P 120-122

3.1.3 Construct CFGs that generate each of these languages.

(c) $\{w \in \{a, b\}^* : w = w^R\}$.

Solution: (c) $G = (V, \Sigma, R, S)$, where:

$$
\begin{aligned}
V &= \{a, b, S\} \\
\Sigma &= \{a, b\} \\
R &= \{S \to aSa, \\
&\quad S \to bSb, \\
&\quad S \to a, \\
&\quad S \to b, \\
&\quad S \to e\}.
\end{aligned}
$$

3.1.9 Show that the following languages are context-free by exhibiting CFGs generating each.

(a) $\{a^m b^n : m \geq n\}$

(b) $\{a^m b^n c^p d^q : m + n = p + q\}$

Solution: (a) $G = (V, \Sigma, R, S)$, where:

$$
\begin{aligned}
V &= \{a, b, S\} \\
\Sigma &= \{a, b\} \\
R &= \{S \to aSb, \\
&\quad S \to aS, \\
&\quad S \to e\}.
\end{aligned}
$$

(b) Let $m+n=p+q=N$,	then $n=N-m$, $p=N-q$.
	$a^m b^n c^p d^q = a^m b^{N-m} c^{N-q} d^q$
In case of $m \geq q$,	$a^m b^n c^p d^q = a^q a^{m-q} b^{N-m} c^{N-m} c^{m-q} d^q$
Therefore, we can obtain	CFG $G=(V,\Sigma,R,S)$, where:
	$V=\{a,b,c,d,S,A,B\}$
	$\Sigma=\{a,b\}$
	$R=\{S \to aSd,$
	$S \to A,$
	$A \to aAc,$
	$A \to B,$
	$B \to bBc,$
	$B \to e\}$
In case of $m < q$,	we can obtain the similar results.

P 135

3.3.2 Construct pushdown automata that accept each of the following.

(c) The language $\{w \in \{a,b\}^* : w = w^R\}$.

(d) The language $\{w \in \{a,b\}^* : w$ has twice as many b's as a's $\}$.

Solution: (c) $M=(K,\Sigma,\Gamma,\Delta,s,F)$, where:

$$K=\{q,r\}$$
$$\Sigma=\{a,b\}$$
$$\Gamma=\{a,b\}$$
$$F=\{r\}$$
$$\begin{aligned}\Delta=\{&((q,a,e),(q,a))\\ &((q,b,e),(q,b))\\ &((q,e,e),(r,e))\\ &((q,a,e),(r,e))\\ &((q,b,e),(r,e))\\ &((q,a,a),(r,e))\\ &((q,b,b),(r,e))\}\end{aligned}$$

(d) $M = (K, \Sigma, \Gamma, \Delta, s, F)$, where:

$$
\begin{aligned}
K &= \{q\} \\
\Sigma &= \{a, b\} \\
\Gamma &= \{A, a, b\} \\
F &= \{q\} \\
\Delta &= \{((q, a, e), (q, A)) \\
&\quad ((q, b, e), (q, b)) \\
&\quad ((q, a, b), (q, a)) \\
&\quad ((q, b, A), (q, a)) \\
&\quad ((q, b, a), (q, e))\}
\end{aligned}
$$

P142

3.4.1 Carry out the construction of Lemma 3.4.1 for the grammar of Example 3.1.4. Trace the operation of the automaton you have constructed on the input string (()()).

Solution:

The new machine is $M = (\{p, q\}, \{(,)\}, \{(,), S\}, \Delta, p, \{q\})$, where

$$
\begin{aligned}
\Delta = \{&((p, e, e), (q, S), ((q, e, S), (q, SS)), ((q, e, S), (q, (S))), \\
&((q, e, S), (q, e)), ((q, (, (), (q, e)), ((q,),)), (q, e))\}
\end{aligned}
$$

Then

$$
\begin{aligned}
(p, (()()), e) &\vdash_M (p, (()()), S) \\
&\vdash_M (p, (()()), (S)) \\
&\vdash_M (p, ()()), S)) \\
&\vdash_M (p, ()()), SS)) \\
&\vdash_M (p, ()()), (S)S)) \\
&\vdash_M (p,)()), S)S)) \\
&\vdash_M (p,)()),)S)) \\
&\vdash_M (p, ()), S)) \\
&\vdash_M (p, ()), (S))) \\
&\vdash_M (p,)), S))) \\
&\vdash_M (p,)),))) \\
&\vdash_M (p,),) \\
&\vdash_M (p, e, e))
\end{aligned}
$$

P 148

3.5.1 Use closure under union to show that the following languages are context-free.

(b) $\{a, b\}^* - \{a^n b^n : n \geq 0\}$
(c) $\{a^m b^n c^p d^q : n = q,$ or $m \leq p$ or $m + n = p + q\}$
(d) $\{a, b\}^* - L$ where L is the language

$$L = \{babaabaaab \ldots ba^{n-1}ba^n b : n \geq 1\}$$

Solution:
(b) This language cab be expressed as $\{a^m b^n \mid m \neq n\} \cup \Sigma^* a \Sigma^* b \Sigma^* a \Sigma^* \cup \Sigma^* b \Sigma^* a \Sigma^* b \Sigma^*$, all three of which are context free, so that their union is also context-free.

(c) This language can be expressed as $L_1 \cup L_2 \cup L_3$, where

$$L_1 = \{a^* b^n c^* d^n : n \geq 0\}$$
$$L_2 = \{a^m b^* c^n d^* : m \leq n\}$$
$$L_3 = \{a^m b^n c^p d^q : m + n = p + q\}$$

Each of these language can easily be recognized by a PDA created by modifying the one which accepts $\{a^n b^n : n \geq 0\}$.

