Chapter 1

INTRODUCTION

1.1 Background of the Study

Honey bees are considered to be one of the most important and most studied insects not only because of their production of honey but also for their vital role in pollination. Almost all of the flowering plants and trees rely on bees for pollination. According to Canadian Honey Council [2], honey bees have been estimated to account for at least 80% of all pollinators. Therefore, honey bees are very essential in sustaining life on earth by their contribution to pollination of crops and other plants [22]. In addition to that, honey bees are, in economical terms, the third most important domestic animals after cattle and pigs and before poultry[26].

1.1.1 American Foulbrood

Pathogens are significant threats to the health of honey bees. These pathogens can be viruses, mites and bacteria. One of the economically most important diseases of honey bees worldwide are the bacterial brood diseases like the American Foulbrood (AFB) [8]. Among the different species of honey bees, European honey bees (*Apis mellifera* L.) are most susceptible to the disease while the Asian honey bees (*Apis*

Chapter 2

THEORETICAL FRAMEWORK

This chapter discusses the existing mathematical tools in analyzing a mathematical model. Section 2.1 mentions two different methods in analyzing system of Ordinary Differential Equations (ODEs) while section 2.2 talks about numerical approach in analyzing stochastic differential equations (SDEs). All the theorems and definitions discussed here came from the following: [24], [12], [11], [13], and [25].

2.1 Qualitative Analysis of Non-linear System of Ordinary Differential Equationy (ODE)

Definition 2.1. An **autonomous system of ODE** is of the form

$$\begin{aligned}
\frac{dX_1}{dt} &= f_1(X_1, X_2, \cdots, X_n) \\
\frac{dX_2}{dt} &= f_2(X_1, X_2, \cdots, X_n) \\
&\vdots \\
\frac{dX_n}{dt} &= f_n(X_1, X_2, \cdots, X_n)
\end{aligned}$$

or in verctor form,

$$\frac{dX_i}{dt} = f_i(X_1, X_2, ..., X_n), i = 1, 2, ..., n, \tag{2.1}$$

with initial condition $X_i(0) = X_0 \ \forall i$.

We assumed that $t \geq 0$ and $f_i : B \to \mathbb{R}$, $i = 1, 2, ..., n$ where $B \subset \mathbb{R}^n$. If we have a non-autonomous system of ODEs, $\frac{dX_i}{dt} = f_i(X_1, X_2, ..., X_n, t)$, $i = 1, 2, ..., n$, then we convert it to an autonomous system by defining $t = X_{n+1}$ and $\frac{X_{n+1}}{dt} = 1$[21].

The point $X = (X_1, X_2, \cdots, X_n)$ is the state of the system. In an epidemic model, this point is composed of the compartments of the model. The collection of these states is the state space. The solution curve of the system starting from a fixed initial condition is called the trajectory or orbit which is usually denoted by $X(t)$. The collection of this trajectories is called the flow of the system which is denoted by $\phi(X_0)$. The concept of of the flow of the differential equation indicates the dependence of the system on its initial condition.

For an ODE model to be useful, it is necessary that it has a solution. Existence of the unique solution for a given initial condition is important to effectively predict the behavior of the system.

The following are the theorems that guarantee local and global exeistence and uniqueness of solutions to ODEs:

Theorem 2.1. Existence theorem (Peano, Cauchy). Consider the autonomous system. Suppose that f is continuous on B. Then the system has a solution (not necessarily unique) on [0,δ] for sufficiently small $\delta > 0$ given any $X_0 \in B$.

Theorem 2.2. Local existence-uniqueness theorem (Picard, Lindelorf, Lipschitz, Cauchy). Consider the autonomous system. Suppose that f is locally Lipschitz continuous on B, that is, F satisfies the following condition: For each point $X_0 \in B$ there is an ϵ-neighborhood of X_0 (denoted as $B_\epsilon(X_0)$ where $B_\epsilon(X_0) \subset B$) and a positive constant m_0 such that $|f(X)-f(Y)| \leq m_0|X-Y|\ \forall X, Y \in B_\epsilon(X_0)$. Then the system has exactly one solution on [0,δ] for sufficiently small $\delta > 0$ given any $X_0 \in B$.

Theorem 2.3. Global existence-uniqueness theorem. If there is a positive constant m such that $|f(X) - f(Y)| \leq m|X - Y|\ \forall X, Y \in B$ (i.e., f is globally Lipschitz continuous on B) then the system has exactly one solution defined for all $t \in \mathbb{R}^{\oplus}$ for any $X_0 \in B$.

If all the partial derivatives $\dfrac{\partial f_i}{\partial X_j}$ $i, j = 1, 2, ..., n$ are continuous on B (i.e., $f \in C^1(B)$) then f is locally Lipschitz continuous on B. Furthermore, if the absolute value of these partial derivatives are also bounded for all $X \in B$ then f is globally Lipschitz continuous on B.

In a qualitative analysis of nonlinear system, determining the steady state of the system (the point where the system is at rest) is very important. After determining the steady state of the system, one should determine if this given point is asymptotically stable or not. Being asymptotically stable means that all the trajectories tend to that point, otherwise unstable.

Definition 2.2. The point $X^* = (X_1^*, X_2^*, \cdots, X_n^*)$ is said to be an Equilibrium state of the system if and only if $f(X^*) = 0$.

From this definition, the equation $X(t) = X^*$ with initial condition $X(0) = X^*$ is a solution to the system.

In an epidemic model, there are two types of equilibrium statess. These are the disease-free equilibrium state and the endemic equilibrium state. Disease-free equilibrium state occurs when there is no infection present in the population while endemic equilibrium state occurs when the infection is introduced to the host population. Our goal in an epidemic model is to determine if this given equilibrium point X^* is locally asymptotically stable or globally asymptotically stable. We say that an equilibrium point is stable if all the trajectory $X(t)$ starting closed to X^* stays closed to X^* as $t \to +\infty$. X^* is locally asymptotically stable if it is stable and every trajectory $X(t)$ starting closed to X^* tend to X^* as $t \to +\infty$. If it is true for all initial condition then X^* is said to be globally asymptotically stable. Moreover, if theX^* is not stable then it is called unstable. The diagrams in Figures 2.1, 2.2 can describe the said definitions above.

Local stability is the local behavior of the system near the equilibrium point. It can be determined using two methods. These are linearization and the second direct method of Lyapunov. Linearization is derived by linearizing the nonlinear system on the neighborhood of equilibrium point X^*. We linearized the system by computing

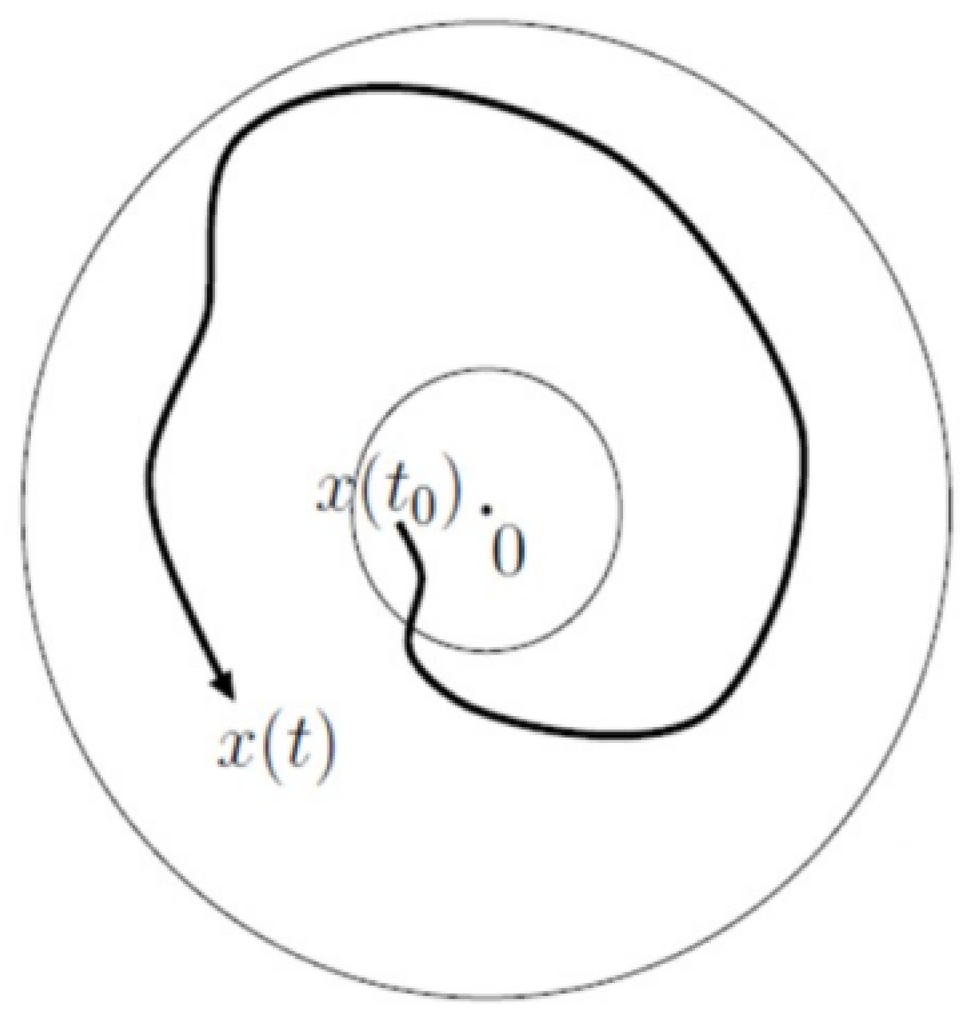

Figure 2.1. Stable Equilibrium Point

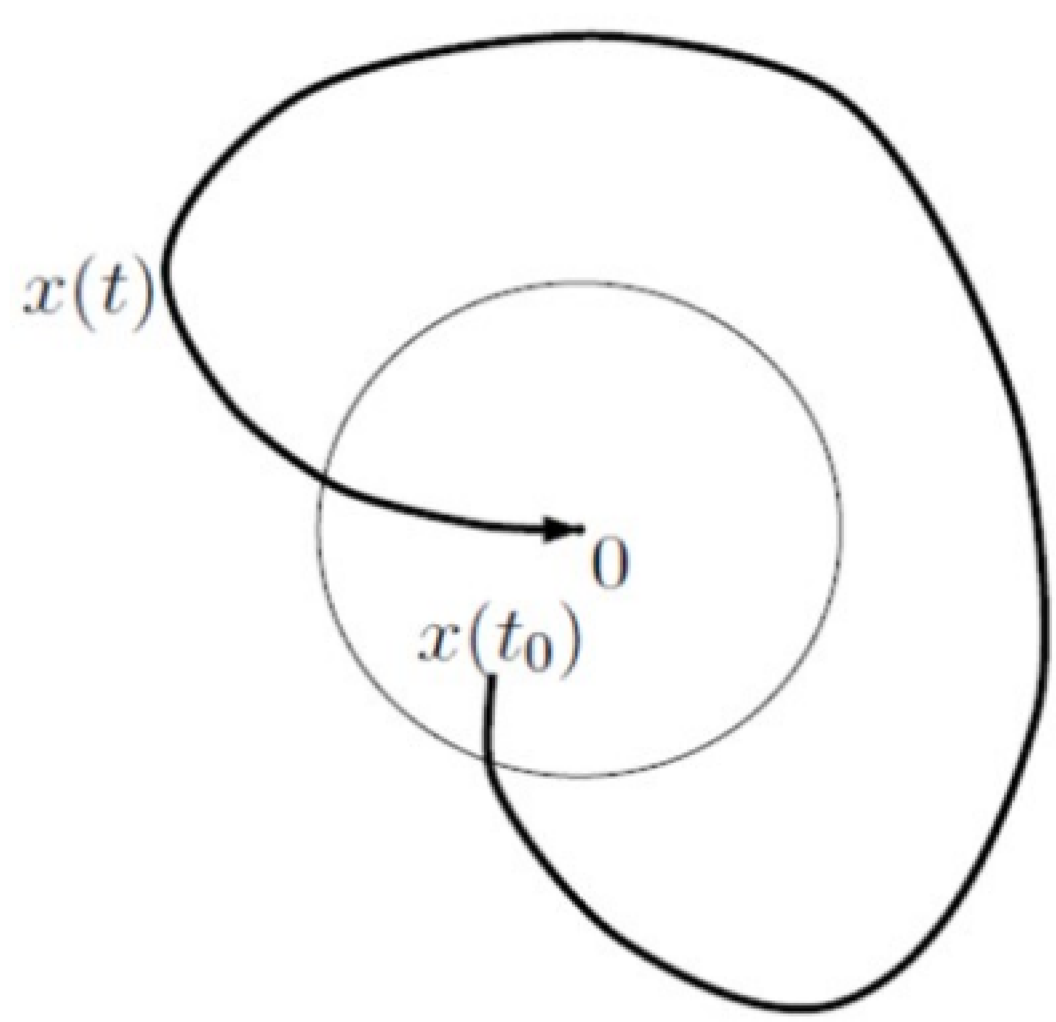

Figure 2.2. Asymtotically Stable Equilibrium Point

the Jacobian matrix $Jf(X)$ of the system, defined as

$$Jf(X) = \begin{pmatrix} \frac{\partial f_1}{\partial X_1} & \frac{\partial f_2}{\partial X_1} & \cdots & \frac{\partial f_n}{\partial X_1} \\ \frac{\partial f_1}{\partial X_2} & \frac{\partial f_2}{\partial X_2} & \cdots & \frac{\partial f_n}{\partial X_2} \\ \vdots & \vdots & \ddots & \vdots \\ \frac{\partial f_1}{\partial X_n} & \frac{\partial f_2}{\partial X_n} & \cdots & \frac{\partial f_n}{\partial X_n} \end{pmatrix}$$

and then evaluating $Jf(X)$ at X^*. The stability of the X^* can now be studied by looking at the stability of matrix $Jf(X^*)$. Eigenvalues have vital role in determining whether a given matrix is stable or not. Hence, we will only look at the eigenvalues of $Jf(X^*)$ to determine if X^* is asymptotically stable or not. If none of the real parts of the eigenvalues of $Jf(X^*)$ is equal to zero then X^* is called an hyperbolic equilibrium point. Otherwise, it is non-hyperbolic. However, in this proposal we will only deal with hyperbolic equilibrium points.

