1. Say whether the following is true or false and support your answer by a proof.
(∃m ∈ N )(∃n ∈ N )(3m + 5n = 12)

The statement is false!
Proof by contradiction:
Let n,m ∈ ℕ be the solution to 3m + 5n = 12. Then

$$3m = 12 - 5n \quad | /4$$
$$m = 4 - \frac{5}{3}n$$

- ⇨ If n is not divisible by 3, then m ∉ ℕ.
- ⇨ n must by divisible by 3.
- ⇨ For n = 3 (first natural number that is divisible by 3) m ∉ ℕ. (m = -1) Clearly for all n ∈ N, divisible by 3 m is negative and thus m ∉ ℕ.

In all cases m ∉ ℕ which is a contradiction to the assumption. So there are no m,n ∈ ℕ, so that 3m + 5n = 12.

2. Say whether the following is true or false and support your answer by a proof: The sum of any five consecutive integers is divisible by 5 (without remainder).

The statement is true:
Proof by induction:
Let S(n) = n + (n+1) + (n+2) + (n+3) + (n+4)
n=1: S(1) = 1+2+3+4+5 = 15, which is divisible by 3.
Let n ∈ ℕ by arbitrary:
S(n+1) = (n+1) + ((n+1) +1) + ((n+1) +2) + ((n+1) +3) + ((n+1) +4)
= (n+1) + (n+2) + (n+3) + (n+4) + (n+5)
= n + (n+1) + (n+2) + (n+3) + (n+4) + 5
= S(n) + 5 | by induction hypothesis
S(n) is by the induction hypothesis divisible by 5 and 5 is also divisible by 5, so S(n+1) is divisible by 5.

3. Say whether the following is true or false and support your answer by a proof: For any integer n, the number $n^2 + n + 1$ is odd.

The statement is true.
Proof by contradiction:
Assume that $n^2 + n + 1$ is even. Then $n^2 + n = n*(n+1)$ is odd. For n*(n+1) to be odd, both cases have to be odd. (Because every multiplication with an even number (divisible by 2) results in an even number. 2 is then a factor to the resulting number)
Assume n is odd, then $\exists k \in \mathbb{N}, n = k * 2 + 1 => n + 1 = k * 2 + 2$. So n+1 is divisible by 2 and thus even.
The case n is even is trivial.
So in all cases n*(n+1) contains at least one even factor. And thus n*(n+1) is even and $n^2+n+1$ is odd, which is a contradiction to the assumption.

4. Prove that every odd natural number is of one of the forms 4n + 1 or 4n + 3, where n is an integer.

Proof by contradiction:
Assume $\exists n \in \mathbb{N}$, so that 4n+1 is even (and thus also 4n+3 is even)

- ⇨ $\exists k \in \mathbb{N} \mid 4n + 1 = 2k$
- ⇨ $\exists k \in \mathbb{N} \mid 2n + 1/2 = k$

2n is by the definition of multiplication on natural numbers a natural number. But ½ is not a natural number. This is a contradiction to the assumption. So every odd number is one of the forms 4n+1 or 4n+3.

5. Prove that for any integer n, at least one of the integers n, n + 2, n + 4 is divisible by 3.

Proof by contradiction:
Assume n, n+2, n+4 are not divisible by 3, then also n*(n+2)+(n+4) must not be divisible by 3.

$n*(n+2)*(n+4) = (n^2 + 2n) * (n+4)$
$= n^3 + 4n^2 + 2n^2 + 8n$
$= n^3 + 6n^2 + 8n = n*(n^2+8) + 6n^2$

$6n^2$ is always divisible by 3 so the term $n*(n^2+8)$ must not by divisible by 3.
Let n be not divisible by 3. (according to assumption) Then n is one of the form 3k+1 or 3k+2, for a specific $k \in \mathbb{N}$.
Case 3k + 1: $n^2+8 = (3k+1)^2 + 8 = 9k^2 + 6k + 1 + 8 = 3 * (3k^2 + 2k + 3)$, which is divisible by 3
Case 3k + 2: $n^2+8 = (3k+2)^2 + 8 = 9k^2 + 12k + 4 + 8 = 3 * (3k^2 + 4k + 4)$, which is divisible by 3
So for all cases $n*(n^2+8)$ is divisible by 3, which contradicts the assumption. So one of n, n+2, n+4 is always divisible by 3.

