Lecture 3 – Asymptotic Analysis and Algorithmic Correctness: Θ/O/Ω, Invariants, and Empirical Validation

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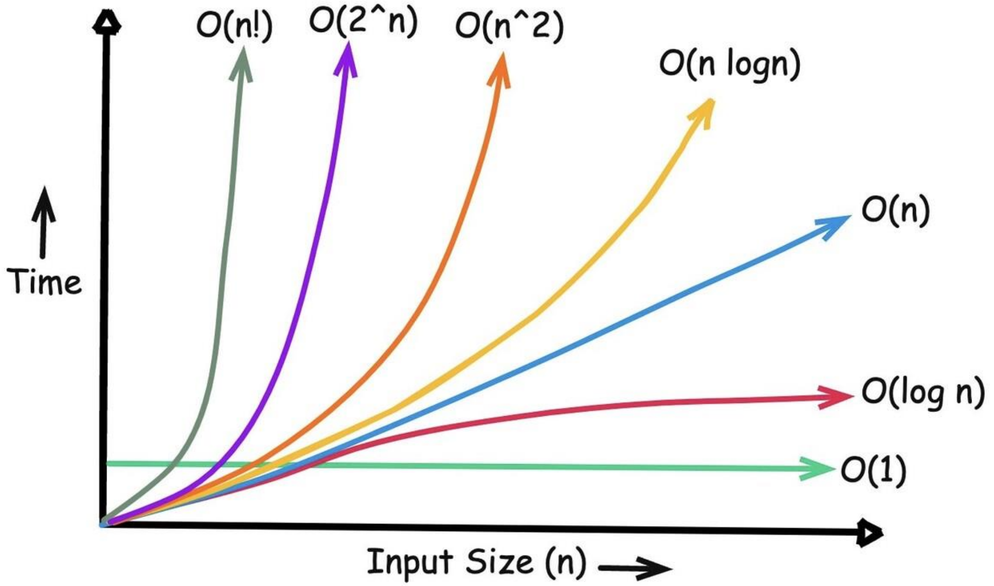
Looking back at the last course



Upper bound on growth rate (worst-case scale behavior)

Compare functions as $n \to \infty$; ignore constants & lower-order terms

$f(n) \in O(g(n)) \Leftrightarrow \exists\ c>0,\ n_0$ s.t. $\forall\ n \geq n_0$: $f(n) \leq c \cdot g(n)$

$7n \to O(n)$; $3n^2 + 10n \to O(n^2)$

$n^3 + 100n \log n + 5 \to O(n^3)$

$O(1)$ (constant), $O(\log n)$, $O(n)$, $O(n \log n)$, $O(n^2)$, $O(2^n)$

Worst-case runtime, space, comparisons, I/O ops (as a function of n)

Example:

- Linear search: $O(n)$
- Binary search: $O(\log n)$
- Insertion sort (worst-case): $O(n^2)$
- Merge sort: $O(n \log n)$

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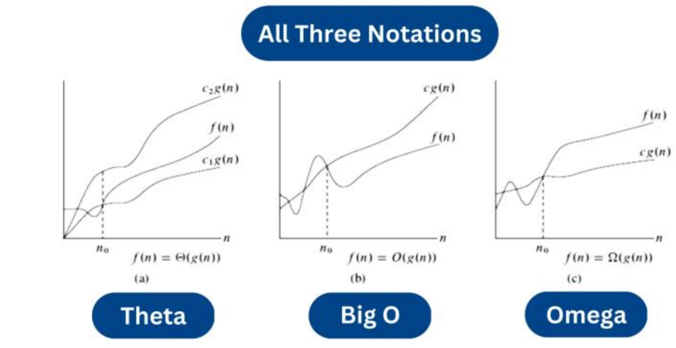
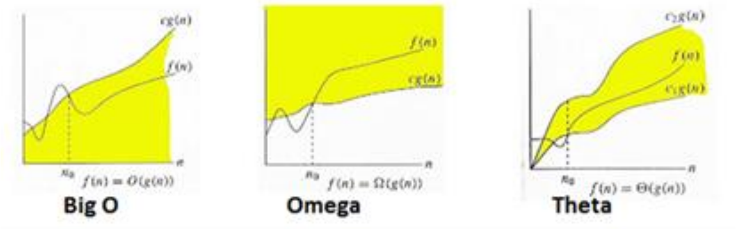
Extending asymptotic notation





O is only an **upper bound,** but we also need **lower** and **tight** bounds.