Definition 2.3. X^* is called sink/attractor if all of the eigenvalues of $Jf(X^*)$ have negative real part. X^* is called source/repeller if all of the eigenvalues of $Jf(X^*)$ have positive real part. X^* is called saddle if it is hyperbolic and $Jf(X^*)$ has at least one eigenvalue with a positive real part and at least one with a negative real part.

Theorem 2.4. If all the eigenvalues of $Jf(X^*)$ have negative real parts then X^* is an asymptotically stable equilibrium point. If at least one of the eigenvalues of $Jf(X^*)$ has a positive real part then X^* is an unstable equilibrium point.

Remark: If at least one of the eigenvalues of $Jf(X^*)$ is equal to zero then the linearization method fails. If the the linearization method fails, we cannot conclude anything about the stability of the equilibrium point. Determining the stability of the equilibrium point may now require other techniques.

In the aforementioned, the stability of X^* by linearization can be determined through studying the stability of matrix $Jf(X^*)$. There are two criteria in determining the stability of matrices. These are the Hurwitz criterion and the use of Lozinski measure.

For the second direct method of lyapunov, the followig theorem will be used:

Theorem 2.5. Let E be an open subset of $\mathbb{R}^n$ containing X^*. Suppose that $f \in C^1(E)$ and that $f(X^*) = 0$. Suppose further that there exists a real valued function $V \in C^1(E)$ satisfying $V(X^*) = 0$ and $V(X) > 0$ if $X \neq X^*$. Then,

1. if $\dfrac{dV}{dt} \leq 0 \forall X \in E$, X^* is stable;
2. if $\dfrac{dV}{dt} \leq 0 \forall X \in E - X^*$, X^* is asymptotically stable;
3. if $\dfrac{dV}{dt} \geq 0 \forall X \in E - X^*$, X^* is unstable.

This function $V : \mathbb{R}^n \to \mathbb{R}$ is called Lyapunov function. The construction of this function has no particular method. That is why the use of this method is basically more difficult compare to the linearization method.

Local stability is just concern with a behavior of the system near the neighborhood of the equilibrium point. It must be better if we can study the global behavior of this system on its entire feasible region. Hence, the global stability of the equilibrium point should be determined. Muldowney gave a sufficient condition for a given equilibrium point to be globally assymtotically stable through a geometric approach. Aside from the geometric approach proposed by Muldowney, another tool can be useful in determining the global stability of the equilibrium point by using the so called Lyapunov function and the LaSalle Invariance principle.

2.2 Numerical Solution to SDEs

The solutions to ODES are functions, while the solutions to SDEs are stochastic process. We define a continuous-time stochastic process Xas a set of random variables$X_{(t)}$ where the index variable ≥ 0 takes a continuous set of values. The index variable t may represent time.

Suppose we have an SDE model of the form $dX = \mu(X_e - X)dt + \sigma dW$ (Ornstein Uhlenbeck process) where W is a stochastic process called Brownian motion (Wiener process). The differential dW is called the white nose. Brownian motion is the continuous version of random walk and has the following properties:

1. For each t, the random variable $W_{(t)}$ is normally distributed with mean zero

and variance$t$.

2. For each $t_i < t_{(i+1)}$, the normal random variable $\Delta W_{(t_i)} = W_{(t_{(i+1)})} - W_{(t_i)}$ is independent of random variables $W_{(t_i)}, 0 \leq j \leq t_i$

3. Brownian motion W can be represented by continuous paths.

Suppose $W_{(t_0)} = 0$. We can simulate a Brownian motion using computers by discretizing time as $0 = t_0 < t_1 < \cdots$ and choosing a random number that would represent $\Delta W_{(t_{i-1})}$ from the normal distribution $N(0, t_i - t_{i-1}) = \sqrt{t_i - t_{i-1}} N(0,1)$. This implies that we obtain $W_{(t_i)}$ by multiplying $\sqrt{t_i - t_{i-1}}$ by standard normal random number and then adding the product to $W_{(t_{i-1})}$.

The solution to an SDE model has different realizations because it is based on random numbers. We can approximate a realization of the system numerically using using different methods. In this thesis, we use the Euler-Maruyama method. The Euler -Maruyama method is similar to the Euler method for ODE.

bees per day. Here, the sum of the probability that broods become spore-carrier and the probability that broods become spore-free adult bee is equal to one.

The term $\alpha_2 \frac{C}{v} A$ in (5.5) and (5.6) is the number of adult bees that become spore-carrier upon contact with the infected cells.

The term $\alpha_3 \frac{A_s}{1 + A + A_s} A$ in (5.5) and (5.6) is the number of spore-free adult bees that become spore-carrier upon contact with the spore-carrier adult bees. From the given assumptions, it is assumed that the number of spore-carrier bees has the indirect effect on the spread of spores in the storage area. The parameters involve in this case will just be embedded in the term $\alpha_3 \frac{A_s}{1 + A + A_s} A$. $\alpha_4 A$ is the number of spore-free adult bees that become spore-carrier upon contact with the food provided by the beekeeper outside the hive. We assumed here that the rate of infection α_4 is constant. Lastly, μ is the death rate of the spore-free and spore-carrier adult bees. It is assumed here that the death rate of spore-free and spore-carrier adult bees are the same.

3.2 Boundedness of the Solution

In this section, we study some basic results on the solutions of the system (3.7) which will be very useful in proving stability and persistence results.

Recall first that in the following we are interested with the the flow, $\phi(X_0)$, of the system, which we defined as the collection of trajectories or orbits given any initial condition X_0.

Lemma 3.1. *The system (3.7) is positivity preserving, that is, the flow* $\phi(X_0)$ *of the system (where* $X_0 = (B(0), B_a(0), B_s(0), C(0), A(0), A_s(0)) \in \mathbb{R}^{\oplus 6}$ *can be any initial condition) is always in* $\mathbb{R}^{\oplus 6}$.

Proof. To prove this lemma we need to observe what will happen to $F_j(X)$ where $X = (x_1 = B, x_2 = B_a, x_3 = B_s, x_4 = C, x_5 = A, x_6 = A_s)$ if $x_j = 0\ \forall j = 1, 2, 3, 4, 5, 6$.

If $B = 0$ then

$\dfrac{dB}{dt} = M - \dfrac{C}{v}M \geq 0$, since $0 \leq \dfrac{C}{v} \leq 1$ for $C \geq 0$, $A \geq 0$, and $A_s \geq 0$.

If $B_a = 0$ then

$\dfrac{dB_a}{dt} = \beta_1 B \geq 0$, for $B \geq 0$

If $B_s = 0$ then

$\dfrac{dB_s}{dt} = \alpha_1 \dfrac{A_s}{1 + A + A_s} B + \dfrac{C}{v}M \geq 0$, for $B \geq 0$, $C \geq 0$, $A \geq 0$, and $A_s \geq 0$.

If $C = 0$ then

$\dfrac{dC}{dt} = \varphi_1 \varphi_2 B_s\,(\sigma v) \geq 0$ for $B_s \geq 0$.

If $A = 0$ then

$\dfrac{dA}{dt} = \beta_2 B_a \geq 0$, for $B_a \geq 0$.

If $A_s = 0$ then

$\dfrac{dA_s}{dt} = \alpha_2 \dfrac{C}{v} A + \alpha_4 A \geq 0$, for $B_a \geq 0$, $C \geq 0$, and $A \geq 0$.

Thus, for every $X_0 \in \mathbb{R}^{\oplus 6}$, the flow $\phi(X_0)$ of the system (3.7) is always in $\mathbb{R}^{\oplus 6}$. $\square$

Lemma 3.2. *Let f be a differentiable function of t satisfying the differential inequality*

$$f'(t) \leq b - af(t) \text{ for all } t \geq 0. \tag{3.8}$$

where a and b are constants. Then

$$f(t) \leq \frac{b}{a} - \frac{b}{a}e^{-at} + f(0)e^{-at} \text{ for all } t \geq 0.$$

Proof. Multiply both sides of (3.8) by e^{at} and transpose all terms on the right side to left side, getting

$$\begin{aligned} 0 &\geq \left(f'(t) + af(t) - b\right)e^{at} \\ &= e^{at}f'(t) + ae^{at}f(t) - be^{at} \\ &= \frac{d}{dt}\left(f(t)e^{at}\right) - \frac{d}{dt}\left(\frac{b}{a}e^{at}\right) \\ &= \frac{d}{dt}\left(f(t) - \frac{b}{a}\right)e^{at} \end{aligned}$$

The function $\left(f(t) - \frac{b}{a}\right)e^{at}$ thus has a negative or zero derivative and so is non-increasing for $t \geq 0$. Therefore, for any $t \geq 0, we have$

$$\begin{aligned} &\left(f(t) - \frac{b}{a}\right)e^{at} \leq \left(f(0) - \frac{b}{a}\right)e^{0} \\ \Rightarrow &\left(f(t) - \frac{b}{a}\right)e^{at} \leq f(0) - \frac{b}{a} \\ \Rightarrow &f(t)e^{at} \leq f(0) - \frac{b}{a} + \frac{b}{a}e^{at} \\ \Rightarrow &f(t) \leq f(0)e^{-at} - \frac{b}{a}e^{-at} + \frac{b}{a}. \end{aligned}$$

Therefore,

$$f(t) \leq \frac{b}{a} - \frac{b}{a}e^{-at} + f(0)e^{-at} \text{ for all } t \geq 0.$$

□

Theorem 3.3. The flow $\phi(X_0)$ $(X_0 \in \Gamma)$ of the system (3.7) is confined on the set $\Gamma = \Bigg\{(B, B_a, B_s, C, A, A_s)\,|0 \le B + B_s \le \dfrac{L}{\gamma} + (B+B_s)(0)e^{-\gamma t},$
$0 \le B_a \le \dfrac{L+\beta_1 B(0)}{\beta_2} + B_a(0)e^{-\beta_2 t}, 0 \le C \le v + C(0)e^{-\varphi_1\varphi_2\left(\frac{L}{\gamma}+(B+B_s)(0)\right)\sigma t},$
$0 \le A + A_s \le \dfrac{L+\beta_1 B(0)+\beta_2 B_a(0)}{\mu} + (A+A_s)(0)e^{-\mu t}\Bigg\}$ for all $t \ge 0$, where $\gamma = \min(\beta_1, \varphi_1\varphi_2)$.

