

Math2

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1 W1C1

Permutation

$$nx(n-1)x\dots x(n-k+1) = \frac{n!}{n-k!} =: P_k^n$$

This is the number of ways to choose and arrange k distinct objects from n distinct objects (order of picking matters). This is because we want them to fill up some unique positions.

permutation without repetition if there are identical objects then we must divide by the number of identical factorial

$$\frac{n!}{n_1!n_2!\dots n_r!}$$

where n_r is the number of identical objects

If order does not matter when we pick our objects then we use

$$\binom{n}{k} = C_k^n = \frac{P_k^n}{k!} = \frac{n!}{k!(n-k)!}$$

Remember the slotting method when you are arranging objects such that no 2 of a type are beside each other.

divide n distinct objects into r diff piles of size n_1, n_2

$$\binom{n}{n_1} \binom{n-n_1}{n_2} = \frac{n!}{n_1!n_2!\dots n_r!}$$

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$$(a+b)^n = \sum_{k=0}^n \binom{n}{k} a^k b^{n-k}$$

counting given n distinct objects find the arrangements of k of them where order does not matter but repetitions are allowed

Multichoose

$$\binom{k+n-1}{n-1}$$

$n-1$ is number of bars

we can think of it as having $k+n-1$ objects and $n-1$ will transform into the bars.

Circular symmetry

$$\frac{n!}{n}$$

1. Remember when counting 2 separate cases make sure they are unique before adding unless minus the intersection
2. Check complements sometimes, its easier to do
3. when you splitting into equal unlabeled piles rmb to divide by $n!$, n being the number of piles after doing choosing.
4. alternative general formula $\frac{n!}{h!^k k!}$

Extra:

Multinomial Theorem

$$(x_1 + x_2 + x_3 \dots x_k)^n = \sum_{B_1 + B_2 \dots B_k = n} \binom{n}{b_1, b_2, b_3, \dots, b_k} \prod_{j=1}^k x_j^{b_j}$$

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Axioms of probability

1. $0 \leq P(E) \leq 1$
2. $P(S)=1$
3. $P(\bigcup_{i=1}^{\infty} E_i) = \sum_{i=1}^{\infty} P(E_i)$

Axiom 3 finite version

$$\mathbb{P}(\bigcup_{i=1}^n E_i) = \sum_{i=1}^n \mathbb{P}E_i$$

inclusion exclusion

$$\mathbb{P}(A \cup B) = \mathbb{P}A + \mathbb{P}B - \mathbb{P}A \cap B$$

$$\mathbb{P}(A \cup B \cup C) = \mathbb{P}(A) + \mathbb{P}(B) + \mathbb{P}(C) - \mathbb{P}(A \cap B) - \mathbb{P}(B \cap C) - \mathbb{P}(C \cap A) + \mathbb{P}(A \cap B \cap C)$$

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Hypergeometric distribution

$$\mathbb{P}(r \text{ men on a committee of } k \text{ people}) = \frac{\binom{n}{r} \binom{m}{k-r}}{\binom{n+m}{k}}$$

Arithmetic Series

$$1 + 2 + 3 + \dots + n = \frac{n(n+1)}{2}$$

Taylor Series for e^x

$$\sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x$$

Geometric Series

$$1 + r + r^2 + \dots + r^{n-1} = \frac{1 - r^n}{1 - r}$$

if $|r| < 1$ and $n \rightarrow \infty$,

$$1 + r + r^2 + \dots = \frac{1}{1 - r}$$

$$P(d, n) = 1 - (1 - \frac{1}{d})(1 - \frac{2}{d}) \dots (1 - \frac{n-1}{d})$$

this equations to

$$P(d, n) \approx 1 - e^{1/d} e^{-2/d} \dots e^{-(n-1)/d}$$

when using the identity $e^{-x} \approx 1 - x$ Derangement

$$D_n = n! (1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \dots + (-1)^n \frac{1}{n!})$$

$$D_n = (n-1)(D_{n-1} + D_{n-2})$$

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Definition(conditional Probability)

if $P(B) > 0$, then

$$\mathbb{P}(A|B) := \frac{\mathbb{P}(A \cap B)}{\mathbb{P}(B)}$$

hence

$$\mathbb{P}(A \cap B) = \mathbb{P}(B)\mathbb{P}(A|B) = \mathbb{P}(A)\mathbb{P}(B|A)$$

which allows us to multiply probabilities

interesting result which is proven from the definition of conditional probability

The answers are different!

There are 6 equally likely ways to draw 2 cards from the deck.

If at least one card is an ace, there are 5 equiprobable events.

If one card is the ace of spades, there are 3 equiprobable events.