6. A classic unsolved problem in number theory asks if there are infinitely many pairs of 'twin primes', pairs of primes separated by 2, such as 3 and 5, 11 and 13, or 71 and 73. Prove that the only prime triple (i.e. three primes, each 2 from the next) is 3, 5, 7.

Prime triples are from the from n, n+2, n+4 for specific $n \in \mathbb{N}$. Using the theorem proven in exercise 5, at least one of those 3 number must be divisible by 3. So the only prime triple can by the one that includes 3 (which is a prime), 3,5,7 .

7. Prove that for any natural number n, 2 + 22 + 23 + . . . + 2n = 2n+1 – 2

Proof by induction:
Let $S(n) = \sum_{i=1}^{n} 2^n$ . By definition: S(n) = $2^{n+1} - 2$
n=1: $S(1) = 2^2 - 2 = 2$
Let $n \in \mathbb{N}$ by given. n+1: $2 + 2^2 + 2^3 + \ldots + 2^{n+1} = \sum_{i=1}^{n} 2^n + 2^{n+1}$
$= 2^{n+1} - 2 + 2^{n+1}$ | by induction hypothesis
$= 2*2^{n+1} - 2 = 2^{n+2} - 2$

Which is the definition of S(n+1). Thus by induction $2 + 2^2 + 2^3 + \ldots + 2^n = 2^{n+1} - 2$, for all $n \in \mathbb{N}$.

8. Prove (from the definition of a limit of a sequence) that if the sequence {an}∞
n=1 tends to limit L as n → ∞, then for any fixed number M > 0, the sequence {M an}∞
n=1 tends to the limit ML.

If {an} tends to the limit L as n→ ∞. Then $\exists\ \varepsilon \in\ \mathbb{R}, so\ that\ |a_n - L| < \epsilon$

= $|M * a_n - M * L| < M * \epsilon$ | * M (as M>0)

Let $\epsilon' = \frac{1}{M} * \epsilon$, then $|M * a_n - M * L| < \epsilon'$

So by the definition of limit and because $\epsilon'$ is arbitrary, M*an tends to M*L for n → ∞.

9. Given an infinite collection An, n = 1, 2, . . . of intervals of the real line, their intersection is defined to be \∞ n=1 An = {x |(∀n)(x ∈ An)}
Give an example of a family of intervals An, n = 1, 2, . . ., such that An+1 ⊂ An for all n and
T∞ n=1 An = ∅. Prove that your example has the stated property.

Let $A_n = \left(0, \frac{1}{n}\right)$.
Clearly all An have the same lower bound. For the upper bound 1/n > 1/(n+1) and thus the upper bound of An > upper bound of An+1. → $A_{n+1} \subset A_n$
As $A_{n+1} \subset A_n$, the intersection of both sets is always s $A_{n+1}$.
For n -> ∞, lim 1/n = 0 ➔ $A_n = (0,0)$ = ∅.

10. Give an example of a family of intervals An, n = 1, 2, . . ., such that An+1 ⊂ An for all n and
T∞ n=1 An consists of a single real number. Prove that your example has the stated property.

Let $A_n = [0, \frac{1}{n}]$.
Clearly all An have the same lower bound. For the upper bound 1/n > 1/(n+1) and thus the upper bound of An > upper bound of An+1. → $A_{n+1} \subset A_n$
As $A_{n+1} \subset A_n$, the intersection of both sets is always s $A_{n+1}$.
For n -> ∞, lim 1/n = 0 → $A_n =$ [0,0] = 0.