Proof. Consider eqution (5.1),

$$\begin{aligned}
&\frac{dB}{dt} = L\frac{A+A_s}{w+A+A_s} - \alpha_1\frac{A_s}{1+A+A_s}B - \frac{C}{v}M - \beta_1 B\\
\Rightarrow&\frac{dB}{dt} \le L\frac{A+A_s}{w+A+A_s} - \beta_1 B\\
\Rightarrow&\frac{dB}{dt} \le L - \beta_1 B
\end{aligned}$$

Using Lemma 3.2, we have

$$\begin{aligned}
&B \le \frac{L}{\beta_1} - \frac{L}{\beta_1}e^{-\beta_1 t} + B(0)e^{-\beta_1 t} \text{ for all } t \ge 0.\\
\Rightarrow&B \le \frac{L}{\beta_1} + B(0)e^{-\beta_1 t} \text{ for all } t \ge 0. \qquad (3.9)
\end{aligned}$$

Consider equation (5.2),

$$\begin{aligned}
&\frac{dB_a}{dt} = \beta_1 B - \beta_2 B_a\\
\Rightarrow&\frac{dB_a}{dt} \le \beta_1\left(\frac{L}{\beta_1} + B(0)e^{-\beta_1 t}\right) - \beta_2 B_a\\
\Rightarrow&\frac{dB_a}{dt} \le \beta_1\left(\frac{L}{\beta_1} + B(0)\right) - \beta_2 B_a, \text{ since } 0 \le e^{-\beta_1 t} \le 1 \text{ for all } t \ge 0\\
\Rightarrow&\frac{dB_a}{dt} \le L + \beta_1 B(0) - \beta_2 B_a
\end{aligned}$$

Using Lemma 3.2, we have

$$B_a \leq \frac{L+\beta_1 B(0)}{\beta_2} - \frac{L+\beta_1 B_a(0)}{\beta_2}e^{-\beta_2 t} + B_a(0)e^{-\beta_2 t} \text{ for all } t \geq 0.$$
$$\Rightarrow B_a \leq \frac{L+\beta_1 B(0)}{\beta_2} + B_a(0)e^{-\beta_2 t} \text{ for all } t \geq 0. \tag{3.10}$$

From equations (5.5) and (5.6), we have

$$\frac{d(A+A_s)}{dt} = \beta_2 B_a - \mu(A+A_s)$$
$$\Rightarrow \frac{d(A+A_s)}{dt} \leq \beta_2\left(\frac{L+\beta_1 B(0)}{\beta_2} + B_a(0)e^{-\beta_2 t}\right) - \mu(A+A_s),$$
$$\text{since } B_a \leq \frac{L+\beta_1 B(0)}{\beta_2} + B_a(0)e^{-\beta_2 t} \text{ for all } t \geq 0.$$
$$\Rightarrow \frac{d(A+A_s)}{dt} \leq L+\beta_1 B(0) + \beta_2 B_a(0)e^{-\beta_2 t} - \mu(A+A_s).$$
$$\Rightarrow \frac{d(A+A_s)}{dt} \leq L+\beta_1 B(0) + \beta_2 B_a(0) - \mu(A+A_s), \text{ since } 0 \leq e^{-\beta_2 t} \leq 1 \text{ for all } t \geq 0$$

Using Lemma 3.2, we have

$$A+A_s \leq \frac{L+\beta_1 B(0)+\beta_2 B_a(0)}{\mu} - \frac{L+\beta_1 B(0)+\beta_2 B_a(0)}{\mu}e^{-\mu t}$$
$$+ (A+A_s)(0)e^{-\mu t} \text{ for all } t \geq 0.$$
$$\Rightarrow A+A_s \leq \frac{L+\beta_1 B(0)+\beta_2 B_a(0)}{\mu} + (A+A_s)(0)e^{-\mu t} \text{ for all } t \geq 0. \tag{3.11}$$

From equations (5.1) and (5.3), we have

$$\frac{d(B+B_s)}{dt} = L\frac{A+A_s}{w+A+A_s} - \varphi_1\varphi_2 B_s - \beta_1 B$$
$$\Rightarrow \frac{d(B+B_s)}{dt} \leq L - \varphi_1\varphi_2 B_s - \beta_1 B$$
$$\Rightarrow \frac{d(B+B_s)}{dt} \leq L - \gamma(B+B_s), \text{ where } \gamma = \min(\varphi_1\varphi_2, \beta_1)$$

Using Lemma 3.2, we have

$$
\begin{aligned}
B+B_s &\leq \frac{L}{\gamma}-\frac{L}{\gamma}e^{-\gamma t}+(B+B_s)(0)e^{-\gamma t} \text{ for all } t\geq 0.\\
\Rightarrow B+B_s &\leq \frac{L}{\gamma}+(B+B_s)(0)e^{-\gamma t} \text{ for all } t\geq 0. \qquad (3.12)
\end{aligned}
$$

Lastly, From the equations (5.4),

$$
\begin{aligned}
\frac{dC}{dt} &= \varphi_1\varphi_2 B_s\left[\sigma\left(v-C\right)\right]\\
\frac{dC}{dt} &= \varphi_1\varphi_2 B_s\sigma v-\varphi_1\varphi_2 B_s\sigma C\\
\frac{dC}{dt} &\leq \varphi_1\varphi_2(B+B_s)\sigma v-\varphi_1\varphi_2(B+B_s)\sigma C\\
\frac{dC}{dt} &\leq \varphi_1\varphi_2\left(\frac{L}{\gamma}+(B+B_s)(0)e^{-\gamma t}\right)\sigma v-\varphi_1\varphi_2\left(\frac{L}{\gamma}+(B+B_s)(0)e^{-\gamma t}\right)\sigma C\\
\frac{dC}{dt} &\leq \varphi_1\varphi_2\left(\frac{L}{\gamma}+(B+B_s)(0)\right)\sigma v-\varphi_1\varphi_2\left(\frac{L}{\gamma}+(B+B_s)(0)\right)\sigma C \text{ for all } t\geq 0
\end{aligned}
$$

Using Lemma 3.2, we have

$$
\begin{aligned}
C \leq\ & \frac{\varphi_1\varphi_2\left(\frac{L}{\gamma}+(B+B_s)(0)\right)\sigma v}{\varphi_1\varphi_2\left(\frac{L}{\gamma}+(B+B_s)(0)\right)\sigma}\\
&-\frac{\varphi_1\varphi_2\left(\frac{L}{\gamma}+(B+B_s)(0)\right)\sigma v}{\varphi_1\varphi_2\left(\frac{L}{\gamma}+(B+B_s)(0)\right)\sigma}e^{-\varphi_1\varphi_2\left(\frac{L}{\gamma}+(B+B_s)(0)\right)\sigma t}\\
&+C(0)e^{-\varphi_1\varphi_2\left(\frac{L}{\gamma}+(B+B_s)(0)\right)\sigma t} \text{ for all } t\geq 0.\\
\Rightarrow C \leq\ & v-ve^{\varphi_1\varphi_2\left(\frac{L}{\gamma}+(B+B_s)(0)\right)\sigma}+C(0)e^{-\varphi_1\varphi_2\left(\frac{L}{\gamma}+(B+B_s)(0)\right)\sigma t} \text{ for all } t\geq 0.\\
\Rightarrow C \leq\ & v+C(0)e^{-\varphi_1\varphi_2\left(\frac{L}{\gamma}+(B+B_s)(0)\right)\sigma t} \text{ for all } t\geq 0. \qquad (3.13)
\end{aligned}
$$

Observe that C cannot exceed v from our assumptions. Also, as the value of C approaches v, $\frac{dC}{dt}$ approaches zero.

Therefore, from (3.10)(3.11)(3.12)(3.13) and Lemma 3.1, the theorem holds. $\square$

Corollary 3.4. The set Γ is an invariant set. That is, for every initial condition $X_0 \in \Gamma$ the flow $\phi(X_0)$ remains in Γ.

Proof. The proof follows from Theorem 3.3. $\square$

This corollary can help us to limit our study on the behavior of the system by considering initial conditions within Γ. That is, we can just simply look at the solutions of the system within the set Γ, since we have shown that all the solutions starting within Γ will just converge in a point on Γ.

Corollary 3.5. If the initial condition of the system (3.7) is $X_0 = (0, 0, 0, 0, A, 0)$, where A is non-zero, then the its flow $\phi(X_0)$ is confined on the set

$$\Gamma_2 = \left\{ (B, B_a, B_s, C, A, A_s) \,|0 \leq B + B_s \leq \frac{L}{\gamma}, 0 \leq B_a \leq \frac{L}{\beta_2}, 0 \leq C \leq v, 0 \leq A + A_s \leq \frac{L}{\mu} A e^{-\mu t} \right\}$$

for all $t \geq 0$.

Proof. If the initial condition of the system (3.7) is $(0, 0, 0, 0, A, 0)$, where A is non-zero then the values of $B(0)$, $B_a(0)$, $B_s(0)$, $C(0)$, and $A_s(0)$ are all zero while the value of $A(0) = A$. from the Theorem 3.3, the corollary follows. $\square$

Qualitative study on the behavior of the system (3.7) is concerned with what will happen to the system as time t approaches ∞. The next theorem will show us what will happen to the solutions of the system (3.7) as the time $t \to \infty$.

Theorem 3.6. The flow $\phi(X_0)$ $(X_0 \in \Gamma_1)$ of the system (3.7) is eventually confined on the set $\Gamma_1 = \left\{ (B, B_a, B_s, C, A, A_s) \,|\, 0 \leq B + B_s \leq \frac{L}{\gamma}, 0 \leq B_a \leq \frac{L + \beta_1 B(0)}{\beta_2}, 0 \leq C \leq v, 0 \leq A + A_s \leq \frac{L + \beta_1 B(0) + \beta_2 B_a(0)}{\mu} \right\}$ as t approaches infinity, where $\gamma = \min(\beta_1, \varphi_1\varphi_2)$.

Proof. From Theorem 3.3, we know that

$$\begin{aligned}
0 \leq B + B_s &\leq \frac{L}{\gamma} + (B + B_s)(0)e^{-\gamma t} \\
0 \leq B_a &\leq \frac{L + \beta_1 B(0)}{\beta_2} + B_a(0)e^{-\beta_2 t} \\
0 \leq C &\leq v + C(0)e^{-\varphi_1\varphi_2\left(\frac{L}{\gamma} + (B+B_s)(0)\right)\sigma t} \\
0 \leq A + A_s &\leq \frac{L + \beta_1 B(0) + \beta_2 B_a(0)}{\mu} + (A + A_s)(0)e^{-\mu t}
\end{aligned}$$

As t approaches ∞ the the values of $e^{-\gamma t}$, $e^{-\beta_2 t}$, $e^{-\varphi_1\varphi_2\left(\frac{L}{\gamma} + (B+B_s)(0)\right)\sigma t}$, and $e^{-\mu t}$ will approach to zero. This will make the last terms of the right hand side of the above inequality equal to zero. Hence, we have

$$\begin{aligned}
0 \leq B + B_s &\leq \frac{L}{\gamma} + (B + B_s)(0) \\
0 \leq B_a &\leq \frac{L + \beta_1 B(0)}{\beta_2} \\
0 \leq C &\leq v
\end{aligned}$$

$$0 \leq A + A_s \leq \frac{L + \beta_1 B(0) + \beta_2 B_a(0)}{\mu}$$

Therefore, The flow $\phi(X_0)$ $(X_0 \in \Gamma_1)$ of the system (3.7) is eventually confined on the set $\Gamma_1 = \left\{ (B, B_a, B_s, C, A, A_s) \,|0 \leq B + B_s \leq \frac{L}{\gamma}, 0 \leq B_a \leq \frac{L + \beta_1 B(0)}{\beta_2}, \right.$ $\left. 0 \leq C \leq v, 0 \leq A + A_s \leq \frac{L + \beta_1 B(0) + \beta_2 B_a(0)}{\mu} \right\}$ as t approaches infinity. □

3.3 Existence and Uniqueness of the Solution

The next results will be used to show the existence and uniqueness of the solution of the system (3.7). The following Lemmas are corollaries of Theorems 2.2 and 2.3.

Lemma 3.7. [21] *Given the system* $\frac{dX}{dt} = F(X)$, *let* $F : U \to \mathbb{R}^n$ *where* $U \in \mathbb{R}^{\oplus n}$ *and* $F(X) = (F_1(X), F_2(X), \cdots, F_n(X))$, $X = (x_1, x_2, \cdots, x_n)$. *Further, suppose* F *is continuous in* U *and have partial derivatives* $\frac{\partial F_j}{\partial x_k}$ *which exist and are continuous in* U *for all* $j, k = 1, 2, \cdots, n$. *Then* F *is locally lipchitz in* U.

Moreover, if the absolute value of these partial derivatives are also bounded for all $X \in U$ *then* F *is globally lipchitz continuous on* U.

Lemma 3.8. [21] *Let* $\frac{dX}{dt} = F(X)$ *where* $X \in U$ *and* $F : U \to \mathbb{R}^n$. *If* F *is globally lipchitz continuous on* U *then* $\frac{dX}{dt} = F(X)$ *has exactly one solution defined for all* $t \in \mathbb{R}^{\oplus}$.

Here, we will look at the behavior of the solutions of system (3.7) along the set Γ since from Theorem (3.3), the flow of the system (3.7) remains in Γ for every initial condition $X_0 \in \Gamma$. Now, consider the function $F : \Gamma \to \Gamma$, where F is the function given in system (3.7). It is clear that F is a continuous function of $X = (B, B_a, B_s, C, A, A_s)$.