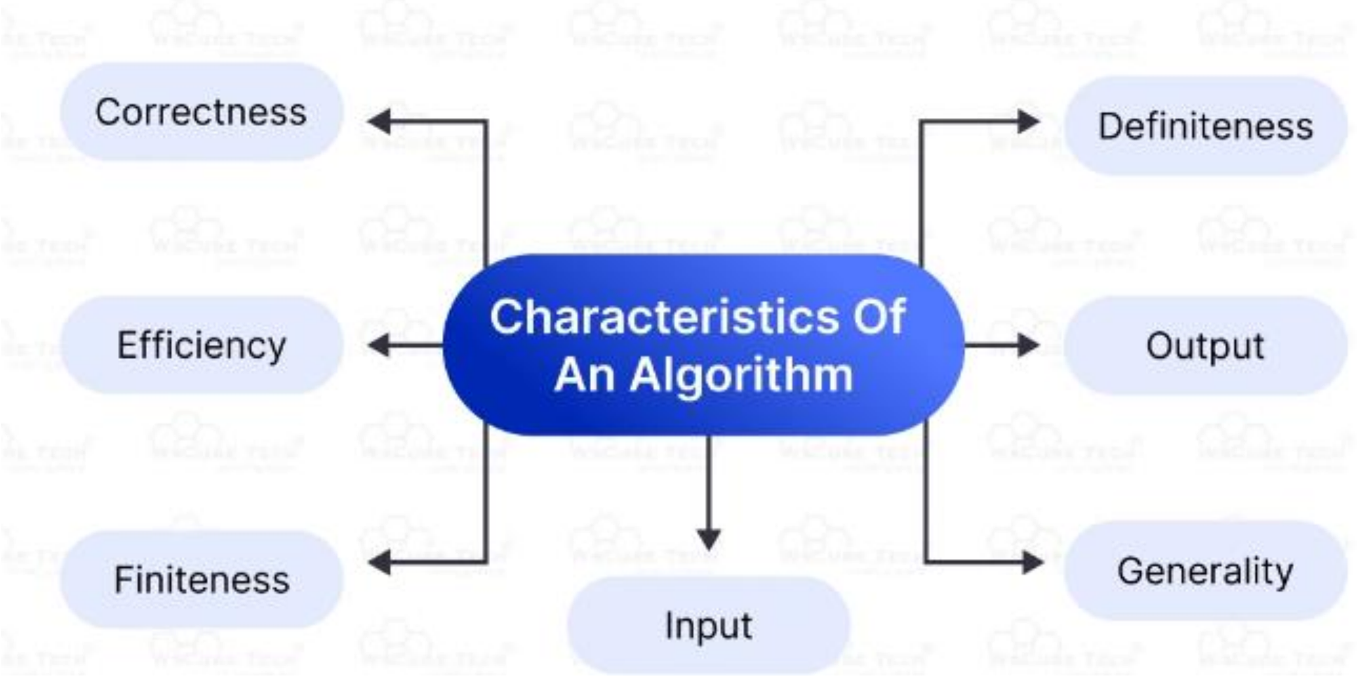
Core set:

- **Ω(g):** lower bound (eventually $\geq c \cdot g$)
- **Θ(g):** tight bound (sandwiched: $c_1 g \leq f \leq c_2 g$)
- **o(g):** strictly smaller; **ω(g):** strictly larger

Understanding:

- **O**: “It won’t be worse than this (eventually).”
- **Ω**: “It can’t be better than this (eventually).”
- **Θ**: “It’s this, up to constants.”