(d) The language is $\Sigma^* bb \Sigma^* \cup \Sigma^* \{ba^n ba^m : n + 1 \neq m\} \Sigma^* \cup a \Sigma^* \cup \Sigma^* a$.
Each of these languages is context-free, so is their union.

3.5.2 Use Theorem 3.5.2 and 3.5.3 to show that the following languages are not context-free.
(c) $\{www : w \in \{a, b\}^*\}$

Solution: (c) Assume L is context-free, So the pumping theorem must hold.

Then there is a number $k > 0$ such that for any $w \in L$ such that $|w| \geq k$ there exist $u, v, x, y, z \in \Sigma^*$ such that $w = uvxyz$, $|vxy| \leq k, vy \neq e$, and $uv^n xy^n z \in L$ for all $n \geq 0$.

Consider the string $w = a^k ba^k ba^k b$. This string is in L and satisfies $|w| \geq k$.

By our assumption, $u, v, x, y,\ z$ exist as above. Neither v nor y can contain more than one b.
— this follows from the fact that $|vxy| \leq k$, so in particular $|v|, |y| \leq k$. So each cannot contain more than one b.
In fact, neither v nor y can contain any instance of b at all.

1) Suppose, without loss of generality, the v contained a b. Then uv^2xy^2z contains four occurrences of b and hence certainly cannot be in L.

2) Similarly, if v and y each contained a b, the string uv^2xy^2z would have five instances of b and by same reasoning could not be in L.

3) So the only case remaining is $v, y \in L(a^*)$. Suppose $v = a^p$, $y \in a^q$, where $p, q \leq k$. Without loss of generality, let us consider the case when v is in the first set of a's and y is in the second set of a's. Then $uv^2xy^2z = a^{k+p}ba^{k+q}ba^kb$, which can not be in L, since at least one of p and q must be nonzero.

Having exhausted all possible cases, we conclude that the context-free pumping fail on w and hence L cannot be context-free.

3.5.14 Which of the following languages are context-free? Explain briefly in each case.

$a)$ $\{a^m b^n c^p \mid m = n$ or $n = p$ or $m = p\}$
$b)$ $\{a^m b^n c^p \mid m \neq n$ or $n \neq p$ or $m \neq p\}$
$c)$ $\{a^m b^n c^p \mid m = n$ and $n = p$ and $m = p\}$
$d)$ $L = \{w \in \{a, b, c\}^* : w$ does not contain equal numbers of occcurrences of a, b and $c\}$
$e)$ $L = \{w \in \{a, b\}^* \mid w = w_1 w_2 \cdots w_m$ for some $m \geq 2$ and $w_1, w_2, \cdots, w_m$ such that $|w_1| = |w_2| = \cdots = |w_m| \geq 2\ \}$

Solution:

(a) **This language is context-free.** Since it can be represented as

$$\{a^n b^n c^m : m, n \in \mathbb{N}\} \cup \{a^n b^m c^n : m, n \in \mathbb{N}\} \cup \{a^m b^n c^n : m, n \in \mathbb{N}\}$$

Each of which, being essentially the language $\{a^n b^n : n \in \mathbb{N}\}$ is context-free.

(b) **This language is context-free.** Since it can be represented as

$$\{a^m b^n c^p : m \neq n\} \cup \{a^m b^n c^p : n \neq p\} \cup \{a^m b^n c^p : m \neq p\}$$

Each of which, being essentially the language $\{a^m b^n : m \neq n\}$ is context-free.

(c) **This language is not context-free.** Since this language is same as language

$$\{a^n b^n c^n : n \in \mathbb{N}\}$$

that was shown not to be context-free by pumping theorem.

(d) **This language is context-free.** Since it can be represented as

$$\{w \in \{a,b,c\}^* : w \text{ has different numbers of } a\text{'s and } b\text{'s}\}$$
$$\cup \{w \in \{a,b,c\}^* : w \text{ has different numbers of } b\text{'s and } c\text{'s}\}$$
$$\cup \{w \in \{a,b,c\}^* : w \text{ has different numbers of } a\text{'s and } c\text{'s}\}.$$

each of which is context-free.

(e) **This language is context-free.**

– This language is really just the set of all strings whose lengths are composite.

It is not context-free for much the same reason that the language of strings of prime length is not context-free:

the "gaps" represented by prime numbers do not follow a simple enough pattern to be context-free.

To see this in action, first apply the homomorphism $h(a) = h(b) = a$. The resulting language — which must be context-free if the original language was(by P148 Ex 3.5.3) — is the language of strings of composite length over the alphabet $\{a\}$.

If this language is CFL, it would be regular, as any CFL over an alphabet of a single symbol is regular [noted on P 147].

If this language were regular, its complement — $\{a^p : p \text{ is prime number}\}$ — would also be regular.

But this language is not even context-free, so we have a contradiction to the assumption that the original language was context-free.

3.5.15 Suppose that L is context-free and R is regular. Is $L - R$ necessarily context-free? What about $R - L$? Justify your answers.

Solution:

1) $L - R$ is context-free.

$L - R = L \cap \bar{R}$ and Theorem 3.5.2 (P144)

2) $R - L$ need not be context-free.

If we restrict to the case $R = \Sigma^*$ (certainly a regular language!) then $R - L = \overline{L}$. Since we know that the class of context-free language is not closed under complementation.