Define,

$$F_1(X) = L\frac{A + A_s}{w + A + A_s} - \alpha_1\frac{A_s}{1 + A + A_s}B - \frac{C}{v}L\frac{A + A_s}{w + A + A_s} - \beta_1 B$$
$$F_2(X) = \beta_1 B - \beta_2 B_a$$
$$F_3(X) = \alpha_1\frac{A_s}{1 + A + A_s}B + \frac{C}{v}L\frac{A + A_s}{w + A + A_s} - \varphi_1\varphi_2 B_s$$
$$F_4(X) = \varphi_1\varphi_2 B_s\left[\sigma\left(v - C\right)\right]$$
$$F_5(X) = \beta_2\left(1 - \alpha_1\frac{A_s}{1 + A + A_s}\right)B_a - \alpha_2\frac{C}{v}A - \alpha_3\frac{A_s}{1 + A + A_s}A - \alpha_4 A - \mu A$$
$$F_6(X) = \alpha_1\beta_2\frac{A_s}{1 + A + A_s}B_a + \alpha_2\frac{C}{v}A + \alpha_3\frac{A_s}{1 + A + A_s}A + \alpha_4 A - \mu A_s$$

where $X = (B, B_a, B_s, C, A, A_s)$.

Now, let us consider the following partial derivatives

$$\frac{\partial F_1}{\partial B} = -\alpha_1\frac{A_s}{1 + A + A_s} - \beta_1,$$
$$\frac{\partial F_1}{\partial B_a} = 0,$$
$$\frac{\partial F_1}{\partial B_s} = 0,$$
$$\frac{\partial F_1}{\partial C} = -L\frac{A + A_s}{v(w + A + A_s)},$$
$$\frac{\partial F_1}{\partial A} = \frac{wL}{(w + A + A_s)^2} + \frac{\alpha_1 A_s B}{(1 + A + A_S)^2} - \frac{wLC}{v(w + A + A_s)^2} \text{ and}$$
$$\frac{\partial F_1}{\partial A_s} = \frac{wL}{(w + A + A_s)^2} - \frac{\alpha_1(A + 1)B}{(1 + A + A_S)^2} - \frac{wLC}{v(w + A + A_s)^2}$$

$$\frac{\partial F_2}{\partial B} = \beta_1,$$

$$\frac{\partial F_2}{\partial B_a} = -\beta_2,$$
$$\frac{\partial F_2}{\partial B_s} = 0,$$
$$\frac{\partial F_2}{\partial C} = 0,$$
$$\frac{\partial F_2}{\partial A} = 0, \text{ and}$$
$$\frac{\partial F_2}{\partial A_s} = 0;$$

$$\frac{\partial F_3}{\partial B} = \alpha_1 \frac{A_s}{1 + A + A_s},$$
$$\frac{\partial F_3}{\partial B_a} = 0,$$
$$\frac{\partial F_3}{\partial B_s} = -\varphi_1\varphi_2,$$
$$\frac{\partial F_3}{\partial C} = L\frac{A + A_s}{v(w + A + A_s)},$$
$$\frac{\partial F_3}{\partial A} = -\frac{\alpha_1 A_s B}{(1 + A + A_S)^2} + \frac{wLC}{v(w + A + A_s)^2}, \text{ and}$$
$$\frac{\partial F_3}{\partial A_s} = \frac{\alpha_1(1 + A)B}{(1 + A + A_S)^2} + \frac{wLC}{v(w + A + A_s)^2};$$

$$\frac{\partial F_4}{\partial B} = 0,$$
$$\frac{\partial F_4}{\partial B_a} = 0,$$
$$\frac{\partial F_4}{\partial B_s} = \varphi_1\varphi_2\sigma v - \varphi_1\varphi_2\sigma C,$$
$$\frac{\partial F_4}{\partial C} = -\varphi_1\varphi_2 B_s\sigma,$$
$$\frac{\partial F_4}{\partial A} = 0, \text{ and}$$
$$\frac{\partial F_4}{\partial A_s} = 0;$$

$$\frac{\partial F_5}{\partial B} = 0,$$
$$\frac{\partial F_5}{\partial B_a} = \beta_2 - \frac{\beta_2\alpha_1 A_s}{1+A+A_s},$$
$$\frac{\partial F_5}{\partial B_s} = 0,$$
$$\frac{\partial F_5}{\partial C} = -\frac{\alpha_2 A}{v},$$
$$\frac{\partial F_5}{\partial A} = \frac{\alpha_1\beta_2 A_s B_a}{(1+A+A_s)^2} - \alpha_2\frac{C}{v} - -\alpha_3\frac{A_s(1+A_s)}{(1+A+A_s)^2} - \alpha_4 - \mu, \text{ and}$$
$$\frac{\partial F_5}{\partial A_s} = -\frac{\beta_2\alpha_1(1+A)B_a}{(1+A+A_s)^2} - \alpha_3\frac{A(1+A)}{(1+A+A_s)^2};$$

$$\frac{\partial F_6}{\partial B} = 0,$$
$$\frac{\partial F_6}{\partial B_a} = \frac{\beta_2\alpha_1 A_s}{1+A+A_s},$$
$$\frac{\partial F_6}{\partial B_s} = 0,$$
$$\frac{\partial F_6}{\partial C} = \frac{\alpha_2 A}{v},$$
$$\frac{\partial F_6}{\partial A} = -\frac{\alpha_1\beta_2 A_s B_a}{(1+A+A_s)^2} + \alpha_2\frac{C}{v} + \alpha_3\frac{A_s(1+A_s)}{(1+A+A_s)^2} + \alpha_4, \text{ and}$$
$$\frac{\partial F_6}{\partial A_s} = \frac{\beta_2\alpha_1(1+A)B_a}{(1+A+A_s)^2} + \alpha_3\frac{A(1+A)}{(1+A+A_s)^2} - \mu.$$

Lemma 3.9. *Let $F : \Gamma \to \Gamma$ where $\Gamma \subset \mathbb{R}^n$ $F(X) = (F_1(X), F_2(X), \cdots, F_6(X))$ and $X = (B, B_a, B_s, C, A, A_s)$. Then F is locally lipchitz continuous on Γ.*

Proof. It is clear that F is a continuous function on Γ. Moreover, from the partial derivatives above, we can say that they all exist and continuous on Γ. Thus, from Lemma 3.7 we can say that F is locally lipchitz continuous on Γ. □

Theorem 3.10. For all $X_0 \in \Gamma$, there exists a unique solution to the system (3.7).

Proof. It is clear that all of the solutions of the system(3.7) are bounded from theorem (3.3). Hence, the partial derivatives $\dfrac{\partial F_j}{\partial X}$ above, will be bounded. Therefore, from Lemmas (3.7), (3.9) and (3.8), the system (3.7) has an exactly one solution defined on $t \in \mathbb{R}^{\oplus}$. $\square$

3.4 Finding the Equilibrium Points

In this section we will find the possible equilibrium states of the system (3.7). The possible equilibrium states of the system (3.7) are the trivial equilibrium state $(0, 0, 0, 0, 0, 0)$, disease-free equilibrium state $(B^*, B_a^*, 0, 0, A^*, 0)$, and the extinction equilibrium state $(0, 0, 0, C^*, 0, 0)$. These equilibrium states will be used to determine the fate of the colony in the long run in a given situation. Stability of these equilibrium states will be determined in section 3.5. Since the colony is expected to be eradicated in the long run once infected by the AFB spores [6], [5], [7],[8], it is better to focus on the stability analysis of the extinction equilibrium state. Also, it is the interest of this thesis to determine the factors that will ensure that a bee colony will be AFB spore-free.

To find the equilibrium states of our system (3.7), we need to equate all $F_j(X)$ to zero for all $j = 1, 2, 3, 4, 5, 6$. Hence, we have the following:

$$L\frac{A + A_s}{w + A + A_s} - \alpha_1 \frac{A_s}{1 + A + A_s} B - L\frac{C}{v}\frac{A + A_s}{w + A + A_s} - \beta_1 B = 0 \tag{3.14}$$

$$\beta_1 B - \beta_2 B_a = 0 \tag{3.15}$$

$$\alpha_1 \frac{A_s}{1 + A + A_s} B + L \frac{C}{v} \frac{A + A_s}{w + A + A_s} - \varphi_1 \varphi_2 B_s = 0 \tag{3.16}$$

$$\varphi_1 \varphi_2 B_s \left[\sigma \left(v - C \right) \right] = 0 \tag{3.17}$$

$$\beta_2 \left(1 - \alpha_1 \frac{A_s}{1 + A + A_s} \right) B_a - \alpha_2 \frac{C}{v} A - \alpha_3 \frac{A_s}{1 + A + A_s} A - \alpha_4 A - \mu A = 0 \tag{3.18}$$

$$\beta_2 \alpha_1 \frac{A_s}{1 + A + A_s} B_a + \alpha_2 \frac{C}{v} A + \alpha_3 \frac{A_s}{1 + A + A_s} A + \alpha_4 A - \mu A_s = 0. \tag{3.19}$$

For the disease free equilibrium point: we set $B_s = C = A_s = 0$, so that the above equations become:

$$L \frac{A}{w + A} - \beta_1 B = 0 \tag{3.20}$$

$$\beta_1 B - \beta_2 B_a = 0 \tag{3.21}$$

$$\beta_2 B_a - \alpha_4 A - \mu A = 0 \tag{3.22}$$

$$\alpha_4 A = 0 \tag{3.23}$$

From (3.23), it will follow that $\alpha_4 = 0$ since A can not be equal to zero. This means that the disease-free equilibrium point will exist only if the rate at which adult bees will be infected upon contact to the infected food outside the hive is equal to zero.

From equations (3.20) (3.21)(3.22) we have,

$$\begin{aligned}
& L \frac{A}{w + A} - \mu A = 0 \\
\Rightarrow & A \left(\frac{L}{w + A} - \mu \right) = 0 \\
\Rightarrow & \frac{L}{w + A} - \mu = 0, \text{ since } A \neq 0 \\
\Rightarrow & \frac{L}{w + A} = \mu \\
\Rightarrow & w + A = \frac{L}{\mu}
\end{aligned}$$

$$\Rightarrow A = \frac{L}{\mu} - w \tag{3.24}$$

$$\Rightarrow A = \frac{L - w(\mu)}{\mu} \tag{3.25}$$

From equations (3.21)(3.22) we have,

$$\mu A = \beta_1 B$$

$$\Rightarrow B = \frac{\mu A}{\beta_1}$$

$$\Rightarrow B = \frac{L - w(\mu)}{\beta_1} \tag{3.26}$$

From equation (3.21) we have,

$$B_a = \frac{\beta_1 B}{\beta_2}$$

$$\Rightarrow B_a = \frac{L - w(\mu)}{\beta_2} \tag{3.27}$$

Thus, the disease free equilibrium point of the system (3.7) is $X^* = \left(\frac{L - w\mu}{\beta_1}, \frac{L - w\mu}{\beta_2}, 0, 0, \frac{L - w\mu}{\mu}, 0\right)$. Aside from the condition that $\alpha_4 = 0$, one thing that we need to consider so that X^* exists is that $\frac{L}{w\mu} > 1$ since $L - w\mu$ must be greater than zero.

From the given assumptions, the colony is asssumed to be healthy from the start. Thus, if $\alpha_4 = 0$ with initial condition $(B, B_a, 0, 0, 0, A, 0)$, the system (3.7) can be reduced into system of ordinary differential equations of three state variables: A, B, B_a.

The reduced system (3.28) is given by

$$\begin{aligned} \frac{dB}{dt} &= L\frac{A}{w+A} - \beta_1 B \\ \frac{dB_a}{dt} &= \beta_1 B - \beta_2 B_a \\ \frac{dA}{dt} &= \beta_2 B_a - \mu A \end{aligned} \tag{3.28}$$

It is more interesting to find out what will happen to our system (3.7) if $\alpha_4 \neq 0$. In this case, it is more practical to study the behavior of the solution of the system (3.7) if $\dfrac{L}{w\mu} > 1$ since if it is the otherwise, the colony as we will see in Corollary (3.18) will die.

The next theorem will tell us that if $\alpha_4 \neq 0$ then trivial equilibrium state and the diseased equilibrium state X^* do not exist.

Theorem 3.11. In system (3.7), if $\alpha_4 \neq 0$ then the solution of C will converge to a non-zero value.

Proof. From equation (5.4), we have

$$\frac{dC}{dt} = \varphi_1 \varphi_2 B_s \left[\sigma \left(v - C\right)\right].$$

It is clear that the value of $\dfrac{dC}{dt}$ is always greater than or equal to zero depending on the the value of B_s. If $\alpha_4 \neq 0$, there is a moment that A_s is non-zero which means that B_s has also a moment that it is non-zero. Hence, C will be increasing and converging to a non-zero value. $\square$

Theorem 3.12. The point $(0,0,B_s,C,0,A_s)$,where B_s, C, A_s are all non-zero, is not an equilibrium point of the system (3.7).

Proof. If $B = 0, B_a = 0$ and $A = 0$ then

$$L\frac{A_s}{w+A_s} - \left(\frac{C}{v}\right)\left(\frac{A_s}{w+A_s}\right) = 0 \tag{3.29}$$

$$\left(\frac{C}{v}\right)\left(\frac{A_s}{w+A_s}\right) - \varphi_1\varphi_2 B_s = 0 \tag{3.30}$$

$$\varphi_1\varphi_2 B_s\left[\sigma\left(v-C\right)\right] = 0 \tag{3.31}$$

$$\mu A_s = 0 \tag{3.32}$$

In (3.32), $\mu = 0$ or $A_s = 0$. However, μ cannot be equal to zero since it is the death rate of adult bees. Hence, A_s will be force to be equal to 0. If $A_s = 0$ then from (3.30) $\varphi_1\varphi_2 B_s = 0$. Since φ_1 and φ_2 cannot be both equal to 0, then $B_s = 0$.