Correctness of algorithms



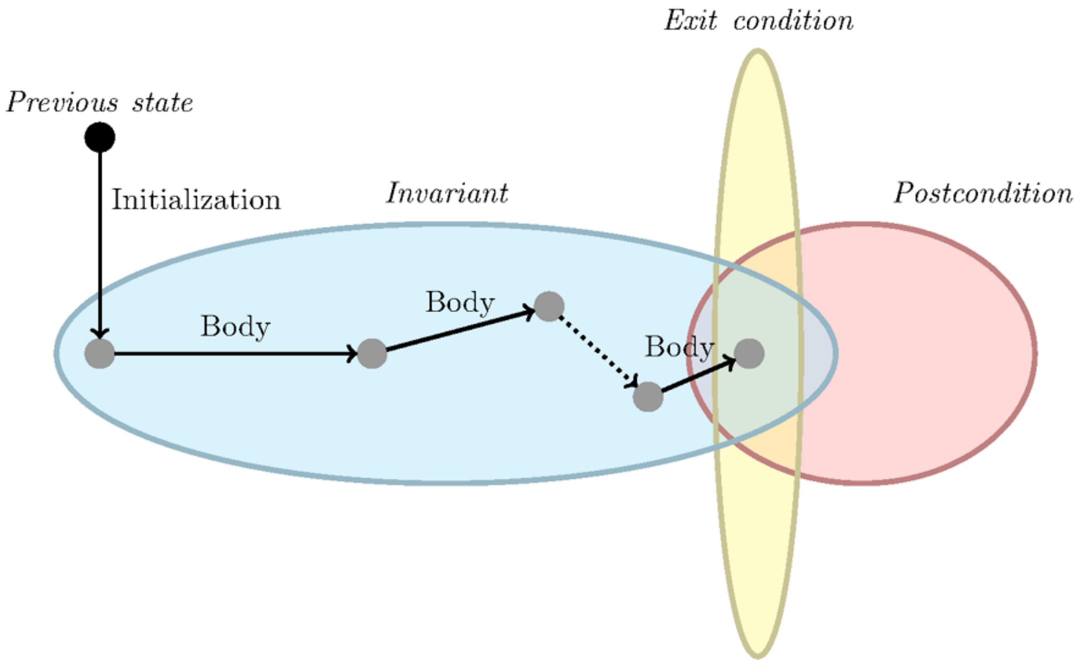
For any allowed input, the algorithm **halts** and the result **meets the spec**.

Write the spec first:

- **Preconditions** – what you assume (e.g., array is sorted, graph is unweighted).
- **Postconditions** – what must be true after (e.g., output is a permutation and is sorted).
- **Effects** – which state may change (no hidden side effects).

Proof obligations:

1. **Initialization** – your invariant holds before the loop/recursion.
2. **Preservation** – one step keeps the invariant true.
3. **Progress/Termination** – some measure strictly decreases (problem size shrinks).
4. **Postcondition** - invariant + exit condition ⇒ required result.



Loop invariants

A loop invariant is a statement about your program's state that is **true before the loop starts** and **remains true after every iteration**.

It captures "what is already correct" at each step, so you can **prove the loop's result** when it finishes.

How it proves correctness:

1. **Initialization:** it's true before the first iteration.
2. **Maintenance:** one iteration keeps it true.
3. **Termination:** when the loop ends, the invariant **plus the exit condition** implies the goal (postcondition).

Formulating a good invariant:

- Describe the **portion already correct** (a sorted prefix)
- Capture **where the work remains** (the unsorted suffix)
- Be **strong enough** to imply the goal at exit, but **simple enough** to check each step

Examples:

- **Insertion Sort:** "Prefix A[0..i] is sorted; all elements belong to final set."
- **Selection Sort:** "Prefix A[0..i-1] holds the i smallest elements in order."

  Binary Search: "If the target exists, it is inside [lo..hi]; each step shrinks this interval."
- **BFS:** "Queue contains the current frontier; discovered nodes have correct levels."

Empirical validation

Empirical validation is the **measurement-based check** that an algorithm's **real-world behavior** matches its **theoretical expectations** (scaling trend, resource use).

Measuring:

- Runtime, CPU time
- Operation counts, allocations/memory
- Cache/branches from profilers

Purpose:

- Confirms or challenges asymptotic predictions in practice
- Reveals **constants, cross-over points, variance**, and **system effects** (cache, JIT/GC, branch prediction, memory layout)
- Builds **credibility and reproducibility** for your results

How to work:

- **Form a hypothesis**: state expected growth (e.g., "~ n log n"), the **case** (worst/average/best), and **metrics** (time, comparisons, memory).
- **Control the environment**: same machine, release/optimized build, stable settings; **warm up** for JITed runtimes; minimize background load.
- **Design inputs**: include **random**, **best/worst**, **adversarial**, and **realistic** datasets; fix **seeds** for reproducibility.
- **Scale & repeat**: grow input size by **doubling**; run **multiple trials** per size; report **median** and **spread**. Avoid I/O in timed code.
- **Analyze trends**: check curve **shape** and **doubling behavior**; optionally normalize (e.g., time per n or per n·log n) to test fit; note **cross-overs**.
- **Report transparently**: list hardware/OS, compiler/JVM and flags, dataset description, seeds, number of trials; include clear plots and a compact table.