A♠, A♣	A♠, A♣	A♠, A♣
A♠, 2♠	A♠, 2♠	A♠, 2♠
A♠, 2♣	A♠, 2♣	A♠, 2♣
A♣, 2♠	A♣, 2♠	
A♣, 2♣	A♣, 2♣	
2♠, 2♣		

Pr(both aces | at least one ace) = 1/5

Law of total probability

$$\mathbb{P}(A) = \mathbb{P}(B_1)\mathbb{P}(A|B_1) + \mathbb{P}(B_2)\mathbb{P}(A|B_2) + \dots + \mathbb{P}(B_n)\mathbb{P}(A|B_n)$$

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Bayes' Law(Bae's)

$$\mathbb{P}(A|B) = \frac{\mathbb{P}(B|A)\mathbb{P}(A)}{\mathbb{P}(B)}$$

Independent

$$\mathbb{P}(A|B) = \mathbb{P}(A)$$

$$\mathbb{P}(B|A) = \mathbb{P}(B)$$

$$\mathbb{P}(A \cap B) = \mathbb{P}(A)\mathbb{P}(B)$$

Three events are independent if they are pairwise independent and

$$\mathbb{P}(A \cap B \cap C) = \mathbb{P}(A)\mathbb{P}(B)\mathbb{P}(C)$$

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$$\mathbb{P}_B(A) := \mathbb{P}(A|B)$$

E_1 and E_2 are said to be conditionally independent if

$$\mathbb{P}_B(E_1 \cap E_2) = \mathbb{P}_B(E_1)\mathbb{P}_B(E_2)$$

or

$$\mathbb{P}((E_1 \cap E_2)|B) = \mathbb{P}(E_1|B)\mathbb{P}(E_2|B)$$

Geometric distribution:

Probability first tail appear on the n toss

$$(1-p)^{n-1}p$$

Binomial distribution

Toss a weighted coin (with $\mathbb{P}(T) = P$) n times; what is the probability exactly n tails are obtained

$$\mathbb{P}(\text{exactly } k \text{ tails}) = \binom{n}{k} p^k (1-p)^{n-k}$$

A before B

$$\begin{aligned} \mathbb{P}(A \text{ before } B) &= a + ca + c^2a + \dots \\ &= \frac{a}{1-c} = \frac{a}{a+b} \end{aligned}$$

Player A needs n points to win and player B needs m points to win

First we assume the maximum length of game to n+m-1 is played and then we limit this to a binomial distribution that counts the number of games that are n points or greater.

This is because as long as A wins n games then B cannot possibly win enough games to win and hence this is equivalent to A winning the normal unextended game. It includes all combinations of how the arrangement of how A can get N games first before B wins.

Those that he wins already and continues playing will cancel out each other. $(1-p)(p)=1$

$$P(n, m) = \sum_{k=n}^{m+n-1} \binom{m+n-1}{k} p^k (1-p)^{m+n-1-k}$$

Gamblers Ruin

First try to make the question into a kinda recursion based form.

Then simplify it using the extremes.

$$P_k = pP_{k+1} + qP_{k-1}$$

$$pP_k + qP_k = pP_{k+1} + qP_{k-1}$$

$$P_{k+1} - P_k = \frac{q}{p}(P_k - P_{k-1})$$

$$P_k - P_1 = P_1 \left[\frac{q}{p} + \frac{q^2}{p^2} + \dots + \frac{q^{k-1}}{p^{k-1}} \right]$$

sub $P_N=1$

$$P_k = \frac{1 - (q/p)^k}{1 - (q/p)^N}$$

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random variable is a variable that represents each outcome of a random experiment by a number.

i.e. No. of heads after 10 coin tosses

Probability mass function (pmf) of X is given by:

$$p(a) := \mathbb{P}(X = a)$$

cdf (cumulative distribution function) is defined as

$$F(x) := \mathbb{P}(X \leq x) = \sum_{all a \leq x} p(a)$$

Is a summation of all pmf below and equal to a maximum of 3 numbers obtained from 3 fair dice roll

$$\mathbb{P}(M \leq i) = \frac{i^3}{6^3}$$

$$\begin{aligned} p(i) &= \mathbb{P}(M = i) = \mathbb{P}(M \leq i) - \mathbb{P}(M \leq i - 1) \\ &= \frac{i^3}{6^3} - \frac{(i - 1)^3}{6^3} \end{aligned}$$

Expected value is the mean. (sum of all values divide by n)

Mean is sensitive to extremely large and small values while median is not

Median is the sorted center value

$$E(X) = \sum_{i=1}^n x_i \mathbb{P}(x = x_i) = \sum_{i=1}^n x_i p(x_i)$$

Theorem 1 of random variable

$$E(g(x)) = \sum_{i=1}^n g(x_i) p(x_i)$$

A Useful result is

$$E(aX + b) = a\mathbb{E}(X) + b$$