Therefore, the theorem holds. $\square$

Note that in the proof of the previous theorem, C can be any value. This means that the point $(0,0,0,C,0,0)$, where $C \in \mathbb{R}$, is an equilibrium point of system (3.7). This equilibrium state occurs only when the value of the parameter $\alpha_4 \neq 0$ or the initial condition of either B_s, C, or A_s is non-zero.

In the next theorem, we will show that $(0,0,0,0,0,0), X^*$ and $(0,0,0,C,0,0)$ are the only equilibrium states of the system (3.7).

Theorem 3.13. The system (3.7) has only three equilibrium states: $(0,0,0,0,0,0)$, X^* and $(0,0,0,C,0,0)$, where $C \in \mathbb{R}$.

Proof. From equations (3.17), we have

$$\varphi_1\varphi_2 B_s \left[\sigma\left(v - C\right)\right] = 0.$$

$$\Rightarrow \varphi_1\varphi_2 B_s = 0 \text{ or } \sigma\left(v - C\right) = 0$$

Case 1: If $\varphi_1\varphi_2 B_s = 0$ and $C = 0$ then Equation 3.16 will be $\alpha_1 \dfrac{A_s}{1 + A + A_s} B = 0$. There are two cases that can be considered in this equation: $B = 0$ or $A_s = 0$. If $B = 0$ then using Equation 3.15, $B_a = 0$. Substituting the values of $B_a = 0$ and $C = 0$, Equation 3.18 will be $-\alpha_3 \dfrac{A_s}{1 + A + A_s} A - \alpha_4 A - \mu A = 0$. This equation will give the value of $A = 0$. Hence, the equilibrium state that can be derived from this case will be the trivial equilibrium state. Now, for the second case, we will take the value of $A_s = 0$. If we substitute the values $C = 0$ and $A_s = 0$ to Equations 3.14, 3.15, 3.16, 3.17, 3.18, and 3.19 then we will have the following equations: $\beta_1 B - \beta_2 B_a = 0$, $\alpha_4 A = 0$, and $\beta_2 B_a - \alpha_4 - \mu A = 0$. From these equations, the disease free equilibrium state will be derived.

Case 2: If $\varphi_1\varphi_2 B_s = 0$ and $C \neq 0$ then $B_s = 0$ since $\varphi_1\varphi_2$ cannot take a zero value. Equation (3.16) will be $\alpha_1 \dfrac{A_s}{1 + A + A_s} B + \dfrac{C}{v} M = 0$. Since the two terms are greater than or equal to zero then $\alpha_1 \dfrac{A_s}{1 + A + A_s} B = 0$ and $L \dfrac{C}{v} \dfrac{A + A_s}{w + A + A_s} = 0$. Since $C \neq 0$ then A_s will be zero. Equation 3.19 will be $\alpha_2 \dfrac{C}{v} A + \alpha_4 A = 0$. This implies that $A = 0$, since $\alpha_4 \neq 0$ if $C \neq 0$. If $A = 0$ then in Equation 3.18, $B_a = 0$. This will make $B = 0$ in Equation 3.15. Hence, for this case the point $(0, 0, 0, C, 0, 0)$ will derived as one of the equilibrium points.

Case 3: If $\varphi_1\varphi_2 B_s \neq 0$ and $C \neq 0$ then $\sigma\left(v - C\right) = 0$.

$\Rightarrow v - C = 0$, since σ cannot be equal to zero.

$\Rightarrow C = v$.

Equation (3.14) will become $\alpha_1 \dfrac{A_s}{1 + A + A_s} B + \beta_1 B = 0$

$\Rightarrow B = 0$ since the two terms on the left side of the above equation cannot be negative.

$\Rightarrow B_a = 0$, from Equation (3.15).

From Equations (3.18) and (3.19), $A = 0$ and $A_s = 0$

$\Rightarrow B_s = 0$, Using Equation (3.16)

$\Rightarrow C = v$.

Thus, for this case the point $(0, 0, 0, v, 0, 0)$ will be the equilibrium state.

Therefore, the theorem holds.

□

As Biological Remark:

If the value of $C = v$ then all of the eggs laid by the queen will be surely infected. If this is the case, there will be no more eggs that will eclose to adult bees so that A_s and A will converge to zero.

From Theorems (3.11) and (3.13) we have the following conclusion.

Theorem 3.14. In system (3.7), if $\alpha_4 \neq 0$ then the point $(0, 0, 0, C, 0, 0)$ is the only equilibrium state.

3.5 Stability Analysis

To study the stability behavior of the equilibrium states, $(0,0,0,0,0,0)$ and $X^* = \left(\frac{L-w\mu}{\beta_1}, \frac{L-w\mu}{\beta_2}, 0, 0, \frac{L-w\mu}{\mu}, 0\right)$, of system (3.7) we can just simply consider the system (3.28). The equilibrium states $(0,0,0,0,0,0)$ and X^* are equivalent to $(0,0,0)$ and $\left(\frac{L-w\mu}{\beta_1}, \frac{L-w\mu}{\beta_2}, \frac{L-w\mu}{\mu}\right)$ in system (3.28) respectively.

Theorem 3.15. In system (3.28), if $\frac{L}{w\mu} > 1$ then the equilibrium state $X^{**} = \left(\frac{L-w\mu}{\beta_1}, \frac{L-w\mu}{\beta_2}, \frac{L-w\mu}{\mu}\right)$ is asymptotically stable.

Proof. We will use the linearization method to show that the equilibrium state X^{**} is assymptotically stable when $\frac{L}{w\mu} > 1$.

The Jacobian matrix of system (3.28) at X^{**} is given by

$$JF\left(X^{**}\right) = \begin{pmatrix} -\beta_1 & 0 & \frac{Lw}{\left(w + \frac{L-w\mu}{\mu}\right)^2} \\ \beta_1 & -\beta_2 & 0 \\ 0 & \beta_2 & -\mu \end{pmatrix}$$

To show that the given equilibrium state is stable, we need to show that the real part of the eigenvalues of $JF\left(X^{**}\right)$ are all negative. Here, we will use the Routh-Hurwitz Criterion.

The characteristic polynomial of $JF\left(X^{**}\right)$ is $\lambda^3 + \left(\mu + \beta_1 + \beta_2\right)\lambda^2 + \left(\mu\left(\beta_1 + \beta_2\right) + \beta_1\beta_2\right)\lambda + \frac{1}{L}\mu\beta_1\beta_2\left(L - w\mu\right)$. Let $a_1 = \mu + \beta_1 + \beta_2$, $a_2 = \mu\left(\beta_2 + \beta_2\right) + \beta_1\beta_2$ and $a_3 =$

$\frac{1}{L}\mu\beta_1\beta_2\left(L - w\mu\right)$. The real part of the eigenvalues are all negative if the following conditions are satisfied:

i.) $a_1 > 0$ and $a_3 > 0$

ii.) $a_1 a_2 > a_3$

It is clear that a_1 is always positive while a_3 is positive only if $\dfrac{L}{w\mu} > 1$. The last step that we need to show is that $a_1 a_2 > a_3$. Now,

$a_1 a_2 = \mu^2\beta_1 + \mu^2\beta_2 + \mu\beta_1^2 + 3\mu\beta_1\beta_2 + \mu\beta_2^2 + \beta_1^2\beta_2 + \beta_1\beta_2^2$ and it is clear that $a_1 a_2 > a_3$ since $\mu\beta_1\beta_2 > \dfrac{w\mu}{L}$ and $a_1 a_2 > 3\mu\beta_1\beta_2$. Thus, all the real part of the eigenvalues of $Jf\left(X^{**}\right)$ are all negative. Therefore, the theorem holds. $\square$

Theorem 3.16. In system (3.28), if $\dfrac{L}{w\mu} \leq 1$ then the equilibrium point $(0,0,0)$ is asymptotically stable.

Proof. The jacobian matrix of system (3.28) at $(0,0,0)$ is given by

$$JF\left(0,0,0\right) = \begin{pmatrix} -\beta_1 & 0 & \dfrac{L}{w} \\ \beta_1 & -\beta_2 & 0 \\ 0 & \beta_2 & -\mu \end{pmatrix}.$$

Its characteristic polynomial is $\lambda^3 + \left(\mu + \beta_1 + \beta_2\right)\lambda^2 + \left(\mu\left(\beta_1 + \beta_2\right) + \beta_1\beta_2\right)\lambda + \Big(\mu\beta_1\beta_2 - \dfrac{L}{w}\beta_1\beta_2\Big)$. Let $a_1 = \mu + \beta_1 + \beta_2$, $a_2 = \mu\left(\beta_2 + \beta_2\right) + \beta_1\beta_2$ and $a_3 = \left(\mu\beta_1\beta_2 - \dfrac{L}{w}\beta_1\beta_2\right)$. a_1 is always positive while a_3 is positive only if $\frac{L}{w\mu} < 1$.

$a_1 a_2 = \mu^2\beta_1 + \mu^2\beta_2 + \mu\beta_1^2 + 3\mu\beta_1\beta_2 + \mu\beta_2^2 + \beta_1^2\beta_2 + \beta_1\beta_2^2$ and it is clear that $a_1 a_2 > a_3$ since $\mu\beta_1\beta_2 > \dfrac{L}{w}\beta_1\beta_2$ and $a_1 a_2 > 3\mu\beta_1\beta_2$. Thus, all the real part of the eigenvalues of $Jf\,(0,0,0)$ are all negative. Therefore, the theorem holds. $\square$

Corollary 3.17. In system (3.7) with initial condition $(B, B_a, 0, 0, A, 0)$, if $\alpha_4 = 0$ and $\dfrac{L}{w\mu} > 1$ then X^* is asymptotically stable.

Corollary 3.18. In system (3.7) with initial condition $(B, B_a, 0, 0, A, 0)$, if $\alpha_4 = 0$ and $\dfrac{L}{w\mu} \leq 1$ then $(0,0,0,0,0,0)$ is asymptotically stable.

The two Corollaries above immediately follow from Theorems (3.15) and (3.16). From Corollary (3.17), we can conclude that the colony will be free of AFB spores if $\alpha_4 = 0$. Moreover, the colony will survive if the value of $\dfrac{L}{w\mu} > 1$. Here, $\dfrac{L}{w\mu}$ serves as the treshold of the system (3.7) if $\alpha_4 = 0$.

In the initial condition of system (3.7), if the values of either $B_s, C_w, C,$ and A_s have non-zero value then the equilibrium points $(0,0,0,0,0,0)$ and X^* are unstable. Illustrations of these cases can be seen in Figures , 3.5, 3.6, and 3.7. Hence, we can say that the domain or basin of attraction of of the spore-free equilibrium state is the set $\{(B, Ba, 0, 0, A, 0)|B, Ba, A \in \mathbb{R}^{\oplus}\}$. We defined the basin or domain of attraction of an equilibrium state X^* as the set $\left\{X_0|\lim_{t\to\infty}\phi(X_0) = X^*\right\}$, that is, it contains all the initial conditions that will lead to the convergence of the solutions to the equilibrium state. As for example, the initial condition $(0,0,1,10000,0)$ is not included in the domain of attraction of the spore-free equilibrium state since the solution of the system (3.7) will converge to (0,0,0,10000,0,0).

Theorem 3.19. The Equilibrium State $(0,0,0,C,0,0)$, where C is non-zero, is non-hyperbolic.

Proof. The Jacobian matrix of the system (3.7) at $(0,0,0,C,0,0)$ is given by

$JF(0,0,0,C,0,0) =$

$$\begin{pmatrix} -\beta_1 & 0 & 0 & 0 & \dfrac{L}{w} - \dfrac{LC}{vw} & \dfrac{L}{w} - \dfrac{LC}{vw} \\ \beta_1 & -\beta_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\varphi_1\varphi_2 & 0 & \dfrac{LC}{vw} & \dfrac{LC}{vw} \\ 0 & 0 & \varphi_1\varphi_2\sigma v - \varphi_1\varphi_2\sigma C & 0 & 0 & 0 \\ 0 & \beta_2 & 0 & 0 & -\alpha_2\dfrac{C}{v} - \alpha_4 - \mu & 0 \\ 0 & 0 & 0 & 0 & \alpha_2\dfrac{C}{v} + \alpha_4 & -\mu \end{pmatrix}.$$

We can observe that one of the columns of JF(0, 0, 0, C, 0, 0) is a zero vector. From this, we can say that one of its eigenvalues will be equal to zero. This makes the equilibrium state $(0,0,0,C,0,0)$ non-hyperbolic. □

Linearization method fails to check the stability of the equilibrium state $(0,0,0,$ $C,0,0)$, since it is non-hyperbolic. There are other method availble like the use of Lyapunov functions, however this method is more difficult. We will simply used numerical approach to study the stability behavior.