Amortized analysis

A method to bound the **average cost per operation over any worst-case sequence** of operations.

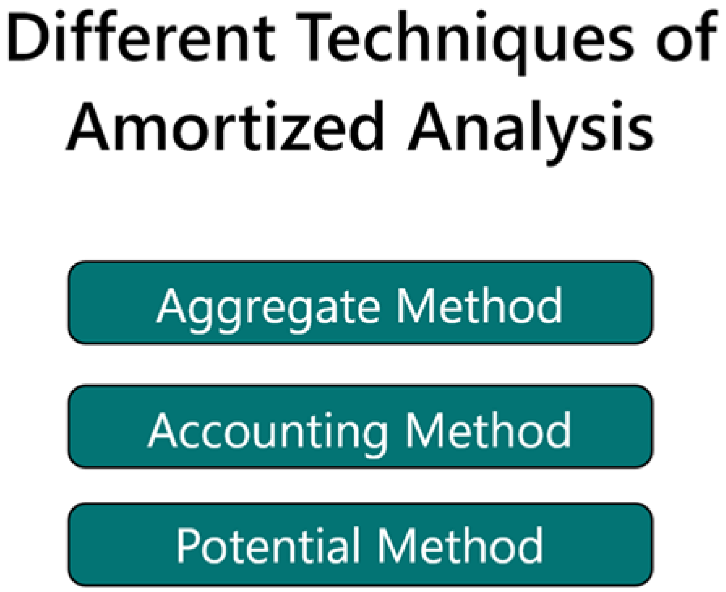
Not probabilistic: **different from average-case** (no input distribution; guarantee holds for every sequence).

Different techniques:

- **Aggregate method:** Bound the **total** cost of *m* operations, then divide by *m*.
- **Accounting method:** Charge some ops extra “credits” that pay for rare expensive ops later; credits never go negative.
- **Potential method:** Define a potential (energy) on the data structure’s state; amortized cost = actual work ± change in potential.

How to work:

1. **Specify the operation sequence & policy** (e.g., array doubles when full; hash table rehashes at load factor α).
2. **Choose a technique** (aggregate/accounting/potential) and a simple **cost model** (e.g., element moves, comparisons).
3. Prove the bound over any sequence:
   - Aggregate: show total work ≤ K·m → amortized ≤ K.
   - Accounting: assign per-op charges so credits cover future expensive steps.
   - Potential: pick a potential that never goes negative and drops when an expensive step happens.
4. **State the result clearly:** worst-case per op vs **amortized** per op; note assumptions (growth factor, load factor).
5. **Check for correctness:** construct adversarial sequences and (optionally) measure to see the predicted average holds.



Literature

Algorithms (4th ed.), Robert Sedgewick & Kevin Wayne

1.4 Analysis of Algorithms – worst-case guarantees, randomized guarantees, and an intro to amortized analysis via resizing arrays

3.2 Binary Search in an Ordered Array – iterative and recursive versions, with correctness notes

Introduction to Algorithms (4th ed.) – Cormen, Leiserson, Rivest, Stein (CLRS)

Ch. 3.1 O-, Ω-, Θ-notation – formal asymptotic notation.

Ch. 2 Insertion Sort – loop invariants and how they prove correctness.

Ch. 2: Best/Worst/Average-case discussion – why worst-case is often the focus.

Algorithms: Design, Techniques, and Analysis – M. H. Alsuwaiyel

1.8 Time complexity – O, Ω, Θ (and related ideas).

1.12 Worst-Case and Average-Case Analyses;

1.13 Amortized Analysis

Grokking Algorithms – Aditya Bhargava (Manning)

Ch. 1 Big-O (intro) and *Ch. 4 Quicksort* – clear average- vs worst-case explanation.

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Thank you