equations, the following terms $M_1 = L_1 \frac{A_1 + A_{1s} + A_{21} + A_{21s}}{w_1 + A_1 + A_{1s} + A_{21} + A_{21s}}$, $M_2 = L_2 \frac{A_2 + A_{2s} + A_{12} + A_{12s}}{w_2 + A_2 + A_{2s} + A_{12} + A_{12s}}$, $P_1 = \frac{A_{1s} + A_{21s}}{1 + A_1 + A_{1s} + A_{21} + A_{21s}}$, and $P_2 = \frac{A_{2s} + A_{12s}}{1 + A_2 + A_{2s} + A_{12} + A_{12s}}$. The additional parameters that were added are d_1 and d_2 for the drifting rate, s_1 and s_2 for the success rate representing the likelihood of adult bees being integrated into the colony they've drifted to, and lastly r_1 and r_2 for the return rate of the drifting adult bee to their original colony if they fail to integrate into the other colony.