From Theorem 3.14, if $\alpha_4 \neq 0$ then the only equilibrium state of the system is $(0,0,0,C,0,0)$. However, we are not yet certain if this equilibrium state is stable, that is, the solution of the system is not converging to a limit cycle and will not blow up in a finite time. Since it is difficult to show analytically that the equilibrium

a_4	B	B_a	B_s	C	A	A_s
0.001	6.40E-12	1.27472	293.149	10000	0.08769	329.451
0.00122	6.59E-12	1.18271	281.557	10000	0.08134	316.043
0.00149	6.82E-12	1.09712	270.367	10000	0.07543	303.131
0.00182	7.11E-12	1.01752	259.572	10000	0.06994	290.702
0.00223	7.42E-12	0.94354	249.164	10000	0.06482	278.743
0.00272	7.81E-12	0.87482	239.134	10000	0.06007	267.244
0.00333	8.27E-12	0.81103	229.477	10000	0.05566	256.194
0.00406	8.84E-12	0.75186	220.186	10000	0.05156	245.583
0.00497	9.57E-12	0.69701	211.254	10000	0.04775	235.401
0.00607	1.06E-11	0.64621	202.678	10000	0.04422	225.642
0.00741	1.20E-11	0.59922	194.453	10000	0.04095	216.299
0.00906	1.40E-11	0.5558	186.575	10000	0.03792	207.366
0.01106	1.61E-11	0.51574	179.043	10000	0.03511	198.837
0.01352	1.84E-11	0.47882	171.855	10000	0.03252	190.71
0.01652	2.08E-11	0.44488	165.011	10000	0.03012	182.983
0.02018	2.35E-11	0.41372	158.51	10000	0.02791	175.654
0.02466	2.65E-11	0.3852	152.355	10000	0.02587	168.724
0.03013	2.99E-11	0.35915	146.546	10000	0.02398	162.192
0.03681	3.38E-11	0.33546	141.088	10000	0.02225	156.061
0.04497	3.83E-11	0.31397	135.983	10000	0.02066	150.334
0.05494	4.30E-11	0.29458	131.235	10000	0.01919	145.011
0.06713	4.80E-11	0.27717	126.846	10000	0.01785	140.097
0.08202	5.34E-11	0.26161	122.819	10000	0.01661	135.591
0.10021	5.92E-11	0.24782	119.153	10000	0.01546	131.493
0.12243	6.51E-11	0.23567	115.848	10000	0.01441	127.8
0.14959	7.10E-11	0.22507	112.898	10000	0.01342	124.507
0.18276	7.68E-11	0.2159	110.294	10000	0.0125	121.603
0.2233	8.25E-11	0.20804	108.026	10000	0.01164	119.073
0.27282	8.78E-11	0.2014	106.074	10000	0.01082	116.898
0.33333	9.26E-11	0.19584	104.42	10000	0.01003	115.055

Table 3.3. Final values of the state variable with different values of α_4: initial condition $(0, 0, 0, 0, 10000, 0)$ and parameters: $L = 1500$, $w = 21000$, $v = 10000$, $\alpha_3 = 0.0005$, $\alpha_2 = 1$, $\alpha_1 = 0.001$, $\varphi_1 = 1/3$, $\varphi_2 = 1/3$ $\beta_1 = 1/3$, $\beta_2 = 1/15$, and $\mu = 1/30$,END-TIME=200

state is stable or not, we will just consider numerical approach to determine what will happen to the system for different values of α_4. In Tables 3.3, 3.4, 3.5, 3.6, 3.7, 3.8, 3.9, 3.10, 3.11, and 3.12 different values of α_4 were considered to check the fate of the state variables. We considered different end-time for the simulation and checked the final values of each state variable.

Based on the Tables 3.3, 3.4, 3.5, 3.6, 3.7, 3.8, 3.9, 3.10, 3.11, and 3.12, we can say that the values of all state variables, except for C, are decreasing and approaching to zero. C on the other hand, is increasing and approaching to a certain value. Hence, from the simulations, we can claim that the solutions of the system for any non-zero value of α_4 is not oscillating. Therefore, we can say that the system (3.7) stabilizes at extinction equilibrium state, since it is the only equilibrium point

a_4	B	B_a	B_s	C	A	A_s
0.001	2.34E-13	0.00162	10.8639	10000	1.12E-04	11.8436
0.00122	2.40E-13	0.00151	10.4196	10000	1.04E-04	11.3587
0.00149	2.48E-13	0.0014	9.99194	10000	9.60E-05	10.892
0.00182	2.58E-13	0.00129	9.5804	10000	8.90E-05	10.4429
0.00223	2.68E-13	0.0012	9.18458	10000	8.25E-05	10.011
0.00272	2.81E-13	0.00111	8.8041	10000	7.65E-05	9.59593
0.00333	2.96E-13	0.00103	8.43857	10000	7.08E-05	9.19718
0.00406	3.13E-13	9.57E-04	8.08768	10000	6.56E-05	8.81441
0.00497	3.33E-13	8.87E-04	7.75112	10000	6.08E-05	8.44731
0.00607	3.57E-13	8.22E-04	7.4286	10000	5.63E-05	8.09555
0.00741	3.85E-13	7.63E-04	7.11991	10000	5.21E-05	7.75889
0.00906	4.17E-13	7.07E-04	6.82484	10000	4.83E-05	7.4371
0.01106	4.55E-13	6.56E-04	6.54322	10000	4.47E-05	7.13001
0.01352	4.99E-13	6.09E-04	6.27494	10000	4.14E-05	6.83747
0.01652	5.51E-13	5.66E-04	6.0199	10000	3.83E-05	6.55939
0.02018	6.11E-13	5.27E-04	5.77806	10000	3.55E-05	6.29572
0.02466	6.80E-13	4.90E-04	5.54941	10000	3.29E-05	6.04644
0.03013	7.60E-13	4.57E-04	5.33398	10000	3.05E-05	5.81157
0.03681	8.50E-13	4.27E-04	5.1318	10000	2.83E-05	5.59117
0.04497	9.53E-13	4.00E-04	4.94295	10000	2.63E-05	5.38531
0.05494	1.07E-12	3.75E-04	4.7675	10000	2.44E-05	5.19406
0.06713	1.19E-12	3.53E-04	4.60551	10000	2.27E-05	5.01749
0.08202	1.33E-12	3.33E-04	4.45701	10000	2.11E-05	4.85563
0.10021	1.47E-12	3.15E-04	4.32198	10000	1.97E-05	4.70846
0.12243	1.62E-12	3.00E-04	4.20032	10000	1.83E-05	4.57586
0.14959	1.78E-12	2.86E-04	4.09182	10000	1.71E-05	4.45761
0.18276	1.93E-12	2.75E-04	3.99615	10000	1.59E-05	4.35333
0.2233	2.08E-12	2.65E-04	3.91281	10000	1.48E-05	4.26251
0.27282	2.22E-12	2.56E-04	3.84118	10000	1.38E-05	4.18445
0.33333	2.35E-12	2.49E-04	3.78047	10000	1.28E-05	4.11828

Table 3.4. Final values of the state variable with different values of α_4: initial condition $(0,0,0,0,10000,0)$ and parameters: $L = 1500$, $w = 21000$, $v = 10000$, $\alpha_3 = 0.0005$, $\alpha_2 = 1$, $\alpha_1 = 0.001$, $\varphi_1 = 1/3$, $\varphi_2 = 1/3$ $\beta_1 = 1/3$, $\beta_2 = 1/15$, and $\mu = 1/30$, END-TIME=300

a_4	B	B_a	B_s	C	A	A_s
0.001	8.36E-15	2.06E-06	0.38811	10000	1.42E-07	0.42262
0.00122	8.59E-15	1.92E-06	0.37222	10000	1.32E-07	0.40532
0.00149	8.88E-15	1.78E-06	0.35692	10000	1.22E-07	0.38866
0.00182	9.22E-15	1.65E-06	0.3422	10000	1.13E-07	0.37263
0.00223	9.59E-15	1.53E-06	0.32805	10000	1.05E-07	0.35722
0.00272	1.01E-14	1.42E-06	0.31444	10000	9.74E-08	0.3424
0.00333	1.06E-14	1.31E-06	0.30137	10000	9.02E-08	0.32817
0.00406	1.12E-14	1.22E-06	0.28883	10000	8.36E-08	0.31451
0.00497	1.19E-14	1.13E-06	0.2768	10000	7.74E-08	0.30141
0.00607	1.28E-14	1.05E-06	0.26527	10000	7.17E-08	0.28886
0.00741	1.38E-14	9.71E-07	0.25424	10000	6.64E-08	0.27685
0.00906	1.49E-14	9.00E-07	0.24369	10000	6.15E-08	0.26536
0.01106	1.63E-14	8.35E-07	0.23363	10000	5.69E-08	0.2544
0.01352	1.79E-14	7.76E-07	0.22404	10000	5.27E-08	0.24396
0.01652	1.97E-14	7.21E-07	0.21493	10000	4.88E-08	0.23404
0.02018	2.18E-14	6.70E-07	0.20629	10000	4.53E-08	0.22463
0.02466	2.43E-14	6.24E-07	0.19812	10000	4.19E-08	0.21574
0.03013	2.72E-14	5.82E-07	0.19042	10000	3.89E-08	0.20736
0.03681	3.04E-14	5.43E-07	0.1832	10000	3.61E-08	0.19949
0.04497	3.41E-14	5.09E-07	0.17646	10000	3.35E-08	0.19214
0.05494	3.82E-14	4.77E-07	0.17019	10000	3.11E-08	0.18532
0.06713	4.27E-14	4.49E-07	0.1644	10000	2.89E-08	0.17902
0.08202	4.75E-14	4.24E-07	0.1591	10000	2.69E-08	0.17324
0.10021	5.27E-14	4.01E-07	0.15428	10000	2.51E-08	0.16799
0.12243	5.80E-14	3.82E-07	0.14993	10000	2.34E-08	0.16326
0.14959	6.35E-14	3.65E-07	0.14606	10000	2.18E-08	0.15904
0.18276	6.90E-14	3.50E-07	0.14264	10000	2.03E-08	0.15532
0.2233	7.43E-14	3.37E-07	0.13966	10000	1.89E-08	0.15208
0.27282	7.93E-14	3.26E-07	0.13711	10000	1.75E-08	0.14929
0.33333	8.39E-14	3.17E-07	0.13494	10000	1.63E-08	0.14693

Table 3.5. Final values of the state variable with different values of α_4: initial condition $(0,0,0,0,10000,0)$ and parameters: $L = 1500$, $w = 21000$, $v = 10000$, $\alpha_3 = 0.0005$, $\alpha_2 = 1$, $\alpha_1 = 0.001$, $\varphi_1 = 1/3$, $\varphi_2 = 1/3$ $\beta_1 = 1/3$, $\beta_2 = 1/15$, and $\mu = 1/30$, END-TIME=400

α_4	B	B_a	B_s	C	A	A_s
0.001	2.99E-16	2.63E-09	0.01385	10000	1.81E-10	0.01508
0.00122	3.07E-16	2.44E-09	0.01328	10000	1.68E-10	0.01446
0.00149	3.17E-16	2.26E-09	0.01273	10000	1.56E-10	0.01387
0.00182	3.29E-16	2.10E-09	0.01221	10000	1.44E-10	0.01329
0.00223	3.43E-16	1.94E-09	0.0117	10000	1.34E-10	0.01274
0.00272	3.59E-16	1.80E-09	0.01122	10000	1.24E-10	0.01222
0.00333	3.78E-16	1.67E-09	0.01075	10000	1.15E-10	0.01171
0.00406	4.00E-16	1.55E-09	0.0103	10000	1.06E-10	0.01122
0.00497	4.26E-16	1.44E-09	0.00987	10000	9.86E-11	0.01075
0.00607	4.56E-16	1.33E-09	0.00946	10000	9.13E-11	0.0103
0.00741	4.92E-16	1.24E-09	0.00907	10000	8.45E-11	0.00988
0.00906	5.33E-16	1.15E-09	0.00869	10000	7.83E-11	0.00947
0.01106	5.81E-16	1.06E-09	0.00833	10000	7.25E-11	0.00908
0.01352	6.37E-16	9.87E-10	0.00799	10000	6.71E-11	0.0087
0.01652	7.03E-16	9.17E-10	0.00767	10000	6.22E-11	0.00835
0.02018	7.80E-16	8.53E-10	0.00736	10000	5.76E-11	0.00801
0.02466	8.69E-16	7.94E-10	0.00707	10000	5.34E-11	0.0077
0.03013	9.70E-16	7.40E-10	0.00679	10000	4.95E-11	0.0074
0.03681	1.09E-15	6.91E-10	0.00654	10000	4.59E-11	0.00712
0.04497	1.22E-15	6.47E-10	0.0063	10000	4.26E-11	0.00685
0.05494	1.36E-15	6.07E-10	0.00607	10000	3.96E-11	0.00661
0.06713	1.52E-15	5.71E-10	0.00587	10000	3.68E-11	0.00639
0.08202	1.70E-15	5.39E-10	0.00568	10000	3.43E-11	0.00618
0.10021	1.88E-15	5.11E-10	0.0055	10000	3.19E-11	0.00599
0.12243	2.07E-15	4.86E-10	0.00535	10000	2.97E-11	0.00582
0.14959	2.27E-15	4.64E-10	0.00521	10000	2.77E-11	0.00567
0.18276	2.46E-15	4.45E-10	0.00509	10000	2.58E-11	0.00554
0.2233	2.65E-15	4.29E-10	0.00498	10000	2.40E-11	0.00543
0.27282	2.83E-15	4.15E-10	0.00489	10000	2.23E-11	0.00533
0.33333	2.99E-15	4.04E-10	0.00481	10000	2.07E-11	0.00524

Table 3.6. Final values of the state variable with different values of α_4: initial condition $(0, 0, 0, 0, 10000, 0)$ and parameters: $L = 1500$, $w = 21000$, $v = 10000$, $\alpha_3 = 0.0005$, $\alpha_2 = 1$, $\alpha_1 = 0.001$, $\varphi_1 = 1/3$, $\varphi_2 = 1/3$ $\beta_1 = 1/3$, $\beta_2 = 1/15$, and $\mu = 1/30$, END-TIME=500

α_4	B	B_a	B_s	C	A	A_s
0.001	1.06E-17	3.34E-12	4.94E-04	10000	2.30E-13	5.38E-04
0.00122	1.09E-17	3.10E-12	4.74E-04	10000	2.14E-13	5.16E-04
0.00149	1.13E-17	2.88E-12	4.54E-04	10000	1.98E-13	4.95E-04
0.00182	1.17E-17	2.67E-12	4.36E-04	10000	1.84E-13	4.74E-04
0.00223	1.22E-17	2.48E-12	4.18E-04	10000	1.70E-13	4.55E-04
0.00272	1.28E-17	2.29E-12	4.00E-04	10000	1.58E-13	4.36E-04
0.00333	1.35E-17	2.13E-12	3.84E-04	10000	1.46E-13	4.18E-04
0.00406	1.43E-17	1.97E-12	3.68E-04	10000	1.35E-13	4.00E-04
0.00497	1.52E-17	1.83E-12	3.52E-04	10000	1.25E-13	3.84E-04
0.00607	1.63E-17	1.70E-12	3.38E-04	10000	1.16E-13	3.68E-04
0.00741	1.75E-17	1.57E-12	3.24E-04	10000	1.08E-13	3.52E-04
0.00906	1.90E-17	1.46E-12	3.10E-04	10000	9.96E-14	3.38E-04
0.01106	2.07E-17	1.35E-12	2.97E-04	10000	9.23E-14	3.24E-04
0.01352	2.27E-17	1.26E-12	2.85E-04	10000	8.54E-14	3.10E-04
0.01652	2.51E-17	1.17E-12	2.74E-04	10000	7.91E-14	2.98E-04
0.02018	2.78E-17	1.09E-12	2.63E-04	10000	7.33E-14	2.86E-04
0.02466	3.10E-17	1.01E-12	2.52E-04	10000	6.80E-14	2.75E-04
0.03013	3.46E-17	9.42E-13	2.42E-04	10000	6.30E-14	2.64E-04
0.03681	3.87E-17	8.80E-13	2.33E-04	10000	5.85E-14	2.54E-04
0.04497	4.34E-17	8.24E-13	2.25E-04	10000	5.43E-14	2.45E-04
0.05494	4.86E-17	7.73E-13	2.17E-04	10000	5.05E-14	2.36E-04
0.06713	5.43E-17	7.28E-13	2.09E-04	10000	4.69E-14	2.28E-04
0.08202	6.05E-17	6.87E-13	2.02E-04	10000	4.37E-14	2.20E-04
0.10021	6.70E-17	6.51E-13	1.96E-04	10000	4.07E-14	2.14E-04
0.12243	7.39E-17	6.19E-13	1.91E-04	10000	3.79E-14	2.08E-04
0.14959	8.09E-17	5.91E-13	1.86E-04	10000	3.53E-14	2.02E-04
0.18276	8.78E-17	5.67E-13	1.82E-04	10000	3.29E-14	1.98E-04
0.2233	9.46E-17	5.47E-13	1.78E-04	10000	3.06E-14	1.94E-04
0.27282	1.01E-16	5.29E-13	1.74E-04	10000	2.85E-14	1.90E-04
0.33333	1.07E-16	5.15E-13	1.72E-04	10000	2.64E-14	1.87E-04

Table 3.7. Final values of the state variable with different values of α_4: initial condition $(0, 0, 0, 0, 10000, 0)$ and parameters: $L = 1500$, $w = 21000$, $v = 10000$, $\alpha_3 = 0.0005$, $\alpha_2 = 1$, $\alpha_1 = 0.001$, $\varphi_1 = 1/3$, $\varphi_2 = 1/3$ $\beta_1 = 1/3$, $\beta_2 = 1/15$, and $\mu = 1/30$, END-TIME=600

α_4	B	B_a	B_s	C	A	A_s
0.001	2.43E-11	3.96E-01	1.55E+02	10000	2.57E-02	1.72E+02
0.00122	2.75E-11	3.69E-01	1.49E+02	10000	2.39E-02	1.65E+02
0.00149	3.13E-11	3.43E-01	1.43E+02	10000	2.22E-02	1.58E+02
0.00182	3.59E-11	3.18E-01	1.37E+02	10000	2.06E-02	1.52E+02
0.00223	4.10E-11	2.96E-01	1.32E+02	10000	1.92E-02	1.46E+02
0.00272	4.69E-11	2.75E-01	1.26E+02	10000	1.78E-02	1.40E+02
0.00333	5.37E-11	2.56E-01	1.21E+02	10000	1.65E-02	1.34E+02
0.00406	6.16E-11	2.38E-01	1.17E+02	10000	1.54E-02	1.29E+02
0.00497	7.06E-11	2.21E-01	1.12E+02	10000	1.43E-02	1.23E+02
0.00607	8.12E-11	2.06E-01	1.07E+02	10000	1.33E-02	1.18E+02
0.00741	9.31E-11	1.91E-01	1.03E+02	10000	1.23E-02	1.14E+02
0.00906	1.07E-10	1.78E-01	9.92E+01	10000	1.15E-02	1.09E+02
0.01106	1.23E-10	1.66E-01	9.53E+01	10000	1.07E-02	1.05E+02
0.01352	1.41E-10	1.55E-01	9.16E+01	10000	9.92E-03	1.01E+02
0.01652	1.61E-10	1.44E-01	8.81E+01	10000	9.23E-03	9.69E+01
0.02018	1.84E-10	1.35E-01	8.48E+01	10000	8.60E-03	9.33E+01
0.02466	2.10E-10	1.26E-01	8.17E+01	10000	8.01E-03	8.98E+01
0.03013	2.38E-10	1.18E-01	7.88E+01	10000	7.47E-03	8.65E+01
0.03681	2.70E-10	1.11E-01	7.60E+01	10000	6.97E-03	8.35E+01
0.04497	3.05E-10	1.05E-01	7.34E+01	10000	6.50E-03	8.06E+01
0.05494	3.43E-10	9.86E-02	7.10E+01	10000	6.08E-03	7.80E+01
0.06713	3.83E-10	9.32E-02	6.88E+01	10000	5.68E-03	7.55E+01
0.08202	4.26E-10	8.84E-02	6.68E+01	10000	5.31E-03	7.33E+01
0.10021	4.71E-10	8.40E-02	6.49E+01	10000	4.97E-03	7.12E+01
0.12243	5.17E-10	8.02E-02	6.32E+01	10000	4.65E-03	6.94E+01
0.14959	5.63E-10	7.68E-02	6.17E+01	10000	4.35E-03	6.77E+01
0.18276	6.09E-10	7.39E-02	6.04E+01	10000	4.07E-03	6.62E+01
0.2233	6.54E-10	7.13E-02	5.92E+01	10000	3.80E-03	6.49E+01
0.27282	6.95E-10	6.91E-02	5.81E+01	10000	3.54E-03	6.37E+01
0.33333	7.34E-10	6.73E-02	5.73E+01	10000	3.29E-03	6.28E+01

Table 3.8. Final values of the state variable with different values of α_4: initial condition $(0,0,0,0,10000,0)$ and parameters: $L = 1500$, $w = 21000$, $v = 10000$, $\alpha_3 = 0.05$, $\alpha_2 = 1$, $\alpha_1 = 0.01$, $\varphi_1 = 1/3$, $\varphi_2 = 1/3$ $\beta_1 = 1/3$, $\beta_2 = 1/15$, and $\mu = 1/30$,END-TIME=200

α_4	B	B_a	B_s	C	A	A_s
0.001	6.31E-13	5.05E-04	5.65E+00	10000	3.30E-05	6.16E+00
0.00122	7.06E-13	4.69E-04	5.42E+00	10000	3.07E-05	5.91E+00
0.00149	7.95E-13	4.36E-04	5.20E+00	10000	2.85E-05	5.67E+00
0.00182	9.01E-13	4.05E-04	4.99E+00	10000	2.65E-05	5.44E+00
0.00223	1.02E-12	3.77E-04	4.79E+00	10000	2.46E-05	5.22E+00
0.00272	1.17E-12	3.50E-04	4.59E+00	10000	2.29E-05	5.00E+00
0.00333	1.34E-12	3.25E-04	4.41E+00	10000	2.13E-05	4.80E+00
0.00406	1.54E-12	3.02E-04	4.23E+00	10000	1.98E-05	4.61E+00
0.00497	1.78E-12	2.81E-04	4.06E+00	10000	1.84E-05	4.42E+00
0.00607	2.06E-12	2.62E-04	3.89E+00	10000	1.71E-05	4.24E+00
0.00741	2.38E-12	2.43E-04	3.74E+00	10000	1.59E-05	4.07E+00
0.00906	2.75E-12	2.27E-04	3.59E+00	10000	1.48E-05	3.91E+00
0.01106	3.19E-12	2.11E-04	3.45E+00	10000	1.37E-05	3.75E+00
0.01352	3.69E-12	1.97E-04	3.31E+00	10000	1.28E-05	3.61E+00
0.01652	4.26E-12	1.84E-04	3.18E+00	10000	1.19E-05	3.47E+00
0.02018	4.91E-12	1.72E-04	3.06E+00	10000	1.11E-05	3.34E+00
0.02466	5.65E-12	1.61E-04	2.95E+00	10000	1.03E-05	3.21E+00
0.03013	6.48E-12	1.51E-04	2.84E+00	10000	9.63E-06	3.10E+00
0.03681	7.41E-12	1.41E-04	2.74E+00	10000	8.98E-06	2.99E+00
0.04497	8.44E-12	1.33E-04	2.65E+00	10000	8.39E-06	2.88E+00
0.05494	9.57E-12	1.25E-04	2.56E+00	10000	7.84E-06	2.79E+00
0.06713	1.08E-11	1.19E-04	2.48E+00	10000	7.33E-06	2.70E+00
0.08202	1.21E-11	1.12E-04	2.41E+00	10000	6.86E-06	2.62E+00
0.10021	1.35E-11	1.07E-04	2.34E+00	10000	6.42E-06	2.55E+00
0.12243	1.49E-11	1.02E-04	2.28E+00	10000	6.01E-06	2.48E+00
0.14959	1.63E-11	9.78E-05	2.22E+00	10000	5.62E-06	2.42E+00
0.18276	1.77E-11	9.40E-05	2.17E+00	10000	5.25E-06	2.37E+00
0.2233	1.91E-11	9.08E-05	2.13E+00	10000	4.90E-06	2.32E+00
0.27282	2.04E-11	8.80E-05	2.09E+00	10000	4.57E-06	2.28E+00
0.33333	2.16E-11	8.56E-05	2.06E+00	10000	4.25E-06	2.24E+00

Table 3.9. Final values of the state variable with different values of α_4: initial condition $(0,0,0,0,10000,0)$ and parameters: $L = 1500$, $w = 21000$, $v = 10000$, $\alpha_3 = 0.05$, $\alpha_2 = 1$, $\alpha_1 = 0.01$, $\varphi_1 = 1/3$, $\varphi_2 = 1/3$ $\beta_1 = 1/3$, $\beta_2 = 1/15$, and $\mu = 1/30$,END-TIME=300

a_4	B	B_a	B_s	C	A	A_s
0.001	2.30E-14	6.42E-07	2.02E-01	10000	4.37E-08	2.20E-01
0.00122	2.58E-14	5.97E-07	1.94E-01	10000	4.07E-08	2.11E-01
0.00149	2.90E-14	5.55E-07	1.86E-01	10000	3.78E-08	2.02E-01
0.00182	3.29E-14	5.16E-07	1.78E-01	10000	3.51E-08	1.94E-01
0.00223	3.74E-14	4.79E-07	1.71E-01	10000	3.27E-08	1.86E-01
0.00272	4.27E-14	4.45E-07	1.64E-01	10000	3.03E-08	1.79E-01
0.00333	4.90E-14	4.14E-07	1.57E-01	10000	2.82E-08	1.71E-01
0.00406	5.64E-14	3.85E-07	1.51E-01	10000	2.62E-08	1.64E-01
0.00497	6.50E-14	3.58E-07	1.45E-01	10000	2.43E-08	1.58E-01
0.00607	7.51E-14	3.33E-07	1.39E-01	10000	2.26E-08	1.51E-01
0.00741	8.68E-14	3.10E-07	1.33E-01	10000	2.10E-08	1.45E-01
0.00906	1.00E-13	2.88E-07	1.28E-01	10000	1.96E-08	1.39E-01
0.01106	1.16E-13	2.69E-07	1.23E-01	10000	1.82E-08	1.34E-01
0.01352	1.34E-13	2.51E-07	1.18E-01	10000	1.69E-08	1.29E-01
0.01652	1.55E-13	2.34E-07	1.14E-01	10000	1.58E-08	1.24E-01
0.02018	1.79E-13	2.19E-07	1.09E-01	10000	1.47E-08	1.19E-01
0.02466	2.06E-13	2.05E-07	1.05E-01	10000	1.37E-08	1.15E-01
0.03013	2.36E-13	1.92E-07	1.01E-01	10000	1.27E-08	1.10E-01
0.03681	2.70E-13	1.80E-07	9.78E-02	10000	1.19E-08	1.07E-01
0.04497	3.08E-13	1.69E-07	9.45E-02	10000	1.11E-08	1.03E-01
0.05494	3.49E-13	1.60E-07	9.14E-02	10000	1.04E-08	9.95E-02
0.06713	3.93E-13	1.51E-07	8.85E-02	10000	9.68E-09	9.64E-02
0.08202	4.40E-13	1.43E-07	8.59E-02	10000	9.05E-09	9.35E-02
0.10021	4.89E-13	1.36E-07	8.34E-02	10000	8.46E-09	9.09E-02
0.12243	5.40E-13	1.30E-07	8.13E-02	10000	7.92E-09	8.85E-02
0.14959	5.91E-13	1.24E-07	7.93E-02	10000	7.40E-09	8.63E-02
0.18276	6.42E-13	1.20E-07	7.75E-02	10000	6.91E-09	8.44E-02
0.2233	6.92E-13	1.16E-07	7.60E-02	10000	6.45E-09	8.28E-02
0.27282	7.39E-13	1.12E-07	7.47E-02	10000	6.00E-09	8.13E-02
0.33333	7.82E-13	1.09E-07	7.35E-02	10000	5.57E-09	8.01E-02

Table 3.10. Final values of the state variable with different values of α_4: initial condition $(0, 0, 0, 0, 10000, 0)$ and parameters: $L = 1500$, $w = 21000$, $v = 10000$, $\alpha_3 = 0.05$, $\alpha_2 = 1$, $\alpha_1 = 0.01$, $\varphi_1 = 1/3$, $\varphi_2 = 1/3$ $\beta_1 = 1/3$, $\beta_2 = 1/15$, and $\mu = 1/30$,END-TIME=400

a_4	B	B_a	B_s	C	A	A_s
0.001	8.26E-16	8.17E-10	7.19E-03	10000	5.63E-11	7.83E-03
0.00122	9.25E-16	7.60E-10	6.90E-03	10000	5.23E-11	7.52E-03
0.00149	1.04E-15	7.06E-10	6.62E-03	10000	4.86E-11	7.21E-03
0.00182	1.18E-15	6.56E-10	6.35E-03	10000	4.52E-11	6.92E-03
0.00223	1.34E-15	6.10E-10	6.10E-03	10000	4.20E-11	6.64E-03
0.00272	1.53E-15	5.67E-10	5.85E-03	10000	3.90E-11	6.37E-03
0.00333	1.76E-15	5.27E-10	5.61E-03	10000	3.62E-11	6.11E-03
0.00406	2.02E-15	4.90E-10	5.38E-03	10000	3.36E-11	5.86E-03
0.00497	2.33E-15	4.56E-10	5.16E-03	10000	3.12E-11	5.62E-03
0.00607	2.69E-15	4.24E-10	4.96E-03	10000	2.90E-11	5.40E-03
0.00741	3.11E-15	3.94E-10	4.76E-03	10000	2.70E-11	5.18E-03
0.00906	3.60E-15	3.67E-10	4.57E-03	10000	2.51E-11	4.97E-03
0.01106	4.17E-15	3.42E-10	4.39E-03	10000	2.33E-11	4.78E-03
0.01352	4.82E-15	3.19E-10	4.22E-03	10000	2.17E-11	4.59E-03
0.01652	5.56E-15	2.98E-10	4.05E-03	10000	2.02E-11	4.41E-03
0.02018	6.42E-15	2.78E-10	3.90E-03	10000	1.88E-11	4.25E-03
0.02466	7.38E-15	2.60E-10	3.75E-03	10000	1.75E-11	4.09E-03
0.03013	8.46E-15	2.44E-10	3.62E-03	10000	1.63E-11	3.94E-03
0.03681	9.67E-15	2.29E-10	3.49E-03	10000	1.52E-11	3.80E-03
0.04497	1.10E-14	2.16E-10	3.37E-03	10000	1.42E-11	3.67E-03
0.05494	1.25E-14	2.03E-10	3.26E-03	10000	1.33E-11	3.55E-03
0.06713	1.41E-14	1.92E-10	3.16E-03	10000	1.24E-11	3.44E-03
0.08202	1.57E-14	1.82E-10	3.06E-03	10000	1.16E-11	3.34E-03
0.10021	1.75E-14	1.73E-10	2.98E-03	10000	1.08E-11	3.24E-03
0.12243	1.93E-14	1.66E-10	2.90E-03	10000	1.01E-11	3.16E-03
0.14959	2.12E-14	1.59E-10	2.83E-03	10000	9.47E-12	3.08E-03
0.18276	2.30E-14	1.53E-10	2.77E-03	10000	8.84E-12	3.01E-03
0.2233	2.47E-14	1.47E-10	2.71E-03	10000	8.25E-12	2.95E-03
0.27282	2.64E-14	1.43E-10	2.66E-03	10000	7.68E-12	2.90E-03
0.33333	2.80E-14	1.39E-10	2.62E-03	10000	7.12E-12	2.86E-03

Table 3.11. Final values of the state variable with different values of α_4: initial condition $(0, 0, 0, 0, 10000, 0)$ and parameters: $L = 1500$, $w = 21000$, $v = 10000$, $\alpha_3 = 0.05$, $\alpha_2 = 1$, $\alpha_1 = 0.01$, $\varphi_1 = 1/3$, $\varphi_2 = 1/3$ $\beta_1 = 1/3$, $\beta_2 = 1/15$, and $\mu = 1/30$,END-TIME=500

a_4	B	B_a	B_s	C	A	A_s
0.001	2.95E-17	1.04E-12	2.57E-04	10000	7.17E-14	2.79E-04
0.00122	3.30E-17	9.67E-13	2.46E-04	10000	6.66E-14	2.68E-04
0.00149	3.72E-17	8.99E-13	2.36E-04	10000	6.19E-14	2.57E-04
0.00182	4.21E-17	8.36E-13	2.27E-04	10000	5.75E-14	2.47E-04
0.00223	4.79E-17	7.77E-13	2.17E-04	10000	5.34E-14	2.37E-04
0.00272	5.47E-17	7.22E-13	2.09E-04	10000	4.97E-14	2.27E-04
0.00333	6.27E-17	6.71E-13	2.00E-04	10000	4.61E-14	2.18E-04
0.00406	7.21E-17	6.24E-13	1.92E-04	10000	4.29E-14	2.09E-04
0.00497	8.31E-17	5.81E-13	1.84E-04	10000	3.98E-14	2.01E-04
0.00607	9.60E-17	5.40E-13	1.77E-04	10000	3.70E-14	1.93E-04
0.00741	1.11E-16	5.03E-13	1.70E-04	10000	3.44E-14	1.85E-04
0.00906	1.28E-16	4.68E-13	1.63E-04	10000	3.20E-14	1.77E-04
0.01106	1.49E-16	4.37E-13	1.57E-04	10000	2.98E-14	1.70E-04
0.01352	1.72E-16	4.08E-13	1.50E-04	10000	2.77E-14	1.64E-04
0.01652	1.99E-16	3.81E-13	1.45E-04	10000	2.58E-14	1.57E-04
0.02018	2.29E-16	3.56E-13	1.39E-04	10000	2.41E-14	1.51E-04
0.02466	2.63E-16	3.34E-13	1.34E-04	10000	2.24E-14	1.46E-04
0.03013	3.02E-16	3.13E-13	1.29E-04	10000	2.10E-14	1.41E-04
0.03681	3.45E-16	2.95E-13	1.25E-04	10000	1.96E-14	1.36E-04
0.04497	3.93E-16	2.78E-13	1.20E-04	10000	1.83E-14	1.31E-04
0.05494	4.45E-16	2.63E-13	1.16E-04	10000	1.72E-14	1.27E-04
0.06713	5.01E-16	2.49E-13	1.13E-04	10000	1.61E-14	1.23E-04
0.08202	5.61E-16	2.37E-13	1.09E-04	10000	1.51E-14	1.19E-04
0.10021	6.24E-16	2.27E-13	1.06E-04	10000	1.42E-14	1.16E-04
0.12243	6.89E-16	2.17E-13	1.03E-04	10000	1.33E-14	1.13E-04
0.14959	7.55E-16	2.09E-13	1.01E-04	10000	1.25E-14	1.10E-04
0.18276	8.20E-16	2.02E-13	9.87E-05	10000	1.17E-14	1.07E-04
0.2233	8.83E-16	1.96E-13	9.67E-05	10000	1.10E-14	1.05E-04
0.27282	9.43E-16	1.91E-13	9.50E-05	10000	1.02E-14	1.03E-04
0.33333	9.99E-16	1.86E-13	9.36E-05	10000	9.55E-15	1.02E-04

Table 3.12. Final values of the state variable with different values of α_4: initial condition $(0, 0, 0, 0, 10000, 0)$ and parameters: $L = 1500$, $w = 21000$, $v = 10000$, $\alpha_3 = 0.05$, $\alpha_2 = 1$, $\alpha_1 = 0.01$, $\varphi_1 = 1/3$, $\varphi_2 = 1/3$ $\beta_1 = 1/3$, $\beta_2 = 1/15$, and $\mu = 1/30$,END-TIME=600

of the system.. The assurance of the stability of the extinction equilibrium state is the fact that the flow of the system (3.7) is eventually confined in the invariant set Γ as $t \to \infty$ from Corollary 3.4. That is, the trajectories of the state variables will not blow up.

It is also noticeable that the state variable C is converging to 10000 in all of the tables. Hence, we can just consider the equilibrium state $(0, 0, 0, v, 0, 0)$ for the stability of the extinction equilibrium state. We have seen that $(0, 0, 0, v, 0, 0)$ is non-hyperbolic, it implies that this equilibrium state cannot be asymptotically stable. However, we can use the next lemma to show that it is stable.

Lemma 3.20. [?] If the Jacobian matrix at the equilibrium state X^* has simple eigenvalues with one equal to zero, and the rest have real part less than zero, then the equilibrium state X^* is stable.

An eigenvalue λ of a matrix is simple if its algebraic multiplicity is equal to one.

Theorem 3.21. The equilibrium state $(0,0,0,v,0,0)$ is globally stable, that is, all trajectories of the system (3.7), with any initial condition, tends toward near the state $(0,0,0,v,0,0)$.

Proof. The Jacobian matrix of the system (3.7) at $(0,0,0,v,0,0)$ is given by

$$JF(0,0,0,C,0,0)=\begin{pmatrix} -\beta_1 & 0 & 0 & 0 & 0 & 0 \\ \beta_1 & -\beta_2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -\varphi_1\varphi_2 & 0 & \dfrac{L}{w} & \dfrac{L}{w} \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & \beta_2 & 0 & 0 & -\alpha_2-\alpha_4-\mu & 0 \\ 0 & 0 & 0 & 0 & \alpha_2+\alpha_4 & -\mu \end{pmatrix}.$$

JF(0, 0, 0, C, 0, 0) has simple eigenvalues with one zero. The eigenvalues are $-\beta_2$, 0, $-\beta_1$, $-\mu$, $-\mu-\alpha_2-\alpha_4$, $-\varphi_1\varphi_2$. Hence from Theorem 3.20, we can say that $0,0,0,C,0,0$ is stable. □